第八章 多元函数微分

例1 利用极限的定义证明

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3+y^3}{x^2+y^2}=0.$$

分析： 由二元函数极限定义,可知$\lim\limits_{\substack{x\to x_0\\ y\to y_0}}f(x,y)=A$

$\Longleftrightarrow\forall\varepsilon>0,\exists\delta>0,$ 当$0<\rho=\sqrt{(x-x_0)^2+(y-y_0)^2}$ $<\delta$时,$|f(x,y)-A|<\varepsilon.$

证明： 由于$|x+y|^2=(x+y)^2\le 2(x^2+y^2)$，又

$$|xy|\le\frac{1}{2}(x^2+y^2),$$

所以

$$|f(x,y)-0|=\left|\frac{(x+y)(x^2+y^2-xy)}{x^2+y^2}\right|$$

$$\le|x+y|\frac{x^2+y^2+|xy|}{x^2+y^2}\le|x+y|(1+\frac{1}{2})\le\frac{3}{2}\sqrt{2(x^2+y^2)},$$

所以$\forall\varepsilon>0$，取$\delta=\frac{\sqrt{2}}{3}\varepsilon$，则当$0<\sqrt{x^2+y^2}<\delta$时，

$|f(x,y)-0|<\varepsilon$恒成立．故

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3+y^3}{x^2+y^2}=0.$$

例2 证明函数$f(x,y)=\begin{cases}\frac{x^2}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

在$(0,0)$点连续，但$f'_x(0,0)$不存在.

分析： 讨论分段函数在分段点处偏导数是否存在要从定义出发.

证明： 先证连续性．由于

$$0\le\left|\frac{x^2}{\sqrt{x^2+y^2}}\right|=|x|\left|\frac{x}{\sqrt{x^2+y^2}}\right|\le|x|,$$

且

$$\lim_{\substack{x\to 0\\ y\to 0}}|x|=0,$$

所以

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2}{\sqrt{x^2+y^2}}=0=f(0,0).$$

根据连续的定义,$f(x,y)$在$(0,0)$处连续．再讨论$f'_x(0,0)$.

根据偏导数定义,考虑

$$\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\frac{\frac{(\Delta x)^2}{\sqrt{(\Delta x)^2+0^2}}-0}{\Delta x}=\frac{\Delta x}{|\Delta x|},$$

而

$$\lim_{\Delta x\to 0^+}\frac{\Delta x}{|\Delta x|}=1,\qquad \lim_{\Delta x\to 0^-}\frac{\Delta x}{|\Delta x|}=-1$$

所以当$\Delta x\to 0$时，$\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}$的极限不存在,

即$f'_x(0,0)$不存在.

例3 证明函数

$$f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

在点$(0,0)$的邻域内有偏导数$f'_x(x,y)$和$f'_y(x,y)$，但是,此函数在点$(0,0)$处不可微.

解： (1) 当$x^2+y^2\neq 0$时,

$$f'_x(x,y)=y\cdot\frac{\sqrt{x^2+y^2}-\frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{y^3}{(x^2+y^2)^{3/2}}$$

$$f'_y(x,y)=x\cdot\frac{\sqrt{x^2+y^2}-\frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{y^3}{(x^2+y^2)^{3/2}}$$

当$x^2+y^2=0$时,

$$f(x,0)=0,\quad f'_x(x,0)=0,\quad f'_x(0,0)=0$$

同理可得$f'_y(0,0)=0$.

(2) 证明函数在点$(0,0)$处不可微.

容易验证函数$f(x,y)$在点$(0,0)$处连续.由于$f'_x(0,0)=0$，$f'_y(0,0)=0$，那么

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{f(x,y)-f(0,0)-xf'_x(0,0)-yf'_y(0,0)}{\sqrt{x^2+y^2}}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}$$

由于$\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}$不存在，因此根据函数可微定义知,函数$f(x,y)$在点$(0,0)$处不可微.

