Smoothed Gradient Clipping and Error Feedback for Decentralized Optimization under Symmetric Heavy-Tailed Noise

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Robustness at the Intersection of Optimization, Statistics, and Machine Learning

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Distributed systems

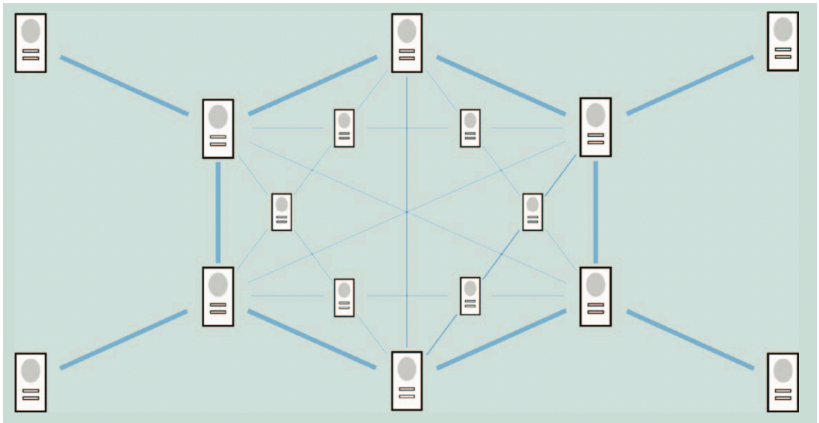


- Cyber-physical systems: power grids, sensor networks.
- Cloud centered devices: smartphones, wearable devices.
- Autonomous vehicle systems: sensors, actuators, multi-vehicle coordination.
- ...
- Goal: robust information processing over distributed systems.

Two distributed schemes

Server/client model

Server coordinates the *global* and *local* information exchange

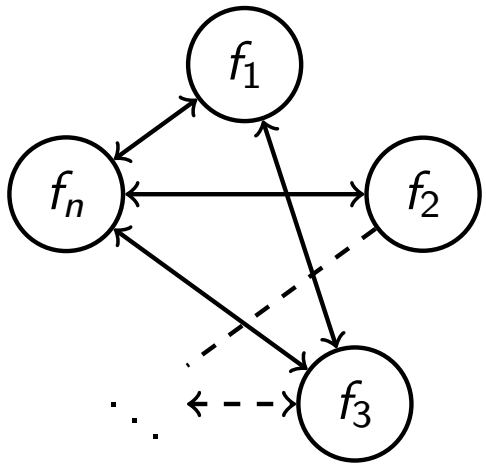


Decentralized model

Agents exchange *local* information with direct neighbors over a graph

- Data are distributed over multiple agents due to privacy and scalability.
- We focus on the more general decentralized model.
  - Flexible: no central server is required.
  - Less communication: communication with neighbors only.

Decentralized optimization



- Consider a network of agents $i = 1, \ldots, n$.
- Agent i holds a local data distribution $\mathcal{D}_i$, on which we define

$$f_i(\boldsymbol{x}) := \mathbb{E}_{\boldsymbol{\xi}_i \sim \mathcal{D}_i} \ell(\boldsymbol{x}, \boldsymbol{\xi}_i).$$

  for some loss function ℓ. Examples include: least-squares, logistic-regression, neural networks.
- Agents communicate over a graph to minimize $f(\boldsymbol{x}) := \frac{1}{n} \sum_{i=1}^{n} f_i(\boldsymbol{x})$.

Decentralized SGD

A basic decentralized gradient method:

- Agent i holds a local decision variable $\boldsymbol{x}_i$;
- Agent i computes a stochastic gradient with random noise $\boldsymbol{\xi}_i$,

$$g_i(\boldsymbol{x}_i) = \nabla f_i(\boldsymbol{x}_i) + \boldsymbol{\xi}_i;$$

- Agent i employs some weight w_{ij}, $w_{ij} > 0$ if agent j is the neighbor of agent i;
- Decentralized stochastic gradient descent (SGD), for some stepsize η,

$$\boldsymbol{x}_i^+ \leftarrow \sum_{j=1}^{n} w_{ij} \left[\boldsymbol{x}_i - \eta(\nabla f_i(\boldsymbol{x}_i) + \boldsymbol{\xi}_i)\right].$$

- $\mathbb{E}[\boldsymbol{\xi}_i|\boldsymbol{x}_i] = \mathbf{0}$ and $\mathbb{E}[\|\boldsymbol{\xi}_i\|^2|\boldsymbol{x}_i] \leq \sigma^2$ for some $\sigma > 0$.

