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Extended phase properties and stability analysis of RKN-type integrators for solving general oscillatory second-order initial value problems

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Abstract In this paper, we study in detail the phase properties and stability of numerical methods for general oscillatory second-order initial value problems whose right-hand side functions depend on both the position y and velocity y'. In order to analyze comprehensively the numerical stability of integrators for oscillatory systems, we introduce a novel linear test model $y''(t) + \omega^2 y(t) + \mu y'(t) = 0$ with $\mu < 2\omega$. Based on the new model, further discussions and analysis on the phase properties and stability of numerical methods are presented for general oscillatory problems. We give the new definitions of dispersion and dissipation which can be viewed as an essential extension of the traditional ones based on the linear test model $y''(t) + \omega^2 y(t) = 0$. The numerical experiments are carried out, and the numerical results show that the analysis of phase properties and stability presented in this paper is more suitable for the numerical methods when they are applied to the general oscillatory second-order initial value problem involving both the position and velocity.

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1 Introduction

In the last decade, a great interest has arisen in the research of numerical methods for systems of initial value problems with oscillatory solutions. These systems are frequently encountered in many fields such as molecular dynamics, orbital mechanics, electronics, and engineering. A lot of theoretical and numerical researches have been made on the modeling and simulation of these oscillations. It is well known that the efficiency of these numerical methods for oscillatory systems in long-term integration is closely connected with their stability, and phase properties: dispersion and dissipation. On one hand, when solving the system of ODEs, especially the stiff ones, it is highly desirable or even obligated to use the numerical methods with good stability. Many conceptions about stability such as A-stability, strongly A-stability, and L-stability are very popular and commonly used. More detailed information on these conceptions on stability of numerical methods can be found in relevant textbooks on numerical solution of systems of ODEs (see, for example, [3, 8, 9, 11, 12]). On the other hand, it is shown that the numerical methods with reduced phase errors (dissipation error and dispersion error) are more suitable for long-term integration of differential equations with oscillations than the conventional numerical methods [20]. Many researchers have developed numerical methods with the purpose of making the phase-lag of the method smaller. In [20], Van der Houwen and Sommeijer proposed second-order m-stage methods (with $m = 4, 5, 6$) and phase-lag order $q = 6, 8, 10$, respectively. Chawla and Rao [4, 5] constructed Numerov-type methods with minimal phase-lag in for the numerical integration of second-order ordinary differential equations. RK methods with minimal phase-lag or phase-lag of order infinity obtained in [16–19] for the numerical approximation of second-order differential equations. For more research on this topic, we refer the reader to [1, 2, 6, 10, 14].

For a numerical method, the analysis of phase and amplitude of oscillations depends on the test model under consideration. It is thus very important to consider an appropriate test equation for the analysis of phase properties and the stability. In the case of classical Runge-Kutta-Nyström (RKN) methods for systems of initial value problems

$$y''(t) = g(y), y(0) = y_0, y'(0) = y'_0, \tag{1}$$

the stability and phase properties are analyzed by using the second-order homogeneous linear test model (see, e.g. [20])

$$y''(t) + \omega^2 y(t) = 0. \tag{2}$$

Based on the linear model, P. J. Van der Houwen and B. P. Sommeijer analyzed in detail the phase properties of RKN methods in [20]. The notions of dispersion and

dissipation errors of numerical methods were originally introduced in [20]. The study of linear stability for a numerical method was based on the homogeneous linear test model (2) as well.

