Mathematical Preliminaries

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Vectors and Norms

- Real numbers: $\mathcal{R}, \quad \mathcal{R}_+, \quad \text{int}\,\mathcal{R}_+$
- n-dimensional Euclidean space $\mathcal{R}^n, \quad \mathcal{R}^n_+, \quad \text{int}\,\mathcal{R}^n_+$
- Component-wise: $\mathbf{x} \geq \mathbf{y}$ means $x_j \geq y_j$ for $j = 1, 2, ..., n$
- $\mathbf{0}$: vector of all zeros; and $\mathbf{e}$: vector of all ones
- Inner-product of two vectors:

$$\mathbf{x} \bullet \mathbf{y} := \mathbf{x}^T\mathbf{y} = \sum_{j=1}^{n} x_j y_j$$

- Euclidean norm: $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}}$,

  Infinity-norm: $\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, ..., |x_n|\}$,

  p-norm: $\|\mathbf{x}\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}$

- The dual of the p norm, denoted by $\|.\|^*$, is the q norm, where $\frac{1}{p}+\frac{1}{q}=1$
- Column vector:

$$\mathbf{x}=(x_1;x_2;\ldots;x_n)$$

Row vector:

$$\mathbf{x}=(x_1,x_2,\ldots,x_n)$$

- Transpose operation: A^T
- A set of vectors $\mathbf{a}_1$, ..., $\mathbf{a}_m$ is said to be linearly dependent if there are scalars λ_1, ..., λ_m, not all zero, such that the linear combination

$$\sum_{i=1}^{m}\lambda_i\mathbf{a}_i=\mathbf{0}$$

- A linearly independent set of vectors that span R^n is a basis.

