第八章

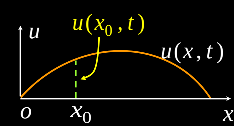
第二节 偏 导 数

一、 偏导数概念及其计算

二 、高阶偏导数

一、 偏导数定义及其计算法

引例: 研究弦在点 x_0 处的振动速度与加速度，就是将振幅 $u(x,t)$中的 x 固定于 x_0 处，求 $u(x_0,t)$关于 t 的一阶导数与二阶导数.



定义1. 设函数 $z=f(x,y)$在点(x_0,y_0) 的某邻域内

极限 $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$

存在，则称此极限为函数 $z=f(x,y)$ 在点 (x_0,y_0) 对 x 的偏导数，记为 $\left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$; $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$; $\left.z_x\right|_{(x_0,y_0)}$;

$f_x(x_0,y_0)$; $f_1'(x_0,y_0)$.

注意: $f_x(x_0,y_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$

$$=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y_0)\Big|_{x=x_0}$$

同样可定义对 y 的偏导数

$$f_y(x_0,y_0)=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

$$=\frac{\mathrm{d}}{\mathrm{d}y}f(x_0,y)\Big|_{y=y_0}$$

若函数 $z=f(x,y)$ 在域 D 内每一点 (x,y) 处对 x 或 y 偏导数存在，则该偏导数称为偏导函数，也简称为**偏导数**，记为 $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial f}{\partial x}$, z_x, $f_x(x,y)$, $f_1'(x,y)$

$\dfrac{\partial z}{\partial y}$, $\dfrac{\partial f}{\partial y}$, z_y, $f_y(x,y)$, $f_2'(x,y)$

偏导数的概念可以推广到二元以上的函数 .

例如, 三元函数 $u=f(x,y,z)$ 在点 (x,y,z) 处对 x 的偏导数定义为

$$f_x(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x}$$

$$f_y(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x,y+\Delta y,z)-f(x,y,z)}{\Delta y}$$

$$f_z(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x,y,z+\Delta z)-f(x,y,z)}{\Delta z}$$

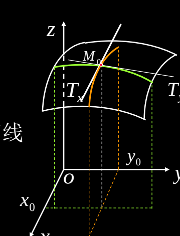
二元函数偏导数的几何意义:

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}}=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y_0)\Big|_{x=x_0}$$

是曲线 $\begin{cases}z=f(x,y)\\ y=y_0\end{cases}$在点 M_0 处的切线 M_0T_x 对 x 轴的斜率.

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}}=\frac{\mathrm{d}}{\mathrm{d}y}f(x_0,y)\Big|_{y=y_0}$$

是曲线 $\begin{cases}z=f(x,y)\\ x=x_0\end{cases}$ 在点M_0 处的切线 M_0T_y 对 y 轴的斜率.



注意: 函数在某点各偏导数都存在，

但在该点**不一定连续**.

例如, $z = f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$

显然 $f_x(0,0) = \dfrac{\mathrm{d}}{\mathrm{d}x} f(x,0)\Big|_{x=0} = 0$

$f_y(0,0) = \dfrac{\mathrm{d}}{\mathrm{d}y} f(0,y)\Big|_{y=0} = 0$

在上节已证 $f(x, y)$ 在点(0 , 0)并不连续!

例1 . 求 $z = x^2 + 3xy + y^2$ 在点(1 , 2) 处的偏导数.

解法1: $\dfrac{\partial z}{\partial x} = 2x + 3y$, $\dfrac{\partial z}{\partial y} = 3x + 2y$

$\therefore \dfrac{\partial z}{\partial x}\Big|_{(1,2)} = 2\cdot 1 + 3\cdot 2 = 8$, $\dfrac{\partial z}{\partial y}\Big|_{(1,2)} = 3\cdot 1 + 2\cdot 2 = 7$

解法2: $z\big|_{y=2} = x^2 + 6x + 4$

$\dfrac{\partial z}{\partial x}\Big|_{(1,2)} = (2x+6)\Big|_{x=1} = 8$

$z\big|_{x=1} = 1 + 3y + y^2$

$\dfrac{\partial z}{\partial y}\Big|_{(1,2)} = (3+2y)\Big|_{y=2} = 7$

例2. 设 $z = x^y$ ($x > 0$, 且 $x \neq 1$), 求证

$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z$$

证: $\because \dfrac{\partial z}{\partial x} = y x^{y-1}$, $\dfrac{\partial z}{\partial y} = x^y \ln x$

$\therefore \dfrac{x}{y}\dfrac{\partial z}{\partial x} + \dfrac{1}{\ln x}\dfrac{\partial z}{\partial y} = x^y + x^y = 2z$

例3. 求 $r = \sqrt{x^2 + y^2 + z^2}$ 的偏导数 .

解: $\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x}{r}$

$\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

例4. 已知理想气体的状态方程 $pV = RT$ (R 为常数) ,

求证: $\dfrac{\partial p}{\partial V}\cdot\dfrac{\partial V}{\partial T}\cdot\dfrac{\partial T}{\partial p} = -1$

证: $p = \dfrac{RT}{V}$, $\dfrac{\partial p}{\partial V} = -\dfrac{RT}{V^2}$

$V = \dfrac{RT}{p}$, $\dfrac{\partial V}{\partial T} = \dfrac{R}{p}$

$T = \dfrac{pV}{R}$, $\dfrac{\partial T}{\partial p} = \dfrac{V}{R}$

$\therefore \dfrac{\partial p}{\partial V}\cdot\dfrac{\partial V}{\partial T}\cdot\dfrac{\partial T}{\partial p} = -\dfrac{RT}{pV} = -1$

说明: 此例表明, 偏导数记号是一个整体记号, 不能看作分子与分母的商 !

