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Expected Value of Distribution Information for the Newsvendor Problem

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This paper extends previous work on the distribution-free newsvendor problem, where only partial information about the demand distribution is available. More specifically, the analysis assumes that the demand distribution f belongs to a class of probability distribution functions (pdf) $\mathscr{F}$ with mean μ and standard deviation σ. While previous work has examined the expected value of distribution information (EVDI) for a particular order quantity and a particular pdf f, this paper aims at computing the maximum EVDI over all $f \in \mathscr{F}$ for any order quantity. In addition, an optimization procedure is provided to calculate the order quantity that minimizes the maximum EVDI.

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1. Introduction

The newsvendor problem is a popular model in the operations management literature. The problem is simple, attractive, and yet rich enough to capture the fundamentals of many business decisions. In fact, the newsvendor problem has served as a building block for numerous models in inventory management (Zheng 1992, Chen and Zheng 1998, Hariga and Ben-Daya 1999), supply chain management and coordination (Shang and Song 2002), yield management (Gallego and van Ryzin 1994), scheduling (Baker and Scudder 1990), option pricing models (Lo 1987), and many other areas. See Khouja (1999) for a survey of various newsvendor-related models and their applications.

In this paper, we study the following variation of the classical newsvendor problem:

$$\min_q G_f(q) \equiv c_u \mathrm{E}(z-q)^+ + c_o \mathrm{E}(q-z)^+, \tag{1}$$

where q is the decision variable (for order size), z is a random variable (for customer demand) described by a probability density function (pdf) f, and $c_u, c_o > 0$ represent "underage" and "overage" costs, respectively. In addition, t^+ represents the positive portion of t, i.e., $t^+ = \max\{t, 0\}$, and $\mathrm{E}(\cdot)$ is the expected value operator. It can be shown that problem (1) is equivalent to the classical newsvendor problem after some parameter transformations, and the optimal decision can be obtained if pdf f is known.

In many practical situations, however, the true demand distribution may not be easily and accurately estimated. In this paper, we are interested in the situation where the pdf f belongs to a class of pdfs $\mathscr{F}$ with a given mean μ and a given standard deviation σ. The order size decision q based only on the partial distribution information is often referred to as the "distribution-free decision." This problem was first studied by Scarf (1958), where he obtained the following upper bound for $G_f(q)$:

$$G_u(q) \equiv \frac{(c_o - c_u)}{2}(q-\mu) + \frac{(c_u + c_o)}{2}[\sigma^2 + (q-\mu)^2]^{1/2}. \tag{2}$$

The bound is shown to be tight and achievable by a two-point pdf in $\mathscr{F}$. Scarf also obtained the following distribution-free solution, often referred to as "Scarf's rule,"

$$q_u^* \equiv \mu + \frac{\sigma}{2}\left[\left(\frac{c_u}{c_o}\right)^{1/2} - \left(\frac{c_o}{c_u}\right)^{1/2}\right],$$

as the solution of the following minimax problem:

$$\min_q \max_{f \in \mathscr{F}} G_f(q) = \min_q G_u(q).$$

Scarf's results were resurrected by Gallego and Moon (1993), who provided a compact new proof for the upper bound $G_u(q)$. Gallego (1991) also identified the range of optimal solutions for (1) over all $f \in \mathscr{F}$, and a lower bound of $G_f(q)$:

$$G_l(q) \equiv c_u(\mu - q)^+ + c_o(q-\mu)^+. \tag{3}$$

In this paper, we show that the lower bound $G_l(q)$ is tight.

Because any distribution-free order size q is not optimal for all pdfs $f \in \mathcal{F}$, it is important to measure its performance before implementing the decision in practice. For a true demand pdf f, the performance of a distribution-free decision q can be measured by the associated expected value of distribution information (EVDI), which is defined as the difference in cost functions between the decision q and the optimal decision under the true pdf f. For example, Gallego and Moon (1993) evaluated the performance of Scarf's rule by calculating its EVDI when f is a normal distribution and concluded that the resulting EVDI is relatively small. However, there are an infinite number of pdfs in $\mathcal{F}$, and it is impossible to calculate EVDIs for all pdfs in $\mathcal{F}$. In this paper, for any decision q, we show how to calculate the associated maximum EVDI over all pdfs in $\mathcal{F}$. Clearly, the maximum EVDI can be used as a robustness measurement for the distribution-free decision q. Furthermore, we also propose an optimization procedure to find the minimax EVDI decision, the most robust decision as measured by the EVDI for the newsvendor problem. It is worthwhile to note that the distribution-free approach, as taken by this paper, aims at a robust solution by bounding the cost space.

2. Two-Point Distributions in $\mathcal{F}$ and Their Properties

The two-point pdfs in $\mathcal{F}$ play a critical role in our calculation for the maximum EVDI. In this section, we will provide a neat representation for these two-point pdfs and investigate their relationships with the upper and lower bounds of the cost function.

Consider a two-point pdf $T(\gamma)$, with parameter $-c_u < \gamma < c_o$, that assigns weights

$$w_1(\gamma) = \frac{c_u+\gamma}{c_o+c_u} \quad \text{and} \quad w_2(\gamma) = \frac{c_o-\gamma}{c_o+c_u}$$

to points

$$q_1(\gamma) = \mu - \sigma\left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2} \quad \text{and} \quad q_2(\gamma) = \mu + \sigma\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2}, \tag{4}$$

respectively. It is straightforward to verify that $T(\gamma) \in \mathcal{F}$ for all $-c_u < \gamma < c_o$. When γ increases from $-c_u$ to c_o, $q_1(\gamma)$ increases from $-\infty$ to μ and $q_2(\gamma)$ increases from μ to $+\infty$. In fact, the set $\{T(\gamma) \mid -c_u < \gamma < c_o\}$ represents all two-point pdfs in $\mathcal{F}$. Based on (1) and (4), the cost function $G_{T(\gamma)}(q)$, after some algebraic calculations, can be simplified as follows:

$$G_{T(\gamma)}(q) = \begin{cases} -c_u(q-\mu) & \text{if } q < q_1(\gamma), \\ \sigma[(c_o-\gamma)(c_u+\gamma)]^{1/2} + \gamma(q-\mu) & \text{if } q_1(\gamma) \leqslant q \leqslant q_2(\gamma), \\ c_o(q-\mu) & \text{if } q > q_2(\gamma). \end{cases} \tag{5}$$