注： 从本例可以看出，函数在某点的邻域内连续,且偏导数存在,不是函数在该点处可微的充分条件,这是多元函数与一元函数的又一本质不同之处(在一元函数中,函数在某点处的可导性与可微性是等价的).

例4 设函数

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$$

证明: (1)在点$(0,0)$的邻域内有偏导数$f'_x(x,y)$, $f'_y(x,y)$; (2) 偏导数$f'_x(x,y)$和$f'_y(x,y)$ 在点$(0,0)$处不连续; (3) 函数$f(x,y)$在点$(0,0)$处可微.

证明: 当$x^2+y^2\neq 0$时,有

$$\begin{aligned}f'_x(x,y)&=2x\sin\frac{1}{x^2+y^2}+(x^2+y^2)\cos\frac{1}{x^2+y^2}\cdot\left[-\frac{2x}{(x^2+y^2)^2}\right]\\&=2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}\end{aligned}$$

同理可得

$$f'_y(x,y)=2y\sin\frac{1}{x^2+y^2}-\frac{2y}{x^2+y^2}\cos\frac{1}{x^2+y^2}$$

当$x^2+y^2=0$时,有

$$f(x,0)=x^2\sin\frac{1}{x^2}$$

$$f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}x\sin\frac{1}{x^2}=0$$

同理可得$f'_y(0,0)=0$.

所以,$f(x,y)$在点$(0,0)$的领域内有偏导数$f'_x(x,y)$和$f'_y(x,y)$.

(2) 因为

$$\lim_{\substack{x\to 0\\ y\to 0}}f'_x(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\left(2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}\right)$$

考虑点(x,y)沿直线$y=x$趋向于点$(0,0)$,有

$$\lim_{\substack{x\to 0\\ y=x\to 0}}2x\sin\frac{1}{x^2+y^2}=\lim_{x\to 0}2x\sin\frac{1}{2x^2}=0$$

$$\lim_{\substack{x\to 0\\ y=x\to 0}}\frac{2y}{x^2+y^2}\cos\frac{1}{x^2+y^2}=\lim_{x\to 0}\frac{1}{x}\cos\frac{1}{2x^2}\text{不存在!}$$

因此,$\lim\limits_{\substack{x\to 0\\ y\to 0}}f(x,y)$ 不存在,则$f'_x(x,y)$在点$(0,0)$处不连续.

同理可证,$f'_y(x,y)$在点$(0,0)$处不连续.

(3) 由于$f'_x(0,0)=f'_y(0,0)=0$,那么

$$\begin{aligned}&\lim_{\substack{x\to 0\\ y\to 0}}\frac{f(x,y)-f(0,0)-xf'_x(0,0)-yf'_y(0,0)}{\sqrt{x^2+y^2}}\\&=\lim_{\substack{x\to 0\\ y\to 0}}\sqrt{x^2+y^2}\cdot\sin\frac{1}{x^2+y^2}=0\end{aligned}$$

于是有$f(x,y)-f(0,0)=xf'_x(0,0)+yf'_y(0,0)+o(\rho)$(其中$\rho=\sqrt{x^2+y^2}$),即函数$f(x,y)$ 在点$(0,0)$处可微.

注: 从本例可以看出,函数在某点处有连续偏导数是它在该点可微的充分条件,而不是必要条件.

例5 设可微函数$f(x,y,z)$恒满足关系式

$$f(tx,ty,tz)=t^kf(x,y,z)$$

试证明

$$xf'_x+yf'_y+zf'_z=kf(x,y,z).$$

证明: 令$u=tx,v=ty,\omega=tz$,由已知条件可得

$$f'_u\cdot x+f'_v\cdot y+f'_\omega\cdot z=kt^{k-1}f(x,y,z).$$

对上式两端同乘t,得

$$txf'_u+tyf'_v+tzf'_\omega=kt^kf(x,y,z)=kf(tx,ty,tz),$$

即$uf'_u+vf'_v+\omega f'_\omega=kf(u,v,\omega)$. 令$u=x,v=y,\omega=z$,即得所证.

