Maths Methods III
Autumn 2017

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These are the lecture notes for Maths Methods III for the Autumn term of 2017 in the Department of Earth Science & Engineering at Imperial College London.
These notes were written by Sam Krevor. The notes and lectures have been significantly aided by useful discussions with Prof. Mike Warner, Dr. Gareth Williams, Dr. Matt Piggott, and Prof. Velisa Vesovic.

CHAPTER 1

Vector Functions and Differentiation

1. Introduction: Vectors, symmetry and physical processes

Vector analysis was developed by physicists from an earlier system known as quaternion analysis to deal with problems in three dimensional space[1]. The often used definition of a vector is an entity which has both magnitude and direction. This is in fact the original definition of a vector given in Gibbs & Wilson, *Vector Analysis*, University Press 1901, available on ESESIS. This has clear application in describing physical phenomena, such as the velocity of wind. Acceleration and force also have magnitude and direction and the laws of physics are well served by a description using vectors.

Less commonly considered outside of the realm of physics, but of equal importance for the utility of vector analysis are that physical laws have a property known as *symmetry* which is also found in the operations of vector analysis[2]. In particular, physical laws are symmetric, or invariant, with translations or rotations in the coordinate axes or in a conversion between curvilinear (Cartesian, polar) co-ordinate systems. We will illustrate this using Newton's second law which tells us that the acceleration of a body is proportional to a force applied to the body.

Taking m to be the mass of an object to which a force, $\mathbf{F} = (F_x, F_y, F_z)$, is applied the law in a Cartesian system is given by three equations

$$\begin{aligned} F_x &= m\frac{d^2x}{dt^2} \\ F_y &= m\frac{d^2y}{dt^2} \\ F_z &= m\frac{d^2z}{dt^2} \end{aligned}$$

Now let's say that we wish to observe this acceleration in a new Cartesian coordinate system, x', y', z' that is the same as the original except for a translation along the x-axis, that is $x' = x - a,\ y' = y,\ z' = z$ where a is some constant. In this coordinate system Newton's second law is

[1]See M.J. Crowe, *A History of Vector Analysis*, Dover 1967, also summarised in an article by the same author available on ESESIS

[2]Much of this discussion is derived from the treatment given in *The Feynman Lectures on Physics* Vol. 1 Ch. 11 available online at www.feynmanlectures.caltech.edu

$$
\begin{aligned}
F'_x &= m\frac{d^2x'}{dt^2}\\
F'_y &= m\frac{d^2y'}{dt^2}\\
F'_z &= m\frac{d^2z'}{dt^2}
\end{aligned}
$$

It is trivial from the definition of our new coordinate system that $F'_y = F_y,\ F'_z = F_z$, but how about F'_x?

$$
\begin{gathered}
F'_x = m\frac{d^2x'}{dt^2}\\
= m\frac{d^2(x-a)}{dt^2} = m\frac{d^2x}{dt^2}\\
= F_x
\end{gathered}
$$

The same could be shown, albeit in a slightly more complicated fashion with a rotation of the axes, or a combination of a translation and rotation. Thus it does not matter what coordinate system we use, a given force will always correspond to a particular acceleration. This is one type of symmetry present in all physical laws. In vector notation, Newton's law is written simply as

$$\mathbf{F} = m\mathbf{a},$$

where $\mathbf{a}$ is the acceleration vector. We could write the law also in terms of the velocity vector, $\mathbf{v}$ as[3]

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

and because of the symmetry of the law $\mathbf{v}$, $\mathbf{a}$ and $\mathbf{F}$ can be written in terms of any coordinate system that is convenient for a given problem.

But this is a class on maths and not physics so why am I bringing this all up? Because it turns out that operations of vectors possess the same symmetry as the physical laws. The above equations in vector form exhibit two such operations, the multiplication of the scalar mass by the acceleration vector. This you already know how to do. The second shows the time derivative of the velocity vector, which we will learn about today. Part of the utility of vector analysis is that these equations are general for any coordinate system and we do not need to specify a particular coordinate system for their manipulation or

[3]We will see later in this lecture the precise meaning of the time derivative of a vector

solution. This is a must for the analysis of physical phenomena which also have this property, i.e. symmetry.

Many more vector operations are required if useful analysis is to be performed. You have already learned how to obtain the magnitude of a vector and to obtain the scalar (dot) and vector (cross) products of two vectors. These operations are easily shown to be invariant under a translation or rotation of coordinate axes or transformation to a polar coordinate system. In this module we will learn about derivative and integral operations of vectors as well.

But today in this lecture I will introduce the concept of a *vector function* and two new operations of differentiation for vector functions, the *divergence* and *curl*, as well as one new operation of differentiation for real functions of multiple independent variables, the *gradient*. We will also show how functions (both real and vector) of multiple independent variables can be represented as a scalar or vector *field* of values. Differentiation of these fields creates a link between vector and real functions as differentiation of one can lead to the creation of the other.

2. Vector Functions

2.1. Vector functions of a real variable t. Recall that a vector is an entity that requires a magnitude and a direction for its complete specification and can be represented by a set of numbers equal in size to the number of spatial dimensions in which the entity is being considered. In this module (and in Earth Sciences in general) we are only interested in three dimensional space.

By analogue, a *vector function* describes a vector whose value (magnitude and direction) is a function of some independent variable (e.g., time, space). Let us first consider a vector function whose components are not dependent on a location in a coordinate space. This is how we describe a vector function that is, for example, dependent on time only and we will call the independent variable in this case t. For example, if $f(t)$ represents a real function of t, then analogously $\mathbf{F}(t) = [f_x(t), f_y(t), f_z(t)]$ will represent a vector function of t. In this case $\mathbf{F}(t)$ is a vector in 3-dimensional space and its magnitude and direction depend on the independent variable t.

2.2. An aside on the visualisation of concepts in vector calculus and the position vector. Many concepts and even some calculations in vector calculus are aided by some kind of visualisation, e.g. of a vector or vector function. When we represent a vector graphically in a way that is numerically accurate with regards to the magnitude and direction of that vector we are said to be creating a *geometric representation* of the vector. The geometric representation of a vector is nearly always performed using an arrow that has length and direction (relative to some coordinate axes) equal to the magnitude and direction of the vector of interest.

A useful subset of this kind of representation is a representation of the vector in a Cartesian (x, y, z) coordinate system where the arrow begins at the origin and ends at the point given by the vector components. This is a long-winded way of saying that if, for example, a vector $\mathbf{b} = (5, 4, 7)$, an often used convention is to visualise that vector with an arrow starting at point (0, 0, 0) and ending at point (5, 4, 7) in a 3-dimensional Cartesian coordinate system. This arrow will have the correct magnitude relative to the origin and direction relative to the axes.

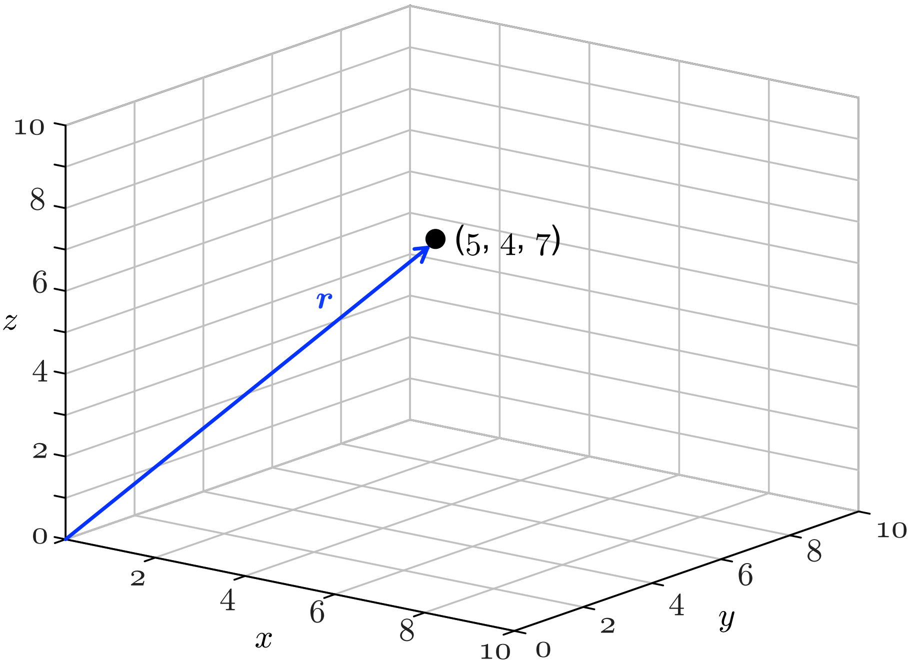


FIGURE 1. The geometric representation of a vector $\mathbf{b} = (5, 4, 7)$ using the position vector $\mathbf{r}$.

When our choice of geometric representation of a vector is that of an arrow starting at the origin and ending at the coordinates specified by the vector components, it is said that we are using the *position vector* to represent the vector. The position vector is simply the name given to any geometric representation of a vector by an arrow that begins at the origin and ends at a point in space as specified by the vector components. It is usually denoted by the symbol $\mathbf{r}$ and is defined by

$$\mathbf{r} \equiv (x, y, z).$$

The position vector will prove to be useful for many aspects of vector calculus but first we will use it in the very simple application of allowing us to visualise the response of a vector function $\mathbf{F}(t)$ to changes in the independent variable t.

2.3. The geometric representation of a vector function. A vector function represents a vector whose magnitude and direction depends on the value of the independent variable t and thus cannot be represented geometrically by a single arrow. The position vector can be employed in the creation of a geometrical representation of a vector function $\mathbf{F}(t) = (f_x(t), f_y(t), f_z(t))$ if the position vector itself is allowed to be a function of t. That is, set

$$\mathbf{F}(t) = \mathbf{r}(x(t), y(t), z(t)),$$

by letting

$$\begin{aligned} x(t) &= f_x(t) \\ y(t) &= f_y(t) \\ z(t) &= f_z(t). \end{aligned}$$

The geometric representation of a vector function is often thus made by the curve "drawn" by the tip of the position vector as t changes (see below example and figure 3). The curve, which I will now refer to as $\mathcal{C}$, is the geometrical representation of our vector function $\mathbf{F}$(t). Also, the equation $\mathbf{F}(t) = (f_x(t), f_y(t), f_z(t))$ is called the *parametric equation* of the curve $\mathcal{C}$.

EXAMPLE: The parametric equation of the straight line passing through the two points $A = (2, 6)$ and $B = (6, 3)$ is given by

$$\begin{aligned} \mathbf{F}(t) &= [f_x(t), f_y(t)] \\ f_x(t) &= 2 + (6 - 2)t = 2 + 4t \\ f_y(t) &= 6 + (3 - 6)t = 6 - 3t \end{aligned}$$

and a point on this line will move from A to B as t varies from 0 to 1.

EXAMPLE: Describe the shape of the geometric representation of the vector function given by $\mathbf{F}(t) = [t + (t/10)\sin t, t + (t/10)\cos t]$ as t increases from 0.

The first term in each of the vector components is simply t which will push the end of the arrow increasingly further from the origin with increasing t. The second term is periodic. The amplitude of both periodic terms is given by the multiplier $t/10$, which is increasing with t. Because the periodic terms of each component are 90 degrees out of phase from one another, these terms will serve to create a circular movement of the arrow and the increasing amplitude will lead to an increasing radius of movement as t increases. Putting it all together, the function should create a curve that corkscrews away from the origin. The radius of curvature will be increasing with distance from the origin.

Think about it: If $\mathbf{F}(t)$ is single valued in t is $\mathcal{C}$ single valued in x, y, z?

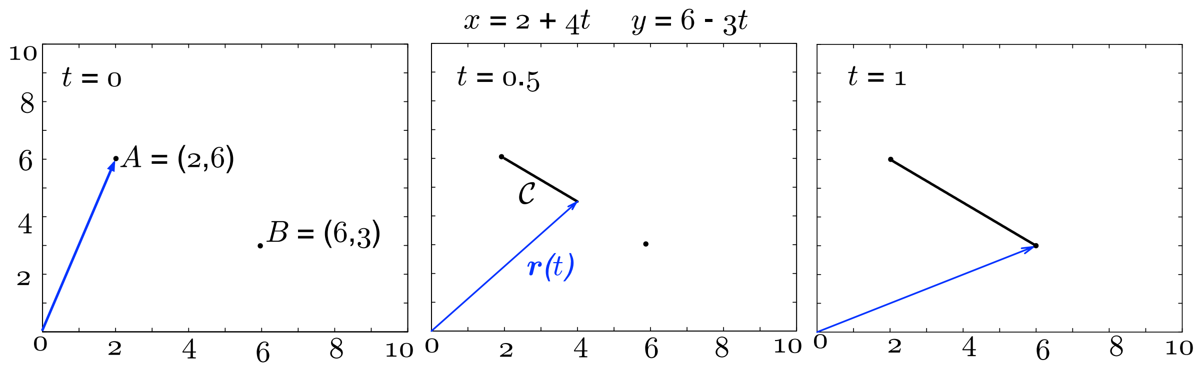


FIGURE 2. A 2-dimensional vector function sketching out a straight-line curve $\mathcal{C}$ from (2,6) to (6, 3) as t goes from zero to one. The vector is shown as the position vector by the blue arrow with magnitude and direction given in reference to the origin.

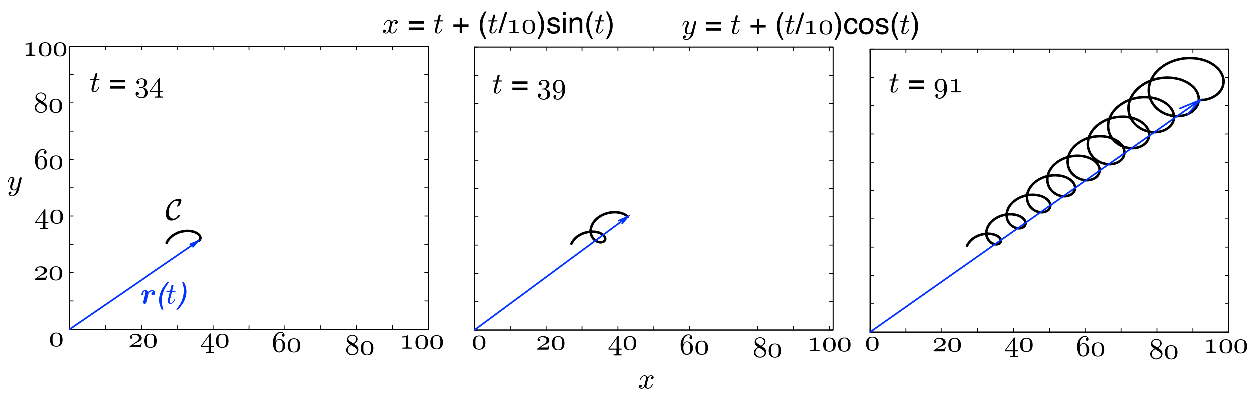


FIGURE 3. A 2-dimensional vector function sketching out a curve $\mathcal{C}$ which corkscrews away from the origin with an increasing radius of curvature with increasing t.

2.4. The differentiation of vector functions in t. The derivative of a vector function $\mathbf{F}(t)$, with respect to t, is given by

$$\frac{d\mathbf{F}}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{F}(t + \delta t) - \mathbf{F}(t)}{\delta t}.$$

This is a direct analogue of the differentiation of real-valued functions.

In terms of the individual components of the vector function, the derivative for a vector in three dimensions corresponds to

$$\frac{d\mathbf{F}}{dt} = \left(\frac{df_x}{dt}, \frac{df_y}{dt}, \frac{df_z}{dt}\right).$$

The derivative of the vector function in three dimensions is itself a function with three vector components. That is to say that the derivative of a vector function is a vector function. Higher order derivatives can be defined analogously.

EXAMPLE: The derivative of the vector function $\mathbf{F}(t) = (2+4t, 6-3t)$ whose geometrical representation $\mathcal{C}$ is given by the straight line, derived above, is

$$\frac{d\mathbf{F}}{dt} = (4, -3).$$

EXAMPLE: The derivative of the vector function $\mathbf{F}(t) = [t + (t/10)\sin t, t + (t/10)\cos t]$ from the previous example is given by

$$\frac{d\mathbf{F}}{dt} = \left[\left(1 + \frac{1}{10}\sin t + \frac{t}{10}\cos t\right), \left(1 + \frac{1}{10}\cos t - \frac{t}{10}\sin t\right)\right].$$

Note that the derivative of a vector function that sketches out a straight line is a constant vector whereas the derivative of the vector function that sketches out the curvy line is another vector function changing in t.

Can you answer this: What is the second derivative of a vector function that has a straight line as its geometrical representation?

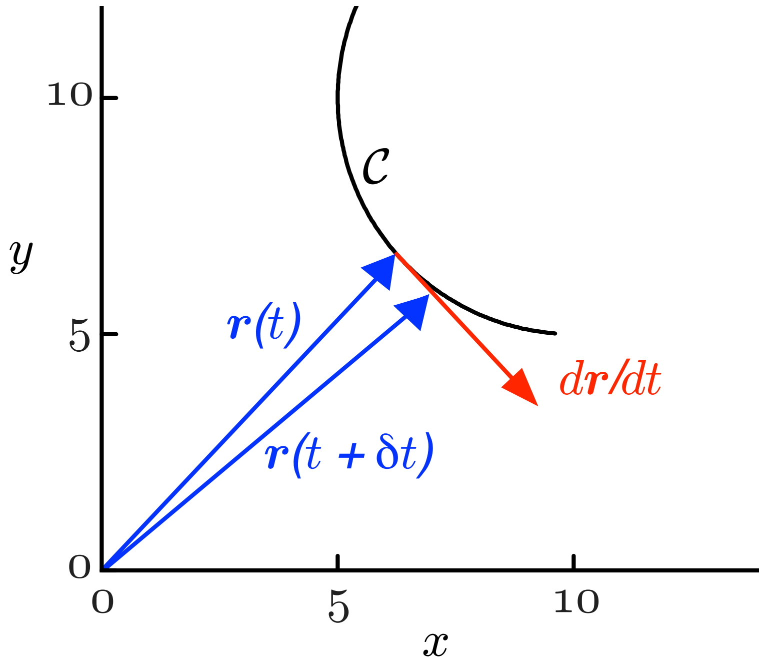
2.5. The geometrical representation of vector differentiation. We can again make use of the position vector $\mathbf{r}$ to visualize $d\mathbf{F}/dt$. Letting $\mathbf{F}(t) = \mathbf{r}(t)$ in a three dimensional Cartesian space,

$$\frac{d\mathbf{F}}{dt} = \frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).$$

We have already seen that $\mathbf{F}(t)$ represented in this way sketches out a curve $\mathcal{C}$ in the coordinate space over some range of values of t. Now, provided $d\mathbf{F}/dt$ exists and is non-zero, it is a vector that points along the curve and is directed in the sense in which t increases. That is, at each point on the curve, $d\mathbf{F}/dt$ is the tangent to $\mathcal{C}$.

