Ph21 Assignment 2

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1 Part I

1.

$$\begin{aligned}
\tilde{h}_k &= \frac{1}{L}\int_0^L \left(\sum_{k'=-\infty}^{\infty} \tilde{h}_{k'} e^{2\pi i (f_k - f_k')x} \right) \ dx \\
&= \frac{1}{L}\int_0^L \tilde{h}_k \ dx + \frac{1}{L}\int_0^L \left(\sum_{\substack{k'=-\infty \\ k'\neq k}}^{\infty} \tilde{h}_{k'} e^{(2\pi i (f_k - f_{k'})x} \right) \ dx \\
&= \frac{1}{L}\int_0^L \tilde{h}_k \ dx + \sum_{\substack{k'=-\infty \\ k'\neq k}}^{\infty} \left(\frac{1}{L}\int_0^L \tilde{h}_{k'} e^{(2\pi i (f_k - f_{k'})x} \right) \ dx \\
&= \tilde{h}_k
\end{aligned}$$

2.

$$A\sin\left(\frac{2\pi x}{L} + \cos\phi\right) = A\left[\sin\left(\frac{2\pi x}{L}\right)\cos\phi + \sin\phi\cos\left(\frac{2\pi x}{L}\right)\right]$$

$$\cos\left(\frac{2\pi x}{L}\right) = \frac{1}{2}\left(e^{\frac{2\pi i x}{L}} + e^{-\frac{2\pi i x}{L}}\right) \ , \qquad \sin\left(\frac{2\pi x}{L}\right) = \frac{1}{2i}\left(e^{\frac{2\pi i x}{L}} - e^{-\frac{2\pi i x}{L}}\right)$$

$$\begin{aligned}
A\sin\left(\frac{2\pi x}{L} + \cos\phi\right) &= \frac{A}{2}\sin\phi\left(e^{\frac{2\pi i x}{L}} + e^{-\frac{2\pi i x}{L}}\right) + \frac{A}{2i}\cos\phi\left(e^{\frac{2\pi i x}{L}} - e^{-\frac{2\pi i x}{L}}\right) \\
&= \frac{A}{2}\left[(\sin\phi - i\cos\phi) \ e^{\frac{2\pi i x}{L}} + (\sin\phi + i\cos\phi) \ e^{-\frac{2\pi i x}{L}}\right]
\end{aligned}$$

3.

$$\begin{aligned}
\tilde{h}_k^* &= \frac{1}{L}\overline{\int_0^L h(x)\ e^{2\pi i f_k x}\ dx} \\
&= \frac{1}{L}\int_0^L \overline{h(x)\ e^{2\pi i f_k x}}\ dx \\
&= \frac{1}{L}\int_0^L \overline{h(x)}\ \overline{e^{2\pi i f_k x}}\ dx \\
&= \frac{1}{L}\int_0^L h(x)\ e^{-2\pi i f_k x}\ dx \\
&= \tilde{h}_{-k}
\end{aligned}$$

4.

$$H(x) = \sum_{k=-\infty}^{\infty} \tilde{H}_k \cdot e^{-2\pi i f_k x}\ ,\ \ f_k = k/L \tag{1}$$

$$h_{k''}^{(1)}(x) = \sum_{k''=-\infty}^{\infty} \tilde{h}_{k''}^{(1)} \cdot e^{-2\pi i f_{k''} x}\ ,\ \ f_{k''} = k''/L$$

$$h_{k'}^{(2)}(x) = \sum_{k'=-\infty}^{\infty} \tilde{h}_{k'}^{(2)} \cdot e^{-2\pi i f_{k'} x}\ ,\ \ f_{k'} = k'/L$$

$$\begin{aligned}
H(x) &= h_{k''}^{(1)}(x) \cdot h_{k'}^{(2)}(x) \\
&= \left(\sum_{k''=-\infty}^{\infty} \tilde{h}_{k''}^{(1)} \cdot e^{-2\pi i f_{k''} x}\right)\left(\sum_{k'=-\infty}^{\infty} \tilde{h}_{k'}^{(2)} \cdot e^{-2\pi i f_{k'} x}\right) \\
&= \sum_{k''=-\infty}^{\infty}\sum_{k''=-\infty}^{\infty} \tilde{h}_{k'}^{(1)} \cdot \tilde{h}_{k'}^{(2)}\ e^{-2\pi i f_{k''+k'} x}
\end{aligned}$$

Let $k = k'' + k'$, then $k'' = k - k'$ and the sum still runs from $-\infty$ to ∞. We get

$$H(x) = \sum_{k=-\infty}^{\infty}\left(\sum_{k'=-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)} \cdot \tilde{h}_{k'}^{(2)}\right)\ e^{2\pi i f_k x}. \tag{2}$$

Equating the coefficients of (1) and (2), we have

$$\tilde{H}_k = \sum_{k'=-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)} \cdot \tilde{h}_{k'}^{(2)}$$

