第二节 换元积分法

第四章

一、第一类换元法

二、第二类换元法

基本思路

设 $F'(u)=f(u)$, $u=\varphi(x)$ 可导, 则有

$$\mathrm{d}F[\varphi(x)]=f[\varphi(x)]\varphi'(x)\mathrm{d}x$$

$$\therefore \int f[\varphi(x)]\varphi'(x)\mathrm{d}x=F[\varphi(x)]+C=F(u)+C\big|_{u=\varphi(x)}$$

$$=\int f(u)\mathrm{d}u\big|_{u=\varphi(x)}$$

$$\int f[\varphi(x)]\varphi'(x)\,\mathrm{d}x \underset{\text{第二类换元法}}{\overset{\text{第一类换元法}}{\rightleftarrows}} \int f(u)\,\mathrm{d}u$$

一、第一类换元法

定理1. 设 $f(u)$ 有原函数, $u=\varphi(x)$ 可导, 则有换元公式

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=\int f(u)\mathrm{d}u\Big|_{u=\varphi(x)}$$

即 $\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=\int f(\varphi(x))\mathrm{d}\,\varphi(x)$

(也称**配元法，凑微分法**)

例1. 求 $\int(ax+b)^m\mathrm{d}x \quad (m\neq-1)$.

解: 令 $u=ax+b$, 则 $\mathrm{d}u=a\mathrm{d}x$, 故

$$\text{原式}=\int u^m\frac{1}{a}\mathrm{d}u=\frac{1}{a}\cdot\frac{1}{m+1}u^{m+1}+C$$

$$=\frac{1}{a(m+1)}(ax+b)^{m+1}+C$$

注: 当 $m=-1$ 时

$$\int\frac{\mathrm{d}x}{ax+b}=\frac{1}{a}\ln|ax+b|+C$$

例2. 求 $\int\frac{\mathrm{d}x}{a^2+x^2}$.

解: $\int\frac{\mathrm{d}x}{a^2+x^2}=\frac{1}{a^2}\int\frac{\mathrm{d}x}{1+(\frac{x}{a})^2}$

令 $u=\frac{x}{a}$, 则 $\mathrm{d}u=\frac{1}{a}\mathrm{d}x$

$$=\frac{1}{a}\int\frac{\mathrm{d}u}{1+u^2}=\frac{1}{a}\arctan u+C$$

$$=\frac{1}{a}\arctan(\frac{x}{a})+C$$

想到公式

$$\int\frac{\mathrm{d}u}{1+u^2}=\arctan u+C$$

例3. 求 $\int\frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\ (a>0)$.

解: $\int\frac{\mathrm{d}x}{\sqrt{a^2-x^2}}=\int\frac{\mathrm{d}x}{a\sqrt{1-(\frac{x}{a})^2}}=\int\frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}}$

$$=\arcsin\frac{x}{a}+C$$

想到 $\int\frac{\mathrm{d}u}{\sqrt{1-u^2}}=\arcsin u+C$

$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=\int f(\varphi(x))\mathrm{d}\varphi(x)$ (直接配元)

例4. 求 $\int \tan x \mathrm{d}x$.

解: $\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos x}$

$= -\ln|\cos x| + C$

类似

$\int \cot x \mathrm{d}x = \int \frac{\cos x \mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}\sin x}{\sin x}$

$= \ln|\sin x| + C$

例5. 求 $\int \frac{\mathrm{d}x}{x^2-a^2}$.

解:

$\because \frac{1}{x^2-a^2} = \frac{1}{2a}\frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$

$\therefore$ 原式 $= \frac{1}{2a}\left[\int \frac{\mathrm{d}x}{x-a} - \int \frac{\mathrm{d}x}{x+a}\right]$

$= \frac{1}{2a}\left[\int \frac{\mathrm{d}(x-a)}{x-a} - \int \frac{\mathrm{d}(x+a)}{x+a}\right]$

$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

常用的几种配元形式:

(1) $\int f(ax+b)\mathrm{d}x = \frac{1}{a}\int f(ax+b)\ \mathrm{d}(ax+b)$

(2) $\int f(x^n)x^{n-1}\mathrm{d}x = \frac{1}{n}\int f(x^n)\ \mathrm{d}x^n$

(3) $\int f(x^n)\frac{1}{x}\mathrm{d}x = \frac{1}{n}\int f(x^n)\frac{1}{x^n}\ \mathrm{d}x^n$

万能凑幂法

(4) $\int f(\sin x)\cos x \mathrm{d}x = \int f(\sin x)\mathrm{d}\sin x$

(5) $\int f(\cos x)\sin x \mathrm{d}x = -\int f(\cos x)\ \mathrm{d}\cos x$

(6) $\int f(\tan x)\sec^2 x \mathrm{d}x = \int f(\tan x)\ \mathrm{d}\tan x$

(7) $\int f(e^x)e^x \mathrm{d}x = \int f(e^x)\ \mathrm{d}e^x$

(8) $\int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\ \mathrm{d}\ln x$

例6. 求 $\int \frac{\mathrm{d}x}{x(1+2\ln x)}$.

解: 原式 $= \int \frac{\mathrm{d}\ln x}{1+2\ln x} = \frac{1}{2}\int \frac{\mathrm{d}(1+2\ln x)}{1+2\ln x}$

$= \frac{1}{2}\ln|1+2\ln x| + C$

例7. 求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$.

解: 原式 $= 2\int e^{3\sqrt{x}}\mathrm{d}\sqrt{x} = \frac{2}{3}\int e^{3\sqrt{x}}\mathrm{d}(3\sqrt{x})$

$= \frac{2}{3}e^{3\sqrt{x}} + C$

例8. 求 $\int \sec^6 x \mathrm{d}x$.

解: 原式 $= \int (\tan^2 x + 1)^2\ \mathrm{d}\tan x$

$= \int (\tan^4 x + 2\tan^2 x + 1)\,\mathrm{d}\tan x$

$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

例9. 求 $\int \frac{\mathrm{d}x}{1+e^x}$.

解法1

$\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{(1+e^x)-e^x}{1+e^x}\mathrm{d}x = \int \mathrm{d}x - \int \frac{\mathrm{d}(1+e^x)}{1+e^x}$

$= x - \ln(1+e^x) + C$

解法2

$\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{e^{-x}}{1+e^{-x}}\mathrm{d}x = -\int \frac{\mathrm{d}(1+e^{-x})}{1+e^{-x}}$

$= -\ln(1+e^{-x}) + C$

$-\ln(1+e^{-x}) = -\ln[e^{-x}(e^x+1)]$ 两法结果一样

例10. 求 $\int \sec x \mathrm{d}x$.

解法1

$$\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$$

$$= \frac{1}{2}\int\left[\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right]\mathrm{d}\sin x$$

$$= \frac{1}{2}\left[\ln|1+\sin x| - \ln|1-\sin x|\right]+C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C$$

解法 2

$$\int \sec x \mathrm{d}x = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\mathrm{d}x$$

$$= \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\mathrm{d}x$$

$$= \int \frac{\mathrm{d}(\sec x+\tan x)}{\sec x+\tan x}$$

$$= \ln|\sec x+\tan x|+C$$

同样可证

$$\int \csc x\mathrm{d}x = \ln|\csc x-\cot x|+C$$

或 $\int \csc x\mathrm{d}x = \ln\left|\tan\frac{x}{2}\right|+C$

例11. 求 $\int \frac{x^3}{(x^2+a^2)^{3/2}}\mathrm{d}x$.

解: 原式 $= \frac{1}{2}\int\frac{x^2\,\mathrm{d}x^2}{(x^2+a^2)^{3/2}} = \frac{1}{2}\int\frac{(x^2+a^2)-a^2}{(x^2+a^2)^{3/2}}\mathrm{d}x^2$

$$= \frac{1}{2}\int(x^2+a^2)^{-1/2}\,\mathrm{d}(x^2+a^2)$$

$$-\frac{a^2}{2}\int(x^2+a^2)^{-3/2}\,\mathrm{d}(x^2+a^2)$$

$$= \sqrt{x^2+a^2}+\frac{a^2}{\sqrt{x^2+a^2}}+C$$

例12 . 求 $\int\cos^4 x\,\mathrm{d}x$.

