Applications of Duality

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Production Problem I

$$\max\ \mathbf{p}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \le \mathbf{r}, \quad \mathbf{x} \ge \mathbf{0}$$

where

- $\mathbf{p}$: profit margin vector
- A: resources consumption rate matrix
- $\mathbf{r}$: available resource vector
- $\mathbf{x}$: production level decision vector

Production Problem II: Liquidation Pricing

- $\mathbf{y}$: the fair price vector
- $A^T\mathbf{y} \geq \mathbf{p}$: competitiveness
- $\mathbf{y} \geq 0$: positivity
- $\min\ \mathbf{r}^T\mathbf{y}$: minimize the total liquidation cost

$$
Primal: \quad \begin{array}{lrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 1 \\
 & & x_2 & \leq 1 \\
 & x_1 & +x_2 & \leq 1.5 \\
 & x_1, & x_2 & \geq 0.
\end{array}
$$

$$
Dual: \quad \begin{array}{lrrrl}
\text{minimize} & y_1 & +y_2 & +1.5y_3 & \\
\text{subject to} & y_1 & & +y_3 & \geq 1 \\
 & & y_2 & +y_3 & \geq 2 \\
 & y_1, & y_2, & y_3 & \geq 0.
\end{array}
$$

Optimal Value Function I

For fixed matrix A and objective coefficient vector $\mathbf{c}$, the optimal value is a function of right-hand-side vector $\mathbf{b}$:

$$f_b(\mathbf{b}) = \begin{array}{rl} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Theorem: $f_b(\mathbf{b})$ is a convex function in $\mathbf{b}$, that is, for any $0 \leq \alpha \leq 1$

$$f_b(\alpha\mathbf{b}_1 + (1-\alpha)\mathbf{b}_2) \leq \alpha f_b(\mathbf{b}_1) + (1-\alpha) f_b(\mathbf{b}_2).$$

Optimal Value Function II

For fixed matrix A and right-hand-side vector $\mathbf{b}$, the optimal value is a function of objective coefficient vector $\mathbf{c}$:

$$\begin{array}{rll} f_c(\mathbf{c}) = & \text{minimize} & \mathbf{c}^T\mathbf{x} \\ & \text{subject to} & A\mathbf{x} = \mathbf{b}, \\ & & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Theorem: $f_c(\mathbf{c})$ is a concave function in $\mathbf{c}$.

Optimal Value Function III

Consider the dual representation of $f_c(\mathbf{c})$:

$$
\begin{array}{rll}
f_c(\mathbf{c}) = & \text{maximize} & \mathbf{b}^T\mathbf{y} \\
 & \text{subject to} & A^T\mathbf{y} \le \mathbf{c}
\end{array}
$$