5. (a)

$$f(t) = A\cos(ft + \phi) + C$$

$$
\begin{aligned}
\mathcal{F}[f(t)] &= \tilde{f}(k) \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)\ e^{2\pi i f_k t}\ dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)\ [\cos(2\pi f_k t) + i\sin(2\pi f_k t)]\ dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)\ [\cos kt + i\sin kt]\ dt
\end{aligned}
$$

Since $\int_0^{2\pi}\sin(mt)\sin(nt)\ dt = \int_0^{2\pi}\cos(mt)\cos(nt)\ dt = \pi\delta_{mn}$ and $\int_0^{2\pi}\sin(mt)\cos(nt)\ dt = 0$ for $m, n \subset \mathbb{Z}$, the only coefficient surviving is at $k = f$,

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}[A\cos(ft+\phi)+C]\cos(ft)\ dt &= \frac{1}{2\pi}\int_0^{2\pi} A[\cos\phi\cos^2(ft) - \sin\phi\sin(ft)\cos(ft)] + C\cos(ft)\ dt \\
&= A\pi\cos\phi + [C\sin(ft)]_0^{2\pi} \\
&= \frac{A\cos\phi}{2}.
\end{aligned}
$$

This gives a peak at $k = f$ with amplitude $A\cos\phi/2$.

For the special case $A = 3, C = 1, \phi = 0, f = 2$, we get $\tilde{f}(2) = 1.5$.

(b)

$$
\begin{aligned}
f(t) &= Ae^{-B(t-\frac{L}{2})^2} \\
\mathcal{F}[f(t)] = \tilde{f}(k) &= \frac{A}{\sqrt{2B}}\ e^{-\frac{k^2}{4B}+\frac{iLk}{2}}\ \text{(another Gaussian)}
\end{aligned}
$$

For the special case $A = 5, B = 10, L = 1$, this gives $\tilde{f}(k) = \frac{5}{\sqrt{10}}\ e^{-\frac{k^2}{40}+\frac{ik}{2}}$.

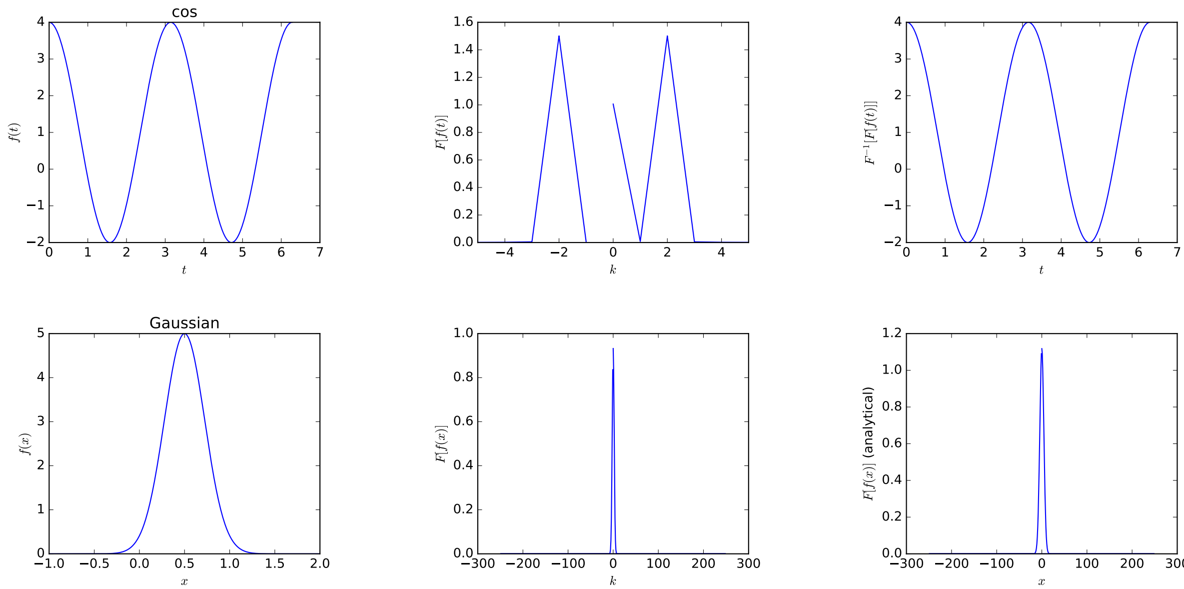

Figure 1: Plots of $A\cos(ft+\phi)$ and the Gaussian $Ae^{-B(t-L/2)^2}$ with their FFTs. For the cosine, $A = 3$, $f = 2$ and $\phi = 0$; for the Gaussian, $A = 5$ and $B = 10$.