二、高阶偏导数

设 $z = f(x, y)$在域 D 内存在连续的偏导数

$$\frac{\partial z}{\partial x} = f_x(x,y), \quad \frac{\partial z}{\partial y} = f_y(x,y)$$

若这两个偏导数仍存在偏导数，则称它们是$z = f(x, y)$ 的二阶偏导数 . 按求导顺序不同, 有下列四个二阶偏导数:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x,y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x,y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y)$$

类似可以定义更高阶的偏导数.

例如， $z = f(x, y)$ 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3}$$

$z = f(x, y)$ 关于 x 的 $n-1$ 阶偏导数 , 再关于 y 的一阶偏导数为

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = \frac{\partial^n z}{\partial x^{n-1} \partial y}$$

例5. 求函数 $z=e^{x+2y}$ 的二阶偏导数及 $\dfrac{\partial^3 z}{\partial y\partial x^2}$.

解： $\boxed{\dfrac{\partial z}{\partial x}=e^{x+2y}}$ $\quad$ $\boxed{\dfrac{\partial z}{\partial y}=2e^{x+2y}}$

$$\frac{\partial^2 z}{\partial x^2}=e^{x+2y}\qquad \frac{\partial^2 z}{\partial x\partial y}=2e^{x+2y}$$

$$\frac{\partial^2 z}{\partial y\partial x}=2e^{x+2y}\qquad \frac{\partial^2 z}{\partial y^2}=4e^{x+2y}$$

$$\frac{\partial^3 z}{\partial y\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\partial x}\right)=2e^{x+2y}$$

注意: 此处 $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y\partial x}$, 但这一结论并不总成立.

例如, $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

$$f_x(x,y)=\begin{cases} \boxed{y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

$$f_y(x,y)=\begin{cases} \boxed{x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

$$\left.\begin{aligned} f_{xy}(0,0)&=\lim_{\Delta y\to 0}\frac{\boxed{f_x(0,\Delta y)}-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1\\ f_{yx}(0,0)&=\lim_{\Delta x\to 0}\frac{\boxed{f_y(\Delta x,0)}-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1\end{aligned}\right\}\text{二者不等}$$

例6. 证明函数 $u=\dfrac{1}{r},\ r=\sqrt{x^2+y^2+z^2}$ 满足拉普拉斯

方程 $\Delta u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$

证: $\dfrac{\partial u}{\partial x}=-\dfrac{1}{r^2}\dfrac{\partial r}{\partial x}=-\dfrac{1}{r^2}\cdot\dfrac{x}{r}$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{1}{r^3}+\frac{3x}{r^4}\cdot\frac{\partial r}{\partial x}=-\frac{1}{r^3}+\frac{3x^2}{r^5}$$

利用对称性, 有 $\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{1}{r^3}+\dfrac{3y^2}{r^5},\quad \dfrac{\partial^2 u}{\partial z^2}=-\dfrac{1}{r^3}+\dfrac{3z^2}{r^5}$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3}{r^3}+\frac{3(\overbrace{\boxed{x^2+y^2+z^2}}^{\boxed{=r^2}})}{r^5}=0$$

定理. 若 $f_{xy}(x,y)$ 和 $f_{yx}(x,y)$ 都在点 (x_0,y_0) 连续, 则

$$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$$ (证明略)

本定理对 n 元函数的高阶混合导数也成立.

例如, 对三元函数 $u=f(x,y,z)$, 当三阶混合偏导数在点 (x,y,z) **连续**时, 有

$$f_{xyz}(x,y,z)=f_{yzx}(x,y,z)=f_{zxy}(x,y,z)$$
$$=f_{xzy}(x,y,z)=f_{yxz}(x,y,z)=f_{zyx}(x,y,z)$$

说明: 因为初等函数的偏导数仍为初等函数, 而初等函数在其定义区域内是连续的, 故求初等函数的高阶导数可以选择方便的求导顺序.

内容小结

1. 偏导数的概念及有关结论
   - 定义; 记号; 几何意义
   - 函数在一点偏导数存在 $\not\Longrightarrow$ 函数在此点连续
   - 混合偏导数连续 $\Longrightarrow$ 与求导顺序无关
2. 偏导数的计算方法
   - 求一点处偏导数的方法 $\begin{cases}\text{先代后求}\\ \text{先求后代}\\ \text{利用定义}\end{cases}$
   - 求高阶偏导数的方法 —— 逐次求导法

   (与求导顺序无关时, 应选择方便的求导顺序)

备用题 设 $z=f(u)$, 方程 $u=\varphi(u)+\int_y^x p(t)\,dt$ 确定 u 是 x,y 的函数, 其中 $f(u),\varphi(u)$ 可微, $p(t),\varphi'(u)$ 连续, 且 $\varphi'(u)\neq 1$, 求 $p(y)\dfrac{\partial z}{\partial x}+p(x)\dfrac{\partial z}{\partial y}$.

解: $\dfrac{\partial z}{\partial x}=f'(u)\dfrac{\partial u}{\partial x},\quad \dfrac{\partial z}{\partial y}=f'(u)\dfrac{\partial u}{\partial y}$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\varphi'(u)\frac{\partial u}{\partial x}+p(x)\\ \frac{\partial u}{\partial y}&=\varphi'(u)\frac{\partial u}{\partial y}-p(y)\end{aligned}\right\}\Longrightarrow\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{p(x)}{1-\varphi'(u)}\\[2mm] \dfrac{\partial u}{\partial y}=\dfrac{-p(y)}{1-\varphi'(u)}\end{cases}$$

$$\therefore\ p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=f'(u)\left[p(y)\frac{\partial u}{\partial x}+p(x)\frac{\partial u}{\partial y}\right]=0$$