解: $\because \cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$

$$= \tfrac{1}{4}(1+2\cos 2x+\cos^2 2x)$$

$$= \tfrac{1}{4}(1+2\cos 2x+\tfrac{1+\cos 4x}{2})$$

$$= \tfrac{1}{4}(\tfrac{3}{2}+2\cos 2x+\tfrac{1}{2}\cos 4x)$$

$\therefore$ $\int\cos^4 x\,\mathrm{d}x = \tfrac{1}{4}\int(\tfrac{3}{2}+2\cos 2x+\tfrac{1}{2}\cos 4x)\,\mathrm{d}x$

$$= \tfrac{1}{4}\left[\tfrac{3}{2}\int\mathrm{d}x+\int\cos 2x\,\mathrm{d}(2x)+\tfrac{1}{8}\int\cos 4x\,\mathrm{d}(4x)\right]$$

$$= \tfrac{3}{8}x+\tfrac{1}{4}\sin 2x+\tfrac{1}{32}\sin 4x+C$$

例13. 求 $\int\sin^2 x\cos^2 3x\,\mathrm{d}x$.

解: $\because \sin^2 x\cos^2 3x = [\tfrac{1}{2}(\sin 4x-\sin 2x)]^2$

$$= \tfrac{1}{4}\sin^2 4x-\tfrac{1}{4}\cdot 2\sin 4x\sin 2x+\tfrac{1}{4}\sin^2 2x$$

$$= \tfrac{1}{8}(1-\cos 8x)-\sin^2 2x\cos 2x+\tfrac{1}{8}(1-\cos 4x)$$

$\therefore$ 原式 $= \tfrac{1}{4}\int\mathrm{d}x-\tfrac{1}{64}\int\cos 8x\,\mathrm{d}(8x)$

$$-\tfrac{1}{2}\int\sin^2 2x\,\mathrm{d}(\sin 2x)-\tfrac{1}{32}\int\cos 4x\,\mathrm{d}(4x)$$

$$= \tfrac{1}{4}x-\tfrac{1}{64}\sin 8x-\tfrac{1}{6}\sin^3 2x-\tfrac{1}{32}\sin 4x+C$$

例14. 求 $\int\frac{x+1}{x(1+xe^x)}\mathrm{d}x$.

解: 原式= $\int\frac{(x+1)\,e^x}{xe^x(1+xe^x)}\mathrm{d}x = \int\left(\frac{1}{xe^x}-\frac{1}{1+xe^x}\right)\mathrm{d}(xe^x)$

$$= \ln|xe^x|-\ln|1+xe^x|+C$$

$$= x+\ln|x|-\ln|1+xe^x|+C$$

分析: $\frac{1}{xe^x(1+xe^x)} = \frac{1+xe^x-xe^x}{xe^x(1+xe^x)} = \frac{1}{xe^x}-\frac{1}{1+xe^x}$

$$(x+1)e^x\,\mathrm{d}x = xe^x\,\mathrm{d}x+e^x\,\mathrm{d}x = \mathrm{d}(xe^x)$$

例15. 求 $\displaystyle\int\left[\frac{f(x)}{f'(x)}-\frac{f''(x)f^2(x)}{f'^3(x)}\right]\mathrm{d}\,x.$

解: 原式 $\displaystyle=\int\frac{f(x)}{f'(x)}\left[1-\frac{f''(x)f(x)}{f'^2(x)}\right]\mathrm{d}x$

$$=\int\frac{f(x)}{f'(x)}\cdot\frac{f'^2(x)-f''(x)f(x)}{f'^2(x)}\mathrm{d}x$$

$$=\int\frac{f(x)}{f'(x)}\,\mathrm{d}\Big(\frac{f(x)}{f'(x)}\Big)$$

$$=\frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2+C$$

小结 常用简化技巧:

(1) 分项积分: 利用积化和差; 分式分项;

