Continuous Random Variable

X is a continuous random variable if it assigns a real number $X(s)$ to an uncountable event space's events.

Cumulative distribution function (cdf)

$$F_X(x) = P(X \leq x), -\infty < x < \infty$$

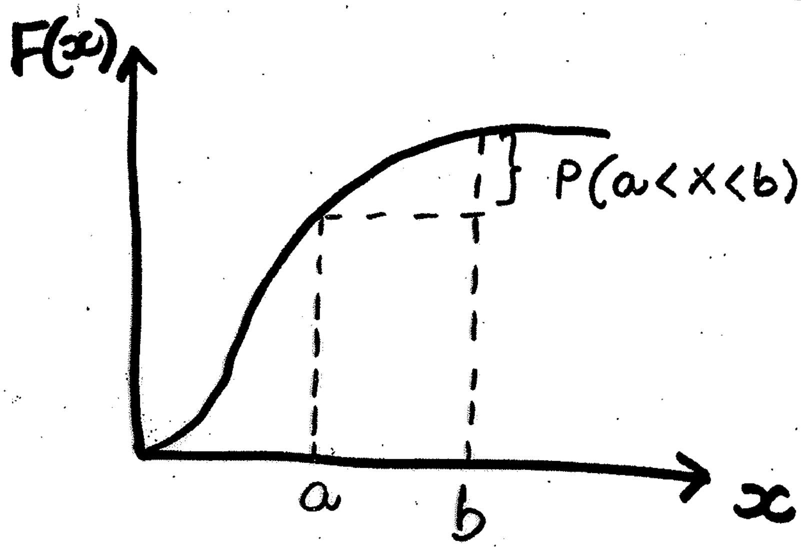
F_X is a smooth distribution function unlike in the case of discrete r.v.s.

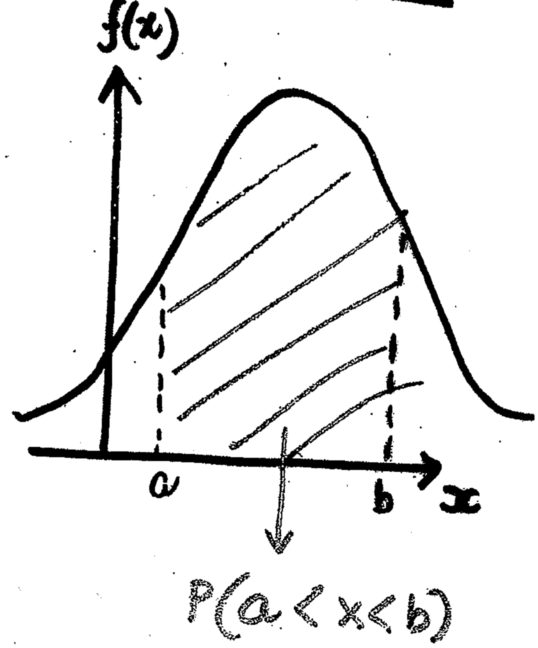
Probability density function (pdf)

$$f(x) = \frac{dF(x)}{dx}$$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

Relation between pdf & cdf





$$P(a < x < b) = \int_a^b f(t)\,dt$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$

$$= F(b) - F(a)$$

Exponential Distribution

"If job arrivals follow a Poisson process, then inter-arrival time follows Exponential distribution.

Example: • Time to failure of a component

• Service time at a server in a queuing network.

$$F(x) = 1 - e^{-\lambda x}, \; x > 0.$$

$$f(x) = \lambda e^{-\lambda x}, \; x > 0$$

$$X \sim EXP(\lambda)$$

Memoryless property

$$P(Y \leq y \mid X > t) = P(X < y)$$

independent of t.

Given a component has survived till time t, what is the probability it will ~~survive till~~ fail in $t+y$.

$$Y = X - t$$

Exponential Distribution

Derivation of memoryless property

$P(Y \leq y \mid X > t) = P(X \leq t+y \mid X > t)$

$\boxed{Y = X - t}$

$$= \frac{P(t < X \leq t+y)}{1 - P(X \leq t)}$$

$$= \frac{F(t+y) - F(t)}{1 - F(t)} = \frac{1 - e^{-\lambda(t+y)} - (1 - e^{-\lambda t})}{1 - (1 - e^{-\lambda t})}$$

$$= \frac{e^{-\lambda t} - e^{-\lambda(t+y)}}{e^{-\lambda t}} = P(X \leq y)$$

Suppose failure rate is λ (constant)

P (component will fail in time t)

$= 1 - P$ (No failure in time t)

$= 1 - P$ ($N_t = 0$ in Poisson process)

$$= 1 - e^{-\lambda t} \frac{(\lambda t)^0}{0!}$$

$= 1 - e^{-\lambda t} \longrightarrow$ Exponential distr.

Reliability & Failure Rate

$$R(t) = 1 - \text{Prob.}(\text{component fails in time } t)$$

$$= 1 - F(t)$$

$$R'(t) = -f(t) \qquad f(t) = F'(t)$$

Say, component survived till time t.

P(component will not survive an addl. time x | survived till time t)

$$= \frac{P(t < X < t+x)}{P(X > t)}$$

$$= \frac{F(t+x) - F(t)}{R(t)}$$

$h(t)$: Instantaneous failure rate

$$= \lim_{x \to 0} \frac{F(t+x) - F(t)}{R(t)} \cdot \frac{1}{x}$$

$$= \frac{1}{R(t)} \lim_{x \to 0} \frac{F(t+x) - F(t)}{x} = \frac{f(t)}{R(t)}$$

$$= \frac{1}{R(t)} F'(t)$$

Failure Rate

For exponential distribution

failure rate $h(t) = \dfrac{f(t)}{R(t)}$

$$= \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

$h(t)\,\Delta t$: Given that component has survived till 't', what is the probability it will fail in $(t, t+\Delta t]$.

