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ORIGINAL PAPER

A new Jacobi-type iteration method for solving M-matrix or nonnegative linear systems

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Abstract

In this paper, based on the exponential integrator, a new Jacobi-type iteration method is proposed for solving linear system $Ax = b$. The traditional Jacobi iteration method can be viewed as a special case of the new method. The convergence and two comparison theorems of the new Jacobi-type method are established for linear system with different type of coefficient matrices. The convergence of the traditional Jacobi iteration method follows immediately from these results. It is shown that for the linear system with coefficient matrix that is M-matrix or nonnegative matrix, the new method is convergent. Under suitable conditions, the spectral radius of iteration matrix for new method is much smaller than traditional Jacobi method for the case of nonnegative coefficient matrix. Numerical experiments are carried out to show the effectiveness of the new method when dealing with the nonnegative matrix system.

Keywords Nonnegative matrix · Linear system · Dynamic system method · M-matrix · Exponential integrator · Jacobi iteration method

Mathematics Subject Classification 65L05 · 65L07 · 65L20 · 65F10 · 65N22

1 Introduction

In this paper, we consider the solution of the following linear system,

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$$Ax = b \tag{1}$$

where $A = (a_{ij}) \in \mathcal{R}^{n\times n}$ is a nonsingular matrix, and $x, b \in \mathcal{R}^n$.

Many scientific problems lead to the requirement to solve linear systems of equations as part of the computations. Several stationary iterative methods including Jacobi, Gauss-Seidel, and successive over-relaxation (SOR) method [1, 6] are designed to approximately solve the linear systems under special conditions. In [11], the relationship between SOR-type methods for linear systems and the discrete gradient methods for gradient systems are discussed. Based on the result, adaptive SOR methods based on the Wolfe conditions are derived [12]. Two accelerative algorithms based on the scheduled relaxation Jacobi method are developed for linear systems in [7]. An effective stationary iterative method is established via double splittings of matrices [8]. In [17], the authors developed a numerical method for solving system of linear equations through taking advantages of properties of repetitive tridiagonal matrices. On some classes of problems, such iterative methods compete well with direct methods. However, these approaches suffer from relatively large prefactors and poor scaling with system size. In comparison, Krylov subspace methods, such as generalized minimal residual and conjugate gradient [15] are preferable to solving large and sparse linear systems. Despite of the disadvantage of the stationary iterative methods, the Jacobi iteration stands out for its tremendous simplicity and potential for massive parallelization [13]. This motivates the development of strategies that are able to significantly accelerate the convergence of the Jacobi method, while maintaining its underlying locality and simplicity to the maximum extent possible.

Since 1970s, the connections between iterative numerical methods in numerical linear algebra and continuous dynamic systems have been studied, due to the fact that many discretization methods for continuous dynamic systems have the iterative nature. The link between iteration methods and continuous dynamic systems leads to the development of so-called dynamic system methods [14] which become fruitfully alternative methods. In contrast to discrete methods, dynamic system methods have some superiors to discrete ones as they can obtain the convergence results more easily under weaker restrictions on the convergence theorems. For example, the method of steepest descent can be achieved by applying explicit Euler methods with variable stepsize to first-order ordinary differential equations $\frac{dx}{dt} = b - Ax, x(0) = 0$. Applying explicit Euler methods with fixed stepsize $h = 1$ to differential equations $\frac{dx}{dt} = A^{-1}(b - Ax), x(0) = 0$ gives Wilkinson's iterative refinement [18].

On the other hand, much attention has been paid on geometric numerical integration (structure-preserving methods) of differential equations in the past few decades. A numerical integration method is called geometric if it exactly preserves one or more physical/geometric properties such as first integrals, symplectic structure, symmetries, phase-space volume, etc. of the system. We refer the reader to [2, 9, 16, 19] for recent surveys of this research. The study of geometric numerical integration provides us a variety of numerical tools for solving the continuous dynamic systems. It is much likely that using state-of-the-art numerical methods could lead to better iterative numerical methods.

In the present work, instead of the explicit Euler method, the so-called exponential integrator [3, 4] will be used as the underling numerical method to solve the dynamic system corresponding to the linear system $Ax = b$. A new Jacobi-type iteration method for solving linear system $Ax = b$ will be presented. The rest of the paper is organized as follows. The new Jacobi-type iteration method is derived in Sect. 2. The convergence of the proposed method and two comparison theorem are studied for linear systems with different type of coefficient matrices in Sect. 3. In Sect. 4, we present some numerical examples to show the effectiveness of the new method. The last section is devoted to the conclusions.