We have for any $0 \le \alpha \le 1$

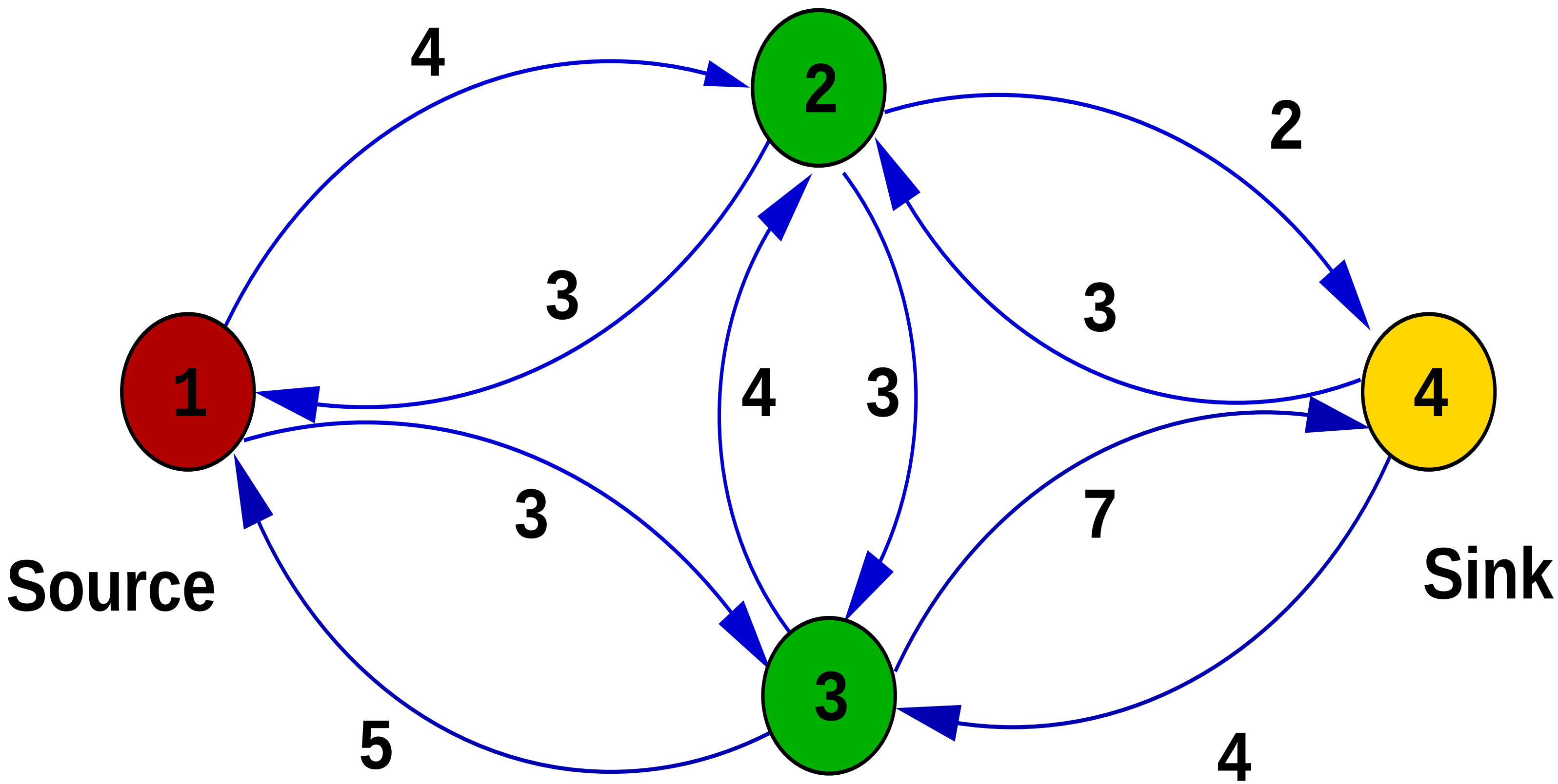
$$f_c(\alpha\mathbf{c}_1 + (1-\alpha)\mathbf{c}_2) \ge \alpha f_c(\mathbf{c}_1) + (1-\alpha) f_c(\mathbf{c}_2).$$

Max-Flow and Min-Cut

Given a directed graph with nodes $1, ..., n$ and edges A, where node 1 is called source and node n is called the sink, and each edge (i, j) has a flow rate capacity u_{ij}. The Max-Flow problem is to find the largest possible flow rate from source to sink.

Let x_{ij} be the flow rate from node i to node j. Then the problem can be formulated as

