例9－1 设$D: x^2+y^2\le a^2\,(a>0)$, 求$\iint\limits_D |xy|\,d\sigma$.

解 由于被积函数是x和y的偶函数,积分域关于x轴和y轴都对称,记$D_1: x^2+y^2\le a^2, x\ge 0, y\ge 0$, 则有

原式 $=4\iint\limits_{D_1} xy\,d\sigma \;=4\int_0^a x\,dx\int_0^{\sqrt{a^2-x^2}} y\,dy.$

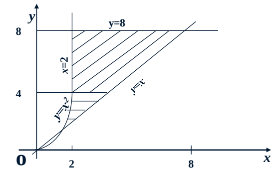
$=2\int_0^a x(a^2-x^2)dx=\frac{1}{2}a^4$.

例9－2 改变积分次序:

$I=\int_1^2 dx\int_x^{x^2} f(x,y)dy+\int_2^8 dx\int_x^8 f(x,y)dy$

解 重新划分积分域如图所示, 改变积分次序,

$I=\int_1^4 dy\int_{\sqrt{y}}^{y} f(x,y)dx+\int_4^8 dy\int_2^{y} f(x,y)dx$



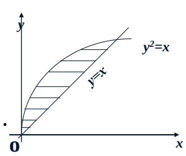
例9－3 求$\iint\limits_D \frac{\sin y}{y}\,d\sigma$, D:由曲线$y^2=x$与直线$y=x$所围成的闭区域.

解 积分域如图所示. 由于$\int\frac{\sin y}{y}dy$ 的原函数不是初等函数, 因此应先对 x 积分, 有

原式$=\int_0^1\frac{\sin y}{y}dy\int_{y^2}^{y} dx$

$=\int_0^1 \frac{\sin y}{y}\cdot(y-y^2)dy$

$=\int_0^1 \sin y\cdot(1-y)dy=1-\sin 1$.

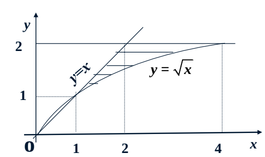


例9－4 求$\iint\limits_D \sin\frac{\pi x}{2y}\,d\sigma$, D:由曲线$y=\sqrt{x}$与直线$y=x$, $y=2$所围成的闭区域.

解 积分域如图所示.

原式$=\int_1^2 dy\int_y^{y^2}\sin\frac{\pi x}{2y}dx=\int_1^2\frac{2y}{\pi}\cdot(\cos\frac{\pi}{2}-\cos\frac{\pi}{2}y)dy$

$=\frac{4}{\pi^3}(\pi+2)$



例9－5 求$\iint\limits_D|\cos(x+y)|\,d\sigma$, D:由直线$x=0$, $y=0$, $x+y=\pi$所围成三角形区域.

解 将积分域D用直线$x+y=\frac{\pi}{2}$分成积分域D_1, D_2.

原式$=\iint\limits_{D_1}\cos(x+y)d\sigma+\iint\limits_{D_2}(-1)\cos(x+y)d\sigma$

因直接计算$\iint\limits_{D_2}\cos(x+y)dxdy$比较复杂,故

原式$=2\iint\limits_{D_1}\cos(x+y)d\sigma-\iint\limits_{D_1\cup D_2}\cos(x+y)d\sigma$

$=2\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}-x}\cos(x+y)dy-\int_0^{\pi}dx\int_0^{\pi-x}\cos(x+y)dy$

$=2\int_0^{\frac{\pi}{2}}(1-\sin x)dx+\frac{1}{2}\int_0^{\pi}\sin x\,dx=\pi$

例9－6 求$\iint\limits_D\sqrt{|y-x^2|}\,d\sigma$, D: $|x|\le 1, 0\le y\le 2$.