$f(t)\,\Delta t$: Unconditional probability that the component will fail in $(t, t+\Delta t]$

$$h(t) \;\; < / = / > \;\; f(t) \; ?$$

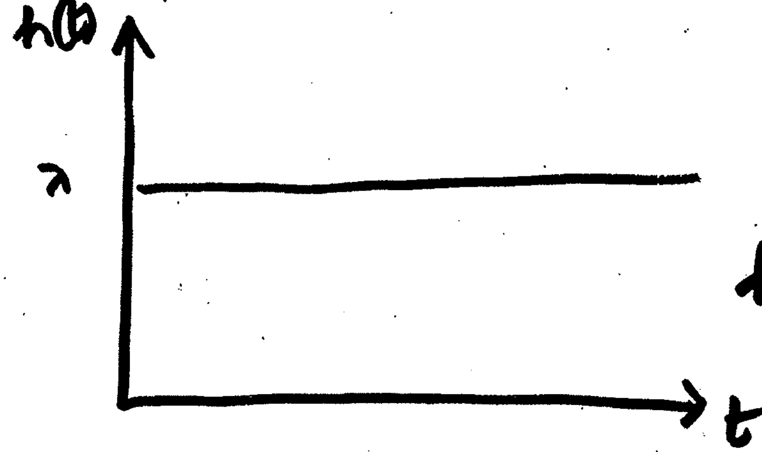
$$R(t) = e^{-\int_{-\infty}^{t} h(x)\,dx}$$

for exponential
$$= e^{-\int_{-\infty}^{t} \lambda\,dx}$$
$$= e^{-\lambda t}$$

$R(t) = 1 - F(t)$
$= 1 - [1 - e^{-\lambda t}]$
$= e^{-\lambda t}$

Failure Rate

With exponential distribution

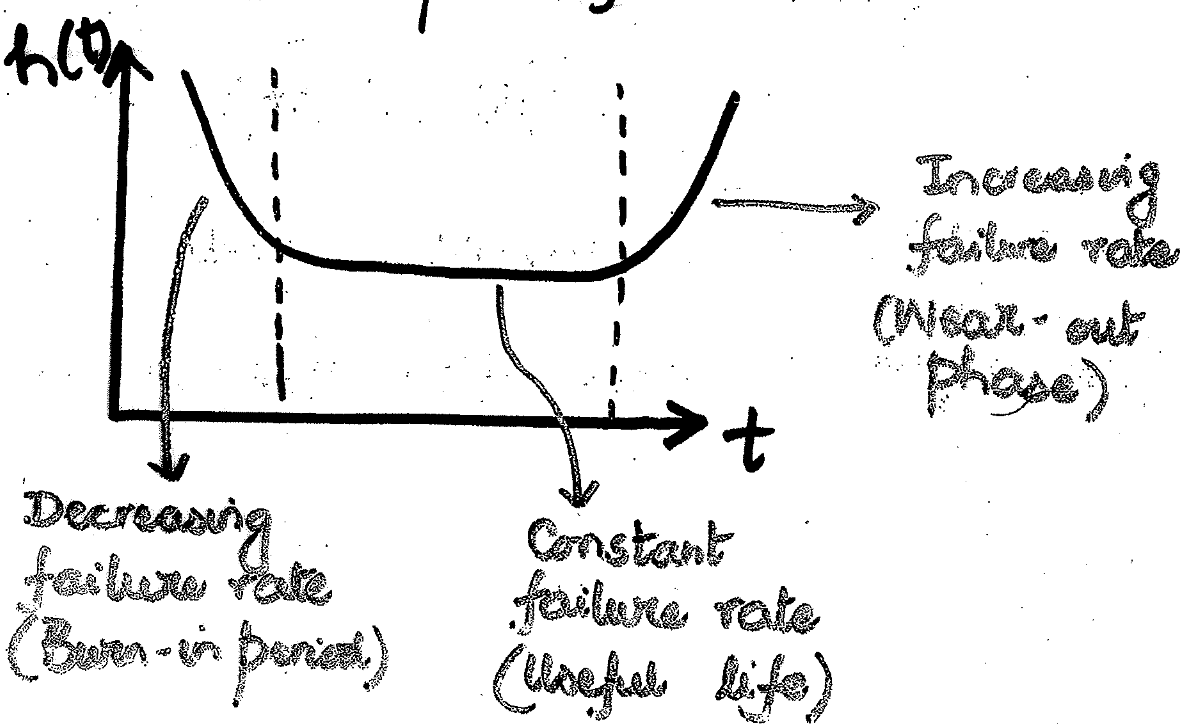


$$f(t) = \lambda e^{-\lambda t}$$

$$R(t) = e^{-\lambda t}$$

$$h(t) = \frac{f(t)}{R(t)} = \lambda$$

Typical computer failure rate distribution



Example problem #1.

Hypoexponential Distribution

A process goes through a sequence of phases. Time it spends at each phase is independent and exponentially distributed. Overall time spent by the process in all phases follows hypoexponential distribution

parameters = # stages

2-stage hypoexponential : $X \sim HYPO(\lambda_1, \lambda_2)$

$$f(t) = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right), \quad t>0$$

Denominator should be \lambda2 - \lambda1

$$F(t) = ? \qquad \int_0^t f(x)\,dx$$

$$h(t) = ? \qquad \frac{f(t)}{R(t)} = \frac{f(t)}{1-F(t)}$$

Which stage of a component's lifetime can this characterize?