Q: $\boldsymbol{\xi}_i$ is typically assumed to be light-tailed, what if it is heavy-tailed?

Heavy-tailed noise

- Deep learning training may involve heavy-tailed gradient noise (Simsekli, Sagun, and Gurbuzbalaban 2019)

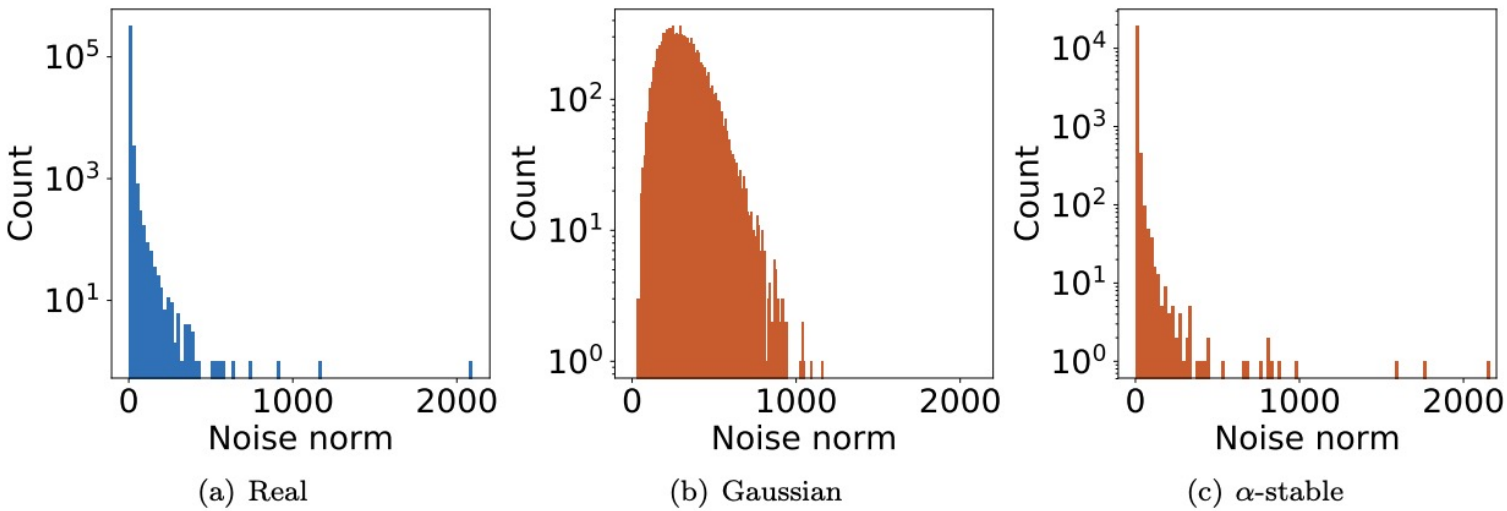
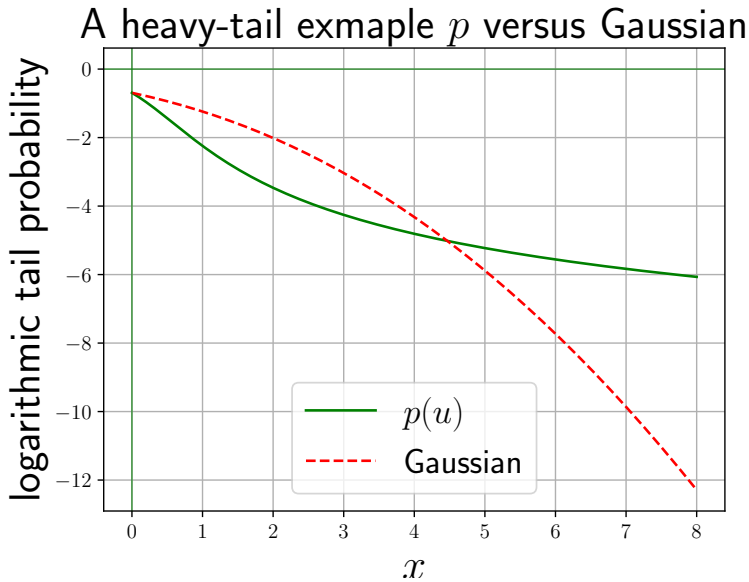


Figure: (a) The histogram of the norm of the gradient noises computed with AlexNet on CIFAR10. (b) and the histograms of the norms of (scaled) Gaussian and α-stable random variables.

