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Stability Analysis for Explicit ERKN Methods Solving General Second-Order Oscillatory Systems

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Abstract

In order to solve the general multidimensional perturbed oscillatory system $y''+\Omega y = f(y, y')$ with $K \in \mathbb{R}^{d\times d}$, the order conditions for the ERKN (extended Runge–Kutta–Nyström) methods and some effective ERKN methods were presented in the literature. These methods integrate exactly the multidimensional unperturbed oscillator $y'' + \Omega y = 0$. In this paper, we analyze the stability of ERKN methods for general oscillatory second-order initial value problems whose right-hand-side functions depend on both y and y'. Based on the linear test model $y''(t)+\omega^2 y(t)+\mu y'(t) = 0$ with $\mu < 2\omega$, further discussion and analysis on the linear stability of ERKN methods for general oscillatory problems are presented. A new conception of α-stability region is proposed to investigate how well the numerical methods respect the damping rate of the general oscillatory systems. It gains more insight to the numerical methods when applied to the systems involving y'. Numerical experiments are carried out to show the significance of the theory.

Keywords Extended Runge–Kutta–Nyström methods · Stability · Damping · Oscillatory systems

Mathematics Subject Classification 65L05 · 65L06

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1 Introduction

In the past decades, the research of numerical methods for differential equations having oscillatory solutions has gained much attention. Those problems arise in many fields such as theoretical physics, quantum physics, theoretical chemistry and electronics. In particular, one needs to solve efficiently an oscillatory system with the form

$$\begin{cases} y''(t) + \Omega y(t) = f(y(t), y'(t)), \qquad t \in [t_0, T], \\ y(t_0) = y_0, \quad y'(t_0) = y'_0, \end{cases} \tag{1}$$

where Ω is a positive semi-definite matrix that implicitly contains the dominant frequencies of the system and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $y \in \mathbb{R}^d$, $y' \in \mathbb{R}^d$. For the case where the right-hand-side function f depends only on position y, i.e., $f \equiv f(y)$, a variety of numerical methods have been developed. Many efforts have been spent to derive and analyze Gautschi-type methods [1,2] after the Gautschi's method [3] was proposed in 1961. In [4], based on G-functions, a new family of explicit Runge–Kutta-type methods denoted by RKGM for the numerical integration of perturbed oscillators is designed. Franco [5,6] presented a new type of adapted Runge–Kutta–Nyström methods (ARKN) specially designed for the numerical integration of perturbed oscillators.

In [7], Wu et al. presented a standard form of the multidimensional ERKN (extended Runge–Kutta–Nyström) methods. The structure brought by Ωy is incorporated into the internal stages and updates of the ERKN methods. An outstanding advantage of ERKN methods is that they could integrate the multidimensional unperturbed problem $y'' + \Omega y = 0$ exactly. In [8], You et al. extended it to the general systems with right-hand-side term f depending on both position y and velocity y'. The ERKN method has become a very powerful tools dealing with oscillatory systems (1). Based on the ERKN methods, many related structure-preserving algorithms are designed for the oscillatory systems of the form (1). Both the Gautschi-type methods, RKGM methods and ARKN methods can be categorized as a special case of ERKN methods. In [9], the author presented a multi-symplectic extended leap-frog methods for Hamiltonian wave equations. An extended discrete gradient formula for oscillatory Hamiltonian systems is given in [10]. Wang et al. [11,12] constructed two types of trigonometric Fourier collocation methods based on the shifted Legendre polynomials and Lagrange basis polynomials. The readers are referred to [13–15] for more details on this topic.

It is well known that a numerical method cannot be used to numerically solve differential equations without the knowledge of its stability behavior. In practice, the stability behavior is also closely related to the efficiency of the numerical method. This fact is particularly obvious when it comes to the oscillatory differential equations. The stepsize of the numerical method is limited by the largest frequency of the system under consideration. The restriction of the stepsize is mainly concerned with the stability of the numerical method. For the particular case $f \equiv f(y)$, in order to investigate the stability of an ARKN method, Franco [16] considered the test equation

$$y''(t) + \omega^2 y(t) = -\varepsilon y(t), \ \omega, \varepsilon > 0, \tag{2}$$

where ε is viewed as the error of the estimation of frequency. Wu [17] pointed out that the assumption $\varepsilon > 0$ leads to incompleteness of the stability region. The author removed the restriction $\varepsilon > 0$ and gave a complete stability analysis based on the modified test equation

$$y''(t) + \omega^2 y(t) = -\varepsilon y(t),\ \omega^2 + \varepsilon > 0. \tag{3}$$