例6 试证明:柯西-黎曼方程$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ 在极坐标(r,θ)下可化为

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta},\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}.$$

证明: 由题设可知,$u=u(x,y),v=v(x,y),x=r\cos\theta,y=r\sin\theta$. 所以

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y};$$

$$\frac{\partial v}{\partial\theta}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial\theta}=-r\sin\theta\frac{\partial v}{\partial x}+r\cos\theta\frac{\partial v}{\partial y};$$

因为$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$, 于是

$$\begin{aligned}\frac{\partial v}{\partial\theta}&=-r\sin\theta\left(-\frac{\partial u}{\partial y}\right)+r\cos\theta\frac{\partial u}{\partial x}\\&=r\left(\sin\theta\frac{\partial u}{\partial y}+\cos\theta\frac{\partial u}{\partial x}\right)=r\frac{\partial u}{\partial r},\end{aligned}$$

因此$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta}$. 同理可得$\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}$.

例7 证明: 若$u=f(x,y,z)$, 其中f具有连续的二阶偏导数, 而$x=r\cos\theta,y=r\sin\theta,z=z$,则有

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=\frac{\partial^2u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2u}{\partial z^2}.$$

分析: 应用多元复合函数微分法,通过变量代换(柱坐标变换)可以将直角坐标系下的拉普拉斯方程化为柱坐标系下的形式.

证明: $\frac{\partial u}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r},\Rightarrow\frac{\partial u}{\partial r}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta,$

$\frac{\partial u}{\partial\theta}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta},\Rightarrow\frac{\partial u}{\partial\theta}=-\frac{\partial f}{\partial x}r\sin\theta+\frac{\partial f}{\partial y}r\cos\theta,$

$\frac{\partial u}{\partial z}=\frac{\partial f}{\partial z}.$

$$\frac{\partial^2 u}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right).$$

$\boxed{\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)(\frac{\partial f}{\partial x}\text{是以}x,y\text{为中间变量的函数})}$ $\boxed{\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)}$

$$\begin{aligned}&=\left(\frac{\partial^2 f}{\partial x^2}\cos\theta+\frac{\partial^2 f}{\partial y\partial x}\sin\theta\right)\cos\theta+\left(\frac{\partial^2 f}{\partial x\partial y}\cos\theta+\frac{\partial^2 f}{\partial y^2}\sin\theta\right)\sin\theta\\&=\left(\frac{\partial^2 f}{\partial x^2}\cos^2\theta+2\frac{\partial^2 f}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 f}{\partial y^2}\sin^2\theta\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2}&=\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\\&=-r\sin\theta\left(\frac{\partial^2 f}{\partial x^2}(-r\sin\theta)+\frac{\partial^2 f}{\partial y\partial x}r\cos\theta\right)-r\frac{\partial f}{\partial x}\cos\theta\\&\quad+r\cos\theta\left(\frac{\partial^2 f}{\partial x\partial y}(-r\sin\theta)+\frac{\partial^2 f}{\partial y^2}r\cos\theta\right)-r\frac{\partial f}{\partial y}\sin\theta\\&=\frac{\partial^2 f}{\partial x^2}r^2\sin^2\theta-2\frac{\partial^2 f}{\partial y\partial x}r^2\sin\theta\cos\theta\\&\quad+\frac{\partial^2 f}{\partial y^2}r^2\cos^2\theta-\frac{\partial f}{\partial x}r\cos\theta-\frac{\partial f}{\partial y}r\sin\theta\end{aligned}$$

$$\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 f}{\partial z^2}.$$

于是有

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}.$$

例8 设$f(x),g(x)$是可微函数,且

$$u(x,y)=f(2x+5y)+g(2x-5y),$$
$$u(x,0)=\sin 2x, u'_y(x,0)=0,$$

求$u(x,y)$的表达式.