2 The new Jacobi-type iteration method for (1)

Consider the following dynamic system

$$\begin{cases} \dfrac{dx}{dt} = b - Ax, \\ x(0) = x^0, x^0 \in \mathcal{R}^n. \end{cases} \tag{2}$$

For now, we restrict A to be symmetric positive definite and let $f(x) = Ax - b$. Then we have the following theorem.

Theorem 1 *The solution $x^* = A^{-1}b$ of linear system (1) is a unique globally asymptotically stable equilibrium point of the dynamic system (2).*

Proof Let

$$H(x) = \frac{1}{2} f(x)^T f(x).$$

We have

$$\frac{dH}{dt} = \frac{dH}{dx}\frac{dx}{dt} = f(x)^T A \frac{dx}{dt} = -f(x)^T A f(x) < 0,$$

provided $x \neq x^*$. Moreover, we have $f(x^*) = 0$ and $f(x) \neq 0$, for any $x \neq x^*$. Therefore, $H(x)$ is a strict Liapunov function of dynamic system (2). Therefore, the unique root of $H(x)$, $x = x^*$ is a unique globally asymptotically stable equilibrium point of dynamic system (2). □

From Theorem 1, it can be seen that if $x(t, x^0)$ is the solution of dynamic system (1), then we have

$$\lim_{t \to \infty} x(t, x^0) = A^{-1}b.$$

Generally speaking, it is difficult to obtain the analytic expression of $x(t, x^0)$ for large n, although the formalization of the solution is available. Therefore, proceeding from an initial point $x^0 \in \mathcal{R}^n$ to the solution $x^* = A^{-1}b$ of linear system (1), via a smooth, curvilinear trajectory is not practical in general. Fortunately, we can employ numerical integrators of ordinary differential equations. Instead of explicit Euler method, which possesses only first order of convergence and has only very limited region of absolute stability, we use exponential integrator as the underling numerical method. To this end, let us express the matrix A as the matrix sum $A = D - L - U$, where D is the diagonal matrix whose diagonal entries are those of A, $-L$ is the strictly lower-triangular part of A, and $-U$ is the strictly upper-triangular part of A. With these notations, dynamic system (2) can be reformulated as

$$\begin{cases} \dfrac{dx}{dt} + Dx = Lx + Ux + b, \\ x(0) = x^0, x^0 \in \mathcal{R}^n. \end{cases} \tag{3}$$

From the well-known variation-of-constants formula, the solution of the system (3) has the form

$$x(t) = \exp(-tD)x^0 + \int_0^t \exp(-(t-\xi)D)(Lx(\xi) + Ux(\xi) + b)\mathrm{d}\xi. \tag{4}$$

It follows immediately that

$$x(t_k + h) = \exp\big(-hD\big)x(t_k) + \int_0^h \exp\big(-\xi D\big)\big(Lx(t_k+\xi) + Ux(t_k+\xi) + b\big)\mathrm{d}\xi, \tag{5}$$

where h is the stepsize used when numerically solving the dynamic system (3) and $t_k = kh, k = 1, 2, \ldots,$ is the grid points where the solution of (3) is approximated.

Replacing $x(t_k + \xi)$ with $x(t_k)$, the integral in (5) can be approximated by:

$$\begin{aligned} &\int_0^h \exp\big(-\xi D\big)\big(Lx(t_k+\xi) + Ux(t_k+\xi) + b\big)\mathrm{d}\ \xi \\ &\quad = h\varphi(-hD)(Lx(t_k) + Ux(t_k) + b), \end{aligned}$$

where the scalar function is given by

$$\varphi(z) = (\exp(z) - 1)/z.$$

Then we obtain the new iteration scheme

$$x^{k+1} = \exp\big(-hD\big)x^k + h\varphi(-hD)(Lx^k + Ux^k + b) \tag{6}$$

or equivalently

$$x^{k+1} = \exp\big(-hD\big)x^k + D^{-1}(I - \exp\big(-hD\big))(Lx^k + Ux^k + b), \tag{7}$$

where I is the identity matrix of dimension n. We refer (6) or (7) as an *exponential Jacobi iteration method* and denote it by EJ.

