第十章

# 第七节 斯托克斯公式 环流量与旋度

一、斯托克斯公式

*二、空间曲线积分与路径无关的条件

三、环流量与旋度

*四、向量微分算子

## 一、斯托克斯( Stokes ) 公式

**定理1.** 设光滑曲面 $\Sigma$ 的边界 $\Gamma$ 是分段光滑曲线, $\Sigma$ 的侧与 $\Gamma$ 的正向符合右手法则, $P,Q,R$ 在包含 $\Sigma$ 在内的一个空间域内具有连续一阶偏导数, 则有

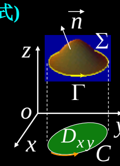
$$\iint_\Sigma \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\,\mathrm{d}x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

$$=\oint_\Gamma P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z \quad \text{(斯托克斯公式)}$$

**证: 情形1** $\Sigma$ 与平行 z 轴的直线只交于一点, 设其方程为

$$\Sigma: z=f(x,y),\quad (x,y)\in D_{xy}$$

为确定起见, 不妨设 $\Sigma$ 取上侧 (如图).

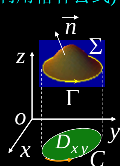



则 $\oint_\Gamma P\,\mathrm{d}x=\oint_C P(x,y,z(x,y))\,\mathrm{d}x$

$$=-\iint_{D_{xy}}\frac{\partial}{\partial y}P(x,y,z(x,y))\,\mathrm{d}x\,\mathrm{d}y \quad \text{(利用格林公式)}$$

$$=-\iint_{D_{xy}}\left[\frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial y}\right]\mathrm{d}x\,\mathrm{d}y$$

$$=-\iint_\Sigma\left[\frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}f_y\right]\cos\gamma\,\mathrm{d}S$$




$\because\ \cos\gamma=\dfrac{1}{\sqrt{1+f_x^2+f_y^2}},\quad \cos\beta=\dfrac{-f_y}{\sqrt{1+f_x^2+f_y^2}},$

$\therefore\ f_y=-\dfrac{\cos\beta}{\cos\gamma}$

因此 $\oint_\Gamma P\,\mathrm{d}x=-\iint_\Sigma\left[\frac{\partial P}{\partial y}-\frac{\partial P}{\partial z}\frac{\cos\beta}{\cos\gamma}\right]\cos\gamma\,\mathrm{d}S$

$$=\iint_\Sigma\left[\frac{\partial P}{\partial z}\cos\beta-\frac{\partial P}{\partial y}\cos\gamma\right]\mathrm{d}S$$

$$=\iint_\Sigma\frac{\partial P}{\partial z}\mathrm{d}z\,\mathrm{d}x-\frac{\partial P}{\partial y}\mathrm{d}x\,\mathrm{d}y$$

同理可证 $\oint_\Gamma Q\,\mathrm{d}y=\iint_\Sigma\frac{\partial Q}{\partial x}\mathrm{d}x\,\mathrm{d}y-\frac{\partial Q}{\partial z}\mathrm{d}y\,\mathrm{d}z$

$$\oint_\Gamma R\,\mathrm{d}x=\iint_\Sigma\frac{\partial R}{\partial y}\mathrm{d}y\,\mathrm{d}z-\frac{\partial R}{\partial x}\mathrm{d}z\,\mathrm{d}x$$

三式相加, 即得斯托克斯公式 ;

**情形2** 曲面 $\Sigma$ 与平行 z 轴的直线交点多于一个, 则可通过作辅助线面把 $\Sigma$ 分成与z 轴只交于一点的几部分, 在每一部分上应用斯托克斯公式, 然后相加, 由于沿辅助曲线方向相反的两个曲线积分相加刚好抵消, 所以对这类曲面斯托克斯公式仍成立. 证毕

**注意:** 如果 $\Sigma$ 是 *xoy* 面上的一块平面区域, 则斯托克斯公式就是格林公式, 故格林公式是斯托克斯公式的特例.

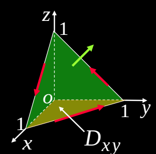
为便于记忆, 斯托克斯公式还可写作:

$$\iint_\Sigma\begin{vmatrix}\mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\oint_\Gamma P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$$

或用第一类曲面积分表示:

$$\iint_\Sigma\begin{vmatrix}\cos\alpha & \cos\beta & \cos\lambda\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}\mathrm{d}S=\oint_\Gamma P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$$

**例1.** 利用斯托克斯公式计算积分 $\oint_\Gamma z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z$ 其中Γ为平面 $x+y+z=1$ 被三坐标面所截三角形的整个边界, 方向如图所示.