Hyper plane and Half-spaces

$$H = \{\mathbf{x} : \ \mathbf{ax} = \sum_{j=1}^{n} a_j x_j = b\}$$

$$H^{+} = \{\mathbf{x} : \ \mathbf{ax} = \sum_{j=1}^{n} a_j x_j \leq b\}$$

$$H^{-} = \{\mathbf{x} : \ \mathbf{ax} = \sum_{j=1}^{n} a_j x_j \geq b\}$$

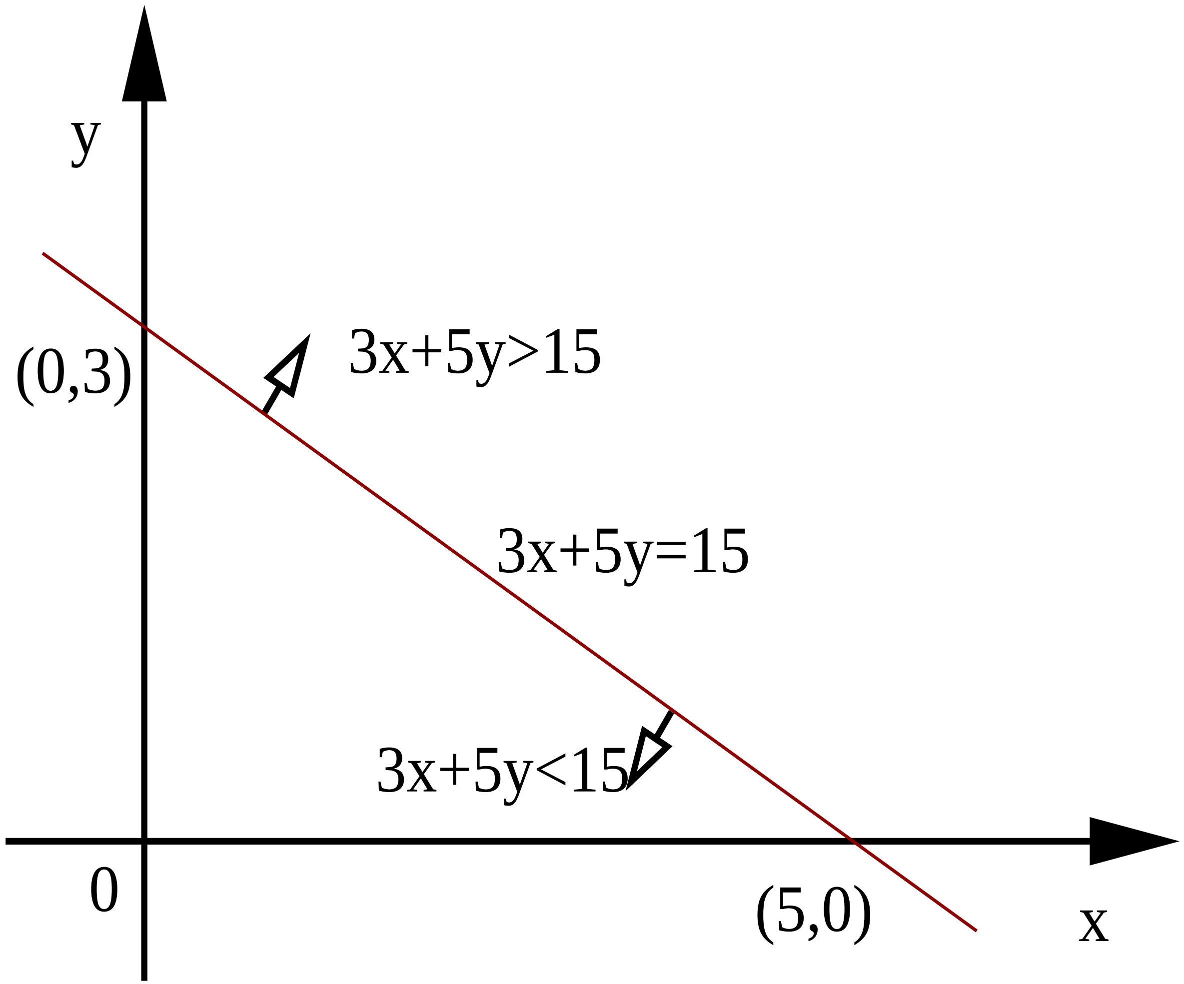

Figure 1: Plane and Half-Spaces

System of Linear Equations

Solve for $\mathbf{x} \in \mathcal{R}^n$ from:

$$\left.\begin{array}{rcl} \mathbf{a}_1\mathbf{x} & = & b_1 \\ \mathbf{a}_2\mathbf{x} & = & b_2 \\ \dots & . & . \\ \mathbf{a}_m\mathbf{x} & = & b_m \end{array}\right. \quad \Rightarrow A\mathbf{x} = \mathbf{b}$$

