第八章

第四节 多元复合函数的求导法则

一元复合函数 $y=f(u),\ u=\varphi(x)$

求导法则 $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}u}\cdot\dfrac{\mathrm{d}u}{\mathrm{d}x}$

微分法则 $\mathrm{d}y=f'(u)\,\mathrm{d}u=f'(u)\varphi'(x)\,\mathrm{d}x$

本节内容:

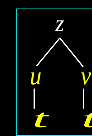
一、多元复合函数求导的链式法则

二、多元复合函数的全微分

一、多元复合函数求导的链式法则

定理. 若函数 $u=\varphi(t),\ v=\psi(t)$ 在点 t 可导, $z=f(u,v)$ 在点 (u,v) 处偏导连续, 则复合函数 $z=f(\varphi(t),\psi(t))$ 在点 t 可导, 且有链式法则

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$



证: 设 t 取增量$\triangle t$, 则相应中间变量有增量$\triangle u$,$\triangle v$,

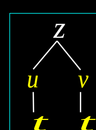
$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho)\quad\left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\right)$$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\frac{\Delta v}{\Delta t}+\frac{o(\rho)}{\Delta t}\quad\left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\right)$$

令 $\Delta t\to 0$, 则有 $\Delta u\to 0,\ \Delta v\to 0$,

$$\frac{\Delta u}{\Delta t}\to\frac{\mathrm{d}u}{\mathrm{d}t},\quad \frac{\Delta v}{\Delta t}\to\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\frac{o(\rho)}{\Delta t}=\frac{o(\rho)}{\rho}\sqrt{\left(\frac{\Delta u}{\Delta t}\right)^2+\left(\frac{\Delta v}{\Delta t}\right)^2}\to 0$$



($\triangle$t<0 时, 根式前加 “–”号)

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}}\quad\text{（全导数公式）}$$

说明: 若定理中 $f(u,v)$ 在点 (u,v) 偏导数连续减弱为偏导数存在, 则定理结论不一定成立.

例如: $z=f(u,v)=\begin{cases}\dfrac{u^2v}{u^2+v^2}, & u^2+v^2\neq 0\\ 0, & u^2+v^2=0\end{cases}$

$u=t,\quad v=t$

易知: $\left.\dfrac{\partial z}{\partial u}\right|_{(0,0)}=f_u(0,0)=0,\quad \left.\dfrac{\partial z}{\partial v}\right|_{(0,0)}=f_v(0,0)=0$

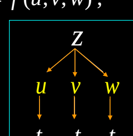
但复合函数 $z=f(t,t)=\dfrac{t}{2}$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{1}{2}\neq\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}=0\cdot 1+0\cdot 1=0$$

推广: 设下面所涉及的函数都可微 .

1) 中间变量多于两个的情形. 例如, $z=f(u,v,w)$, $u=\varphi(t),\ v=\psi(t),\ w=\omega(t)$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\partial z}{\partial w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t}$$
$$=f_1'\varphi'+f_2'\psi'+f_3'\omega'$$

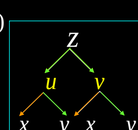


2) 中间变量是多元函数的情形.例如,

$z=f(u,v),\quad u=\varphi(x,y),\quad v=\psi(x,y)$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=f_1'\varphi_1'+f_2'\psi_1'$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=f_1'\varphi_2'+f_2'\psi_2'$$

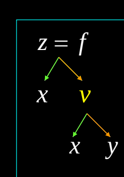


又如, $z=f(x,v),\ v=\psi(x,y)$

当它们都具有可微条件时, 有

$$\boxed{\frac{\partial z}{\partial x}}=\boxed{\frac{\partial f}{\partial x}}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=f_1'+f_2'\psi_1'$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=f_2'\psi_2'$$



注意: 这里 $\dfrac{\partial z}{\partial x}$ 与 $\dfrac{\partial f}{\partial x}$ 不同,

$\dfrac{\partial z}{\partial x}$ 表示固定 y 对 x 求导, $\dfrac{\partial f}{\partial x}$ 表示固定 v 对 x 求导

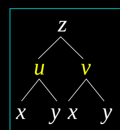
口诀: 分段用乘, 分叉用加, 单路全导, 叉路偏导

例1. 设 $z=e^u\sin v$, $u=xy$, $v=x+y$, 求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

解: $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$

$=e^u\sin v\cdot y+e^u\cos v\cdot 1$

$=e^{xy}[y\cdot\sin(x+y)+\cos(x+y)]$



$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$

$=e^u\sin v\cdot x+e^u\cos v\cdot 1$

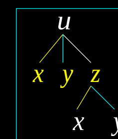
$=e^{xy}[x\cdot\sin(x+y)+\cos(x+y)]$

例2. $u=f(x,y,z)=e^{x^2+y^2+z^2}$, $z=x^2\sin y$, 求 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$

解: $\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$

$=2xe^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot 2x\sin y$

$=2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$



$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$

$=2ye^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot x^2\cos y$

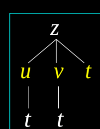
$=2(y+x^4\sin y\cos y)e^{x^2+y^2+x^4\sin^2 y}$

例3. 设 $z=uv+\sin t$, $u=e^t$, $v=\cos t$, 求全导数 $\frac{\mathrm{d}z}{\mathrm{d}t}$.