The exponential Jacobi iteration method (7) can be written, in a component-wise fashion, as

$$x_i^{k+1} = e^{-ha_{ii}}x_i^k + \frac{1}{a_{ii}}(1 - e^{-ha_{ii}})\left(- \sum_{j<i} a_{ij}x_j^k - \sum_{j>i} a_{ij}x_j^k + b_i \right), i = 1, \dots, n. \quad (8)$$

Remark 1 Notice that $a_{ii} > 0, i = 1, \dots, n$ since the matrix A is positive definite. The new iteration method (6) reduces to the traditional Jacobi iteration method as $h \to +\infty$:

$$x^{k+1} = D^{-1}(L + U)x^k + D^{-1}b. \quad (9)$$

It can be seen from the proof of Theorem 1 that the unique globally asymptotically stable equilibrium point of the dynamic system (2) is the unique optimal solution to the unconditioned optimization problem $\min\limits_{x \in \mathcal{R}^n} H(x)$, or equivalently, the solution to $\nabla H(x) = 0$, i.e., $x^* = A^{-1}b$. Furthermore, $H(x(t, x_0))$ decreases to its minimum as $t \to \infty$ along the exact flow. Therefore if the numerical method preserves the decay of $H(x)$ when applied to the dynamic system system (2), then the corresponding iteration method converges to the unique solution $x^* = A^{-1}b$.

In order to simplify the presentation, for the rest of the paper we assume without loss of generality that $D = I$. That is, the coefficient matrix A is preconditioned by $D^{-1/2}AD^{-1/2}$ such that its diagonal matrix coincides with the identity matrix. Then the exponential Jacobi iteration method (7) reduces to

$$x^{k+1} = e^{-h}x^k + (1 - e^{-h})(Lx^k + Ux^k + b). \quad (10)$$

The iteration method (10) is coincided with the the weighted Jacobi method proposed in [13].

Theorem 2 *Assume that the coefficient matrix A of the linear system* (1) *is strictly diagonally dominant and positive definite. The sequence $\{x_n\}_{n=0}^{\infty}$ generated by the exponential Jacobi iteration method* (7) *with any stepsize $h > 0$ converges to the unique solution $x^* = A^{-1}b$ of the linear system* (1) *for any initial value x_0.*

Proof Multiplying both sides of (10) by A yields

$$Ax^{k+1} = e^{-h}Ax^k + (1 - e^{-h})A(Lx^k + Ux^k + b).$$

Noticing that $A = I - L - U$, we have

$$Ax^{k+1} = e^{-h}Ax^k + (1 - e^{-h})A(x^k - Ax^k + b).$$

Therefore,

$$\begin{aligned}Ax^{k+1}-b=&e^{-h}Ax^k+(1-e^{-h})A(x^k-Ax^k+b)-b\\=&e^{-h}Ax^k+(1-e^{-h})Ax^k-b-(1-e^{-h})A(Ax^k-b)\\=&(I-(1-e^{-h})A)(Ax^k-b).\end{aligned}$$

Let $\omega(h)=1-e^{-h}$. Then $0<\omega(h)<1$ for $h>0$ and we have

$$\begin{aligned}H(x^{k+1})&=\frac{1}{2}(Ax^{k+1}-b)^T(Ax^{k+1}-b)=\frac{1}{2}\|Ax^{k+1}-b\|_2^2\\&=\frac{1}{2}\|(I-\omega(h)A)(Ax^k-b)\|_2^2\leqslant\|I-\omega(h)A\|_2^2H(x^k).\end{aligned}$$

Here, $\|\cdot\|_2$ is the Euclidean norm of $\mathcal{R}^n$. Since A is strictly diagonally dominant, the spectral radius $\rho(A)<2$. Since A is positive definite, all the eigenvalues of A are positive. Let the eigenvalues of the matrix A be $0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n<2$. Then the eigenvalues of the matrix $I-\omega(h)A$ are

$$-1<1-\omega(h)\lambda_n\leqslant\cdots\leqslant1-\omega(h)\lambda_2\leqslant1-\omega(h)\lambda_1<1$$

for all $h>0$. Therefore, we have $\|I-\omega(h)A\|_2<1$. Hence,

$$H(x^{k+1})\leqslant\|I-\omega(h)A\|_2^2H(x^k)<H(x^k).$$

The function $H(x)$ is strictly decreasing with respect to the sequence $\{x^k\}_{k=0}^{\infty}$ generated by (7). The theorem follows immediately by the above discussion. □

Corollary 1 *The Jacobi iteration method is convergent for linear system with strictly diagonally dominant and positive definite coefficient matrix.*

Proof The result follows from Theorem 2 and Remark 1. □

3 Convergence and comparison theorem

It should be noted that the iteration method (7) can also be applied to any linear systems with coefficient matrices whose diagonal entries are positive. More precisely, if the exponential Jacobi iteration method (7) converges to x^*, we have

$$x^*=\exp\big(-hD\big)x^*+D^{-1}(I-\exp\big(-hD\big))(Lx^*+Ux^*+b),$$

the solution of which is $x^*=A^{-1}b$.