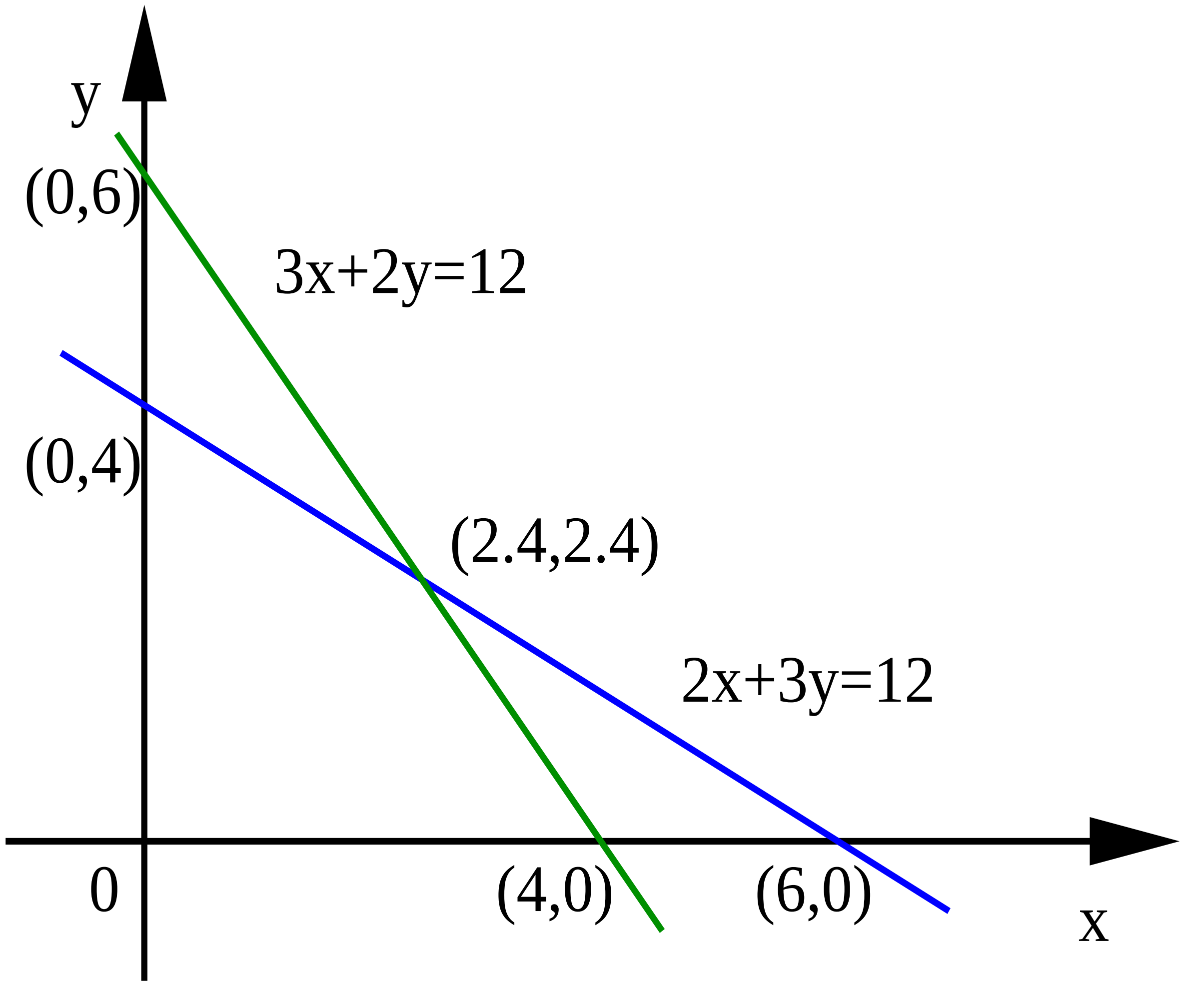

Figure 2: System of Linear Equations

Fundamental theorem of linear equations

Theorem 1 *Let $A \in \mathcal{R}^{m \times n}$ and $\mathbf{b} \in \mathcal{R}^m$. The system $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}\}$ has a solution if and only if that $A^T\mathbf{y} = \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} \neq 0$ has no solution.*

A vector $\mathbf{y}$, with $A^T\mathbf{y} = 0$ and $\mathbf{b}^T\mathbf{y} \neq 0$, is called an infeasibility certificate for the system.

It is also called the alternative system theorem, that is, exactly one of the two systems, $\{\mathbf{x} : \ A\mathbf{x} = \mathbf{b}, \}$ and $\{\mathbf{y} : \ A^T\mathbf{y} = \mathbf{0}, \ \mathbf{b}^T\mathbf{y} \neq 0\}$, is feasible.

Example Let $A = (1; -1)$ and $\mathbf{b} = (1; 1)$. Then, $\mathbf{y} = (1/2; 1/2)$ is an infeasibility certificate.

Gaussian elimination method

$$
\begin{pmatrix} a_{11} & A_{1.} \\ 0 & A' \end{pmatrix} \begin{pmatrix} x_1 \\ x' \end{pmatrix} = \begin{pmatrix} b_1 \\ b' \end{pmatrix}.
$$

$$
A = L \begin{pmatrix} U & C \\ 0 & 0 \end{pmatrix}
$$

Matrices and Norms

- Matrix: $\mathcal{R}^{m\times n}$, ith row: $a_{i.}$, jth column: $a_{.j}$, ijth element: a_{ij}
- A_I denotes the submatrix of A whose rows belong to index set I, A_J denotes the submatrix whose columns belong to index set J, A_{IJ} denotes the submatrix whose rows belong to index set I and columns belong to index set J.
- Determinant: $\det(A)$, Trace: tr(A)
- The operator norm of $\|A\|$,

$$\|A\|^2 := \max_{0\neq x\in\mathcal{R}^n} \frac{\|Ax\|^2}{\|x\|^2}$$

- All-zero matrix: $\mathbf{0}$, and identity matrix: I
- Diagonal matrix: $X = \text{diag}(\mathbf{x})$

- Symmetric matrix: $Q = Q^T$
- Positive Definite: $Q \succ 0$ iff $\mathbf{x}^T Q \mathbf{x} > 0, \quad \text{for all} \quad \mathbf{x} \neq \mathbf{0}$
- Positive Semidefinite: $Q \succeq 0$ iff $\mathbf{x}^T Q \mathbf{x} \geq 0, \quad \text{for all} \quad \mathbf{x}$
- Null space: $\mathcal{N}(A)$, Range space: $\mathcal{R}(A)$:

  Theorem 2 *The null space and range space of a matrix are perpenticular to each other.*

Symmetric Matrix Space

- n-dimensional symmetric matrix space: $\mathcal{M}^n$
- Inner Product:

$$X \bullet Y = \text{tr} X^T Y = \sum_{i,j} X_{i,j} Y_{i,j}$$

- Frobenius norm:

$$\|X\|_f = \sqrt{\text{tr} X^T X}$$

- Positive semidefinite matrix set: $\mathcal{M}^n_+$, Positive definite matrix set: $\text{int}\, \mathcal{M}^n_+$

Affine and Convex Combination

$S \subset R^n$ is affine if

$$[\, \mathbf{x}, \mathbf{y} \in S \text{ and } \alpha \in R\,] \Longrightarrow \alpha x + (1-\alpha) y \in S.$$

When $\mathbf{x}$ and $\mathbf{y}$ are two distinct points in R^n and α runs over R ,

$$\{\mathbf{z} : \mathbf{z} = \alpha \mathbf{x} + (1-\alpha)\mathbf{y}\}$$

is the line set determined by $\mathbf{x}$ and $\mathbf{y}$.