Clearly, $G_{T(\gamma)}(q)$ is a piecewise-linear convex function, and the slopes of the three linear functions are $-c_u$, γ, and c_o, respectively. Denote $q_m(\gamma)$ as the middle point between $q_1(\gamma)$ and $q_2(\gamma)$:

$$q_m(\gamma) \equiv \frac{1}{2}[q_1(\gamma)+q_2(\gamma)] = \mu + \frac{\sigma}{2}\left[\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2} - \left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right], \tag{6}$$

it is easy to verify (see the appendix) that

$$G_{T(\gamma)}(q) = \begin{cases} G_l(q) & \text{if } q \leqslant q_1(\gamma) \text{ or } q \geqslant q_2(\gamma), \\ G_u(q) & \text{if } q = q_m(\gamma). \end{cases} \tag{7}$$

Therefore, the upper bound $G_u(q)$ and the lower bound $G_l(q)$ of $G_f(q)$ are naturally linked through the two-point pdf $T(\gamma)$. See Figure 1 for a geometric illustration. Such a relationship also allows us to provide a simple proof for the tightness of the upper bound $G_u(q)$ and the lower bound $G_l(q)$. Indeed, for any given q, there exists a $-c_u < \gamma < c_o$ such that $q_m(\gamma) = q$ and the upper bound $G_u(q)$ is achieved by the two-point pdf $T(\gamma)$. On the other hand, for any q, there exists a $-c_u < \gamma < c_o$ such that either $q \leqslant q_1(\gamma)$ or $q \geqslant q_2(\gamma)$. Therefore, the lower bound $G_l(q)$ is also achieved by the two-point pdf $T(\gamma)$. Furthermore, the upper and lower bounds can be constructed through $G_{T(\gamma)}(q)$; as γ increases from $-c_u$ to c_o, $G_{T(\gamma)}(q_1(\gamma))$ and $G_{T(\gamma)}(q_2(\gamma))$ forms a V-shaped lower-bound curve $G_l(q)$. In the meantime, $G_{T(\gamma)}(q_m(\gamma))$ forms the shallow U-shaped upper-bound curve $G_u(q)$. More precisely, we have

$$G_u(q) = \sup_{-c_u<\gamma<c_o} G_{T(\gamma)}(q) \quad \text{and}$$
$$G_l(q) = \inf_{-c_u<\gamma<c_o} G_{T(\gamma)}(q).$$

In other words, the upper and lower envelopes of $G_{T(\gamma)}(q)$ form the upper and lower bounds of $G_f(q)$, respectively. In Figure 1, the U-shaped upper bound $G_u(q)$ attains its minimum at q_u^* (Scarf's rule). The V-shaped lower bound $G_l(q)$ attains its minimum at the mean μ. $G_{T(\gamma)}(q)$ represents the cost function when f is a two-point distribution $T(\gamma)$ $(-c_u < \gamma < c_o)$. It coincides with the lower bound $G_l(q)$ when $q \leqslant q_1(\gamma)$ or $q \geqslant q_2(\gamma)$ and tangents the upper bound $G_u(q)$ at $q_m(\gamma) = [q_1(\gamma)+q_2(\gamma)]/2$.

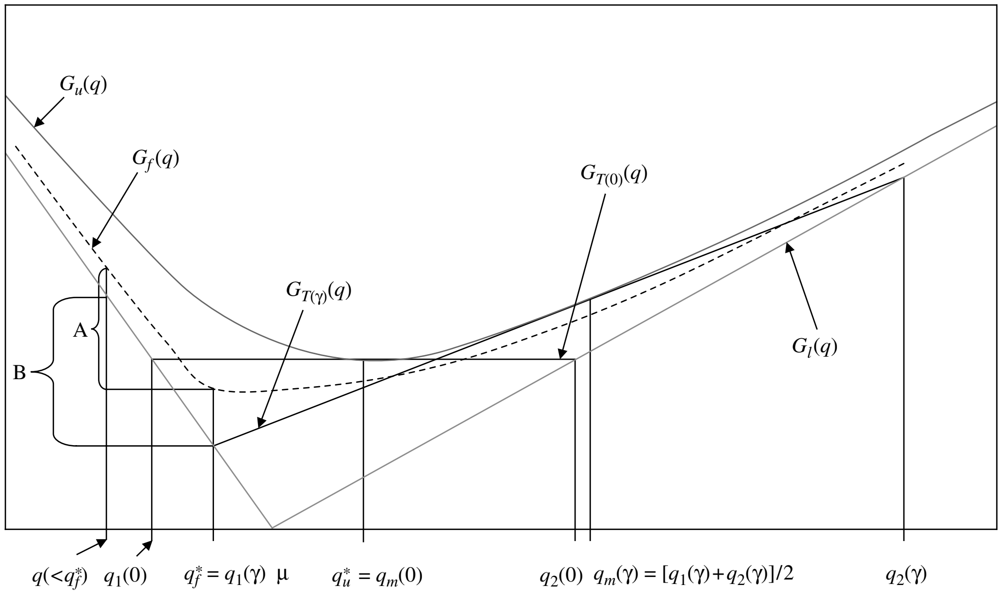
For any given pdf $f \in \mathcal{F}$, there is an associated optimal solution q_f^* to problem (1). Gallego (1991) showed the following range of optimal solutions based on Cantelli and Marshall's inequality:

$$q_1(0) \leqslant q_f^* \leqslant q_2(0) \quad \text{for all } f \in \mathcal{F}. \tag{8}$$

A more direct proof is possible based on our analysis of the two-point pdfs $T(\gamma)$ and the geometry shown in Figure 1: Consider the two-point pdf $T(0)$. By definition,

$$G_f(q_f^*) \leqslant G_f(q_u^*) \leqslant G_u(q_u^*) \quad \text{for all } f \in \mathcal{F}.$$

Figure 1. The bounds of objective function, optimal solution range, and proof of Theorem 1: Case 1.