The exponential Jacobi iteration method (7) is a stationary iterative method of the form

$$x^{n+1}=Gx^n+c. \tag{11}$$

The iteration sequence $\{x^k\}$ converges to the unique solution of $x=Gx+c$ for any starting value x^0 if and only if $\rho(G)<1$.

Keeping $e^x = \varphi(x)x + 1$ and the assumption $D = I$ in mind, the iterative matrix G and vector c for the exponential Jacobi iteration method are expressed as

$$G_{EJ}(h) = I - h\varphi(-h)A, \qquad c_{EJ}(h) = h\varphi(-h)b. \tag{12}$$

Correspondingly, the iterative matrix and vector of the Jacobi iteration methods are expressed as

$$G_J = I - A, \qquad c_J = b. \tag{13}$$

In this section, we will present some theoretical results concerning the convergence of the proposed method for some linear systems with special coefficient matrices.

3.1 For M-matrix systems

For convenience, we first give some of the notations, definitions and lemmas which will be used in what follows.

For $A = (a_{ij}), B = (b_{ij}) \in \mathcal{R}^{n\times n}$, we say that $A \geqslant B$ if $a_{ij} \geqslant b_{ij}$ for all $i, j = 1, \dots, n$. A is called nonnegative if each $a_{ij} \geqslant 0$, and this is denoted by writing $A \geqslant 0$. A is a positive matrix when each $a_{ij} > 0$.

Definition 1 [20] A matrix A is a L-matrix if $a_{ii} > 0, i = 1, \dots, n$ and $a_{ij} \leqslant 0$ for all $i, j = 1, \dots, n, i \neq j$. A nonsingular L-matrix A is a nonsingular M-matrix if $A^{-1} \geqslant O_{n\times n}$.

Definition 2 [21] Let A be a real matrix. Then $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called

- (i) regular if $M^{-1} > 0$ and $N \geqslant O_{n\times n}$;
- (ii) weakly regular if $M^{-1} > 0$ and $M^{-1}N \geqslant O_{n\times n}$;
- (iii) M-splitting if M is a nonsingular M-matrix and $N \geqslant O_{n\times n}$.
- (iv) convergent splitting if $\rho(M^{-1}N) < 1$.

Definition 3 [10] A matrix A is said to be a reducible matrix when there exists a permutation matrix P such that

$$P^TAP = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix},$$

where X and Z are both square. Otherwise, A is said to be an irreducible matrix.

Lemma 1 [21] *Let $A = M - N$ be an M-splitting of A, then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.*

Lemma 2 [21] *Let A be a nonsingular M-matrix, and let $A = M_1 - N_1 = M_2 - N_2$ be two convergent splittings, the first one regular and the second one weakly regular. If $M_1^{-1} \leqslant M_2^{-1}$, then*

$$\rho(M_1^{-1}N_1) \geqslant \rho(M_2^{-1}N_2).$$

As a matter of fact, based on the splitting $A = M(h) - N(h)$, where $M(h) = (h\varphi(-h))^{-1}I$ and $N(h) = (h\varphi(-h))^{-1}I - A$, the iterative matrix of the exponential Jacobi iteration method can be written as $G_{EJ}(h) = M(h)^{-1}N(h)$.

Theorem 3 *Asumme that the matrix A is a nonsingular M-matrix. Then we have*

$$\rho(G_{EJ}(h)) < 1 \quad \text{for} \quad h > 0$$

and

$$\rho(G_{EJ}(h_1)) \geqslant \rho(G_{EJ}(h_2)) \quad \text{for} \quad 0 < h_1 < h_2.$$

Proof Let $M = M(h)$ and $N = N(h)$. Since A is a M-matrix, it can be observed that for $h > 0$, $M^{-1} = h\varphi(-h)I > O_{n\times n}$ is a nonsingular M-matrix and $N(i,i) = \dfrac{e^{-h}}{1-e^{-h}} > 0$ for $i = 1,\dots,n$ and $N(i,j) = -a_{ij} \geqslant 0$ for $i,j = 1,\dots,n, i \neq j$. Therefore, $A = M - N$ is an M-splitting of matrix A. According to Lemma 1, $\rho(G_{EJ}) < 1 \quad \text{for} \quad h > 0$.

