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# Multidimensional ARKN methods for general oscillatory second-order initial value problems

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ABSTRACT

Based on B-series theory, the order conditions of the multidimensional ARKN methods are presented for the general multi-frequency and multidimensional oscillatory second-order initial value problems by Wu et al. (2009). These multidimensional ARKN methods exactly integrate the multi-frequency and multidimensional unperturbed oscillators. In this paper, we pay attention to the analysis of the concrete multidimensional ARKN methods for the general multi-frequency oscillatory second-order initial value problems whose right-hand side functions *depend on both* $y$ *and* $y'$ (the class of physical problems which fall within its scope is broader). Numerical experiments are carried out to show that the new multidimensional ARKN methods are more efficient compared with some well-known methods for dealing with the oscillatory problems in the scientific literature.



## 1. Introduction

Differential equations having oscillatory solutions are of particular interest. Those problems are usually encountered in many fields of the applied sciences and engineering, such as celestial mechanics, theoretical physics, chemistry, electronics, control engineering. A lot of theoretical and numerical researches have been made on the modeling and simulation of these oscillations. Among typical topics is the numerical integration of an oscillatory system associated with an initial value problem of the form

$$\begin{cases} y''(t) + Ky(t) = f(y(t), y'(t)), & t \in [t_0, T], \\ y(t_0) = y_0, \qquad y'(t_0) = y'_0, \end{cases} \tag{1}$$

where $K$ is a $d \times d$ positive semi-definite matrix (stiffness matrix, not necessarily symmetric) that implicitly contains the frequencies of the oscillatory problem and $f : R^d \times R^d \to R^d, y \in R^d, y' \in R^d$.

The system (1) can be integrated with general purpose methods or using other codes adapted to the special structure of the problem. However, it has become a common understanding that, for the problems with a structure of particular interest, numerical algorithms should respect that structure of the problems. An outstanding advantage of multidimensional ARKN methods for (1) is that their updates take into account the special structure of the system (1) brought by the linear term $Ky$ so that they naturally integrate the multi-frequency and multidimensional unperturbed system $y'' + Ky = 0$ exactly. For the work on this topic, we refer the reader to [1–11].

In [9], the authors constructed the multidimensional ARKN methods for the oscillatory systems whose right-hand side functions are independent of $y'$. In this paper we present the analysis of the concrete multidimensional ARKN methods for the oscillatory systems with right-hand side functions depending on both $y$ and $y'$, based on the order conditions for the ARKN methods derived by Wu et al. [8]. The class of physical problems which fall within its scope is broader in applications. We will discuss the construction of novel multidimensional ARKN methods for the oscillatory system (1) in detail.

The rest of the paper is organized as follows. In Section 2, we restate the basic idea and order conditions of multidimensional ARKN methods for the multi-frequency oscillatory system (1). In Section 3, some multidimensional ARKN methods are proposed based on the order conditions derived in [8]. The stability of the new multidimensional ARKN methods are analyzed. Section 4 gives the numerical experiments. Section 5 is devoted to conclusions and discussions.

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## 2. Multidimensional ARKN methods and the corresponding order conditions for (1)

In this section, we first introduce the following matrix-valued functions defined in [8]:

$$\phi_j(K) := \sum_{i=0}^{\infty} \frac{(-1)^i K^i}{(2i+j)!}, \quad j = 0, 1, 2, \ldots, \tag{2}$$

where $K$ is a $d \times d$ matrix.

In the recent paper [8], we presented the following matrix-variation-of-constants formula for the exact solution and its derivative for the multi-frequency oscillatory system (1).

**Theorem 2.1.** *If $K \in R^{m\times m}$ is a symmetric and positive semi-definite matrix and $f : R^m \times R^m \to R^m$ is continuous in (1), then the solution of (1) and its derivative satisfy the following equations*

$$\begin{cases} y(t) = \phi_0((t-t_0)^2K)y_0 + (t-t_0)\phi_1((t-t_0)^2K)y_0' + \displaystyle\int_{t_0}^{t} (t-\xi)\phi_1((t-\xi)^2K)\hat{f}(\xi)d\xi, \\ y'(t) = -(t-t_0)K\phi_1((t-t_0)^2K)y_0 + \phi_0((t-t_0)^2K)y_0' + \displaystyle\int_{t_0}^{t} \phi_0((t-\xi)^2K)\hat{f}(\xi)d\xi \end{cases} \tag{3}$$

*for any real number $t_0$, $t$, where $\hat{f}(\xi) = f(y(\xi), y'(\xi))$.*

**Remark 2.1.** We point out that Theorem 2.1 is also true when matrix $K \in R^{d\times d}$ is not symmetric from the proof of the theorem stated in [8].

To obtain a numerical integrator for (1) we approximate the integrals by some suitable quadrature formulas. This leads to the following multidimensional ARKN methods proposed in [8].

**Definition 2.1.** An $s$-stage multidimensional ARKN method for the numerical integration of the multi-frequency oscillatory initial value problem (1) is defined as the following scheme

$$\begin{cases} Y_i = y_n + hc_iy_n' + h^2 \displaystyle\sum_{j=1}^{s} \bar{a}_{ij}\big(f(Y_j, Y_j') - KY_j\big), \quad i = 1, \ldots, s, \\ Y_i' = y_n' + h \displaystyle\sum_{j=1}^{s} a_{ij}\big(f(Y_j, Y_j') - KY_j\big), \quad i = 1, \ldots, s, \\ y_{n+1} = \phi_0(V)y_n + h\phi_1(V)y_n' + h^2 \displaystyle\sum_{i=1}^{s} \bar{b}_i(V)f(Y_i, Y_i'), \\ y_{n+1}' = \phi_0(V)y_n' - hK\phi_1(V)y_n + h \displaystyle\sum_{i=1}^{s} b_i(V)f(Y_i, Y_i'). \end{cases} \tag{4}$$