When $0 \leq \alpha \leq 1$, it is called the convex combination of $\mathbf{x}$ and $\mathbf{y}$ and it is the line segment between $\mathbf{x}$ and $\mathbf{y}$.

Convex Sets

- Set notations: $x \in \Omega$, $y \notin \Omega$ $S \cup T$, $S \cap T$
- Ω is said to be a convex set if for every $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$ and every real number $\alpha \in [0, 1]$, the point $\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2 \in \Omega$.
- The convex hull of a set Ω is the intersection of all convex sets containing Ω
- Intersection of convex sets is convex

Proof of convex set

- All solutions to the system of linear equations, $\{\mathbf{x} : \; A\mathbf{x} = \mathbf{b}\}$, form a convex set.

- All solutions to the system of linear inequalities,

$$\{\mathbf{x} : \; A\mathbf{x} \leq \mathbf{b}\}$$

  , form a convex set.

- All solutions to the system of linear equations and inequalities, $\{\mathbf{x} : \; A\mathbf{x} = \mathbf{b}, \; \mathbf{x} \geq \mathbf{0}\}$, form a convex set.

- Ball is a convex set: for $\mathbf{y} \in \mathcal{R}^n$ and $r > 0$,

$$B(\mathbf{y}, r) = \{\mathbf{x} : \; \|\mathbf{x} - \mathbf{y}\| \leq r\}$$

- Ellipsoid is a convex set: for a positive definite matrix Q,

$$E(\mathbf{y}, Q) = \{\mathbf{x} : \ (\mathbf{x} - \mathbf{y})^T Q(\mathbf{x} - \mathbf{y}) \le 1\}$$

More proof of convex set

Consider the set B of all $\mathbf{b}$, for fixed A and $\mathbf{c}$, such that the following linear program is feasible:

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Show that B is a convex set.

Convex Cones

- A set C is a cone if $\mathbf{x} \in C$ implies $\alpha\mathbf{x} \in C$ for all $\alpha > 0$
- A convex cone is cone plus convex-set.
- Dual cone:

$$C^* := \{\mathbf{y} : \mathbf{y} \bullet \mathbf{x} \geq 0 \quad \text{for all} \quad \mathbf{x} \in C\}$$

$-C^*$ is also called the polar of C.

Cone Examples

- Example 2.1: The n-dimensional non-negative orthant, $\mathcal{R}^n_+ = \{\mathbf{x} \in \mathcal{R}^n : \ \mathbf{x} \geq \mathbf{0}\}$, is a convex cone.
- Example 2.2: The set of all positive semi-definite matrices in $\mathcal{M}^n$, $\mathcal{M}^n_+$, is a convex cone, called the positive semi-definite matrix cone
- Example 2.3: The set $\{(t; \mathbf{x}) \in \mathcal{R}^{n+1} : \ t \geq \|\mathbf{x}\|\}$ is a convex cone in $\mathcal{R}^{n+1}$, called the second-order cone.
- Example 2.4: The set $\{(t; \mathbf{x}) \in \mathcal{R}^{n+1} : \ t \geq \|\mathbf{x}\|_p\}$ is a convex cone in $\mathcal{R}^{n+1}$, called the p-order cone.

Polyhedral Convex Cones

- A cone C is (convex) polyhedral if C can be represented by

$$C = \{\mathbf{x} : A\mathbf{x} \leq 0\}$$

for some matrix A.