解: 令$y=0$, 则$u(x,0)=f(2x)+g(2x)=\sin 2x$,

即
$$f(x)+g(x)=\sin x$$

又 $u'_y(x,0)=[f'(2x+5y)\cdot 5+g'(2x-5y)(-5)]|_{y=0}$
$$=5[f'(2x)-g'(2x)]=0,$$

所以$f'(x)-g'(x)=0$.

对上式积分得

$$f(x)-g(x)=C$$

这样我们得到

$$f(x)=\frac{1}{2}(\sin x+C),\quad g(x)=\frac{1}{2}(\sin x-C),$$

于是

$$\begin{aligned}u(x,y)&=f(2x+5y)+g(2x-5y)\\&=\frac{1}{2}[\sin(2x+5y)+C]+\frac{1}{2}[\sin(2x-5y)-C]\\&=\sin 2x\cos 5y\end{aligned}$$

例9 设函数$u=u(x)$由方程组

$$\begin{cases}u=f(x,y)\\g(x,y,z)=0\\h(x,z)=0\end{cases}$$

所确定,且$h'_z\neq 0$, $g'_y\neq 0$. 试求$\frac{du}{dx}$.

解: 由四个未知量三个方程知其中只有一个变量为自变量. 再由所求结果$\frac{du}{dx}$可知, x是自变量,y,z,u均为x的一元函数.

分别对已知方程组中每个方程关于x求导,得

$$\frac{du}{dx}=f'_x+f'_y\frac{dy}{dx}$$
$$g'_x+g'_y\frac{dy}{dx}+g'_z\frac{dz}{dx}=0$$
$$h'_x+h'_z\frac{dz}{dx}=0$$

于是,得
$$\frac{dy}{dx}=\frac{-g'_xh'_z+g'_zh'_x}{g'_yh'_z}.$$

所以
$$\frac{du}{dx}=f'_x+\frac{-g'_xh'_z+g'_zh'_x}{g'_yh'_z}f'_y=f'_x-\frac{g'_xf'_y}{g'_y}+\frac{g'_zh'_xf'_y}{g'_yh'_z}.$$

例9a 求球面$x^2+y^2+z^2=14$与椭球面$3x^2+y^2+z^2=16$在点$(-1,-2,3)$处交角(两曲面在交点处的交角定义为它们在该点处的切平面的交角).

解: 令$F(x,y,z)=x^2+y^2+z^2-14,G(x,y,z)=3x^2+y^2+z^2-16$.

因为
$$\vec{n_1}=\{F'_x,F'_y,F'_z\}|_{(-1,-2,3)}=2\{-1,-2,3\},$$
$$\vec{n_2}=\{G'_x,G'_y,G'_z\}|_{(-1,-2,3)}=2\{-3,-2,3\},$$

故所求交角θ满足

$$\cos\theta=\frac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}=\frac{|3+4+9|}{\sqrt{1+4+9}\sqrt{9+4+9}}=\frac{8}{\sqrt{77}},$$

于是$\theta=\arccos\frac{8}{\sqrt{77}}$.

例10 已知曲面$e^{2x-z}=f(\pi y-\sqrt{2}z)$, 且$f$可微, 证明该曲面为柱面.

分析: 要证明曲面是柱面,只须证明过曲面上任意一点的切平面平行于一条定直线,即证明曲面上任意一点的法向量垂直于定向量.