EXAMPLE: Find the tangent, $\mathbf{t}(t)$, to the curve created by $\mathbf{r}(t) = [10+5\sin t, 10+5\cos t]$ as t goes from 0 to 2π.

$$\frac{d\mathbf{r}}{dt} = (5\cos t, -5\sin t)$$



We can define a *unit tangent* $\hat{\mathbf{t}}$ to the curve $\mathcal{C}$ as

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}$$

The unit tangent has the same direction as the vector derivative, but has a magnitude of unity (that is, one).

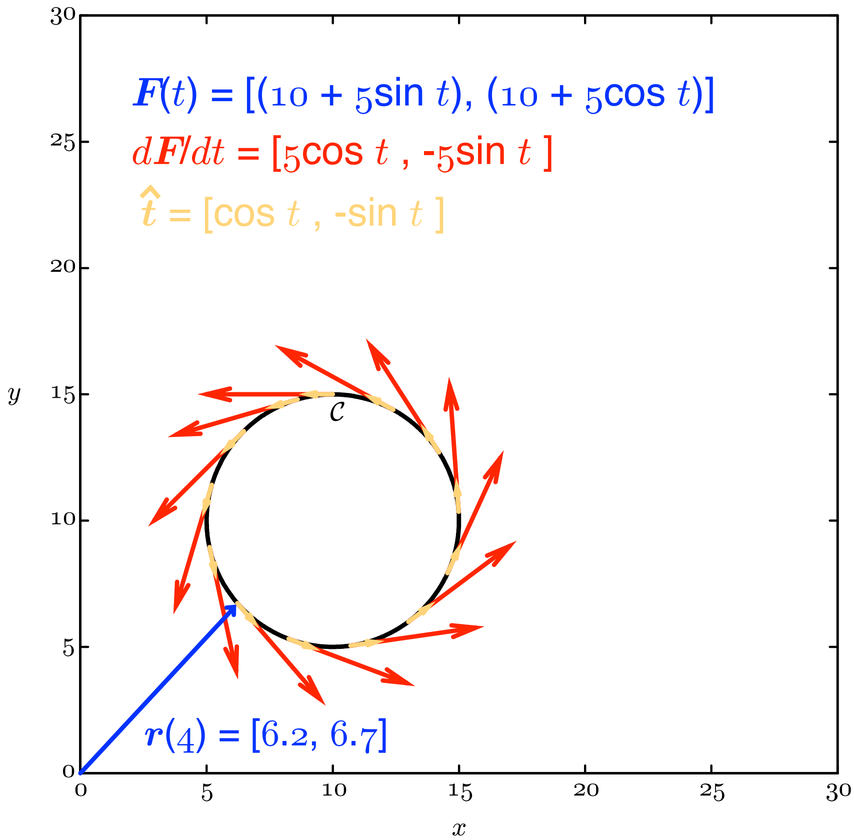
EXAMPLE: Find the unit tangent, $\hat{\mathbf{t}}(t)$, to the curve $\mathbf{r}(t) = (3, t, t^2)$

$$\frac{d\mathbf{r}}{dt} = (0, 1, 2t)$$
$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + 4t^2}$$

Thus

$$\hat{\mathbf{t}}(t) = \left(0, \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\right)$$

Extra: Check that the magnitude equals one.



2.6. Differentiation rules. If $\mathbf{a}$, $\mathbf{b}$ and λ are functions of t, then

$$\frac{d}{dt}(\mathbf{a}+\mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$$
$$\frac{d}{dt}(\lambda\mathbf{a}) = \frac{d\lambda}{dt}\mathbf{a} + \lambda\frac{d\mathbf{a}}{dt}$$
$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \frac{d\mathbf{a}}{dt}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{dt}$$
$$\frac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \frac{d\mathbf{a}}{dt}\times\mathbf{b} + \mathbf{a}\times\frac{d\mathbf{b}}{dt}$$

NOTE: that in the case of a cross product the order of **a** and **b** must be maintained.

2.7. Arc length. When we use $\mathbf{F}(t)$ to define a curve $\mathcal{C}$, the parameter t can be any convenient variable that increases continuously as the function sketches out the curve. In these notes I initially used time as an example (thus the choice of t). Another useful way to define position along a curve is to measure the distance along it from some starting point. We call this variable *arc length* and it is usually given the symbol s.

FIGURE 4. Differing approaches to labelling hiking distances. Swiss signs (left) are parametric while the American ones (right) are intrinsic.

In the way that we can construct the equation of a curve sketched out over time, we can write the equation of a curve $\mathcal{C}$ in terms of the arc length s. For example, if a curve C represented the vector function $\mathbf{F}$ that sketched out the path you walked on a hike in the mountains (figure 4) the function $\mathbf{F}(t)$ would show how the path developed as a function of the amount time you were hiking whereas $\mathbf{F}(s)$ would show the path developing as a function of the distance you had travelled.

The equation

$$\mathbf{F}(s) = \mathbf{r}(s) = (f_x(s), f_y(s), f_z(s))$$

is called the *intrinsic equation* of the curve $\mathcal{C}$.

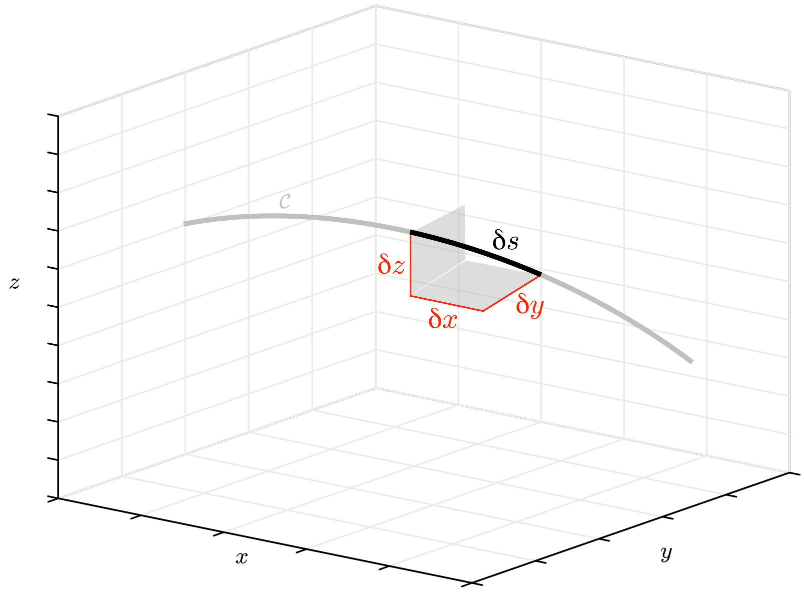
If s measures arc length along the curve $\mathcal{C}$, then for a small element of the curve

$$\delta s \approx \left[\delta x^2 + \delta y^2 + \delta z^2\right]^{\frac{1}{2}}$$

If a vector function may be parameterized by an independent variable, t, such as time or the length of the curve, s, sketched out by the position vector, there must be a relationship between t and s. Note that a small step in arc length, δs, will be produced by a small increment in the independent variable, δt. In the limit as $\delta t \to 0$ we get

$$\frac{ds}{dt} = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{\frac{1}{2}} = \left|\frac{d\mathbf{r}}{dt}\right| = \left|\frac{d\mathbf{F}}{dt}\right|$$

Thus if we represent a vector function geometrically by drawing a curve in space the rate of change of the arc length of that curve with respect to an independent variable gives the magnitude of the derivative of the vector function with respect to that independent variable.



This provides another formula for obtaining the unit tangent $\hat{\mathbf{t}}$:

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|} = \frac{d\mathbf{r}/dt}{d\mathbf{s}/dt} = \frac{d\mathbf{r}}{ds}$$

In other words, the tangent to the geometrical representation of a function $\mathbf{F}(t)$ is given by $d\mathbf{F}/dt$ whereas the unit tangent is given by $d\mathbf{F}/ds$.

3. Scalar and Vector Fields

3.1. Scalar fields. A *scalar field* is a scalar quantity that varies as a function of position, it has a value everywhere within some volume of space. It can be written as $\Omega(x, y, z)$. Since the position vector $\mathbf{r}$ represents a point in space, we can write $\Omega(x, y, z) = \Omega(\mathbf{r})$. That is, the scalar Ω is a function of position $\mathbf{r}$. Possible examples of scalar fields would be temperature, density or the magnitude of the magnetic field measured at all points within the earth at some instant in time.

There are two common methods for visualising a 3D scalar field shown in figure 5. We can take 2D slices through 3D space, and plot contours of equal values of the scalar (e.g. isobars on a weather map). To visualise the full 3D field, we need many 2D slices. Alternatively, we can plot surfaces of constant value (iso-surfaces) in 3D. In this case several iso-surfaces will be required to visualise the full field.

3.2. Vector fields. A vector field is a vector quantity that varies as a function of position; it is defined everywhere within some volume of space. It can be written as $\mathbf{F}(x, y, z)$. Since the position vector $\mathbf{r}$ represents a point in space, we can also write $\mathbf{F}(x, y, z) = \mathbf{F}(\mathbf{r})$. That is, $\mathbf{F}$ is a vector function of position $\mathbf{r}$. Examples of vector fields

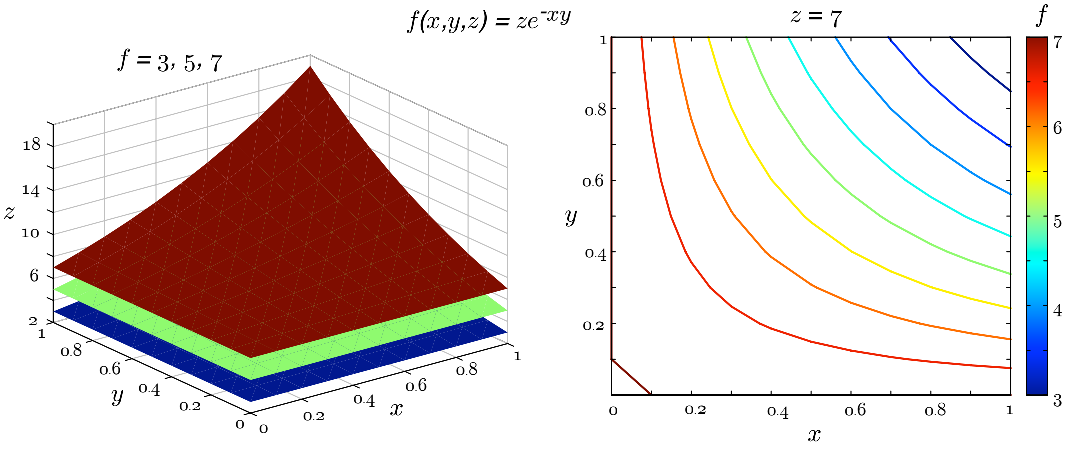


FIGURE 5. Isosurfaces (left) and contours (right) of the scalar field defined by $f(x, y, z) = ze^{-xy}$.

include the velocity of ocean currents (figure 7) and the magnetic field measured at all points within the earth at some instant in time.

To visualize a vector field, we need to visualize the vector at every point in space. One common technique is to draw arrows at each location in space with length and direction showing the magnitude and direction of the vector (e.g. wind strength and direction on a weather map). This type of plot is known as a *quiver plot*. This is easier to do in two dimensions than three dimensions, but quiver plots are commonly used in both situations.

Another type of plot often used to show a vector field is the *streamline plot*. In this kind of plot, also known as a field line or flow line plot, curves are drawn such that the vector field is everywhere tangent to the curves as shown in figure 6. Mathematically, this is represented by parameterizing x, y and z with another independent variable, e.g. t. If $\mathbf{F}$ is a vector field, a flow line in the vector field is a curve $\mathbf{c}(t)$ such that

$$\frac{d\mathbf{c}}{dt} = \mathbf{F}(\mathbf{c}(t)).$$

Note that this differential equation represents a family of lines as there will be arbitrary constants and associated with the integration. Another way to think about this is that the vector fields shows the velocity of the stream-function. If the vector field represents the velocity of fluid flow in the ocean, for example, the stream lines would show a path taken by a particle dropped into the flow field (figure 7).

EXAMPLE: Show that the path $\mathbf{c}(t) = (\cos t, \sin t)$ is a flow line of $\mathbf{F}(x, y) = (-y, x)$

We need to show that $\mathbf{c}' = \mathbf{F}(\mathbf{c}(t))$. Using the definitions of $\mathbf{c}$ and $\mathbf{F}$,

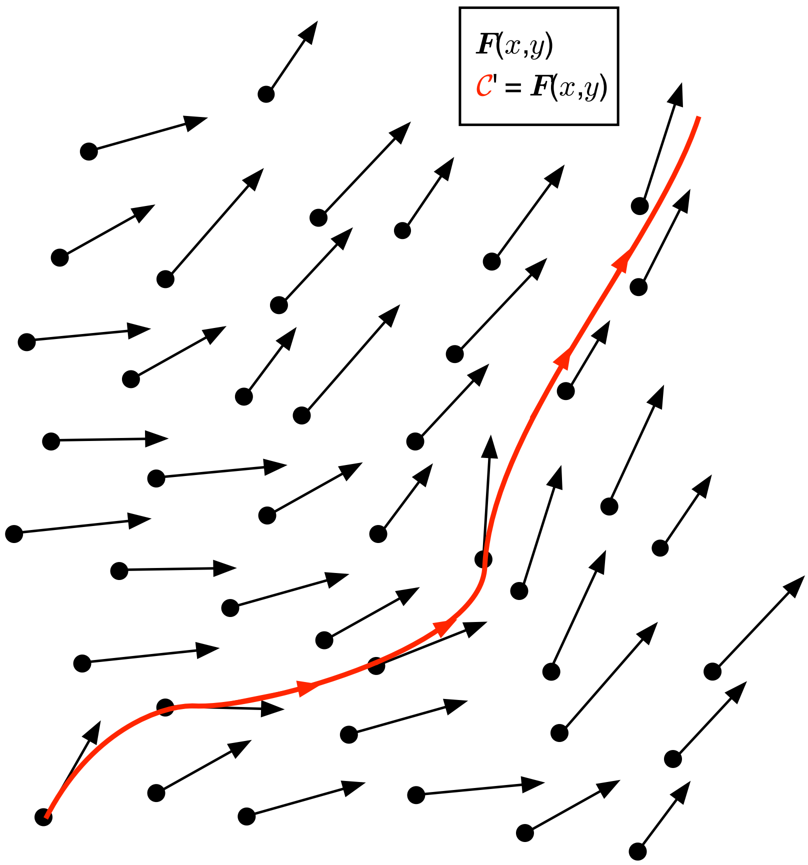


FIGURE 6. A quiver plot of some vector field $\mathbf{F}$ and one streamline $\mathbf{c}$ that meets the requirement $d\mathbf{c}/dt = \mathbf{F}$.

$$\mathbf{F}(\mathbf{c}(t)) = \mathbf{F}(\cos t, \sin t) = (-\sin t, \cos t).$$

Now taking the derivative of $\mathbf{c}$,

$$\frac{d\mathbf{c}}{dt} = (-\sin t, \cos t) = \mathbf{F}(\mathbf{c}(t)).$$

4. Derivatives of scalar and vector fields and the ∇ operator

If there exists a scalar or vector field, $\Omega(x, y, z)$ or $\mathbf{F}(x, y, z)$, it is straightforward to take the derivative of the function with respect to a non-spatial variable, e.g. time, $\partial\Omega/\partial t, \partial\mathbf{F}/\partial t$. Taking the derivative with respect to space, however, is less straightforward as there are three spatial dimension; Thus does one take the derivative with respect to one of the dimensions? All?

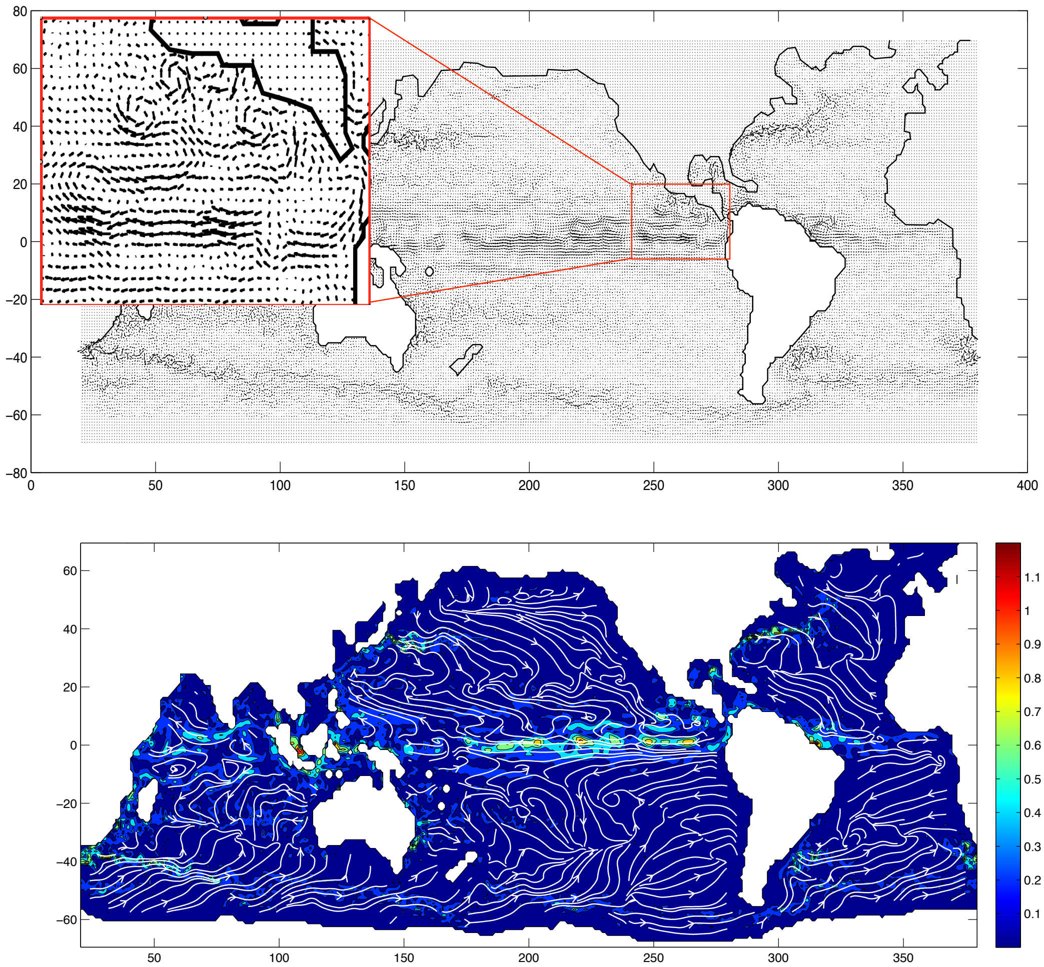


FIGURE 7. A quiver plot (above) showing the magnitude and direction of ocean currents at 0m depth and a streamline plot (below) imposed onto a heat map showing the direction and magnitude respectively for the vector field. Data from http://www.esr.org/oscar_index.html.

The answer lies in the search for an operation of derivatives where the values, as with the physical world, do not depend on the coordinate axes which have been chosen to frame the problem. Vectors and their algebraic manipulations (addition, subtraction, the dot and cross product), defined in a 3-D space, for example, are invariant with respect to coordinate axes. If the coordinate space is translated (e.g. the x-axis moved by a value, a), then the vector components all change in a way that serves to preserve the value of the vector or operations between vectors. Given that vectors and vector operations themselves serve this purpose provides a clue - the most useful derivative operations on scalar and vector fields will be those that are similar to vector operations.

