例8-1 设 $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, x^2+y^2\neq 0\\ 0, \quad x^2+y^2=0\end{cases}$ 试讨论$f(x, y)$在点(0, 0)处的连续性. 偏导数存在性及函数的可微性.

解 $\forall\varepsilon>0$, 取$\delta=2\varepsilon$, 当 $0<\sqrt{x^2+y^2}<\delta$ 时, 恒有

$$\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|\le\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{x^2+y^2}{2}=\frac{1}{2}\sqrt{x^2+y^2}$$

$$<\frac{1}{2}\delta=\frac{1}{2}\cdot 2\varepsilon=\varepsilon$$

故$f(x, y)$在点(0, 0)处连续.

$$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x}=0$$

类似地 $f_y(0, 0)=0$. $f(x, y)$在(0, 0)处不可微. ▮

例8-2 试讨论$f(x, y)$在(0, 0)处的连续性与偏导数存在性

$$f(x,y)=\begin{cases}1-x-y, & x\ge 0, y\ge 0;\\ 1+x-y, & x<0, y\ge 0;\\ 1+x+y, & x<0, y<0;\\ 1-x+y, & x\ge 0, y<0.\end{cases}$$

解 显然 $\lim\limits_{(x,y)\to(0,0)}f(x,y)=1=f(0,0)$故$f(x, y)$在(0, 0)处连续

$$f(x,0)=\begin{cases}1-x, x\ge 0,\\ 1+x, x<0.\end{cases}$$

$$\lim_{x\to+0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to+0}\frac{1-x-1}{x}=-1,$$

$$\lim_{x\to-0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to-0}\frac{1+x-1}{x}=1.$$

故$f_x(0, 0)$不存在. 类似可知$f_y(0, 0)$也不存在. ▮

例8-3 设 $f(x,y)=\begin{cases}xy\cdot\dfrac{x^2-y^2}{x^2+y^2}, x^2+y^2\neq 0\\ 0, \quad x^2+y^2=0\end{cases}$

试求$f_{xy}(0, 0)$和$f_{yx}(0, 0)$.

解 容易求得$f_x(0, 0)=0$, $f_y(0, 0)=0$.

当$y\neq 0$时

$$f_x(0,y)=\lim_{x\to 0}\frac{f(x,y)-f(0,y)}{x-0}=\lim_{x\to 0}\frac{y(x^2-y^2)}{x^2+y^2}=-y,$$

$$f_{xy}(0,0)=\lim_{y\to 0}\frac{f_x(0,y)-f_x(0,0)}{y-0}=\lim_{y\to 0}\frac{-y}{y}=-1;$$

当$x\neq 0$时

$$f_y(x,0)=\lim_{y\to 0}\frac{f(x,y)-f(x,0)}{y-0}=\lim_{y\to 0}\frac{x(x^2-y^2)}{x^2+y^2}=x.$$

类似于$f_{xy}(0, 0)$的求法, 得$f_{yx}(0, 0)=1$. ▮

例8-4 $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, x^2+y^2\neq 0,\\ 0, \quad x^2+y^2=0.\end{cases}$

证明: $f(0, 0)$在(0, 0)处偏导数不连续但可微分.

证 $f_x(0,0)=\lim\limits_{x\to+0}\dfrac{f(x,0)-f(0,0)}{x}=\lim\limits_{x\to 0}\dfrac{x^2\sin\frac{1}{x^2}}{x}$

$$=\lim_{x\to 0}x\sin\frac{1}{x}=0,$$

类似地 $f_y(0, 0)=0$;

$$f_x(x,y)=\begin{cases}2x\sin\dfrac{1}{x^2+y^2}-\dfrac{2x}{x^2+y^2}\cos\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0;\end{cases}$$

$$f_y(x,y)=\begin{cases}2y\sin\dfrac{1}{x^2+y^2}-\dfrac{2y}{x^2+y^2}\cos\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

$\lim\limits_{y=x,x\to 0}f_x(x,y)=\lim\limits_{x\to 0}(2x\sin\dfrac{1}{2x^2}-\dfrac{1}{x}\cos\dfrac{1}{2x^2})$ 不存在,

故偏导数不连续.

$$\Delta z=f_x(0,0)\Delta x+f_y(0,0)\Delta y+\alpha$$

$$\alpha=(\Delta x^2+\Delta y^2)\sin\frac{1}{\Delta x^2+\Delta y^2},$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\alpha}{\rho}=\lim_{(\Delta x,\Delta y)\to(0,0)}\sqrt{\Delta x^2+\Delta y^2}\cdot\sin\frac{1}{\Delta x^2+\Delta y^2}=0.$$

故可微分. ▮

例8-5 $z=f(x^2-y^2, xy)$, 求 $\dfrac{\partial^2 z}{\partial x\partial y},\dfrac{\partial^2 z}{\partial x^2},\dfrac{\partial^2 z}{\partial y^2}$,

已知$f(u, v)$二阶偏导数连续.

