F PROOF OF PROPOSITION 4.3

Let us first simplify each agent i's loss in (1) of our mechanism. The system's global model of our mechanism is $\hat{\theta}(\{\hat{\theta}_i(m_i(\theta_i))\}_{i=1}^N) = \sum_{i=1}^N w_i\hat{\theta}_i(m_i(\theta_i))$. Substituting this to (1), we have

$$\begin{aligned}
&\ell_i(\theta_i,\hat{\theta}(\{m_i(\theta_i)\}_{i=1}^N))\\
=&\mathbb{E}_{\{\theta_j\}_{j=1,j\neq i}^N,\{m_i(\theta_i)\}_{i=1}^N}\left[\left(\theta_i-\hat{\theta}(\{m_i(\theta_i)\}_{i=1}^N)\right)^2\right]\\
=&\mathbb{E}_{\{\theta_j\}_{j=1,j\neq i}^N,\{m_i(\theta_i)\}_{i=1}^N}\left[\left(\theta_i-w_i\hat{\theta}_i(m_i(\theta_i))-\sum_{j\neq i}w_j\hat{\theta}_j(m_j(\theta_j))\right)^2\right]\\
=&\left(\theta_i-w_i\hat{\theta}_i(m_i(\theta_i))\right)^2+\mathbb{E}_{\{\theta_j\}_{j=1,j\neq i}^N,\{m_i(\theta_i)\}_{i=1}^N}\left[\left(\sum_{j\neq i}w_j\hat{\theta}_j(m_j(\theta_j))\right)^2\right]-2\left(\theta_i-w_i\hat{\theta}_i(m_i(\theta_i))\right)\sum_{j\neq i}w_j\mathbb{E}_{\theta_j,m_j(\theta_j)}\left[\hat{\theta}_j(m_j(\theta_j))\right].
\end{aligned}\tag{32}$$

Since

$$\begin{aligned}
&\mathbb{E}_{\{\theta_j\}_{j=1,j\neq i}^N,\{m_i(\theta_i)\}_{i=1}^N}\left[\left(\sum_{j\neq i}w_j\hat{\theta}_j(m_j(\theta_j))\right)^2\right]\\
=&\mathbb{E}_{\{\theta_j\}_{j=1,j\neq i}^N,\{m_i(\theta_i)\}_{i=1}^N}\left[\left(\sum_{j\neq i}w_j^2\hat{\theta}_j^2(m_j(\theta_j))\right)+\sum_{j\neq i}\sum_{k\neq j,i}w_jw_k\hat{\theta}_j(m_j(\theta_j))\hat{\theta}_k(m_k(\theta_k))\right]\\
=&\sum_{j\neq i}w_j^2\mathbb{E}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j^2(m_j(\theta_j))]+\sum_{j\neq i}\sum_{k\neq j,i}w_jw_k\mathbb{E}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]\mathbb{E}_{\theta_k,m_k(\theta_k)}[\hat{\theta}_k(m_k(\theta_k))]\\
=&\sum_{j\neq i}w_j^2\left(\mathbb{E}^2_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]+\mathbb{V}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]\right)+\sum_{j\neq i}\sum_{k\neq j,i}w_jw_k\mathbb{E}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]\mathbb{E}_{\theta_k,m_k(\theta_k)}[\hat{\theta}_k(m_k(\theta_k))]\\
=&\sum_{j\neq i}\left(w_j\mathbb{E}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]\right)^2+\sum_{j\neq i}\sum_{k\neq j,i}w_jw_k\mathbb{E}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]\mathbb{E}_{\theta_k,m_k(\theta_k)}[\hat{\theta}_k(m_k(\theta_k))]+\sum_{j\neq i}w_j^2\left(\mathbb{V}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]\right)\\
=&\left(\sum_{j\neq i}w_j\mathbb{E}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]\right)^2+\sum_{j\neq i}w_j^2\left(\mathbb{V}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]\right).
\end{aligned}\tag{33}$$

Substitute (33) to (32), we have

$$\begin{aligned}
&\ell_i(\theta_i,\hat{\theta}(\{m_i(\theta_i)\}_{i=1}^N))\\
=&\left(\theta_i-w_i\hat{\theta}_i(m_i(\theta_i))\right)^2+\left(\sum_{j\neq i}w_j\mathbb{E}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))\right)^2-2\left(\theta_i-w_i\hat{\theta}_i(m_i(\theta_i))\right)\sum_{j\neq i}w_j\mathbb{E}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]+\sum_{j\neq i}w_j^2\left(\mathbb{V}_{\theta_j,m_j(\theta_j)}[\hat{\theta}_j(m_j(\theta_j))]\right)\\
=&\left(\theta_i-w_i\hat{\theta}_i(m_i(\theta_i))-\sum_{j\neq i}^N w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\right)^2+\sum_{j\neq i}^N w_j\mathbb{V}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))].
\end{aligned}\tag{34}$$

In the following, we first prove the truthfulness of another mechanism, and will then show that it is equivalent to the mechanism in the proposition.