Let $M_1 = M(h_1)$, $N_1 = N(h_1)$ and $M_1 = M(h_2)$, $N_1 = N(h_2)$. Then $A = M_1 - N_1$ and $A = M_2 - N_2$ are two convergent splittings. Moreover, M_1^{-1} and M_2^{-1} are two diagonal matrices with the nonnegative diagonal elements $M_1^{-1}(i,i) = 1 - e^{-h_1}$ and $M_2^{-1}(i,i) = 1 - e^{-h_2}$, for $i = 1,\dots,n$. Hence $M_1^{-1} < M_2^{-1}$. All the conditions in Lemma 2 have been verified. The proof is completed. □

According to Theorem 3 and Remark 1, even though the exponential Jacobi iteration method is convergent for the M-matrix system, the spectral radius of the exponential Jacobi iteration method is larger than that of the traditional Jabobi iteration method. It is not a good option when dealing with the M-matrix system. However, for the nonnegative matrix system, the proposed iteration method performs a better convergence than traditional Jacobi iteration method as shown in the next subsection.

3.2 For nonnegative matrix systems

According to Perron-Frobenius Theorem [10], we have the following theorem dealing with non-negativity of a matrix and the spectral radius.

Theorem 4 *If $A \geqslant 0$ is irreducible. Then each of the following is true.*

(i) *A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;*
(ii) *$\rho(A)$ is a simple eigenvalue of A.*

Definition 4 A nonnegative irreducible matrix A having only one eigenvalue, $r = \rho(A)$, on its spectral circle is said to be a primitive matrix.

Lemma 3 [21] *If A is a nonnegative $n \times n$ matrix, then*

$$\min_i \sum_{j=1}^{n} a_{ij} \leqslant \rho(A) \leqslant \max_i \sum_{j=1}^{n} a_{ij}.$$

The comparison matrix of A is denoted by matrix $< A >= (m_{ij})$, where

$$m_{ii} = \left|a_{ii}\right|, \quad m_{ij} = -\left|a_{ij}\right| \qquad (i \neq j).$$

A matrix A is an H-matrix if and only if its comparison matrix is an M-matrix.

Lemma 4 [5] *If A is an H-matrix, then $\rho(A) < 2\max_i \left|a_{ii}\right|$.*

For the spectral radii of $G_{EJ}(h)$ and G_J, we have the following comparison theorem.

Theorem 5 *If A is an irreducible nonnegative $n \times n$ matrix with unit diagonal elements, then $\rho(G_J) = \rho(A) - 1$. Moreover, if A is primitive, then there exists $h_0 > 0$ such that for all $h > h_0$, $\rho(G_{EJ}(h)) = (1 - e^{-h})\rho(A) - 1$. Therefore, $\rho(G_{EJ}(h)) < \rho(G_J)$, for all $h > h_0$.*

Proof By Lemma 3, we have $\rho(A) > 1$. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of G_J. Since $A \geqslant 0$, then $G_J = I - A \leqslant 0$. Therefore, corresponding to $\rho(G_J)$, there exists a negative eigenvalue of G_J. Denote $\rho(G_J) = -\mu_k$. Then

$$\rho(A) = \rho(I - G_J) = \max_i \left|1 - \mu_i\right| = 1 - \mu_k = 1 + \rho(G_J).$$

Noticing that $G_{EJ}(h) = (1 - e^{-h})G_J + e^{-h}I$, we have

$$\rho(G_{EJ}(h)) = \max_i \left|(1 - e^{-h})\mu_i + e^{-h}\right|.$$

If A is primitive matrix, then there exists an $\alpha > 0$ such that $\left|\mu_i\right| \leqslant \alpha < \rho(G_J)$, for all $i \neq k$. Therefore, we have

$$\begin{aligned}
&\left|(1 - e^{-h})\mu_i + e^{-h}\right| \\
&\leqslant (1 - e^{-h})\alpha + e^{-h} \\
&= (1 - e^{-h})\rho(G_J) - e^{-h} + (1 - e^{-h})(\alpha - \rho(G_J)) + 2e^{-h}
\end{aligned}$$

for all $i \neq k$. Since $\alpha < \rho(G_J)$, there exists an $h_0 > 0$ such that $(1 - e^{-h})\rho(G_J) - e^{-h} > 0$ and $(1 - e^{-h})(\alpha - \rho(G_J)) + 2e^{-h} < 0$ for all $h > h_0$. Thus,