4.1. The gradient of a scalar field. Starting with a scalar field, $\Omega(x, y, z)$, as mentioned previously we have the choice of taking spatial derivatives in either of three dimensions, x, y and z. Given that there are three spatial derivatives, three components makes a vector, and we are generally looking for derivative operations that are vector-like, one possible derivative is known as *the gradient*, defined as

$$\nabla\Omega \equiv \left(\frac{\partial\Omega}{\partial x}, \frac{\partial\Omega}{\partial y}, \frac{\partial\Omega}{\partial z}\right).$$

The gradient is sometimes referred to as *grad* so that in this case one could refer to this as "grad omega". The gradient looks like a vector (or vector field) and in fact its invariance with respect to a change of coordinate axes can be shown, so this operation meets our needs for a useful derivative operation for scalar fields.

I have used a new symbol, ∇ referred to as "nabla" or "del", without explaining why or precisely its meaning. I will come back to this after finishing the discussion of the gradient operation.

4.2. Geometrical interpretation of $\nabla\Omega$. The vector $\nabla\Omega$ represents the rate of change of Ω in space. It follows that the x-component of the vector represents the rate of change in the x direction, the y component the rate of change in the y-direction and so on. Any direction could be defined regardless of the axes and a component of $\nabla\Omega$ in that direction would represent the rate of change of Ω in that direction. The direction in which the largest possible component (the largest rate of change) can be found is of course the direction of the vector itself. This is to say that the direction of the vector $\nabla\Omega$ is the direction in which Ω increases most rapidly. The magnitude $|\nabla\Omega|$ gives the rate of change of Ω in this direction.

EXAMPLE: Find $\nabla\Omega$ when $\Omega = x^2 + xy + y^2$

$$\begin{aligned}\nabla\Omega &= (2x + y, x + 2y) \\ &= (2x + y)\hat{\mathbf{i}} + (x + 2y)\hat{\mathbf{j}}\end{aligned}$$

4.3. Directional derivatives and the unit normal. In general Ω will vary at a different rate in different directions. The directional derivative of Ω in the direction of a unit vector $\hat{\mathbf{a}}$ is written as $\partial\Omega/\partial a$, and it measures the rate of change of Ω in the direction of $\hat{\mathbf{a}}$.

An important property of $\nabla\Omega$ is

$$\frac{\partial\Omega}{\partial a} = \hat{\mathbf{a}} \cdot \nabla\Omega$$

which allows one to calculate the rate of change in any direction of $\nabla\Omega$.

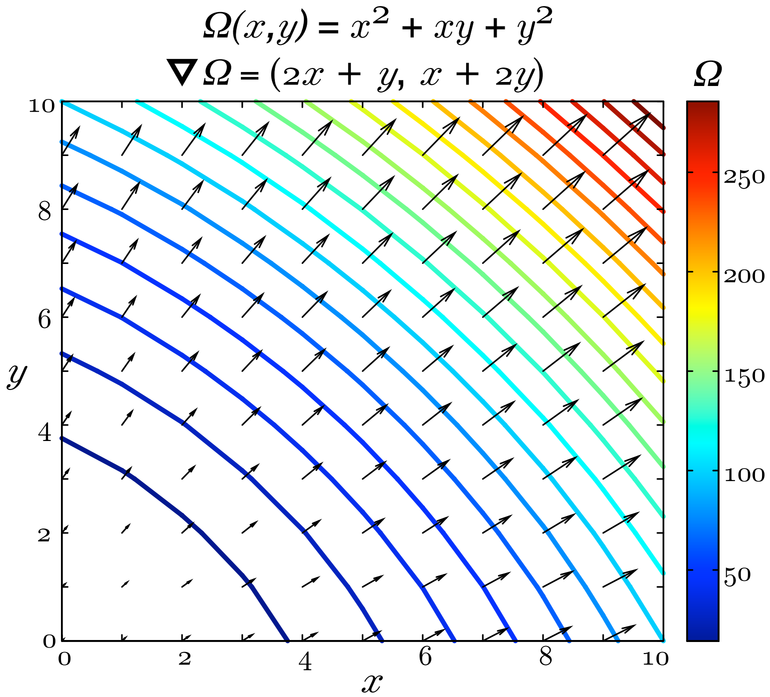


FIGURE 8. Contour plot of the scalar field Ω and quiver plot of the vector field $\nabla\Omega$.

EXAMPLE: Find the derivative of the scalar field $\Omega = x^2yz + 4xz^2$ in the direction of the vector $\mathbf{a} = (2, -1, -1)$ at the point $P = (1, -2, -1)$.

First find $\hat{\mathbf{a}}$

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{(2,-1,-1)}{\sqrt{4+1+1}} = \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

Next find $\nabla\Omega$

$$\nabla\Omega = \left(\frac{\partial\Omega}{\partial x}, \frac{\partial\Omega}{\partial y}, \frac{\partial\Omega}{\partial z}\right) = \left(2xyz + 4z^2, x^2z, x^2y + 8xz\right)$$

The directional derivative is given by

$$\frac{\partial\Omega}{\partial a} = \hat{\mathbf{a}}\cdot\nabla\Omega = \frac{2}{\sqrt{6}}\left(2xyz + 4z^2\right) - \frac{1}{\sqrt{6}}\left(x^2z\right) - \frac{1}{\sqrt{6}}\left(x^2y + 8xz\right)$$

Finally, evaluating this at the point P,

$$\frac{\partial\Omega}{\partial a}(1,-2,-1) = \frac{2}{\sqrt{6}}(8) + \frac{1}{\sqrt{6}} + \frac{10}{\sqrt{6}} = \frac{27}{\sqrt{6}}.$$

The use of directional derivatives allows us to see another geometrical feature of the gradient operation. If $\Omega(x, y, z)$ is a scalar field than the equation $\Omega(x, y, z) = \text{constant}$ defines a surface of constant value. If the unit vector $\hat{\mathbf{a}}$ is in the direction of the surface then $\hat{\mathbf{a}} \cdot \nabla\Omega = 0$ because the gradient of Ω is zero in any direction along a surface of constant value. This implies that $\hat{\mathbf{a}}$ and $\nabla\Omega$ are perpendicular to each other and thus $\nabla\Omega$ is always normal to the iso-surfaces of Ω. This can be seen in two dimension in figure 8 where the vectors in the field defined by the gradient are always perpendicular to the countour lines of constant value, i.e. $\nabla\Omega$ is everywhere perpendicular to the lines defined by the equation $\Omega(x, y) = \text{constant}$. From this it follows that the unit normal is defined as $\frac{\nabla\Omega}{|\nabla\Omega|}$.

EXAMPLE: Find the unit normal vector to the surface $x^3y + 4xz^2 + xy^2z + 2 = 0$ at the point (1, 3, -1).

First we find $\nabla\Omega$

$$\nabla\Omega = (3x^2y + 4z^2 + y^2z)\mathbf{i} + (x^3 + 2xyz)\mathbf{j} + (8xz + xy^2)\mathbf{k}$$

Next, $\nabla\Omega(1, 3, -1) = (4, -5, 1)$ and $|\nabla\Omega| = \sqrt{42}$ therefore the unit normal is given by

$$\frac{\nabla\Omega}{|\nabla\Omega|} = \frac{1}{\sqrt{42}}(4, -5, 1).$$

4.4. The ∇ operator. Before moving on to derivative operations for vector fields, it is helpful to describe more precisely the meaning of the symbol ∇, referred to as "nabla" or "del", that was used in the gradient operation above. I have been using the term "operation" somewhat loosely but it turns out to be a very important and useful concept for vector calculus. I will not provide a formal definition of an operation or operator here but hopefully the meaning becomes clear with a few examples.

Taking a derivative is an example of an operation. Thus for a function f, the total derivative operation with respect to an independent variable t is indicated by df/dt and a partial derivative operation is given by $\partial f/\partial t$. In these two examples, the *operators* are given by d/dt and $\partial/\partial t$ respectively. The important thing is that there is a certain algebra associated with the use of the operators. For example, often the operator d/dt is referred to simply as D. The derivative operation can be performed on many types of functions and could be represented by the product of the operator, D, with the function. Thus if we have functions $f(t), g(t), h(t)$ etc., we could represent their derivatives by the products of the operator with the function, Df, Dg, Dh.

Similarly, the operator ∇ is defined as a vector in the following way:

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Because the operator is defined as a vector, the rules of vector algebra apply in its use. From this it follows that the gradient of a scalar field represents the product of the operator with the scalar field, and also that the product will itself be a vector (the product of a vector with a scalar is a vector). The notation I used to represent the gradient operation used the product of the ∇ operator with the scalar field Ω. The gradient is represented by $\nabla\Omega$, and from this it is obvious that the gradient is a vector field. This operator can be used on any differentiable scalar field. The gradients for scalar fields $f(x, y, z), g(x, y, z), h(x, y, z)$ are given by $\nabla f, \nabla g$ and ∇h respectively.

You will see that derivative operations for vector fields also use the ∇ operator. In this case, however, the operations will involve the algebra of vectors - the dot and cross products. The last part of these notes defines three operations using the nabla operator, the divergence, curl and laplacian. In the next lecture we will discuss the geometrical and physical meaning of these operations.

Before we get too carried away it is important to note that operators do not follow all of the rules of algebra. In particular, the order of multiplication matters. The operator is always "hungry" for an argument and only operates on functions on its right hand side. Thus $Df \neq fD$. Whereas Df represents the derivative of f, the expression $fD = f\frac{d}{dt}$ is still an operation waiting to happen; it is still hungry for its operand.

4.5. The divergence of a vector field. The first derivative operation that we will look at using vector fields is called the *divergence* and it is the dot product of the ∇ operator and the vector field function. Suppose $\mathbf{F}$ is a vector field where $\mathbf{F} = (f_x, f_y, f_z)$, then the divergence of $\mathbf{F}$ is defined as

$$\text{div}\mathbf{F} \equiv \nabla \cdot \mathbf{F} = \frac{\partial f_x}{\partial_x} + \frac{\partial f_y}{\partial_y} + \frac{\partial f_z}{\partial_z}$$

The dot product of two vectors is also known as the scalar product and always results in a scalar. Note that it is not possible to take the divergence of a scalar field (why not?).

EXAMPLE: Find the divergence of the vector field $\mathbf{F} = (x^2, 3y, x^3)$

$$\nabla \cdot \mathbf{F} = 2x + 3$$

4.6. The geometrical and physical meaning of the divergence. The $(\nabla\cdot)$ operation is referred to as the divergence because, as I will show here, when the operator is applied to the flow velocity field of some fluid it represents the net amount of fluid that is entering or leaving a particular point. If the fluid is incompressible, as is often assumed for water, then the divergence is zero. A positive divergence suggests that the density of the fluid at that location is decreasing while a negative divergence suggests that it is increasing.

Consider the movement of some fluid through a small cube with a velocity field given by $\mathbf{V}(x, y, z)$ as shown in figure 9. The cube is oriented so that its faces are parallel to the xz, yz and xy planes of the coordinate system. First we consider fluid exiting the cube through the faces parallel to the yz plane. The volumetric flux exiting at the point (x, y, z) is given by:

$$-\hat{\mathbf{i}} \cdot \mathbf{V}(x, y, z)dy\, dz.$$

The volumetric flux exiting at the face at point $(x + dx, y, z)$ is given by

$$\hat{\mathbf{i}} \cdot \mathbf{V}(x + dx, y, z)dy\, dz = \hat{\mathbf{i}} \cdot \left[\mathbf{V}(x, y, z)\ + \frac{\partial \mathbf{V}}{\partial x}dx\right]dy\, dz.$$

Summing these together provides the net flux in the $x-$direction:

$$\begin{aligned}
-\hat{\mathbf{i}} \cdot \mathbf{V}(x, y, z)dy\, dz + \hat{\mathbf{i}} \cdot &\left[\mathbf{V}(x, y, z)\ + \frac{\partial \mathbf{V}}{\partial x}dx\right]dy\, dz \\
&= \hat{\mathbf{i}} \cdot \frac{\partial \mathbf{V}}{\partial x}dx\, dy\, dz \\
&= \frac{\partial V_x}{\partial x}dx\, dy\, dz.
\end{aligned}$$

Similarly, the fluxes in the y and z directions are given by

$$\begin{aligned}
&= \frac{\partial V_y}{\partial y}dx\, dy\, dz \\
&= \frac{\partial V_z}{\partial z}dx\, dy\, dz
\end{aligned}$$

respectively. The total flux out of the cube is thus given by the sum of the fluxes in each direction:

$$\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)dx\, dy\, dz = \big(\nabla \cdot \mathbf{V}\big)dx\, dy\, dz.$$

The term $dx\, dy\, dz$ is simply the volume of our small cube and thus the divergence of the fluid velocity, $\nabla \cdot \mathbf{V}$, represents the net flux of fluid per unit volume at a given point. As suggested earlier, if the net flux is positive, the implication is that there is a net flux of fluid away from the point. By the rules of mass balance, if there is no

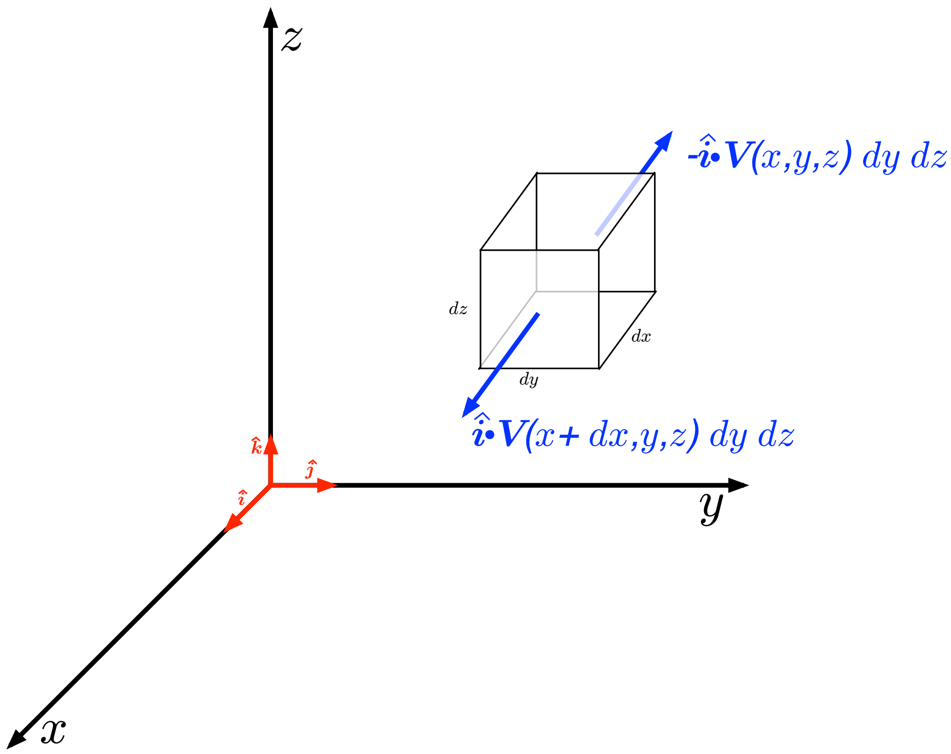


FIGURE 9. The flux of fluid out of a small cube.

addition of fluid at a location (e.g. injection of water underground into an oil reservoir) a positive divergence must correspond to a decreasing density of fluid at that point. Similarly a negative divergence corresponds to an increasing density. A divergence of zero is indicative of an incompressible system as is often assumed for water.

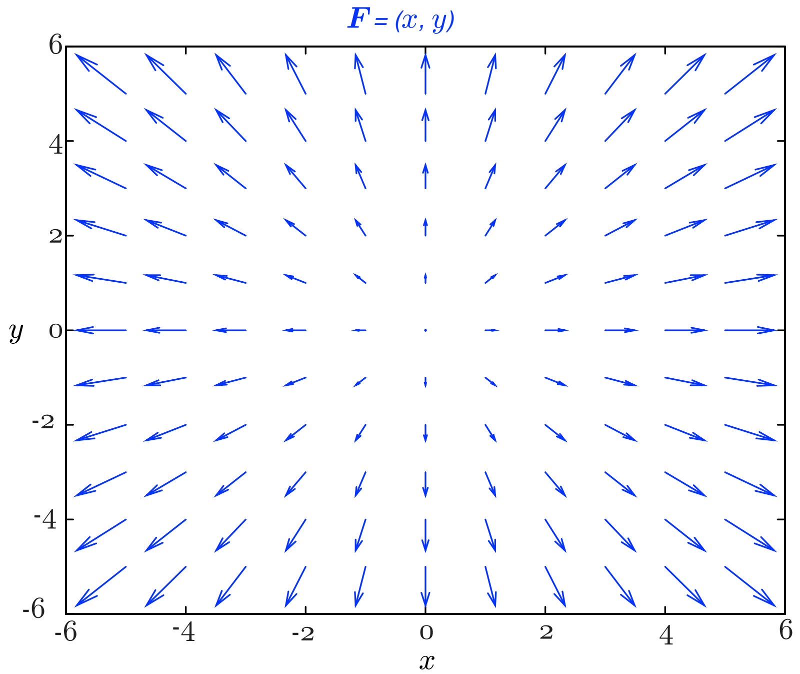
EXAMPLE: Consider the vector field $\mathbf{F}(x, y) = (x, y)$. First draw a quiver plot of $\mathbf{F}$ around the origin. Based on the plot is the divergence of the field positive, negative or zero? Next, calculate the divergence to prove this.

The quiver plot is easily drawn, the components of the vector at each point are in the direction and equal in magnitude to the coordinates x and y at that point.

If the vectors represented flow velocities, at each point and in each direction, there would be a greater flow away from the point than to that point and thus the divergence should be positive. This is particularly easy to see at the origin but applies to every point.

Calculating the divergence demonstrates this:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y = 2.$$



4.7. The curl of a vector field. The next derivative operation on vector fields involves the cross product of the nabla operator and the vector field and it is known as the *curl*.

The curl of $\mathbf{F}$ is defined as

$$\text{curl } \mathbf{F} \equiv \nabla \times \mathbf{F} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\mathbf{k}$$

We can also write curl $\mathbf{F}$ as a determinant,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

As with the divergence, it is only possible to take the curl of a *vector* field (why?), and the result is another *vector* field .

EXAMPLE: Find the curl of the vector field $\mathbf{F} = (z, x, y)$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$
$$= (1-0)\hat{\mathbf{i}} + (1-0)\hat{\mathbf{j}} + (1-0)\hat{\mathbf{k}}$$
$$= \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$
$$= (1,1,1)$$

EXAMPLE: Find the curl of the vector field $\mathbf{F} = (xy, -\sin z, 1)$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{vmatrix}$$
$$= (0 + \cos z)\hat{\mathbf{i}} + (0)\hat{\mathbf{j}} + (0 - x)\hat{\mathbf{k}}$$
$$= (\cos z, 0, -x)$$

As with the divergence, the curl has an important geometrical and physical significance. It is less straightforward to demonstrate than the divergence and this will be covered in the next lecture when we get to the integral theorems for vector calculus.