解 原式$=\int_{-1}^1 dx\int_{x^2}^2\sqrt{y-x^2}\,dy+\int_{-1}^1dx\int_0^{x^2}\sqrt{x^2-y}\,dy$

$=\int_{-1}^1\frac{2}{3}(y-x^2)^{\frac{3}{2}}\Big|_{x^2}^{2}dx-\int_{-1}^1\frac{2}{3}(x^2-y)^{\frac{3}{2}}\Big|_{0}^{x^2}dx$

$=\frac{5}{3}+\frac{\pi}{2}$

例9－7 求$\int_0^1 dx\int_0^{\sqrt{x}} e^{-\frac{y^2}{2}}dy$

解 由于先对y积分时,被积函数的原函数不是初等函数,故应改变积分次序.

$$\therefore \text{原式}=\int_0^1 dy\int_{y^2}^1 e^{-\frac{y^2}{2}}dx=\int_0^1(1-y^2)e^{-\frac{y^2}{2}}dy$$

$$\because \int y^2e^{-\frac{y^2}{2}}dy=-\int ye^{-\frac{y^2}{2}}d(-\frac{y^2}{2})=-\int yde^{-\frac{y^2}{2}}$$

$$=-ye^{-\frac{y^2}{2}}+\int e^{-\frac{y^2}{2}}dy$$

$$\therefore \text{原式}=\int_0^1(1-y^2)e^{-\frac{y^2}{2}}dy=ye^{-\frac{y^2}{2}}\Big|_0^1=\frac{1}{\sqrt{e}}$$ ▌

例9－8 设$f(x)$为恒正连续函数，$D:x^2+y^2\le R^2$ $(R>0)$,求$\iint\limits_D\frac{af(x)+bf(y)}{f(x)+f(y)}d\sigma$

解 若$\varphi(x,y)$为连续函数,积分域**D**关于直线$y=x$对称,则有$\iint\limits_D\varphi(x,y)d\sigma=\iint\limits_D\varphi(y,x)d\sigma$

$$\text{原式}=\frac{1}{2}\iint\limits_D\left[\frac{af(x)+bf(y)}{f(x)+f(y)}+\frac{af(y)+bf(x)}{f(x)+f(y)}\right]dxdy$$

$$=\frac{1}{2}\iint\limits_D(a+b)dxdy=\frac{1}{2}(a+b)\pi R^2$$ ▌

例9－9 $D:\{(x,y)\big|0\le x\le\frac{\pi}{2},0\le y\le\frac{\pi}{2}\}$,

求$\iint\limits_D\sqrt{1-\sin^2(x+y)}d\sigma$

解 原式$=\iint\limits_D\left|\cos(x+y)\right|dxdy$

$$=\int_0^{\frac{\pi}{2}}dx\int_0^{\frac{\pi}{2}-x}\cos(x+y)dy+\int_0^{\frac{\pi}{2}}dx\int_{\frac{\pi}{2}-x}^{\frac{\pi}{2}}(-1)\cos(x+y)dy$$

$$=\int_0^{\frac{\pi}{2}}(1-\sin x)dx+\int_0^{\frac{\pi}{2}}(1-\cos x)dx=\pi-2$$ ▌

例9－10 设$f(t)$是连续函数,$D:|x|\le\frac{A}{2},|y|\le\frac{A}{2}$

试证:$\iint\limits_D f(x-y)d\sigma=\int_{-A}^A f(t)(A-|t|)dt$ (**A**为常数)

证 $\iint\limits_D f(x-y)d\sigma=\int_{-\frac{A}{2}}^{\frac{A}{2}}dx\int_{-\frac{A}{2}}^{\frac{A}{2}}f(x-y)dy$

$$\underline{\underline{\text{令}x-y=t}}\int_{-\frac{A}{2}}^{\frac{A}{2}}dx\int_{x+\frac{A}{2}}^{x-\frac{A}{2}}f(t)d(-t)=\int_{-\frac{A}{2}}^{\frac{A}{2}}dx\int_{x-\frac{A}{2}}^{x+\frac{A}{2}}f(t)dt$$

$$=\int_{-A}^0 f(t)dt\int_{-\frac{A}{2}}^{t+\frac{A}{2}}dx+\int_0^A f(t)dt\int_{t-\frac{A}{2}}^{\frac{A}{2}}dx$$

$$=\int_{-A}^0 f(t)(t+A)dt+\int_0^A f(t)(A-t)dt$$

$$=\int_{-A}^0 f(t)(A-|t|)dt+\int_0^A f(t)(A-|t|)dt$$

$$=\int_{-A}^A f(t)(A-|t|)dt$$ ▌

例9－11 $D:\{(x,y)\big|0\le x\le1,0\le y\le1\}$,

计算$\iint\limits_D e^{\max\{x^2,y^2\}}d\sigma$

解 设$\boldsymbol{D_1}=\{(x,y)\big|0\le x\le1,0\le y\le x\}$;