In this paper, we focus ourselves on the stability analysis of ERKN methods for general oscillatory systems (1). In most of the studies, the test model was used for the stability analysis of ERKN methods. However, it should be noted that when it comes to the general case $f \equiv f(y, y')$, neither (2) nor (3) is appropriate to analyze the stability of the numerical methods because the two models do not involve y', which contradicts the fact that the methods are for general problems explicitly containing velocity y'. To remedy this shortcoming, a natural idea is to add the linear damping term $\mu y'$ to the test equation. Thus, in order to deal with the stability of ERKN methods for general second-order oscillatory systems (1), we introduce the following linear damped oscillator

$$y''(t) + \omega^2 y(t) + \mu y'(t) = 0, \tag{4}$$

where $\omega > 0$ is called the undamped natural frequency of the system, and $\mu > 0$ is called the damping. Here, for simplicity, we assume that the natural frequency ω can be estimated exactly. It is always assumed that $\dfrac{\mu}{2\omega} < 1$, in which case the solution oscillates with a amplitude gradually decreasing to zero.

The rest of the paper is organized as follows. In Sect. 2, based on (4), we state the elementary analysis of stability for multidimensional ERKN methods for general oscillatory systems. In Sect. 3, the stability regions for some available multidimensional ERKN methods are shown based on the definition given in Sect. 2. In Sect. 4, the conception of $\alpha-$stability region is introduced and a numerical experiment is carried out to illustrate our theory of stability. Section 5 is devoted to conclusions.

2 The Stability Analysis for ERKN Methods for General Oscillatory Systems

Before describing the stability analysis, we define a series of matrix-valued functions which were given in [18] as follows:

$$\phi_j(K) := \sum_{i=0}^{\infty} \frac{(-1)^i K^i}{(2i+j)!}, \quad j = 0, 1, \ldots. \tag{5}$$

Under the assumption that K is a symmetric and positive semi-definite matrix, we have the decomposition of V as

$$K = P^T W^2 P = \Gamma^2 \text{ with } \Gamma = P^T W P,$$

where P is an orthogonal matrix and W is a diagonal matrix whose diagonal entries are the square roots of the eigenvalues of K. It can be verified that

$$\begin{aligned}\phi_0(K) &= \cos(\Gamma) = P^T \cos(W)P,\\ \phi_1(K) &= \Gamma^{-1}\sin(\Gamma) = P^T W^{-1}\sin(W)P,\end{aligned}$$

and the functions $\phi_j(K)$ $(j = 0, 1, \cdots)$ have the following recurrence relations

$$\phi_{j+2}(K) = V^{-1}\left(\frac{I}{j!} - \phi_j(K)\right),\ \ j = 0, 1, \cdots .$$

The formulation of the ERKN methods and the corresponding order conditions were presented in [8] for the general oscillatory system (1).

Definition 2.1 An s-stage ERKN method for the numerical integration of the general oscillatory system (1) is defined by the following scheme

$$\begin{cases} Y_i = \phi_0(c_i^2K)y_n + c_i\phi_1(c_i^2K)hy_n' + h^2\sum\limits_{j=1}^{s}\bar{a}_{ij}(K)f(Y_j, Y_j'), & i = 1, \ldots, s,\\ Y_i' = -c_ih\Omega\phi_1(c_i^2V)y_n + \phi_0(c_i^2V)y_n' + h\sum\limits_{j=1}^{s}a_{ij}(K)f(Y_j, Y_j'), & i = 1, \ldots, s,\\ y_{n+1} = \phi_0(K)y_n + \phi_1(K)hy_n' + h^2\sum\limits_{i=1}^{s}\bar{b}_i(K)f(Y_i, Y_i'),\\ y_{n+1}' = -h\Omega\phi_1(K)y_n + \phi_0(K)y_n' + h\sum\limits_{i=1}^{s}b_i(K)f(Y_i, Y_i'), \end{cases} \tag{6}$$

where $\phi_j(K)$, $j = 0, 1, \ldots,$ are given by (5), and $\bar{a}_{ij}(K), a_{ij}(K)$, $\bar{b}_i(K), b_i(K), i, j = 1, \ldots, s$ are matrix-valued functions which can be expanded into series in powers of $K = h^2\Omega$ with real coefficients.