Heavy-tailed noise

- A random variable X is heavy-tailed if $\forall a > 0$, $\limsup_{x \to \infty} \mathbb{P}(X > x) e^{ax} = \infty$.



- Alternatively, a random variable $\boldsymbol{\xi}$ is called heavy-tailed if it satisfies the *bounded moment condition (BM)*: $\mathbb{E}\|\boldsymbol{\xi}\|^{\alpha} \leq \sigma^{\alpha}$ for $\alpha < 2$ and $\sigma > 0$.
- In non-distributed case, vanilla SGD, i.e., $\boldsymbol{x}^{+} \leftarrow \boldsymbol{x} - \eta g(\boldsymbol{x})$, diverges under heavy-tailed noise (Zhang et al. 2020).

Non-distributed case

- Clipped SGD with step η, ℓ_2-norm threshold τ, stochastic gradient $g(\boldsymbol{x})$:

$$\boldsymbol{x}^+ \leftarrow \boldsymbol{x} - \eta \operatorname{clip}_\tau(g(\boldsymbol{x})), \ \operatorname{clip}_\tau(g(\boldsymbol{x})) = \min\left\{\frac{\tau}{\|g(\boldsymbol{x})\|}, 1\right\}$$

- Gradient clipping is effective in handling heavy-tailed gradient noise (Zhang et al. 2020).
- More general nonlinearities Ψ are effective in handling heavy-tailed noise (Jakovetić et al. 2023)

$$\boldsymbol{x}^+ \leftarrow \boldsymbol{x} - \eta \Psi\big(g(\boldsymbol{x})\big).$$

  Examples of Ψ include:
  - gradient clipping (Nguyen et al. 2023),
  - component-wise clipping (Zhang et al. 2020),
  - component-wise quantization,
  - sign,
  - normalized gradient (Hübler, Fatkhullin, and He 2024).

Fixed point issue

Q: How can we extend the gradient clipping to the distributed case?

- A direct extension of gradient clipping to server/client case: $\boldsymbol{x}^+ \leftarrow \boldsymbol{x} - \frac{\eta}{n}\sum_{i=1}^n \mathrm{clip}_\tau(g_i(\boldsymbol{x}))$.
- Consider minimizing $f(x) = \frac{1}{3}\sum_{i=1}^3 f_i(x), f_i(x) = \frac{1}{2}(x - a_i)^2$ where $a_1 = a_2 = -2, a_3 = 7$. Then $x^* = 1$, but for $\tau = 1$,

$$\frac{1}{3}\sum_{i=1}^3 \mathrm{clip}_\tau(f_i'(1)) = \frac{1}{3}.$$

  The global minimizer $x^* = 1$ is not a fixed point.
- Consider minimizing $f(x) = \frac{1}{2}\sum_{i=1}^2 f_i(x), f_i(x) = \frac{1}{2}(x - a_i)^2$ where $a_1 = 2, a_2 = -2$. Then $x^* = 0$, but for $\tau = 1$,

$$\frac{1}{2}\sum_{i=1}^2 \mathrm{clip}_\tau(f_i'(1)) = 0.$$

  A Non-minimizer $x = 1$ is a fixed point.

Distributed methods

Comparisons with existing distributed methods robust to heavy-tailed noise.

- Distributed setup: SC (sever/client), D (decentralized network).
- Gradient and noise condition: UG (unbounded gradient).
- c_s is a constant independent of moment index α.

	SC?	D?	UG?	α	Rate	Cvg
Yang, Qiu, and Liu 2022	✓	✗	✗	$\alpha > 1$	$O(1/t^{2-2/\alpha})$	MSE
gorbunov2023high	✓	✗	✓	$\alpha > 1$	$O(1/t^{2-2/\alpha})$	h.p.
Sun 2023	✓	✓	✗	$\alpha > 1$	no rate	a.s.
Ours	✓	✓	✓	$\alpha \geq 1$	$\mathcal{O}(1/t^{\min(c_s,\ 2/5)})$	MSE

Our method

Q: How to deal with heavy-tailed noise in decentralized gradient method?