Note. A: EVDI for pdf f; B: EVDI for two-point pdf $T(\gamma)$.

On the other hand, for any $f \in \mathcal{F}$, we have

$$G_f(q) \geqslant G_l(q) = G_{T(0)}(q) > G_{T(0)}(q_2(0)) = G_{T(0)}(q_1(0)) \\ = G_u(q_u^*) \quad \text{for all } q \notin [q_1(0), q_2(0)],$$

where the first equality follows from (7), the strict inequality and the remaining equalities follow from (5) with $\gamma = 0$. Therefore, the range of optimal solutions is given by (8). Furthermore, because all $q_1(0) \leqslant q \leqslant q_2(0)$ are optimal solutions of (1) when $f = T(0)$, the optimal solution range is tight.

Note that Scarf's rule happens to be the midpoint of the optimal solution range $[q_1(0), q_2(0)]$, i.e.,

$$q_u^* = q_m(0) = \frac{q_1(0) + q_2(0)}{2}.$$

Therefore, Scarf's rule is not only the optimal solution for the worst-case scenario, but also a "robust" solution with respect to the optimal solution range.

3. Largest EVDI for Any Decision q

Because any distribution-free order size q is not optimal for all pdfs $f \in \mathcal{F}$, a natural follow-up question is how to measure its performance. In this section, we develop a procedure to calculate the maximum EVDI for any given decision q over all pdfs in $\mathcal{F}$.

Let q be a chosen decision. Define the EVDI of q under distribution f as

$$\text{EVDI}_f(q) \equiv G_f(q) - G_f(q_f^*).$$

Clearly, EVDI_f represents the amount of improvement in the cost function G when the information on the pdf f is available and is used to optimize the decision variable q. Define the maximum EVDI as

$$\text{EVDI}_{\max}(q) \equiv \max_{f \in \mathcal{F}} \text{EVDI}_f(q).$$

Our calculation of $\text{EVDI}_{\max}(q)$ follows a two-step process. First, we prove in Theorem 1 that for any given q, the maximum EVDI is achieved by a two-point pdf in $\mathcal{F}$. We then locate the worst-case pdf by searching among the subset of $\mathcal{F}$ that consists of only two-point pdfs.

Theorem 1. *For any q and $f \in \mathcal{F}$, there exists a two-point pdf $T(\gamma) \in \mathcal{F}$ with $-c_u < \gamma < c_o$ such that*

$$EVDI_f(q) \leqslant EVDI_{T(\gamma)}(q).$$

Proof. Let $f \in \mathcal{F}$ be an arbitrary but fixed pdf. Because problem (1) is symmetric in structure, we may assume, without loss of generality, that $q_f^* < \mu$. Because $q_f^* < \mu$, there exists a two-point pdf $T(\gamma)$ such that $\gamma > 0$ and $q_1(\gamma) = q_f^* = q_{T(\gamma)}^*$. In addition, by (7), we have $G_{T(\gamma)}(q_f^*) = G_l(q_f^*) \leqslant G_f(q_f^*)$.

Function $G_{T(\gamma)}(q)$ represents a piecewise-linear convex approximation of the convex function $G_f(q)$ and both functions attain the same minimum at $q_1(\gamma) = q_f^*$. For any given q, we prove that Theorem 1 is true under each of the following three cases: $q < q_f^*$, $q_f^* \leqslant q \leqslant q_m(\gamma)$, and $q > q_m(\gamma)$.

Case 1. $q < q_f^*$. We have

$$\text{EVDI}_f(q) \\ = G_f(q) - G_f(q_f^*) = \left.\frac{dG_f(q)}{dq}\right|_{q=\tilde{q}} (q - q_f^*) \leqslant (-c_u)(q - q_f^*) \\ = G_{T(\gamma)}(q) - G_{T(\gamma)}(q_f^*) = \text{EVDI}_{T(\gamma)}(q).$$

In the above derivation, the second equality follows the mean value theorem, where $q < \tilde{q} < q_f^*$; the inequality follows from the fact that

$$\frac{dG_f(q)}{dq} \geqslant \frac{dG_{T(\gamma)}(q)}{dq} = -c_u \quad \text{for all } q < q_f^*;$$

the third and fourth equalities follow from the fact that q_f^* is also a minimum of $G_{T(\gamma)}(q)$. See Figure 1 for a geometric interpretation. Note that q_f^* is a minimum of the cost function $G_f(q)$. The two-point distribution $T(\gamma)$ is constructed such that $q_1(\gamma)$, which minimizes the associated cost function $G_{T(\gamma)}(q)$, is the same as q_f^*.

Case 2. $q_f^* \leqslant q \leqslant q_m(\gamma)$. Note that $G_f(q)$ is a convex function and $G_{T(\gamma)}(q)$ is a linear function in this range. Because

$$G_{T(\gamma)}(q_f^*) \leqslant G_f(q_f^*) \quad \text{and}$$
$$G_{T(\gamma)}(q_m(\gamma)) = G_u(q_m(\gamma)) \geqslant G_f(q_m(\gamma)),$$

G_f and $G_{T(\gamma)}$ intersect only once at some point $\bar{q}$ with $q_f^* \leqslant \bar{q} \leqslant q_m(\gamma)$. If $q_f^* \leqslant q \leqslant \bar{q}$, we have

$$\frac{dG_f(q)}{dq} \leqslant \frac{dG_{T(\gamma)}(q)}{dq} = \gamma$$

because $G_f(q)$ is a convex function. Following the similar argument as in Case 1, we have $\text{EVDI}_f(q) \leqslant \text{EVDI}_{T(\gamma)}(q)$. If $\bar{q} \leqslant q \leqslant q_m(\gamma)$, we have $G_{T(\gamma)}(q) \geqslant G_f(q)$ and the result follows immediately. See Figure 2 for a geometric interpretation.

