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A new analytical formula for the wave equations with variable coefficients☆

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ABSTRACT

This article presents a new analytical formula for the Cauchy problem of the wave equation with variable coefficients, which is a much simpler solution than that given by the Poisson formula. The derivation is based on the variation-of-constants formula and the theory of pseudodifferential operator. The formula is applied to an example to illustrate the feasibility.

1. Introduction

The study of partial differential equations can date back to the 18th century in the context of the movements of models in the physics of continuous media [1]. The history of partial differential equations began with the one-dimensional wave equation $u_{tt} = u_{xx}$ introduced by d'Alembert as a model of a vibrating string [2]. Giving its initial displacement $u(x, 0) = \varphi(x)$ and velocity $u_t(x, 0) = \psi(x)$, it is known to all that an explicit solution to the Cauchy, or initial value, problem of the wave equation is

$$u(x,t) = \frac{1}{2}(\varphi(x-t) + \varphi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)d\xi.$$

For the Cauchy problem of the wave equation in higher dimensions, the derivation of the solution is very technical. Various methods have been utilized to solve the Cauchy problem for higher dimensional wave

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equation. For example, the Poisson formula for $n = 3$ is proved by the method of spherical means and the solution for $n = 2$ is derived by the Hadamard method of descent [3]. A formal series solution has been obtained separately using a classical power series in [4] and the Adomian decomposition method in [5]. A direct derivation of the spherical means solution for arbitrary dimension n is presented using Fourier transform in [6]. In [7–9], formulating the wave equation as an abstract ordinary differential equations and applying Duhamel principle, the authors gave the so-called operator–variation-of-constants formula for semilinear wave equations with constant coefficients under different types of boundary conditions. Motivated by [7,8], the use of variation-of-constants formula and the interpretation of differential operator with variable coefficients as pseudodifferential operator are the essential ingredients in this paper.

Consider the Cauchy problem for the following linear homogeneous wave equation with variable coefficients in $\mathbb{R}^n$

$$\begin{cases} u_{tt} - Lu = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $L = \sum_{i=1}^n \sum_{1=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$ is a second order linear differential operator with variable coefficients satisfying that the matrix of coefficients $A = (a_{ij}(x))_{n\times n}$ is symmetric and positive semi-definite for all $x \in \mathbb{R}^n$. It is noted that the form of wave equation (1) is nearly the same as the following oscillatory system of ordinary differential equations with the right-hand side $f \equiv 0$.

$$\begin{cases} q_{tt}(t) + Mq(t) = f(q(t)), \\ q(0) = q_0, q_t(0) = p_0, \end{cases} \tag{2}$$

where $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$ and M is a symmetric and positive semi-definite matrix.

In recent years, there has been an enormous advance in numerically solving the oscillatory system (2). Many useful approaches to constructing Runge–Kutta–Nyström (RKN)-type integrators have been proposed (see, e.g. [10–14]). Taking account of the special structure introduced by the linear term Mq, Wu et al. [14] formulated a standard form of the multidimensional extended RKN (ERKN) integrators. The ERKN integrators naturally integrate exactly the unperturbed linear equation $q_{tt} + Mq = 0$, which has the same form as that of the equation concerned in the paper. The following variation-of-constants formula is established in [13] which is essential for ERKN integrators. It gives the exact solution and its derivative for the oscillatory system (2):

$$\begin{cases} q(t) = \phi_0(t^2M)q_0 + t\phi_1(t^2M)p_0 + \displaystyle\int_0^t (t-s)\phi_1((t-s)^2M) f\big(q(s)\big) ds, \\ q_t(t) = -tM\phi_1(t^2M)q_0 + \phi_0(t^2M)p_0 + \displaystyle\int_0^t \phi_0((t-s)^2M) f\big(q(s)\big) ds, \end{cases} \tag{3}$$

where the analytical functions $\phi_j(\cdot), j = 0, 1, \dots$ are defined as

$$\phi_j(x) := \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{(2i+j)!}, \quad j = 0, 1, \dots. \tag{4}$$

For more details on the ERKN integrator, we refer the reader to [15–19].

Letting the right-hand-side function $f \equiv 0$, we have that the exact solution of the linear homogeneous ordinary differential equations

$$\begin{cases} q_{tt}(t) + Mq(t) = 0, \\ q(0) = q_0, q_t(0) = p_0 \end{cases} \tag{5}$$

is

$$q(t) = \phi_0(t^2M)q_0 + t\phi_1(t^2M)p_0. \tag{6}$$

In this paper, we will show that the linear homogeneous wave equation with variable coefficients (1) also admits solution of the form (6). The remainder of the paper is organized as follows. Some preliminaries and notations are introduced in Section 2. In Section 3, the main result of the paper are presented. An analytical formula for the linear wave equations with variable coefficients is derived. In Section 4, the feasibility of the formula is illustrated by an example.