The method can be expressed in partitioned Butcher's tableau of coefficients

$$\begin{array}{c|cc} c & A & \bar{A} \\ \hline & b^T(K) & \bar{b}^T(K) \end{array} = \begin{array}{c|ccc|ccc} c_1 & a_{11}(K) & \ldots & a_{1s}(K) & \bar{a}_{11}(K) & \ldots & \bar{a}_{1s}(K)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ c_s & a_{s1}(K) & \cdots & a_{ss}(K) & \bar{a}_{s1}(K) & \cdots & \bar{a}_{ss}(K)\\ \hline & b_1(K) & \cdots & b_s(K) & \bar{b}_1(K) & \cdots & \bar{b}_s(K) \end{array}$$

The order conditions for ERKN methods (6) are given in [8].

Applying an ERKN method (6) to the test equation (4), letting $\nu = h\omega, \sigma = h\mu$ gives

$$\begin{cases} Y_i = \phi_0(c_i^2\nu^2)y_n + c_i h\phi_1(c_i^2\nu^2)y_n' + h^2\sum_{j=1}^{s}\bar{a}_{ij}(\nu^2)\big(-\mu Y_i'\big), \quad i = 1, \cdots, s, \\ hY_i' = \phi_0(c_i^2\nu^2)hy_n' - c_i\nu^2\phi_1(c_i^2\nu^2)y_n + h^2\sum_{j=1}^{s}a_{ij}(\nu^2)\big(-\mu Y_i'\big), \quad i = 1, \cdots, s, \\ y_{n+1} = \phi_0(\nu^2)y_n + h\phi_1(\nu^2)y_n' + h^2\sum_{i=1}^{s}\bar{b}_i(\nu^2)\big(-\mu Y_i'\big), \\ hy_{n+1}' = \phi_0(\nu^2)hy_n' - \nu^2\phi_1(\nu^2)y_n + h^2\sum_{i=1}^{s}b_i(\nu^2)\big(-\mu Y_i'\big), \end{cases} \tag{7}$$

or in the compact form

$$\begin{cases} Y = \phi_0(c^2\nu^2) \otimes y_n + c\phi_1(c^2\nu^2) \otimes hy_n' - \sigma\bar{A}hY', \\ hY' = -c\phi_1(c^2\nu^2)\nu^2 \otimes y_n + \phi_0(c^2\nu^2) \otimes hy_n' - A\sigma hY', \\ y_{n+1} = \phi_0 y_n + \phi_1 hy_n' - \sigma\bar{b}^T hY', \\ hy_{n+1}' = \phi_0 hy_n' - \nu^2\phi_1 y_n - \sigma b^T hY', \end{cases} \tag{8}$$

where $\phi_i(c^2\nu^2) = (\phi_i(c_1^2\nu^2), \cdots, \phi_i(c_s^2\nu^2))^\intercal, i = 0, 1$ and the arguments (ν^2) in ϕ-functions are suppressed.

By the second equation of (8), we have

$$hY' = (I + \sigma A)^{-1}\big(-c\phi_1(c^2\nu^2)\nu^2 \otimes y_n + \phi_0(c^2\nu^2) \otimes hy_n'\big).$$

Substituting it in the last two equations of (8) obtains

$$\begin{cases} y_{n+1} = \phi_0 y_n + \phi_1 hy_n' - \sigma\bar{b}^T(I + \sigma A)^{-1}\big(-c\phi_1(c^2\nu^2)\nu^2 \otimes y_n + \phi_0(c^2\nu^2) \otimes hy_n'\big), \\ hy_{n+1}' = \phi_0 hy_n' - \nu^2\phi_1 y_n - \sigma b^T(I + \sigma A)^{-1}\big(-c\phi_1(c^2\nu^2)\nu^2 \otimes y_n + \phi_0(c^2\nu^2) \otimes hy_n'\big), \end{cases}$$