$$\left|(1 - e^{-h})\mu_i + e^{-h}\right| < (1 - e^{-h})\rho(G_J) - e^{-h}$$

for all $h > h_0$ and $i \neq k$. Furthermore, for $i = k$ and $h > h_0$, it is easy to see that

$$\left|(1-e^{-h})\mu_i + e^{-h}\right| = (1-e^{-h})\rho(G_J) - e^{-h}.$$

Hence, for $h > h_0$,

$$\rho(G_{EJ}(h)) = (1-e^{-h})\rho(G_J) - e^{-h} = (1-e^{-h})\rho(A) - 1.$$

The proof of $\rho(G_{EJ}(h)) < \rho(G_J)$, for all $h > h_0$ is trivial. □

Corollary 2 *Let A be a positive matrix with unit diagonal elements. Then there exists an* $h_0 > 0$ *such that* $\rho(G_{EJ}(h)) < \rho(G_J)$, *for all* $h > h_0$.

Proof A positive matrix is primitive [10]. □

Suppose the nonnegative matrix A is strictly diagonally dominant. By Lemma 3, we have $1 < \rho(A) < 2$. Therefore, we have the following corollary.

Corollary 3 *Let A be a strictly diagonally dominant nonnegative matrix with unit diagonal elements. Assume that A is a primitive matrix. Then there exists an* $h_0 > 0$ *such that*

$$\rho(G_{EJ}(h)) < \rho(G_J) < 1$$

for all $h > h_0$.

Corollary 4 *Let A be a nonnegative H-matrix with unit diagonal elements. Assume that A is a primitive matrix. Then there exists an* $h_0 > 0$ *such that*

$$\rho(G_{EJ}(h)) < \rho(G_J) < 1$$

for all $h > h_0$.

Proof By Lemma 3 and Lemma 4, $1 < \rho(A) < 2$. The rest of the proof is trivial. □

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the matrix A. It can be seen from the proof that under the conditions in Theorem 5, the best choice of h would be

$$h^* = \arg\min_h \max_i \left|(1-e^{-h})(1-\lambda_i) + e^{-h}\right|.$$

In practice, it costs much effort or even is impossible to evaluate all the eigenvalues of the matrices A. Therefore, the best choice of h is not obtainable in general. The information that can be obtained from the proof of Theorem 5 regards to the choice of h is that $(1-e^{-h})\rho(G_J) - e^{-h} > 0$ and $(1-e^{-h})(\alpha - \rho(G_J)) + 2e^{-h} < 0$, which still relies on the first and second largest eigenvalues of the matrix A. Generally speaking, the choice of h is quite tricky. In the next section, stepszies h with moderate magnitude are used in the numerical experiments.

Remark 2 At first glance, the matrix A has very restrict restrictions that it must be M-matrix or nonnegative irreducible matrix or primitive matrix, etc.. However, it should be noted that linear systems with these types of coefficient matrices arise in many applications such as the Leontief's Input-Output Economic Model and Leslie Population Age Distribution Model (see [10] for more details).

4 Numerical experiments

In this section, we perform two numerical examples of nonnegative matrix systems to illustrate the effectiveness of our method in this paper. In all the numerical experiments, the initial approximation x^0 is taken as the zero vector, and the right-hand-side vector b is chosen so that $x^* = (1, \dots, 1)^T$ is the solution of the considered system. All experiments were executed by using the MATLAB programming package with double precision by default. The traditional Jacobi and exponential Jacobi iteration methods are applied to solving the linear system. The comparison between two methods are illustrated by plotting the logarithm of the approximation errors $\|x^k - x^*\|_\infty$ against the number of iterations. We also plot the spectral radius $\rho(G_{EJ}(h))$ against the stepsize h by computing the maximum eigenvalues in magnitude in the numerical experiments.

Example 1 [5] Let

$$A = \begin{pmatrix} 1.0 & 0.1 & 0.2 & 0.0 & 0.3 & 0.5 \\ 0.2 & 1.0 & 0.3 & 0.0 & 0.4 & 0.1 \\ 0.0 & 0.3 & 1.0 & 0.6 & 0.2 & 0.0 \\ 0.2 & 0.3 & 0.1 & 1.0 & 0.1 & 0.3 \\ 0.0 & 0.3 & 0.2 & 0.1 & 1.0 & 0.2 \\ 0.2 & 0.3 & 0.0 & 0.3 & 0.1 & 1.0 \end{pmatrix}.$$

Then

$$\rho(A) = 1.9711, \rho(G_J) = \rho(A) - 1 = 0.9711.$$

The stepsizes are chosen to be $h = 1$ and $h = 1.5$. The corresponding spectral radii are $\rho(G_{EJ}(1)) = 0.6126$ and $\rho(G_{EJ}(1.5)) = 0.5313$, respectively. The numerical results are shown in Fig. 1.

