例10－1 设$L:\begin{cases} x^2+y^2+z^2=a^2, \\ y+z=a \end{cases}(a>0)$, 求$\oint_L y^2 ds$.

解 将$z=a-y$代入$x^2+y^2+z^2=a^2$,得

$$x^2+y^2+(y-a)^2=a^2 \Rightarrow x^2+(\sqrt{2}y-\frac{1}{\sqrt{2}}a)^2=(\frac{1}{\sqrt{2}}a)^2$$

令$x=\frac{a}{\sqrt{2}}\cos\theta, \sqrt{2}y-\frac{a}{\sqrt{2}}=\frac{a}{\sqrt{2}}\sin\theta$,则将$L$化成参数式:

$$x=\frac{a}{\sqrt{2}}\cos\theta,\ y=\frac{a}{2}+\frac{a}{2}\sin\theta,\ z=\frac{a}{2}-\frac{a}{2}\sin\theta\ (0\le\theta\le 2\pi)$$

$$\therefore ds=\sqrt{(-\frac{a}{\sqrt{2}}\sin\theta)^2+(\frac{a}{2}\cos\theta)^2+(-\frac{a}{2}\cos\theta)^2}\,d\theta=\frac{a}{\sqrt{2}}d\theta$$

$$\therefore \oint_L y^2 ds=\int_0^{2\pi}(\frac{a}{2}+\frac{a}{2}\sin\theta)^2\frac{a}{\sqrt{2}}d\theta$$

$$=\frac{a^3}{4\sqrt{2}}\int_0^{2\pi}(1+2\sin\theta+\sin^2\theta)^2\frac{a}{\sqrt{2}}d\theta$$

$$=\frac{3\pi a^3}{4\sqrt{2}}.$$

例10－2 求$\int_L xdy-ydx$, L:沿摆线$x=t-\sin t$, $y=1-\cos t$,从$O(0,0)$到$A(2\pi,0)$的一段.

解 $$\int_L xdy-ydx=\int_0^{2\pi}\left[(t-\sin t)\bullet\sin t-(1-\cos t)^2\right]dt$$

$$=\int_0^{2\pi}(t\sin t-2+2\cos t)dt$$

$$=-6\pi.$$

例10－3 求$\int_L \frac{x}{2}dx+ydy+zdz$,

L: 圆周$\begin{cases} x^2+y^2+z^2=1, \\ y=z \end{cases}$在第一卦限从$A(1,0,0)$到$B(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$一段.

解 $x=\cos t,\ y=\frac{1}{\sqrt{2}}\sin t,\ z=\frac{1}{\sqrt{2}}\sin t\quad t:0\to\frac{\pi}{2}$.

原式$$=\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}\cos t\bullet(-\sin t)+\frac{1}{2}\sin t\cos t+\frac{1}{2}\sin t\cos t\right]dt$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin t\cos t dt=\frac{1}{4}$$

例10－4 求$\oint_L xy^2dy-x^2ydx$,

L: 圆$x^2+y^2=a^2(a>0)$依逆时针一周.

解 $P(x,y)=-x^2y,\ Q(x,y)=xy^2$,

记$D: x^2+y^2\le a^2$,利用格林公式,得

原式$$=\iint_D\left[\frac{\partial}{\partial x}(xy^2)-\frac{\partial}{\partial y}(-x^2y)\right]dxdy$$

$$=\iint_D(x^2+y^2)dxdy=\int_0^{2\pi}d\theta\int_0^a r^2\bullet rdr=\frac{1}{2}\pi a^4$$

例10－5 求$\int_{ANO}(e^x\sin y-my)dx+(e^x\cos y-m)dy$,

ANO: 从$A(a,0)$到$O(0,0)$的上半圆周$(a>0)$.

解 补充$\overline{OA}$. 显然在$\overline{OA}$上的积分值为0,于是有

$$\int_{ANO}(e^x\sin y-my)dx+(e^x\cos y-m)dy$$

$$=\oint_{ANOA}(e^x\sin y-my)dx+(e^x\cos y-m)dy$$

$$=\iint_{x^2+y^2\le ax}\left[\frac{\partial}{\partial x}(e^x\cos y-m)-\frac{\partial}{\partial y}(e^x\sin y-my)\right]dxdy$$

$$=\iint_{\substack{x^2+y^2\le ax\\ y\ge 0}}(e^x\cos y-e^x\cos y+m)dxdy$$

$$=m\bullet\frac{1}{2}\pi\left(\frac{a}{2}\right)^2=\frac{\pi a^2 m}{8}$$

例10－6 求$\oint_L \frac{xdy-ydx}{x^2+y^2}$,

$L: x^2+y^2=a^2(a>0)$逆时针一周.