or in a matrix form,

$$
\begin{aligned}
&\begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} \phi_0 & \phi_1 \\ -\nu^2\phi_1 & \phi_0 \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix} \\
&\quad + \begin{pmatrix} \sigma\nu^2\bar{b}^T(I+\sigma A)^{-1}c\phi_1(c^2\nu^2) & -\sigma\bar{b}^T(I+\sigma A)^{-1}\phi_0(c^2\nu^2) \\ \sigma\nu^2 b^T(I+\sigma A)^{-1}c\phi_1(c^2\nu^2) & -\sigma b^T(I+\sigma A)^{-1}\phi_0(c^2\nu^2) \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix} \\
&= \begin{pmatrix} \phi_0 + \sigma\nu^2\bar{b}^T(I+\sigma A)^{-1}c\phi_1(c^2\nu^2) & \phi_1 - \sigma\bar{b}^T(I+\sigma A)^{-1}\phi_0(c^2\nu^2) \\ -\nu^2\phi_1 + \sigma\nu^2 b^T(I+\sigma A)^{-1}c\phi_1(c^2\nu^2) & \phi_0 - \sigma b^T(I+\sigma A)^{-1}\phi_0(c^2\nu^2) \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix}.
\end{aligned}
\tag{9}
$$

Namely, the characteristic matrix of an ERKN method is given by

$$
M(\nu,\sigma) = \begin{pmatrix} \phi_0 + \sigma\nu^2\bar{b}^T(I+\sigma A)^{-1}c\phi_1(c^2\nu^2) & \phi_1 - \sigma\bar{b}^T(I+\sigma A)^{-1}\phi_0(c^2\nu^2) \\ -\nu^2\phi_1 + \sigma\nu^2 b^T(I+\sigma A)^{-1}c\phi_1(c^2\nu^2) & \phi_0 - \sigma b^T(I+\sigma A)^{-1}\phi_0(c^2\nu^2) \end{pmatrix}
$$

The stability behavior of an ERKN method is characterized by the spectral radius $\rho(M)$. If $\rho(M) \le 1$, the ERKN method is stable; otherwise, the method is unstable. We are now ready to give the definition of stability for an ERKN method for solving the general second-order oscillatory system (1).

Definition 2.2 Let $M(\nu,\sigma)$ denote the characteristic matrix of an ERKN method when applied to the linear test model (4). The nomenclature of stability for ERKN methods is given as follows:

(i) $R_s = \{(\nu,\sigma) |\ \nu > 0,\ \sigma > 0 \text{ and } \rho(M) < 1\}$ is called the *stability region* of the ERKN method.

(ii) If $R_s = (0,\infty)\times(0,\infty)$, the ERKN method is called *A-stable*.

Remark 2.1 It is noted that the characteristic matrix $M(\nu,\sigma)$ depends on the variables ν and σ; therefore, the stability region is two-dimensional in the (ν,σ)-plane. Since the linear test equation (4) is damped, there is no need to consider the periodicity region of the numerical method for general second-order systems.

3 Stability Regions for Some Existing Multidimensional ERKN Methods

In this section, we present the traditional stability regions of the three ERKN methods ERKN2, ERKN3 and ERKN4 proposed in [8]. The three methods are of order two, three and four, respectively, and can be expressed in Butcher's tableaus as follows:

ERKN2:

$$
\begin{array}{c|cc|cc}
0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & \frac{2}{3}I & 0 & 0 & 0 \\
\hline
 & \phi_1(K) - \frac{3}{2}\phi_2(K) & \frac{3}{2}\phi_2(K) & \phi_2(K) - \frac{3}{2}\phi_3(K) & \frac{3}{2}\phi_3(K)
\end{array}
$$

ERKN3:

$$\begin{array}{c|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2} & \frac{1}{2}I & 0 & 0 & 0 & 0 & 0\\
1 & -I & 2I & 0 & I & 0 & 0\\
\hline
 & b_1(K) & b_2(K) & b_3(K) & \bar{b}_1(K) & \bar{b}_2(K) & \bar{b}_3(K)
\end{array}$$

with

$$\begin{aligned}
&b_1(K) = \phi_1(K) - 3\phi_2(K) + 4\phi_3(K),\ \ b_2(K) = 4\phi_2(K) - 8\phi_3(K),\\
&b_3(K) = -\phi_2(K) + 4\phi_3(K),\\
&\bar{b}_1(K) = \phi_2(K) - 2\phi_3(K),\ \ \bar{b}_2(K) = 2\phi_3(K),\ \ \bar{b}_3(K) = 0.
\end{aligned}$$