**解:** 记三角形域为Σ, 取上侧, 则

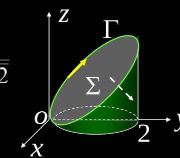
$$\oint_\Gamma z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z = \iint_\Sigma \begin{vmatrix} \mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \iint_\Sigma \mathrm{d}y\,\mathrm{d}z + \mathrm{d}z\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}y = \underbrace{3\iint_{D_{xy}} \mathrm{d}x\,\mathrm{d}y}_{\text{利用对称性}} = \frac{3}{2}$$

**例2.** Γ 为柱面 $x^2+y^2=2y$ 与平面 $y = z$ 的交线,从 z 轴正向看为顺时针, 计算 $I=\oint_\Gamma y^2\,\mathrm{d}x + xy\,\mathrm{d}y + xz\,\mathrm{d}z$.

**解:** 设Σ为平面 $z = y$ 上被 Γ 所围椭圆域，且取下侧, 则其法线方向余弦

$\cos\alpha = 0$, $\cos\beta = \frac{1}{\sqrt2}$, $\cos\gamma = -\frac{1}{\sqrt2}$




利用斯托克斯公式得

$$I = \iint_\Sigma \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} \mathrm{d}S = \frac{1}{\sqrt2}\iint_\Sigma (y-z)\,\mathrm{d}S = 0$$

## *二、空间曲线积分与路径无关的条件

**定理2.** 设 $G$ 是空间一维单连通域, 函数 $P,Q,R$ 在$G$内具有连续一阶偏导数, 则下列四个条件相互等价:

(1) 对$G$内任一分段光滑闭曲线 Γ, 有

$$\oint_\Gamma P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = 0$$

(2) 对$G$内任一分段光滑曲线 Γ, $\int_\Gamma P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ 与路径无关

(3) 在$G$内存在某一函数 $u$, 使 $\mathrm{d}u = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$

(4) 在$G$内处处有

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},\quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

**证:** (4) ⇒ (1) 由斯托克斯公式可知结论成立;

(1) ⇒ (2) (自证)

(2) ⇒ (3) 设函数

$$u(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

则

$$\frac{\partial u}{\partial x} = \lim_{\Delta x\to 0}\frac{u(x+\Delta x,y,z)-u(x,y,z)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_{(x,y,z)}^{(x+\Delta x,y,z)} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_{x}^{x+\Delta x} P\,\mathrm{d}x = \lim_{\Delta x\to 0} p(x+\Delta x, y, z)$$

$$= P(x,y,z)$$

同理可证 $\frac{\partial u}{\partial y} = Q(x,y,z)$, $\frac{\partial u}{\partial z} = R(x,y,z)$

故有 $\mathrm{d}u = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$

(3) ⇒ (4) 若(3)成立, 则必有

$$\frac{\partial u}{\partial x} = P,\quad \frac{\partial u}{\partial y} = Q,\quad \frac{\partial u}{\partial z} = R$$

因$P$, $Q$, $R$ 一阶偏导数连续, 故有

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial Q}{\partial x}$$

同理 $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$ 证毕

**例3.** 验证曲线积分 $\int_\Gamma (y+z)\,\mathrm{d}x + (z+x)\,\mathrm{d}y + (x+y)\mathrm{d}z$ 与路径无关, 并求函数

$$u(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} (y+z)\,\mathrm{d}x + (z+x)\,\mathrm{d}y + (x+y)\,\mathrm{d}z$$