解 设$u=x^2-y^2$, $v=xy$, 则 $\dfrac{\partial z}{\partial x}=2x\dfrac{\partial z}{\partial u}+y\dfrac{\partial z}{\partial v}$, 注意到 $\dfrac{\partial z}{\partial u},\dfrac{\partial z}{\partial v}$ 仍然是u与v的函数. 于是

$$\frac{\partial^2 z}{\partial x\partial y}=2x\left[\frac{\partial^2 z}{\partial u^2}(-2y)+\frac{\partial^2 z}{\partial u\partial v}\cdot x\right]+\frac{\partial z}{\partial v}+y\left[\frac{\partial^2 z}{\partial u\partial v}\cdot(-2y)+\frac{\partial^2 z}{\partial v^2}\cdot x\right]$$

$$=\frac{\partial z}{\partial v}-4xy\frac{\partial^2 z}{\partial u^2}+(2x^2-2y^2)\frac{\partial^2 z}{\partial u\partial v}+xy\frac{\partial^2 z}{\partial v^2}.$$

类似的可求得

$$\frac{\partial^2 z}{\partial x^2}=2\frac{\partial z}{\partial u}+4x^2\frac{\partial^2 z}{\partial u^2}+4xy\frac{\partial^2 z}{\partial u\partial v}+y^2\frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial^2 z}{\partial y^2}=-2\frac{\partial z}{\partial u}+4y^2\frac{\partial^2 z}{\partial u^2}-4xy\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}.$$ ▮

例8-6 设$u = yf\left(\frac{x}{y}\right) + xg\left(\frac{y}{x}\right)$, f与g都具有连续二阶偏导数, 求$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y}$.

解 $\frac{\partial u}{\partial x} = yf'(\frac{x}{y}) \cdot \frac{1}{y} + g(\frac{y}{x}) + xg'(\frac{y}{x})(-\frac{y}{x^2})$

$= f'(\frac{x}{y}) + g(\frac{y}{x}) - \frac{y}{x}g'(\frac{y}{x})$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{y}f''(\frac{x}{y}) - \frac{y}{x^2}g'(\frac{y}{x}) + \frac{y}{x^2}g'(\frac{y}{x}) + \frac{y^2}{x^3}g''(\frac{y}{x})$$

$$= \frac{1}{y}f''(\frac{x}{y}) + \frac{y^2}{x^3}g''(\frac{y}{x}),$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{x}{y^2}f''(\frac{x}{y}) + \frac{1}{x}g'(\frac{y}{x}) - \frac{1}{x}g'(\frac{y}{x}) - \frac{y}{x}g''(\frac{y}{x}) \cdot \frac{1}{x}$$

$$= -\frac{x}{y^2}f''(\frac{x}{y}) - \frac{y}{x^2}g''(\frac{y}{x}),$$

$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y}$

$$= x\left[\frac{1}{y}f''(\frac{x}{y}) + \frac{y^2}{x^3}g''(\frac{y}{x})\right] + y\left[-\frac{x}{y^2}f''(\frac{x}{y}) - \frac{y}{x^2}g''(\frac{y}{x})\right]$$

$$= \frac{x}{y}f''(\frac{x}{y}) + \frac{y^2}{x^2}g''(\frac{y}{x}) - \frac{x}{y}f''(\frac{x}{y}) - \frac{y^2}{x^2}g''(\frac{y}{x}) = 0.$$

例8-7 设$w=f(u)$二阶可导, 且$u = \ln\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$, 求$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$.

解 令$r = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$, 于是$u=\ln r$, 则

$$\frac{\partial w}{\partial x} = f'(u)\frac{\partial u}{\partial x} = f'(u)\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(u)\frac{1}{r}\frac{x-a}{r}$$

$$= f'(u)(x-a)r^{-2}$$

$$\frac{\partial^2 w}{\partial x^2} = f''(u)\frac{(x-a)^2}{r^4} + \frac{f'(u)}{r^2} - \frac{2(x-a)^2}{r^4}f'(u)$$

由对称性, 有

$$\frac{\partial^2 w}{\partial y^2} = f''(u)\frac{(y-b)^2}{r^4} + \frac{f'(u)}{r^2} - \frac{2(y-b)^2}{r^4}f'(u)$$

$$\frac{\partial^2 w}{\partial z^2} = f''(u)\frac{(z-c)^2}{r^4} + \frac{f'(u)}{r^2} - \frac{2(z-c)^2}{r^4}f'(u)$$

$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2}$

$$= \frac{f''(u)\left[(x-a)^2+(y-b)^2+(z-c)^2\right]}{r^4} + \frac{3f'(u)}{r^2} - \frac{2\left[(x-a)^2+(y-b)^2+(z-c)^2\right]}{r^4}f'(u)$$

$$= \frac{f''(u)}{r^2} + \frac{3f'(u)}{r^2} - \frac{2f'(u)}{r^2} = \frac{f''(u)+f'(u)}{r^2}$$

例8-8 设函数$f(\xi,\eta)$具有二阶连续偏导数, 且满足拉普拉斯方程$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} = 0$. 试证: 函数$z=f(x^2-y^2, 2xy)$也满足拉普拉斯方程$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

证 令$\xi=x^2-y^2$, $\eta=2xy$, 则$z=f(\xi,\eta)$.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = 2x\frac{\partial f}{\partial \xi} + 2y\frac{\partial f}{\partial \eta}.$$

类似地有$\frac{\partial z}{\partial y} = -2y\frac{\partial f}{\partial \xi} + 2x\frac{\partial f}{\partial \eta}$.