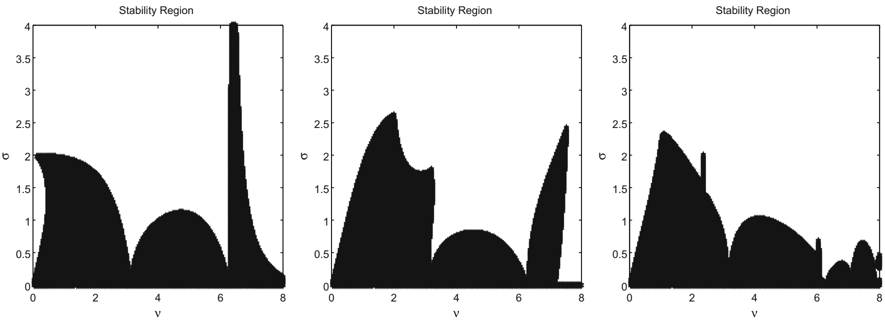
ERKN4:

$$\begin{array}{c|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
c_2 & a_{21}(K) & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
c_3 & a_{31}(K) & a_{32}(K) & 0 & 0 & \bar{a}_{31}(K) & 0 & 0 & \\
c_4 & a_{41}(K) & a_{42}(K) & a_{43}(K) & 0 & \bar{a}_{41}(K) & \bar{a}_{42}(K) & 0 & 0\\
\hline
 & b_1(K) & b_2(K) & b_3(K) & b_4(K) & \bar{b}_1(K) & \bar{b}_2(K) & \bar{b}_3(K) & \bar{b}_4(K)
\end{array}$$

The coefficients of ERKN4 are given below:

$$\begin{aligned}
&c_2 = \frac{1}{5},\ c_3 = \frac{2}{3},\ c_4 = 1, a_{21}(K) = \frac{1}{2}c_2\big(\phi_0(c_2^2K) + \phi_1(c_2^2K)\big),\\
&a_{31}(K) = -\frac{17}{27}I,\ a_{32}(K) = \frac{1}{2}\Big(-2a_{31}(K) + c_3\big(\phi_0(c_3^2K) + \phi_1(c_3^2K)\big)\Big)\big(\phi_0(c_2^2K)\big)^{-1},\\
&a_{41}(K) = \frac{13}{5}I,\ a_{42}(K) = -\frac{22}{7}I,\\
&a_{43}(K) = \frac{1}{2}\Big(-2a_{41}(K) - 2a_{42}(K)\phi_0(c_2^2K) + c_4\big(\phi_0(c_4^2K) + \phi_1(c_4^2K)\big)\Big)(\phi_0(c_3^2K))^{-1},\\
&\bar{a}_{31}(K) = a_{32}(K)a_{21}(K),\ \bar{a}_{41}(K) = a_{42}(K)a_{21}(K) + a_{43}(K)a_{31}(K),\\
&\bar{a}_{42}(K) = a_{43}(K)a_{32}(K).
\end{aligned} \tag{10}$$

$$\begin{aligned}
&b_1(K) = \phi_1(K) + \frac{-(c_3c_4 + c_2c_3 + c_2c_4)\phi_2(K) + 2(c_2 + c_3 + c_4)\phi_3(K) - 6\phi_4(K)}{c_2c_3c_4},\\
&b_2(K) = \frac{c_3c_4\phi_2(K) - 2\big((c_3 + c_4)\phi_3(K) - 3\phi_4(K)\big)}{c_2(c_2 - c_3)(c_2 - c_4)},\\
&b_3(K) = \frac{-c_2c_4\phi_2(K) + 2\big((c_2 + c_4)\phi_3(K) - 3\phi_4(K)\big)}{c_3(c_2 - c_3)(c_3 - c_4)},\\
&b_4(K) = \frac{c_2c_3\phi_2(K) - 2(c_2 + c_3)\phi_3(K) + 6\phi_4(K)}{c_4(c_2 - c_4)(c_3 - c_4)},\\
&\bar{b}_1(K) = \phi_2(K) + \big(\beta_1\phi_3(K) + \gamma_1\phi_4(K)\big)M_1^{-1},\\
&\bar{b}_2(K) = \big(\beta_2\phi_3(K) + \gamma_2\phi_4(K)\big)M_2^{-1},\\
&\bar{b}_3(K) = \big(\beta_3\phi_3(K) + \gamma_3\phi_4(K)\big)M_3^{-1},\\
&\bar{b}_4(K) = \big(\beta_4\phi_3(K) + \gamma_4\phi_4(K)\big)M_4^{-1},
\end{aligned} \tag{11}$$