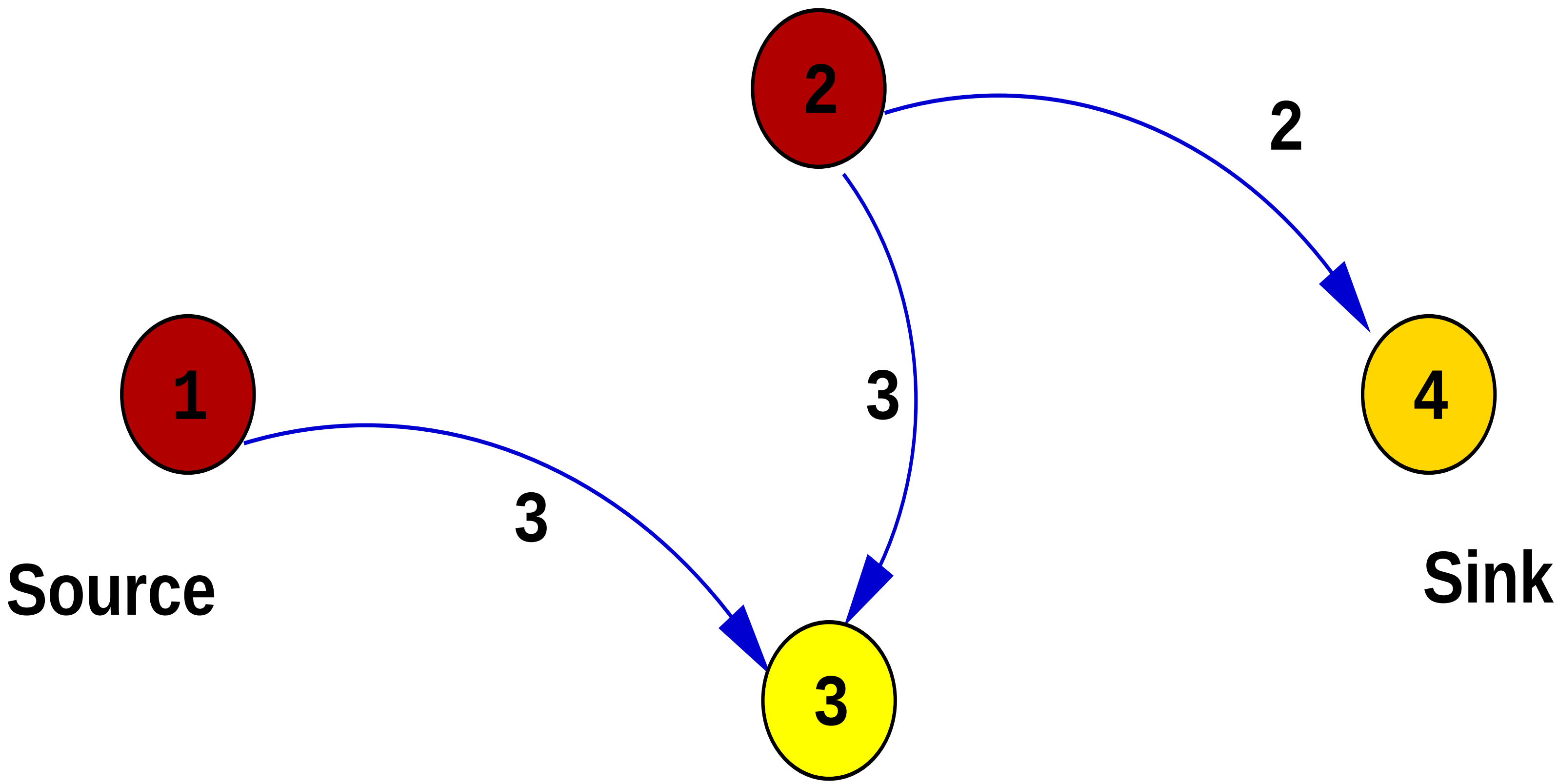
$$
\begin{array}{ll}
\text{maximize} & x_{n1} \\
\text{subject to} & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} - x_{n1} = 0,\ \forall i = 1, \\
& \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0, \forall i = 2, ..., n-1, \\
& \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} + x_{n1} = 0, \forall i = n, \\
& 0 \le x_{ij} \le u_{ij},\ \forall (i, j) \in A.
\end{array}
$$


4
2
2
3
3
4
3
1
4
3
7
Source
Sink
3
5
4

The dual of the Max-Flow problem

$$
\begin{array}{ll}
\text{minimize} & \sum_{(i,j)\in A} u_{ij} z_{ij} \\
\text{subject to} & y_i - y_j + z_{ij} \geq 0, \ \forall (i,j) \in A, \\
& -y_1 + y_n = 1, \\
& z_{ij} \geq 0, \ \forall (i,j) \in A.
\end{array}
$$

This problem is called the Min-Cut problem.


1
2
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2
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3
Source
Sink

Two-Person Zero-Sum Game

Let P be the payoff matrix of a two-person, ”column” and ”row”, zero-sum game.

$$P = \begin{pmatrix} +3 & -1 & -4 \\ -3 & +1 & +4 \end{pmatrix}$$

Players usually use randomized strategies in such a game. A randomized strategy is a vector of probabilities, each associated with a particular decision.