解 先将x^2+y^2替换为a^2,再用格林公式.

$$原式=\oint_L \frac{xdy-ydx}{a^2}=\frac{1}{a^2}\oint_L xdy-ydx$$

$$=\frac{1}{a^2}\iint_{x^2+y^2\le a^2} 2dxdy=\frac{1}{a^2}\cdot 2\pi a^2=2\pi$$ ▌

例10－7 求$\oint_L \frac{xdy-ydx}{4x^2+y^2}$, L: 以点$(1,0)$为中心,$R(R>0)$为半径的圆周,取逆时针方向.

解 $P=\frac{-y}{4x^2+y^2}$, $Q=\frac{x}{4x^2+y^2}$,

$$\frac{\partial Q}{\partial x}=\frac{y^2-4x^2}{(4x^2+y^2)^2}=\frac{\partial P}{\partial y}\quad(4x^2+y^2\neq 0)$$

取$\gamma: 4x^2+y^2=\delta^2(\delta>0)$使此椭圆含于$L$内,取逆时针一周,记$D: 4x^2+y^2\le\delta^2$,则有

$$原式=\oint_L \frac{xdy-ydx}{4x^2+y^2}=\frac{1}{\delta^2}\oint_L xdy-ydx=\frac{1}{\delta^2}\iint_D 2dxdy$$

$$=\frac{1}{\delta^2}\cdot 2\cdot\pi\cdot\frac{\delta}{2}\cdot\delta=\pi$$ ▌

例10－8 求$\int_L \frac{yzdx+zxdy+xydz}{1+x^2y^2z^2}$,

L: 从$(1,1,1)$到$(1,1,\sqrt{3})$的直线段.

解 $\because \frac{yzdx+zxdy+xydz}{1+x^2y^2z^2}=\frac{d(xyz)}{1+(xyz)^2}$

$$=d\left[\arctan(xyz)+C\right]$$

$\therefore u=\arctan(xyz)+C$

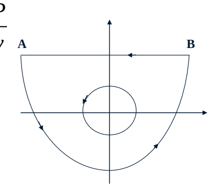
$$\therefore 原式=\arctan(xyz)\Big|_{(1,1,1)}^{(1,1,\sqrt{3})}=\frac{\pi}{12}$$ ▌

例10－9 求$\int_L \frac{(x-y)dx+(x+y)dy}{x^2+y^2}$,

L: 抛物线$y=2x^2-1$从$A(-1,1)$到$B(1,1)$一段弧.

解 $P=\frac{x-y}{x^2+y^2}$, $Q=\frac{x+y}{4x^2+y^2}$,

$$\frac{\partial Q}{\partial x}=\frac{(x^2+y^2)-2x(x+y)}{(x^2+y^2)^2}=\frac{\partial P}{\partial y}$$



补充直线段$\overline{BA}$,在$\overline{BA}$上,

$y=1$, x从1变到-1,

取以原点为心,半径$\varepsilon<\frac{1}{\sqrt{2}}$的圆$\gamma$逆时针一周,则

$$\int_{\overline{BA}}=\int_1^{-1}\frac{x-1}{x^2+1}dx=\frac{\pi}{2}$$

$$\therefore 原式=\oint_\gamma-\int_{\overline{BA}}$$

$$=\int_0^{2\pi}\frac{\left[(\varepsilon\cos t-\varepsilon\sin t)(-\varepsilon\sin t)+(\varepsilon\cos t+\varepsilon\sin t)(\varepsilon\cos t)\right]}{\varepsilon^2}dt-\frac{\pi}{2}$$

$$=2\pi-\frac{\pi}{2}=\frac{3}{2}\pi$$ ▌

例10－10 求$\iint_\Sigma zds$, Σ: 锥面$z=\sqrt{x^2+y^2}$在柱面$x^2+y^2=2x$内部分.

解 记 $D: (x-1)^2+y^2\le 1$

$\because z=\sqrt{x^2+y^2}$ $\therefore \frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$

$$ds=\sqrt{1+{z_x}^2+{z_y}^2}dxdy=\sqrt{2}dxdy$$

$$\iint_\Sigma zdS=\iint_D\sqrt{x^2+y^2}\cdot\sqrt{2}dxdy$$

$$=\sqrt{2}\cdot 2\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^2dr$$

$$=\frac{16}{3}\sqrt{2}\int_0^{\frac{\pi}{2}}\cos^3\theta d\theta=\frac{32}{9}\sqrt{2}$$ ▌

例10－11 求$\oint\!\!\oint_{\Sigma}\left(x^2+\frac{1}{2}y^2+\frac{1}{4}z^2\right)dS$，