Here, the weight functions $b_i : R^{d\times d} \to R^{d\times d}$ and $\bar{b}_i : R^{d\times d} \to R^{d\times d}$, $i = 1, \ldots, s$ in the updates are matrix-valued functions of $V = h^2K$. The scheme (4) can also be denoted by the Butcher tableau

$$\begin{array}{c|cc} c & A & \bar{A} \\ \hline & b^T(V) & \bar{b}^T(V) \end{array} = \begin{array}{c|ccccccc} c_1 & a_{11} & \cdots & a_{1s} & & \bar{a}_{11} & \cdots & \bar{a}_{1s} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} & & \bar{a}_{s1} & \cdots & \bar{a}_{ss} \\ \hline & b_1(V) & \cdots & b_s(V) & & \bar{b}_1(V) & \cdots & \bar{b}_s(V) \end{array}$$

In the special case, where the right-hand side function of (1) is independent of $y'$, the scheme (4) reduces to the corresponding ARKN method for the special oscillatory system (see [9])

$$\begin{cases} y''(t) + Ky(t) = f(y(t)), \quad t \in [t_0, T], \\ y(t_0) = y_0, \qquad y'(t_0) = y_0', \end{cases} \tag{5}$$

namely,

$$\begin{cases} Y_i = y_n + hc_iy_n' + h^2 \displaystyle\sum_{j=1}^{s} \bar{a}_{ij}\big(f(Y_j) - KY_j\big), \quad i = 1, \ldots, s, \\ y_{n+1} = \phi_0(V)y_n + h\phi_1(V)y_n' + h^2 \displaystyle\sum_{i=1}^{s} \bar{b}_i(V)f(Y_i), \\ y_{n+1}' = \phi_0(V)y_n' - hK\phi_1(V)y_n + h \displaystyle\sum_{i=1}^{s} b_i(V)f(Y_i). \end{cases} \tag{6}$$

The scheme (6) can also be denoted by the Butcher tableau

$$\begin{array}{c|c} c & \bar{A} \\ \hline & \bar{b}^T(V) \\ \hline & b^T(V) \end{array} = \begin{array}{c|ccc} c_1 & \bar{a}_{11} & \cdots & \bar{a}_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & \bar{a}_{s1} & \cdots & \bar{a}_{ss} \\ \hline & \bar{b}_1(V) & \cdots & \bar{b}_s(V) \\ \hline & b_1(V) & \cdots & b_s(V) \end{array}$$

A multidimensional ARKN method (4) has *order* $p$, if for a sufficiently smooth problem (1) the conditions

$$e_{n+1} := y_{n+1} - y(t_n + h) = \mathcal{O}(h^{p+1}) \quad \text{and} \quad e'_{n+1} := y'_{n+1} - y'(t_n + h) = \mathcal{O}(h^{p+1}) \tag{7}$$

are satisfied simultaneously, where $y(t_n + h)$ and $y'(t_n + h)$ are the exact solutions of (1) and its derivative at $t_n + h$, respectively, and $y_{n+1}$ and $y'_{n+1}$ are the one step numerical results obtained by the method from the exact starting values $y_n = y(t_n)$ and $y'_n = y'(t_n)$ (the local assumptions). In paper [8], the order conditions are achieved for the multidimensional ARKN method (4). The following theorem gives the order conditions of multidimensional ARKN method (4).

**Theorem 2.2** (*See Wu et al. [8]*). *The necessary and sufficient conditions for a multidimensional ARKN method (4) to be of order p are given by*

$$\big(\Phi(\tau)^T \otimes I_m\big)\bar{b}(V) - \frac{\rho(\tau)!}{\gamma(\tau)}\phi_{\rho(\tau)+1}(V) = \mathcal{O}(h^{p-\rho(\tau)}), \quad \rho(\tau) = 1, \ldots, p-1,$$
$$\big(\Phi(\tau)^T \otimes I_m\big)b(V) - \frac{\rho(\tau)!}{\gamma(\tau)}\phi_{\rho(\tau)}(V) = \mathcal{O}(h^{p+1-\rho(\tau)}), \quad \rho(\tau) = 1, \ldots, p,$$

*where $\tau$ is the Nyström tree associated with an elementary differential $\mathcal{F}(\tau)(y_n, y'_n)$ of the function $\tilde{f}(y, y') = f(y, y') - Ky$ at $(y_n, y'_n)$ and $\Phi_j(\tau), j = 1, \ldots, s$ are weight functions defined in [12].*

## 3. New multidimensional ARKN methods for oscillatory system (1) with $f$ depending on both $y$ and $y'$

In this section, we construct three explicit ARKN methods for the multi-frequency oscillatory problem (1) and analyze the stability and phase properties of the new methods.

### 3.1. Construction of multidimensional ARKN methods for (1)

First, we construct an explicit multidimensional explicit ARKN method of order three. Since $f$ depends on both $y$ and $y'$ in (1), at least three stages are required to meet all the order conditions. Therefore, we consider the following Butcher tableau:

$$\begin{array}{c|cc} c & A & \bar{A} \\ \hline & b^T(V) & \bar{b}^T(V) \end{array} = \begin{array}{c|ccccccc} c_1 & 0 & 0 & 0 & & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 & & \bar{a}_{21} & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 & & \bar{a}_{31} & \bar{a}_{32} & 0 \\ \hline & b_1(V) & b_2(V) & b_3(V) & & \bar{b}_1(V) & \bar{b}_2(V) & \bar{b}_3(V) \end{array}$$

