Linear & Nonlinear Programming

Deng Chang

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Example 3.8

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0	0	0	0	0	2	2	0	1
1	1	0	0	1/2	1/2	-1/2	0	1/2
1/2	0	1	0	-3/4	1/4	1/4	0	-3/4
0	0	0	0	0	-1	-1	1	0
1/3	0	0	1	1/3	0	0	0	1/3

remove redundant constraint and artificial variables x_5, x_6, x_7, x_8

	x_1	x_2	x_3	x_4
0	1	1	1	0
1	1	0	0	1/2
1/2	0	1	0	-3/4
1/3	0	0	1	1/3

(notice, the first row is 1 1 1 0). By easy procedure, we can get therefore, the optimal

	x_1	x_2	x_3	x_4
-7/4	0	0	1/4	0
1/2	1	0	-3/2	0
5/4	0	1	9/4	0
1	0	0	3	1

cost is $\frac{7}{4}$, when $x = (1/2, 5/4, 0, 1)$.

Exercise 3.12

(a) In this problem, we introduce a slack variables $x_3, x_4 \geq 0$, then it is standard form and $(0, 0, 2, 6)$ is a basic solution satisfies requirement.

	x_1	x_2	x_3	x_4
0	-2	-1	0	0
2	1	-1	1	0
6	1	1*	0	1

(b)

	x_1	x_2	x_3	x_4
6	-1	0	0	1
8	2*	0	1	1
6	1	1	0	1

then optimal cost is -10, and $x = (4\ 2\ 0\ 0)^T$

	x_1	x_2	x_3	x_4
10	0	0	1/2	3/2
4	1	0	1/2	1/2
2	0	1	-1/2	1/2

(c) Please draw picture carefully.

Exercise 3.17

Original table is listed below.

	x_1	x_2	x_3	x_4	x_5
0	2	3	3	1	-2
2	1	3	0	4	1
2	1	2	0	-3	1
1	-1	-4	3	0	0

Notice, there are 3 constraints, therefore, we introduce y_1, y_2, y_3.

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3
-5	0	0	0	0	0	1	1	1
2	1	3	0	4	1	1	0	0
2	1	2	0	-3	1	0	1	0
1	-1	-4	3	0	0	0	0	1

Phase I
Let us eliminate coefficients of y_1, y_2, y_3

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3
-5	-1	-1	-3	-1	-2	0	0	0
2	1	3	0	4	1	1	0	0
2	1	2	0	-3	1	0	1	0
1	-1	-4	3	0	0	0	0	1

then we bring x_3 into basis,

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3
-4	-2	-5	0	-1	-2	0	0	1
2	1	3	0	4	1	1	0	0
2	1	2	0	-3	1	0	1	0
1/3	-1/3	-4/3	1	0	0	0	0	1/3

then we bring x_1 into basis,

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3
0	0	1	0	6	0	2	0	1
2	1	3	0	4	1	1	0	0
0	0	1	0	7	0	1	-1	0
1	0	-1/3	1	4/3	1/3	1/3	0	1/3

then we bring x_2 into basis,

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3
0	0	0	0	-1	0	1	1	1
2	1	0	0	-17	1	-2	3	0
0	0	1	0	7	0	1	-1	0
1	0	0	1	11/3	1/3	2/3	-1/3	1/3

Phase II

	x_1	x_2	x_3	x_4	x_5
0	2	3	3	1	-2
2	1	0	0	-17	1
0	0	1	7	0	0
1	0	0	1	11/3	1/3

then

	x_1	x_2	x_3	x_4	x_5
-7	0	0	0	3	-5
2	1	0	0	-17	1
0	0	1	7	0	0
1	0	0	1	11/3	1/3

we let x_5 enter into basis.

	x_1	x_2	x_3	x_4	x_5
3	5	0	0	-82	0
2	1	0	0	-17	1
0	0	1	0	7	0
1/3	-1/3	0	1	28/3	0

Therefore, we know optimal solution is $(0, 0, 1/3, 0, 2)$, and optimal value is -3.