Case 3. $q > q_m(\gamma)$. In this case, we construct another two-point pdf $T(\gamma')$ with $q = q_m(\gamma')$. Because $G_{T(\gamma')}$ tangents the upper-bound curve G_u at q by construction, $G_f(q) \leqslant G_u(q) = G_{T(\gamma')}(q)$. In addition, because $q = q_m(\gamma') > q_m(\gamma)$, we have

$$\begin{aligned} &\gamma' > \gamma \geqslant 0, \quad q_1(\gamma) < q_1(\gamma'), \quad \text{and} \\ &G_{T(\gamma)}(q_1(\gamma)) > G_{T(\gamma')}(q_1(\gamma')), \end{aligned} \tag{9}$$

where the above inequalities follow from (6), (4), and (5), respectively. Therefore, function $G_{T(\gamma')}(q)$ is another piecewise-linear convex approximation of the convex function $G_f(q)$, which attains its minimum at $q_1(\gamma') > q_f^*$. We have

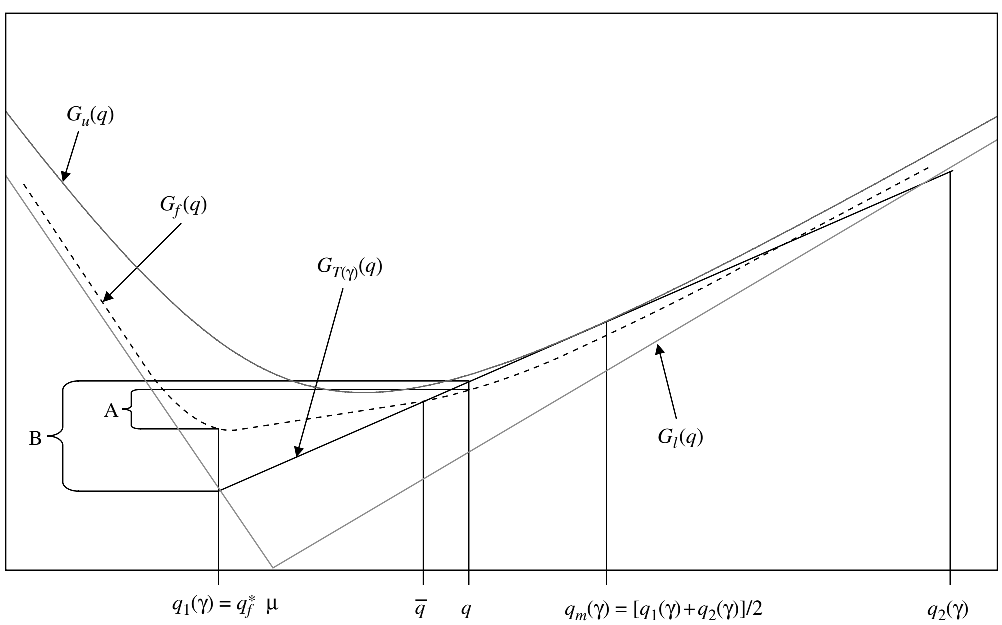
$$\begin{aligned} \text{EVDI}_f(q) = G_f(q) - G_f(q_f^*) &\leqslant G_{T(\gamma')}(q) - G_{T(\gamma)}(q_1(\gamma)) \\ &< G_{T(\gamma')}(q) - G_{T(\gamma')}(q_1(\gamma')) \\ &= \text{EVDI}_{T(\gamma')}(q). \end{aligned} \tag{10}$$

The first inequality in (10) is true because $G_f(q) \leqslant G_u(q) = G_{T(\gamma')}(q)$ by construction of q and $G_f(q_f^*) \geqslant G_l(q_f^*) = G_{T(\gamma)}(q_1(\gamma))$. The strict inequality in (10) is from (9). Therefore, the conclusion is also true for Case 3. See Figure 3 for a geometric interpretation. In Figure 3, the two-point distribution $T(\gamma')$, with the associated cost function $G_{T(\gamma')}(q)$, is chosen such that $q_m(\gamma') = q$.

In summary, for any q and $f \in \mathcal{F}$, there is a two-point pdf in $\mathcal{F}$ that leads to an EVDI that is at least as large as that associated with f.

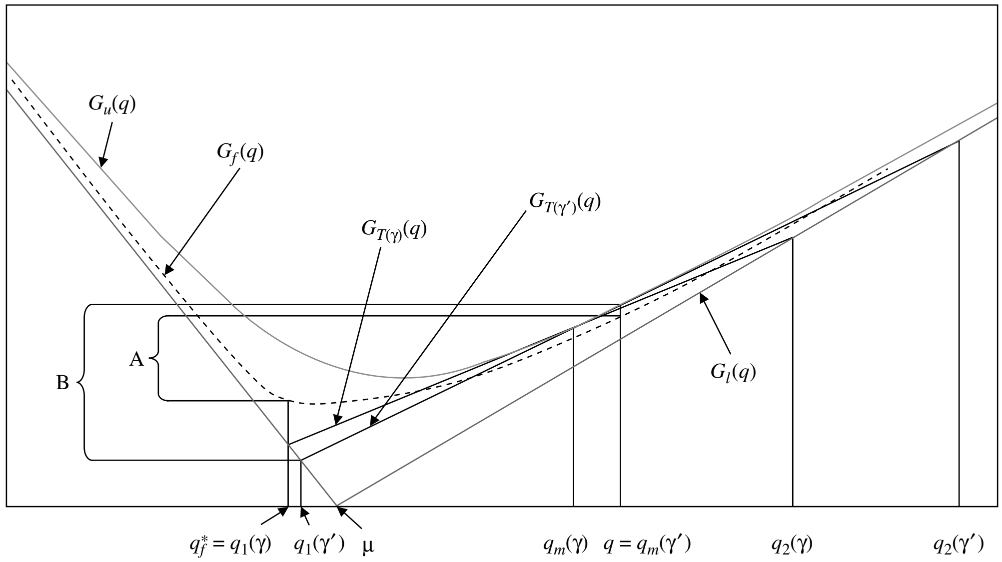
In view of Theorem 1, for any given decision q, we only have to search among two-point pdfs $T(\gamma) \in \mathcal{F}$ to find its largest EVDI, i.e., to find a parameter $-c_u \leqslant \gamma \leqslant c_o$ that

Figure 2. Proof of Theorem 1: Case 2.