From Theorem 2.2, a three-stage ARKN method is of order three if its coefficients satisfy

$$\begin{array}{ll} (e^T \otimes I)b(V) = \phi_1(V) + \mathcal{O}(h^3), & (c^T \otimes I)b(V) = \phi_2(V) + \mathcal{O}(h^2), \\ ((c^2)^T \otimes I)b(V) = 2\phi_3(V) + \mathcal{O}(h), & (e^T \otimes I)\bar{b}(V) = \phi_2(V) + \mathcal{O}(h^2), \\ (c^T \otimes I)\bar{b}(V) = \phi_3(V) + \mathcal{O}(h), & ((\bar{A}e)^T \otimes I)b(V) = \phi_3(V) + \mathcal{O}(h), \\ ((Ae)^T \otimes I)\bar{b}(V) = \phi_3(V) + \mathcal{O}(h), & ((Ae)^T \otimes I)b(V) = \phi_2(V) + \mathcal{O}(h^2), \\ ((Ac)^T \otimes I)b(V) = \phi_3(V) + \mathcal{O}(h), & ((c \cdot Ae)^T \otimes I)b(V) = 2\phi_3(V) + \mathcal{O}(h), \\ ((A^2e)^T \otimes I)b(V) = \phi_3(V) + \mathcal{O}(h), & ((Ae \cdot Ae)^T \otimes I)b(V) = 2\phi_3(V) + \mathcal{O}(h), \end{array} \tag{8}$$

where $e = (1, 1, 1)^T$.

Choosing $c_1 = 0, c_2 = \frac{1}{2},\ c_3 = 1$, and solving all the equations in (8), we obtain

$$\begin{array}{lll} a_{21} = \dfrac{1}{2}, & a_{31} = -1, & a_{32} = 2, \\ \bar{a}_{21} = \dfrac{1}{8}, & \bar{a}_{31} = \dfrac{1}{2}, & \bar{a}_{32} = 0, \end{array} \tag{9}$$

and

$$\begin{array}{ll} b_1(V) = \phi_1(V) - 3\phi_2(V) + 4\phi_3(V), & \bar{b}_1(V) = \phi_2(V) - \dfrac{3}{2}\phi_3(V), \\ b_2(V) = 4\phi_2(V) - 8\phi_3(V), & \bar{b}_2(V) = \phi_3(V), \\ b_3(V) = -\phi_2(V) + 4\phi_3(V), & \bar{b}_3(V) = \dfrac{1}{2}\phi_3(V). \end{array} \tag{10}$$

(9)–(10) give an explicit multidimensional three-stage ARKN method of order three. The Taylor expansions of the coefficients $b_i(V)$ and $\bar{b}_i(V)$ of this method are given by

$$\begin{aligned}
b_1(V) &= \frac{1}{6}I - \frac{3}{40}V + \frac{5}{1008}V^2 - \frac{7}{51840}V^3 + \frac{1}{492800}V^4 + \cdots,\\
b_2(V) &= \frac{2}{3}I - \frac{1}{10}V + \frac{1}{252}V^2 - \frac{1}{12960}V^3 + \frac{1}{1108800}V^4 + \cdots,\\
b_3(V) &= \frac{1}{6}I + \frac{1}{120}V - \frac{1}{1680}V^2 + \frac{1}{72576}V^3 - \frac{1}{5702400}V^4 + \cdots,\\
\bar{b}_1(V) &= \frac{1}{4}I - \frac{7}{240}V + \frac{11}{10080}V^2 - \frac{1}{48348}V^3 + \frac{19}{79833600}V^4 + \cdots,\\
\bar{b}_2(V) &= \frac{1}{6}I - \frac{1}{120}V + \frac{1}{5040}V^2 - \frac{1}{362880}V^3 + \frac{1}{39916800}V^4 + \cdots,\\
\bar{b}_3(V) &= \frac{1}{12}I - \frac{1}{240}V + \frac{1}{10080}V^2 - \frac{1}{725760}V^3 + \frac{1}{79833600}V^4 + \cdots.
\end{aligned} \tag{11}$$

We denote this method as ARKN3s3.

We then construct an explicit multidimensional ARKN method of order four. In this case, four stages are needed. The method can be denoted by the following Butcher tableau:

$$\begin{array}{c|cc} c & A & \bar{A} \\ \hline & b^T(V) & \bar{b}^T(V) \end{array} = \begin{array}{c|cccccccc}
c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
c_2 & a_{21} & 0 & 0 & 0 & \bar{a}_{21} & 0 & 0 & 0\\
c_3 & a_{31} & a_{32} & 0 & 0 & \bar{a}_{31} & \bar{a}_{32} & 0 & 0\\
c_4 & a_{41} & a_{42} & a_{43} & 0 & \bar{a}_{41} & \bar{a}_{42} & \bar{a}_{43} & 0\\ \hline
 & b_1(V) & b_2(V) & b_3(V) & b_4(V) & \bar{b}_1(V) & \bar{b}_2(V) & \bar{b}_3(V) & \bar{b}_4(V)
\end{array}$$

If we let $Ae = c, \bar{A} = A^2$, then the order conditions up to order four reduce to

$$\begin{aligned}
&(e^T \otimes I)b(V) = \phi_1(V) + \mathcal{O}(h^4), && (c^T \otimes I)b(V) = \phi_2(V) + \mathcal{O}(h^3),\\
&((c^2)^T \otimes I)b(V) = 2\phi_3(V) + \mathcal{O}(h^2), && ((Ac)^T \otimes I)b(V) = \phi_3(V) + \mathcal{O}(h^2),\\
&((A^2c)^T \otimes I)b(V) = \phi_4(V) + \mathcal{O}(h), && ((Ac^2)^T \otimes I)b(V) = 3\phi_4(V) + \mathcal{O}(h),\\
&((c^3)^T \otimes I)b(V) = 6\phi_4(V) + \mathcal{O}(h), && ((c \cdot Ac)^T \otimes I)b(V) = 2\phi_4(V) + \mathcal{O}(h),\\
&(e^T \otimes I)\bar{b}(V) = \phi_2(V) + \mathcal{O}(h^3), && (c^T \otimes I)\bar{b}(V) = \phi_3(V) + \mathcal{O}(h^2),\\
&((c^2)^T \otimes I)\bar{b}(V) = 2\phi_4(V) + \mathcal{O}(h), && ((Ac)^T \otimes I)\bar{b}(V) = \phi_4(V) + \mathcal{O}(h),
\end{aligned} \tag{12}$$

where $e = (1, 1, 1, 1)^T$.