SClip-EF-Network

Require: $\varphi_t, \epsilon_t, \beta_t, \eta_t; \boldsymbol{x}_i^0 = \boldsymbol{x}_j^0,\ \forall i, j$; component-wise: $\boldsymbol{\Psi}_t(\boldsymbol{x}) = \frac{\varphi_t}{\sqrt{\boldsymbol{x}^2+\epsilon_t}}$.

1: $\boldsymbol{m}_i^0 \leftarrow \boldsymbol{0},\ \forall i = 1, \ldots, n$
2: **for** $t = 0, \ldots, T-1$ **do**
3: **for** node $i = 1, \ldots, n$ in parallel **do**
4: $\boldsymbol{m}_i^{t+1} \leftarrow \beta_t \boldsymbol{m}_i^t + (1-\beta_t)\boldsymbol{\Psi}_t(g_i(\boldsymbol{x}^t) - \boldsymbol{m}_i^t)$
5: Send $\boldsymbol{x}_i^t - \eta_t \boldsymbol{m}_i^{t+1}$ to all neighbors of agent i
6: $\boldsymbol{x}_i^{t+1} \leftarrow \sum_{j=1}^n w_{ij}(\boldsymbol{x}_j^t - \eta_t \boldsymbol{m}_j^{t+1})$
7: **end for**
8: **end for**

Main ideas

- Local gradient estimator $\boldsymbol{m}_i^t$:

$$\boldsymbol{m}_i^{t+1} \leftarrow \beta_t \boldsymbol{m}_i^t + (1-\beta_t)\boldsymbol{\Psi}_t(g_i(\boldsymbol{x}^t) - \boldsymbol{m}_i^t).$$

- Use estimation error as a feedback to compensate the current estimate.
- Apply $\boldsymbol{\Psi}_t$ on the estimation error, as $g_i(\boldsymbol{x}^t) - \boldsymbol{m}_i^t \to \boldsymbol{0}$.
- Use smooth clipping operator to avoid discontinuity/non-differentiability.

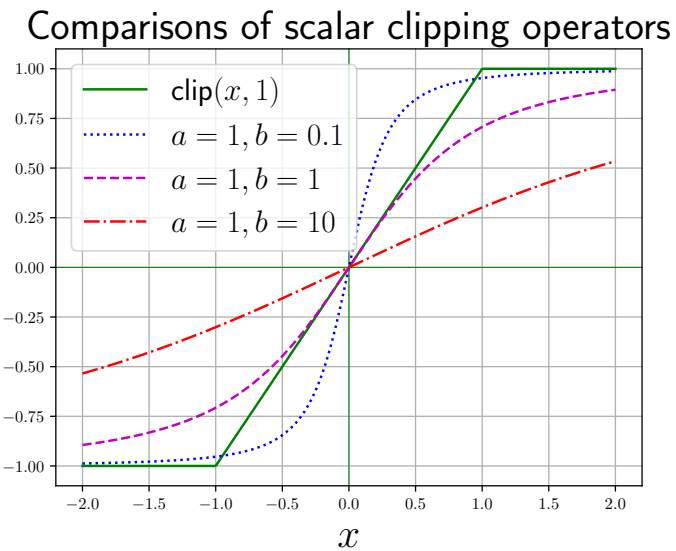


Figure: An illustration of $\Psi(x) = \frac{ax}{\sqrt{x^2+b}}$

Problem model

Functions:

- $\forall i,\ \mu \boldsymbol{I} \preceq \nabla^2 f_i(\boldsymbol{x}) \preceq L\boldsymbol{I}$ for some constants $L \geq \mu > 0$.
- Heterogeneity: Let $\boldsymbol{x}^* := \arg\min_{\boldsymbol{x}\in\mathbb{R}^d} f(\boldsymbol{x})$. $\|\nabla f_i(\boldsymbol{x}^*)\|_\infty \leq c_* < \infty$.

Noise:

- $\{\{\boldsymbol{\xi}_i^t\}_{i\in[n]}\}_{t\geq 0}$ are independently distributed over all iterations $t \geq 0$. Every component of $\boldsymbol{\xi}_i^t$ has marginal density p.
  - p is symmetric: $\forall u \in \mathbb{R}, p(u) = p(-u)$.
  - p has bounded first absolute moment: $\int_{-\infty}^{\infty} |u|p(u)du \leq \sigma$ for some $\sigma > 0$.