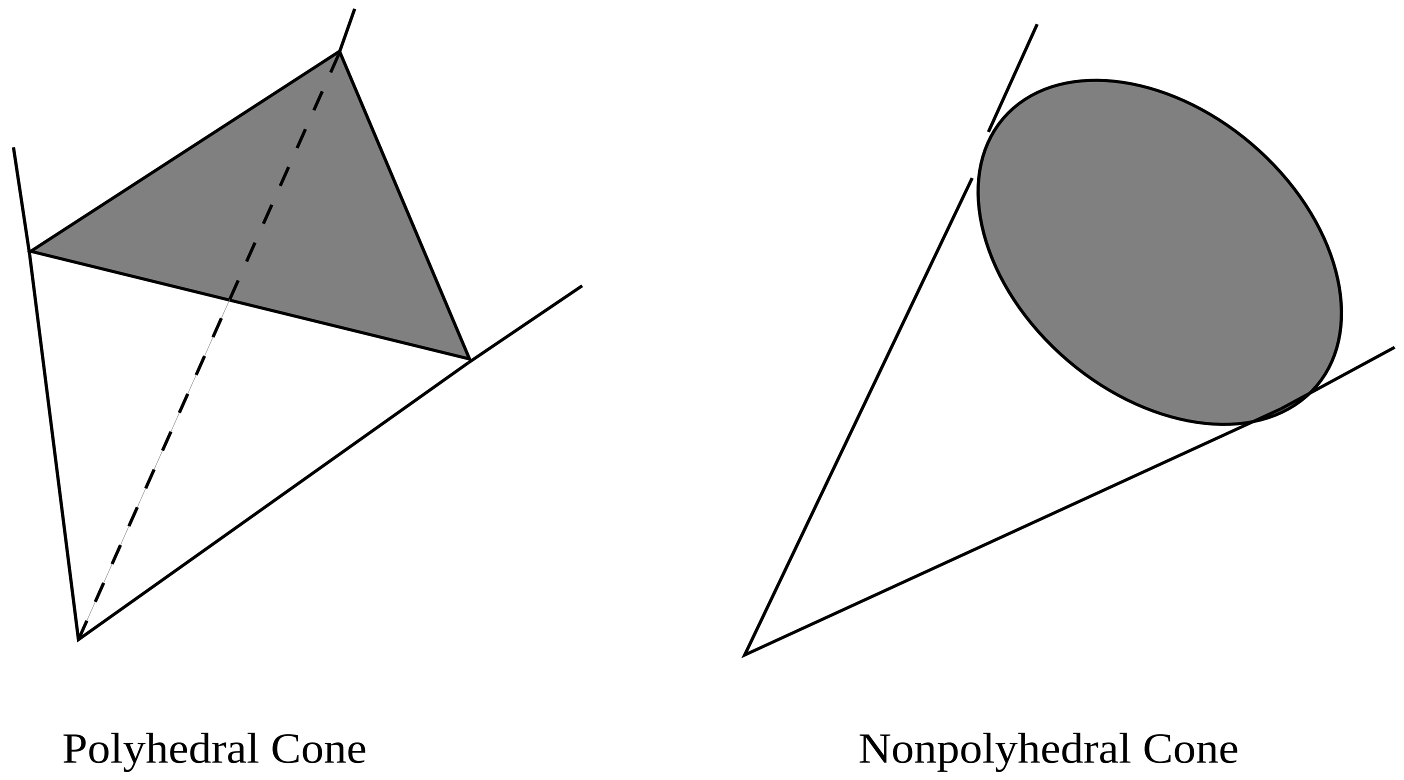

Figure 3: Polyhedral and non-polyhedral cones.

- The nonnegative orthant is a polyhedral cone but the second-order cone is not polyhedral.

Real Functions

- Continuous functions C
- Weierstrass theorem: a continuous function $f(\mathbf{x})$ defined on a compact set (bounded and closed) $\Omega \subset \mathcal{R}^n$ has a minimizer in Ω.
- The least upper bound or supremum of f over Ω

$$\sup\{f(\mathbf{x}) : \mathbf{x} \in \Omega\}$$

and the greatest lower bound or infimum of f over Ω

$$\inf\{f(\mathbf{x}) : \mathbf{x} \in \Omega\}$$

- A function $f(\mathbf{x})$ is called homogeneous of degree k if $f(\alpha \mathbf{x}) = \alpha^k f(\mathbf{x})$ for all $\alpha \geq 0$.

  Let $\mathbf{c} \in \mathcal{R}^n$ be given and $\mathbf{x} \in \text{int}\, \mathcal{R}^n_+$. Then $\mathbf{c}^T\mathbf{x}$ is homogeneous of degree 1 and

$$\phi(\mathbf{x}) = n\log(\mathbf{c}^T\mathbf{x}) - \sum_{j=1}^{n} \log x_j$$

is homogeneous of degree 0.

Let $C \in \mathcal{M}^n$ be given and $X \in \text{int}\,\mathcal{M}^n_+$. Then $\mathbf{x}^T C\mathbf{x}$ is homogeneous of degree 2, $C \bullet X$ and $\det(X)$ are homogeneous of degree 1 and n, respectively; and

$$\Phi(X) = n\log(C \bullet X) - \log\det(X)$$

is homogeneous of degree 0.

- The gradient vector C^1:

$$\nabla f(\mathbf{x}) = \{\partial f/\partial x_i\}, \quad \text{for} \quad i = 1, ..., n.$$

- The Hessian matrix C^2:

$$\nabla^2 f(\mathbf{x}) = \left\{ \frac{\partial^2 f}{\partial x_i \partial x_j} \right\} \quad \text{for} \quad i = 1, ..., n; \; j = 1, ..., n.$$

- Vector function: $\mathbf{f} = (f_1; f_2; ...; f_m)$
- The Jacobian matrix of $\mathbf{f}$ is

$$\nabla \mathbf{f}(\mathbf{x}) = \begin{pmatrix} \nabla f_1(\mathbf{x}) \\ ... \\ \nabla f_m(\mathbf{x}) \end{pmatrix}.$$

Convex Functions

- f convex function iff for $0 \leq \alpha \leq 1$,

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}).$$

- The level set of convex function f

$$L(z) = \{\mathbf{x} : \quad f(\mathbf{x}) \leq z\}$$

is a convex set.