When the right-hand side function $g(y)$ in (1) has the form $g(y) = -Ky + f(y)$, where K is a positive semi-definite matrix that implicitly contains the dominant frequencies of the oscillatory problem, the special structure introduced by the linear term Ky has been taken account of in the design of numerical methods for this kind of systems in the recent researches. In [23], the authors proposed the multi-frequency and multidimensional ARKN (Adapted Runge-Kutta-Nyström) methods. An outstanding advantage of the multi-frequency and multidimensional ARKN methods is that their updates are incorporated with the special structure of the original system so that they naturally integrate exactly the multi-frequency oscillatory homogeneous system $y'' + Ky = 0$. For references on this topic, we refer the reader to [7, 13, 15, 22–25]. Since the ARKN methods can exactly integrate $y'' + Ky = 0$, it is pointless to consider the phase properties of the methods on the basis of the conventional linear test Eq. 2 because they automatically are zero-dispersive and zero-dissipative.

When it comes to numerical methods for general second-order initial value problems whose right-hand side functions depend on both y and y', i.e., $y''(t) = g(y, y')$, it is not appropriate again to use the linear model (2) to analyze the phase properties and stability of the numerical methods. The main reason for this point is that the model (2) for the analysis of phase properties and stability does not involve y', therefore, the internal stages Y_i' are never used when the numerical methods are applied to (2). Thus, the traditional analysis reflects only partly the phase properties and stability of the numerical methods.

Based on the fact stated above, in this paper, we present a new test model which tends to be more suitable for the phase properties and stability analysis of numerical methods for general oscillatory second-order systems involving both y and y'. Moreover, even though ARKN methods can exactly integrate $y'' + Ky = 0$, when they are applied to the oscillatory system of the form $y''(t) + Ky = f(y, y')$, where involving y' usually means damping, phase properties and stability of the numerical methods should be reevaluated in the new outline.

The rest of the paper is organized as follows. In Section 2, a new test model is proposed and examined. Based on the new model, we present the phase properties and stability of numerical methods for general oscillatory second-order initial value problems whose right-hand side functions depend on both y and y'. In Section 3, the characteristic matrices of RKN methods and ARKN methods for general problems are analyzed and derived. On the basis of the characteristic matrices, numerical experiments are carried out to verify our theory established in Section 4. The last section is devoted to conclusions.

2 Analysis of dissipation and dispersion through a new test model

The test Eq. 2 describes a simple harmonic oscillator. However, if y' is involved, for many oscillatory systems, it usually means that there exists a damping force which is related to the velocity of oscillations. Assume that the force is linearly related to the

velocity of the oscillations and consider an ideal mass-spring-damper system with mass m, spring constant k and viscous damper of damping coefficient c. Newton's law gives

$$\begin{cases} my''(t) = -ky(t) - cy'(t), \\ y(0) = y_0, y'(0) = y'_0. \end{cases}$$

Let

$$\omega = \sqrt{\frac{k}{m}}, \mu = \frac{c}{m}.$$

The equation can be rearranged into the following linear initial value problem of the form

$$\begin{cases} y''(t) + \omega^2 y(t) + \mu y'(t) = 0, \\ y(0) = y_0, y'(0) = y'_0, \end{cases} \tag{3}$$

where ω is called the (undamped) natural frequency of the system. and μ is called the damping. From now on, we consider only the case where μ and ω satisfy the condition: $\mu < 2\omega$, then the exact solution to (3) is given by

$$y(t) = c_+ e^{\left(-\frac{\mu}{2} + \frac{\sqrt{4\omega^2-\mu^2}}{2} i\right)t} + c_- e^{\left(-\frac{\mu}{2} - \frac{\sqrt{4\omega^2-\mu^2}}{2} i\right)t}, \tag{4}$$

where

$$c_\pm = \frac{y_0}{2} \pm \frac{\frac{\mu}{2} y_0 + y'_0}{\sqrt{4\omega^2 - \mu^2}} i.$$

In this case, the solution of the system oscillates at the natural damped frequency which is a function of the natural frequency and the damping.