Note. A: EVDI for pdf f; B: EVDI for two-point pdf $T(\gamma)$.

Figure 3. Proof of Theorem 1: Case 3.



Note. A: EVDI for pdf f; B: EVDI for two-point pdf $T(\gamma')$.

maximizes $\mathrm{EVDI}_{T(\gamma)}(q)$:

$$\mathrm{EVDI}_{\max}(q) = \max_{c_o \geqslant \gamma \geqslant -c_u} \{\mathrm{EVDI}_{T(\gamma)}(q)\}.$$

To simplify calculations for the maximum EVDI, define

$$V_+(q,\gamma) \equiv \gamma[q - q_1(\gamma)] = \gamma\left[q - \mu + \sigma\left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right] \quad \text{for all } c_o \geqslant \gamma \geqslant 0 \quad (11)$$

$$V_-(q,\gamma) \equiv \gamma[q - q_2(\gamma)] = \gamma\left[q - \mu - \sigma\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2}\right] \quad \text{for all } 0 \geqslant \gamma \geqslant -c_u. \quad (12)$$

By definition, $V_+(q, c_o) = c_o(q-\mu)$ and $V_-(q,-c_u) = -c_u(q-\mu)$.

Let us first consider the situation where $c_o \geqslant \gamma \geqslant 0$ and $q^*_{T(\gamma)} = q_1(\gamma)$. We have

$$\mathrm{EVDI}_{T(\gamma)}(q) = G_{T(\gamma)}(q) - G_{T(\gamma)}(q_1(\gamma))$$
$$= \begin{cases} \gamma[q-q_1(\gamma)] = V_+(q,\gamma) \\ \quad \text{if } q_1(\gamma) \leqslant q \leqslant q_2(\gamma), \\ c_o(q-\mu) - G_{T(\gamma)}(q_1(\gamma)) \leqslant V_+(q,c_o) \\ \quad \text{if } q \geqslant q_2(\gamma), \\ -c_u(q-\mu) - G_{T(\gamma)}(q_1(\gamma)) \leqslant V_-(q,-c_u) \\ \quad \text{if } q \leqslant q_1(\gamma). \end{cases}$$

On the other hand, because $G_{T(\gamma)}(q)$ is a convex function, we have

$$\mathrm{EVDI}_{T(\gamma)}(q) = G_{T(\gamma)}(q) - G_{T(\gamma)}(q_1(\gamma)) \geqslant \gamma[q-q_1(\gamma)] = V_+(q,\gamma).$$

It follows that

$$V_+(q,\gamma) \leqslant \mathrm{EVDI}_{T(\gamma)}(q) \leqslant \max\{V_+(q,\gamma), V_+(q,c_o), V_-(q,-c_u)\}. \quad (13)$$

In case $0 \geqslant \gamma \geqslant -c_u$, based on a similar argument, we have

$$V_-(q,\gamma) \leqslant \mathrm{EVDI}_{T(\gamma)}(q) \leqslant \max\{V_-(q,\gamma), V_+(q,c_o), V_-(q,-c_u)\}. \quad (14)$$

Combining (13) and (14), we have

$$\max_{c_o \geqslant \gamma \geqslant -c_u}\{\mathrm{EVDI}_{T(\gamma)}(q)\} = \max\left\{\max_{c_o \geqslant \gamma_1 \geqslant 0} V_+(q,\gamma_1), \max_{0 \geqslant \gamma_2 \geqslant -c_u} V_-(q,\gamma_2)\right\}.$$

Because $V_-(q,\gamma_2) \leqslant 0$ for all $q \geqslant q_2(0)$ and $V_+(q,\gamma_1) \leqslant 0$ for all $q \leqslant q_1(0)$, we have the following simplified formula for the maximum EVDI:

$$\mathrm{EVDI}_{\max}(q) = \begin{cases} \max\limits_{c_o \geqslant \gamma_1 \geqslant 0} V_+(q,\gamma_1) & \text{if } q \geqslant q_2(0), \\ \max\left\{\max\limits_{c_o \geqslant \gamma_1 \geqslant 0} V_+(q,\gamma_1), \max\limits_{0 \geqslant \gamma_2 \geqslant -c_u} V_-(q,\gamma_2)\right\} & \text{if } q_1(0) < q < q_2(0), \\ \max\limits_{0 \geqslant \gamma_2 \geqslant -c_u} V_-(q,\gamma_2) & \text{if } q \leqslant q_1(0). \end{cases} \quad (15)$$

The above formula can be further simplified by introducing the minimax EVDI decision q^e, which minimizes the maximum EVDI:

$$q^e = \arg\min_q \mathrm{EVDI}_{\max}(q). \quad (16)$$

Note that $V_+(q,\gamma_1)$ and $\max_{c_o\geqslant\gamma_1\geqslant 0} V_+(q,\gamma_1)$ are both increasing functions of q. Similarly, $V_-(q,\gamma_2)$ and $\max_{0\geqslant\gamma_2\geqslant -c_u} V_-(q,\gamma_2)$ are both decreasing functions of q. It follows that

$$q_1(0)\leqslant q^e\leqslant q_2(0) \tag{17}$$

and

$$\max_{c_o\geqslant\gamma_1\geqslant 0} V_+(q^e,\gamma_1)=\max_{0\geqslant\gamma_2\geqslant -c_u} V_-(q^e,\gamma_2). \tag{18}$$

Based on observations (17) and (18), we can simplify the formula for $\mathrm{EVDI}_{\max}(q)$ in (15) as follows:

$$\mathrm{EVDI}_{\max}(q)=\begin{cases}\max\limits_{c_o\geqslant\gamma_1\geqslant 0} V_+(q,\gamma_1) & \text{if } q\geqslant q^e,\\ \max\limits_{0\geqslant\gamma_2\geqslant -c_u} V_-(q,\gamma_2) & \text{if } q\leqslant q^e.\end{cases} \tag{19}$$

A procedure to calculate q^e will be provided in the next section.