故$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial f}{\partial \xi} + 2y\frac{\partial f}{\partial \eta}\right)$

$$= 2\frac{\partial f}{\partial \xi} + 2x\frac{\partial^2 f}{\partial \xi^2} \cdot \frac{\partial \xi}{\partial x} + 2x\frac{\partial^2 f}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial x} + 2y\frac{\partial^2 f}{\partial \xi \partial \eta} \cdot \frac{\partial \xi}{\partial x} + 2y\frac{\partial^2 f}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial x}$$

$$= 2\frac{\partial f}{\partial \xi} + 4x^2\frac{\partial^2 f}{\partial \xi^2} + 8xy\frac{\partial^2 f}{\partial \xi \partial \eta} + 4y^2\frac{\partial^2 f}{\partial \eta^2}.$$

类似地有

$$\frac{\partial^2 z}{\partial y^2} = -2\frac{\partial f}{\partial \xi} + 4y^2\frac{\partial^2 f}{\partial \xi^2} - 8xy\frac{\partial^2 f}{\partial \xi \partial \eta} + 4x^2\frac{\partial^2 f}{\partial \eta^2}.$$

从而$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2}\right) = 0.$

例8-9 设$z=z(x, y)$由方程 $F=\left(x+\frac{z}{y}, y+\frac{z}{x}\right)=0$

所确定, 且$F(u, v)$具有连续偏导数, 则

$$z=xy+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}.$$

证 令 $u=x+\frac{z}{y}, v=y+\frac{z}{x}$. 则$F(u, v)=0$,

$$Fu\cdot\left(1+\frac{1}{y}\frac{\partial z}{\partial x}\right)+Fv\cdot\left(\frac{\frac{\partial z}{\partial x}\cdot x-z}{x^2}\right)=0.$$

解出$\frac{\partial z}{\partial x}=\frac{zFv-x^2Fu}{xFu+yFv}\cdot\frac{y}{x}$. 类似地有$\frac{\partial z}{\partial y}=\frac{zFu-y^2Fv}{xFu+yFv}\cdot\frac{x}{y}$,

于是

$$xy+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$$

$$=xy+x\cdot\frac{zFv-x^2Fu}{xFu+yFv}\cdot\frac{y}{x}+y\cdot\frac{zFu-y^2Fv}{xFu+yFv}\cdot\frac{x}{y}$$

$$=xy+\frac{yzFv-x^2yFu+xzFu-y^2xFv}{xFu+yFv}$$

$$=xy+\frac{z(yFv+xFu)-xy(xFu+yFv)}{xFu+yFv}=z.$$ ▮

例8-10 设$x^2+2y^2+3z^2+xy-z=0$, 当$x=1, y=-2, z=1$时

求$\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial x\partial y},\frac{\partial^2 z}{\partial y^2}$的值.

解 将所给方程两边对x及y分别求偏导数, 得

$$\begin{cases}2x+6z\frac{\partial z}{\partial x}+y-\frac{\partial z}{\partial x}=0, & (1)\\ 4y+6z\frac{\partial z}{\partial y}+x-\frac{\partial z}{\partial y}=0. & (2)\end{cases}$$

以$x=1, y=-2, z=1$代入式(1), (2)得 $\frac{\partial z}{\partial x}=0, \frac{\partial z}{\partial y}=\frac{7}{5}$.

式(1)及(2)再对x及y求偏导数得

$$\begin{cases}2+6z\frac{\partial^2 z}{\partial x^2}+6\left(\frac{\partial z}{\partial x}\right)^2-\frac{\partial^2 z}{\partial x^2}=0, & (3)\\ 6z\frac{\partial^2 z}{\partial x\partial y}+6\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+1-\frac{\partial^2 z}{\partial x\partial y}=0, & (4)\\ 4+6z\frac{\partial^2 z}{\partial y^2}+6\left(\frac{\partial z}{\partial y}\right)^2-\frac{\partial^2 z}{\partial y^2}=0. & (5)\end{cases}$$

再将$x=1, y=-2, z=1$, $\frac{\partial z}{\partial x}=0, \frac{\partial z}{\partial y}=\frac{7}{5}$. 代入式(3), (4)及(5)

解出$\frac{\partial^2 z}{\partial x^2}=-\frac{2}{5}, \frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{5}, \frac{\partial^2 z}{\partial y^2}=-\frac{394}{125}$. ▮

例8-11 $z=x^2+y^2$中$y=y(x)$由方程$x^2-xy+y^2=1$定义,

求$\frac{dz}{dx}$.

解 将$x^2-xy+y^2=1$两边对x求导数, 得

$$2x-y-x\frac{dy}{dx}+2y\frac{dy}{dx}=0,$$

解出 $\frac{dy}{dx}=\frac{2x-y}{x-2y}$.

将$z=x^2+y^2$两边对x求导数, 得 $\frac{dz}{dx}=2x+2y\frac{dy}{dx}$,

再将 $\frac{dy}{dx}$ 表示式代入此式, 得

$$\frac{dz}{dx}=\frac{2(x^2-y^2)}{x-2y}.$$ ▮

例8-12 设$\begin{cases}x^2+y^2+z^2=50,\\ x+2y+3z=4\end{cases}$. 确定$y$与$z$为$x$的函数,

求$\frac{dy}{dx},\frac{dz}{dx}$.