As far as the analysis of dissipation and dispersion is concerned, when a numerical method is applied to the general second-order differential equation involving first derivative of the oscillator, it is natural to consider (3) as the test equation for the analysis of phase and amplitude of the oscillation. We now follow the procedure and notations in [20]. Applying a one-step method to the test Eq. 3 obtains the numerical solution

$$\begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix} = M \begin{pmatrix} y_n \\ hy'_n \end{pmatrix}, M := \begin{pmatrix} A_m(\nu^2, \sigma) & B_m(\nu^2, \sigma) \\ A^*_m(\nu^2, \sigma) & B^*_m(\nu^2, \sigma) \end{pmatrix} \tag{5}$$

with

$$\nu = h\omega, \quad \sigma = h\mu, \tag{6}$$

where A_m, A^*_m and B_m, B^*_m are polynomials in ν^2 and σ, completely determined by parameters of the numerical method. M is the *characteristic matrix* of the numerical method. The eigenvalues of M are called the *amplification factors* and are denoted by $\tilde{a}_+$ and $\tilde{a}_-$; the corresponding eigenvectors are given by

$$\mathbf{z} = (1, z_\pm)^\intercal, \qquad z_\pm := \frac{A^*_m(\nu^2, \sigma)}{\tilde{a}_\pm - B^*_m(\nu^2, \sigma)}.$$

In terms of $\tilde{a}_\pm$ and $z_\pm$, the numerical solution y_n is given by

$$y_n = \tilde{c}_+ (\tilde{a}_+)^n + \tilde{c}_- (\tilde{a}_-)^n, \tag{7}$$

where

$$\tilde{c}_+ := \frac{z_- y_0 - hy'_0}{z_+ - z_-}, \quad \tilde{c}_- := \frac{z_+ y_0 - hy'_0}{z_+ - z_-}. \tag{8}$$

By (4), the continuous solution reads

$$y(t_n) = c_+ a_+^n + c_- a_-^n, \tag{9}$$

where

$$c_\pm = \frac{y_0}{2} \pm \frac{\frac{\mu}{2} y_0 + y_0'}{\sqrt{4\omega^2 - \mu^2}} i, \quad a_\pm = e^{-\frac{\sigma}{2} \pm \frac{\sqrt{4\nu^2 - \sigma^2}}{2} i}. \tag{10}$$

Assuming that the amplification factors $\tilde{a}_+$ and $\tilde{a}_-$ are complex conjugate we may write

$$\tilde{c}_\pm = |\tilde{c}| e^{\pm i \tilde{\psi}}, \qquad \tilde{a}_\pm = |\tilde{a}| e^{\pm i \tilde{\nu}}$$

and similarly

$$c_\pm = |c| e^{\pm i \psi}, \qquad a_\pm = |a| e^{\pm i \bar{\nu}}.$$

Substituting them into (7) and (9) gives

$$y_n = 2|\tilde{c}||\tilde{a}|^n \cos(\tilde{\psi} + n\tilde{\nu}), \tag{11}$$

$$y(t_n) = 2|c||a|^n \cos(\psi + n\bar{\nu}). \tag{12}$$

These expressions lead to the following definition:

Definition 2.1 When applying a one-step method to the test Eq. 3, the quantities

$$\phi_0(\nu, \sigma) := \psi - \tilde{\psi}, \quad \phi(\nu, \sigma) := \bar{\nu} - \tilde{\nu}, \quad \alpha_0(\nu, \sigma) := |c| - |\tilde{c}|, \quad \alpha(\nu, \sigma) := |a| - |\tilde{a}|$$

are respectively called initial dispersion (or phase lag), (propagated) dispersion, initial dissipation (or amplification error) and (propagated) dissipation.

As mentioned in [20], the initial dispersion and the initial amplification are determined by the initial values y_0 and y_0'. These errors are not propagated in the numerical computations. Thus, we will concentrate on the propagated dispersion and dissipation.