$1=\sin^2 x+\cos^2 x$ 等

(2) 降低幂次: 利用倍角公式 , 如

$\cos^2 x=\frac{1}{2}(1+\cos 2x)$; $\sin^2 x=\frac{1}{2}(1-\cos 2x)$;

万能凑幂法 $\begin{cases}\int f(x^n)x^{n-1}\mathrm{d}x=\frac{1}{n}\int f(x^n)\,\mathrm{d}\,x^n\\ \int f(x^n)\frac{1}{x}\mathrm{d}x=\frac{1}{n}\int f(x^n)\frac{1}{x^n}\,\mathrm{d}\,x^n\end{cases}$

(3) 统一函数: 利用三角公式 ; 配元方法

(4) 巧妙换元或配元

思考与练习 1. 下列各题求积方法有何不同?

(1) $\displaystyle\int\frac{\mathrm{d}x}{4+x}=\int\frac{\mathrm{d}(4+x)}{4+x}$ (2) $\displaystyle\int\frac{\mathrm{d}x}{4+x^2}=\frac{1}{2}\int\frac{\mathrm{d}(\frac{x}{2})}{1+(\frac{x}{2})^2}$

(3) $\displaystyle\int\frac{x}{4+x^2}\mathrm{d}x=\frac{1}{2}\int\frac{\mathrm{d}(4+x^2)}{4+x^2}$

(4) $\displaystyle\int\frac{x^2}{4+x^2}\mathrm{d}x=\int\Big[1-\frac{4}{4+x^2}\Big]\mathrm{d}\,x$

(5) $\displaystyle\int\frac{\mathrm{d}x}{4-x^2}=\frac{1}{4}\int\Big[\frac{1}{2-x}+\frac{1}{2+x}\Big]\mathrm{d}x$

(6) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{4x-x^2}}=\int\frac{\mathrm{d}(x-2)}{\sqrt{4-(x-2)^2}}$

2. 求 $\displaystyle\int\frac{\mathrm{d}\,x}{x\,(x^{10}+1)}$.

提示:

法1 $\displaystyle\int\frac{\mathrm{d}\,x}{x\,(x^{10}+1)}=\int\frac{(x^{10}+1)-x^{10}}{x\,(x^{10}+1)}\mathrm{d}\,x$

法2 $\displaystyle\int\frac{\mathrm{d}\,x}{x\,(x^{10}+1)}=\frac{1}{10}\int\frac{\mathrm{d}\,x^{10}}{x^{10}(x^{10}+1)}$

法3 $\displaystyle\int\frac{\mathrm{d}\,x}{x\,(x^{10}+1)}=\int\frac{\mathrm{d}\,x}{x^{11}(1+x^{-10})}=\frac{-1}{10}\int\frac{\mathrm{d}\,x^{-10}}{1+x^{-10}}$

二、第二类换元法

第一类换元法解决的问题

$$\underset{\text{难求}}{\int f[\varphi(x)]\varphi'(x)\mathrm{d}x}=\underset{\text{易求}}{\int f(u)\mathrm{d}u}\Big|_{u=\varphi(x)}$$

若所求积分 $\int f(u)\mathrm{d}u$ 难求,

$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$ 易求,

则得第二类换元积分法 .

定理2 . 设 $x=\psi(t)$ 是单调可导函数 , 且 $\psi'(t)\neq 0$, $f[\psi(t)]\psi'(t)$ 具有原函数 , 则有换元公式

$$\int f(x)\,\mathrm{d}x=\int f[\psi(t)]\psi'(t)\,\mathrm{d}\,t\Big|_{t=\psi^{-1}(x)}$$

其中 $t=\psi^{-1}(x)$ 是 $x=\psi(t)$ 的反函数 .

证: 设 $f[\psi(t)]\psi'(t)$ 的原函数为 $\Phi(t)$, 令

$$F(x)=\Phi[\psi^{-1}(x)]$$

则 $\displaystyle F'(x)=\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,t}\cdot\frac{\mathrm{d}\,t}{\mathrm{d}x}=f[\psi(t)]\cancel{\psi'(t)}\cdot\frac{1}{\cancel{\psi'(t)}}=f(x)$

$\therefore$ $\displaystyle\int f(x)\,\mathrm{d}x=F(x)+C=\Phi[\psi^{-1}(x)]+C$

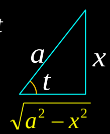
$$=\int f[\psi(t)]\psi'(t)\,\mathrm{d}\,t\Big|_{t=\psi^{-1}(x)}$$

例16. 求 $\int \sqrt{a^2-x^2}\,\mathrm{d}x \ (a>0)$.