解 将上面方程两边对x求导, 得

$$\begin{cases}2x+2y\frac{dy}{dx}+2z\frac{dz}{dx}=0,\\ 1+2\frac{dy}{dx}+3\frac{dz}{dx}=0.\end{cases}$$

解出$\begin{cases}\frac{dy}{dx}=\frac{z-3x}{3y-2x},\\ \frac{dz}{dx}=\frac{2x-y}{3y-2z}.\end{cases}$ ▮

例8-13 $z=(x^2+y^2)e^{\frac{x^2+y^2}{xy}}$，求$dz$.

解 $\frac{\partial z}{\partial x}=2xe^{\frac{x^2+y^2}{xy}}+(x^2+y^2)(\frac{1}{y}-\frac{y}{x^2})e^{\frac{x^2+y^2}{xy}}$

$=e^{\frac{x^2+y^2}{xy}}(2x+\frac{x^2}{y}-\frac{y^3}{x^2})$，

类似地有$\frac{\partial z}{\partial y}=e^{\frac{x^2+y^2}{xy}}(2y+\frac{y^2}{x}-\frac{x^3}{y^2})$

故$dz=e^{\frac{x^2+y^2}{xy}}\left[(2x+\frac{x^2}{y}-\frac{y^3}{x^2})dx+(2y+\frac{y^2}{x}-\frac{x^3}{y^2})\,dy\right]$. ▌

例8-14 $z=uv+\arcsin w, u=e^x, v=\cos y, w=\frac{x}{\sqrt{x^2+y^2}}$，求$dz$.

解 $\frac{\partial z}{\partial x}=ve^x+\frac{1}{\sqrt{1-w^2}}\cdot\frac{y^2}{(x^2+y^2)^{3/2}}, \frac{\partial z}{\partial y}=-u\sin y+\frac{1}{\sqrt{1-w^2}}\cdot\frac{-xy}{(x^2+y^2)^{3/2}}$.

故$dz=\left[e^x\cos y+ve^x+\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}}\cdot\frac{y^2}{(x^2+y^2)^{3/2}}\right]dx+$

$\left[-e^x\sin y+\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}}\cdot\frac{-xy}{(x^2+y^2)^{3/2}}\right]dy$

$=\left(e^x\cos y+\frac{y}{x^2+y^2}\right)dx-\left(e^x\sin y+\frac{x}{x^2+y^2}\right)dy$. ▌

例8-15 设 $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

试讨论$f(x,y)$在点$M_0(0,0)$处的可微性及偏导数的连续性.

解 $f_x(0,0)=\lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}$

$=\lim\limits_{x\to 0}\frac{(x^2+0^2)\sin\frac{1}{\sqrt{x^2+0^2}}-0}{x}=\lim\limits_{x\to 0}x\sin\frac{1}{\sqrt{x^2}}=0.$

同样$f_y(0,0)=0$.

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta z}{\rho}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left[(\Delta x)^2+(\Delta y)^2\right]\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\sqrt{(\Delta x)^2+(\Delta y)^2}\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=0.$$

故$f(x,y)$在$M_0(0,0)$处可微分.

当$(x,y)\neq(0,0)$时，

$f_y(x,y)=2x\sin\frac{1}{\sqrt{x^2+y^2}}-\frac{x}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}$.

沿直线$y=0$，令$x\to 0$，有

$$\lim_{x=0,y=0}f_x(x,y)=\lim_{x\to 0}\left[2x\sin\frac{1}{|x|}-\frac{x}{|x|}\cos\frac{1}{|x|}\right]$$

不存在. 故$f_x(x,y)$在$M_0(0,0)$处不连续.

类似地可证$f_y(x,y)$在$M_0(0,0)$处也不连续. ▌

例8-16 $z=e^{x-2y}, x=\sin t, y=t^3$，求$\frac{dz}{dt}$.

解法一 $\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$

$=e^{x-2y}\cdot\cos t+(-2e^{x-2y})\cdot 3t^2$

$=e^{\sin t-2t^3}(\cos t-6t^2)$.

解法二 先将$x=\sin t, y=t^3$代入 $z=e^{x-2y}$，得

$z=e^{\sin t-2t^3}$.

再对t求导，有 $\frac{dz}{dt}=e^{\sin t-2t^3}\cdot(\sin t-2t^3)'$

$=e^{\sin t-2t^3}(\cos t-6t^2)$. ▌

例8-17 求曲线 $\Gamma:\begin{cases}x=\int_0^t e^u\cos u\,du,\\ y=2\sin t+\cos t,\\ z=1+e^{3t}.\end{cases}$

在$t=0$处的切线方程和法平面方程.

解 当$t=0$时，对应Γ上点$P(0,1,2)$. 切向量为

$T=(x'(t),y'(t),z'(t))=(e^t\cos t, 2\cos t-\sin t, 3e^{3t})$

$T|_P=(e^t\cos t,\ 2\cos t-\sin t,\ 3e^{3t})\big|_{t=0}=(1,2,3)$.

所求切线方程为$\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-2}{3}$;

所求法平面方程为$x+2(y-1)+3(z-2)=0$,

化简为$x+2y+3z-8=0$. ▌

例8-18 求曲线$\Gamma:\begin{cases}2x^2+y^2+z^2=45,\\ x^2+2y^2=z.\end{cases}$

在点$P(-2, 1, 6)$的切线方程和法平面方程.