$\boldsymbol{D_2}=\{(x,y)\big|0\le x\le1,x\le y\le1\}$

$$\text{原式}=\iint\limits_{D_1}e^{\max\{x^2,y^2\}}d\sigma+\iint\limits_{D_2}e^{\max\{x^2,y^2\}}d\sigma$$

$$=\iint\limits_{D_1}e^{x^2}d\sigma+\iint\limits_{D_2}e^{y^2}d\sigma=\int_0^1dx\int_0^xe^{x^2}dy+\int_0^1dy\int_0^ye^{y^2}dx$$

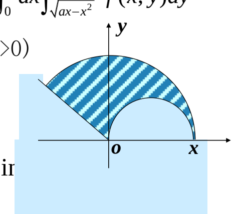
$$=\int_0^1xe^{x^2}dx+\int_0^1ye^{y^2}dy=e-1$$ ▌

例9－12 设$f(x,y)$连续, 试将

$$\mathrm{I}=\int_{-\frac{\sqrt{2}}{2}a}^0dx\int_{-x}^{\sqrt{a^2-x^2}}f(x,y)dy+\int_0^adx\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}}f(x,y)dy$$

写成极坐标下的累次积分(a>0)

解 由积分域,有



$$\mathrm{I}=\int_0^{\frac{\pi}{2}}d\theta\int_{a\cos\theta}^af(r\cos\theta,r\sin$$

$$+\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}d\theta\int_0^{a\cos\theta}f(r\cos\theta,r\sin\theta)rdr$$ ▌

例9−13 求$\iint\limits_D (x^2+4y^2+9+xy)d\sigma$, $D: x^2+y^2\le 4$

解 原式$=\iint\limits_D (x^2+4y^2)d\sigma+\iint\limits_D 9d\sigma+\iint\limits_D xyd\sigma$

$$=\iint\limits_D (x^2+4y^2)d\sigma+36\pi+0$$

$$=\frac{1}{2}\iint\limits_D (x^2+y^2)d\sigma+2\iint\limits_D (x^2+y^2)d\sigma+36\pi$$

$$=\frac{5}{2}\iint\limits_D (x^2+y^2)d\sigma+36\pi$$

$$=\frac{5}{2}\int_0^{2\pi}d\theta\int_0^2 r^3dr+36\pi=20\pi+36\pi=56\pi$$ ∎

例9−14 求$\int_0^a dx\int_{-x}^{-a+\sqrt{a^2-x^2}}\frac{dy}{\sqrt{x^2+y^2}\cdot\sqrt{4a^2-x^2-y^2}}$

解 原式$=\lim\limits_{\varepsilon\to+0}\int_{-\frac{\pi}{4}}^{0}d\theta\int_{0+\varepsilon}^{-2a\sin\theta}\frac{rdr}{r\cdot\sqrt{4a^2-r^2}}$

$$=\int_{-\frac{\pi}{4}}^{0}[\arcsin\frac{r}{2a}]\Big|_{0+\varepsilon}^{-2a\sin\theta}d\theta$$

$$=\int_{-\frac{\pi}{4}}^{0}(-\theta)d\theta=\frac{\pi^2}{32}$$ ∎

例9−15 求$\iint\limits_D e^{\frac{y-x}{y+x}}d\sigma$, D:以(0, 0), (1, 0), (0, 1)为顶点的三角形内部区域.

解 令$y-x=u$, $y+x=v$,解出

$$x=\frac{1}{2}(v-u),\quad y=\frac{1}{2}(v+u)\Rightarrow J=-\frac{1}{2}$$

$\therefore$ 原式$=\frac{1}{2}\int_0^1 dv\int_{-v}^{v}e^{\frac{u}{v}}du=\frac{1}{4}(e-e^{-1})$ ∎

