第8章 多元函数微分法及其应用

一、内容提要

（一）主要定义

1．二元函数的极限

设函数$z=f(x,y)$在点$P_0(x_0,y_0)$的附近有定义(点P_0可除外)，点P_0的任一个邻域内都有使z有定义的点$P(x,y)$异于P_0，当点P以任意方式趋近于P_0时，函数$f(x,y)$相应地趋于一个确定的常数A，则称A为$f(x,y)$当$x\to x_0,y\to y_0$时的极限，记作

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=A$$

2．二元函数在一点连续

设函数$z=f(x,y)$在点$P_0(x_0,y_0)$的某领域内有定义，如果有$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$，则称函数$z=f(x,y)$在点$P_0$处连续.

3．偏导数

设函数$z=f(x,y)$在点$P_0(x_0,y_0)$的某邻域内有定义，如果极限

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

或 $$\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

存在，则称此极限为$z=f(x,y)$在点P_0处对x的偏导数，称极限

$$\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

或 $$\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

为$f(x,y)$在P_0处对y的偏导数. 分别记作

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\y=y_0}},\ f_x(x_0,y_0)\text{ 与 }\left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_0\\y=y_0}},\ f_y(x_0,y_0)\text{等.}$$

4．全微分

如果函数$z=f(x,y)$在点$P_0(x_0,y_0)$处的全增量

$$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$$

可表示为$\Delta z=A\Delta x+B\Delta y+o(\rho)$，其中$A,B$不依赖于$\Delta x,\Delta y$，$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，则称$z=f(x,y)$在点$(x,y)$处可微.

此时表达式$A\Delta x+B\Delta y$叫做$z=f(x,y)$在点(x_0,y_0)处的全微分，记作dz，即$dz=A\Delta x+B\Delta y$或$dz=Adx+Bdy$.

可以证明$dz=f_x(x_0,y_0)dx+f_y(x_0,y_0)dy$.

5．方向导数

设$z=f(x,y)$在包含$P(x,y)$，$P(x+\Delta x,y+\Delta y)$的邻域内有定义，$l=(\Delta x,\Delta y)$，则$f(x,y)$在$P(x,y)$处沿l方向的方向导数定义为

$$\frac{df}{dl}=\lim_{\rho\to 0}\frac{f(x+\Delta x,\ y+\Delta y)-f(x,\ y)}{\rho}$$

$$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$$

类似地可以定义空间上的方向导数为

$$\frac{df}{dl}=\lim_{\rho\to 0}\frac{f(x+\Delta x,\ y+\Delta y,\ z+\Delta z)-f(x,\ y,\ z)}{\rho}$$

$$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$$

6. 梯度(gradient)

设函数$z=f(x,y)$在点$P(x,y)$的某邻域内有连续的一阶偏导数，则向量

$$\frac{df}{dx}i+\frac{df}{dy}j$$

称为$z=f(x,y)$在点$P(x,y)$处的梯度，记作$\mathrm{grad}f(x,y)$，即

$$\mathrm{grad}f(x,y)=\frac{df}{dx}i+\frac{df}{dy}j$$

注 $$\mathrm{grad}f(x,y,z)=\frac{df}{dx}i+\frac{df}{dy}j+\frac{df}{dz}k$$

（二）主要结论

1. 可微与可偏导的关系

函数$z=f(x,y)$在点$P_0(x_0,y_0)$处可微，则必可偏导，即$f_x(x_0,y_0)$，$f_y(x_0,y_0)$存在，反之不真. 特别地，即使$f_x(x_0,y_0)$，$f_y(x_0,y_0)$存在，函数$z=f(x,y)$在点$P_0(x_0,y_0)$处也不一定连续，当然也不一定可微.

2. 多元复合函数求导法则

(1)如果$u=u(x,y)$，$v=v(x,y)$在点(x,y)处有偏导数，$z=f(u,v)$在点(u,v)处有连续偏导数，则$z=f[u(x,y),v(x,y)]$在点$P(x,y)$处也有关于x或y的偏导数，则

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x},\quad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

在相应的条件下，还有下列求导公式:

(2)若$z=f(u, v, w)$, $u=u(x, y)$, $v=v(x, y)$, $w=w(x, y)$, 则

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$$
$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}$$

(3)若$z=f(u, x, y)$, $u = u(x, y)$, 则

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x},\quad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y}$$

(4)若$z=f(u, v, w)$, $u=u(t)$, $v=v(t)$, $w=w(t)$, 则

$$\frac{dz}{dt}=\frac{df}{du}\cdot\frac{du}{dt}+\frac{df}{dv}\cdot\frac{dv}{dt}+\frac{df}{dw}\cdot\frac{dw}{dt}$$