4.8. The gradient of the curl of a vector field. Using the nabla operator we can derive several useful identities, many of which are listed in appendix B. One of the most important involves the curl of a vector field defined as the gradient of a scalar field. If $\Omega(x,y,z)$ is a scalar field,

$$\nabla \times (\nabla\Omega)$$
$$= \nabla \times \left(\frac{\partial\Omega}{\partial x}, \frac{\partial\Omega}{\partial y}, \frac{\partial\Omega}{\partial z}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\Omega}{\partial x} & \frac{\partial\Omega}{\partial y} & \frac{\partial\Omega}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2\Omega}{\partial y\partial z} - \frac{\partial^2\Omega}{\partial y\partial z}\right)\mathbf{i} + \left(\frac{\partial^2\Omega}{\partial x\partial z} - \frac{\partial^2\Omega}{\partial x\partial z}\right)\mathbf{j} + \left(\frac{\partial^2\Omega}{\partial y\partial x} - \frac{\partial^2\Omega}{\partial y\partial x}\right)\mathbf{k}$$
$$= 0$$

In other words the curl of the gradient of a scalar field is always zero.

4.9. Multiple operators and the Laplacian. We can use the del operator to define further operators. The most useful of these is the *Laplacian* which is defined as

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

NOTE: The analogy with $\mathrm{a}^2 = \mathbf{a} \cdot \mathbf{a}$

The Laplacian can operate on a scalar or vector field. For a vector field

$$\nabla^2 \mathbf{F} = (\nabla^2 f_x, \nabla^2 f_y, \nabla^2 f_z)$$

EXAMPLE [4]: If $\phi = xyz - 2y^2z + x^2z^2$ find $\nabla^2\phi$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$
$$= \frac{\partial(yz + 2xz^2)}{\partial x} + \frac{\partial(xz - 4yz)}{\partial y} + \frac{\partial(xy - 2y^2 + 2x^2z)}{\partial z}$$
$$= 2z^2 - 4z + 2x^2$$

EXAMPLE: If $\mathbf{F} = x^2yz\mathbf{i} + xyz^2\mathbf{j} + (xy - 2y^2 + 2x^2z)\mathbf{k}$ find $\nabla^2\mathbf{F}$

$$\nabla^2 \mathbf{F} = (\nabla^2 f_x, \nabla^2 f_y, \nabla^2 f_z)$$
$$\nabla^2 f_x = \frac{\partial}{\partial x}(2xyz) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(x^2y) = 2yz$$
$$\nabla^2 f_y = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(xz^2) + \frac{\partial}{\partial z}(2xyz) = 2xy$$
$$\nabla^2 f_z = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(x - 4y) + \frac{\partial}{\partial z}(2x^2) = 4z - 4$$

The Laplacian operator appears in several important differential equations which govern physical phenomena; these include

Laplace's equation

$$\nabla^2 U = 0$$

The wave equation

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[4] Stroud Programme 22 Frame 88

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

Diffusion equation

$$\nabla^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t}$$

CHAPTER 2

Vector integration over lines, surfaces and volumes

1. Ordinary integrals of a vector function of a real variable t.

As with the discussion of differentiation, I start with the simpler concept of integration of a vector function of a single independent variable. If $\mathbf{F}(t) = [f_x(t), f_y(t), f_z(t)]$ is a vector function of a single variable t, then we define the integral over t as

$$\int_a^b \mathbf{F}(t)dt \equiv \mathbf{i}\int_a^b f_x(t)dt + \mathbf{j}\int f_y(t)dt + \mathbf{k}\int_a^b f_z(t)dt.$$

The integral of a vector function is a vector function.

EXAMPLE: If $\mathbf{F} = (3t^2 + 4t)\mathbf{i} + (2t - 5)\mathbf{j} + 4t^3\mathbf{k}$ the

$$\begin{aligned}\int_1^3 \mathbf{F}dt &= \mathbf{i}\int_1^3 (3t^2 + 4t)dt + \mathbf{j}\int_1^3 (2t - 5)dt + \mathbf{k}\int_1^3 4t^3dt \\ &= \Big[(t^3 + 2t^2)\mathbf{i} + (t^2 - 5t)\mathbf{j} + (t^4)\mathbf{k}\Big]_1^3 \\ &= 42\mathbf{i} - 2\mathbf{j} + 80\mathbf{k}\end{aligned}$$

2. Line integrals

For functions that represent scalar or vector fields it is possible to integrate the functions over a defined curve, surface or volume in which the field is defined. I will first review the meaning of a line integral and discuss the integration of scalar and vector fields along a curve. In subsequent sections I will introduce surface and volume integrals of field functions.

2.1. Line integrals of scalar fields. Consider a curve C in an $x - y - z$ space that runs between points P_1 and P_2 as shown in figure 1. We can break up the curve into tiny line arc segments, Δs_i. If there is a scalar field $\Omega(x, y, z)$ defined over the space we can obtain values of Ω at each location i and add up the products $\Omega_i \Delta s_i$. The line integral of Ω from points P_1 to P_2 is defined as the sum of these products in the limit of small Δs,

$$\int_{P_1}^{P_2} \Omega ds = \lim_{\Delta s \to 0} \sum_i \Omega_i \Delta s_i.$$

The key concept here is that the function is being integrated across a smoothly changing combination of values for x, y and z that is defined by the curve C rather than integration as some independent variable moves between two values.

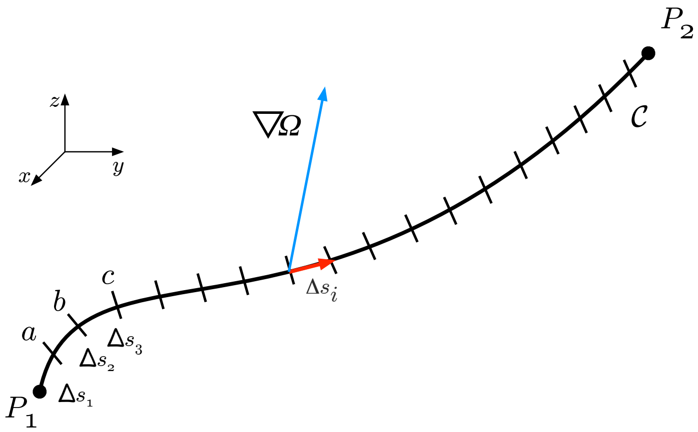


FIGURE 1. A geometrical representation of the integration of a scalar ($\Omega(x, y, z)$) and vector ($\nabla\Omega(x, y, z)$) field along a curve $\mathcal{C}$.

The easiest way to evaluate a line integral is to obtain every variable, including the limits of integration, in terms of a single variable. This is illustrated in the following example:

EXAMPLE: Consider a scalar field defined by $\Omega = z^2 - x^2$ and a curve $\mathcal{C}$ defined by the intrinsic equation $\mathbf{r} = (1, s^2, (1 + s^2)^{\frac{1}{2}})$ with $0 \leq s \leq 1$. This is illustrated in figure 2. Find the line integral $\int_{\mathcal{C}} \Omega ds$.

The intrinsic equation for the curve (the curve as a function of arc length s) is given, and this allows for the substitution of x, y and z in addition to the limits (0 and 1) in terms of the arc length s. Thus in this case it is easiest to obtain every variable in terms of the arc length, s.

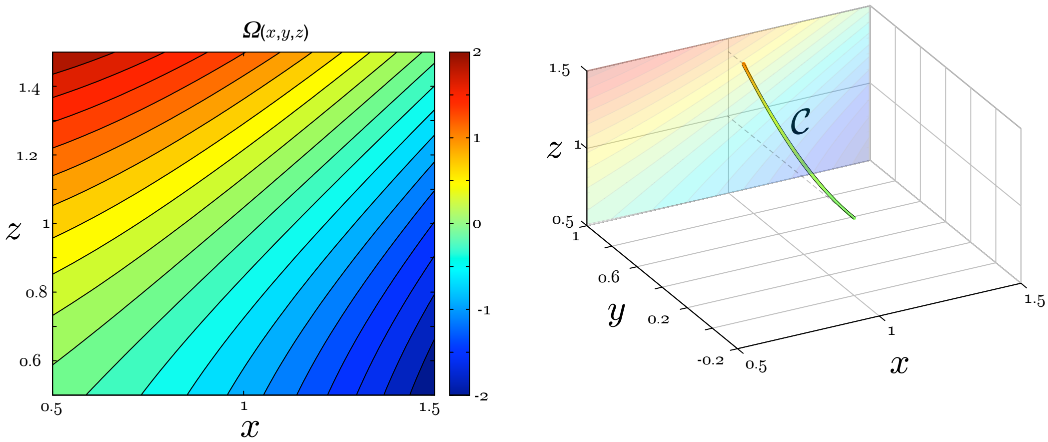


FIGURE 2. The left shows a contour heat map showing Ω in the x-z plane (Ω is constant with y). On the right the curve $\mathcal{C}$ is shown in 3D space with colours corresponding to the values of Ω at each location on the curve.

$$\begin{aligned}\int_{\mathcal{C}} \Omega ds &= \int_{s=0}^{1} (z^2 - x^2) ds \\ &= \int_{s=0}^{1} [(1+s^2) - 1] ds \\ &= \int_{0}^{1} s^2 ds \\ &= \left.\frac{s^3}{3}\right|_0^1 = \frac{1}{3}.\end{aligned}$$

Choosing the single variable to use for performing the integration is simply a matter of choosing which variable is the most useful in terms of substitution and re-definition of the curve and limits of integration. Sometimes the intrinsic equation of the curve is not available or convenient to obtain and it is more convenient to describe the curve with another independent variable, e.g. t, θ. The single variable may be s, t, x, y, z or any other useful variable. We do, however, need an expression to convert the differential from ds to something in terms of the target independent variable, e.g. dt. Continuing with t as our independent variable, taking advantage of the chain rule, $ds = \frac{ds}{dt} dt$. Thus if we can find $\frac{ds}{dt}$ we can perform the conversion. Since s is arc length we can use

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

where t is any useful variable that varies continuously along the curve $\mathcal{C}$.

EXAMPLE: Evaluate the line integral of $\int_{\mathcal{C}} \Omega ds$ with $\Omega = (a^2y^2/b^2 + b^2x^2/a^2)^{1/2}$ around the ellipse $\mathcal{C}$ defined by the equation $x^2/a^2 + y^2/b^2 = 1, z = 0$.

We are not given an intrinsic equation for the curve (e.g. x, y and z on the curve as a function of arc length s) but there is a simple way to develop a parametric equation for ellipses using an angle θ as the independent variable. The parametric equation for $\mathcal{C}$ is given by

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \\ z &= 0 \\ 0 \leq \theta &\leq 2\pi \end{aligned}$$

We now have x, y and z in terms of the variable θ and we know the limits (0 to 2π). The only remaining work is to convert the differential so that we can evaluate the integral,

$$\int_{\mathcal{C}} \Omega ds = \int_{\theta=0}^{2\pi} \Omega \frac{ds}{d\theta} d\theta.$$

First substituting for x, y and z in Ω:

$$\begin{aligned} \Omega &= \left(\frac{a^2y^2}{b^2} + \frac{b^2x^2}{a^2}\right)^{\frac{1}{2}} \\ &= (a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{1}{2}}. \end{aligned}$$

Next we obtain $\frac{ds}{d\theta}$

$$\begin{aligned} \frac{ds}{d\theta} &= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2} \\ &= (a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{1}{2}}. \end{aligned}$$

The integral is then

$$\begin{aligned} &\int_{\theta=0}^{2\pi} \Omega \frac{ds}{d\theta} d\theta. \\ &= \int_{\theta=0}^{2\pi} (a^2\sin^2\theta + b^2\cos^2\theta) d\theta \\ &= \pi a^2 + \pi b^2 = \pi(a^2 + b^2) \end{aligned}$$

We have developed this concept for purely scalar integrals, but in general line integrals of any form can be reduced to an expression in one variable only. This is inherent in the one dimensional nature of a line. Thus in the following section on line integrals involving vectors the goal of reducing the integral to expressions of a single variable still applies.

2.2. Line integrals of vector fields. Consider a vector field $\mathbf{F}(x, y, z)$ and a curve $\mathcal{C}$. With a vector field there are three possibilities for integration along the curve corresponding to the three vector multiplication operations - a simple summation of the vectors existing along the curve is given by the integral over $\mathbf{F}ds$ where ds is a component of arc length along the curve. A summation of the component of the vectors in the field that are in the direction of the curve is given by integrating over $\mathbf{F} \cdot \hat{\mathbf{t}}ds$ where $\hat{\mathbf{t}}$ is the unit tangent to the curve. Finally, a summation of the component of the vectors in the field orthogonal to the direction of the curve is given by integrating over the cross product $\mathbf{F} \times \hat{\mathbf{t}}ds$. The integration over the dot product is the most widely used and we will cover that here. It is known as the *scalar line integral* of the vector field.

Consider a vector field $\mathbf{F}(x, y, z)$ and a curve $\mathcal{C}$. The scalar line integral is concerned with the component of the vector field in the direction of (tangent to) the curve at any given point, $\mathbf{F} \cdot \hat{\mathbf{t}}ds$. Recall that $\hat{\mathbf{t}} = d\mathbf{r}/ds$ where $d\mathbf{r}$ is a vector element of arc length. The scalar line integral of $\mathbf{F}$ along $\mathcal{C}$ is

$$\int_{\mathcal{C}} \mathbf{F} \cdot \hat{\mathbf{t}}ds = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

EXAMPLE: Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = x^2y\mathbf{i} + xz\mathbf{j} - 2yz\mathbf{k}$ the curve is defined by $\mathbf{r} = (4t, 2t^2, t^3)$ from points A (0, 0, 0) and B (4, 2, 1).

For this example, it is easiest to First, find $d\mathbf{r}$ in terms of t,

$$\begin{aligned} d\mathbf{r} &= \frac{d\mathbf{r}}{dt}dt \\ &= (4dt, 4tdt, 3t^2dt). \end{aligned}$$

Next, express $\mathbf{F}$ in terms of t,

$$\begin{aligned} \mathbf{F} &= x^2y\mathbf{i} + xz\mathbf{j} - 2yz\mathbf{k} \\ &= (4t)^2 2t^2\mathbf{i} + 4tt^3\mathbf{j} - 2 \cdot 2t^2t^3\mathbf{k} \\ &= 32t^4\mathbf{i} + 4t^4\mathbf{j} - 4t^5\mathbf{k} \\ &= (32t^4, 4t^4, -4t^5) \end{aligned}$$

Next find the dot product

FIGURE 3. Sisyphus has a lot of time to think about $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= (32t^4, 4t^4, -4t^5) \cdot (4dt, 4tdt, 3t^2dt) \\ &= (128t^4 + 16t^5 - 12t^7)dt\end{aligned}$$

Note that at point A, $t = 0$ and at point B, $t = 1$. Finally integrate with respect to t.

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{t=1} (128t^4 + 16t^5 - 12t^7)dt \\ &= \frac{128}{5} + \frac{8}{3} - \frac{3}{2} = 26.77\end{aligned}$$

EXAMPLE: Calculate $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{dr}$ Where $\mathcal{C}$ is the part of the spiral defined by $\mathbf{r} = (a\cos\theta, a\sin\theta, a\theta)$ corresponding to $0 \leq \theta \leq \frac{1}{2}\pi$, and $\mathbf{F} = r^2\hat{\mathbf{i}}$

$$
\begin{aligned}
\mathbf{r} &= a\cos\theta\hat{\mathbf{i}} + a\sin\theta\hat{\mathbf{j}} + a\theta\hat{\mathbf{k}} \\
d\mathbf{r} &= -a\sin\theta d\theta\hat{\mathbf{i}} + a\cos\theta d\theta\hat{\mathbf{j}} + ad\theta\hat{\mathbf{k}} \\
\mathbf{F}\cdot d\mathbf{r} &= r^2\hat{\mathbf{i}}\cdot(-a\sin\theta d\theta\hat{\mathbf{i}} + a\cos\theta d\theta\hat{\mathbf{j}} + ad\theta\hat{\mathbf{k}}) \\
&= -ar^2\sin\theta d\theta \\
&= -a^3(\cos^2\theta + \sin^2\theta + \theta^2)\sin\theta d\theta \\
&= -a^3(1+\theta^2)\sin\theta d\theta
\end{aligned}
$$

Recalling in the previous derivation that $r = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta + \theta^2}$.

$$
\begin{aligned}
\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} &= -a^3\int_0^{\pi/2}(1+\theta^2)\sin\theta d\theta \\
&= -a^3[\cos\theta + 2\theta\sin\theta - \theta^2\cos\theta]_0^{2\pi} \\
&= -a^3(\pi - 1)
\end{aligned}
$$

You should see from the examples that performing line integrals with vector fields or treating the curve itself as made of vector components is not particularly different than the line integrals performed with scalar fields, other than the use of a dot or cross product operation. The general approach for line integrals of vector functions is as follows:

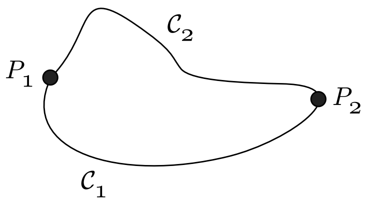
(1) Express the three major components of the integral in terms of a single variable.
   (a) The functional form of the field
   (b) The differential, ds
   (c) The limits of the integration
(2) Perform any vector operations (scalar or vector product). Note that some times it is easier to perform this step before step 1 is performed.
(3) Perform the integration.

For many of these examples, the intrinsic equation for $\mathcal{C}$ was given in terms of the single variable of interest. Sometimes, as with the example of the ellipse previously, an intrinsic or parametric equation is not provided and must be obtained. In this case it is generally useful to keep in mind that straight lines are easily parameterized in a cartesian coordinate system by an independent variable t going from 0 to 1 as the line goes from one point to the next. If the shape of the curve is circular or elliptical (even in 3D), it is usually easiest to parameterize the curve in a polar coordinate system with θ as the independent variable of interest. It is also possible to parameterize the equation in terms of one of the spatial variables, x, y or z, e.g., as $x, y(x), z(x)$. This is particularly useful if the curve is parallel to one of the coordinate axes.

This general discussion applies in two and three dimensions to surface and volume integrals as you will see in following sections. In these cases, especially in the case of surface integrals, defining the shape itself in a form convenient for integrating is usually the most difficult part. Thus below I provide a review of polar coordinate systems and defining line, surface and volume elements in these systems.

Finally, a note about notation. Sometimes a curve being used for integration is closed, which is to say that the end point of integration is the same as the beginning. In this case, a small circle is often added to the integral sign to denote that the integration is taking place along a closed line,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}\bigg|_{\mathcal{C}_1} + \int_{P_2}^{P_1} \mathbf{F} \cdot d\mathbf{r}\bigg|_{\mathcal{C}_2}.$$



3. Conservative vector fields

At this early stage with line integrals (keep in mind we have yet to cover surface and volume integrals) we can cover a topic that is very important for physical systems, the concept of a *conservative field*. One physical concept that is well represented by the scalar line integral is that of work. If $\mathbf{F}$ is a vector field representing force then the integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ will calculate the work done in moving something along the curve C. The work done in moving an object from a point P_1 to P_2 often depends upon the path $\mathcal{C}$ taken. For some special fields the work done depends upon only the location of the points, and is independent of the path C used to get from one to the other. Such fields are called *conservative fields*.