Choosing $c_1 = 0, c_2 = \frac{1}{2}, c_3 = \frac{1}{2}$ and $c_4 = 1$, we solve the above equations and obtain

$$\begin{aligned}
&a_{21} = \frac{1}{2}, \quad a_{31} = 0, \quad a_{32} = \frac{1}{2}, \quad a_{41} = 0, \quad a_{42} = 0, \quad a_{43} = 1,\\
&\bar{a}_{21} = 0, \quad \bar{a}_{31} = \frac{1}{4}, \quad \bar{a}_{32} = 0, \quad \bar{a}_{41} = 0, \quad \bar{a}_{42} = \frac{1}{2}, \quad \bar{a}_{43} = 0,
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&b_1(V) = \phi_1(V) - 3\phi_2(V) + 4\phi_3(V), && \bar{b}_1(V) = \phi_2(V) - 3\phi_3(V) + 4\phi_4(V),\\
&b_2(V) = 2\phi_2(V) - 4\phi_3(V), && \bar{b}_2(V) = 2\phi_3(V) - 4\phi_4(V),\\
&b_3(V) = 2\phi_2(V) - 4\phi_3(V), && \bar{b}_3(V) = 2\phi_3(V) - 4\phi_4(V),\\
&b_4(V) = -\phi_2(V) + 4\phi_3(V), && \bar{b}_4(V) = -\phi_3(V) + 4\phi_4(V).
\end{aligned} \tag{14}$$

(13)–(14) give an explicit four-stage ARKN method of order four. The Taylor expansions of the coefficients $b_i(V)$ and $\bar{b}_i(V)$ of this method are:

$$\begin{aligned}
&b_1(V) = \frac{1}{6}I - \frac{3}{40}V + \frac{5}{1008}V^2 - \frac{7}{51840}V^3 + \cdots, && \bar{b}_1(V) = \frac{1}{6}I - \frac{1}{45}V + \frac{1}{1120}V^2 - \frac{1}{56700}V^3 + \cdots,\\
&b_2(V) = \frac{1}{3}I - \frac{1}{20}V + \frac{1}{504}V^2 - \frac{1}{25920}V^3 + \cdots, && \bar{b}_2(V) = \frac{1}{6}I - \frac{1}{90}V + \frac{1}{3360}V^2 - \frac{1}{226800}V^3 + \cdots,\\
&b_3(V) = \frac{1}{3}I - \frac{1}{20}V + \frac{1}{504}V^2 - \frac{1}{25920}V^3 + \cdots, && \bar{b}_3(V) = \frac{1}{6}I - \frac{1}{90}V + \frac{1}{3360}V^2 - \frac{1}{226800}V^3 + \cdots,\\
&b_4(V) = \frac{1}{6}I + \frac{1}{120}V - \frac{1}{1680}V^2 + \frac{1}{72576}V^3 + \cdots, && \bar{b}_4(V) = \frac{1}{360}V - \frac{1}{10080}V^2 + \frac{1}{604800}V^3 + \cdots.
\end{aligned} \tag{15}$$

In the single-frequency case where $K = \omega^2 I$, the resulting method reduces to the four-stage fourth-order method given in [2]. We denote this method as ARKN4s4.

In what follows we analyze the construction of multidimensional ARKN methods of order five. In this case, at least six stages are required. Thus, we consider the following Butcher tableau:

$$
\begin{array}{c|cccccccccccc}
c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_2 & a_{21} & 0 & 0 & 0 & 0 & 0 & \bar{a}_{21} & 0 & 0 & 0 & 0 & 0 \\
c_3 & a_{31} & a_{32} & 0 & 0 & 0 & 0 & \bar{a}_{31} & \bar{a}_{32} & 0 & 0 & 0 & 0 \\
c_4 & a_{41} & a_{42} & a_{43} & 0 & 0 & 0 & \bar{a}_{41} & \bar{a}_{42} & \bar{a}_{43} & 0 & 0 & 0 \\
c_5 & a_{51} & a_{52} & a_{53} & a_{54} & 0 & 0 & \bar{a}_{51} & \bar{a}_{52} & \bar{a}_{53} & \bar{a}_{54} & 0 & 0 \\
c_6 & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & 0 & \bar{a}_{61} & \bar{a}_{62} & \bar{a}_{63} & \bar{a}_{64} & \bar{a}_{65} & 0 \\
\hline
 & b_1(V) & b_2(V) & b_3(V) & b_4(V) & b_5(V) & b_6(V) & \bar{b}_1(V) & \bar{b}_2(V) & \bar{b}_3(V) & \bar{b}_4(V) & \bar{b}_5(V) & \bar{b}_6(V)
\end{array}
$$

If we impose the order conditions up to order five, by using the simplifying assumption $Ae = c, \bar{A} = A^2$, the coefficients must satisfy

$$
\begin{array}{ll}
(e^T \otimes I)b(V) = \phi_1(V) + \mathcal{O}(h^5), & (c^T \otimes I)b(V) = \phi_2(V) + \mathcal{O}(h^4), \\
((c^2)^T \otimes I)b(V) = 2\phi_3(V) + \mathcal{O}(h^3), & ((Ac)^T \otimes I)b(V) = \phi_3(V) + \mathcal{O}(h^3), \\
((A^2c)^T \otimes I)b(V) = \phi_4(V) + \mathcal{O}(h^2), & ((Ac^2)^T \otimes I)b(V) = 2\phi_4(V) + \mathcal{O}(h^2), \\
((c^3)^T \otimes I)b(V) = 6\phi_4(V) + \mathcal{O}(h^2), & ((c \cdot Ac)^T \otimes I)b(V) = 3\phi_4(V) + \mathcal{O}(h^2), \\
((A^3c)^T \otimes I)b(V) = 6\phi_5(V) + \mathcal{O}(h), & ((c \cdot A^2c)^T \otimes I)b(V) = 4\phi_5(V) + \mathcal{O}(h), \\
((c^4)^T \otimes I)b(V) = 24\phi_5(V) + \mathcal{O}(h), & ((c^2 \cdot Ac)^T \otimes I)b(V) = 12\phi_5(V) + \mathcal{O}(h), \\
((Ac \cdot Ac)^T \otimes I)b(V) = 6\phi_5(V) + \mathcal{O}(h), & ((c \cdot Ac^2)^T \otimes I)b(V) = 8\phi_5(V) + \mathcal{O}(h), \\
((A^2c^2)^T \otimes I)b(V) = 2\phi_5(V) + \mathcal{O}(h), & ((A(c \cdot Ac))^T \otimes I)b(V) = 3\phi_5(V) + \mathcal{O}(h), \\
((Ac^3)^T \otimes I)b(V) = 6\phi_5(V) + \mathcal{O}(h), &
\end{array}
\tag{16}
$$