解: 令 $x = a\sin t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2 t} = a\cos t$$

$$\mathrm{d}x = a\cos t\,\mathrm{d}t$$



$$\therefore\ 原式 = \int a\cos t\cdot a\cos t\,\mathrm{d}t = a^2\int\cos^2 t\,\mathrm{d}t$$

$$= a^2\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)+C$$

$$\sin 2t = 2\sin t\cos t = 2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$$

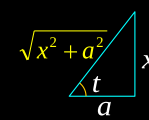
例17. 求 $\int\frac{\mathrm{d}x}{\sqrt{x^2+a^2}}\ (a>0)$.

解: 令 $x = a\tan t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t+a^2} = a\sec t$$

$$\mathrm{d}x = a\sec^2 t\,\mathrm{d}t$$

$$\therefore\ 原式 = \int\frac{a\sec^2 t}{a\sec t}\mathrm{d}t = \int\sec t\,\mathrm{d}t$$



$$= \ln|\sec t+\tan t|+C_1$$

$$= \ln\left[\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right]+C_1$$

$$= \ln\left[x+\sqrt{x^2+a^2}\right]+C \qquad (C = C_1-\ln a)$$

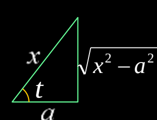
例18. 求 $\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\ (a>0)$.

解: 当 $x>a$ 时, 令 $x = a\sec t,\ t\in(0,\frac{\pi}{2})$, 则

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2 t-a^2} = a\tan t$$

$$\mathrm{d}x = a\sec t\tan t\,\mathrm{d}t$$

$$\therefore\ 原式 = \int\frac{a\sec t\tan t}{a\tan t}\mathrm{d}t = \int\sec t\,\mathrm{d}t$$



$$= \ln|\sec t+\tan t|+C_1$$

$$= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|+C \qquad (C = C_1-\ln a)$$

当 $x<-a$ 时, 令 $x=-u$, 则 $u>a$, 于是

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = -\int\frac{\mathrm{d}u}{\sqrt{u^2-a^2}} = -\ln\left|u+\sqrt{u^2-a^2}\right|+C_1$$

$$= -\ln\left|-x+\sqrt{x^2-a^2}\right|+C_1$$

$$= -\ln\left|\frac{a^2}{-x-\sqrt{x^2-a^2}}\right|+C_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|+C \qquad (C = C_1-2\ln a)$$

$x>a$ 时, $\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|+C$

说明:

被积函数含有 $\sqrt{x^2+a^2}$ 或 $\sqrt{x^2-a^2}$ 时, 除采用三角代换外, 还可利用公式

$$\mathrm{ch}^2 t-\mathrm{sh}^2 t = 1$$

采用双曲代换

$$x = a\,\mathrm{sh}\,t \quad 或 \quad x = a\,\mathrm{ch}\,t$$

消去根式, 所得结果一致.

例19. 求 $\int\frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x$.

解: 令 $x=\frac{1}{t}$, 则 $\mathrm{d}x = -\frac{1}{t^2}\mathrm{d}t$

$$原式 = \int\frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\cdot\frac{-1}{t^2}\mathrm{d}t = -\int(a^2t^2-1)^{\frac{1}{2}}|t|\,\mathrm{d}t$$

当 $x>0$ 时,

$$原式 = -\frac{1}{2a^2}\int(a^2t^2-1)^{\frac{1}{2}}\,\mathrm{d}(a^2t^2-1)$$

$$= -\frac{(a^2t^2-1)^{\frac{3}{2}}}{3a^2}+C = -\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C$$

当 $x<0$ 时, 类似可得同样结果.