**解:** 令 $P = y+z$, $Q = z+x$, $R = x+y$

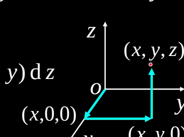
$$\because\ \frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x},\quad \frac{\partial Q}{\partial z} = 1 = \frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x} = 1 = \frac{\partial P}{\partial y}$$

∴ 积分与路径无关, 因此

$$u(x,y,z) = \int_0^x 0\,\mathrm{d}x + \int_0^y x\,\mathrm{d}y + \int_0^z (x+y)\,\mathrm{d}z$$

$$= xy + (x+y)z$$

$$= xy + yz + zx$$



## 三、 环流量与旋度

斯托克斯公式

$$\iint_\Sigma \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\,\mathrm{d}x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

$$=\oint_\Gamma P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$$

设曲面 Σ 的法向量为 $\vec n=(\cos\alpha,\cos\beta,\cos\gamma)$

曲线 Γ的单位切向量为 $\vec\tau=(\cos\lambda,\cos\mu,\cos\nu)$

则斯托克斯公式可写为

$$\iint_\Sigma \left[\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\alpha+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cos\beta+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\gamma\right]\mathrm{d}S$$

$$=\oint_\Gamma (P\cos\lambda+Q\cos\mu+R\cos\nu)\,\mathrm{d}s$$

令 $\vec A=(P,Q,R)$, 引进一个向量

$$\left(\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right),\ \left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right),\ \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\right)=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

记作 $\operatorname{rot}\vec A$ *rotation*

于是得斯托克斯公式的向量形式：

$$\iint_\Sigma \operatorname{rot}\vec A\cdot\vec n\,\mathrm{d}S=\oint_\Gamma \vec A\cdot\vec\tau\,\mathrm{d}s$$

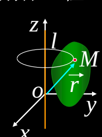
或

$$\iint_\Sigma (\operatorname{rot}A)_n\,\mathrm{d}S=\oint_\Gamma A_\tau\,\mathrm{d}s \qquad ①$$

**定义:** $\oint_\Gamma P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z=\oint_\Gamma A_\tau\,\mathrm{d}s$ 称为向量场$\vec A$沿有向闭曲线 Γ的**环流量**. 向量 $\operatorname{rot}\vec A$ 称为向量场 $\vec A$ 的**旋度** .

**旋度的力学意义:**

设某刚体绕定轴 $l$ 转动, 角速度为 $\vec\omega$, $M$为刚体上任一点, 建立坐标系如图, 则




$$\vec\omega=(0,0,\omega),\quad \vec r=(x,y,z)$$

点 $M$ 的线速度为

$$\vec v=\vec\omega\times\vec r=\begin{vmatrix}\vec i & \vec j & \vec k\\ 0 & 0 & \omega\\ x & y & z\end{vmatrix}=(-\omega y,\omega x,0)$$

$$\operatorname{rot}\vec v=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -\omega y & \omega x & 0\end{vmatrix}=(0,0,2\omega)=2\vec\omega$$

(此即“旋度”一词的来源)

**斯托克斯公式①的物理意义:**

$$\iint_\Sigma (\operatorname{rot}A)_n\,\mathrm{d}S=\oint_\Gamma A_\tau\,\mathrm{d}s$$

向量场 $\vec A$ 产生的旋度场穿过 Σ 的通量

为向量场 $\vec A$ 沿 Γ的环流量

注意 Σ 与 Γ 的方向形成右手系!

**例4.** 求电场强度 $\vec E=\frac{q}{r^3}\vec r$ 的旋度 .

**解:** $$\operatorname{rot}\vec E=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{qx}{r^3} & \frac{qy}{r^3} & \frac{qz}{r^3}\end{vmatrix}=(0,0,0)\quad(\text{除原点外})$$

这说明, 在除点电荷所在原点外, 整个电场无旋.