例9−16 求$\iint\limits_D (x+y)d\sigma$, $D: x^2+y^2\le x+y$

解 D的边界曲线为$x^2+y^2\le x+y$,改写为

$(x-\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{1}{2}$, 则

原式$=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}d\theta\int_0^{\cos\theta+\sin\theta}(r\cos\theta+r\sin\theta)\,rdr$

$$=\frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}[\sqrt{2}\sin(\frac{\pi}{4}+\theta)]^4d\theta$$

$$\overset{\frac{\pi}{4}+\theta=t}{=}\frac{4}{3}\int_0^{\pi}\sin^4 tdt=\frac{8}{3}\int_0^{\frac{\pi}{2}}\sin^4 tdt=\frac{\pi}{2}$$ ∎

例9−17 求$\iiint\limits_\Omega y\sqrt{1-x^2}dv$, Ω:平面$y=1$与曲面$y=-\sqrt{1-x^2-z^2}$, $x^2+z^2=1$所围成的闭区域.

解 按$y\to z\to x$的次序:

原式$=\int_{-1}^{1}\sqrt{1-x^2}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dz\int_{-\sqrt{1-x^2-z^2}}^{1}ydy$

$$=\int_{-1}^{1}\sqrt{1-x^2}dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{x^2+z^2}{2}dz.$$

$$=\int_{-1}^{1}(-\frac{2}{3}x^4+\frac{1}{3}x^2+\frac{1}{3})dx=\frac{28}{45}.$$ ∎

例9−18 求$\iiint\limits_\Omega e^z dv$,

Ω:平面$x+y+z=1$与三坐标面所围成的闭区域.

解 原式$=\int_0^1dx\int_0^{1-x}dy\int_0^{1-x-y}e^zdz$

$$=\int_0^1dx\int_0^{1-x}(e^{1-x-y}-1)dy.$$

$$=\int_0^1e^{1-x}dx-\frac{3}{2}=e-\frac{5}{2}\ .$$ ∎

例9－19　求 $\iiint\limits_{\Omega} y\cos(z+x)dv$，$\Omega$：由抛物柱面 $y=\sqrt{x}$ 及平面 $y=0,\ z=0,\ x+z=\dfrac{\pi}{2}$ 所围成的闭区域.

解　原式 $=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}dy\int_0^{\frac{\pi}{2}-x}y\cos(z+x)dz$

$$=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y(1-\sin x)dy.$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{2}x(1-\sin x)dx=\frac{\pi^2}{16}-\frac{1}{2}\quad .\ \blacksquare$$

例9－20　证明：当 $f(z)$ 连续时，有

$$\iiint\limits_{x^2+y^2+z^2\le 1} f(z)dv=\pi\int_{-1}^{1}f(t)(1-t^2)dt,$$

并用此公式计算 $\iiint\limits_{x^2+y^2+z^2\le 1}(z^3+z^2+z+1)dv$ 的值

解　左 $=\int_{-1}^{1}[f(z)\cdot\iint\limits_{x^2+y^2\le 1-z^2}d\sigma]dz$

$$=\int_{-1}^{1}f(z)\bullet\pi(1-z^2)dz=\pi\int_{-1}^{1}f(t)\bullet(1-t^2)dt=\text{右}$$

$$\iiint\limits_{x^2+y^2+z^2\le 1}(z^3+z^2+z+1)dv$$

$$=\pi\int_{-1}^{1}(t^3+t^2+t+1)(1-t^2)dt=\frac{8}{5}\pi\ \blacksquare$$

例9－21　求 $\iiint\limits_{\Omega}(x+z)dv$，$\Omega$：由锥面 $z=\sqrt{x^2+y^2}$ 与半球面 $z=\sqrt{1-x^2-y^2}$ 所围成的闭区域.

解　原式 $=\iiint\limits_{\Omega}xdv+\iiint\limits_{\Omega}zdv=0+\iiint\limits_{\Omega}zdv$

$$=\int_0^{\frac{\sqrt{2}}{2}}zdz\iint\limits_{D_1(z)}d\sigma+\int_{\frac{\sqrt{2}}{2}}^{1}zdz\iint\limits_{D_2(z)}d\sigma$$

$$=\int_0^{\frac{\sqrt{2}}{2}}z\pi z^2dz+\int_{\frac{\sqrt{2}}{2}}^{1}z\pi(1-z^2)dz=\frac{\pi}{8}\ \blacksquare$$