解 记$F=2x^2+y^2+z^2-45$, $G=x^2+2y^2-z$则切向量为

$$\boldsymbol{T}=\begin{vmatrix}\vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}}\\ F_x & F_y & F_z\\ G_x & G_y & G_z\end{vmatrix}=\begin{vmatrix}\vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}}\\ 4x & 2y & 2z\\ 2x & 4y & -1\end{vmatrix}$$

$$=-(2y+8yz)\vec{\boldsymbol{i}}+(4x+4xz)\vec{\boldsymbol{j}}+12xy\vec{\boldsymbol{k}},$$

$$T\big|_P=(-2y-8yz,4x+4xz,12xy)\big|_P=(-50,-56,-24).$$

可取$T_1=(25,28,12)$作为切向量.

所求切线方程为 $\frac{x+2}{25}=\frac{y-1}{28}=\frac{z-6}{12}$;

所求法平面方程为$25x+28y+12z-50=0$. ▮

例8-19 求曲线$\Gamma:\begin{cases}x^2+y^2+z^2-3x=0,\\ 2x-3y+5z-4=0.\end{cases}$

在点$P(1, 1, 1)$的切线方程和法平面方程.

解 将曲线方程对x求导, 得$\Gamma:\begin{cases}2x+2y\dfrac{dy}{dx}+2z\dfrac{dz}{dx}-3=0,\\ 2-3\dfrac{dy}{dx}+5\dfrac{dz}{dx}=0.\end{cases}$

解出$\frac{dy}{dx}=\frac{-15+10x-4z}{-10y-6z}$, $\frac{dz}{dx}=\frac{4y+6x-9}{-10y-6z}$,

$$\left.\frac{dy}{dx}\right|_{(1,1,1)}=\frac{9}{16},\ \left.\frac{dz}{dx}\right|_{(1,1,1)}=-\frac{1}{16}.$$

切线向量为$(1, 9/16, -1/16)$或$(16, 9\ -1)$. 则切线方程为

$$\frac{x-1}{16}=\frac{y-1}{9}=\frac{z-1}{-1}.$$

法平面方程为$16x+9y-z-24=0$. ▮

注 此处使用的曲线的切向量为$T=\left(1,\frac{dy}{dx},\frac{dz}{dx}\right)$,

这使得运算十分简便, 但当 $\left.\frac{dy}{dx}\right|_M=\infty$ $\left(\text{或}\left.\frac{dz}{dx}\right|_M=\infty\right)$

需选用其他变量为参数, 例如: 求曲线$\begin{cases}x^2+y^2+z^2=4a^2,\\ x^2+y^2=2ay.\end{cases}$

在$M_0(a,a,\sqrt{2}a)$处的切线方程及法平面方程时, $\left.\frac{dy}{dx}\right|_{M_0}=\infty$,

故改选y为参数, 经计算可得 $T_1=\left(0,1,-1/\sqrt{2}\right)$

故选取$T=(0,\sqrt{2},-1)$, 可得切线方程为

$$\frac{x-a}{0}=\frac{y-a}{\sqrt{2}}=\frac{z-\sqrt{2}a}{-1},$$

法平面方程为$\sqrt{2}y-z=0$.

使用此方法避免了许多繁琐的计算, 不妨用求

$$T=\left(\begin{vmatrix}F_y & F_z\\ G_y & G_z\end{vmatrix},\begin{vmatrix}F_z & F_x\\ G_z & G_x\end{vmatrix},\begin{vmatrix}F_x & F_y\\ G_x & G_y\end{vmatrix}\right)\Bigg|_{M_0}$$

的方法试试看, 一经比较, 便知优劣. ▮

例8-20 试证螺旋线$x=a\cos t$, $y=a\sin t$, $z=bt$ $(a>0, b>0)$

上任一点的切线与z轴成定角.

证 切向量$T=(-a\sin t, a\cos t, b)$, z轴由$s=(0, 0, 1)$表示,

设交角为φ, 则有

$$\cos\varphi=\frac{(-a\sin t)\cdot 0+(a\cos t)\cdot 0+b\cdot 1}{\sqrt{(-a\sin t)^2+(a\cos t)^2+b^2}\cdot\sqrt{0^2+0^2+1^2}}$$

$$=\frac{b}{\sqrt{a^2+b^2}}.$$

故交角为φ常数, 有

$$\varphi=\arccos\frac{b}{\sqrt{a^2+b^2}}.$$ ▮

例8-21 在椭圆抛物面$z=x^2+\frac{1}{4}y^2-1$上求一点P, 使过点P的切平面与平面$2x+y+z=0$平行, 并求过P点的切平面与法线.

解 曲面上任一点$P(x_0, y_0, z_0)$处的法向量为

$$\boldsymbol{n}_1=\left(2x_0,\frac{1}{2}y_0,-1\right).$$

已知平面的法向量为$n_2=(2, 1, 1)$. 当且仅当$n_1/\!/n_2$, 即当

$\frac{2x_0}{2}=\frac{\frac{1}{2}y_0}{1}=\frac{-1}{1}$时, 两平面平行.

将$x_0=-1$, $y_0=-2$, 代入椭圆抛物面方程中, 得$z_0=1$.