Σ：球面$x^2+y^2+z^2=a^2$

解 注意到 $\oint\!\!\oint_{\Sigma}x^2dS=\oint\!\!\oint_{\Sigma}y^2dS=\oint\!\!\oint_{\Sigma}z^2dS$

$$原式=\left(1+\frac{1}{2}+\frac{1}{4}\right)\oint\!\!\oint_{\Sigma}x^2dS$$
$$=\left(1+\frac{1}{2}+\frac{1}{4}\right)\bullet\frac{1}{3}\oint\!\!\oint_{\Sigma}(x^2+y^2+z^2)dS$$
$$=\left(1+\frac{1}{2}+\frac{1}{4}\right)\bullet\frac{1}{3}\oint\!\!\oint_{\Sigma}a^2dS$$
$$=\left(1+\frac{1}{2}+\frac{1}{4}\right)\bullet\frac{1}{3}\bullet a^2\bullet 4\pi a^2=\frac{7}{3}\pi a^4 \quad \blacksquare$$

例10－12 求$\oint\!\!\oint_{\Sigma}\left(|x|+|y|\right)^2dS$，

Σ：八面体$|x|+|y|+|z|\le 1$ 的表面.

解 设Σ_1是Σ在第一卦限的部分，则Σ_1的方程为

$z=1-x-y$， Σ_1在xOy面上投影域

$D:0\le x\le 1,\quad 0\le y\le 1-x,$

则$dS=\sqrt{3}dxdy$

$$\therefore \oint\!\!\oint_{\Sigma}(|x|+|y|)^2dS=8\oint\!\!\oint_{\Sigma_1}(|x|+|y|)^2dS$$
$$=8\iint_{\Sigma_1}(x^2+y^2+2xy)dS$$

$$=8\left[\iint_{\Sigma_1}(x^2+xy)dS+\iint_{\Sigma_1}(y^2+xy)dS\right]$$
$$=16\iint_{\Sigma_1}(x^2+xy)dS$$
$$=16\sqrt{3}\iint_{D}(x^2+xy)dxdy$$
$$=16\sqrt{3}\int_0^1dx\int_0^{1-x}(x^2+xy)dy$$
$$=16\sqrt{3}\int_0^1\frac{1}{2}(x-x^3)dx=2\sqrt{3} \quad \blacksquare$$

例10－13 求$\iint_{\Sigma}\frac{e^zdxdy}{\sqrt{x^2+y^2}}$， Σ：锥面$z=\sqrt{x^2+y^2}$及平面$z=1$，$z=2$所围成的立体表面的外侧.

解 将Σ分成$\Sigma_1:z=2\quad\left(x^2+y^2\le 4\right)$，取上侧；

$\Sigma_2:z=\sqrt{x^2+y^2}\ \left(1\le x^2+y^2\le 4\right)$，取下侧；

$\Sigma_3:z=1\quad\left(x^2+y^2\le 1\right)$，取下侧，

再记

$D_1:x^2+y^2\le 4$ ，$D_2:1\le x^2+y^2\le 4$ ，$D_3:x^2+y^2\le 1$

$$\iint_{\Sigma}\frac{e^zdxdy}{\sqrt{x^2+y^2}}=\iint_{\Sigma_1}\frac{e^zdxdy}{\sqrt{x^2+y^2}}+\iint_{\Sigma_2}\frac{e^zdxdy}{\sqrt{x^2+y^2}}+\iint_{\Sigma_3}\frac{e^zdxdy}{\sqrt{x^2+y^2}}$$
$$=\iint_{D_1}\frac{e^2dxdy}{\sqrt{x^2+y^2}}-\iint_{D_2}\frac{e^{\sqrt{x^2+y^2}}dxdy}{\sqrt{x^2+y^2}}-\iint_{\Sigma_3}\frac{edxdy}{\sqrt{x^2+y^2}}$$
$$=e^2\int_0^{2\pi}d\theta\int_0^2\frac{1}{r}\bullet rdr+\left(-\int_0^{2\pi}d\theta\int_0^2e^rdr\right)+\left(-e\int_0^{2\pi}d\theta\int_0^2dr\right)$$
$$=4\pi e^2+2\pi\left(e-e^2\right)+\left(-2\pi e\right)=2\pi e^2 \quad \blacksquare$$

例10－14 求$\oint\!\!\oint_{\Sigma}\frac{1}{x}dydz+\frac{1}{y}dzdx+\frac{1}{z}dxdy$，

Σ：椭球面$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$的外侧$(a>0,b>0,c>0)$

解 此题不可直接用高斯公式，因为不满足公式条件.