Nash Equilibrium

In a Nash Equilibrium, if your (column) strategy is a pure strategy (one where you always play a single action), the expected payout for the (dominating) action that you are playing should be greater than or equal to the expected payout for any other action. If you are playing a randomized strategy, the expected payout for each action included in your strategy should be the same (if one were lower, you won't want to ever choose that action) and these payouts should be greater than or equal to the actions that aren't part of your strategy.

LP formulation of Nash Equilibrium

”Column” strategy:

$$\begin{array}{ll} \max & v \\ s.t. & v\mathbf{e} \leq P\mathbf{x} \\ & \mathbf{e}^T\mathbf{x} = 1 \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

”Row” strategy:

$$\begin{array}{ll} \min & u \\ s.t. & u\mathbf{e} \geq P^T\mathbf{y} \\ & \mathbf{e}^T\mathbf{y} = 1 \\ & \mathbf{y} \geq \mathbf{0}. \end{array}$$

They are dual to each other.

Multi-Firm LP Alliance I

Consider a finite set I of firms each of whom has operations that have representations as linear programs. Suppose the linear program representing the operations of firm i in I entails choosing an n-column vector $\mathbf{x} \geq \mathbf{0}$ of activity levels that maximize the firm's profit

$$\mathbf{c}^T\mathbf{x}$$

subject to the constraint that its consumption $A\mathbf{x}$ of resources minorizes its available resource vector $\mathbf{b}^i$, that is,

$$A\mathbf{x} \leq \mathbf{b}^i.$$

Multi-Firm LP Alliance II

An alliance is a subset of the firms. If an alliance S pools its resource vectors, the linear program that S faces is that of choosing an n-column vector $\mathbf{x} \geq \mathbf{0}$ that maximizes the profit $\mathbf{c}^T\mathbf{x}$ that S earns subject to its resource constraint

$$A\mathbf{x} \leq \mathbf{b}^S = \sum_{i \in S} \mathbf{b}^i.$$

Let V^S be the resulting maximum profit of S. The grand alliance is the set I of all firms.

$$\begin{array}{rll} V^S := & \max & \mathbf{c}^T\mathbf{x} \\ & \text{s.t.} & A\mathbf{x} \leq \sum_{i \in S} \mathbf{b}^i, \\ & & \mathbf{x} \geq \mathbf{0}, \end{array}$$

Multi-Firm LP Alliance III: Core

Core is the set of payment vector $\mathbf{z} = (z_1, \ ..., \ z_{|I|})$ to each company such that

$$\sum_{i \in I} z_i = V^I$$

and

$$\sum_{i \in S} z_i \geq V^S, \ \forall S \subset I.$$

Theorem 1 *For each optimal dual price vector for the linear program of the grand alliance, allocating each firm the value of its resource vector at those prices yields a profit allocation vector in the core.*

Combinatorial Auction Pricing I

Given the m different states that are mutually exclusive and exactly one of them will be true at the maturity.

A contract on a state is a paper agreement so that on maturity it is worth a notional $\$w$ if it is on the winning state and worth $\$0$ if is not on the winning state. There are n orders betting on one or a combination of states, with a price limit and a quantity limit.

Combinatorial Auction Pricing II: an order

The jth order is given as $(\mathbf{a}_j \in R^m_+,\ \pi_j \in R_+,\ q_j \in R_+)$: $\mathbf{a}_j$ is the combination betting vector where each component is either 1 or 0

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \ldots \\ a_{mj} \end{pmatrix},$$

where 1 is winning and 0 is non-winning; π_j is the price limit for one such a contract, and q_j is the maximum number of contracts the better like to buy.

Combinatorial Auction Pricing III: Pricing each state

Let x_j be the number of contracts awarded to the jth order. Then, the jth better will pay the amount

$$\pi_j \cdot x_j$$

and the total collected amount is

$$\sum_{j=1}^{n} \pi_j \cdot x_j = \pi^T \mathbf{x}$$

If the ith state is the winning state, then the auction organizer need to pay back

$$w \cdot \left(\sum_{j=1}^{n} a_{ij} x_j \right)$$

The question is, how to decide $\mathbf{x} \in R^n$.