Since $\tilde{a}_+$ and $\tilde{a}_-$ are eigenvalues of M and $\tilde{a}_\pm = |\tilde{a}| e^{\pm i \tilde{\nu}}$, it follows that

$$\tilde{a}_+ + \tilde{a}_- = trace(M) = A_m(\nu^2, \sigma) + B_m^*(\nu^2, \sigma) := S(\nu^2, \sigma),$$

$$\tilde{a}_+ \tilde{a}_- = det(M) = A_m(\nu^2, \sigma) B_m^*(\nu^2, \sigma) - A_m^*(\nu^2, \sigma) B_m(\nu^2, \sigma) := P(\nu^2, \sigma)$$

or equivalently

$$2|\tilde{a}| cos(\tilde{\nu}) = S(\nu^2), \quad |\tilde{a}|^2 = P(\nu^2).$$

By (10), we have

$$|a| = e^{-\frac{\sigma}{2}}, \qquad \bar{\nu} = \frac{\sqrt{4\nu^2 - \sigma^2}}{2},$$

and form which it follows that

$$\phi(\nu, \sigma) = \frac{\sqrt{4\nu^2 - \sigma^2}}{2} - \arccos \frac{S(\nu^2, \sigma)}{2\sqrt{P(\nu^2, \sigma)}}, \quad \alpha(\nu, \sigma) = e^{-\frac{\sigma}{2}} - \sqrt{P(\nu^2, \sigma)}. \tag{13}$$

To define the orders of dispersion and dissipation, we set

$$\zeta = \frac{\mu}{2\omega}. \tag{14}$$

For the linear damped oscillator, ζ is called *the damping ratio* which critically determines the behavior of the system.

- Overdamped ($\zeta > 1$): The system exponentially decays to steady state without oscillating.
- Critically damped ($\zeta = 1$): The system also returns to steady state without oscillating but is slower than the case $\zeta > 1$.
- Underdamped ($\zeta < 1$): The system oscillates with a slightly different frequency than the undamped case) and the amplitude gradually decreases to zero.

In our case, it is always assumed that $\zeta < 1$. Since $\nu = h\omega, \sigma = h\mu$, we have $\sigma = 2\zeta\nu$. Substituting it into (13) yields

$$\begin{aligned} \phi(\nu) &\equiv \phi(\nu, 2\zeta\nu) = \sqrt{1-\zeta^2}\nu - \arccos \frac{S(\nu^2, 2\zeta\nu)}{2\sqrt{P(\nu^2, 2\zeta\nu)}}, \\ \alpha(\nu) &\equiv \alpha(\nu, 2\zeta\nu) = e^{-\zeta\nu} - \sqrt{P(\nu^2, 2\zeta\nu)}. \end{aligned} \tag{15}$$

If

$$\phi(\nu) = \mathcal{O}(\nu^{q+1}) \quad \text{and} \quad \alpha(\nu) = \mathcal{O}(\nu^{p+1}),$$

then the method is called *dispersive of order q* and *dissipative of order p*, respectively. If $\phi(\nu) = 0$ and $\alpha(\nu) = 0$,the method is called *zero-dispersive* (or *phase-fitted*) and *zero-dissipative* (or *amplification-fitted*), respectively. Note that when $\zeta \to 0$ which means $\sigma \to 0$, the new definitions of dissipation and dispersion reduce to the traditional ones which are based on (2) (see [20]).

It is known to all that that it could cause disaster to use a numerical method without the knowledge of its stability behaviour. Besides, in practice, the efficiency of the numerical method is closely related to its stability behaviour. Therefore, the study of stability properties of numerical methods is of great importance.

The characteristic matrix $M(\nu, \sigma)$ is also called stability matrix since the spectral radius $\rho(M)$ represents the stability of the numerical method. It can be observed that the characteristic matrix $M(\nu, \sigma)$ depends on the variables ν and σ, therefore, the stability region is two-dimensional in the (ν, σ)-plane. Accordingly, we have the following definitions of stability for the numerical method for general second-order systems.

(i) $R_s = \{(\nu, \sigma) |\ \nu > 0,\ \sigma > 0$ and $\rho(M) < 1\}$ is called the *stability region* of the numerical method.