For a conservative field $\mathbf{F}$ integrated between points P_1 and P_2,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for all possible curves C_1 and C_2. This also implies that for any closed curve C,

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0.$$

EXAMPLE[1]: If $\mathbf{F} = (2xyz,\ x^2z,\ x^2y)$ evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ between points $(0,0,0)$ and $(2,4,6)$ along the following two paths:

(1) The curve C parameterized by $\mathbf{r}(u) = (u, u^2, 3u)$.

First express $\mathbf{F}$ in terms of u

$$\begin{aligned} \mathbf{F} &= (2xyz,\ x^2z,\ x^2y) \\ &= (2uu^2 3u,\ u^2 3u,\ u^2u^2) = (6u^4,\ 3u^3,\ u^4) \end{aligned}$$

Next, find $d\mathbf{r}$ in terms of u,

$$d\mathbf{r} = \frac{d\mathbf{r}}{du} du = (1, 2u, 3) du$$

Note that the boundaries are $u = 0$ to $u = 2$. From here the final integration can be set up.

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{u=0}^{2} (6u^4,\ 3u^3,\ u^4) \cdot (1, 2u, 3)\ du \\ &= \int_{u=0}^{2} (6u^4 + 6u^4 + 3u^4)\ du = \int_{u=0}^{2} 15u^4\ du \\ &= 3u^5 \Big|_0^2 = 96 \end{aligned}$$

(2) The curve made of the three straight lines, first in the x-direction from (0, 0, 0) to (2, 0, 0), next in the y-direction from (2,0,0) to (2,4,0) and finally in the z-direction from (2, 4, 0) to (2, 4, 6).

Here the full integration can be broken up into the sum of three integrals in the x, y and z directions respectively. Starting with the first segment,

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_{x=0}^{2} (2xyz,\ x^2z,\ x^2y) \cdot (dx,\ 0,\ 0) \\ &= \int_{x=0}^{2} 2xyz\ dx = 0 \end{aligned}$$

Because $y, z = 0$. The second segment

[1] Stroud, Programme 23 Frame 43

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_{y=0}^{4} (2xyz,\ x^2z,\ x^2y) \cdot (0,\ dy,\ 0) \\ &= \int_{y=0}^{4} x^2z\ dy = 0\end{aligned}$$

because $z = 0$. Finally

$$\begin{aligned}\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_{z=0}^{6} (2xyz,\ x^2z,\ x^2y) \cdot (0,\ 0,\ dz) &= \int_{z=0}^{6} x^2y\ dz \\ &= x^2yz\Big|_{z=0}^{6}\Big|x = 2, y = 4 \\ &= 16z\Big|_{z=0}^{6} = 96\end{aligned}$$

Repeating the exercise with more curves would reveal that the answer is always the same and independent of the path and that the vector field $\mathbf{F}$ is conservative.

It follows from the definition of a conservative field that when calculating the line integral for a conservative field between two points P_1 and P_2, we can chose any path from P_1 to P_2 for the integration. Sometimes it can be recognised that a particular path will simplify the integration. In the example above, the the second path reduced the integral to integration of the z-component of the field in the z direction. Other times it is just more convenient to define the curve of integration along a path that is not complicated.

An important relationship also arises from the concept of a conservative vector field. First, consider a scalar field, $\Omega(x, y, z)$ and the scalar line integral of the gradient of that field, $\nabla\Omega$, from one arbitrary point P_1 to another P_2 along a curve C,

$$\int_{P_1}^{P_2} \nabla\Omega \cdot d\mathbf{r}$$

Figure 1 shows a curve in such a field broken up into infinitesimal sections Δs_i with the blue arrow showing $\nabla\Omega$ at one point on the curve. Recall from the first lecture that the dot product $\nabla\Omega_i \cdot \hat{\mathbf{t}}_i\Delta s_i$ represents the magnitude of the rate of change of Ω in the direction tangent to the curve, $\hat{\mathbf{t}}$, multiplied by the distance Δs with the value of all of those parameters specific to point i on the curve. Thus it can be inferred that a reasonable approximate for the value of Ω at point a on the curve could be found by simply adding this change to Ω at P_1.

$$\Omega_1 \approx \Omega_{P_1} + \nabla\Omega_{P_1} \cdot \hat{\mathbf{t}}_1\Delta s_1$$

Moving on to point b on the curve a similar expression could be used relative to point a,

$$\Omega_2 \approx \Omega_1 + \nabla\Omega_1 \cdot \hat{\mathbf{t}}_2 \Delta s_2.$$

You could continue on with this exercise until you got all of the way to the value of Ω at point P_2,

$$\Omega_{P_2} \approx \Omega_{P_1} + \sum_i \nabla\Omega_i \cdot \hat{\mathbf{t}}_i \Delta s_i.$$

The second term on the right of course becomes an integral in the limit that $\Delta s_i \to 0$,

$$\lim_{\Delta s \to 0} \sum_i \nabla\Omega_i \cdot \hat{\mathbf{t}}_i \Delta s_i = \int_C \nabla\Omega \cdot \hat{\mathbf{t}} ds = \int_C \nabla\Omega \cdot d\mathbf{r}$$

and thus

$$\Omega_{P_2} = \Omega_{P_1} + \int_C \nabla\Omega \cdot d\mathbf{r}$$

or rearranged,

$$\int_C \nabla\Omega \cdot d\mathbf{r} = \Omega_{P_2} - \Omega_{P_1}.$$

Thus integrating the gradient of a scalar field along a curve is only dependent on the values of the scalar field at the beginning and end point of the integration. Note that no particular curve or scalar field were defined in this derivation and this result is general for any curve and the gradient of any scalar field. This also tells you that the gradient of a scalar field is a conservative vector field because the scalar line integral of the field is not dependent on the path of integration. It can be shown that any conservative vector field $\mathbf{F}$ can be expressed as the gradient of a scalar field. That is, if $\mathbf{F}$ is a conservative vector field there exists some scalar field Ω such that $\mathbf{F} = \nabla\Omega$. Expressing a given vector field as the gradient of a scalar field is not always trivial, as it involves solving coupled differential equations, but at times it can be done. If you are asked for this in a problem set or test the solution will be straightforward. The following provides an example:

EXAMPLE [2]: Express the vector field from the previous example, $\mathbf{F} = (2xyz,\ x^2z,\ x^2y)$, as the gradient of a scalar field.

[2]Stroud Programme 23, Frame 49

We found the vector field to be conservative so this should be possible. Letting $\nabla\Omega = \mathbf{F}$,

$$\frac{\partial\Omega}{\partial x} = 2xyz, \ \frac{\partial\Omega}{\partial y} = x^2z, \ \frac{\partial\Omega}{\partial z} = x^2y$$

Partial integration of each of these equations results in

$$\Omega = x^2yz + f(y,z), \ \Omega = x^2yz + g(x,z), \ \Omega = x^2yz + h(x,y)$$

Thus in this case it is clear that a valid expression for Ω is given by $\Omega = x^2yz$.

Finally, recall from section 4.8 in chapter 1 (or simply convince yourself) that the curl of the gradient of a scalar field is always zero, $\nabla\times\nabla\Omega = 0$. Thus, if a vector field $\mathbf{F}$ is conservative, it follows that the curl of the vector field must be zero, $\nabla\times\mathbf{F} = \nabla\times\nabla\Omega = 0$. It can also be shown that if the curl of a vector field is zero, the vector field is always conservative.

EXAMPLE: Using the curl relationship, show that the vector field from the previous two examples is conservative.

$$\mathbf{F} = (2xyz, \ x^2z, \ x^2y)$$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix}$$

$$= \Big(x^2 - x^2\Big)\mathbf{i} + \Big(2xy - 2xy\Big)\mathbf{j} + \Big(2xz - 2xz\Big)\mathbf{k} = 0.$$

Taking the curl is probably the most straightforward way to identify if a vector field is conservative.

In summary, each of the following statements implies all of the rest of the statements

(1) $\mathbf{F}$ is a conservative vector field

(2) $\oint \mathbf{F}\cdot d\mathbf{r} = 0$

(3) $\mathbf{F} = \nabla\Omega$

(4) $\nabla\times\mathbf{F} = 0$

Next we will move on to surface and volume integrals. First I provide a review of defining integral elements in polar coordinate systems.

4. Review: Differentials in Cartesian and polar coordinate systems

See figure 4. Recall that the cylindrical coordinate system is defined in relation to the Cartesian coordinate system as

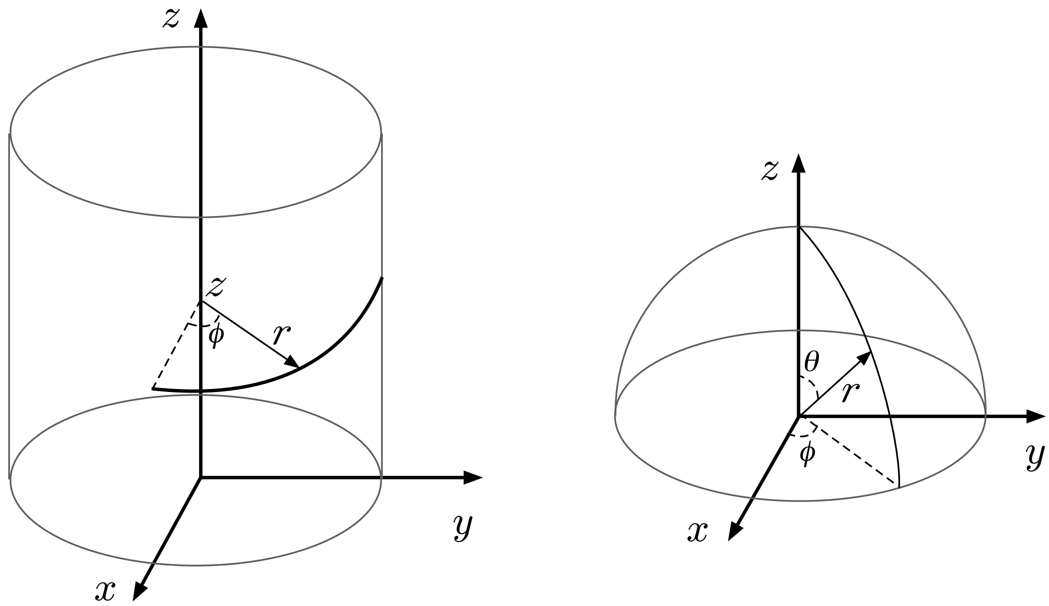


FIGURE 4. Cylindrical (left) and spherical (right) polar coordinate systems in relation to the Cartesian coordinate system

$$
\begin{aligned}
x &= \mathrm{R}\cos\phi \\
y &= \mathrm{R}\sin\phi \\
z &= z \\
\mathrm{R} &= \sqrt{x^2 + y^2}
\end{aligned}
$$

and that the spherical coordinate system is defined as

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta \\
r &= \sqrt{x^2 + y^2 + z^2}.
\end{aligned}
$$

An element of curve arc, $ds = \sqrt{dx^2 + dy^2 + dz^2}$. In cylindrical coordinates, using ϕ as the differential this becomes

$$
\begin{aligned}
ds = \frac{ds}{d\phi} d\phi &= \left(\sqrt{\frac{dx}{d\phi}^2 + \frac{dy}{d\phi}^2 + \frac{dz}{d\phi}^2} \right) d\phi \\
&= \left(\sqrt{\mathrm{R}^2 \cos^2 \phi + \mathrm{R}^2 \sin^2 \phi} \right) d\phi \\
&= \mathrm{R} d\phi.
\end{aligned}
$$

Similarly, in spherical coordinates using θ as the differential,

$$
\begin{aligned}
\frac{ds}{d\theta} d\theta &= \left(\sqrt{r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta} \right) d\theta \\
&= r \left(\sqrt{\cos^2 \theta + \sin^2 \theta} \right) d\theta = r d\theta.
\end{aligned}
$$

Using ϕ as the differential the same analysis results in $ds = r \sin \theta d\phi$.

EXAMPLE: Find an expression for the line element ds for the following three curves using the specified parameterization and coordinate system:

- The straight line from point (1, 1, 1) to (1, 4, 5) parameterized by t in a Cartesian coordinate system

  Here assume that t goes from 0 to 1 as (x, y, z) goes from (1, 1, 1) to (1, 4, 5).

$$
\begin{aligned}
\Big[x(t), y(t), z(t) \Big] &= \Big[1, 1 + (4-1)t, 1 + (5-1)t \Big] \\
&= \Big[1, 1 + 3t, 1 + 4t \Big]
\end{aligned}
$$

  Thus

$$
\begin{aligned}
ds &= \sqrt{dx^2 + dy^2 + dz^2} \\
= \frac{ds}{dt} dt &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\
&= \sqrt{9 + 16} = 5 dt
\end{aligned}
$$

- The curve described by $x^2 + y^2 = 7$ in the plane $z = 5$ using cylindrical coordinates

  This is a matter of identifying that the curve is a circle centred around the z-axis with a radius $r = \sqrt{7}$. Thus

$$ds = rd\phi = \sqrt{7}d\phi$$

- The curve where $x = y$ on the surface described by $x^2+y^2+z^2 = 15$ in spherical coordinates.

  Similar to the previous example this is a matter of identifying that the curve lies on the surface of a sphere with a centre at the origin and radius $r = \sqrt{15}$. In this case x and y are fixed relative to each other along a straight line and $\phi = \pi/2$ is a constant. Thus the integration is best described along θ and

$$ds = rd\theta = \sqrt{15}d\theta$$

With line integrals we are integrating over a one-dimensional object (the line) in a three dimensional space. Thus, if you want to (if it is convenient to) integrate with respect to one of the spatial coordinates, e.g., θ, then the expression for ds is converted accordingly but you have the choice of three spatial coordinates to chose from, whether that is in the Cartesian or polar coordinate systems. The choice of coordinate system or parameterization that is used to describe the curve can be very important for making the integration tractable. It is important to keep in mind, however, that the mathematical concept is independent of the coordinate system. Thus the line integral of some vector field $\int_C \mathbf{F} \cdot d\mathbf{r}$ will be the same regardless of the coordinate system used. In other words, if the same expression is integrated using any number of different coordinate systems, the answer will always be the same.

For integration over surfaces, the infinitesimal element of integration is usually noted dS. The surface is a two dimensional object and thus integration over two dimensions is required. In a Cartesian system, surface elements parallel to one of the axes planes can be described by $dS = dx\ dy$, $dS = dx\ dz$ and $dS = dy\ dz$.

Figure 5 shows surface elements for the cylindrical and spherical coordinate systems in addition to the Cartesian elements. As with the Cartesian system there are three combinations of possible surface elements. For the cylindrical system, the element of interest is usually $dS = rd\phi\ dz$. For the spherical system it is $dS = r^2 \sin\theta\ d\theta\ d\phi$.

EXAMPLE: Find an expression for the surface element dS for the following three surfaces:

- The planar surface defined by $x = 5$ with y going from 3 to 6 and z going from 0 to 10 in cartesian coordinates.

  In this case x is constant and thus the integral is taken over a range of y and z, thus $dS = dy\ dz$.

- The surface defined by $x^2 + y^2 = 8$ along the range of z from 0 to 15 in the 3rd quadrant in cylindrical coordinates

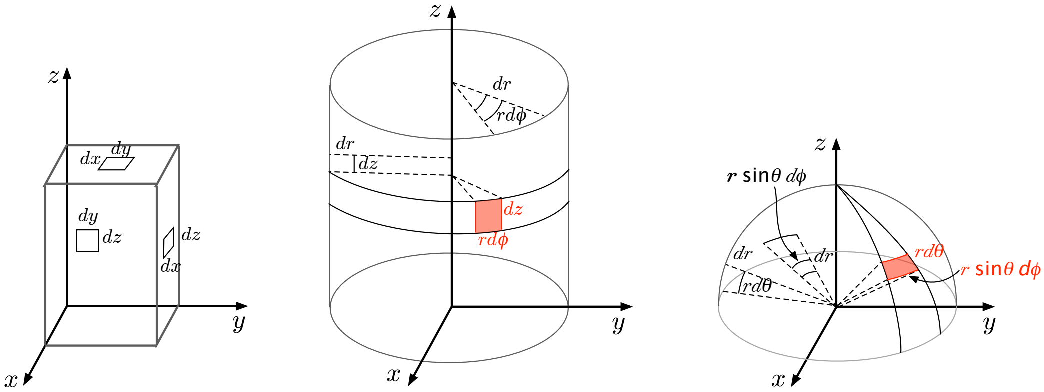


FIGURE 5. Surface elements in Cartesian (left), cylindrical (centre) and spherical (right) coordinate systems. The elements shown in red are the elements most commonly used in these systems.

In this case the surface is a cylinder of radius $\sqrt{8}$. The extra information is useful for defining the boundaries of integration but not necessary here. Thus

$$dS = rd\phi\ dz = \sqrt{8}d\phi\ dz$$

- The surface defined by $x^2 + y^2 + z^2 = 19$ in spherical coordinates

  This is a sphere centred at the origin with radius $r = \sqrt{19}$. Thus

$$\begin{aligned} dS &= r^2 \sin\theta\ d\theta\ d\phi \\ &= 19 \sin\theta\ d\theta\ d\phi \end{aligned}$$

Of course most surfaces are not spherical, cylindrical or planar and a more general way of describing surfaces is needed. Analogous to the reduction of the variable space to one dimension with line integrals, the work of describing the surfaces involves reducing the parameters to an expression in two dimensions. Thus the surface (as well as the other parts of the integral) can be parameterized by non-spatial variables, e.g. $\phi(x, y, z) = \phi(x(u, v), y(u, v), z(u, v))$ whereby $dS \propto du\ dv$. Alternatively, the spatial variables can be solved in terms of each other, eliminating one of the variables, e.g. $\phi(x, y, z) = \phi(x, y, z(x, y))$ and $dS \propto dx\ dy$. This latter case is shown geometrically in figure 6 as a projection of a surface S in $x - y - z$ space onto the $x - y$ plane as the surface H.

Through simple geometrical considerations you can see that

$$dH = \cos\gamma\ dS$$

In the case shown in the figure where the surface can be described as single valued in z, i.e. $z = f(x, y)$ it can also be shown that

$$\begin{aligned}
\frac{1}{\cos\gamma} = \sec\gamma &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \\
dS &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dH \\
&= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx\ dy
\end{aligned}$$

A similar process can be followed projecting the surface into the $x - z$ or $y - z$ planes if the surface is single valued in y or x, respectively. To generalise, a surface element dS in a 3-dimensional Cartesian space may be mapped onto an arbitrary 2-D space with coordinate axes u and v by the following

$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\ dv$$

where the surface is defined using a position vector function parameterized by u and v, $\mathbf{r}[x(u, v),\ y(u, v),\ z(u, v)]$, and limits $u_1 \leq u \leq u_2$ and $v_1 \leq v \leq v_2$.

