Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.3800 Introduction to Inference
Fall 2023

Computational Lab III

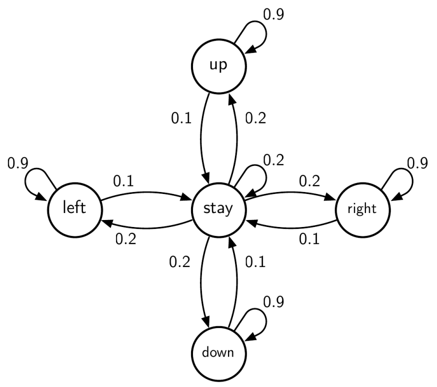
Issued: Wednesday, October 18 **Due:** Friday, November 3, 10pm

Robot Localization

Your pet robot is stuck on Mars somewhere on a 12×8 grid. You need to figure out where your robot is over time so that you can rescue it.

Let random variable $Z_i \in \{0, 1, \ldots, 11\} \times \{0, 1, \ldots, 7\}$ represent the robot's position at time i. For example, $Z_2 = (5, 4)$ means that at time step 2, the robot is in column 5, row 4. Luckily, the robot is quite predictable. At each time step, it makes one of five actions: it stays put, goes left, goes up, goes right, or goes down.

We can visually represent these transitions with the transition diagram[1] shown on the right. The robot action at any time step depends on its previous action. In particular, if the robot's previous action was a movement, it moves in the same direction with probability 0.9 and stays put with probability 0.1. If the robot's previous action was to stay put, it stays again with probability 0.2, or moves, with each direction chosen equally likely from the four options.



On the boundary of the grid, the robot's behavior must adjust but is consistent with above. For instance, at the top of the grid, the robot cannot go any higher. When the robot is at one of the boundary locations, the probabilities of the possible actions are renormalized so that they sum to 1. Such boundary cases suggest that the transition probabilities depend on the robot's previous action *and* its current location. Thus, we model the robot's hidden state X_i at time i as a super variable that includes both the robot's location Z_i and its most recent action A_i, $X_i = (Z_i, A_i)$ as depicted in Fig. 1. The robot's initial position Z_0 is equally likely to be any of the grid locations, and its initial action A_0 =`stay`.

Unfortunately, you cannot directly observe the robot's hidden state X_i. Instead, you have access to a noisy sensor that puts a uniform distribution on valid grid positions within one grid cell of the robot's current true position. The sensor cannot tell you what actions the robot takes. In other words, at time i we observe random variable $Y_i \in \{0, 1, \ldots, 11\} \times \{0, 1, \ldots, 7\}$ as depicted in Fig. 2. Y_i is uniformly distributed over the possible locations determined by Z_i.

[1]A transition diagram, while graphical and with probabilities is *not* the same as a probabilistic graphical model as introduced in class. Each node in a probabilistic graphical model represents a random variable. For a transition diagram, each node is *not* a random variable, but is a value that a random variable can take on.

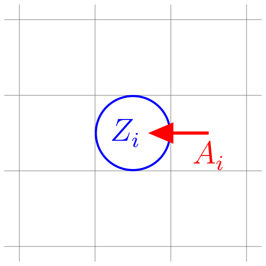


Figure 1: Robot's hidden state $X_i = (Z_i, A_i)$ where a blue circle indicates robot's position Z_i and a red arrow indicates its action A_i.

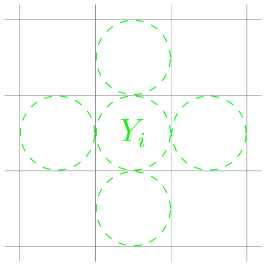
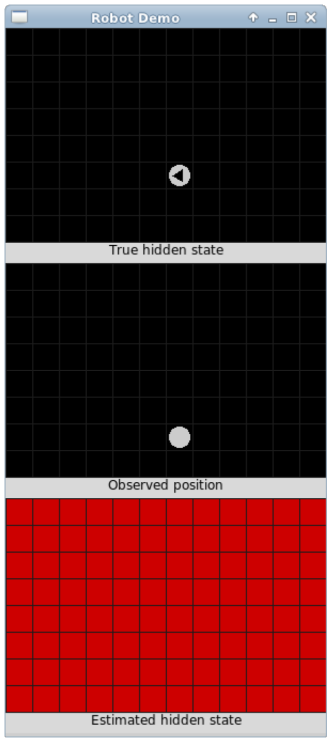


Figure 2: Distribution over observations Y_i given the true location Z_i shown in Fig. 1. Each of the observations shown in green is equally likely.

Code, Data, and Visualization. File `robot.zip` contains five files:

- File `inference.py` will contain all of your code. It also shows how to generate HMM samples. This is the only file you will modify.
- File `robot.py` contains functions for generating the initial distribution, the transition probabilities given a current hidden state, and the observation probabilities given a current hidden state, so you need not re-implement these.
- File `test.txt` contains data for part (b).
- File `test_missing.txt` contains data for parts (c) and (e).
- File `graphics.py` provides graphics code. You do not need to read this file.



When you run `python inference.py`, you should be able to see your robot move around as shown on the right. The top pane shows the true (hidden) state of the robot, which includes location and the most recent action, represented by an arrow. The middle pane shows the observed position of the robot. The bottom pane shows the estimated state of the robot, with red signifying missing data. After you implement your inference algorithms, the bottom pane will automatically show your estimate. This visualization will help you see what your code does. You can also turn the visualization off by setting variable `enable_graphics` to `false` in `inference.py` file. Whether you use visualization or not will *not* affect grading.