To solve the optimization problems in (19), we introduce the following variable transformations:

$$\theta=\frac{q-\mu}{\sigma},\qquad \alpha=\frac{c_u}{c_o},\qquad x=\frac{\gamma_1}{c_o},\qquad y=-\frac{\gamma_2}{c_o}. \tag{20}$$

Denote

$$g_+(\theta,x)\equiv x\left[\theta+\left(\frac{1-x}{\alpha+x}\right)^{1/2}\right]\quad\text{and}$$

$$g_-(\theta,y)\equiv y\left[-\theta+\left(\frac{\alpha-y}{1+y}\right)^{1/2}\right].$$

With the above variable transformation, the formula for the maximum EVDI (19) reduces to

$$\mathrm{EVDI}_{\max}(q)=(c_o\sigma)\begin{cases}\max\limits_{1\geqslant x\geqslant 0} g_+(\theta,x) & \text{if } q\geqslant q^e,\\ \max\limits_{\alpha\geqslant y\geqslant 0} g_-(\theta,y) & \text{if } q\leqslant q^e.\end{cases} \tag{21}$$

Let $x^*(\theta)$ and $y^*(\theta)$ be the optimal solution of $\max_{1\geqslant x\geqslant 0} g_+(\theta,x)$ and $\max_{\alpha\geqslant y\geqslant 0} g_-(\theta,y)$, respectively. The next result shows that these optimal solutions are unique and can be conveniently computed from two unconstrained optimization problems.

Proposition 1. *Let q be any given decision.*

1. *If $q>q^e$, then $g_+(\theta,x)$ is a concave function of x for all $1>x\geqslant 0$. In addition, the optimal solution $x^*(\theta)$ is unique, and it can be obtained by solving the following polynomial equation*:

$$4\theta^2(1-x)(\alpha+x)^3=(2\alpha-3\alpha x+x-2x^2)^2. \tag{22}$$

2. *If $q<q^e$, then $g_-(\theta,y)$ is a concave function of y for all $\alpha>x\geqslant 0$. In addition, the optimal solution $y^*(\theta)$ is unique and it can be obtained by solving the following polynomial equation*:

$$4\theta^2(\alpha-y)(1+y)^3=(2\alpha-3y+\alpha y-2y^2)^2. \tag{23}$$

Proof. If $q>q^e$, then $q>q_1(0)$ because $q_1(0)\leqslant q^e\leqslant q_2(0)$. By the variable transformation (20), this implies $-\alpha^{-1/2}<\theta$.

The first derivative of $g_+(\theta,x)$ with respect to x is

$$\frac{dg_+(\theta,x)}{dx}=\theta+\left(\frac{1-x}{\alpha+x}\right)^{1/2}-\frac{(1+\alpha)x}{2(1-x)^{1/2}(\alpha+x)^{3/2}}.$$

In addition, we have

$$\lim_{x\to 0}\frac{dg_+(\theta,x)}{dx}=(\alpha)^{-1/2}+\theta>0 \tag{24}$$

and

$$\lim_{x\to 1}\frac{dg_+(\theta,x)}{dx}\to-\infty. \tag{25}$$

The second derivative of $g_+(\theta,x)$ with respect to x is

$$\frac{d^2g_+(\theta,x)}{d^2x}=-\frac{(1+\alpha)[3\alpha(1-x)+(\alpha+x)]}{4(1-x)^{3/2}(\alpha+x)^{5/2}}.$$

Clearly, it is negative for all $1>x\geqslant 0$. Therefore, $g_+(\theta,x)$ is a strictly concave function of $1>x\geqslant 0$, and the optimization problem $\max_{1\geqslant x\geqslant 0} g_+(\theta,x)$ has a unique optimal solution $x^*(\theta)$. Moreover, conditions (24) and (25) imply that $1>x^*(\theta)>0$. Consequently, the optimal solution $x^*(\theta)$ can be obtained by setting the first derivative of $g_+(\theta,x)$ to 0, which, after some algebraic calculations, reduces to Equation (22).

The proof for the second part of the theorem is omitted because it is symmetric to the above proof for the first part. □

In summary, the maximum EVDI for any decision q over all pdfs $f\in\mathcal{F}$ is given by

$$\mathrm{EVDI}_{\max}(q)=\begin{cases}(c_o\sigma)g_+[\theta,x^*(\theta)] & \text{if } q\geqslant q^e,\\ (c_o\sigma)g_-[\theta,y^*(\theta)] & \text{if } q\leqslant q^e,\end{cases}$$

where $x^*(\theta)$ and $y^*(\theta)$ are solutions to the polynomial Equations (22) and (23), respectively. Both polynomial Equations (22) and (23) have closed-form solutions in theory. However, they are too complicated to write down explicitly.

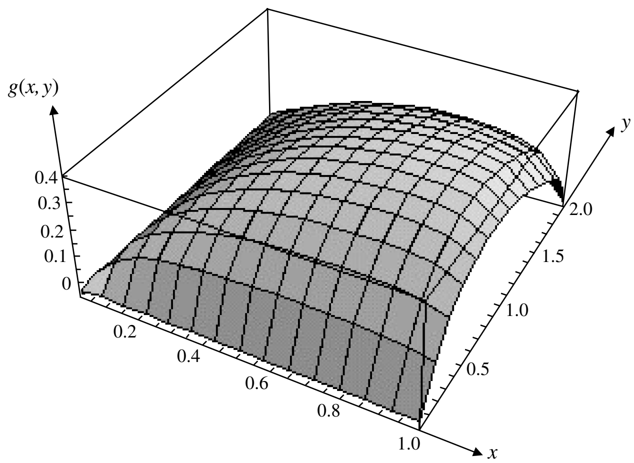
4. The Minimax EVDI Distribution-Free Decision

In the previous section, a procedure is provided to calculate the maximum EVDI for any given decision q. However, the calculation is based on the knowledge of the minimax EVDI decision q^e. In this section, we provide an optimization procedure to calculate q^e defined in (16).

In view of the characterization of q^e obtained in (18), q^e can be obtained by the following optimization problem:

$$\begin{aligned}&\max_{q,\,c_o\geqslant\gamma_1\geqslant 0,\,0\geqslant\gamma_2\geqslant -c_u} V_+(q,\gamma_1)\\ &\text{s.t. } V_+(q,\gamma_1)=V_-(q,\gamma_2).\end{aligned} \tag{26}$$

Figure 4. Function $g(x, y)$ with $\alpha = 2$.