Example 2 [5] Let

$$A = \begin{pmatrix} 1 & q & r & s & q & \cdots \\ s & 1 & q & r & \ddots & q \\ r & s & \ddots & \ddots & \ddots & s \\ q & \ddots & \ddots & 1 & q & r \\ s & \ddots & r & s & 1 & q \\ \cdots & s & q & r & s & 1 \end{pmatrix}$$

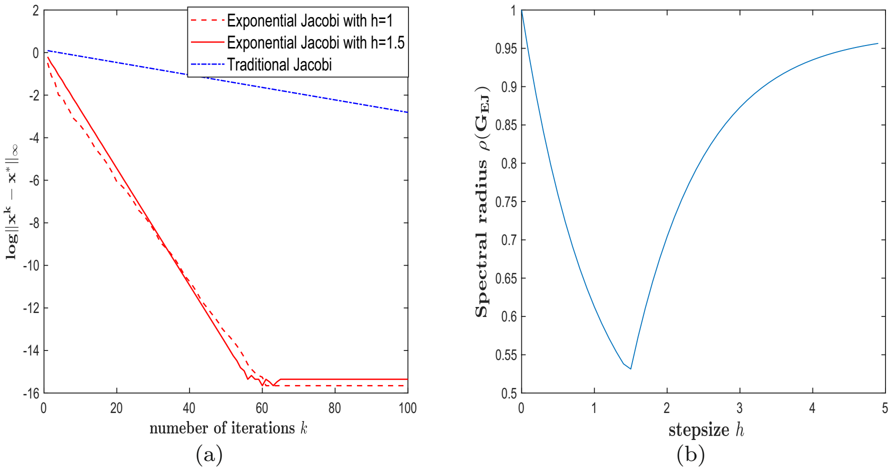

Fig. 1 Example 1: **a** The logarithm of the iteration errors $\|x^k - x^*\|_\infty$ versus number of iterations. **b** Spectral radius of $G_{EJ}(h)$ versus stepsize h

where $q = \frac{1}{n-1}$, $r = \frac{1}{n}$, and $s = \frac{1}{n+1}$.

Setting $n = 1000$, then

$$\rho(A) = 1.999, \rho(G_J) = \rho(A) - 1 = 0.999.$$

The stepsizes are chosen to be $h = 1$ and $h = 1.5$. The corresponding spectral radii are $\rho(G_{EJ}(1)) = 0.369$ and $\rho(G_{EJ}(1.5)) = 0.553$, respectively. The numerical results are shown in Fig. 2. We observe from the numerical results of the two examples that

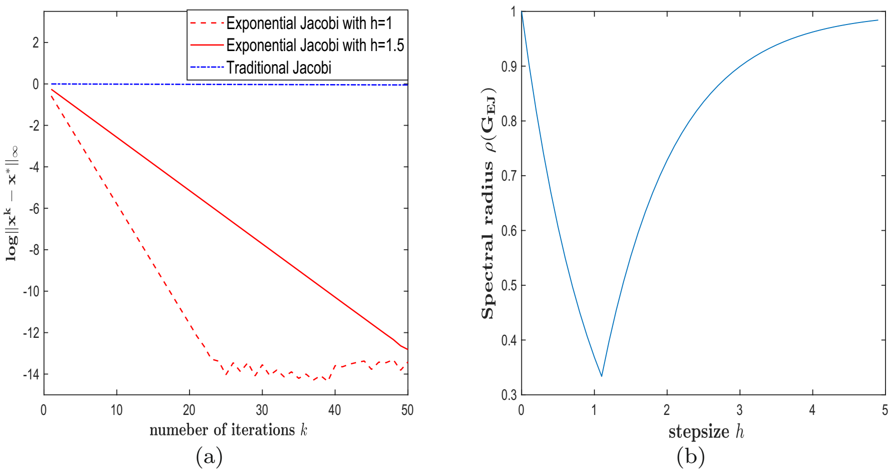

Fig. 2 Example 2: **a** The logarithm of the iteration errors $\|x^k - x^*\|_\infty$ versus number of iterations. **b** Spectral radius of $G_{EJ}(h)$ versus stepsize h

the traditional Jacobi method converges very slowly since the spectral radii are large. The convergence of the exponential Jacobi iteration method is much faster than the traditional Jacobi method. If we choose the stepsize h properly, the spectral radius of the iteration matrix can be reduced significantly.