证明: 设$F(x,y,z)=e^{2x-z}-f(\pi y-\sqrt{2}z)$, 则有

$$F'_x=2e^{2x-z},\quad F'_y=-\pi f',\quad F'_z=-e^{2x-z}+\sqrt{2}f'$$

于是,曲面$e^{2x-z}=f(\pi y-2\sqrt{2}z)$在任意一点$(x,y,z)$的法向量为

$$\vec{n}=(2e^{2x-z},-\pi f',-e^{2x-z}+\sqrt{2}f')$$

设定向量为$\vec{a}=(l,m,n)$,要使$\vec{a}\cdot\vec{n}=0$, 即

$$2le^{2x-z}-m\pi f'-ne^{2x-z}+\sqrt{2}nf'=0$$

只须使
$$2l = n, \quad m\pi = \sqrt{2}n$$
由此,若取$l = \pi, n = 2\pi, m = 2\sqrt{2}$, 则有$\overrightarrow{a} \cdot \overrightarrow{n} = 0$.

纵上所述可以看出,曲面$e^{2x-z} = f(\pi y - \sqrt{2}z)$在任意一点$(x, y, z)$的法向量$\overrightarrow{n}$垂直于定向量$\overrightarrow{a} = (\pi, 2\sqrt{2}, 2\pi)$, 从而过曲面上任意一点的切平面平行于以$\overrightarrow{a}$为方向向量的定直线,即该曲面为柱面.

例11 设曲面$z = f(x, y)$二次可微,且$f'_y \neq 0$. 证明: 对任意给定的常数c, $\begin{cases} f(x,y) = c \\ z = c \end{cases}$ 为一条直线的充要条件是
$$(f'_y)^2 f''_{xx} - 2f'_x f'_y f''_{xy} + (f'_x)^2 f''_{yy} = 0.$$
证明: 必要性: 若$\begin{cases} f(x,y) = c \\ z = c \end{cases}$ 表示一条直线, 则$f(x, y)$一定是关于x, y的一次式. 因此$\frac{d^2y}{dx^2} = 0$.

因为$f(x, y) = c$,于是$\frac{dy}{dx} = -\frac{f'_x}{f'_y}$,
$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{\left(f''_{xx}+f''_{xy}\frac{dy}{dx}\right)f_y - f_x\left(f''_{yx}+f''_{yy}\frac{dy}{dx}\right)}{(f'_y)^2} \\ &= -\frac{f''_{xx}(f'_y)^2 - 2f''_{xy}f'_x f'_y + f''_{yy}(f'_x)^2}{(f'_y)^2} \qquad (1)\end{aligned}$$
从而有,
$$(f'_y)^2 f''_{xx} - 2f'_x f'_y f''_{xy} + (f'_x)^2 f''_{yy} = 0.$$
充分性: 由已知条件及(1)可知:$\frac{d^2y}{dx^2} = 0$, 故$f(x, y) = c$必为关于x, y的一次式, 因此$\begin{cases} f(x,y) = c \\ z = c \end{cases}$ 表示一条直线.

例12 求函数$z = (x^2 + y^2)e^{-(x^2+y^2)}$的极值.

解: 求驻点. 由
$$\begin{cases} \frac{\partial z}{\partial x} = 2x(1 - x^2 - y^2)e^{-(x^2+y^2)} = 0, \\ \frac{\partial z}{\partial y} = 2y(1 - x^2 - y^2)e^{-(x^2+y^2)} = 0, \end{cases}$$
得驻点$(0,0)$和$x^2 + y^2 = 1$.

又

$A = \frac{\partial^2 z}{\partial x^2} = [2(1 - y^2 - 3x^2) - 4x^2(1 - x^2 - y^2)]e^{-(x^2+y^2)}$;
$B = \frac{\partial^2 z}{\partial x \partial y} = -4xy(2 - x^2 - y^2)e^{-(x^2+y^2)}$;
$C = \frac{\partial^2 z}{\partial y^2} = [2(1 - x^2 - 3y^2) - 4y^2(1 - x^2 - y^2)]e^{-(x^2+y^2)}$.

因为$B^2 - AC|_{(0,0)} = -4 < 0$, 且$A|_{(0,0)} = 2 > 0$, 故$f(0,0)$为极小值.