(ii) If $R_s = (0, \infty) \times (0, \infty)$, the method is called *A-stable*.

Remark 2.1 In the multidimensional case, the linear test equation becomes

$$y'' + Ay' + By = 0. \tag{16}$$

We assume that A and B are two symmetric and positive definite matrices. Then, there exists a nonsingular matrix P such that A and B can be expressed as

$$A = P^{-1}\Lambda_A P,\ B = P^{-1}\Lambda_B P,$$

where Λ_A and Λ_B are diagonal matrices.

With the variable substitution $z(t) = P^{-1}y(t)$, the Eq. 16 is identical to a transformed equation

$$z'' + \Lambda_A z' + \Lambda_B z = 0. \tag{17}$$

Thus, we need only consider the scalar linear test equation.

Remark 2.2 It can be seen from the definitions of stability that the stability of numerical methods for general oscillatory second-order initial value problems involving both the position and velocity depends on both the frequency and damping. This means that when a numerical method is applied to a practical problem, the stepsize h is restricted in two directions. Therefore, even h is chosen so that the method is stable in traditional sense, it might not be so for the new test equation.

Since the linear test equation involves the damping term, periodicity region and P-stable property [9] of the numerical method for general second-order systems don't need to be addressed. Moreover, when dealing with a damped system, it is not sufficient to consider only the stability of the numerical method. A further requirement is that the numerical solution should respect the damping rate of the system as much as possible. Therefore, the α-stability region of a numerical method is defined as follows, where $\alpha > 0$.

Definition 2.2 $R_s(\alpha) = \left\{ (\nu, \sigma) \in R_s \;\middle|\; \left| \frac{\rho(M) - e^{-\frac{\sigma}{2}}}{e^{-\frac{\sigma}{2}}} \right| \leq \alpha \right\}$ is called the α-*stability region* of a numerical method.

The α-stability region gains an insight into how well a numerical method can keep in step with the exact damping rate when it is applied to a damping system.

3 The characteristic matrices of RKN methods and ARKN methods for general oscillatory systems

3.1 RKN method

Due to Newton' second law, i.e., the forces are proportional to acceleration, many differential equations which appear in practice are systems of the second-order initial value problems

$$\begin{cases} y''(t) = g(y(t), y'(t)), \\ y(0) = y_0, y'(0) = y'_0, \end{cases} \tag{18}$$

where the right-hand side function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $y \in \mathbb{R}^d$, $y' \in \mathbb{R}^d$.

The definition of RKN methods for (18) is as follows.

Definition 3.1 An s-stage RKN method for the numerical integration of the initial value problem (18) is defined by the following scheme

$$\begin{cases} Y_i &= y_n + hc_i y_n' + h^2 \sum_{j=1}^{s} \bar{a}_{ij} g(Y_j, Y_j'), \qquad i = 1, \cdots, s, \\ Y_i' &= y_n' + h \sum_{j=1}^{s} a_{ij} g(Y_j, Y_j'), \qquad i = 1, \cdots, s, \\ y_{n+1} &= y_n + hy_n' + h^2 \sum_{i=1}^{s} \bar{b}_i g(Y_i, Y_i'), \\ y_{n+1}' &= y_n' + h \sum_{i=1}^{s} b_i g(Y_i, Y_i'). \end{cases} \tag{19}$$

The scheme (19) can also be denoted by the Butcher tableau as

$$\begin{array}{c|cc} c & A & \bar{A} \\ \hline & b^{\mathsf{T}} & \bar{b}^{\mathsf{T}} \end{array} = \begin{array}{c|ccc|ccc} c_1 & a_{11} & \cdots & a_{1s} & \bar{a}_{11} & \cdots & \bar{a}_{1s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} & \bar{a}_{s1} & \cdots & \bar{a}_{ss} \\ \hline & b_1 & \cdots & b_s & \bar{b}_1 & \cdots & \bar{b}_s \end{array}$$