(a) $ERKN2$ **(b)** $ERKN3$ **(c)** $ERKN4$

Fig. 1 Stability regions of the methods ERKN2, ERKN3 and ERKN4

where

$$
\begin{aligned}
\beta_1 &= -\big(a_{42}(K)c_2 + a_{43}(K)c_3\big)(c_2^2 - c_3^2) + a_{32}(K)c_2(c_2^2 - c_4^2),\\
\beta_2 &= -a_{42}(K)c_2c_3^2 - a_{43}(K)c_3^3 + a_{32}(K)c_2c_4^2,\\
\beta_3 &= -c_2\big(a_{42}(K)c_2 + a_{43}(K)c_3\big), \qquad \beta_4 = a_{32}(K)c_2^2,\\
\gamma_1 &= -2a_{32}(K)c_2(c_2 - c_4) + (c_2 - c_3)\\
&\quad \big(2a_{42}(K)c_2 + 2a_{43}(K)c_3 - (c_2 - c_4)(c_3 - c_4)I\big),\\
\gamma_2 &= 2a_{42}(K)c_2c_3 + 2a_{43}(K)c_3^2 + c_4\big(-2a_{32}(K)c_2 + c_3^2 I - c_3c_4 I\big),\\
\gamma_3 &= 2a_{42}(K)c_2 + 2a_{43}(K)c_3 + (c_2 - c_4)c_4 I,\\
\gamma_4 &= -2a_{32}(K)c_2 + c_3(-c_2 + c_3)I,\\
M_1 &= a_{42}(K)c_2^2c_3(c_2 - c_3) + a_{43}(K)c_2c_3^2(c_2 - c_3) - a_{32}(K)c_2^2c_4(c_2 - c_4),\\
M_2 &= c_2\big(a_{42}(K)c_2(c_2 - c_3)c_3 + a_{43}(K)(c_2 - c_3)c_3^2 + a_{32}(K)c_2c_4(-c_2 + c_4)\big),\\
M_3 &= a_{42}(K)c_2c_3(-c_2 + c_3) + a_{43}(K)c_3^2(-c_2 + c_3) + a_{32}(K)c_2(c_2 - c_4)c_4,\\
M_4 &= a_{42}(K)c_2c_3(-c_2 + c_3) + a_{43}(K)c_3^2(-c_2 + c_3) + a_{32}(K)c_2(c_2 - c_4)c_4.
\end{aligned}
$$

The stability regions of ERKN2, ERKN3 and ERKN4 are shown in Fig. 1.

It can be observed from Fig. 1 that all three explicit ERKN methods have large stability regions and the stability regions can go on if we continue drawing. Does it mean that a large stepsize can be chosen when applied to the general oscillatory problem? The answer is negative. The main reason is that the stability of numerical methods only assures the error propagation in linearized equations is bounded. However, when dealing with damping systems involving y', just the linear stability of the numerical method does not guarantee a good behavior of the method. A further requirement for the numerical method is that the numerical solution and the exact solution exhibit the same damping rate as much as possible, which leads us to the conception of α-stability.

4 α-Stability Regions and Numerical Experiments

The exact solution to (4) with initial conditions $y(0) = y_0$, $y'(0) = y_0'$ is given by

$$y(t) = c_+ e^{\left(-\frac{\mu}{2} + \frac{\sqrt{4\omega^2 - \mu^2}}{2} i\right)t} + c_- e^{\left(-\frac{\mu}{2} - \frac{\sqrt{4\omega^2 - \mu^2}}{2} i\right)t}, \tag{12}$$

where

$$c_\pm = \frac{y_0}{2} \pm \frac{\frac{\mu}{2} y_0 + y_0'}{\sqrt{4\omega^2 - \mu^2}} i.$$

Let $c_\pm = |c| e^{\pm i\psi}$. Then, for $t_n = nh$, we have

$$y(t_n) = 2|c| e^{-\frac{\sigma}{2} n} \cos\left(\psi + \frac{\sqrt{4\nu^2 - \sigma^2}}{2} n\right).$$

Hence, $e^{-\frac{\sigma}{2}}$ can be viewed as the damping rate for every step and the solution oscillatorily decays at the rate $e^{-\frac{\sigma}{2}}$. The numerical counterpart of damping rate is $\rho(M)$. This can be seen from the fact that if M has two complex conjugate eigenvalues $\lambda_\pm = e^{a \pm bi}$ ($a < 0$ if the numerical method is stable), $\rho(M) = e^a$ is called the amplification factor of the numerical method which represents the damping rate of the method. Therefore, we introduce the α-stability region of a numerical method as follows.