and

$$
\begin{array}{ll}
(e^T \otimes I)\bar{b}(V) = \phi_2(V) + \mathcal{O}(h^4), & (c^T \otimes I)\bar{b}(V) = \phi_3(V) + \mathcal{O}(h^3), \\
((c^2)^T \otimes I)\bar{b}(V) = 2\phi_4(V) + \mathcal{O}(h^2), & ((Ac)^T \otimes I)\bar{b}(V) = \phi_4(V) + \mathcal{O}(h^2), \\
((A^2c)^T \otimes I)\bar{b}(V) = \phi_5(V) + \mathcal{O}(h), & ((Ac^2)^T \otimes I)\bar{b}(V) = 2\phi_5(V) + \mathcal{O}(h), \\
((c^3)^T \otimes I)\bar{b}(V) = 6\phi_5(V) + \mathcal{O}(h), & ((c \cdot Ac)^T \otimes I)\bar{b}(V) = 3\phi_5(V) + \mathcal{O}(h),
\end{array}
\tag{17}
$$

where $e = (1, 1, 1, 1, 1, 1)^T$.

Choosing $c_1 = 0, c_2 = \frac{1}{6}, c_3 = \frac{2}{6}, c_4 = \frac{3}{6}, c_5 = \frac{4}{6}$ and $c_6 = 1$ and solving the above equations, we obtain

$$
\begin{array}{llllllll}
a_{21} = \dfrac{1}{6}, & a_{31} = 0, & a_{32} = \dfrac{1}{3}, & a_{41} = -\dfrac{1}{4}, & a_{42} = \dfrac{3}{4}, & a_{43} = 0, & a_{51} = -\dfrac{1}{27}, & a_{52} = \dfrac{2}{9}, \\
a_{53} = \dfrac{1}{3}, & a_{54} = \dfrac{4}{27}, & a_{61} = -\dfrac{2}{11}, & a_{62} = \dfrac{3}{11}, & a_{63} = \dfrac{27}{11}, & a_{64} = -4, & a_{65} = \dfrac{27}{11}, & \\
\bar{a}_{21} = 0, & \bar{a}_{31} = \dfrac{1}{18}, & \bar{a}_{32} = 0, & \bar{a}_{41} = \dfrac{1}{8}, & \bar{a}_{42} = 0, & \bar{a}_{43} = 0, & \bar{a}_{51} = 0, & \bar{a}_{52} = \dfrac{2}{9}, \\
\bar{a}_{53} = 0, & \bar{a}_{54} = 0, & a_{61} = \dfrac{21}{22}, & \bar{a}_{62} = -\dfrac{18}{11}, & \bar{a}_{63} = \dfrac{9}{11}, & \bar{a}_{64} = \dfrac{4}{11}, & \bar{a}_{65} = 0, &
\end{array}
\tag{18}
$$

and

$$
\begin{array}{ll}
b_1(V) = \phi_1(V) - \dfrac{15}{2}\phi_2(V) + 40\phi_3(V) - 135\phi_4(V) + 216\phi_5(V), & \bar{b}_1(V) = \phi_2(V) - 5\phi_3(V) + \dfrac{64}{5}\phi_4(V) - 13\phi_5(V), \\
b_2(V) = 0, & \bar{b}_2(V) = 0, \\
b_3(V) = 27\big(\phi_2(V) - 9\phi_3(V) + 39\phi_4(V) - 72\phi_5(V)\big), & \bar{b}_3(V) = 9\phi_3(V) - \dfrac{171}{5}\phi_4(V) + 45\phi_5(V), \\
b_4(V) = -32\big(\phi_2(V) - 11\phi_3(V) + 54\phi_4(V) - 108\phi_5(V)\big), & \bar{b}_4(V) = -4\phi_3(V) + \dfrac{64}{5}\phi_4(V) - 16\phi_5(V), \\
b_5(V) = \dfrac{27}{2}\big(\phi_2(V) - 12\phi_3(V) + 66\phi_4(V) - 144\phi_5(V)\big), & \bar{b}_5(V) = \dfrac{54}{5}\phi_4(V) - 27\phi_5(V), \\
b_6(V) = -\phi_2(V) + 13\phi_3(V) - 81\phi_4(V) + 216\phi_5(V), & \bar{b}_6(V) = -\dfrac{11}{5}\phi_4(V) + 11\phi_5(V).
\end{array}
\tag{19}
$$