Lemma F.1. *By choosing $\theta_{i,0}=A_i$, $\theta_{i,K_i}=B_i$ and $\{\theta_{i,k}\}_{k=1}^{K_i-1}$ for any $K_i\geq 2$ as the solutions to*

$$\begin{aligned}
&\left(\theta_{i,k}-w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k-1},\theta_{i,k}]]-\sum_{j\neq i}^N w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\right)^2\\
=&\left(\theta_{i,k}-w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k},\theta_{i,k+1}]]-\sum_{j\neq i}^N w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\right)^2,
\end{aligned}\tag{35}$$

the system's partial information disclosure mechanism in Definition 4.2 is truthful. Each agent i truthfully messages the index of the partition containing his local model parameter realization, i.e.,

$$m_i^*(\theta_i)=\{k|\theta_i\in[\theta_{i,k-1},\theta_{i,k}],1\leq k\leq K_i.\}$$

Proof. We simplify (35) for further analysis. Since $\{\theta_{i,k}\}_{k=0}^{K_i}$ is a strictly increasing sequence in k, we have

$$\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k-1},\theta_{i,k}]]\leq\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k},\theta_{i,k+1}]].\tag{36}$$

We simplify (35) due to (36) as

$$\begin{aligned}&\theta_{i,k}-w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k-1},\theta_{i,k}]]-\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\\=&-\theta_{i,k}+w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k},\theta_{i,k+1}]]+\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]>0.\end{aligned}\tag{37}$$

In the following, we first show that each agent i will not deviate from messaging $m_i(\theta_i)=k$ to $m_i(\theta_i)=k+1$ if his $\theta_i\in[\theta_{i,k-1},\theta_{i,k}]$. Since $\theta_i\leq\theta_{i,k}$, according to (37), we have

$$\begin{aligned}&\theta_i-w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k-1},\theta_{i,k}]]-\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\\\leq&-\theta_i+w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k},\theta_{i,k+1}]]+\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))].\end{aligned}\tag{38}$$

His loss in (34) with his message $m_i(\theta_i)=k$ is

$$\begin{aligned}&\ell_i(\theta_i|m_i(\theta_i)=k)\\=&\Big(\theta_i-w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k-1},\theta_{i,k}]])-\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\Big)^2\\&+\sum_{j\neq i}^{N}w_j\mathbb{V}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\end{aligned}\tag{39}$$

His loss in (34) with his message $m_i(\theta_i)=k+1$ is

$$\begin{aligned}&\ell_i(\theta_i|m_i(\theta_i)=k+1)\\=&\Big(\theta_i-w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k},\theta_{i,k+1}]])-\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\Big)^2\\&+\sum_{j\neq i}^{N}w_j\mathbb{V}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\end{aligned}\tag{40}$$

To show $\ell_i(\theta_i|m_i(\theta_i)=k)\leq\ell_i(\theta_i|m_i(\theta_i)=k+1)$, according to (39) and (40), we only need to show that

$$\begin{aligned}&\Big(\theta_i-w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k-1},\theta_{i,k}]])-\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\Big)^2\\\leq&\Big(\theta_i-w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k},\theta_{i,k+1}]])-\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\Big)^2,\end{aligned}$$

which holds due to (38). We further have $\ell_i(\theta_i|m_i(\theta_i)=k)\leq\ell_i(\theta_i|m_i(\theta_i)=k')$ for any $k'\geq k+1$ due to

$$\begin{aligned}&\theta_i-w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k-1},\theta_{i,k}]]-\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\\\leq&-\theta_i+w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k},\theta_{i,k+1}]]+\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\\\leq&-\theta_i+w_i\mathbb{E}[\theta_i|\theta_i\in[\theta_{i,k'-1},\theta_{i,k'}]]+\sum_{j\neq i}^{N}w_j\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))].\end{aligned}$$

With a similar analysis as above, we can obtain that $\ell_i(\theta_i|m_i(\theta_i)=k)\leq\ell_i(\theta_i|m_i(\theta_i)=k-1)$ and $\ell_i(\theta_i|m_i(\theta_i)=k)\leq\ell_i(\theta_i|m_i(\theta_i)=k')$ for any $k'\leq k-1$. Therefore, each agent i incurs the smallest loss by truthful messaging:

$$m_i^*(\theta_i)=\{k|\theta_i\in[\theta_{i,k-1},\theta_{i,k}],1\leq k\leq K_i.\}$$

We then finish the proof. ◻

According to Lemma F.1, each agent i truthfully messages the partition containing his θ_i realization with the mechanism in (35). Therefore, we have

$$\begin{aligned}&\mathbb{E}_{\theta_j,m_j|\theta_j}[\hat{\theta}_j(m_j(\theta_j))]\\&=\sum_{k=1}^{K_i}\mathrm{Pr}(\theta_j\in[\theta_{j,k-1},\theta_{j,k}])\mathbb{E}[\theta_j|\theta_j\in[\theta_{j,k-1},\theta_{j,k}]]=\mathbb{E}[\theta_j]=\mu_j.\end{aligned}\tag{41}$$

Substituting (41) to (35), we can obtain (6) in the proposition, which implies that our mechanism in (6) is truthful. We then finish the proof.