Combinatorial Auction Pricing IV: LP model

$$\begin{array}{lrl} \max & \pi^T \mathbf{x} - w \cdot s & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot s & \leq 0, \\ & \mathbf{x} & \leq \mathbf{q}, \\ & \mathbf{x} & \geq 0. \end{array}$$

$\pi^T \mathbf{x}$: the optimistic amount can be collected.

$w \cdot s$: the worst-case amount need to pay back.

Combinatorial Auction Pricing V: The dual of the model

$$
\begin{array}{rrl}
\min & \mathbf{q}^T\mathbf{y} & \\
\text{s.t.} & A^T\mathbf{p}+\mathbf{y} & \geq \pi, \\
 & \mathbf{e}^T\mathbf{p} & = w, \\
 & (\mathbf{p},\mathbf{y}) & \geq 0.
\end{array}
$$

$x_j > 0$	$\mathbf{a}_j^T\mathbf{p}+y_j=\pi_j$ so that $\mathbf{a}_j^T\mathbf{p}\leq\pi_j$
$0 < x_j < q_j$	$y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p}=\pi_j$
$x_j = 0$	$y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p}\geq\pi_j$

$\mathbf{p}$ represents the state price and it is Fair.

$$\mathbf{p}^T(A\mathbf{x}-\mathbf{e}\cdot s)=0 \quad \text{implies} \quad \mathbf{p}^A\mathbf{x}=\mathbf{p}^T\mathbf{e}\cdot s=w\cdot s.$$

Robust Optimization I

Consider a linear program

$$\begin{array}{ll} \text{minimize} & (\mathbf{c} + C\mathbf{u})^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \;=\; \mathbf{b}, \\ & \mathbf{x} \;\geq\; \mathbf{0}, \end{array}$$

where $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{u} \leq \mathbf{e}$ is a state of Nature and beyond decision maker's control.

Robust Model:

$$\begin{array}{ll} \text{minimize} & \max_{\{\mathbf{u} \geq \mathbf{z},\ \mathbf{u} \leq \mathbf{e}\}} (\mathbf{c} + C\mathbf{u})^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \;=\; \mathbf{b}, \\ & \mathbf{x} \;\geq\; \mathbf{0}. \end{array}$$

Robust Optimization II

Nature's (primal) problem:

$$\begin{array}{rrcl} \text{maximize}_{\mathbf{u}} & \mathbf{c}^T\mathbf{x} + \mathbf{x}^T C\mathbf{u} & & \\ \text{subject to} & \mathbf{u} & \leq & \mathbf{e}, \\ & \mathbf{u} & \geq & \mathbf{0}. \end{array}$$

Dual of Nature's problem:

$$\begin{array}{rrcl} \text{minimize}_{\mathbf{y}} & \mathbf{c}^T\mathbf{x} + \mathbf{e}^T\mathbf{y} & & \\ \text{subject to} & \mathbf{y} & \geq & C^T\mathbf{x}, \\ & \mathbf{y} & \geq & \mathbf{0}. \end{array}$$

Robust Optimization III

Decision Maker's Robust Model:

$$\begin{array}{rrcl} \text{minimize}_{\mathbf{x},\mathbf{y}} & \mathbf{c}^T\mathbf{x}+\mathbf{e}^T\mathbf{y} & & \\ \text{subject to} & \mathbf{y} & \geq & C^T\mathbf{x}, \\ & A\mathbf{x} & = & \mathbf{b}, \\ & \mathbf{x},\ \mathbf{y} & \geq & \mathbf{0}. \end{array}$$

Fisher's equilibrium price

Player $i \in B$'s optimization problem for given prices p_j, $j \in G$.

$$\begin{array}{ll} \text{maximize} & \mathbf{u}_i^T \mathbf{x}_i := \sum_{j \in G} u_{ij} x_{ij} \\ \text{subject to} & \mathbf{p}^T \mathbf{x}_i := \sum_{j \in G} p_j x_{ij} \leq w_i, \\ & x_{ij} \geq 0, \quad \forall j, \end{array}$$

Without losing generality, assume that the amount of each good is 1. The equilinitum price vector is the one that for all $j \in G$

$$\sum_{i \in B} x(\mathbf{p})_{ij} = 1$$

Equilibrium price conditions

Player $i \in B$'s dual problem for given prices p_j, $j \in G$.