小结:

1. 第二类换元法常见类型:

(1) $\int f(x, \sqrt[n]{ax+b})\,\mathrm{d}x$， 令 $t=\sqrt[n]{ax+b}$

(2) $\int f(x, \sqrt[n]{\frac{ax+b}{cx+d}})\,\mathrm{d}x$， 令 $t=\sqrt[n]{\frac{ax+b}{cx+d}}$

$\}$ 第四节讲

(3) $\int f(x, \sqrt{a^2-x^2})\,\mathrm{d}x$，令 $x=a\sin t$ 或 $x=a\cos t$

(4) $\int f(x, \sqrt{a^2+x^2})\,\mathrm{d}x$，令 $x=a\tan t$ 或 $x=a\,\mathrm{sh}\,t$

(5) $\int f(x, \sqrt{x^2-a^2})\,\mathrm{d}x$，令 $x=a\sec t$ 或 $x=a\,\mathrm{ch}\,t$

(6) $\int f(a^x)\,\mathrm{d}x$，令 $t=a^x$

(7) 分母中因子次数较高时, 可试用**倒代换**

2. 常用基本积分公式的补充

(16) $\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C$

(17) $\int \cot x\,\mathrm{d}x = \ln|\sin x| + C$

(18) $\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + C$

(19) $\int \csc x\,\mathrm{d}x = \ln|\csc x - \cot x| + C$

(20) $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$

(21) $\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(22) $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C$

(23) $\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln(x+\sqrt{x^2+a^2}) + C$

(24) $\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \ln|x+\sqrt{x^2-a^2}| + C$

例20. 求 $\int \frac{\mathrm{d}x}{x^2+2x+3}$.

解: 原式 $= \int \frac{1}{(x+1)^2+(\sqrt{2})^2}\,\mathrm{d}(x+1)$

$= \frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C$

例21. 求 $I = \int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$.

解: $I = \frac{1}{2}\int \frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}} = \frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right| + C$

例22. 求 $\int \frac{\mathrm{d}x}{\sqrt{1+x-x^2}}$.

解: 原式 $= \int \frac{\mathrm{d}(x-\frac{1}{2})}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}} = \arcsin\frac{2x-1}{\sqrt{5}} + C$

例23. 求 $\int \frac{\mathrm{d}x}{\sqrt{e^{2x}-1}}$.

解: 原式 $= -\int \frac{\mathrm{d}\,e^{-x}}{\sqrt{1-e^{-2x}}} = -\arcsin e^{-x} + C$

例24. 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}}$.

解: 令 $x=\frac{1}{t}$，得

原式 $= -\int \frac{t}{\sqrt{a^2t^2+1}}\,\mathrm{d}t$

$= -\frac{1}{2a^2}\int \frac{\mathrm{d}(a^2t^2+1)}{\sqrt{a^2t^2+1}} = -\frac{1}{a^2}\sqrt{a^2t^2+1} + C$

$= -\frac{\sqrt{x^2+a^2}}{a^2x} + C$

例25. 求 $\int \frac{\mathrm{d}x}{(x+1)^3\sqrt{x^2+2x}}$.

解: 原式 $=\int \frac{\mathrm{d}x}{(x+1)^3\sqrt{(x+1)^2-1}}$ 　　$\boxed{令\ x+1=\frac{1}{t}}$

$$=\int \frac{t^3}{\sqrt{\frac{1}{t^2}-1}}\left(-\frac{1}{t^2}\right)\mathrm{d}t=-\int\frac{t^2}{\sqrt{1-t^2}}\mathrm{d}t$$

$$=\int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\mathrm{d}t=\boxed{\int\sqrt{1-t^2}\,\mathrm{d}t}-\int\frac{1}{\sqrt{1-t^2}}\mathrm{d}t$$

$$=\frac{1}{2}t\sqrt{1-t^2}+\frac{1}{2}\arcsin t-\arcsin t+C$$ 例16

$$=\frac{1}{2}\frac{\sqrt{x^2+2x}}{(x+1)^2}-\frac{1}{2}\arcsin\frac{1}{x+1}+C$$

思考与练习

1. 下列积分应如何换元才使积分简便？

(1) $\int\frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x$ 　令 $t=\sqrt{1+x^2}$

(2) $\int\frac{\mathrm{d}x}{\sqrt{1+e^x}}$ 　令 $t=\sqrt{1+e^x}$

(3) $\int\frac{\mathrm{d}x}{x(x^7+2)}$ 　令 $t=\frac{1}{x}$

2. 已知 $\int x^5 f(x)\mathrm{d}x=\sqrt{x^2-1}+C$, 求 $\int f(x)\mathrm{d}x$.