Finally, infinitesimal volume elements are denoted dV. Volumes are three-dimensional objects and thus there is only one option for their definition in any given three dimensional coordinate system.

For the Cartesian system

$$dV = dx\ dy\ dz,$$

in the cylindrical coordinate system

$$dV = r\ dr\ d\phi\ dz$$

and in the spherical coordinate system

$$dV = r^2 \sin\theta\ dr\ d\phi\ d\theta.$$

5. Surface integrals of vector functions

As was the case with the integration of vector functions along a line, there are several types of operations that can be performed for integration across a surface. If a vector field is given by $\mathbf{F}(x, y, z)$ and there exists a surface S, the simple summation of vectors existing at the locations on the surface is given by $\int_S \mathbf{F} dS$.

Scalar (dot) and vector (cross) product operations can be defined by associating a direction to a surface. In this case the surface element is represented as a vector, $d\mathbf{S}$ whose

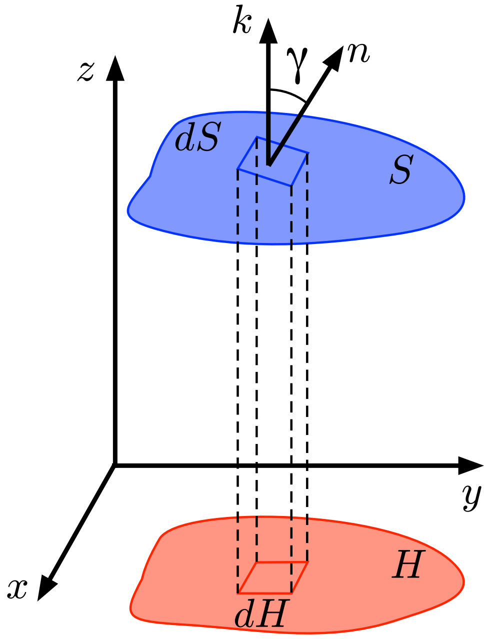


FIGURE 6. A projection of the surface element $d\mathbf{S}$ onto the $x - y$ plane.

direction is the unit normal to the surface. This is equivalent to a scalar element of the surface dS multiplied by the unit normal to the surface $\hat{\mathbf{n}}$,

$$d\mathbf{S} = \hat{\mathbf{n}} dS.$$

Recall that surfaces can be defined as the range of locations where a scalar field is constant, $\Omega(x, y, z) =$ constant, and that the unit normal of the surface is then found by $\hat{\mathbf{n}} = \frac{\nabla\Omega}{|\nabla\Omega|}$.

We will be interested primarily in scalar surface integrals, e.g.,

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS.$$

In this integral we are integrating the component of the vector field that is normal to the surface S. In other words we are summing up the magnitude of the components of all of the vectors in the direction normal to the surface.

EXAMPLE: Evaluate the integral $\int_S \mathbf{V} \cdot d\mathbf{S}$ where $\mathbf{V} = (z, x, -3y^2 z)$ and S is the surface of the cylinder $x^2 + y^2 = 16$ in the first octant with $0 \leq z \leq 5$.

The problem lends itself to the use of cylindrical coordinates, but we will obtain the scalar product before converting.

$$d\mathbf{S} = \mathbf{n}dS$$
$$\nabla\Omega = (2x, 2y, 0)$$
$$|\nabla\Omega| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{16} = 8$$
$$\mathbf{n} = \frac{\nabla\Omega}{|\nabla\Omega|} = \frac{1}{4}(x, y, 0)$$
$$\mathbf{V} \cdot \mathbf{n}dS = (z, x, -3y^2z) \cdot \frac{1}{4}(x, y, 0)dS = \frac{1}{4}(zx + xy)dS$$

Now converting to cylindrical coordinates

$$= \frac{1}{4}(4z\cos\phi + 16\cos\phi\sin\phi)dS.$$

Recalling that $dS = Rd\phi\ dz$

$$= 4(z\cos\phi + 4\cos\phi\sin\phi)d\phi\ dz$$

Thus the final integration becomes

$$\int_S \mathbf{V} \cdot d\mathbf{S} = 4\int_{z=0}^{5}\int_{\phi=0}^{\pi/2}(z\cos\phi + 4\cos\phi\sin\phi)d\phi\ dz.$$

Making use of the relation $\sin 2\phi = 2\sin\phi\cos\phi$,

$$= 4\int_{z=0}^{5}\left[z\sin\phi - \cos 2\phi\right]_{\phi=0}^{\pi/2} dz$$
$$= 4\int_{z=0}^{5}(z+2)dz$$
$$= 4\left(\frac{z^2}{2} + 2z\right)_0^5$$
$$= 90$$

An example of a vector surface integral would be

$$\int_S \mathbf{F} \times d\mathbf{S} = \int_S \mathbf{F} \times \hat{\mathbf{n}}\, dS$$

Similar to the vector line integral, this type of integral is not generally of interest in physical systems. There are, however, integrals involving the cross product that are very important to physical systems and that we will explore in detail in the following sections. In particular, the scalar surface integral of the curl of a vector field is very important for physical systems,

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$

and its evaluation follows the same techniques just covered for evaluating scalar surface integrals.

You can see from the examples that the general approach to performing the integrals is nearly the same as with the line integrals with the added complication that the unit normal must at times be found:

(1) Express the three major components of the integral in terms of two variables.
   (a) The functional form of the vector field
   (b) The differential, dS or $d\mathbf{S}$. If $d\mathbf{S}$ this involves using an expression for the surface, $\Omega(x, y, z)$, and obtaining the unit normal, $\frac{\nabla\Omega}{|\nabla\Omega|}$.
   (c) The limits of the integration
(2) Perform any vector operations (scalar or vector product). Note that some times it is easier to perform this step before step 1 is fully performed.
(3) Perform the integration.

As a final and somewhat unimportant note on surface integrals, in many texts surface integrals are written with double integral signs to explicate that the integration is taking place over the domain of two variables. Additionally, if a surface is closed, e.g., the surface of a sphere, sometimes a small circle is also written in the middle of the integral sign,

$$\oint_S .$$

6. Volume integrals of vector functions

The integration of vector fields over a volume is simpler than the surface integration process as there is no useful reason to assign directionality to the infinitesimal volume elements. Volume integrals are simply summations of the vectors in the field over the domain defined by the volume and there is no need to perform vector operations or obtain unit normals or any of the other complications associated with line and surface integrals. The single integral operation involving a vector field $\mathbf{F}$ over a volume V is

$$\int_V \mathbf{F} dV.$$

EXAMPLE[3]: Evaluate $\int_V \mathbf{F} dV$ where V is the region bounded by the planes $x = 0$, $x = 2$, $y = 0$, $y = 3$, $z = 0$, $z = 4$ and $\mathbf{F} = (xy, z, -x^2)$.

The planes are all parallel to the axes planes and thus this is a straightforward integration over x, y and z:

$$\begin{aligned}
\int_V \mathbf{F} dV &= \int_{z=0}^{4} \int_{y=0}^{3} \int_{x=0}^{2} (xy, z, -x^2)\ dx\ dy\ dz \\
&= \int_{z=0}^{4} \int_{y=0}^{3} \left(\frac{x^2 y}{2},\ zx,\ -\frac{x^3}{3} \right)_{x=0}^{2} dy\ dz \\
&= \int_{z=0}^{4} \int_{y=0}^{3} \left(2y,\ 2z,\ -\frac{8}{3} \right) dy\ dz \\
&= \int_{z=0}^{4} \left(y^2,\ 2zy,\ -\frac{8}{3} y \right)_{y=0}^{3} dz \\
&= \int_{z=0}^{4} \left(9,\ 6z,\ -8 \right) dz \\
&= \left(9z,\ 3z^2,\ -8z \right)_{z=0}^{4} \\
&= (36,\ 48,\ -32)
\end{aligned}$$

As with the surface integrals, some apparently need explicit help in remembering that a volume exists as a 3D object and thus use a triple integral when writing out volume integrals,

$$\iiint_V \mathbf{F} dV.$$

There is no need to memorize the expressions. If you need them, you can look them up, and they will be provided if needed in a closed examination.

[3]Stroud, Programme 23 Frame 21

CHAPTER 3

The integral theorems and applications

At this stage we have covered the differentiation and integration of vector functions. Next we will learn two new theorems of vector calculus involving relationships between line, surface and volume integrals of scalar and vector fields. Finally we will review some important applications in physical sciences and of particular interest in geophysics for which the use of vector calculus is essential.

In elementary calculus, integration can be regarded as the inverse of differentiation, with the fundamental theorem of calculus given by

$$\int_a^b \frac{df}{dx} dx \equiv f(b) - f(a).$$

The integral theorems for vector calculus present analogous rules for line, surface and volume integrals. They each state that the integral of a differential is related to the value of the undifferentiated function evaluated at the boundary of the region of integration. For integration along an open curve, the boundary is represented by two points on the curve. For integration over an open surface, the boundary is represented by a closed curve. And for integration over a finite volume, the boundary is represented by a closed surface.

In this module we have learned three new differentiation operations, the gradient, the divergence and the curl. The three integral theorems covering line, surface and volume integrals each use one of these operations - the theorem for integration along the line utilises the gradient (we already covered this during the last class), the theorem for integration over a surface involves the curl and the integration over a volume involves the divergence.

1. Integral theorem for $\nabla\Omega$

In the last lecture we already covered the integral theorem relating the line integral of the gradient of a scalar field to the values at the starting and end points.

$$\int_{\mathcal{C}} \nabla\Omega \cdot d\mathbf{r} = \Omega_2 - \Omega_1$$

2. Integral theorem for $\nabla \cdot \mathbf{F}$ - Gauss' (the divergence) theorem

The integral theorem involving the divergence relates the integral of the vector field over a volume to the integration of that field across the closed surface that defines the volume. We will show the following:

$$\int_V \nabla \cdot \mathbf{V} dV = \int_S \mathbf{V} \cdot d\mathbf{S}$$

We already did most of the work in deriving this theorem when we discussed the physical meaning of the divergence in the first lecture. Consider again figure 9 from chapter 1 showing the flux of a fluid out of a small box. We showed in the first lecture how this flux was given by

$$(\nabla \cdot \mathbf{V}) dx\ dy\ dz = (\nabla \cdot \mathbf{V}) dV$$

For this volume element you can also see from the figure how this is equal to the sum of the fluxes at each surface,

$$(\nabla \cdot \mathbf{V}) dV = \sum_i \mathbf{V}_i \cdot d\mathbf{S}_i$$

A large volume can be broken up into infinitesimal volume elements dV, each with its own flux in and out of the element where this relationship applies. Summing up the infinitesimal elements, the flux out of one surface is cancelled out by the flux into an adjacent surface and thus the only flux that matters is that at the outer bounding surface,

$$\int_V \nabla \cdot \mathbf{V} dV = \int_S \mathbf{V} \cdot d\mathbf{S}.$$

Integrating the divergence of a vector field over a volume is equal to the scalar surface integral across the bounding surface of the volume. This is known as Gauss' theorem.

EXAMPLE[1]: Use Gauss' theorem to integrate the vector field $\mathbf{F} = (x, 2, z^2)$ over the surface of the region bound by the planes $z = 0,\ z = 4,\ x = 0,\ y = 0$ and the surface $x^2 + y^2 = 4$ in the first octant.

Using Gauss' theorem we can integrate the divergence of the vector field across the volume. The divergence is given by

[1] Stroud Programme 23 Frame 58

$$\nabla \cdot \mathbf{F} = 1 + 2z.$$

The problem lends itself to cylindrical co-ordinates with $0 \leq r \leq 2$, $0 \leq \phi \leq \pi/2$ and $0 \leq z \leq 4$.

$$\begin{aligned}
&\int_{z=0}^{4}\int_{r=0}^{2}\int_{\phi=0}^{\pi/2}(1+2z)r\ d\phi\ dr\ dz \\
&= \int_{z=0}^{4}\int_{r=0}^{2}(1+2z)r\phi\Big|_{0}^{\pi/2}\ dr\ dz \\
&= \int_{z=0}^{4}\int_{r=0}^{2}\frac{\pi}{2}(1+2z)r\ dr\ dz \\
&= \int_{z=0}^{4}\frac{\pi}{4}(1+2z)r^2\Big|_{0}^{2}\ dz \\
&= \int_{z=0}^{4}\pi(1+2z)\ dz = \pi(z+z^2)\Big|_{z=0}^{4} = 20\pi
\end{aligned}$$

You can verify by performing the surface integral.

3. Integral theorem for $\nabla \times \mathbf{F}$ - Stokes' theorem

The final integral theorem is the one utilising the curl and it relates a vector integral over a two dimensional surface to the line integral along the closed loop that bounds the surface. In particular, I will show the following to be true,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Consider a two sided surface S bounded by a simple closed curve C as shown in figure 1 defined over a three dimensional space in which there also exists some vector field $\mathbf{F}$. The scalar line integral of $\mathbf{F}$ around C is given by $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

If we drew a curve C_3 on the surface that split the closed curve into two closed curves, $C_2 + C_3$ and $C_1 + C_3$ you can also see that the sum of the line integrals around both loops is equivalent to the original line integral around C,

$$\begin{aligned}
&\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_3} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r}
\end{aligned}$$

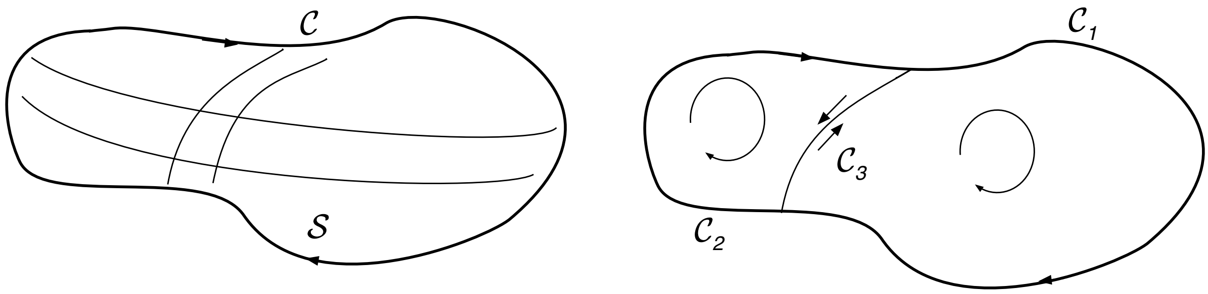


FIGURE 1. A surface S bound by the closed curve C.

If we continued this business and split up the surface into many small elements $d\mathbf{S}_i$ each with bounding curves C_i (figure 2) and summed up their circulations, we would see that the common boundaries would continue to cancel themselves out such that the sum would in the end be equivalent to the circulation around the bounding curve.

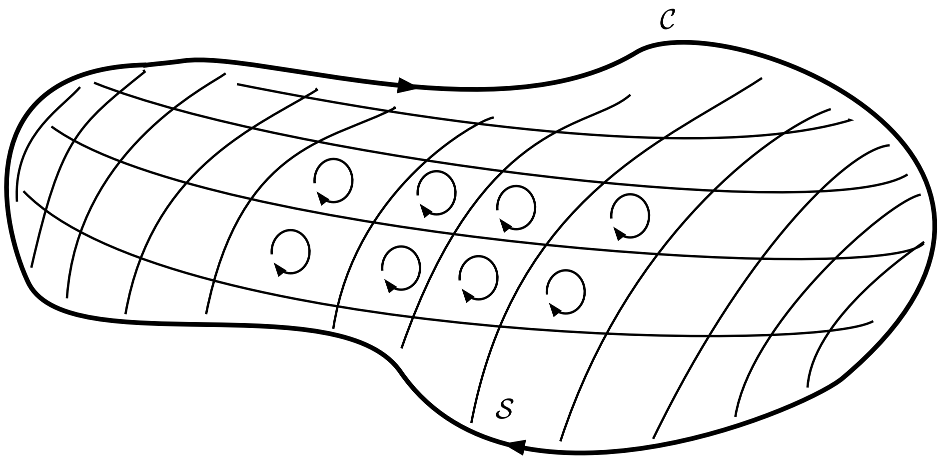


FIGURE 2. The surface S broken up into smaller surfaces, $d\mathbf{S}_i$.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \sum_i \oint_{C_i} \mathbf{F} \cdot d\mathbf{r}$$

Our next step involves finding a formula for the integrals of these individual elements. Because we can chose the shape of the elements and the orientation of the surface (or the orientation of the coordinate system), I will say that at least one of the elements approximates a square surface in the $x - y$ plane as shown in figure 3. In other words, this element is described by $d\mathbf{S}_i = \Delta x \Delta y \mathbf{k}$.

For this element, the circulation can be explicitly written as

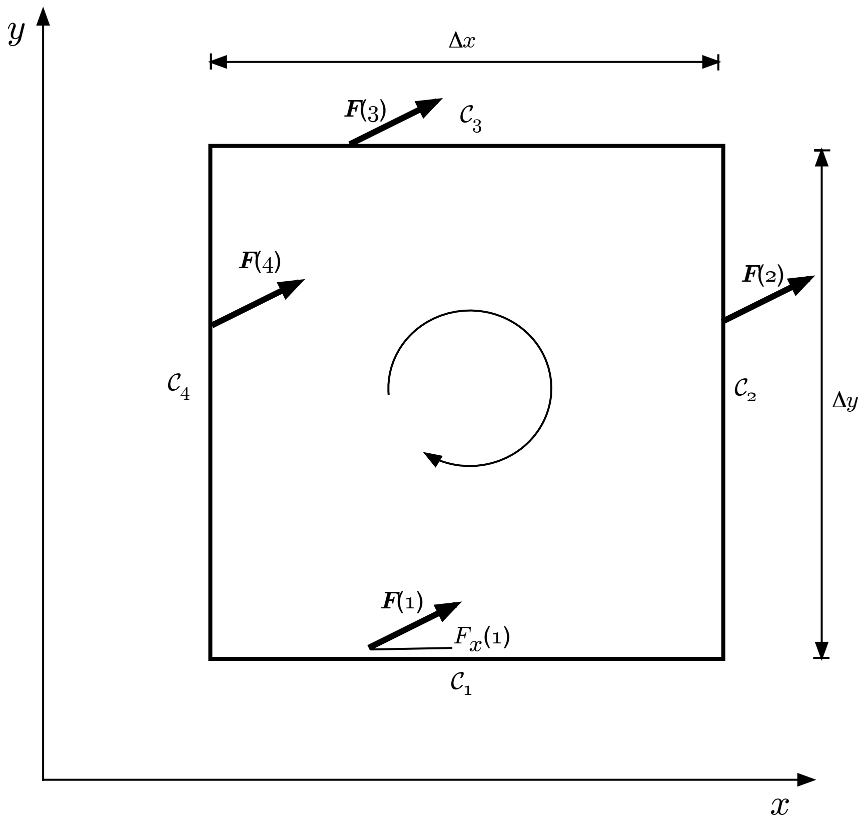


FIGURE 3. The circulation in a single element.

$$\begin{aligned}\int_{C_i} \mathbf{F} \cdot d\mathbf{r} &= F_x(1)\Delta x + F_y(2)\Delta y - F_x(3)\Delta x - F_y(4)\Delta y \\ &= [F_x(1) - F_x(3)]\Delta x + [F_y(2) - F_y(4)]\Delta y\end{aligned}$$

But we can approximate $F_x(3) = F_x(1) + \frac{\partial F_x}{\partial y}\Delta y$ and similarly $F_y(2) = F_y(4) + \frac{\partial F_y}{\partial x}\Delta x$ so that

$$\begin{aligned}\int_{C_i} \mathbf{F} \cdot d\mathbf{r} &= \frac{\partial F_y}{\partial x}\Delta x \Delta y - \frac{\partial F_x}{\partial y}\Delta x \Delta y \\ &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Delta x \Delta y\end{aligned}$$

But this is nothing more than the z component of the curl multiplied by the area of the surface element, $\Delta x \Delta y$, that is

$$\begin{aligned}\int_{C_i} \mathbf{F} \cdot d\mathbf{r} &= (\nabla \times \mathbf{F}) \cdot \mathbf{k} \Delta x \Delta y \\ &= (\nabla \times \mathbf{F}) \cdot d\mathbf{S}_i\end{aligned}$$

To generalise this to any surface element on S (we do not restrict all of the elements to being oriented with the z-axis), we simply replace the vector $\mathbf{k}$ with the unit normal $\hat{\mathbf{n}}$ so that the circulation around any elemental of surface can be given by

$$\int_{C_i} \mathbf{F} \cdot d\mathbf{r} = (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dS$$

The circulation around the bounding curve of the whole surface is thus given by the integration over the circulations of the individual elements,

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

This is known as Stoke's theorem and it relates the surface integral of $\nabla \times \mathbf{F}$ over the surface S to the line integral of a vector field $\mathbf{F}$ around the closed curve C.