又因为$B^2 - AC|_{x^2+y^2=1} = (-4xye^{-1})^2 - (4x^2e^{-1})(4y^2e^{-1}) = 0$.因此, 用通常的方法无法判定, 当$x^2 + y^2 = 1$时, z是否取得极值. 因此需要用其它方法.

令$x^2 + y^2 = t (t \geq 0)$, 则$z = te^{-t}$.

现在利用一元函数求极值的方法, 由$\frac{dz}{dt} = e^{-t}(1 - t) = 0$得驻点$t = 1$.

又
$$\frac{d^2z}{dt^2}\Big|_{t=1} = (t - 2)e^{-t}\Big|_{t=1} = -e^{-1} < 0.$$
所以$z = te^{-t}$在$t = 1$处取得极大值, 即函数$z = (x^2 + y^2)e^{-(x^2+y^2)}$在圆周$x^2 + y^2 = 1$上取极大值$e^{-1}$.

例13 求函数$z = x^2 + y^2 - 12x + 16y$在有界闭区域$x^2 + y^2 \leq 25$上的最大值和最小值.

解: 函数$z = x^2 + y^2 - 12x + 16y$在有界闭区域$x^2 + y^2 \leq 25$上连续, 故必在该区域上取得最大值和最小值.

因为
$$\begin{cases} \frac{\partial z}{\partial x} = 2x - 12 = 0 \\ \frac{\partial z}{\partial y} = 2y + 16 = 0 \end{cases}$$
此方程在区域$x^2 + y^2 < 25$内无解,故最大值和最小值必在边界$x^2 + y^2 = 25$上达到.

现在所要解决的问题变为函数$z = x^2 + y^2 - 12x + 16y$在边界$x^2 + y^2 = 25$上的条件极值问题.

设$L(x, y, \lambda) = x^2 + y^2 - 12x + 16y - \lambda(x^2 + y^2 - 25)$, 解方程组
$$\begin{cases} \frac{\partial L}{\partial x} = 2x - 12 - 2\lambda x = 0 \\ \frac{\partial L}{\partial y} = 2y + 16 - 2\lambda y = 0 \\ \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 25 = 0 \end{cases}$$
解得$x = \frac{6}{1-\lambda}, y = \frac{-8}{1-\lambda}$, $\lambda = -1, 3$; 进一步可得驻点: $P_1(3, -4)$和$P_2(-3, 4)$.

经计算得$z(3, -4) = -75$, $z(-3, 4) = 125$, 因此
$$z_{\max} = z(-3, 4) = 125, \quad z_{\min} = z(3, -4) = -75.$$
例14 求函数$f(x, y, z) = \ln x + \ln y + 3\ln z$在球面$x^2 + y^2 + z^2 = 5r^2 (x > 0, y > 0, z > 0)$上的最大值, 并证明对任何正数$a, b, c$成立不等式
$$abc^3 \leq 27\left(\frac{a+b+c}{3}\right)^5.$$

证明: 这是一个条件极值问题, 作拉格朗日函数
$L(x,y,z,\lambda)=\ln x+\ln y+3\ln z-\lambda(x^2+y^2+z^2-5r^2)$
令
$$\begin{cases}\frac{\partial L}{\partial x}=\frac{1}{x}-2\lambda x=0\\ \frac{\partial L}{\partial y}=\frac{1}{y}-2\lambda y=0\\ \frac{\partial L}{\partial z}=\frac{3}{z}-2\lambda z=0\\ \frac{\partial L}{\partial \lambda}=x^2+y^2+z^2-5r^2=0\end{cases}$$
可解得$\lambda=\frac{1}{2r^2}$,$x=y=|r|,z=\sqrt{3}|r|$. 于是得到驻点$(|r|,|r|,\sqrt{3}|r|)$.