To use the visualization functions, you need to have the `tkinter` module installed. You can install it on Linux by running `sudo apt-get install python python3-tk`. If you are using `ssh` to login to Athena, be sure to use the `-X` flag with `ssh` to enable visualization on your computer.

Our solution runs in 2 seconds. While we are not grading performance, please double check your code if it runs significantly faster or slower.

(a) Suppose we get two observations, $y_0 = (5, 4)$ and $y_1 = (6, 5)$. Compute the forward message $m_{0\to1}(\cdot)$ and the backward message $m_{1\to0}(\cdot)$ in the forward-backward algorithm, and the marginal distributions of X_0 and X_1 given these observations. We recommend you solve this part by hand and use the results later to verify that your answers here and in part (b) are consistent.

(b) Complete the function `forward_backward()` to implement the forward-backward algorithm that takes as input observations $y_0, \ldots y_{n-1}$ and computes the marginal distribution $p_{X_i|Y_0,\ldots,Y_{n-1}}(\cdot|y_0, \ldots, y_{n-1})$ for $i = 0, \ldots, n-1$. Run `python inference.py --load=test.txt` and determine the marginal distribution $p_{X_0|Y_0,\ldots,Y_{99}}(\cdot \mid y_0, \ldots, y_{99})$ of the robot's initial state and the marginal distribution $p_{X_{99}|Y_0,\ldots,Y_{99}}(\cdot \mid y_0, \ldots, y_{99})$ of the robot's final state for this sequence of observations. Only include state values (i.e., position-action configurations) with non-zero probability in your answer.

Hint: You may find it helpful to work with log probabilities to avoid arithmetic underflow and to add special handling of state values whose probability is zero.

Sanity checks: 1) If you run your code on test_small.txt, you should obtain the answer you got in part (a).

2) For the full data file test.txt, the marginal distribution of the robot's state at time $i = 1$ is

$$p_{X_1|Y_0,\ldots,Y_{99}}(x|y_0, \ldots, y_{99}) = \begin{cases} 0.5 & \text{if } x = \texttt{(6, 5, down)}, \\ 0.5 & \text{if } x = \texttt{(6, 5, right)}, \\ 0 & \text{otherwise.} \end{cases}$$

3) The distribution of X_{99} assigns non-zero probability to exactly three states and has a mode at `(11, 0, stay)`. The value of the distribution at the mode is 0.81.

Optional: You need not submit an answer for this part, but it could improve the performance of your code. The forward-backward algorithm takes $\mathcal{O}(nk^2)$ operations, where n is the total number of time steps, and k is the number of possible hidden states. However, this problem has additional structure such that it is possible to compute all the marginal distributions in $\mathcal{O}(nk)$ operations — $\mathcal{O}(nk)$ operations for preprocessing and $\mathcal{O}(n)$ operations for message passing.

Why is this the case? How should you modify the forward-backward algorithm to achieve this faster performance? Note that even without the performance optimization suggested here, your code should run on only a few seconds. We recommend prioritizing the required parts first, and coming back to this optional part later if you have time.

(c) Some of the observations were lost when they were transmitted from Mars to Earth. Modify the `forward_backward()` function so that it can handle missing observations, i.e., where at some time steps, there is no observation. In a list of observations, a missing observation is listed as `None` rather than the usual tuple `(x, y)`. Run `python inference.py --load=test_missing.txt` and determine the marginal distributions for X_0 and X_{99} given the available observations. Only include state values (i.e., position-action configurations) with non-zero probability in your answer. Explain your modifications to the `forward_backward()` function.

Sanity check: The mode of the distribution of X_{99} should be `(3, 0, 'right')`, which has probability 0.9.

(d) Suppose there are a total of $n = 100$ time steps, i.e., our HMM contains hidden states $X_0, \dots, X_{99}$. If you only observe $Y_0, \dots, Y_9$ and seek to determine the marginal distribution $p_{X_9|Y_0,\dots,Y_9}(\cdot|y_0, \dots, y_9)$, why is it that you can safely ignore nodes $X_{10}, \dots, X_{99}$ and $Y_{10}, \dots, Y_{99}$ in the graphical model and get the correct solution?

Hint: Consider the distribution $p_{X_0,\dots,X_9|Y_0,\dots,Y_9}(\cdot|y_0, \dots, y_9)$ of the first ten hidden states given the first ten observations. Provide a formula for this marginal distribution, and show that it does not depend on observations from times past $i = 9$.

(e) Unsatisfied with maximizing marginal distributions, you now seek the most likely *sequence* of states the robot has visited given the observations of its location. Complete the function `Viterbi()` to implement the Viterbi algorithm. Your implementation should be able to handle missing observations. Run `python inference.py --load=test_missing.txt` and determine the last ten states in the MAP estimate.

Sanity check: The first state of the MAP estimate should be `(0, 5, up)`. The last 3 states of the MAP estimate should be: `(1, 0, right)`, `(2, 0, right)`, `(3, 0, right)`.

Hint: You might find it helpful to first implement the Viterbi algorithm that assumes we have all observations and test it by running `python inference.py --load=test.txt`

(f) **Optional (not graded):** You need not submit a solution for this part. Implement a modified version of the Viterbi algorithm that outputs the second-best solution to the MAP problem rather than the best solution by completing the function `second_best()`. Run `python inference.py --load=test.txt` and determine the last ten states in the second-best sequence. Explain your modifications.

What to hand in: Upload to Gradescope your writeup as a single PDF file and your code as `inference.py`. The writeup should include your answers for all parts. Scans/images of handwritten work are fine but please write neatly — we can't grade what we can't decipher!

Gradescope will run a number of tests which are visible to you. These are to make sure your code runs correctly in the Gradescope system. Please make sure you pass these tests as we will use the autograder to check your submission.