2. Preliminaries and some notations

We will begin by introducing some notations. A multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_+^n$ is an n-tuple of nonnegative integers. The length $|\alpha|$ of α is $|\alpha| = \sum_{i=1}^n \alpha_i = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, the differential operator D^α is defined as

$$D^\alpha f(x) = \frac{1}{i^{|\alpha|}} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f(x).$$

With the notations above, the linear differential operator L could be written in the form

$$L \equiv L(x, D) = -\sum_{|\alpha|=2} a_\alpha(x) D^\alpha, \tag{7}$$

where $a_\alpha(x) = a_{ij}(x)$ if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i = \alpha_j = 1$ and $\alpha_k = 0, k \neq i, j$.

Denote the Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^n$ as

$$\mathcal{S}(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty, \forall \alpha, \beta \in \mathbb{Z}_+^n\}$$

with $\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f|$. The Fourier transform of a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and its inversion formula [20,21] are given by

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-ix\cdot\xi} dx \quad \text{and} \quad \varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi) e^{i\xi\cdot x} d\xi, \tag{8}$$

where $x \cdot \xi$ is the inner product of x and ξ in $\mathbb{R}^n$. By integral by parts, it follows immediately that

$$\widehat{D^\alpha \varphi}(\xi) = \xi^\alpha \hat{\varphi}(\xi) \quad \text{and} \quad D^\alpha \widehat{\varphi}(\xi) = \widehat{x^\alpha \varphi}(\xi). \tag{9}$$

Therefore, if $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{\varphi}(\xi) \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, if

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x - y)\psi(y) dy,$$

then

$$\widehat{(\varphi * \psi)}(\xi) = \widehat{\varphi}(\xi)\widehat{\psi}(\xi), \tag{10}$$

where $*$ denote the convolution operation.

According to Eqs. (9) and (10), we have, for $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$

$$L\varphi(x) = L(x, D)\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} L(x, \xi)\hat{\varphi}(\xi) e^{i\xi\cdot x} d\xi \tag{11}$$

where $L(x, \xi) = -\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha$. The right-hand side of Eq. (11) is also well defined for general functions $\phi(x, \xi)$, not only for polynomials. More precisely, if $\phi(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies

$$\left|D_\xi^\alpha D_x^\beta \phi(x, \xi)\right| \leqslant C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}, \quad \forall \alpha, \beta \in \mathbb{Z}_+^n \tag{12}$$

for some $m \in \mathbb{R}_+$, then we have the following definition [22].

Definition 2.1.

$$(\Phi\varphi)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \phi(x,\xi)\hat{\varphi}(\xi)e^{i\xi\cdot x}d\xi \tag{13}$$

is said to be pseudodifferential operator. The function $\phi(x,\xi)$ is called symbol of the pseudodifferential operator Φ.

3. Solution to the wave equation

Before presenting our main results, we first address some useful properties of the ϕ-functions (4). It can be verified that

$$\phi_0(x) = \cos(\sqrt{x}), \quad \phi_1(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}},$$

and the functions $\phi_j(x)$ $(j = 0,1,\ldots)$ have the following recurrence relations:

$$\phi_{j+2}(x) = x^{-1}\left(\frac{I}{j!} - \phi_j(x)\right), \; j = 0,1,\ldots.$$

It should be noted that the function $\phi_1(x)$ is also well defined at $x = 0$ since $\phi_1(x) \to 1$ as $x \to 0^+$.

With a tedious computation, the following equations can be obtained.

$$\begin{aligned}
&\frac{d}{dt}\phi_0(t^2) = -t\phi_1(t^2), \quad \frac{d}{dt}\left(t\phi_1(t^2)\right) = -\phi_0(t^2),\\
&\phi_0^2(x) + x\phi_1^2(x) = 1, \quad x\left(\phi_1(x)^2 - \phi_0(x)\phi_2(x)\right) = 1 - \phi_0(x),\\
&\phi_0(x) + x\phi_2(x) = 1, \quad \phi_1(x)^2 + x\phi_2(x)^2 = 2\phi_2(x).
\end{aligned} \tag{14}$$

Since it is assumed that the matrix $A = (a_{ij}(x))_{n\times n}$ is symmetric and positive semi-definite for all $x \in \mathbb{R}^n$, we have that the symbol of the pseudodifferential operator $-L$, $L(x,\xi) \geqslant 0$, for $\forall x,\xi \in \mathbb{R}^n$. Suppose that the initial data $\varphi,\psi \in \mathcal{S}(\mathbb{R}^n)$. With reasonable restrictions on the smoothness and growth of coefficients $a_{ij}(x), i,j = 1,\ldots,n$ such that (12) holds for $L(x,\xi)$, the following two pseudodifferential operators

$$\begin{aligned}
(\phi_0(-t^2L)\varphi)(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \phi_0(-t^2L(x,\xi))\hat{\varphi}(\xi)e^{i\xi\cdot x}d\xi,\\
(\phi_1(-t^2L)\varphi)(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \phi_1(-t^2L(x,\xi))\hat{\varphi}(\xi)e^{i\xi\cdot x}d\xi
\end{aligned} \tag{15}$$

are well defined for $\forall t \in \mathbb{R}_+$ and the symbols of the two pseudodifferential operators are $\phi_0(-t^2L(x,\xi))$ and $\phi_1(-t^2L(x,\xi))$, respectively.