Network:

- Directed and strongly connected network that admits a non-negative and doubly stochastic weight matrix $\boldsymbol{W} = [w_{ij}]$.

Main result

Theorem (Yu, Jakovetic, and Kar 2023)

Let the above assumptions hold and $\kappa := L/\mu$. Take clipping parameters φ_t, ϵ_t, error feedback wight β_t, and step size η_t as

$$\varphi_t = \frac{c_\varphi}{\sqrt{t+1}},\ \epsilon_t = \tau(t+1)^{3/5},\ \beta_t = \frac{c_\beta}{\sqrt{t+1}},\ \eta_t = \frac{c_\eta}{(t+1)^{1/5}}, \tag{1}$$

where constants $c_\varphi, \tau, c_\beta, c_\eta$ satisfy that $c_\eta^2 c_\varphi^2 = \Theta(\sigma^2/\mu^2)$, $0 \le c_\beta < 1$, and $\tau = \Theta(d\sigma^2\kappa^{5/2})$. Then, there exists some constant $c_s = \Omega\big(d^{-1/2}\kappa^{-5/4}\big)$ such that, for each $i \in [n]$, the iterates $\{\boldsymbol{x}_i^t\}$ satisfy that for any $0 < \delta < (c_s, 2/5)$, $\lim_{t\to\infty}(t+1)^\delta \mathbb{E}\big[\|\boldsymbol{x}_i^t - \boldsymbol{x}^*\|^2\big] = 0$.

Corollary

Let $\boldsymbol{W} = (1/n)\mathbf{1}\mathbf{1}^\top$, and $\boldsymbol{x}^t = \boldsymbol{x}_i^t, \forall i \in [n]$, then the update of $\boldsymbol{x}^t$ specializes into the server/client setup and the above result applies.

Key idea

- Linearize the nonlinear operator:

$$\boldsymbol{\Phi}_t(\nabla f_i(\boldsymbol{x}_i^t) - \boldsymbol{m}_i^t) = \boldsymbol{H}_{\Phi,i}^t(\nabla f_i(\boldsymbol{x}_i^t) - \boldsymbol{m}_i^t).$$

- Let $\zeta_{k,i}^t := [\nabla f_i(\boldsymbol{x}_i^t) - \boldsymbol{m}_i^t]_k$. The k-th diagonal $[\boldsymbol{H}_{\Phi,i}^t]_{kk}$ is

$$\int_{-\infty}^{\infty} \frac{c_\varphi}{\sqrt{t+1}} \frac{p(u)}{\sqrt{\left(\zeta_{k,i}^t + u\right)^2 + \epsilon_t}} du + \frac{1}{\zeta_{k,i}^t} \int_{-\infty}^{\infty} \frac{c_\varphi}{\sqrt{t+1}} \frac{up(u)}{\sqrt{\left(\zeta_{k,i}^t + u\right)^2 + \epsilon_t}} du.$$

- All the diagonal entries of $\boldsymbol{H}_{\Phi,i}^t$ fall into the interval $\left[c_1/(t+1)^{4/5}, c_2/(t+1)^{4/5}\right]$ for sufficiently large t and $c_1 < c_2$.

Discussions

Contributions:

- As a decentralized heavy-tailed noise robust method, in contrast to the prior art (Sun and Chen 2024):
  - Relax the bounded gradient condition $\|\nabla f_i(\boldsymbol{x}_i^t)\| \leq C_0$ for some $C_0 > 0$.
  - Allow any moment bound $\alpha \geq 1$.
  - Provide convergence rate analysis.
- For distributed gradient clipping in general, we provide the first MSE convergence method without assuming bounded gradient or bounded noise (Gorbunov et al. 2024).

Limitations and future directions:

- Noise symmetry may not hold in general.
- Convergence for non-convex functions.

Symmetric noise

- Empirical resemblance between layer norm and stable α-distribution (Barsbey et al. 2021).