例9－22　求 $\iiint\limits_{\Omega}x^2yzdv$，

Ω：$0\le z\le 1,\ 0\le y\le\sqrt{4-x^2},\ -2\le x\le 0$

解　原式 $=\int_{-2}^{0}dx\int_0^{\sqrt{4-x^2}}dy\int_0^1 x^2yzdz$

$$=\frac{1}{2}\int_{-2}^{0}x^2dx\int_0^{\sqrt{4-x^2}}ydy$$

$$=\frac{1}{4}\int_{-2}^{0}(4x^2-x^4)dx=\frac{16}{15}\ \blacksquare$$

例9－23　求 $\iiint\limits_{\Omega}(ax+by+cz)dv$，

Ω：$x^2+y^2+z^2\le 2z,\ (a>0,b>0,c>0)$

解　$\because \iiint\limits_{\Omega}xdv=0,\ \iiint\limits_{\Omega}ydv=0$，　球体形心坐标为 $(0,0,1)$.

$\therefore$ 由形心坐标公式 $1=\bar{z}=\dfrac{\iiint\limits_{\Omega}zdv}{V}$，

有　$\iiint\limits_{\Omega}zdv=V=\dfrac{4\pi}{3}\bullet 1=\dfrac{4\pi}{3}$

$\therefore$ 原式 $=0+0+c\bullet\dfrac{4\pi}{3}=\dfrac{4c\pi}{3}\ \blacksquare$

例9－24　设 Ω：抛物面 $x^2+y^2=2z$ 及平面 $z=1$ 与 $z=2$ 所围成，$f(x,y,z)$ 在 Ω 上连续. 试将 $\iiint\limits_{\Omega}f(x,y,z)dv$ 写成柱面坐标下累次积分.

解　$\iiint\limits_{\Omega}f(x,y,z)dv$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}rdr\int_1^2 f(r\cos\theta,r\sin\theta,z)dz$$

$$+\int_0^{2\pi}d\theta\int_{\sqrt{2}}^{2}rdr\int_{\frac{r^2}{2}}^{2}f(r\cos\theta,r\sin\theta,z)dz$$

$\blacksquare$

例9－25 求$\int_0^2 dx\int_0^{\sqrt{2x-x^2}} dy\int_0^a z\sqrt{x^2+y^2}dz$

解 原式$=\int_0^{\frac{\pi}{2}} d\theta\int_0^{2\cos\theta} dr\int_0^a z\bullet r\bullet rdz$

$$=\int_0^{\frac{\pi}{2}} d\theta\int_0^{2\cos\theta} \frac{a^2}{2}\bullet r^2dr$$

$$=\frac{a^2}{2}\int_0^{\frac{\pi}{2}}\frac{8}{3}\cos^3\theta d\theta=\frac{8}{9}a^2$$

例9－26 求$\iiint\limits_{\Omega}(x^2+y^2)dv$, Ω: 由曲线$\begin{cases} y^2=2z \\ x=0 \end{cases}$绕$z$轴旋转一周所成的曲面与平面$z=2, z=8$所围成空间区域

解 旋转曲面方程为$z=\frac{1}{2}(x^2+y^2)$,积分域为由曲面$z=\frac{1}{2}(x^2+y^2)$与平面$z=2, z=8$所围成闭区域.

利用柱坐标计算,

原式$=\int_0^{2\pi} d\theta\int_0^2 dr\int_2^8 r^2\bullet rdr+\int_0^{2\pi} d\theta\int_2^4 dr\int_{\frac{r^2}{2}}^8 r^2\bullet rdr$

$$=48\pi+288\pi=336\pi$$

例9－27 求$\iiint\limits_{\Omega} zdv$, Ω: 由球面$z=\sqrt{4-x^2-y^2}$与抛物面$z=\frac{1}{3}(x^2+y^2)$所围成的闭区域.

解 两曲面相交于平面z=1上,交线在$x0y$面上的投影区域为D:$x^2+y^2\le 3$.

$\therefore$原式$=\int_0^{2\pi} d\theta\int_0^{\sqrt{3}} dr\int_{\frac{r^2}{3}}^{\sqrt{4-r^2}} z\bullet rdr$

$$=2\pi\int_0^{\sqrt{3}}\frac{1}{2}r(4-r^2-\frac{1}{9}r^4)dr=\frac{13\pi}{4}$$

例9－28 求$\iiint\limits_{\Omega} zdv$, Ω: $x^2+y^2+z^2\le 2a^2$, $\frac{1}{a}(x^2+y^2)\ge z$ $(a>0)$.