解: $\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\partial z}{\partial t}$

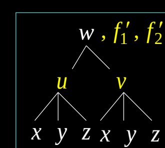
$=ve^t-u\sin t+\cos t$

$=e^t(\cos t-\sin t)+\cos t$



注意：多元抽象复合函数求导在偏微分方程变形与验证解的问题中经常遇到, 下列两个例题有助于掌握这方面问题的求导技巧与常用导数符号.

例4. 设 $w=f(x+y+z,xyz)$, f 具有二阶连续偏导数, 求 $\frac{\partial w}{\partial x},\frac{\partial^2 w}{\partial x\partial z}$.



解: 令 $u=x+y+z$, $v=xyz$, 则

$w=f(u,v)$

$\frac{\partial w}{\partial x}=f_1'\cdot 1+f_2'\cdot yz$

$=f_1'(x+y+z,xyz)+yz\,f_2'(x+y+z,xyz)$

$\frac{\partial^2 w}{\partial x\partial z}=f_{11}''\cdot 1+f_{12}''\cdot xy+y\,f_2'+yz[\,f_{21}''\cdot 1+f_{22}''\cdot xy]$

$=f_{11}''+y(x+z)f_{12}''+xy^2z\,f_{22}''+y\,f_2'$

二、多元复合函数的全微分

设函数 $z=f(u,v)$, $u=\varphi(x,y)$, $v=\psi(x,y)$ 都可微, 则复合函数 $z=f(\varphi(x,y),\psi(x,y))$ 的全微分为

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y$$

$$=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}y$$

$$=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y\right)$$

$$=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v$$

可见无论 u,v 是自变量还是中间变量, 其全微分表达形式都一样, 这性质叫做**全微分形式不变性.**

例 6. 利用全微分形式不变性再解例1.

解: $\mathrm{d}z=\mathrm{d}(e^u\sin v)$

$=e^u\sin v\,\mathrm{d}u+e^u\cos v\,\mathrm{d}v$

$=e^{xy}[\sin(x+y)\,\mathrm{d}(xy)+\cos(x+y)\mathrm{d}(x+y)]$

$=e^{xy}[\sin(x+y)(y\,\mathrm{d}x+x\mathrm{d}y)+\cos(x+y)(\mathrm{d}x+\mathrm{d}y)]$

$=e^{xy}[y\sin(x+y)+\cos(x+y)]\mathrm{d}x$

$\quad+e^{xy}[x\sin(x+y)+\cos(x+y)]\mathrm{d}y$

所以 $\frac{\partial z}{\partial x}=e^{xy}[y\cdot\sin(x+y)+\cos(x+y)]$

$\frac{\partial z}{\partial y}=e^{xy}[x\cdot\sin(x+y)+\cos(x+y)]$

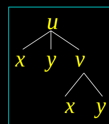
内容小结

1. 复合函数求导的链式法则

“分段用乘, 分叉用加, 单路全导, 叉路偏导”

例如, $u=f(x,y,v),\ v=\varphi(x,y)$,

$$\frac{\partial u}{\partial x}=f_1'+f_3'\,\varphi_1';\qquad \frac{\partial u}{\partial y}=f_2'+f_3'\cdot\varphi_2'$$



2. 全微分形式不变性

对 $z=f(u,v)$, 不论 u, v 是自变量还是因变量,

$$\mathrm{d}z=f_u(u,v)\,\mathrm{d}u+f_v(u,v)\,\mathrm{d}v$$

备用题

1. 已知 $f(x,y)\big|_{y=x^2}=1$, $f_1'(x,y)\big|_{y=x^2}=2x$, 求 $f_2'(x,y)\big|_{y=x^2}$.

解: 由 $f(x,x^2)=1$ 两边对 x 求导, 得

$$f_1'(x,x^2)+f_2'(x,x^2)\cdot 2x=0$$

$$f_1'(x,x^2)=2x$$

$$f_2'(x,x^2)=-1$$

2. 设函数 $z=f(x,y)$ 在点$(1,1)$处可微, 且

$$f(1,1)=1,\quad \frac{\partial f}{\partial x}\Big|_{(1,1)}=2,\quad \frac{\partial f}{\partial y}\Big|_{(1,1)}=3,$$

$\varphi(x)=f(x,f(x,x))$, 求 $\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\Big|_{x=1}$. **(2001考研)**

解: 由题设 $\varphi(1)=f(1,f(1,1))=f(1,1)=1$

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\Big|_{x=1}=3\varphi^2(x)\frac{\mathrm{d}\varphi}{\mathrm{d}x}\Big|_{x=1}$$

$$=3\Big[f_1'(x,f(x,x))+f_2'(x,f(x,x))\big(f_1'(x,x)+f_2'(x,x)\big)\Big]\Big|_{x=1}$$

$$=3\cdot\big[2+3\cdot(2+3)\big]=51$$