EXAMPLE: Integrate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = 2x(x+y)\mathbf{i} + (x^2 + xy + y^2)\mathbf{j}$ and C is the square with vertices at (0,0), (1,0), (1,1) and (0,1).

Stokes' theorem tells us that we can obtain the equivalent information by taking the surface integral of the curl,

$$\int_{S} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{C} \mathbf{F} \cdot d\mathbf{r}.$$

First we find the curl of $\mathbf{F}$,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x(x+y) & x^2 + xy + y^2 & 0 \end{vmatrix}$$

$$= [(2x + y) - 2x]\mathbf{k} = \mathbf{k}y.$$

Next we define $d\mathbf{S}$ in terms of variables of interest. Because the surface exists in the $x-y$ plane, we can define $d\mathbf{S} = \mathbf{k}\ dx\ dy$. Taking the scalar product and integrating,

$$\begin{aligned}\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_S y\mathbf{k} \cdot \mathbf{k}\ dx\ dy \\ &= \int_0^1 ydy \int_0^1 dx = 1/2\end{aligned}$$

EXAMPLE[2]: Verify Stokes' theorem for the vector field $\mathbf{F} = (2x - y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$ Where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

We will first find the integral around the bounding curve, $x^2 + y^2 = 1,\ z = 0$. The problem lends itself to a spherical coordinate system. In the $x - y$ plane $r = 1,\ \theta = \pi/2$ and

$$\begin{aligned}d\mathbf{r} &= \frac{d\mathbf{r}}{d\phi}d\phi \\ &= (-\sin\phi, \cos\phi)d\phi \\ \mathbf{F} \cdot d\mathbf{r} &= [(2x - y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}] \cdot [-\sin\phi\mathbf{i} + \cos\phi\mathbf{j}]d\phi \\ &= [(2x - y)\mathbf{i}] \cdot [-\sin\phi\ \mathbf{i}]d\phi \\ &= (-2\cos\phi\sin\phi + \sin^2\phi)d\phi \\ &= (-\sin 2\phi + \frac{1}{2}(1 - \cos 2\phi))d\phi \\ \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_{\phi=0}^{2\pi} (-\sin 2\phi + \frac{1}{2}(1 - \cos 2\phi))d\phi \\ &= \frac{1}{2}\cos 2\phi + \frac{1}{2}\phi - \frac{1}{4}\sin 2\phi\Big|_0^{2\pi} \\ &= \pi\end{aligned}$$

Now integrating the curl across the surface,

[2]Glyn James Example 3.32

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = (-2yz + 2yz)\mathbf{i} - (0 - 0)\mathbf{k} + (0 + 1)\mathbf{k} = \mathbf{k}$$

$$\mathbf{n}dS = \frac{\nabla\Omega}{|\nabla\Omega|} = (x, y, z)dS = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\sin\theta d\phi d\theta$$

$$\begin{aligned} \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \; d\theta \; d\phi \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \frac{1}{2} \sin 2\theta \; d\theta \; d\phi \\ &= \int_{\phi=0}^{2\pi} -\frac{1}{4} \cos 2\phi \Big|_0^{\pi/2} d\phi \\ &= \int_{\phi=0}^{2\pi} \frac{1}{2} d\phi = \frac{1}{2}\phi \Big|_0^{2\pi} = \pi \end{aligned}$$

4. Applications

4.1. Scalar potential. Consider the vector field generated by taking the gradient of a scalar field Ω. Recall that the line integral of this field between two points P_1 and P_2 is given by

$$\int_C \nabla\Omega \cdot d\mathbf{r} = \Omega_2 - \Omega_1,$$

independent of the path C and that the vector field $\mathbf{F} = \nabla\Omega$ is a conservative field. Recall also that for any conservative field $\mathbf{F}$ we can always find some scalar field Ω such that $\mathbf{F} = \nabla\Omega$.

The scalar field Ω is called the *scalar potential* or *potential* of the field $\mathbf{F}$, a concept which has wide use in physical sciences. You have probably come across many variations of the concept of potential energy when studying gravitational, magnetic and electrostatic fields. The potential energy is nothing more than the scalar potential of the force field for which it is the potential, e.g., the gravitational, magnetic and/or electrostatic force fields, all of which are conservative vector force fields.

Where $\mathbf{F}$ represents a force field, then Ω is related to the potential energy at each point in the field. The potential, Ω, at a point P is given by

$$\Omega = \int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is any continuous curve that runs from a reference point where $\Omega = 0$ to the point P.

4.1.1. *Work.* The work, W, done by a moving force, $\mathbf{F}$, in moving a particle is given by force applied over a distance, and you will see that it is just another term for potential. If the force and the displacement are not parallel, then only the component of the force parallel to the displacement does work,

$$W = F\,d\cos\theta = \mathbf{F}\cdot\mathbf{d}$$

If the force varies with position, or if the direction of the displacement varies, we can write for an infinitesimal displacement $d\mathbf{r}$,

$$dW = \mathbf{F}\cdot d\mathbf{r}$$

To find the total work performed in moving the particle along a trajectory, C, we can integrate dW along the path C,

$$W = \int_C \mathbf{F}\cdot d\mathbf{r}$$

This is the scalar line integral of a vector field. Thus work is a measure of potential. Sometimes the potential with respect to a given force field is described as a potential to do work in that field.

4.1.2. *Gravitational potential.* Newton's law of gravitation in Cartesian coordinates describes the mutual attraction between bodies of mass m and m_0 centred at points $A = (x', y', z)$ and $B = (x, y, z)$ respectively. If r is the distance between the points, $r = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$, the force is given by

$$\mathbf{F} = G\frac{mm_0}{r^2}\hat{\mathbf{r}}.$$

The force is directed along the line between the two bodies. More often the gravitational acceleration $\mathbf{g}$ is used. In this case the force is divided by m_0 which is assumed to be of unit mass,

$$\mathbf{g} = -G\frac{m}{r^2}\hat{\mathbf{r}},$$

and $\mathbf{g}$ describes the force felt due to gravitational attraction of a particle of unit mass a distance r from a body of mass m. The negative sign is the common convention used for $\mathbf{g}$. It is possible to show that $\nabla\times\mathbf{g} = 0$, that is the gravitational field is conservative. The gravitational potential, U, such that $\mathbf{g} = \nabla U$ is given by

$$U = G\frac{m}{r}$$

4.2. Flux. The divergence of a vector field, at a point, gives a measure of whether the field is divergent or convergent at that point. Many problems involve the flow of some substance, for example the flow of a gas, liquid, heat, electricity, or sub-atomic particles.

Consider the example of a fluid flowing. If the density of the fluid at any point is ρ and the velocity at any point is $\mathbf{v}$, then we can define a vector field $\mathbf{F} = \mathbf{v}\rho$ throughout the fluid. Suppose we chose some surface $\mathcal{S}$, with unit normal $\hat{\mathbf{n}}$, through which the fluid flows, then the quantity $\mathbf{F} \cdot \hat{\mathbf{n}}$ measures the mass (or *flux*) of fluid which flows through the surface $\mathcal{S}$ per unit area in unit time.

$$\mathbf{F} \cdot \hat{\mathbf{n}} = \rho\mathbf{v} \cdot \hat{\mathbf{n}} = \rho\mathbf{v}\cos\theta$$

The integral

$$\Phi = \oint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

will give the mass of fluid which flows, in unit time, through a closed surface $\mathcal{S}$. This is the amount of fluid that flows out of the volume $\mathcal{V}$ which is enclosed by $\mathcal{S}$.

For any vector field $\mathbf{F}$ the flux Φ through a surface $\mathcal{S}$ is similarly defined. The concept of flux is particularly useful for problems in electromagnetism.

4.3. Equation of continuity. If $\mathbf{F}$ is the vector field of $\rho\mathbf{v}$, then, if the divergence at a point is non-zero, either fluid must be created or destroyed at that point, or the density of the fluid must be changing.

$$\nabla \cdot (\rho\mathbf{v}) = \psi - \frac{\partial \rho}{\partial t}$$

where $\partial\rho/\partial t$ is the rate of change of density, and ψ is the net mass of fluid being created per unit time per unit volume (i.e. the source density minus the sink density).

If the substance in question is conserved, then this reduces to the equation of continuity

$$\nabla \cdot (\rho\mathbf{v}) + \frac{\partial \rho}{\partial t} = 0$$

If, in addition the fluid is incompressible, i.e. $\rho =$ constant, then this reduces further to

$$\nabla \cdot \mathbf{v} = 0$$

that is, the field $\mathbf{v}$ is solenoidal.

4.4. Laplace's equation. Recall the identity $\nabla^2\mathbf{F} \equiv \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$. If a vector field $\mathbf{F}$ is both irrotational and solenoidal, i.e.

$$\nabla \cdot \mathbf{F} = 0 \text{ and } \nabla \times \mathbf{F} = 0$$

then $\mathbf{F}$ obeys the equation

$$\nabla^2\mathbf{F} = 0$$

Recall also that an irrotational vector field $\mathbf{F}$ may be written as the gradient of a scalar potential field, i.e.

$$\mathbf{F} = \nabla\Omega$$

The identity

$$\nabla \cdot \nabla\Omega = \nabla^2\Omega$$

shows that the potential also obeys the equation

$$\nabla^2\Omega = 0$$

Thus a vector field which is both irrotational and solenoidal obeys Laplace's equation, and further, it can be written in terms of a scalar potential which also obey's Laplace's equation.

The steady gravitational and electrostatic fields in free space, except at points where there is mass or charge, and the velocity field of an incompressible fluid under laminar flow, all obey Laplace's equation.

CHAPTER 7

Convolution

The *convolution* of two functions $a(t)$ and $b(t)$ is denoted by $a(t) * b(t)$ and is defined as follows

$$a(t) * b(t) = \int_{-\infty}^{\infty} a(u)b(t-u)du$$

where u is a dummy variable that disappears on integration.[1] The convolution results in a new function of t, that is $c(t) = a(t) * b(t)$.

EXAMPLE: Find the convolution of two truncated exponential decay functions, $a(t) = H(t)5e^{-t}$ and $b(t) = H(t)3e^{-3t}$, where $H(t)$ is the Heaviside step function.

$$c(t) = a(t) * b(t)$$
$$= \int_{-\infty}^{\infty} H(u)5e^{-u}H(t-u)3e^{-3(t-u)}du$$

We simplify the problem by narrowing the limits of integration, but in this case the limits will go from $u = 0$ (the first Heaviside term is non zero when $u > 0$) to t (the second Heavisde term is non-zero only at values $u < t$). Thus the integral is simplified to

$$= \int_{0}^{t} 5e^{-u}3e^{-3(t-u)}du$$
$$= 15e^{-3t}\int_{0}^{t} e^{2u}du$$
$$= \frac{15}{2}e^{-3t+2u}\Big|_{0}^{t}$$
$$= \frac{15}{2}\left(e^{-t} - e^{-3t}\right)$$

The convolution operation is difficult to visualise geometrically but it can be done by breaking up the operation into its various components. Consider figure 1 comprised of graphs showing the convolution of the the truncated exponential decay functions from the example. In the first graph the functions $a(t)$ and $b(t)$ are shown. The convolution

[1]Note that it is easy to confuse the $*$ used to represent convolution with the superscript * used to indicate the complex conjugate. For that reason, in Fourier analysis, the complex conjugate is often written instead using a bar above the conjugated variable, $a^* \equiv \bar{a}$.

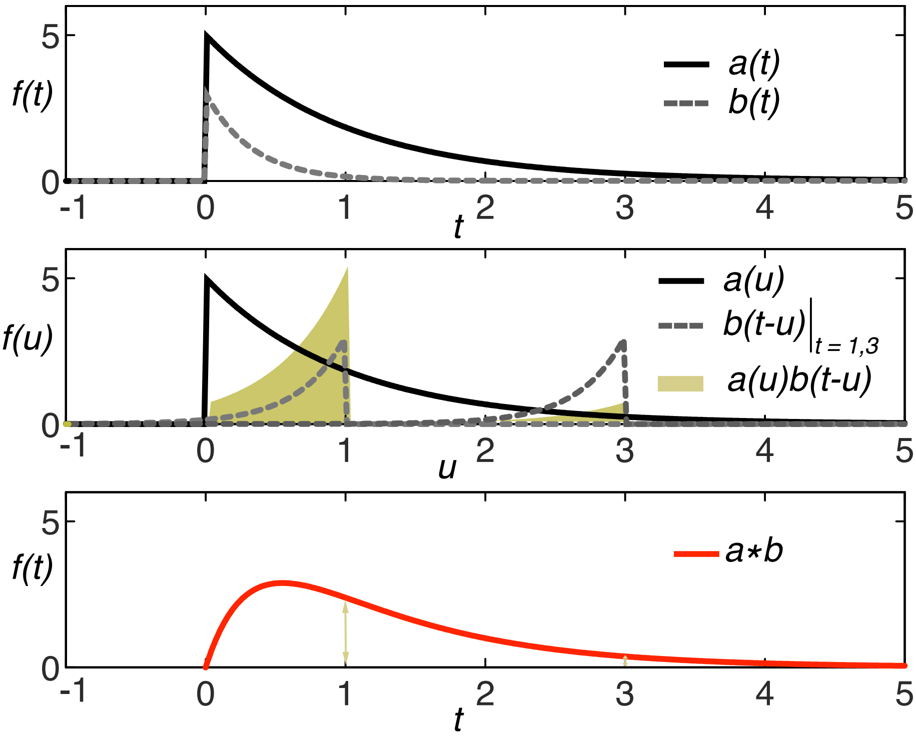


FIGURE 1. The components of the convolution operation from the above example. Top: the functions $a(t)$ and $b(t)$ to be convolved. Middle: $a(u)$ and $b(t-u)$ for two values of t. The solid colouring shows the area under the curve (the integral) of the product function $a(u)b(t-u)$ for two values of t. Lower: The function defined by the convolution $a * b$. It is a function of t with the value at each point given by the integral of the product function $a(u)b(t-u)$ for each value of t.

at a point t involves the integration of a function which is comprised of the product of the two functions, but where one is reversed and displaced, i.e. integration over u of the function $a(u)b(t-u)$. Thus the second graph shows two instances of the function b reversed and displaced, one at $t = 1$ and one at $t = 3$. The solid shaded area is outlined by the product function and the shaded area represents the value of the convolution at the given value of t. Thus the final graph showing the convolution has values at each point t corresponding the to the value of the integral (the shaded area) for each value of t.

A geometric approach to performing a convolution, while qualitative, provides some intuition behind the meaning of the operation. See figure 2. Draw a graph of the first function and then draw a graph of the second function reversed but on a small piece of paper that can be moved below the graph of the first. The overlapping non-zero areas of each function are multiplied by each other to create a third function. The area under the curve of the third function is the value of the convolution at that point. Thus as the area under overlapping part of the curves of both functions increases, the convolution is necessarily increasing. As the area is decreasing the convolution is likewise decreasing. Where there is zero overlap, the convolution is zero.

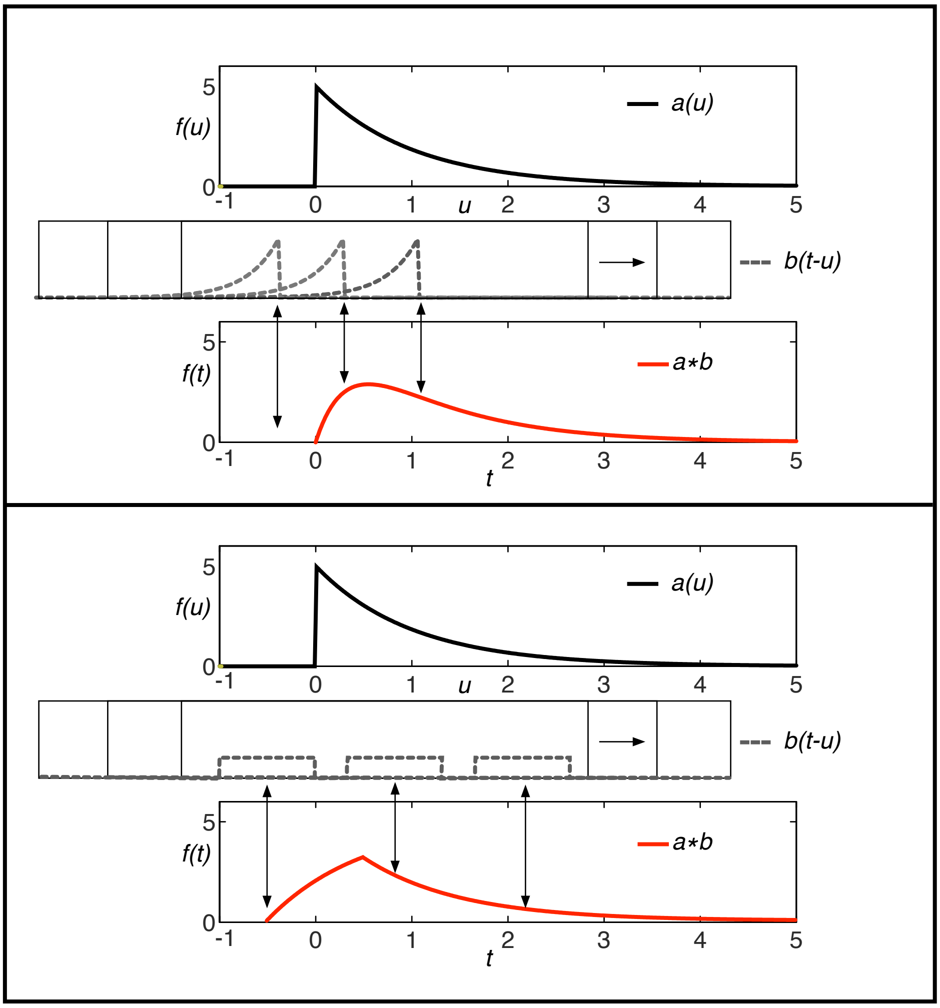


FIGURE 2. Showing how the convolution of two functions may at times be obtained qualitatively by hand by moving a piece of paper with a graph of the second function reversed below a graph of the first function. Upper: The convolution $a * b$ from the above example. Lower: The convolution $a * \Pi_1$ with a from the above example.