设Σ_1，Σ_2为上半椭球面的上侧和下半球面的下侧，

则两曲面在xOy面上投影域为$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$.

$$\oint\!\!\oint_{\Sigma}\frac{1}{z}dxdy=\oint\!\!\oint_{\Sigma_1}\frac{1}{z}dxdy+\oint\!\!\oint_{\Sigma_2}\frac{1}{z}dxdy$$

$$=\frac{2}{c}\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}\frac{dxdy}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}=\frac{2}{c}\cdot 4\int_0^a dx\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\frac{dy}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}$$

$$=\frac{8}{c}\int_0^a\left[\arcsin\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\Bigg|_0^{b\sqrt{1-\frac{x^2}{a^2}}}\right]dx=\frac{4\pi abc}{c^2}$$

类似地,有$\oiint\limits_{\Sigma}\frac{dydz}{x}=\frac{4\pi abc}{a^2}$, $\oiint\limits_{\Sigma}\frac{dzdx}{y}=\frac{4\pi abc}{b^2}$.

$\therefore$ 原式$=4\pi abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$.

例10−15 求$\oiint\limits_{\Sigma}\left|x-\frac{a}{3}\right|dydz+\left|y-\frac{2b}{3}\right|dzdx+\left|z-\frac{c}{4}\right|dxdy$,

Σ:六面体$0\le x\le a$, $0\le y\le b$, $0\le z\le c$的外表面.

解 $I_1=\oiint\limits_{\Sigma}\left|x-\frac{a}{3}\right|dydz=\iint\limits_{\substack{0\le y\le b\\0\le z\le c}}\left(a-\frac{a}{3}\right)dydz-\iint\limits_{\substack{0\le y\le b\\0\le z\le c}}\left|0-\frac{a}{3}\right|dydz$

$$=\frac{2}{3}a\cdot bc-\frac{1}{3}a\cdot bc=\frac{1}{3}abc\ .$$

类似地,有$I_2=\oiint\limits_{\Sigma}\left|y-\frac{2b}{3}\right|dzdx=-\frac{abc}{3}$,

$$I_3=\oiint\limits_{\Sigma}\left|z-\frac{c}{4}\right|dxdy=\frac{abc}{2}.$$

$\therefore$ 原式$=I_1+I_2+I_3=\frac{1}{2}abc$.

例10−16 设Σ为椭球面$\frac{x^2}{2}+\frac{y^2}{2}+z^2=1$的上半部分,点$P(x,y,z)\in\Sigma$, Π为Σ在点P处的切平面,$\rho(x,y,z)$为点$O(0,0,0)$到平面Π的距离,求$\iint\limits_{\Sigma}\frac{z}{\rho(x,y,z)}dS$.

解 $\Sigma:\frac{x^2}{2}+\frac{y^2}{2}+z^2=1$, $\vec{n}=(x,y,2z)$.

过点$P(x,y,z)$的切平面为

$$x(X-x)+y(Y-y)+2z(Z-z)=0.$$

注意到 $x^2+y^2+2z^2=2$

上述方程写成 $\frac{xX}{2}+\frac{yY}{2}+zZ=1$

原点到此平面的距离为

$$\rho(x,y,z)=\left(\frac{x^2}{4}+\frac{y^2}{4}+z^2\right)^{-\frac{1}{2}},$$

代入$z^2=1-\frac{x^2}{2}-\frac{y^2}{2}$,则

$$\rho(x,y,z)=\frac{1}{\sqrt{1-\frac{x^2}{4}-\frac{y^2}{4}}}=\frac{2}{\sqrt{4-x^2-y^2}},$$

$$dS=\frac{\sqrt{4-x^2-y^2}\,dxdy}{2\sqrt{1-\left(\frac{x^2}{2}+\frac{y^2}{2}\right)}}$$

又 $z=\sqrt{1-\left(\frac{x^2}{2}+\frac{y^2}{2}\right)}$, $D:x^2+y^2\le 2$

$\therefore\ \iint\limits_{\Sigma}\frac{z}{\rho(x,y,z)}dS$

$$=\iint\limits_{D}\frac{\sqrt{4-x^2-y^2}}{2}\cdot\sqrt{1-\left(\frac{x^2}{2}+\frac{y^2}{2}\right)}\cdot\frac{\sqrt{4-x^2-y^2}}{2\sqrt{1-\left(\frac{x^2}{2}+\frac{y^2}{2}\right)}}d\sigma$$

$$=\frac{1}{4}\iint\limits_{D}(4-x^2-y^2)dxdy$$

$$=\frac{1}{4}\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}(4-r^2)rdr=\frac{3}{2}\pi$$