$$\begin{array}{ll} \text{minimize} & w_i y_i \\ \text{subject to} & \mathbf{p} y_i \geq \mathbf{u}_i, \; y_i \geq 0 \end{array}$$

The necessary and sufficient conditions for an equilibrium point $\mathbf{x}_i, \mathbf{p}$ are:

$$\begin{array}{rl} \mathbf{p}^T \mathbf{x}_i \leq w_i, \; \mathbf{x}_i \geq \mathbf{0}, & \forall i, \\ p_j y_i \geq u_{ij}, \; y_i \geq 0, & \forall i, j, \\ \mathbf{u}_i^T \mathbf{x}_i = w_i y_i, & \forall i, \\ \sum_i x_{ij} = 1, & \forall j. \end{array}$$

Equilibrium price conditions continued

These conditions can be represented by

$$\begin{array}{ll} \sum_i p_i \le \sum_i w_i, \; \mathbf{x}_i \ge \mathbf{0}, & \forall i, \\ \frac{\mathbf{u}_i^T \mathbf{x}_i}{w_i} \cdot p_j \ge u_{ij}, & \forall i, j, \\ \sum_i x_{ij} = 1, & \forall j. \end{array}$$

since from the second inequality (after multiplying $x_i j$ to both sides and take sum over j) we have

$$\mathbf{p}^T \mathbf{x}_i \ge w_i, \; \forall i.$$

Then, from the rest conditions

$$\sum_i w_i \ge \sum_i p_i = \sum_i \mathbf{p}^T \mathbf{x}_i \ge \sum_i w_i.$$

Thus, these conditions imply $\mathbf{p}^T \mathbf{x}_i = w_i, \; \forall i$.

Equilibrium price property

If u_{ij} has at least one positive coefficient for every j, then we must have $p_j > 0$ for every j at every equilibrium. Morevoer, The second inequality can be rewritten as

$$\log(\mathbf{u}_i^T\mathbf{x}_i) + \log(p_j) \geq \log(w_i) + \log(u_{ij}),\ \forall i, j,\ u_{ij} > 0.$$

The function on the left is (strictly) concave in $\mathbf{x}_i$ and p_j. Thus,

Theorem 2 *The equilibrium set of the Fisher Market is convex, and the equilbrium price vector is unique.*

Example of Fisher's equilibrium price

Buyer $1, 2$'s optimization problems for given prices p_x, p_y.

$$\begin{array}{ll} \text{maximize} & 2x_1 + y_1 \\ \text{subject to} & p_x \cdot x_1 + p_y \cdot y_1 \le 5, \\ & x_1, y_1 \ge 0; \\ \text{maximize} & 3x_2 + y_2 \\ \text{subject to} & p_x \cdot x_2 + p_y \cdot y_2 \le 8, \\ & x_2, y_2 \ge 0. \end{array}$$

$$p_x = \frac{26}{3}, \quad p_y = \frac{13}{3}$$

$$x_1 = \frac{1}{13},\ y_1 = 1,\ x_2 = \frac{12}{13},\ y_2 = 0$$

Equilibrium price of the Arrow-Debreu market

Similary, the equilibrium conditions of the Arrow-Debreu market are

$$\begin{array}{rl} p_i > 0,\ \mathbf{x}_i \geq \mathbf{0}, & \forall i, \\ \frac{\mathbf{u}_i^T \mathbf{x}_i}{p_i} \cdot p_j \geq u_{ij}, & \forall i, j, \\ \sum_i x_{ij} = 1, & \forall j. \end{array}$$

Morevoer, The second inequality can be rewritten as

$$\log(\mathbf{u}_i^T \mathbf{x}_i) + \log(p_j) - \log(p_i) \geq \log(u_{ij}),\ \forall i, j,\ u_{ij} > 0.$$

Treat $\log(p_i)$ as variable y_i, then it becomes

$$\log(\mathbf{u}_i^T \mathbf{x}_i) + y_j - y_i \geq \log(u_{ij}),\ \forall i, j,\ u_{ij} > 0.$$

The function on the left is concave in $\mathbf{x}_i$ and y_j. Thus,

Theorem 3 *The equilibrium set of the Arrow-Debreu Market is convex in the logarithmic of price.*