解 记D_{xy}:$x^2+y^2\le a^2$,有

原式$=\iint\limits_{D_{xy}} dxdy\int_{\frac{1}{a}(x^2+y^2)}^{\sqrt{2a^2-x^2-y^2}} zdz$

$$=\frac{1}{2}\iint\limits_{D_{xy}}\left[(2a^2-x^2-y^2)-\frac{1}{a}(x^2+y^2)^2\right]dxdy$$

$$=\frac{1}{2}\int_0^{2\pi} d\theta\int_0^a (2a^2-r^2-\frac{1}{a^2}r^4)rdr=\frac{7\pi a^4}{12}$$

例9－29 求$\iiint\limits_{\Omega} xe^{\frac{x^2+y^2+z^2}{a^2}}dv$, Ω: $x^2+y^2+z^2\le a^2$在第一卦限的部分.

解 原式$=\int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} d\varphi\int_0^a r\cos\theta\sin\varphi\cdot e^{\frac{r^2}{a^2}}\cdot r^2\sin\varphi dr$

$$=\int_0^{\frac{\pi}{2}}\cos\theta d\theta\int_0^{\frac{\pi}{2}}\sin^2\varphi d\varphi\int_0^a e^{\frac{r^2}{a^2}}\cdot r^3dr$$

$$=\frac{\pi}{8}a^4$$

例9－30 求$\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}dv$, Ω: $x^2+y^2+z^2\le x$

解 原式$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_0^{\pi} d\varphi\int_0^{\sin\varphi\cos\theta} r\cdot r^2\sin\varphi dr$

$$=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta d\theta\int_0^{\pi}\sin^5\varphi d\varphi$$

$$\underline{\underline{\varphi=\frac{\pi}{2}+t}}\ \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta d\theta\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5 tdt$$

$$=\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta\int_0^{\frac{\pi}{2}}\cos^5 tdt$$

$$=\frac{3}{4}\bullet\frac{1}{2}\bullet\frac{\pi}{2}\bullet\frac{4}{5}\bullet\frac{2}{3}\bullet 1=\frac{\pi}{10}$$

例9－31 求$\iiint\limits_{\Omega}(\frac{x^4+y^4}{2}+x^2y^2)dv$， Ω：两半球
$z=\sqrt{A^2-x^2-y^2}$，$z=\sqrt{a^2-x^2-y^2}$及平面$z=0$所围成
$(0<a<A)$

[解] $\because \frac{x^4+y^4}{2}+x^2y^2=\frac{1}{2}(x^2+y^2)^2$

$$\therefore 原式=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_a^A(r^2\sin^2\varphi)^2\bullet r^2\sin\varphi dr$$

$$=\pi\left[\int_0^{\frac{\pi}{2}}\sin^5\varphi d\varphi\right]\left[\int_a^A r^6dr\right]$$

$$=\pi\bullet\frac{8}{15}\bullet\frac{1}{7}(A^7-a^7)=\frac{8\pi}{105}(A^7-a^7)$$ ▌

例9－32 求$\iiint\limits_{\Omega}zdv$， Ω：曲面$z=\sqrt{x^2+y^2}$，
$z=\sqrt{1-x^2-y^2}$及平面$z=2$所围成的闭区域.

解 $$原式=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_1^{\frac{2}{\cos\varphi}}r\cos\varphi\bullet r^2\sin\varphi dr$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{4}}(\sin\varphi\cos\varphi)(\frac{16}{\cos^4\varphi}-1)d\varphi$$

$$=8\pi\int_0^{\frac{\pi}{4}}\tan\varphi d\tan\varphi-\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\sin\varphi d\sin\varphi$$

$$=8\pi\bullet\frac{1}{2}\tan^2\varphi\Big|_0^{\frac{\pi}{4}}-\frac{\pi}{2}\bullet\frac{1}{2}\sin^2\varphi\Big|_0^{\frac{\pi}{4}}=\frac{31\pi}{8}$$ ▌