满足条件的点是$P(-1, -2, 1)$. 所求切平面方程为

$$2x+y+z+3=0.$$

所求法线方程为$\frac{x+1}{2}=\frac{y+2}{1}=\frac{z-1}{1}$. ▮

例8-22 在椭球面$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$上, 求一个截取各坐标轴正半轴为相等线段的切平面.

解 $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$, 切点为$M_0(x_0, y_0, z_0)$,

$$F_x(M_0)=\frac{2x_0}{a^2}, F_y(M_0)=\frac{2y_0}{b^2}, F_z(M_0)=\frac{2z_0}{c^2}.$$

切平面方程为$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0$,

即$\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y+\frac{2z_0}{c^2}z=1$. 各轴上的截距为

$$x=\frac{x_0}{a^2}, y=\frac{y_0}{b^2}, z=\frac{z_0}{c^2}.$$

依题意应有$x=y=z=k$ ($k>0$),

故$x_0=\frac{a^2}{k}, y_0=\frac{b^2}{k}, z_0=\frac{c^2}{k}$. 有$\frac{\frac{a^4}{k}}{a^2}+\frac{\frac{b^4}{k}}{b^2}+\frac{\frac{c^4}{k}}{c^2}=1$,

即$\frac{a^2}{k^2}+\frac{b^2}{k^2}+\frac{c^2}{k^2}=1, k=\sqrt{a^2+b^2+c^2}$,

$$x_0=\frac{a^2}{\sqrt{a^2+b^2+c^2}}, y_0=\frac{b^2}{\sqrt{a^2+b^2+c^2}}, z_0=\frac{c^2}{\sqrt{a^2+b^2+c^2}}.$$

代入切平面方程, 有

$$\frac{x}{\sqrt{a^2+b^2+c^2}}+\frac{y}{\sqrt{a^2+b^2+c^2}}+\frac{z}{\sqrt{a^2+b^2+c^2}}=1,$$

即$x+y+z=\sqrt{a^2+b^2+c^2}$. ▌

例8-23 证明曲面$xyz=a^3$的切平面与三坐标面围成的四面体的体积为一定常数.

证明 $z=\frac{a^3}{xy}, \frac{\partial z}{\partial x}=-\frac{a^3}{x^2y}, \frac{\partial z}{\partial y}=-\frac{a^3}{xy^2}$. 过$M_0(x_0, y_0, z_0)$的

切线平面方程为$z-z_0=-\frac{a^3}{x_0^2y_0}(x-x_0)-\frac{a^3}{x_0y_0^2}(y-y_0)$.

在三坐标轴上的截距为 $A=\frac{3a^3}{y_0z_0}, B=\frac{3a^3}{x_0z_0}, C=\frac{3a^3}{x_0y_0}$.

即$V=\frac{1}{6}ABC=\frac{1}{6}\frac{27a^9}{x_0^2y_0^2z_0^2}=\frac{1}{6}\frac{a^9}{\left(a^3\right)^2}=\frac{2}{9}a^3$. ▌

例8-24 求函数$u=x+y+z$在点$M_0(0, 0, 1)$处沿球面$x^2+y^2+z^2=1$的外法线方向的方向导数.

解 显然球面在点M_0处的外法线即是OM. (O为坐标原点$O(0, 0, 0)$). 即$n=(0, 0, 1)$, $\cos\alpha=\cos\beta=0$, $\cos\gamma=1$.

$$\left.\frac{\partial u}{\partial n}\right|_{M_0}=\left.\frac{\partial u}{\partial x}\right|_{M_0}\cos\alpha+\left.\frac{\partial u}{\partial y}\right|_{M_0}\cos\beta+\left.\frac{\partial u}{\partial z}\right|_{M_0}\cos\gamma$$
$$=1\times0+1\times0+1\times1=1.$$ ▌

例8-25 设有数量场$u=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$, 问$a, b, c$, 满足什么条件时才能使$u(x, y, z)$, $P(x, y, z)$处($x^2+y^2+z^2\neq0$)沿矢径方向的方向导数最大?

解 $\operatorname{grad}u=\frac{\partial u}{\partial x}i+\frac{\partial u}{\partial y}j+\frac{\partial u}{\partial z}k=\frac{2x}{a^2}i+\frac{2y}{b^2}j+\frac{2z}{c^2}k$.

点$P(x, y, z)$的矢径为$r=OP=xi+yj+zk$. 只有当$r//\operatorname{grad}u$时, $\frac{\partial u}{\partial r}$才取最大值. 即$\frac{\frac{2x}{a^2}}{x}=\frac{\frac{2y}{b^2}}{y}=\frac{\frac{2z}{c^2}}{z}$,

即当$|a|=|b|=|c|$时, $\frac{\partial u}{\partial r}$最大. ▌

例9-26 求函数$u=\arctan\sqrt{x^2+y^2+z^2}$在点$P(1, 1, 1)$处的梯度, 并求梯度的大小和方向余弦.