Applying the variable transformation (20) to the constraint, we have

$$c_o\sigma x\left[\theta + \left(\frac{1-x}{\alpha+x}\right)^{1/2}\right] = c_o\sigma y\left[-\theta + \left(\frac{\alpha-y}{1+y}\right)^{1/2}\right].$$

Therefore,

$$\theta = \frac{1}{x+y}\left[y\left(\frac{\alpha-y}{1+y}\right)^{1/2} - x\left(\frac{1-x}{\alpha+x}\right)^{1/2}\right].$$

Substituting the above θ into the cost function, we can reduce the optimization problem (26) to

$$\max_{0\leqslant x\leqslant 1,\, 0\leqslant y\leqslant \alpha} g(x, y) \equiv \frac{xy}{x+y}\left[\left(\frac{\alpha-y}{1+y}\right)^{1/2} + \left(\frac{1-x}{\alpha+x}\right)^{1/2}\right]. \quad (27)$$

One can verify that $g(x, y)$ is a concave function of (x, y) by calculating its Jacobian. The plot of $g(x, y)$ when $\alpha = 2$ is given in Figure 4.

It follows that $g(x, y)$ has a unique maximum and the optimal solution of (27) can be determined by the first-order optimality condition. We now summarize the results in Theorem 2.

THEOREM 2. *Let (x^*, y^*) be the optimal solution of* (27). *The minimax EVDI and the associated decision q^e are given by*

$$EVDI_{\max}(q^e) = (\sigma c_o) g(x^*, y^*) \quad \textit{and} \quad q^e = \mu + \theta^e\sigma,$$

where

$$\theta^e = \frac{1}{x^*+y^*}\left[y^*\left(\frac{\alpha-y^*}{1+y^*}\right)^{1/2} - x^*\left(\frac{1-x^*}{\alpha+x^*}\right)^{1/2}\right].$$

In particular, if $\alpha = 1$ ($c_u = c_o$), we have

$$x^* = y^* = \frac{\sqrt{5}-1}{2} \approx 0.618,$$

$$g(x^*, y^*) = \frac{\sqrt{\sqrt{5}-2}\cdot(\sqrt{5}-1)}{2} \approx 0.3003, \qquad \theta^e = 0.$$

In this case, the minimax EVDI decision is $q^e = \mu$ and the corresponding minimax EVDI is $0.3003\sigma c_o$. Table 1 summarizes the optimal solutions of problem (27) for other values of $\alpha \geqslant 1$.

Clearly, when $c_u \geqslant c_o$ (or $\alpha \geqslant 1$), $q^e \geqslant \mu$. Note that the results for $\alpha \leqslant 1$ are inversely symmetrical. More specifically, parameters $1/\alpha$ and α generate the same $g(x^*, y^*)$ and the same θ^e value, but with different signs.

5. Numerical Examples

In this section, we provide two numerical examples to illustrate various results of this paper. The first example calculates the maximum EVDI for Scarf's rule q_u^* and compares its performance with that of the minimax EVDI decision q^e. The second example compares the robustness of several popular distribution-free decisions.

Our first example calculates the maximum EVDI associated with Scarf's rule q_u^*. Using the variable transformation (20), we have

$$\theta^u \equiv \frac{q_u^* - \mu}{\sigma} = \frac{1}{2}(\alpha^{1/2} - \alpha^{-1/2}).$$

The maximum EVDI of Scarf's rule q_u^* for various $\alpha \geqslant 1$ is summarized in Table 2. Note again that results for $\alpha \leqslant 1$ are inversely symmetrical. As expected, the maximum EVDI for q^e is smaller than that of q_u^*. Because $\theta^u \geqslant \theta^e$ for all $\alpha \geqslant 1$ we selected, we have $q_u^* \geqslant q^e$. Note that

$$\mathrm{EVDI}_{\max}(q_u^*) = (c_o\sigma) g_+[\theta^u, x^*(\theta^u)] \quad \text{for all } c_u \geqslant c_o$$

by formula (21).

Our second numerical experiment is based on the following newsvendor example introduced in Silver and Peterson (1985): The unit product cost is \$35.10, the unit selling price is \$50.30, and the unit salvage value is \$25.00. The mean and the standard deviation of the demand distribution are 900 and 122, respectively. The problem is equivalent to our problem (1) after some parameter transformations. The resulting underage and overage costs, c_u and c_o, are \$15.20 and \$10.10, respectively.

Based on the optimal solution range (8), the optimal order quantity should be in between 801 and 1,049. If f is a

Table 1. The θ^e and $g(x^*, y^*)$ for various $\alpha = c_u/c_o \geqslant 1$.

$\alpha = c_u/c_o$	1	2	3	4	5	6	7	8	9	10
θ^e	0	0.2770	0.4504	0.5832	0.6939	0.7906	0.8773	0.9565	1.030	1.099
$g(x^*, y^*)$	0.3003	0.4356	0.5513	0.6540	0.7468	0.8318	0.9105	0.9841	1.053	1.119

Table 2. The θ^u and $g_+[\theta^u, x^*(\theta^u)]$ for various $\alpha = c_u/c_o \geqslant 1$.

$\alpha = c_u/c_o$	1	2	3	4	5	6	7	8	9	10
θ^u	0	0.3536	0.5774	0.7500	0.8944	1.021	1.339	1.237	1.333	1.423
$g_+[\theta^u, x^*(\theta^u)]$	0.3003	0.4971	0.6637	0.8086	0.9373	1.054	1.160	1.259	1.352	1.439

Table 3. Comparison of several order quantities.