3. 隐函数的求导公式

(1)设$y=y(x)$是由方程$F(x, y)=0$所确定的隐函数. 且二元函数$F(x, y)$有连续的偏导数, $F_y(x, y)\neq 0$, 则

$$\frac{dy}{dx}=-\frac{F_x(x,\ y)}{F_y(x,\ y)}$$

(2)设$z=z(x, y)$是由方程$F(x, y, z)=0$所确定的隐函数, 三元函数$F(x, y, z)$有连续的偏导数, 且$F_z(x, y, z)\neq 0$, 则

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,\ y,\ z)}{F_z(x,\ y,\ z)},\quad \frac{\partial z}{\partial y}=-\frac{F_y(x,\ y,\ z)}{F_z(x,\ y,\ z)}$$

(3)方向导数的计算公式

函数$z=z(x, y)$(或$u=f(x, y, z)$)在其可微点处沿任何方向l的方向导数都存在, 且有下列计算公式

$$\frac{\partial f}{\partial l}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta$$

空间为 $\frac{\partial f}{\partial l}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta+\frac{\partial f}{\partial z}\cos\gamma$

其中α, β为l与x轴和y轴正向的夹角; α, β, γ, 为方向l的方向角.

（三）结论补充

1. $\frac{\partial z}{\partial y}, \frac{\partial z}{\partial x}$在$P_0(x_0, y_0)$点连续, 则$z=f(x, y)$在$P_0(x_0, y_0)$处全微分存在.

2. 在$P_0(x_0, y_0)$处$\frac{\partial^2 z}{\partial x\partial y}$与$\frac{\partial^2 z}{\partial y\partial x}$连续, 则二者相等.

3. $z=f(x, y)$在$P_0(x_0, y_0)$某邻域内连续, 且有一阶及二阶连续偏导数, 又$f_x(x_0, y_0)=f_y(x_0, y_0)= 0$.
记$u(x, y)=f_{xx}f_{yy}-f_{xy}^2$, 则

$u(P_0)> 0$, $f_{xx}(P_0)< 0$时取极大值;

$u(P_0)> 0$, $f_{xx}(P_0)> 0$时取极小值;

$u(P_0)< 0$时不取极值;

$u(P_0)= 0$时不能断定.

4. 可微函数$z=f(x, y)$在可微函数$\varphi(x, y)=0$条件下取极值的必要条件是, 令$F(x, y)=f(x, y)+\lambda\varphi(x, y)$, 满足

$$\begin{cases} f_x(x,y)+\lambda\varphi_x(x,y)=0, \\ f_y(x,y)+\lambda\varphi_y(x,y)=0, \\ \varphi(x,y)=0. \end{cases}$$

5. 曲线$\begin{cases} x=\varphi(t), \\ y=\psi(t), \\ z=\omega(t) \end{cases}$在$P_0(x_0, y_0, z_0)$处的切线方程和法平面方程分别为:

$$\frac{x-x_0}{\varphi'(t_0)}=\frac{y-y_0}{\psi'(t_0)}=\frac{z-z_0}{\omega'(t_0)},$$
$$\varphi'(t_0)(x-x_0)+\psi'(t_0)(y-y_0)+\omega'(t_0)(z-z_0)=0$$

6. 曲面$F(x, y, z)=0$在$P_0(x_0, y_0, z_0)$处的切平面方程和法线方程分别为:

$$F_x(P_0)(x-x_0)+F_y(P_0)(y-y_0)+F_z(P_0)(z-z_0)=0$$
$$\frac{x-x_0}{F_x(P_0)}=\frac{y-y_0}{F_y(P_0)}=\frac{z-z_0}{F_z(P_0)}$$

7. 全微分的几何意义

曲面$z=f(x, y)$在点$M_0(x_0, y_0, z_0)$处切平面上z坐标的增量就是全微分.

注 切平面$z-z_0=f_x(x_0, y_0)(x-x_0)+f_y(x_0, y_0)(y-y_0)$,
记$\Delta x=x-x_0$, $\Delta y= y-y_0$, 则全微分$dz=f_x(x_0, y_0)\Delta x+f_y(x_0, y_0)\Delta y$

8. 由两空间曲面决定的空间曲线$\Gamma:\begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0 \end{cases}$

的切向量$T=\begin{vmatrix} i & j & k \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix}$.

9. 记$\boldsymbol{e}=\cos\alpha\,\boldsymbol{i}+\cos\beta\,\boldsymbol{j}$, α, β为$\boldsymbol{l}$的方向角, 则

$$\frac{\partial f}{\partial l}=\operatorname{grad} f\cdot e$$