**例5.** 设 $\vec A=(2y,3x,z^2)$, $\Sigma: x^2+y^2+z^2=4$, $\vec n$ 为Σ的外法向量, 计算 $I=\iint_\Sigma \operatorname{rot}\vec A\cdot\vec n\,\mathrm{d}S$ .

**解:** $$\operatorname{rot}\vec A=\begin{vmatrix}\vec i & \vec j & \vec k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2y & 3x & z^2\end{vmatrix}=(0,0,1)$$

$$\vec n=(\cos\alpha,\cos\beta,\cos\gamma)$$

$$\therefore\quad I=\iint_\Sigma \cos\gamma\,\mathrm{d}S=2\iint_{D_{xy}}\mathrm{d}x\,\mathrm{d}y=8\pi$$

## *四、向量微分算子

定义向量微分算子:

$$\nabla=\frac{\partial}{\partial x}\vec i+\frac{\partial}{\partial y}\vec j+\frac{\partial}{\partial z}\vec k$$

它又称为▽( Nabla )算子, 或哈密顿( Hamilton ) 算子.

(1) 设 $u=u(x,y,z)$, 则

$$\nabla u=\frac{\partial u}{\partial x}\vec i+\frac{\partial u}{\partial y}\vec j+\frac{\partial u}{\partial z}\vec k=\operatorname{grad}u$$

$$\nabla^2 u=\nabla\cdot\nabla u=\nabla\cdot\operatorname{grad}u$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\Delta u$$

(2) $\vec{A}=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+R(x,y,z)\vec{k}$, 则

$$\nabla\cdot\vec{A}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\operatorname{div}\vec{A}$$

$$\nabla\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\operatorname{rot}\vec{A}$$

高斯公式与斯托克斯公式可写成:

$$\iiint_\Omega \nabla\cdot\vec{A}\,\mathrm{d}v=\oiint_\Sigma A_n\,\mathrm{d}S$$

$$\iint_\Sigma(\nabla\times\vec{A})_n\,\mathrm{d}S=\oint_\Gamma A_\tau\,\mathrm{d}s$$

## 内容小结

**1. 斯托克斯公式**

$$\int_\Gamma P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z=\iint_\Sigma\begin{vmatrix}\mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

$$=\iint_\Sigma\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}\mathrm{d}S$$

**2. 空间曲线积分与路径无关的充要条件**

设 $P, Q, R$ 在$\Omega$内具有一阶连续偏导数, 则

$\int_\Gamma P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$ 在$\Omega$内与路径无关

$\Longleftrightarrow$ 在$\Omega$内处处有

$$\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z},\quad \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

$\Longleftrightarrow$ 在$\Omega$内处处有

$$\operatorname{rot}(P,\ Q,\ R)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\vec{0}$$

**3. 场论中的三个重要概念**

设 $u=u(x,y,z)$, $\vec{A}=(P,Q,R)$, $\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$, 则

**梯度:** $\operatorname{grad}u=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})=\nabla u$

**散度:** $\operatorname{div}\vec{A}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\nabla\cdot\vec{A}$

**旋度:** $\operatorname{rot}\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\nabla\times\vec{A}$

## 思考与练习

设 $r=\sqrt{x^2+y^2+z^2}$, 则

$\operatorname{div}(\operatorname{grad}r)=$ $\underline{2/r}$ ; $\operatorname{rot}(\operatorname{grad}r)=$ $\underline{\vec{0}}$ .

**提示:** $\operatorname{grad}r=(\frac{x}{r},\frac{y}{r},\frac{z}{r})$

$$\frac{\partial}{\partial x}(\frac{x}{r})=\frac{r-x\cdot x/r}{r^2}=\frac{r^2-x^2}{r^3},\quad \frac{\partial}{\partial y}(\frac{y}{r})=\frac{r^2-y^2}{r^3}$$

$$\frac{\partial}{\partial z}(\frac{z}{r})=\frac{r^2-z^2}{r^3}$$ 三式相加即得$\operatorname{div}(\operatorname{grad}r)$

$$\operatorname{rot}(\operatorname{grad}r)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r}\end{vmatrix}=(0,0,0)$$