	Quantity	Profit range	Max EVDI (%)	Profit (normal) (%)	EVDI (normal) (%)
μ	900	[$12,135, $13,678]	643.41	12,446.59	39.50
q^e	920	[$12,165, $13,476]	460.68	12,481.14	4.95
q_u^*	925	[$12,166, $13,426]	497.70	12,484.61	1.48
q^N	931	[$12,165, $13,365]	543.39	12,486.09	0

normal distribution, the optimal order quantity is $q^N = 931$ and the associated expected profit is $12,485. Table 3 compares the performance of several order quantity choices, including average demand μ, Scarf's rule q_u^*, the minimax EVDI order quantity q^e, the optimal order quantity under the normal distribution q^N, in terms of their profit range, expected profit under the normal distribution, the maximum EVDI, and the EVDI under the normal distribution. The profit range for any order quantity q is calculated based on the upper bound $G_u(q)$ formula (2) and the lower bound $G_l(q)$ formula (3).

6. Concluding Remarks

In this paper, we study the expected value of distribution information for the classical newsvendor problem with limited information on its demand distribution. By investigating the properties of the class of two-point distributions, we are able to provide compact proofs for the tightness of the lower bound of the cost function as well as the optimal solution range. In addition, for any distribution-free decision, we obtain the maximum EVDI, which can be used to measure the robustness of the decision. The results presented in this paper are closely related to the three criteria for decision making under uncertainty, namely, minimax, minmin, and minimax regret. For the newsvendor problem, Scarf's rule q_u^* corresponds to the minimax decision. One can easily verify that the minmin decision is always the mean μ because it minimizes the tight lower bound G_l. The minimax EVDI decision q^e we obtained in this paper corresponds to the minimax regret decision, where the EVDI is the "regret" in the newsvendor problem setting. In practice, both q_u^* and q^e can be used as conservative distribution-free decisions and each has its own merits.

While our paper aims at a robust solution by bounding the cost space, there is another approach, with an emphasis on matching supply and demand, that bounds the order decision in the solution space. We refer to Agrawal and Seshadri (2000) for this approach. We have also tried to extend the results of this paper to the case where the order quantity q is restricted to be nonnegative as well as the case where the cost function is quadratic. However, the technique developed in this paper does not seem to work. Because the newsvendor problem serves as a building block for numerous models in operations management and, more recently, in supply chain management, an interesting question would be to extend the minimax EVDI decision and the corresponding analysis to those more complicated settings. It would be interesting to find out to what extent the technique developed in this paper remains applicable.

Appendix

Proof. $G_{T(\gamma)}(q_m(\gamma)) = G_u(q_m(\gamma))$ for all $-c_u < \gamma < c_o$. Because $q_1(\gamma) \leqslant q_m(\gamma) \leqslant q_2(\gamma)$, by (5), we have

$$\begin{aligned}
G_{T(\gamma)}(q_m(\gamma)) &= \sigma[(c_o-\gamma)(c_u+\gamma)]^{1/2} \\
&\quad + \frac{\sigma\gamma}{2}\left[\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2} - \left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right] \\
&= \frac{\sigma}{2}\left\{[(c_o-\gamma)(c_u+\gamma)]^{1/2} + \gamma\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2}\right. \\
&\quad \left. + [(c_o-\gamma)(c_u+\gamma)]^{1/2} - \gamma\left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right\} \\
&= \frac{\sigma}{2}\left[c_o\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2} + c_u\left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right].
\end{aligned}$$

On the other hand, substituting $q_m(\gamma)$ into (2), we have

$$\begin{aligned}
&G_u(q_m(\gamma)) \\
&= \left(\frac{c_o-c_u}{2}\right)\frac{\sigma}{2}\left[\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2} - \left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right] \\
&\quad + \left(\frac{c_u+c_o}{2}\right)\left\{\sigma^2 + \frac{\sigma^2}{4}\left[\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2} - \left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right]^2\right\}^{1/2} \\
&= \left(\frac{c_o-c_u}{2}\right)\frac{\sigma}{2}\left[\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2} - \left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right] \\
&\quad + \left(\frac{c_u+c_o}{2}\right)\frac{\sigma}{2}\left[\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2} + \left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right] \\
&= \frac{\sigma}{2}\left[c_o\left(\frac{c_u+\gamma}{c_o-\gamma}\right)^{1/2} + c_u\left(\frac{c_o-\gamma}{c_u+\gamma}\right)^{1/2}\right].
\end{aligned}$$

It follows that $G_{T(\gamma)}(q_m(\gamma)) = G_u(q_m(\gamma))$ for any $-c_u < \gamma < c_o$. □

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References

Agrawal, V., S. Seshadri. 2000. Distribution free bounds for fill rate optimization of (r, Q) inventory policies. *Naval Res. Logist.* **47** 635–656.

Baker, K., G. Scudder. 1990. Sequencing with earliness and tardiness penalties: A review. *Oper. Res.* **38** 22–36.

Chen, F., Y. S. Zheng. 1998. Near-optimal echelon-stock (R, nQ) policies in a multistage serial system. *Oper. Res.* **46** 592–602.

Gallego, G. 1991. The distribution-free newsboy problem: Review and extensions. Working paper, Columbia University, New York.

Gallego, G., I. Moon. 1993. The distribution-free newsvendor problem: Review and extensions. *J. Oper. Res. Soc.* **44** 825–834.

Gallego, G., G. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Sci.* **40** 999–1020.

Hariga, M., M. Ben-Daya. 1999. Some stochastic inventory models with deterministic variable lead time. *Eur. J. Oper. Res.* **113** 42–51.

Khouja, M. 1999. The single-period (news-vendor) problem: Literature review and suggestions for future research. *Omega: Internat. J. Management Sci.* **27** 537–553.

Lo, A. 1987. Semi-parametric upper bounds for option prices and expected payoffs. *J. Financial Econom.* **19** 373–388.

Scarf, H. 1958. A min-max solution of an inventory problem. *Studies in the Mathematical Theory of Inventory and Production*. Stanford University Press, Stanford, CA, 201–209.

Shang, K., J. S. Song. 2002. Newsvendor bounds and heuristic for optimal policies in serial supply chains. *Management Sci.* **49** 618–638.

Silver, E. A., R. Peterson. 1985. *Decision Systems for Inventory Management and Production Planning*, 2nd ed. John Wiley and Sons, New York.

Zheng, Y. S. 1992. On properties of stochastic inventory systems. *Management Sci.* **38** 87–103.