Now, we are ready to give the analytical formula for the solution to the wave equation (1).

Theorem 3.1. *Assume that the initial data $\varphi,\psi \in \mathcal{S}(\mathbb{R}^n)$. Then the solution to the Cauchy problem for the wave equation (1) is given by*

$$u(x,t) = (\phi_0(-t^2L)\varphi)(x) + t(\phi_1(-t^2L)\psi)(x). \tag{16}$$

Proof. Keeping (14) in mind and differentiating both sides of (16) twice with respect to t gives

$$u_t(x,t) = t(L\phi_1(-t^2L)\varphi)(x) + (\phi_0(-t^2L)\psi)(x)$$

and

$$u_{tt}(x,t) = (L\phi_0(-t^2L)\varphi)(x) + t(L\phi_1(-t^2L)\psi)(x) = (Lu)(x,t).$$

The proof is completed.

Remark 3.1. Substituting the series expansions of ϕ_0 and ϕ_1 gives a series solution to the Cauchy problem of wave equation

$$u(x,t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} L^k \varphi(x) + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} L^k \psi(x), \tag{17}$$

where $L^k\varphi$ is interpreted as $L(L^{k-1}\varphi)$. This solution has been obtained previously in [4,5] with L the Laplacian operator. In this paper, we extend it to the general case that L is dependent on variable coefficients.

4. Example

It is worth noting that the formula (16) or (17) is independent of the space dimension. From the computation point of view, the formula (17) only involves differentiation instead of surface integration as in Poisson formula, where the calculation is pretty lengthy. Now, we illustrate our formula through a simple example.

Example. Consider the linear homogeneous wave equation with $N = 3$

$$\begin{cases} u_{tt}(x,t) - \dfrac{\partial^2 u}{\partial x_1 \partial x_1}(x,t) - \sin^2(x_1+x_2+x_3)\dfrac{\partial^2 u}{\partial x_1 \partial x_2}(x,t) - \dfrac{\partial^2 u}{\partial x_2 \partial x_2}(x,t) \\ \quad - \cos^2(x_1+x_2+x_3)\dfrac{\partial^2 u}{\partial x_3 \partial x_3}(x,t) = 0, \\ u(x,0) = e^{x_1+4x_2+2x_3}, \\ u_t(x,0) = (x_1-x_2)^2 - x_3^2. \end{cases} \tag{18}$$

The linear differential operator in space is

$$L = \frac{\partial^2}{\partial x_1 \partial x_1} + \sin^2(x_1+x_2+x_3)\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_2} + \cos^2(x_1+x_2+x_3)\frac{\partial^2}{\partial x_3 \partial x_3}.$$

It can be verified that the matrix

$$A = \begin{pmatrix} 1 & \dfrac{\sin^2(x_1+x_2+x_3)}{2} & 0 \\ \dfrac{\sin^2(x_1+x_2+x_3)}{2} & 1 & 0 \\ 0 & 0 & \cos^2(x_1+x_2+x_3) \end{pmatrix}$$

is symmetric and positive semi-definite for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. It can be seen that

$$\begin{aligned} &L\left(e^{x_1+4x_2+2x_3}\right) = 21 e^{x_1+4x_2+2x_3}, \qquad && L^k\left(e^{x_1+4x_2+2x_3}\right) = 21^k e^{x_1+4x_2+2x_3}, \quad k = 2,3,\ldots \\ &L\left((x_1-x_2)^2 - x_3^2\right) = 2, && L^k\left((x_1-x_2)^2 - x_3^2\right) = 0, \quad k = 2,3,\ldots. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi_0(-t^2 L)\left(e^{x_1+4x_2+2x_3}\right) &= \cosh(\sqrt{21}t) e^{x_1+4x_2+2x_3}, \\ \phi_1(-t^2 L)\left((x_1-x_2)^2 - x_3^2\right) &= \left(1 + \frac{t^2}{6}\right)\left((x_1-x_2)^2 - x_3^2\right). \end{aligned}$$

The solution of the Cauchy problem for the wave equation (18) is given by

$$u(x,t) = \cosh(\sqrt{21}t) e^{x_1+4x_2+2x_3} + \left(t + \frac{t^3}{6}\right)\left((x_1-x_2)^2 - x_3^2\right).$$

In should be noted that, generally speaking, the closed form solutions for concrete problems are not easily obtainable. In this case, truncated series in the form of the sequence of partial sum

$$u_K(x,t) = \sum_{k=0}^{K} \frac{t^{2k}}{(2k)!} L^k \varphi(x) + \sum_{k=0}^{K} \frac{t^{2k+1}}{(2k+1)!} L^k \psi(x)$$

can be used to approximate the solution.

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