因为驻点惟一,且问题的最大值存在,且$f_{max}=\ln(3\sqrt{3}|r|^5)$.
于是有
$$\ln(xyz^3)=\ln x+\ln y+3\ln z\le\ln(3\sqrt{3}|r|^5)=\ln\left[3\sqrt{3}\left(\frac{x^2+y^2+z^2}{5}\right)^{\frac{5}{2}}\right],$$

令$a=x^2,b=y^2,c=z^2$,则
$$abc^3\le 27\left(\frac{a+b+c}{3}\right)^5.$$

例15 将正数a分成n份,问如何分法才能使这n份的乘积最大?并由此证明不等式
$$\sqrt[n]{x_1x_2\cdots x_n}\le\frac{1}{n}(x_1+x_2+\cdots+x_n).$$

解: 将所分的n份记为$x_1,x_2,\cdots,x_n$. 并设它们的乘积为z, 则问题变为函数
$$z=x_1x_2\cdots x_n$$
在条件$x_1+x_2+\cdots+x_n=a$下的最大值.

令$F=x_1x_2\cdots x_n+\lambda(x_1+x_2+\cdots+x_n-a)$. 由
$$\begin{cases}F'_{x_1}=x_2x_3\cdots x_n+\lambda=0\\ F'_{x_2}=x_1x_3\cdots x_n+\lambda=0\\ \cdots\\ F'_{x_n}=x_1x_2\cdots x_{n-1}+\lambda=0\\ x_1+x_2+\cdots x_n=a\end{cases}$$
得
$$x_1=x_2=\cdots=x_n=\frac{a}{n}.$$

因为驻点$\left(\frac{a}{n},\frac{a}{n},\cdots,\frac{a}{n}\right)$惟一且该问题的最大值必定存在,故将正数$a$等分为$n$等分, 可得这$n$等份的乘积最大.且$z_{max}=\left(\frac{a}{n}\right)^n$.

由于$z_{max}\ge z=x_1x_2\cdots x_n$, 故
$$x_1x_2\cdots x_n\le\left(\frac{a}{n}\right)^n=\left(\frac{x_1+x_2+\cdots x_n}{n}\right)^n.$$

即
$$\sqrt[n]{x_1x_2\cdots x_n}\le\frac{1}{n}(x_1+x_2+\cdots+x_n).$$

例16 当n个正数$x_1,x_2,\cdots,x_n$之和为常数时,求它们的乘积开n次根的最大值.

解: 问题就是求$f=\sqrt[n]{x_1x_2\cdots x_n}$, 在条件$x_1+x_2+\cdots+x_n=c$下的最大值.设辅助函数
$F(x_1,x_2,\cdots,x_n,\lambda)=\sqrt[n]{x_1x_2\cdots x_n}+\lambda(x_1+x_2+\cdots+x_n-c)$,

令
$$F'_{x_1}=\frac{1}{n}x_1^{\frac{1}{n}-1}x_2^{\frac{1}{n}}\cdots x_n^{\frac{1}{n}}+\lambda=0,$$
$$F'_{x_2}=\frac{1}{n}x_1^{\frac{1}{n}}x_2^{\frac{1}{n}-1}\cdots x_n^{\frac{1}{n}}+\lambda=0,$$
$$\cdots$$
$$F'_{x_n}=\frac{1}{n}x_1^{\frac{1}{n}}x_2^{\frac{1}{n}}\cdots x_n^{\frac{1}{n}-1}+\lambda=0,$$
$$F'_{\lambda}=x_1+x_2+\cdots+x_n-c=0.$$

前两式相减得
$$\frac{1}{n}x_1^{\frac{1}{n}}x_2^{\frac{1}{n}}\cdots x_n^{\frac{1}{n}}(x_1^{-1}-x_2^{-1})=0,$$
因为$x_1,x_2,\cdots,x_n$均为正数,故
$$x_1=x_2,$$
同理,将前n个式子两两相减,即可得
$$x_1=x_2=\cdots=x_n.$$
代入最后一个式子可得
$$x_1=x_2=\cdots=x_n=\frac{c}{n}.$$
此为惟一驻点,必为最大值点,故最大值为$f_{max}=\frac{c}{n}$.