例9－33 求$\iiint\limits_{\Omega}\left|x^2+y^2+z^2-1\right|dv$， Ω：$x^2+y^2+z^2\le 2$

解 记$\Omega_1:x^2+y^2+z^2\le 1$， $\Omega_2:x^2+y^2+z^2\le 2$
原式

$$=\iiint\limits_{\Omega_1}\left[1-(x^2+y^2+z^2)\right]dv+\iiint\limits_{\Omega_2}\left[(x^2+y^2+z^2)-1\right]dv$$

$$=\frac{4\pi}{3}-\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^1 r^2\bullet r^2\sin\varphi dr$$

$$+\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_1^{\sqrt{2}}r^2\bullet r^2\sin\varphi dr-\left[\frac{4\pi}{3}2\sqrt{2}-\frac{4\pi}{3}\right]$$

$$=\frac{4\pi}{3}-2\pi\int_0^{\pi}\sin\varphi d\varphi\int_0^1 r^4dr+2\pi\int_0^{\pi}\sin\varphi d\varphi\int_1^{\sqrt{2}}r^4dr-\frac{7\pi}{3}$$

$$=\frac{8\sqrt{2}}{15}\pi$$ ▌

例9－34 利用球坐标计算$\iiint\limits_{\Omega}(x+y+\frac{1}{\sqrt{z}})dv$，
Ω：$x^2+y^2+z^2\le 2$，$x^2+y^2\le 1$，$z\ge 1$

解 $\because \iiint\limits_{\Omega}xdv=0$， $\iiint\limits_{\Omega}ydv=0$

$$\therefore 原式=\iiint\limits_{\Omega}\frac{1}{\sqrt{z}}dv=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_{\frac{1}{\cos\varphi}}^{\sqrt{2}}\frac{r^2\sin\varphi}{\sqrt{r\cos\varphi}}dr$$

$$=2\pi\int_0^{\frac{\pi}{4}}\frac{\sin\varphi}{\sqrt{\cos\varphi}}d\varphi\int_{\frac{1}{\cos\varphi}}^{\sqrt{2}}r^{\frac{3}{2}}dr$$

$$=\frac{8\pi}{5}(2^{\frac{5}{4}}-\frac{9}{4})$$ ▌

例9－35 半径为R的球面S的球心在定球面$S_0:x^2+y^2+z^2=a^2\ (a>0)$上,问$R$取何值时,$S$在定球面$S_0$内的那部分面积最大?

解 设球面S的球心坐标为$(0,0,a)$,则S的方程为
$x^2+y^2+(z-a)^2=R^2$， 两球面交线为

$$\begin{cases}x^2+y^2+z^2=a^2\\ x^2+y^2+(z-a)^2=R^2\end{cases}\Rightarrow x^2+y^2=R^2(1-\frac{R^2}{4a^2})$$

两球面交线在xOy面上的投影域为

$$D:x^2+y^2\le R^2(1-\frac{R^2}{4a^2})$$ ▌

令 $\delta^2=R^2(1-\frac{R^2}{4a^2})$， 则 $D:x^2+y^2\le\delta^2$

S在S_0内部的部分球面方程为$z=a-\sqrt{R^2-x^2-y^2}$

面积微元为$dS=\sqrt{1+z_x^2+z_y^2}dxdy=\frac{R}{\sqrt{R^2-x^2-y^2}}dxdy$

$\therefore S$在S_0内的面积为

$$A(R)=\iint\limits_D\frac{R}{\sqrt{R^2-x^2-y^2}}dxdy=R\int_0^{2\pi}d\theta\int_0^{\delta}\frac{rdr}{\sqrt{R^2-r^2}}$$

$$=2\pi R(-\sqrt{R^2-r^2})\Big|_0^{\delta}=2\pi R(R-\frac{R^2}{2a})$$

令$A'(R)=0$，得$R=\frac{4a}{3}$， $A''(R)=-4\pi<0$.

取$R=\frac{4}{3}a$时,S在S_0内的那部分面积最大. ▌

例9－36 曲面$x^2+y^2=az$将球体$x^2+y^2+z^2\le 4az$ $(a>0)$分成两部分,求这两部分的体积.