(18)–(19) gives an explicit multidimensional six-stage ARKN method of order five. The Taylor expansions of the coefficients (19) are given by:

$$
\begin{aligned}
b_1(V) &= \frac{11}{120}I - \frac{3}{70}V + \frac{25}{8064}V^2 - \frac{259}{2851200}V^3 + \cdots, & \bar{b}_1(V) &= \frac{11}{120}I - \frac{383}{25200}V + \frac{1231}{1814400}V^2 - \frac{2839}{199584000}V^3 + \cdots,\\
b_2(V) &= 0, & \bar{b}_2(V) &= 0,\\
b_3(V) &= \frac{27}{40}I - \frac{99}{560}V + \frac{9}{896}V^2 - \frac{17}{70400}V^3 + \cdots, & \bar{b}_3(V) &= \frac{9}{20}I - \frac{51}{1400}V + \frac{107}{100800}V^2 - \frac{61}{3696000}V^3 + \cdots,\\
b_4(V) &= -\frac{8}{15}I + \frac{4}{35}V - \frac{1}{126}V^2 + \frac{19}{89100}V^3 + \cdots, & \bar{b}_4(V) &= -\frac{4}{15}I + \frac{59}{3150}V - \frac{59}{113400}V^2 + \frac{197}{24948000}V^3 + \cdots,\\
b_5(V) &= \frac{27}{40}I - \frac{9}{140}V + \frac{3}{896}V^2 - \frac{3}{35200}V^3 + \cdots, & \bar{b}_5(V) &= \frac{9}{40}I - \frac{27}{2280}V + \frac{13}{67200}V^2 - \frac{17}{7392000}V^3 + \cdots,\\
b_6(V) &= \frac{11}{120}I + \frac{1}{336}V - \frac{1}{4480}V^2 + \frac{47}{7983360}V^3 + \cdots, & \bar{b}_6(V) &= \frac{11}{12600}V - \frac{11}{453600}V^2 + \frac{1}{3024000}V^3 + \cdots.
\end{aligned}
\tag{20}
$$

We denote this method as ARKN6s5.

### 3.2. Stability analysis of the multidimensional ARKN methods for the oscillatory system (1)

In this subsection, we are concerned with the stability of the multidimensional ARKN method (4). The numerical methods are for the general oscillatory problem with the perturbations depending on both $y$ and $y'$. Thus, we consider the following second-order homogeneous linear test equation

$$y''(t) + \omega^2 y(t) + \mu y'(t) = 0, \quad \mu < 2\omega, \tag{21}$$

where $\omega$ is called the (undamped) natural frequency of the system and $\mu$ is called the damping. Applying an ARKN integrator to (21) produces

$$
\begin{cases}
Y_i = y_n + hc_i y'_n + h^2 \sum\limits_{j=1}^{s} \bar{a}_{ij}\big(-\omega^2 Y_j - \mu Y'_j\big), \quad i = 1, \dots, s,\\
hY'_i = hy'_n + h^2 \sum\limits_{j=1}^{s} a_{ij}\big(-\omega^2 Y_j - \mu Y'_j\big), \quad i = 1, \dots, s,\\
y_{n+1} = \phi_0(\nu^2)y_n + h\phi_1(\nu^2)y'_n + h^2 \sum\limits_{i=1}^{s} \bar{b}_i(\nu^2)\big(-\mu Y'_i\big),\\
hy'_{n+1} = \phi_0(\nu^2)hy'_n - \nu^2\phi_1(\nu^2)y_n + h^2 \sum\limits_{i=1}^{s} b_i(\nu^2)\big(-\mu Y'_i\big)
\end{cases}
\tag{22}
$$

or in the compact form

$$
\begin{cases}
Y = e \otimes y_n + c \otimes hy'_n + \bar{A}\big(-\nu^2 Y - \sigma hY'\big),\\
hY' = e \otimes hy'_n + A\big(-\nu^2 Y - \sigma hY'\big),\\
y_{n+1} = \phi_0 y_n + \phi_1 hy'_n - \sigma \bar{b}^{\intercal} hY',\\
hy'_{n+1} = \phi_0 hy'_n - \nu^2\phi_1 y_n - \sigma b^{\intercal} hY',
\end{cases}
\tag{23}
$$

where $\nu = h\omega$, $\sigma = h\mu$ and the arguments $(\nu^2)$ is suppressed. With some tedious computations, it follows from (23) that

$$\begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix} = R(\nu, \sigma) \begin{pmatrix} y_n \\ hy'_n \end{pmatrix},$$

where the stability matrix $R(\nu, \sigma)$ is given by

$$R(\nu, \sigma) = \begin{pmatrix} \phi_0 + \sigma\nu^2\bar{b}^{\intercal}AC^{-1}e & \phi_1 + \sigma\nu^2\bar{b}^{\intercal}AC^{-1}c + \sigma^2\bar{b}^{\intercal}AC^{-1}e - \sigma\bar{b}^{\intercal}e \\ -\nu^2\phi_1 + \sigma\nu^2 b^{\intercal}AC^{-1}e & \phi_0 + \sigma\nu^2 b^{\intercal}AC^{-1}c + \sigma^2 b^{\intercal}AC^{-1}e - \sigma b^{\intercal}e \end{pmatrix}.$$

The spectral radius $\rho(R(\nu, \sigma))$ represents the stability of an ARKN method. It can be observed that the stability matrix $R(\nu, \sigma)$ depends on the variables $\nu$ and $\sigma$, therefore, the characterization of stability becomes two-dimensional regions in the $(\nu, \sigma)$-plane, geometrically. Accordingly, we have the following definitions of stability for an ARKN method.

(i) $R_s = \{(\nu, \sigma) \mid \nu > 0,\ \sigma > 0 \text{ and } \rho(R) < 1\}$ is called the *stability region* of the numerical method.

(ii) If $R_s = (0, \infty) \times (0, \infty)$, the method is called *A-stable*.

The stability regions for the methods derived in this section are depicted in Fig. 1.