解: 两边求导, 得 $x^5 f(x)=\frac{x}{\sqrt{x^2-1}}$, 则

$$\int f(x)\,\mathrm{d}x=\int\frac{\mathrm{d}x}{x^4\sqrt{x^2-1}}\quad (令\ t=\frac{1}{x})$$

$$=\int\frac{-t^3\,\mathrm{d}t}{\sqrt{1-t^2}}=\frac{1}{2}\int\frac{(1-t^2)-1}{\sqrt{1-t^2}}\mathrm{d}t^2$$

$$=\frac{-1}{2}\int(1-t^2)^{\frac{1}{2}}\mathrm{d}(1-t^2)+\frac{1}{2}\int(1-t^2)^{-\frac{1}{2}}\mathrm{d}(1-t^2)$$

$$=\frac{-1}{3}(1-t^2)^{\frac{3}{2}}+(1-t^2)^{\frac{1}{2}}+C=\cdots$$ (代回原变量)

备用题 1. 求下列积分:

1) $\int x^2\frac{1}{\sqrt{x^3+1}}\mathrm{d}x=\frac{1}{3}\int\frac{1}{\sqrt{x^3+1}}\mathrm{d}(x^3+1)$

$$=\frac{2}{3}\sqrt{x^3+1}+C$$

2) $\int\frac{2x+3}{\sqrt{1+2x-x^2}}\mathrm{d}x=\int\frac{-(2-2x)+5}{\sqrt{1+2x-x^2}}\mathrm{d}x$

$$=-\int\frac{\mathrm{d}(1+2x-x^2)}{\sqrt{1+2x-x^2}}+5\int\frac{\mathrm{d}(x-1)}{\sqrt{2-(x-1)^2}}$$

$$=-2\sqrt{1+2x-x^2}+5\arcsin\frac{x-1}{\sqrt{2}}+C$$

2. 求不定积分 $\int\frac{2\sin x\cos x\sqrt{1+\sin^2 x}}{2+\sin^2 x}\mathrm{d}x$.

解: 利用凑微分法，得

原式 $=\int\frac{\sqrt{1+\sin^2 x}}{2+\sin^2 x}\mathrm{d}(1+\sin^2 x)$

令 $t=\sqrt{1+\sin^2 x}$

$$=\int\frac{2t^2}{1+t^2}\mathrm{d}t=2\int\left(1-\frac{1}{1+t^2}\right)\mathrm{d}t$$

$$=2t-2\arctan t+C$$

$$=2\left[\sqrt{1+\sin^2 x}-\arctan\sqrt{1+\sin^2 x}\right]+C$$

3. 求不定积分 $\int\frac{1}{(1+x^2)\sqrt{1-x^2}}\mathrm{d}x$.

解: 令 $x=\sin t$, $1+x^2=1+\sin^2 t$, $\mathrm{d}x=\cos t\,\mathrm{d}t$

原式 $=\int\frac{\cos t}{(1+\sin^2 t)\cos t}\mathrm{d}t=\int\frac{1}{1+\sin^2 t}\mathrm{d}t$

分子分母同除以 $\cos^2 t$

$$=\int\frac{\sec^2 t}{\sec^2 t+\tan^2 t}\mathrm{d}t=\int\frac{1}{1+2\tan^2 t}\mathrm{d}\tan t$$

$$=\frac{1}{\sqrt{2}}\int\frac{1}{1+(\sqrt{2}\tan t)^2}\mathrm{d}\sqrt{2}\tan t$$

$$=\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan t)+C=\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{2}x}{\sqrt{1-x^2}}+C$$