Definition 4.1 Let $M(\nu, \sigma)$ denote the characteristic matrix of an ERKN method when applied to the linear test model (4).

$$R_s(\alpha) = \left\{ (\nu, \sigma) \in R_s \,\middle|\, \left| \frac{\rho(M) - e^{-\frac{\sigma}{2}}}{e^{-\frac{\sigma}{2}}} \right| \le \alpha \right\}$$

is called the α-*stability region* of the ERKN method, where $0 < \alpha < 1$.

The 0.01-stability regions of methods ERKN2, ERKN3 and ERKN4 are shown in Fig. 2.

In order to illustrate the significance of the new definitions of stability regions, we apply the three ERKN methods to the following Van de Pol equation:

$$\begin{cases} y''(t) + \omega^2 y(t) + \delta(y^2(t) - 1) y'(t) = 0, \\ y(0) = 2 + \dfrac{1}{96}\delta^2 + \dfrac{1033}{552960}\delta^4 + \dfrac{1019689}{55738368000}\delta^6, \quad y'(0) = 0. \end{cases}$$

Taking $\omega = 2, \delta = 1, \dfrac{1}{3}, \dfrac{1}{3^2}$, we integrate the equation in the interval $[0, 40]$ by methods ERKN2, ERKN3 and ERKN4 with stepsize $h = 1$. A reference exact solution is obtained by the classical four-stage RKN method of order four given in II.14 of [19] with a very small stepsize. The reference exact solution and the numerical solutions are plotted in Fig. 3.