Exercise 3.18

a False. Since A has full rank, every BFS is nondegenerate. The cost of every successive BFS visited by the simplex is strictly less than the cost of the previous one because the algorithm is moving along a feasible direction $\mathbf{d}$ such that $\mathbf{c}\top\mathbf{d} < \mathbf{0}$. The basic direction chosen is given by the index j for which $\bar{c}_j < 0$ where $j \in N$. Otherwise, the algorithm would have terminated. The only way the simplex iterates and leaves the cost unchanged is in the case that cycling occurs; in this case, we remain at the same BFS and thus, the simplex is not moving the feasible solution by a positive distance, or at all. **see example 3.6 P104**

b True. $\bar{c}_i = 0$ for the basic variable that is leaving. The update rule for the zero$^{\text{th}}$ row is as follows:

0	a(<0)
1	b(>0)

First column leaves basis, and second column enter basis, then

$-\frac{a}{b}$	0
$\frac{1}{b}$	1

where $-\frac{a}{b} > 0$, Therefore, x_i will not enter in the next iteration.In the next iteration of the simplex, if there exists a nonbasic column for which $\bar{c}_j < 0$, we choose that column. Otherwise, the termination condition for the simplex method is met.
c False. Consider Example 3.8 in textbook.Pay attention to variable x_4
d False.

$$\begin{aligned} \text{minimize} \quad & x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Here, both basis matrices are nondegenerate, but they yield non-unique optimal solutions where the objective function wishes to minimize x_2.
e True. An optimal solution $\mathbf{x}^*$ from the simplex is a basic feasible solution. Theorem 2.4 states that x is a basic solution if and only if we have $\mathbf{Ax} = \mathbf{b}$ satisfied where the basis matrix has m linearly independent columns and for the $n - m$ nonbasic variables, $x_j = 0$. Since $\mathbf{x}^* \geq 0$, and $n - m$ components are 0, then at most m components can be positive.

Exercise 3.19

-10	δ	-2	0	0	0
4	-1	η	1	0	0
1	α	-4	0	1	0
β	γ	3	0	0	1

1.

2. Let $\delta < 0, \alpha \leq 0, \gamma \leq 0$, this imply we find a vector $\mathbf{d} > 0$ with reduced cost $\delta < 0$, then $\bar{\mathbf{x}} = \mathbf{x} + \theta\mathbf{d} > 0$, in this direction, we can make sure objective function is decreasing forever.we see that the optimal cost is $-\infty$.

3. Let $\beta > 0$.if we bring x_2 into the basis.

Exercise 3.20

0	0	0	0	δ	3	γ	ξ
β	0	1	0	α	1	0	3
2	0	0	1	-2	2	η	-1
3	1	0	0	0	-1	2	1

(a) First, we need this initial tableau is feasible.that implies $\beta \geq 0$.
(b)$\beta < 0, \alpha \geq 0$ implies $x_2 < 0$.
(c) Feasibility amounts to $\beta \geq 0$. At least one of the reduced costs δ, γ or ξ must be negative which make sure it is not optimal.Plus, $\beta = 0, \alpha \leq 0, \delta < 0$
(d) We need $\beta \geq 0$. (feasibility), in this situation, we require there exist one column's reduced cost less than 0,and every element is not bigger than 0 too. $\delta < 0$ and $\alpha \leq 0$(x_4 enter the basis ,no positive pivot element found in corresponding column).
(e) We need $\beta \geq 0$ (feasibility), $\gamma < 0$ (x_6 is a candidate to enter the basis). Note that x_3 is the second basic variable. For x_3 to leave the basis, the second row of the tableau must be the winner in the ratio test, that is $\eta > 0$, and $2/\eta < 3/2$, which results in $\eta > 4/3$.
(f) We have $\beta \geq 0, \xi < 0$. For the objective value to remain unchanged,we will add 0 to 0,and first column is $\beta, 2, 3 \therefore \beta = 0$.