解 $r=\sqrt{x^2+y^2+z^2}$, $u=\arctan r$.

$$\frac{\partial u}{\partial x}=\frac{x}{(1+r^2)r}, \frac{\partial u}{\partial y}=\frac{y}{(1+r^2)r}, \frac{\partial u}{\partial z}=\frac{z}{(1+r^2)r}.$$

$$r|_P=\sqrt{3}, \left.\frac{\partial u}{\partial x}\right|_P=\frac{1}{4\sqrt{3}}, \left.\frac{\partial u}{\partial y}\right|_P=-\frac{1}{4\sqrt{3}}, \left.\frac{\partial u}{\partial z}\right|_P=\frac{1}{4\sqrt{3}}.$$

所求梯度为

$$\left.\operatorname{grad}u\right|_P=\left(\frac{1}{4\sqrt{3}},-\frac{1}{4\sqrt{3}},\frac{1}{4\sqrt{3}}\right).$$

其大小为

$$\left|\text{grad}\boldsymbol{u}|_P\right| = \sqrt{\left(\frac{1}{4\sqrt{3}}\right)^2 + \left(-\frac{1}{4\sqrt{3}}\right)^2 + \left(\frac{1}{4\sqrt{3}}\right)^2} = \frac{1}{4}.$$

$$(\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{1}{4\sqrt{3}}, -\frac{1}{4\sqrt{3}}, \frac{1}{4\sqrt{3}}\right)\Big/\frac{1}{4}$$

$$= \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

方向余弦为 $\cos\alpha = \dfrac{1}{\sqrt{3}}, \cos\beta = -\dfrac{1}{\sqrt{3}}, \cos\gamma = \dfrac{1}{\sqrt{3}}$. ▌

例8-27 求由方程$x^2+y^2+z^2-xz-yz+2x+2y+2z-2=0$所确定的变量$x$与$y$的隐函数$z$的极值.

解 方程两边关于x求偏导数, 有

$$2x + 2z\frac{\partial z}{\partial x} - z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} + 2 + 2\frac{\partial z}{\partial x} = 0,$$

$$\frac{\partial z}{\partial x} = \frac{z - 2x - 2}{2z - x - y + 2}.$$

由轮换对称性, 有 $\dfrac{\partial z}{\partial y} = \dfrac{z - 2y - 2}{2z - x - y + 2}$,

令 $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$, 得驻点

$P_1\left(-3+\sqrt{6}, -3+\sqrt{6}, 4+2\sqrt{6}\right)$和$P_2\left(-3-\sqrt{6}, -3-\sqrt{6}, -4-2\sqrt{6}\right)$.

进一步可求得

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{P_1} = -\frac{1}{\sqrt{6}}, \quad \left.\frac{\partial^2 z}{\partial x^2}\right|_{P_2} = \frac{1}{\sqrt{6}},$$

$$\left.\frac{\partial^2 z}{\partial y^2}\right|_{P_1} = -\frac{1}{\sqrt{6}}, \quad \left.\frac{\partial^2 z}{\partial y^2}\right|_{P_2} = \frac{1}{\sqrt{6}},$$

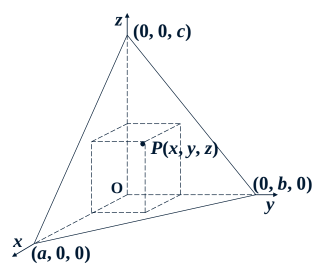
$$\left.\frac{\partial^2 z}{\partial x \partial y}\right|_{P_1} = \left.\frac{\partial^2 z}{\partial x \partial y}\right|_{P_2} = 0.$$

在P_1处$AC-B^2=1/6>0$, $A<0$, 故P_1为极大值点,

类似地P_2为极小值点.

极大值为$z_{max} = -4 + 2\sqrt{6}$; 极小值为 $z_{min} = -4 - 2\sqrt{6}$. ▌

例8-28 平面 $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1(a>0, b>0, c>0)$ 截三轴于A, B, C. $P(x,y,z)$为ΔABC上一点, 以OP为对角线, 以三个坐标面为三面作一个长方体, 试求其最大体积.



解法一 如图所示

设长方体体积为V,

则$V = xyz$, 限制条件为

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

做辅助函数, 有

$$F(x,y,z) = xyz + \lambda\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1\right),$$

$$\begin{cases} F_x = yz + \dfrac{\lambda}{a} = 0, \\ F_y = xz + \dfrac{\lambda}{b} = 0, \\ F_z = xy + \dfrac{\lambda}{c} = 0, \\ \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} - 1 = 0. \end{cases} \Rightarrow \begin{cases} xyz + \dfrac{\lambda x}{a} = 0, \\ xyz + \dfrac{\lambda y}{b} = 0, \\ xyz + \dfrac{\lambda z}{c} = 0, \\ \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} - 1 = 0. \end{cases}$$

从前三式得 $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, 利用最后一式, 得

$$x = \frac{a}{3}, y = \frac{b}{3}, z = \frac{c}{3}, \quad V_{max} = \frac{abc}{27}.$$

解法二 此题还可以利用初等方法求解.

记$x=a\xi$, $y=b\eta$, $z=c\zeta$, 则$V=abc\xi\eta\zeta$.

$$\xi + \eta + \zeta = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

当$\xi=\eta=\zeta=1/3$时 $\left(\text{即} x = \dfrac{a}{3}, y = \dfrac{b}{3}, z = \dfrac{c}{3}\right)$

$$\sqrt[3]{\xi\eta\zeta} \le \frac{\xi + \eta + \zeta}{3}$$

的等号成立.

$$V_{max} = abc\xi\eta\zeta\Big|_{\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)} = \frac{abc}{27}.$$ ▌

例8-29 求内接于椭球面$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$的体积最大的长方体.