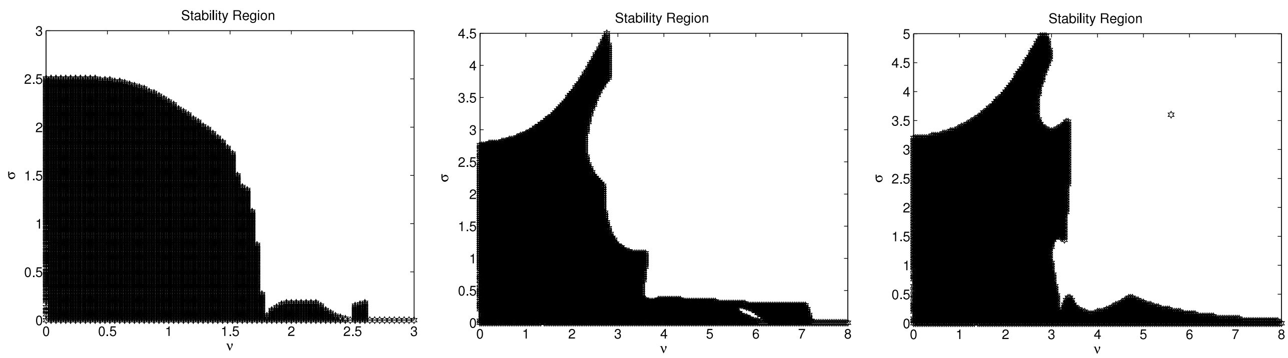


**Fig. 1.** Stability regions for the methods ARKN3s3 (left), ARKN4s4 (middle), and ARKN6s5 (right).

## 4. Numerical experiments

In order to show the robustness and efficiency of the novel ARKN methods in comparison with the existing methods in the scientific literature, we use four problems. The methods used for comparison are:

- ARKN3s3: the three-stage ARKN method of order three given in this paper;
- ARKN4s4: the four-stage ARKN method of order four given in this paper;
- ARKN6s5: the six-stage ARKN method of order five given in this paper;
- ARKNVG5: the fifth-order and six-stage ARKN method ARKNVG5 defined by Franco [2] where the weights are the linear combination of $\phi$ functions without truncation;
- ARKNG5: the fifth-order and six-stage ARKN method ARKNG5 defined by Franco [2] with truncated weights;
- RKN4s4: the classical four-stage RKN method of order four given in II.14 of [12];
- RKN6s5: a six-stage RKN method of order five.

For each experiment, we will display the efficiency curves: accuracy versus the computational cost measured by the number of function evaluations required by each method.

**Remark 4.1.** Actually, for the one-dimensional case, namely, $M$ now is a scalar, the scalar functions $\phi_l(M)$ can be computed by means of the recurrence relation

$$\phi_{j+2}(M) = M^{-1}\left(\frac{I}{j!} - \phi_j(M)\right), \quad j = 0, 1, \ldots$$

with

$$\phi_0(M) = \cos(\sqrt{M}), \qquad \phi_1(M) = \frac{\sin\sqrt{M}}{\sqrt{M}}.$$

Thus, for single-frequency problems, the weights in the numerical methods can be evaluated exactly. For this reason, in the following, we apply methods ARKN3s3, ARKN4s4, ARKN6s5, ARKNVG5, RKN4s4, and RKN6s5 with the weights computed exactly to single-frequency problems. For the multidimensional case, we approximate the weights using Taylor expansion, and methods ARKN3s3, ARKN4s4, ARKN6s5, ARKNG5, RKN4s4, and RKN6s5 with the weights computed with truncation are used for comparison.

**Remark 4.2.** In [3], the author suggests that $\phi_0(V)$ and $\phi_1(V)$ are approximated by polynomials with order no less than the order of the numerical method for the oscillatory systems whose right-hand side functions are independent of $y'$. In this paper, we calculate $\phi_0(V)$ and $\phi_1(V)$ in a more accurate way. Since we have $\phi_0(4V) = 2\phi_0^2(V) - I$, $\phi_1(4V) = \phi_0(V)\phi_1(V)$, which are analogues of the relation between sine and cosine, we use the algorithm similar to Algorithm 1.1 in [14] in our numerical experiments for multi-frequency and multidimensional problems.

**Problem 1.** We consider the linear problem

$$\begin{cases} y''(t) + \omega^2 y(t) = -\delta y'(t), \\ y(0) = 1, \qquad y'(0) = -\dfrac{\delta}{2}. \end{cases}$$

The analytic solution of this initial value problem is given by $y(t) = \exp(-\frac{\delta}{2}t)\cos(\sqrt{\omega^2 - \frac{\delta^2}{4}}\, t)$. In this test we choose the parameter values $\omega = 1,\ \delta = 10^{-3}$. We integrate the problem on the interval $[0, 100]$ with the stepsizes $h = 1/2^j, j = 1, 2, 3, 4$. The numerical results are stated in Fig. 2(left).

**Problem 2.** Consider the Vander Pol equation:

$$\begin{cases} y''(t) + y(t) = \delta(1 - y(t)^2)y'(t), \\ y(0) = 2 + \dfrac{1}{96}\delta^2 + \dfrac{1033}{552960}\delta^4 + \dfrac{1019689}{55738368000}\delta^6, \qquad y'(0) = 0. \end{cases}$$

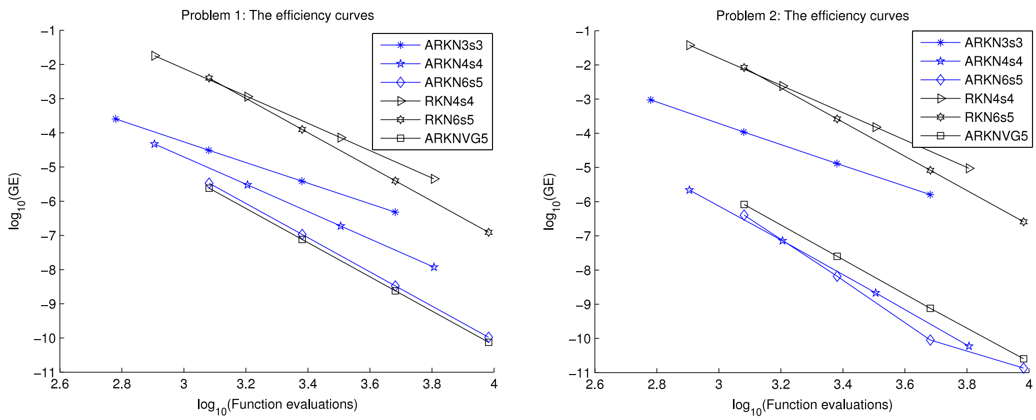


**Fig. 2.** Results for Problems 1 and 2: the number of function evaluations against $\log_{10}(GE)$, the logarithm of the maximum global error over the integration interval.