2 Part II

1. Signal frequency = 137.02 Hz.
2. The FFT of the first dataset can be interpreted as the Fourier Transform of a perfect sinusoid mulitplied by a Gaussian envelope of the form $A\sin(ft+\phi)\,e^{-(t-t_0)^2/\Delta t^2}$. By using `scipy.optimize.curve_fit`, the optimal parameters were found to be:

$$\begin{aligned} t_0 &= 2.364 \text{ s} \\ \Delta t &= 1.084 \text{ s} \\ A &= 0.101 \\ f &= -0.620 \\ \phi &= 4.647. \end{aligned}$$

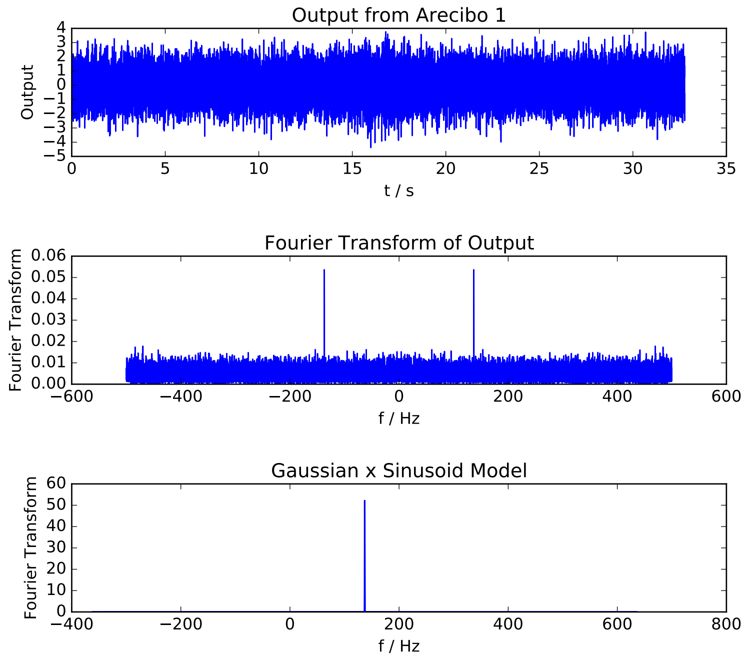

Figure 2: Plots of the output from Arecibo 1, FFT of the output, and FFT of a perfect sinusoid multiplied by a Gaussian envelope, $A\sin(ft+\phi)\ e^{-(t-t_0)^2/\Delta t^2}$, with $t_0 = 2.362$ ms, $\Delta t = 1.084$ ms, $A = 0.101$, $f = -0.620$, $\phi = 4.647$.

3 Part III

1. Other significant frequencies that might arise in astronomical data from an observatory might be due to:

   (a) Dark currents, caused by thermally excited electrons and is temperature dependent;

   (b) Electronic switching noise from digital electronics and power supply. This noise enters the CCD camera circuitry by radiation or by being conducted through the electronic connections;

   (c) Pixel non-uniformity as each pixel has a slightly different sensitivity to light.

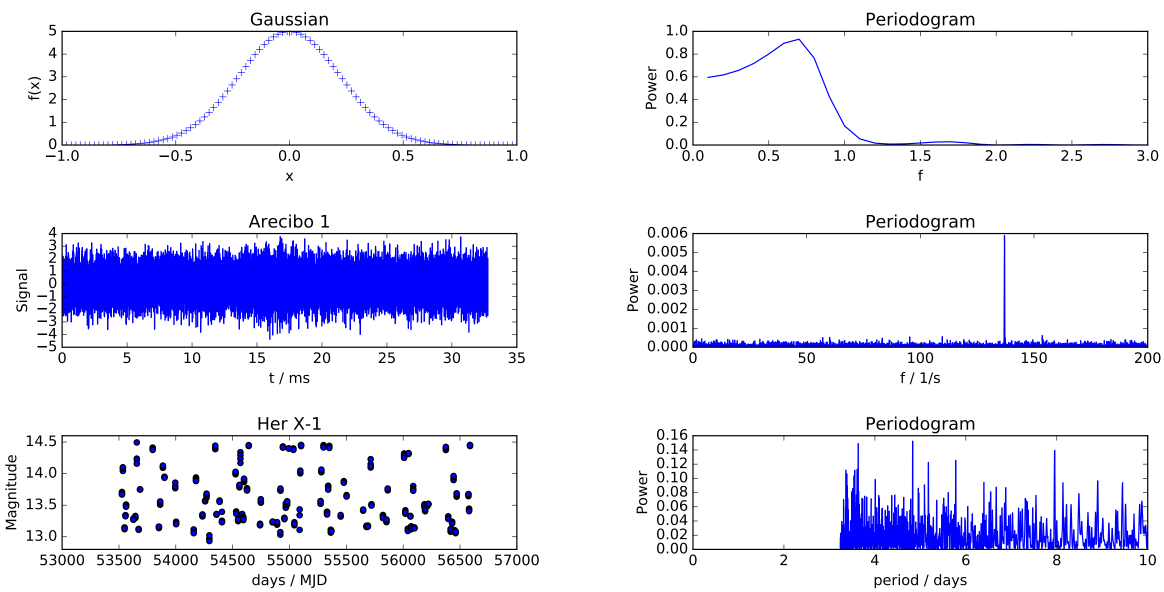

Figure 3: Fourier transforms of a Gaussian, the data from Arecibo 1 and data from Her X-1 computed with the Lomb-Scargle algorithm. The Gaussian is of the form Ae^{-Bx^2} with $A = 5$ and $B = 10$.