Example 3 [5] Let

$$A = \begin{pmatrix} 1.0000 & 0.3223 & 0.5991 & 0.9006 & 0.2877 & 0.2466 \\ 0.2025 & 1.0000 & 0.7049 & 0.4321 & 0.5529 & 0.4485 \\ 0.2606 & 0.5607 & 1.0000 & 0.1585 & 0.3458 & 0.4241 \\ 0.7645 & 0.2312 & 0.2173 & 1.0000 & 0.4107 & 0.1261 \\ 0.3618 & 0.1317 & 0.2375 & 0.5046 & 1.0000 & 0.5141 \\ 0.3837 & 0.2592 & 0.4588 & 0.4186 & 0.2294 & 1.0000 \end{pmatrix}.$$

Then

$$\rho(A) = 2.9373, \rho(G_J) = \rho(A) - 1 = 1.9373.$$

The stepsizes are chosen to be $h = 0.8$ and $h = 1$. The corresponding spectral radii are $\rho(G_{EJ}(0.8)) = 0.9512$ and $\rho(G_{EJ}(1)) = 0.9440$, respectively. The results are shown in Fig. 3. In this example, the Jacobi iteration method fails since $\rho(G_J) > 1$. However, the exponential Jacobi iteration methods with different stepszies still give a good performance.

According to Theorem 5, if the coefficient matrix A satisfy suitable conditions, there exists $h_0 > 0$ such that for all $h > h_0$, $\rho(G_{EJ}(h)) = (1 - e^{-h})\rho(A) - 1$. Therefore, $\rho\big(G_{EJ}(h_1)\big) \le \rho\big(G_{EJ}(h_2)\big)$ for $h_1 \le h_2$ if h_1 and h_2 are large. This result is numerically confirmed by the two examples.

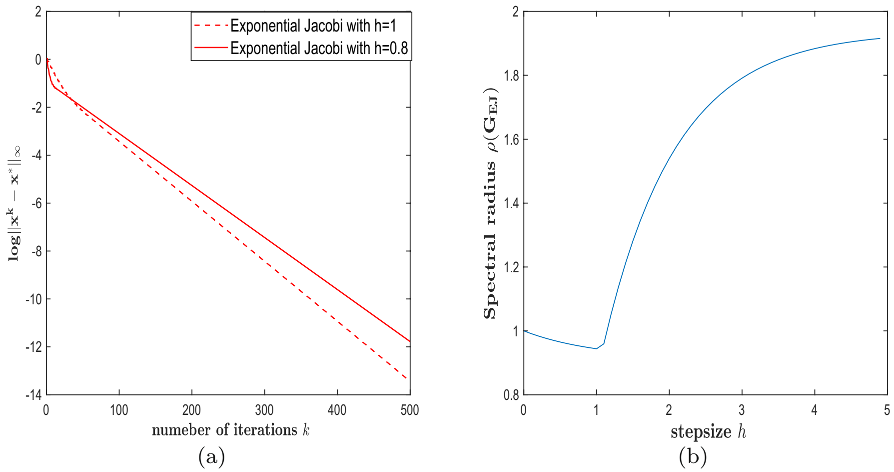

Fig. 3 Example 3: **a** The logarithm of the iteration errors $\|x^k - x^*\|_\infty$ versus number of iterations. **b** Spectral radius of $G_{EJ}(h)$ versus stepsize h

5 Conclusions

Since the Jacobi iteration method is suitable for massive parallelization, it is very necessary to study the acceleration of the convergence for the Jacobi method. In this paper, a new Jacobi-type iteration method is proposed for solving linear systems. The new method requires almost the same computational efforts with the traditional Jacobi iteration method in every step. Some properties concerning with the convergence and spectral radius are analysed for the new method. It is shown that the convergence of the new method is much faster than that of the traditional Jacobi method for nonnegative matrix systems. We prove that the spectral radius of iteration matrix for new method is much smaller than traditional Jacobi method under suitable assumption. Numerical experiments are carried out to illustrate the effectiveness of the new method.

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