解 联立$\begin{cases} x^2+y^2=az \\ x^2+y^2+z^2=4az \end{cases} \Rightarrow z=0, z=3a$

设较大的一部分立体体积为V_1,采用"平行截面法"

$$V_1=\int_0^{3a}dz\iint\limits_{x^2+y^2\le az}dxdy+\int_{3a}^{4a}dz\iint\limits_{x^2+y^2\le 4az-z^2}dxdy$$

$$=\int_0^{3a}\pi azdz+\int_{3a}^{4a}\pi(4az-z^2)dz=\frac{37}{6}\pi a^3$$

另一部分立体的体积为

$$V_2=\frac{4}{3}\pi(2a)^3-\frac{37}{6}\pi a^3=\frac{27}{6}\pi a^3$$

例9－37 设$f(u)$可微分，$f(0)=0$,求

$$\lim_{t\to+0}\frac{1}{\pi t^3}\iint\limits_{x^2+y^2\le t^2}f(x^2+y^2)d\sigma$$

解 原式$=\lim\limits_{t\to+0}\dfrac{\int_0^{2\pi}d\theta\int_0^t f(r)rdr}{\pi t^3}$

$$=\lim_{t\to+0}\frac{2\pi\int_0^t f(r)rdr}{\pi t^3}=\lim_{t\to+0}\frac{2tf(t)}{3t^2}$$

$$=\lim_{t\to+0}\frac{2}{3}\frac{f(t)-f(0)}{t-0}=\frac{2}{3}f'(0)$$

例9－38 在平面$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$与坐标面所围成的四面体内,作一个以该平面为顶面,求在xOy坐标面上的投影为长方形的六面体中体积最大者$(a,b,c>0)$.

解 六面体体积为

$$V=\iint\limits_D zd\sigma=\iint\limits_D c(1-\frac{x}{a}-\frac{y}{b})d\sigma=c\int_0^x dx\int_0^y(1-\frac{x}{a}-\frac{y}{b})dy$$

直线AB($\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$与$xOy$面的交线)方程为$\frac{x}{a}+\frac{y}{b}=1$

令 $F(x,y,\lambda)=c\int_0^x dx\int_0^y(1-\frac{x}{a}-\frac{y}{b})dy+\lambda(\frac{x}{a}+\frac{y}{b}-1)$

$$=xy-\frac{x^2y}{2a}-\frac{xy^2}{2b}+\lambda(\frac{x}{a}+\frac{y}{b}-1)$$

解方程$\begin{cases} F_x=y-\dfrac{y^2}{2b}-\dfrac{xy}{a}+\dfrac{\lambda}{a}=0 \\ F_y=x-\dfrac{xy}{b}-\dfrac{x^2}{2a}+\dfrac{\lambda}{b}=0 \\ F_\lambda=\dfrac{x}{a}+\dfrac{y}{b}-1=0 \end{cases}$

$$\Rightarrow x=\frac{a}{2}, y=\frac{b}{2}$$

$$\therefore V_{\max}=c\int_0^{\frac{a}{2}}dx\int_0^{\frac{b}{2}}(1-\frac{x}{a}-\frac{y}{b})dy=\frac{1}{8}abc$$

例9－39 设$f(t)$连续,$f(0)=0$, $\Omega:0\le z\le h, x^2+y^2\le t^2$,

$F(t)=\iiint\limits_\Omega\left[z^2+f(x^2+y^2)\right]dv$,求$\lim\limits_{t\to+0}\dfrac{F(t)}{t^2}$

解 $\because F(t)=\int_0^h dz\int_0^{2\pi}d\theta\int_0^t\left[z^2+f(r^2)\right]rdr$

$$=2\pi\left[\int_0^h z^2dz\int_0^t rdr+h\int_0^t f(r^2)rdr\right]$$

$$=2\pi\left[\frac{1}{6}h^3t^2+h\int_0^t f(r^2)rdr\right]$$

$$\therefore\lim_{t\to+0}\frac{F(t)}{t^2}=\frac{\pi}{3}h^3+\lim_{t\to+0}\frac{2\pi h\int_0^t f(r^2)rdr}{t^2}$$

$$=\frac{\pi}{3}h^3+\lim_{t\to+0}\frac{2\pi hf(t^2)t}{2t}=\frac{\pi}{3}h^3$$