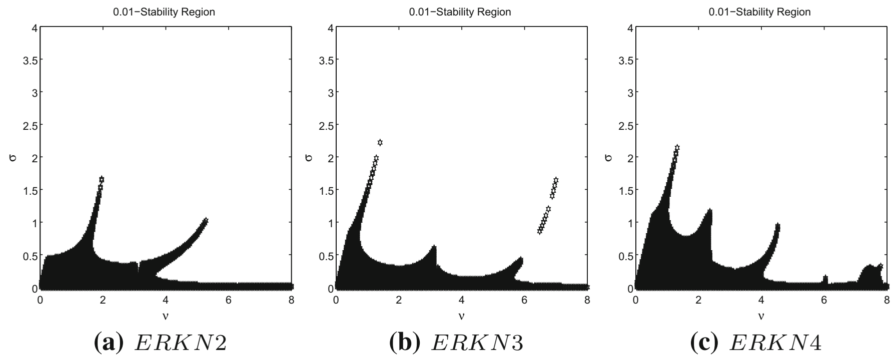


Fig. 2 0.01-Stability regions of the methods ERKN2, ERKN3 and ERKN4

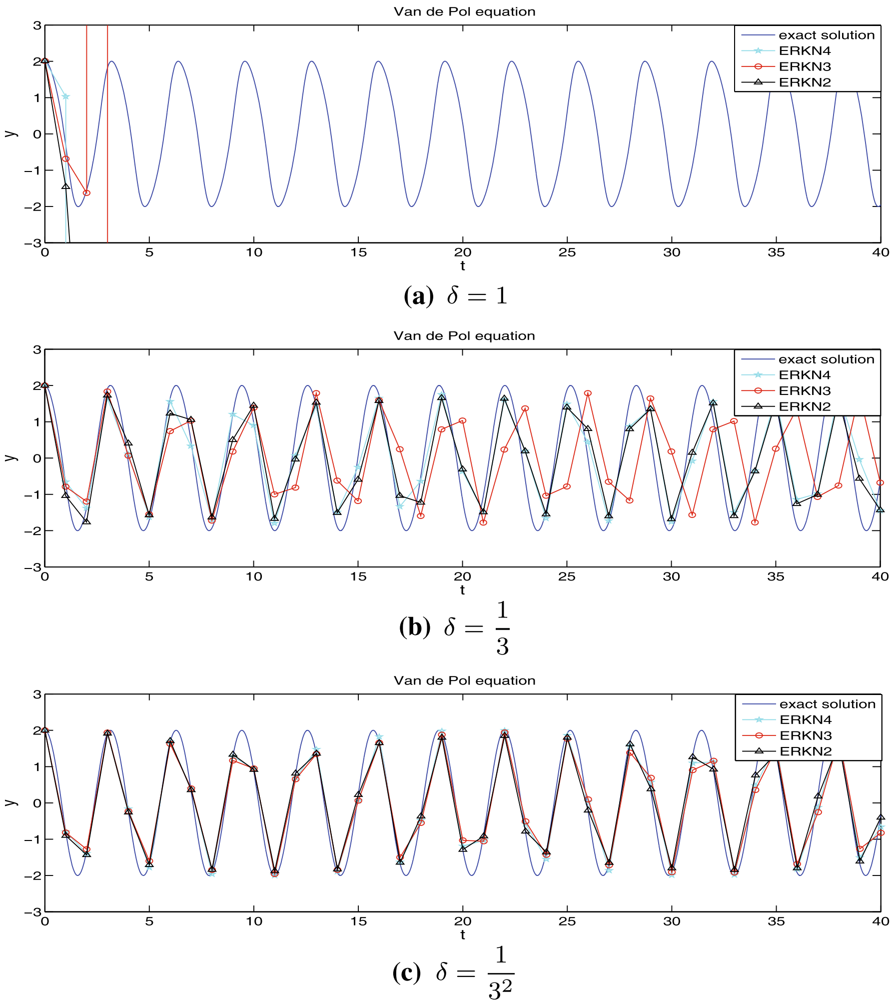


Fig. 3 Numerical solutions of Van de Pol equation

It can be seen from Fig. 3 that for $\omega = 2, \delta = 1, \frac{1}{3}, \frac{1}{3^2}$, the exact solution of Van de Pol equation approximately oscillates from $[-2, 2]$ with frequency $\omega = 2$. The damping term is $\delta(y^2 - 1)y'$. Therefore, we might consider $\mu = \delta(y^2 - 1)$ as the damping of the equation. For $\omega = 2, \delta = 1$, $(h\omega, h\nu)$ frequently escapes from the stability regions of the three methods, which leads to the numerical solutions increasing dramatically as shown in Fig. 3a. For $\omega = 2, \delta = \frac{1}{3}$, noticing that $\mu = \delta(y^2 - 1) \leq 3\delta$, $(h\omega, h\nu)$ lies within the stability regions but frequently escapes the 0.01-stability regions of the three methods. Therefore, the errors of the three numerical methods are bounded and this gives oscillatory numerical solutions as shown in Fig. 3b. However, the approximations are not very accurate since the damping of numerical solutions is not in accordance with that of the exact solution. For $\omega = 2, \delta = \frac{1}{3^2}$, $(h\omega, h\nu)$ lies within the 0.01-stability regions of the three methods, which means that the numerical solutions respect the damping rate of the equation very well. Therefore, in this case, the three methods give satisfactory approximations as shown in Fig. 3c.

It can be observed from the numerical results that the traditional stability region only reflects the boundness of the error propagation of the numerical method. However, the α-stability region gives an insight into how well a numerical method respects the exact damping when numerically solving an oscillatory system with damping. The α-stability region is more feasible than the traditional stability region when dealing with numerical methods for general oscillatory system.

5 Conclusions

For the general multi-frequency oscillatory system $y'' + \Omega y = f(y, y')$, the order conditions for the ERKN methods and some effective ERKN methods were presented in [8]. However, the stability of the methods has not yet been considered. In this paper, based on the linear test model $y''(t) + \omega^2 y(t) + \mu y'(t) = 0$ with $\mu < 2\omega$, the stability of ERKN methods was analyzed for general oscillatory second-order initial value problems whose right-hand-side functions depend on both y and y'. Furthermore, the α-stability of the ERKN methods was introduced to describe how well the numerical methods respect the damping rate of the general oscillatory systems. Numerical result shows that the new stability analysis gives much insight into the ERKN methods for general oscillatory systems (1).

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