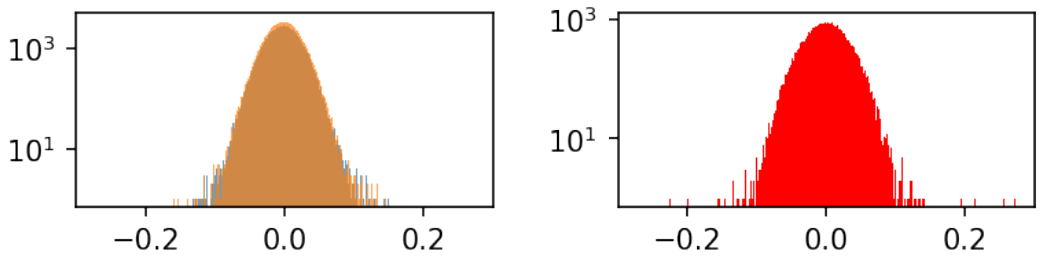
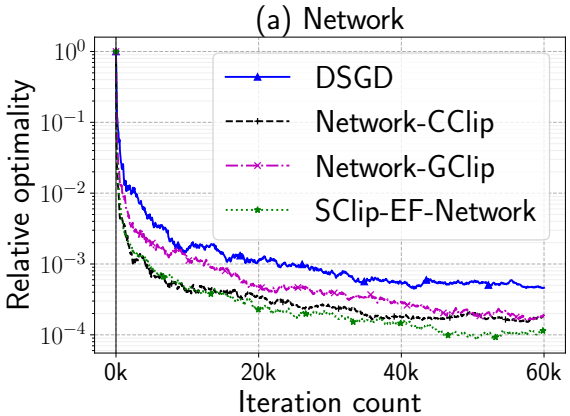


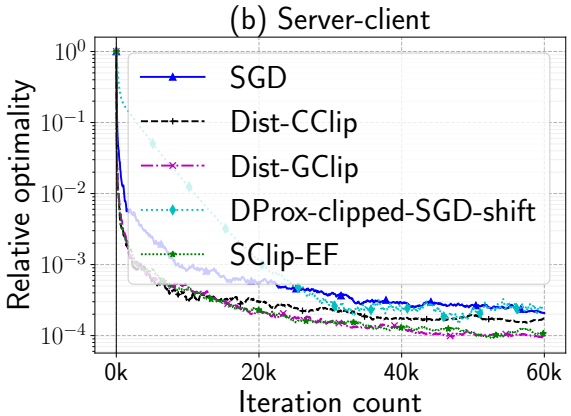
Figure: Empirical distribution of a CNN layer trained on MNIST. (Left) Overlaid histograms of a random partition of the weights, showing an identical distribution. (Right) Comparing the network weights to samples simulated i.i.d. from a symmetric α-stable distribution with the same tail index $\alpha \approx 1.95$.

- Theoretical evidence: gradient noise converges to a heavy-tailed α-stable random variable using generalized CLT (Simsekli, Sagun, and Gurbuzbalaban 2019).

Experiments

- Function: $\min_{\boldsymbol{x}\in\mathbb{R}^d} \sum_{i=1}^n \left(\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i^\top \boldsymbol{x}\right)$.
- Noise: $p(u) = \frac{c_p}{(u^2+2)\ln^2(u^2+2)}$ for normalization constant c_p.





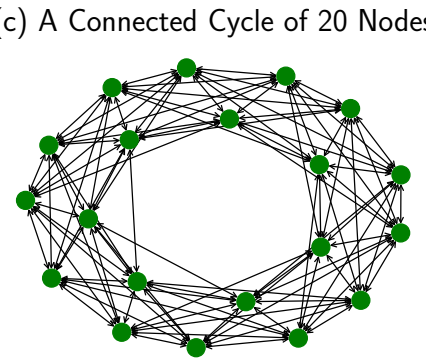


Figure: Average relative optimality $\log_{10}(f(\boldsymbol{x}^t) - f(\boldsymbol{x}^*))/(f(\boldsymbol{x}^0) - f(\boldsymbol{x}^*))$ out of 10 runs in network and server-client cases, and network graph, from left to right.

Thanks

- Shuhua Yu, Dusan Jakovetic, and Soummya Kar. "Smoothed Gradient Clipping and Error Feedback for Distributed Optimization under Heavy-Tailed Noise." arXiv preprint arXiv:2310.16920 (2023).
- Aleksandar Armacki, Shuhua Yu, Sharma, Pranay Gauri Joshi, Dragana Bajovic, Dusan Jakovetic, Soummya Kar "Nonlinear Stochastic Gradient Descent and Heavy-tailed Noise: A Unified Framework and High-probability Guarantees." arXiv preprint arXiv:2410.13954 (2024).