A brief inspection of the convolution graph (lower graphs of figures 1 and 2) shows that it has some resemblance to the original functions $a(t)$ and $b(t)$. In the case of the two exponential decay functions it looks like a smoother and smeared out version of $a(t)$. The convolution can be used very simply to produce a running windowed average of a given function. Convolving a function $f(t)$ with the top hat function would result in a function, $c(t) = f(t) * \Pi_a(t)$ that is the running average of $f(t)$ sampled over the width, a, of the top hat function being used. But as we have already seen, the convolution allows for more sophisticated "running averages" to be generated as the functions can take on any form as long as they can be integrated so what is the meaning or utility of this? We will revisit this in the second half of the lecture where the use of the convolution theorem in Fourier analysis shows just how powerful this tool can be.

In practice, as with the Fourier transform, performing the convolution on functions involves keeping various tricks of calculus in mind. In particular it is important to know how to redefine the boundaries of integration over the area in which the integral is non-zero. Additionally, the operation involves the product of two functions and thus integration by parts is often helpful. I will show another example, but first a handful of properties result from the definition of the convolution:

$$\begin{aligned} a * b &= b * a \\ (\alpha a) * (\beta b) &= \alpha\beta(a * b) \\ (a_1 + a_2) * b &= (a_1 * b) + (a_2 * b) \end{aligned}$$

where a and b are functions and α and β are scalar constants.

Additionally, from the definition of the delta function we can obtain the following results:

$$\begin{aligned} f(t) * \delta(t) &= f(t) \\ f(t) * \delta(t - t_0) &= f(t - t_0) \\ f(t - t_0) * \delta(t) &= f(t - t_0) \end{aligned}$$

EXAMPLE[2]: Show that $f*g = g*f$ for the functions $f(t) = tH(t)$ and $g(t) = H(t)\sin 2t$. Integration by parts is used.

$$\begin{aligned} f * g &= \int_{-\infty}^{\infty} uH(u)H(t-u)\sin 2(t-u)du \\ &= \int_0^t u \sin 2(t-u)du \\ &= \left[\frac{u\cos 2(t-u)}{2} - \int \frac{\cos 2(t-u)}{2} du\right]_0^t \\ &= \left[\frac{u\cos 2(t-u)}{2} + \frac{\sin 2(t-u)}{4}\right]_0^t \\ &= \frac{t}{2} - \frac{\sin 2t}{4} \end{aligned}$$

[2]Glyn James Example 5.58

Performing the convolution in the other direction

$$
\begin{aligned}
g * f &= \int_{-\infty}^{\infty} H(u)\sin 2u(t-u)H(t-u)du \\
&= \int_0^t (t-u)\sin 2u\, du \\
&= \left[-\frac{1}{2}(\cos 2u)(t-u) - \int \frac{1}{2}\cos 2u\, du\right]_0^t \\
&= \left[-\frac{1}{2}(\cos 2u)(t-u) - \frac{1}{4}\sin 2u\, du\right]_0^t \\
&= -\frac{1}{4}\sin 2t + \frac{1}{2}t \\
&= \frac{t}{2} - \frac{\sin 2t}{4}
\end{aligned}
$$

CHAPTER 8

The convolution theorem and Fourier analysis

The importance of convolution arises directly from its relationship to the Fourier transform and specifically the convolution theorem. In fact this theorem is so important that it could be said that much of the importance of Fourier analysis itself is due to its relationship with convolution. This theorem states that the Fourier transform of the convolution of two functions is given by the product of the Fourier transforms of those functions individually,

$$\mathcal{F}[f_1(t) * f_2(t)] = F_1(\omega)F_2(\omega).$$

In the inverse direction, the theorem states that the Fourier transform of the product of two functions is given by the convolution of the Fourier transform of those functions individually,

$$\mathcal{F}[f_1(t)f_2(t)] = \frac{1}{2\pi}F_1(\omega) * F_2(\omega).$$

EXAMPLE: Find the inverse transform of

$$\begin{aligned} F(\omega) &= \frac{1}{(a+i\omega)^2} \\ &= \frac{1}{(a+i\omega)}\frac{1}{(a+i\omega)} \\ &= F_1(\omega)F_1(\omega) \end{aligned}$$

but from previous work we know that

$$F_1(\omega) = \frac{1}{(a+i\omega)} \iff f_1(t) = u(t)e^{-at}$$

where $u(t)$ is the unit step or Heaviside function.

Thus

$$
\begin{aligned}
\mathcal{F}^{-1}[F(\omega)] &= \mathcal{F}^{-1}[F_1(\omega)F_1(\omega)] \\
&= f_1(t) * f_1(t) \\
&= \int_{-\infty}^{\infty} u(\tau)e^{-a\tau}u(t-\tau)e^{-a(t-\tau)}d\tau \\
&= e^{-at}\int_{-\infty}^{\infty} u(\tau)e^{-a\tau}u(t-\tau)e^{a\tau}d\tau \\
&= e^{-at}\int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau
\end{aligned}
$$

$u(t)u(t-\tau) = 0$ when $\tau < 0$ and $\tau - t < 0$, i.e. when $0 < \tau < t$, and $u(t)u(t-\tau) = 1$ otherwise. So the integration can be performed from 0 to t.

$$
\begin{aligned}
&= e^{-at}\int_0^t d\tau \\
&= te^{-at}
\end{aligned}
$$

So,

$$
f(t) = \begin{cases} te^{-at} & t > 0 \\ 0 & t < 0 \end{cases}
$$

which is the same as

$$
f(t) = u(t)te^{-at}
$$

EXAMPLE: Fourier transform of a truncated sinusoid

We can represent a truncated cosine wave of duration T, as a cosine of infinite duration multiplied by a rectangular pulse of width T.

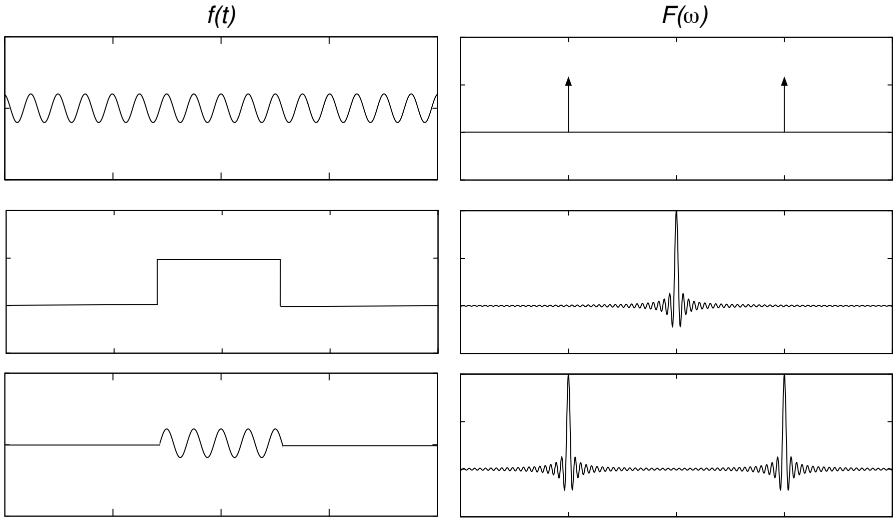
The function $T\Pi_T(t)$ a rectangular pulse of unit height and width T and the Fourier transform is given by $T\text{sinc}(\frac{1}{2}T\omega)$. The Fourier transform of a cosine wave of infinite duration, and frequency ω_0, is given by $\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$. Thus the Fourier transform of a finite duration cosine wave is given by

$$
\mathcal{F}\left\{T\Pi_T(t)\cos\omega_0 t\right\} \iff T\text{sinc}\frac{\omega T}{2} * \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]
$$

By the properties of convolution with the delta function, this becomes

$$= \pi T \left[\mathrm{sinc} \frac{T}{2}(\omega - \omega_0) + \mathrm{sinc} \frac{T}{2}(\omega + \omega_0) \right]$$

In other words the effect of truncating a sinusoid is to broaden its frequency spectrum. This is another example of reciprocal broadening.



APPENDIX A

Trigonometric identities

Below are some trigonometric identities that may be useful in performing integrations.

$$
\begin{aligned}
\sin 2\theta &= 2 \sin\theta \cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
&= 2\cos^2\theta - 1 \\
\sin^2\theta &= \frac{1}{2}(1 - \cos 2\theta) \\
\cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta)
\end{aligned}
$$

APPENDIX B

Identities using ∇

If Ω and Ψ are scalar fields, and $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then the following identities also hold

$$
\begin{gathered}
\nabla\cdot(\nabla\times\mathbf{F})\equiv 0\\
\nabla(\Omega\Psi)\equiv\Omega\nabla\Psi+\Psi\nabla\Omega\\
\nabla\cdot(\Omega\mathbf{F})\equiv\Omega\nabla\cdot\mathbf{F}+\mathbf{F}\cdot\nabla\Omega\\
\nabla\times(\Omega\mathbf{F})\equiv\Omega\nabla\times\mathbf{F}-\mathbf{F}\times\nabla\Omega\\
\nabla(\mathbf{F}\cdot\mathbf{G})\equiv\mathbf{F}\times(\nabla\times\mathbf{G})+\mathbf{G}\times(\nabla\times\mathbf{F})+(\mathbf{F}\cdot\nabla)\mathbf{G}+(\mathbf{G}\cdot\nabla)\mathbf{F}\\
\nabla\cdot(\mathbf{F}\times\mathbf{G})\equiv\mathbf{G}\cdot(\nabla\times\mathbf{F})-\mathbf{F}\cdot(\nabla\times\mathbf{G})\\
\nabla\times(\mathbf{F}\times\mathbf{G})\equiv\mathbf{F}(\nabla\cdot\mathbf{G})-\mathbf{G}(\nabla\cdot\mathbf{F})+(\mathbf{G}\cdot\nabla)\mathbf{F}-(\mathbf{F}\cdot\nabla)\mathbf{G}\\
\nabla^2\Omega\equiv\nabla\cdot(\nabla\Omega)\\
\nabla^2\mathbf{F}\equiv\nabla(\nabla\cdot\mathbf{F})-\nabla\times(\nabla\times\mathbf{F})
\end{gathered}
$$

Note that $(\mathbf{F}\cdot\nabla)$ is a new vector operator defined by

$$
(\mathbf{F}\cdot\nabla)\equiv\left(\mathbf{f_x}\frac{\partial}{\partial\mathbf{x}}+\mathbf{f_y}\frac{\partial}{\partial\mathbf{y}}+\mathbf{f_z}\frac{\partial}{\partial\mathbf{z}}\right)
$$

APPENDIX C

Polar Coordinate Systems

1. Cylindrical polar coordinates

In this coordinate system z remains as before, by x and y are replaced by R, the distance from the z-axis, and ϕ, the angle between the xz-plane and the plane containing the z-axis and the position vector.

$$\begin{aligned}
\mathrm{R} &= \sqrt{x^2 + y^2} \\
\phi &= \tan^{-1}(y/x) \\
z &= z \\
x &= \mathrm{R}\cos\phi \\
y &= \mathrm{R}\sin\phi
\end{aligned}$$

Cylindrical polar coordinates are useful in problems which have rotational symmetry.

2. Spherical polar coordinates

In this coordinate system r gives the distance of the point from the origin, θ is co-latitude and ϕ is longitude

$$\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \cos^{-1}(z/r) \\
\phi &= \tan^{-1}(y/x) \\
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}$$

Spherical polar coordinates are useful in problems with radial symmetry, and are especially useful in whole-Earth problems.

3. Useful expressions

Expressions for important quantities are different in the various coordinate systems:

3.1. Basis vectors. The unit vectors parallel to the coordinate axes are

(1) Rectangular Cartesian coordinates

$$\begin{aligned}\mathbf{e_x} &= \mathbf{i}\\ \mathbf{e_y} &= \mathbf{j}\\ \mathbf{e_z} &= \mathbf{k}\end{aligned}$$

(2) Cylindrical polar coordinates

$$\begin{aligned}\mathbf{e_R} &= \mathbf{i}\cos\phi + \mathbf{j}\sin\phi\\ \mathbf{e}_\phi &= -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi\\ \mathbf{e_z} &= \mathbf{k}\end{aligned}$$

(3) Spherical polar coordinates

$$\begin{aligned}\mathbf{e_r} &= \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta\\ \mathbf{e}_\theta &= \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta\\ \mathbf{e}_\phi &= -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi\end{aligned}$$

3.2. Volume. An element of volume dV is given by

(1) Rectangular Cartesian coordinates

$$dV = dx\,dy\,dz$$

(2) Cylindrical polar coordinates

$$dV = \mathrm{R}\,d\mathrm{R}\,d\phi\,dz$$

(3) Spherical polar coordinates

$$dV = r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

3.3. Arc length. An element of arc length ds is given by

(1) Rectangular Cartesian coordinates

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

(2) Cylindrical polar coordinates

$$ds = \sqrt{d\mathrm{R}^2 + \mathrm{R}^2 d\phi^2 + dz^2}$$

(3) Spherical polar coordinates

$$ds = \sqrt{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2}$$

3.4. Vector arc length. The vector element of arc length, $d\mathbf{r} \equiv \hat{\mathbf{t}} ds$ is given by

(1) Rectangular Cartesian coordinates

$$d\mathbf{r} = \mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz$$

(2) Cylindrical polar coordinates

$$d\mathbf{r} = \mathbf{e}_R\, d\mathrm{R} + \mathrm{R}\mathbf{e}_\phi\, d\phi + \mathbf{k}\, dz$$

(3) Spherical polar coordinates

$$d\mathbf{r} = \mathbf{e}_R\, dr + r\mathbf{e}_\theta\, d\theta + r\sin\theta\, \mathbf{e}_\phi\, d\phi$$

3.5. Grad. The gradient of a scalar field Ω is given by

(1) Rectangular Cartesian coordinates

$$\nabla\Omega = \frac{\partial\Omega}{\partial x}\mathbf{i} + \frac{\partial\Omega}{\partial y}\mathbf{j} + \frac{\partial\Omega}{\partial z}\mathbf{k}$$

(2) Cylindrical polar coordinates

$$\nabla\Omega = \frac{\partial\Omega}{\partial \mathrm{R}}\mathbf{e}_\mathrm{R} + \frac{1}{\mathrm{R}}\frac{\partial\Omega}{\partial \phi}\mathbf{e}_\phi + \frac{\partial\Omega}{\partial z}\mathbf{k}$$

(3) Spherical polar coordinates

$$\nabla\Omega = \frac{\partial\Omega}{\partial \mathrm{r}}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Omega}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Omega}{\partial \phi}\mathbf{e}_\phi$$

3.6. Div. The divergence of a vector field $\mathbf{F}$ is given by

(1) Rectangular Cartesian coordinates

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

(2) Cylindrical polar coordinates

$$\nabla\cdot\mathbf{F} = \frac{1}{\mathrm{R}}\frac{\partial}{\partial \mathrm{R}}(\mathrm{R}F_\mathrm{R}) + \frac{1}{\mathrm{R}}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

(3) Spherical polar coordinates

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

3.7. Curl. The curls of a vector field $\mathbf{F}$ is given by

(1) Rectangular Cartesian coordinates

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

(2) Cylindrical polar coordinates

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_{\mathrm{R}} & \mathrm{R}\mathbf{e}_\phi & \mathbf{k} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_R & \mathrm{R}F_\phi & F_z \end{vmatrix}$$

(3) Spherical polar coordinates

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_{\mathrm{r}} & \mathrm{r}\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

3.8. The Laplacian. The Laplacian operator can be written as

(1) Rectangular Cartesian coordinates

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

(2) Cylindrical polar coordinates

$$\nabla^2 \equiv \frac{1}{\mathrm{R}}\frac{\partial}{\partial \mathrm{R}}(\mathrm{R}\frac{\partial}{\partial \mathrm{R}}) + \frac{1}{\mathrm{R}^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

(3) Spherical polar coordinates

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial \mathrm{r}}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial}{\partial \theta}) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial \phi^2}$$

APPENDIX F

Tricks of integration and complex numbers

1. Integration

Integrating the product of sine and cosine terms where one of the terms is a higher power:

$$\int \sin^3\theta \cos\theta \; d\theta = (1/4)\sin^4\theta$$
$$\int \sin^2\theta \cos\theta \; d\theta = (1/3)\sin^3\theta$$

Integrating a sinusoidal term squared over a full period, T:

$$\int_0^T \sin^2(\frac{2\pi}{T}\theta)d\theta = T/2$$
$$\int_0^T \cos^2(\frac{2\pi}{\theta}) \; d\theta = T/2.$$

If the period $T = 2\pi$,

$$\int_0^{2\pi} \sin^2\theta \; d\theta = \pi$$
$$\int_0^{2\pi} \cos^2\theta \; d\theta = \pi.$$

The integration does not have start at 0 and end at T, it could be any range that is a full period, e.g. $-T/2 < \theta < T/2$.

Integration by parts:

$$\int u \; dv = uv - \int v \; du.$$

This is generally useful when you can express the function $u \; dv$ as the product of two functions, with u being a function that simplifies as it is differentiated and v being a function that is straightforward to integrate.

Another time when integration by parts is useful is when the derivatives and the integrals of the constituent functions u and v are left unchanged after one or two integrations or derivatives. In particular $\frac{d}{dt}e^t = e^t$ and $\frac{d^2}{dt^2}\cos t = -\cos t$. In this case a clever trick can be used with integration by parts. I will illustrate with an example:

$$\int e^t \cos t \, dt$$
$$= e^t \sin t - \int e^t \sin t \, dt$$

Performing integration by parts again,

$$= e^t \sin t + e^t \cos t - \int e^t \cos t \, dt.$$

But now the right hand term is the original integral that we were after. We can add this to both sides to obtain:

$$2 \int e^t \cos t \, dt = e^t \sin t + e^t \cos t$$

And thus the integral is given by

$$\int e^t \cos t \, dt = \frac{1}{2}(e^t \sin t + e^t \cos t).$$

2. Complex numbers

Fourier transforms create functions in the complex domain. Recall

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$\frac{1}{i} = -i$$
$$\frac{1}{a+ib} = \frac{1}{(a+ib)}\frac{(a-ib)}{(a-ib)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

The expression of sine and cosine functions in terms of complex exponentials is key to Fourier analysis.

$$\cos t = \frac{1}{2}(e^{it} + e^{-it})$$
$$\sin t = \frac{1}{2i}(e^{it} - e^{-it})$$