Proof of convex function

Consider the minimal-objective function of $\mathbf{b}$ for fixed A and $\mathbf{c}$:

$$
\begin{aligned}
z(\mathbf{b}) := \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0}.
\end{aligned}
$$

Show that $z(\mathbf{b})$ is a convex function in $\mathbf{b}$ for all feasible $\mathbf{b}$.

Theorems on functions

Taylor's theorem or the mean-value theorem:

Theorem 3 *Let* $f \in C^1$ *be in a region containing the line segment* $[\mathbf{x}, \mathbf{y}]$*. Then there is a* α*,* $0 \le \alpha \le 1$*, such that*

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})(\mathbf{y} - \mathbf{x}).$$

Furthermore, if $f \in C^2$ *then there is a* α*,* $0 \le \alpha \le 1$*, such that*

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y}-\mathbf{x}) + (1/2)(\mathbf{y}-\mathbf{x})^T \nabla^2 f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})(\mathbf{y}-\mathbf{x}).$$

Theorem 4 *Let* $f \in C^1$*. Then* f *is convex over a convex set* Ω *if and only if*

$$f(\mathbf{y}) \ge f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{x},\ \mathbf{y} \in \Omega$*.*

Theorem 5 *Let $f \in C^2$. Then f is convex over a convex set Ω if and only if the Hessian matrix of f is positive semi-definite throughout Ω.*

Known Inequalities

- Cauchy-Schwarz: given $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, $\mathbf{x}^T\mathbf{y} \le \|\mathbf{x}\|\|\mathbf{y}\|$.
- Arithmetic-geometric mean: given $\mathbf{x} > \mathbf{0}$,

$$\frac{\sum x_j}{n} \ge \left(\prod x_j\right)^{1/n}.$$

- Harmonic: given $\mathbf{x} > \mathbf{0}$,

$$\left(\sum x_j\right)\left(\sum 1/x_j\right) \ge n^2.$$

Linear least-squares problem

Given $A \in \mathcal{R}^{m \times n}$ and $\mathbf{c} \in \mathcal{R}^n$,

$$\begin{array}{lll} (LS) & \text{minimize} & \|A^T\mathbf{y} - \mathbf{c}\|^2 \\ & \text{subject to} & \mathbf{y} \in \mathcal{R}^m. \end{array}$$

$$AA^T\mathbf{y} = A\mathbf{c} \quad \text{or} \quad \mathbf{y} = (AA^T)^{-1}A\mathbf{c}$$

with the projection:

$$A^T\mathbf{y} = A^T(AA^T)^{-1}A\mathbf{c}$$

Projection matrix: $P = A^T(AA^T)^{-1}A$ or $P = I - A^T(AA^T)^{-1}A$

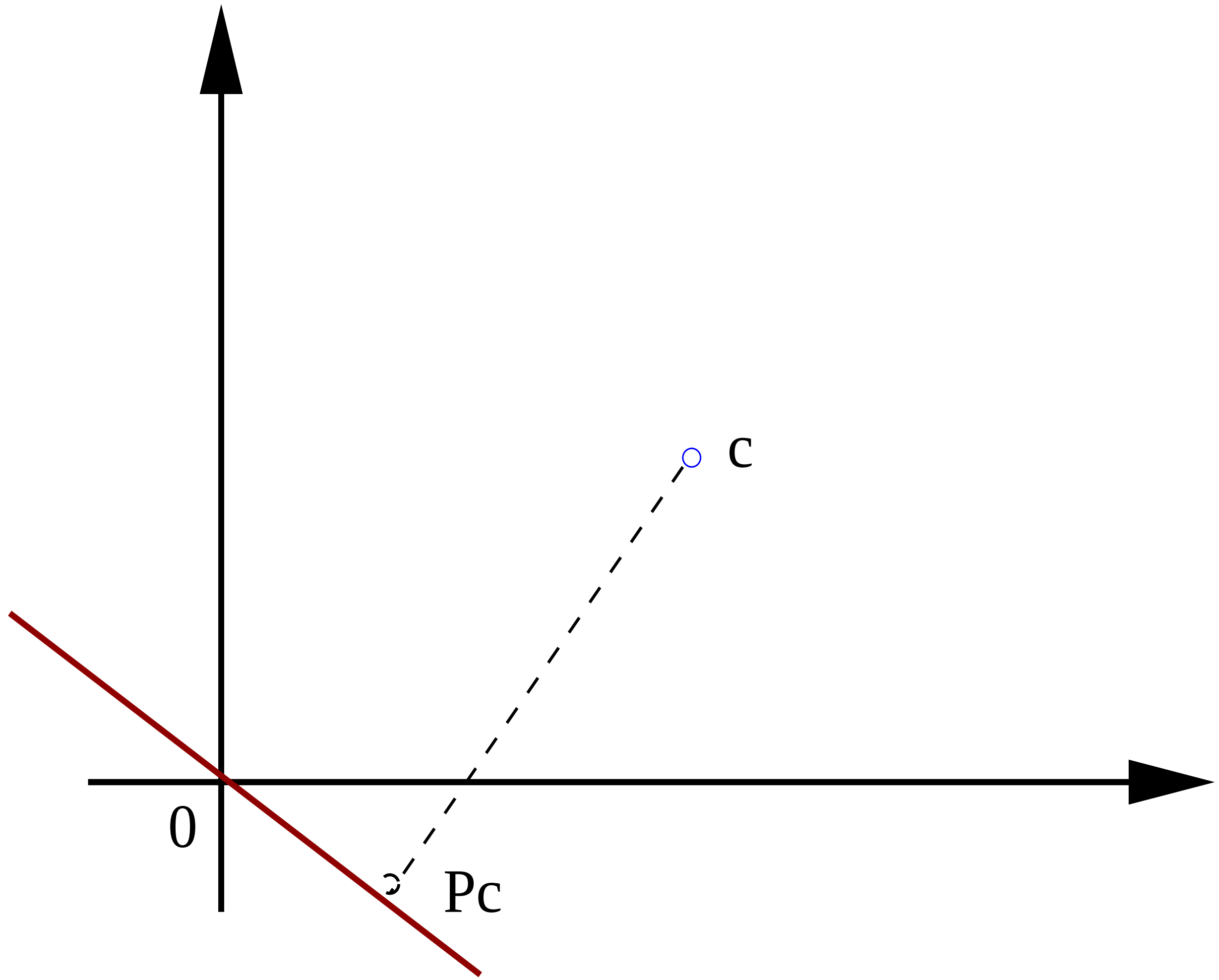

Figure 4: Projection of $\mathbf{c}$ onto a subspace

Choleski decomposition method

$$AA^T = L\Lambda L^T$$

$$L\Lambda L^T y^* = Ac$$

Solving ball-constrained linear problem

$$
\begin{array}{lll}
(BP) & \text{minimize} & \mathbf{c}^T\mathbf{x} \\
& \text{subject to} & A\mathbf{x} = 0,\ \|\mathbf{x}\|^2 \le 1,
\end{array}
$$

$\mathbf{x}^*$ minimizes (BP) if and only if there always exists a $\mathbf{y}$ such that they satisfy

$$AA^Ty = Ac,$$

and if $c - A^Ty \neq \mathbf{0}$ then

$$\mathbf{x}^* = -(\mathbf{c} - A^T\mathbf{y})/\|\mathbf{c} - A^T\mathbf{y}\|;$$

otherwise any feasible $\mathbf{x}$ is a solution.

Solving ball-constrained linear problem

$$
\begin{array}{lll}
(BD) & \text{minimize} & \mathbf{b}^T\mathbf{y} \\
 & \text{subject to} & \|A^T\mathbf{y}\|^2 \le 1.
\end{array}
$$

The solution $\mathbf{y}^*$ for (BD) is given as follows: Solve

$$AA^T\bar{\mathbf{y}} = b$$

and if $\bar{\mathbf{y}} \neq \mathbf{0}$ then set

$$\mathbf{y}^* = -\bar{\mathbf{y}}/\|A^T\bar{\mathbf{y}}\|;$$

otherwise any feasible $\mathbf{y}$ is a solution.

System of nonlinear equations

Given $\mathbf{f}(\mathbf{x}) : \mathcal{R}^n \rightarrow \mathcal{R}^n$, the problem is to solve n equations for n unknowns:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}.$$

Given a point $\mathbf{x}^k$, Newton's Method sets

$$f(\mathbf{x}) \simeq f(\mathbf{x}^k) + \nabla f(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k) = \mathbf{0}.$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k - (\nabla f(\mathbf{x}^k))^{-1} f(\mathbf{x}^k)$$

or solve for direction vector $\mathbf{d}_x$:

$$\nabla f(\mathbf{x}^k)\mathbf{d}_x = -f(\mathbf{x}^k) \quad \text{and} \quad \mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}_x.$$