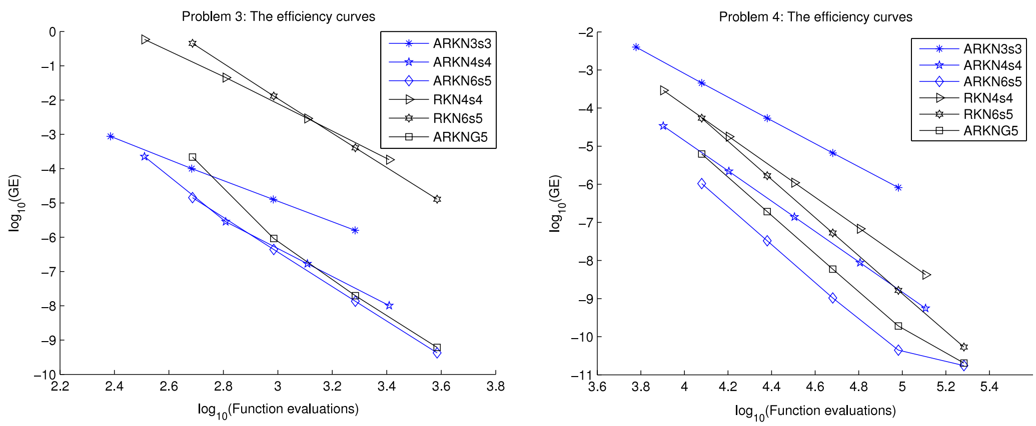


**Fig. 3.** Results for Problems 3 and 4: the number of function evaluations against $\log_{10}(GE)$, the logarithm of the maximum global error over the integration interval.

Here we take $\delta = 0.8 \times 10^{-4}$. We integrate the problem in the interval $[0, 100]$ with the stepsizes $h = 1/2^j, j = 1, 2, 3, 4$. In order to estimate the error for each method, a reference numerical solution is obtained by method RKN4s4 given in II.14 of [12] with a very small stepsize. The numerical results are stated in Fig. 2(right).

**Problem 3.** Consider the initial value problem

$$\begin{cases} y''(t) + \begin{pmatrix} 13 & -12 \\ -12 & 13 \end{pmatrix} y(t) = \dfrac{12\varepsilon}{5} \begin{pmatrix} 3 & 2 \\ -2 & -3 \end{pmatrix} y'(t) + \varepsilon^2 \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \quad 0 < t \le t_{end}, \\ y(0) = \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix}, \qquad y'(0) = \begin{pmatrix} -4 \\ 6 \end{pmatrix} \end{cases}$$

with

$$f_1(t) = \frac{36}{5}\sin(t) + 24\sin(5t), \qquad f_2(t) = -\frac{24}{5}\sin(t) - 36\sin(5t).$$

The analytic solution to this problem is given by

$$y(t) = \begin{pmatrix} \sin(t) - \sin(5t) + \varepsilon\cos(t) \\ \sin(t) + \sin(5t) + \varepsilon\cos(5t) \end{pmatrix}.$$

In this test, we integrate the problem in the interval $[0, 20]$ with the stepsizes $h = 1/2^j, j = 2, 3, 4, 5$ and $\varepsilon = 10^{-3}$. The numerical results are displayed in Fig. 3(left).

**Problem 4.** Consider the damped wave equation with periodic boundary conditions

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \delta\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f(u), \quad -1 < x < 1, \ t > 0, \\ u(-1, t) = u(1, t). \end{cases}$$

The problem stands for the wave propagation in a medium [13]. A semi-discretization in the spatial variable by second-order symmetric differences leads to the following system of second-order ODEs in time

$$\ddot{U} + KU = F(U, \dot{U}), \qquad 0 < t \le t_{end},$$

where $U(t) = (u_1(t), \ldots, u_N(t))^T$ with $u_i(t) \approx u(x_i, t)$, $i = 1, \ldots, N$,

$$K = \frac{1}{\Delta x^2}\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

with $\Delta x = 2/N$ and $x_i = -1 + i\Delta x$, and $F(U, \dot{U}) = \big(f(u_1) - \delta\dot{u}_1, \ldots, f(u_N) - \delta\dot{u}_N\big)^T$.

In this experiment, we take $f(u) = -\sin u$, $\delta = 0.08$ and the initial conditions as

$$U(0) = (\pi)_{i=1}^N, \qquad U_t(0) = \sqrt{N}\left(0.01 + \sin\left(\frac{2\pi i}{N}\right)\right)_{i=1}^N.$$

We choose $N = 40$ and integrate the system in the interval $[0, 100]$ with the stepsizes $h = 0.1/2^j, j = 1, 2, 3, 4$. The reference numerical solution is obtained by method RKN4s4 with a very small stepsize. The numerical results are shown in Fig. 3(right).

## 5. Conclusions

It is noted that the concrete multidimensional ARKN methods for the general multi-frequency and multidimensional oscillatory second-order initial value problem (1) whose right-hand side functions *depend on both y and y′* were not constructed in the existing scientific literature, although the order conditions were presented by Wu et al. [8]. In this paper, we make further discussions and analysis on multidimensional ARKN methods for the multi-frequency oscillatory system (1) proposed in paper [8]. We construct three novel multidimensional ARKN methods for the multi-frequency oscillatory system (1) based on the order conditions presented in [8] and analyze the stability of the novel methods. We compare the novel multidimensional ARKN methods with some existing effective methods in the scientific literature by numerical experiments. It can be observed that our new multidimensional ARKN methods perform very well as compared with the effective methods appeared in the scientific literature and are more suitable for the numerical integration of general multi-frequency oscillatory problem (1).

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