解 设长方体的长、宽、高分别为$2x, 2y, 2z$,

则体积$v=8xyz$, 取$u=v^2=64x^2y^2z^2$, 设$x=a\xi, y=b\eta, z=c\zeta$,

则问题转化为$u=64x^2y^2z^2\xi^2\eta^2\zeta^2$在条件$\xi^2+\eta^2+\zeta^2=1$

下的极值. 当$\xi>0, \eta>0, \zeta>0$时 $\sqrt[3]{\xi^2\eta^2\zeta^2}\le\frac{\xi^2+\eta^2+\zeta^2}{3}$,

故$\xi=\eta=\zeta=\frac{1}{\sqrt{3}}$时, 等号成立, u取最大值$\frac{64a^2b^2c^2}{27}$.

即$x=\frac{a}{\sqrt{3}}, y=\frac{b}{\sqrt{3}}, z=\frac{c}{\sqrt{3}}$时, V取最大值$V=\frac{8}{9}\sqrt{3}abc$. ▌

例8-30 求二元函数$z=f(x, y)=x^2y(4-x-y)$在直线$x+y=6$, x轴和y轴所围成的闭区域D上的最大与最小值.

解 先求函数在D内的驻点(2, 1), 解方程组

$$\begin{cases}F_x(x,y)=2xy(4-x-y)-x^2y=0,\\ F_y(x,y)=x^2(4-x-y)-x^2y=0,\end{cases}$$

得$x=0$, $(0\le y\le 6)$及点(4, 0), (2, 1), 函数在D内有唯一驻点(2, 1), 在该点处$f(2, 1)=4$.

再求$f(x, y)$在D的边界上的最值. 在边界$x=0$, $(0\le y\le 6)$和$y=0$, $(0\le x\le 6)$上$f(x, y)=0$.

在边界$x+y=6$上, $y=6-x$. 代入$f(x, y)$, 得

$$f(x, y)=x^2(6-x)(-2)=2x^2(x-6),$$

$$f'_x=4x(x-6)+2x^2=6x^2-24x=0.$$

解出$x=0$和$x=4$, 从而

$$y=6-x|_{x=4}=2,\ f(4, 2)=x^2y(4-x-y)|_{(4,2)}=-64$$

比较后可知, $f(2, 1)=4$为最大值, $f(4, 2)=-64$为最小值. ▌

例8-31 设有一小山, 取它的底面所在的平面为xOy坐标面, 其底部所占的区域为$D=\{(x, y)|x^2+y^2-xy\le 75\}$. 小山的高度函数为$h(x, y)=75-x^2-y^2+xy$.

(1)$M(x_0, y_0)$为区域D上一点, 问$h(x, y)$在该点沿平面上什么方向的方向导数最大? 若记此方向导数的最大值为$g(x_0, y_0)$, 试写出$g(x_0, y_0)$的表达式.

(2)现欲利用此小山开展攀岩活动, 需要在山脚寻找一上山坡度最大的点作为攀登的起点. 也就是说, 要在D的边界曲线$x^2+y^2-xy=75$上找出使(1)中的$g(x, y)$达到最大值的点. 试确定攀登起点的位置.

解 (1)由梯度的几何意义, $h(x, y)$在点$M(x_0, y_0)$处沿梯度$\mathrm{grad}\, h(x, y)|_{(x_0, y_0)}=(y_0-2x_0)\boldsymbol{i}+(x_0-2y_0)\boldsymbol{j}$方向的方向导数最大, 方向导数的最大值为该梯度的最大值, 所以

$$g(x_0,y_0)=\sqrt{(y_0-2x_0)^2+(x_0-2y_0)^2}=\sqrt{5x_0^2+5y_0^2-8x_0y_0}.$$

(2) 令$f(x, y)=g^2(x, y)=5x^2+5y^2-8xy$,

由题意, 只需求$f(x, y)$在约束条件$75-x^2-y^2+xy=0$下的最大值点.

令$L(x, y, \lambda)=5x^2+5y^2-8xy+\lambda(75-x^2-y^2+xy)$, 则

$$\begin{cases}L_x=10x-8y+\lambda(y-2x)=0, & (1)\\ L_y=10y-8x+\lambda(x-2y)=0, & (2)\\ L_\lambda=75-x^2-y^2+xy=0. & (3)\end{cases}$$

式(1)式(2)相加可得$(x+y)(2-\lambda)=0$, 从而$y=-x$或$\lambda=2$.

若λ=2, 则由式(1)得y=x, 再由式(3)得

$$x=\pm 5\sqrt{3},\quad y=\pm 5\sqrt{3}.$$

若y=-x, 则由式(3)得,

$$x=\pm 5,\quad y=\mp 5.$$

于是得到4个可能的极值点为

$$M_1(5,-5),\qquad M_2(-5,5),$$

$$M_3\left(5\sqrt{3},5\sqrt{3}\right),\quad M_4\left(-5\sqrt{3},-5\sqrt{3}\right).$$

由于$f(M_1)=f(M_2)=450$, $f(M_3)=f(M_4)=150$, $M_1(5, -5)$ 和$M_2(-5, 5)$均可作为攀登的起点. ▌