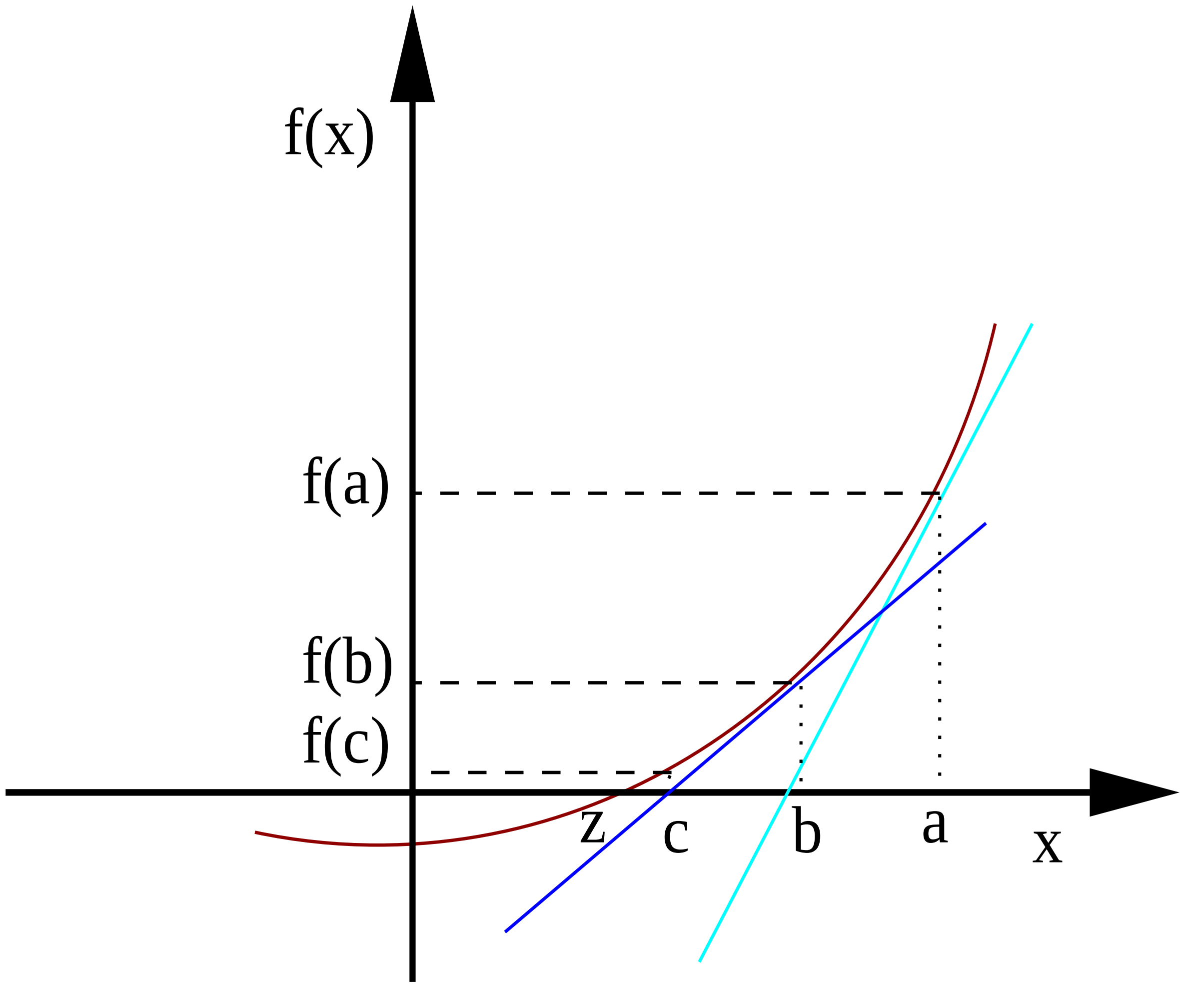

Figure 5: Newton's method for root finding

The quasi Newton method

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha(\nabla f(\mathbf{x}^k))^{-1} f(\mathbf{x}^k)$$

where scalar $\alpha \geq 0$ is called step-size. More generally

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha M^k f(\mathbf{x}^k)$$

where M^k is an $n \times n$ symmetric matrix. In particular, if $M^k = I$, the method is called the gradient method, where f is viewed as the gradient vector of a real function.

Convergence and Big O

- $\{\mathbf{x}^k\}_0^\infty$ denotes a seqence $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^k, ...$
- $\mathbf{x}^k \longrightarrow \bar{\mathbf{x}}$ iff

$$\|\mathbf{x}^k - \bar{\mathbf{x}}\| \longrightarrow 0$$

- $g(x) \geq 0$ is a real valued function of a real nonnegative variable, the notation $g(x) = O(x)$ means that $g(x) \leq \bar{c}x$ for some constant $\bar{c}$;
- $g(x) = \Omega(x)$ means that $g(x) \geq \underline{c}x$ for some constant $\underline{c}$;
- $g(x) = \theta(x)$ means that $\underline{c}x \leq g(x) \leq \bar{c}x$.
- $g(x) = o(x)$ means that $g(x)$ goes to zero faster than x does:

$$\lim_{x \to 0} \frac{g(x)}{x} = 0$$