Applying an RKN method (19) to the new test equation (3) yields

$$\begin{cases} Y_i &= y_n + hc_i y_n' + h^2 \sum_{j=1}^{s} \bar{a}_{ij} \big(-\omega^2 Y_j - \mu Y_j' \big), \qquad i = 1, \cdots, s, \\ hY_i' &= hy_n' + h^2 \sum_{j=1}^{s} a_{ij} \big(-\omega^2 Y_j - \mu Y_j' \big), \qquad i = 1, \cdots, s, \\ y_{n+1} &= y_n + hy_n' + h^2 \sum_{i=1}^{s} \bar{b}_i \big(-\omega^2 Y_i - \mu Y_i' \big), \\ hy_{n+1}' &= hy_n' + h^2 \sum_{i=1}^{s} b_i \big(-\omega^2 Y_i - \mu Y_i' \big). \end{cases} \tag{20}$$

Let $\nu = h\omega, \sigma = h\mu$, we can write (20) in the compact form

$$\begin{cases} Y &= e \otimes y_n + c \otimes hy_n' + \bar{A} \big(-\nu^2 Y - \sigma hY' \big), \\ hY' &= e \otimes hy_n' + A \big(-\nu^2 Y - \sigma hY' \big), \\ y_{n+1} &= y_n + hy_n' + \bar{b}^{\mathsf{T}} \big(-\nu^2 Y - \sigma hY' \big), \\ hy_{n+1}' &= hy_n' + b^{\mathsf{T}} \big(-\nu^2 Y - \sigma hY' \big), \end{cases} \tag{21}$$

where $Y = (Y_1, \cdots, Y_s)^{\mathsf{T}}, Y' = (Y_1', \cdots, Y_s')^{\mathsf{T}}, e = (1, \cdots, 1)^{\mathsf{T}}, c = (c_1, \cdots, c_s)^{\mathsf{T}}$.

From the first two equations in (21), we get

$$\begin{pmatrix} I + \nu^2 \bar{A} & \sigma \bar{A} \\ \nu^2 A & I + \sigma A \end{pmatrix} \begin{pmatrix} Y \\ hY' \end{pmatrix} = \begin{pmatrix} e \otimes y_n + c \otimes hy_n' \\ e \otimes hy_n' \end{pmatrix}, \tag{22}$$

where I is the identity matrix. We now establish a lemma.

Lemma 3.1 *Given two $n \times n$ matrices A, B and two non-zero real number α, β, if $I + \alpha A + \beta B$ is invertible, then $\begin{pmatrix} I+\alpha A & \beta A \\ \alpha B & I+\beta B \end{pmatrix}$ is invertible, and*

$$\begin{aligned}&\begin{pmatrix} I+\alpha A & \beta A \\ \alpha B & I+\beta B \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (I+\beta B)(I+\alpha A+\beta B)^{-1} & -\beta/\alpha I + (\beta/\alpha I + \beta^2/\alpha B)(I+\alpha A+\beta B)^{-1} \\ -\alpha B(I+\alpha A+\beta B)^{-1} & I-\beta B(I+\alpha A+\beta B)^{-1} \end{pmatrix}\end{aligned} \tag{23}$$

Proof By a straightforward computation, (23) can be verified. □

For sufficiently small stepsize h, $I + \nu^2\bar{A} + \sigma A$ is invertible. For simplicity, let $C = I + \nu^2\bar{A} + \sigma A$, by Lemma 3.1, it follows from (22) that

$$\begin{pmatrix} Y \\ hY' \end{pmatrix} = \begin{pmatrix} (I+\sigma A)C^{-1} & -\sigma/\nu^2 I + (\sigma/\nu^2 I + \sigma^2/\nu^2 A)C^{-1} \\ -\nu^2 AC^{-1} & I - \sigma AC^{-1} \end{pmatrix} \begin{pmatrix} e\otimes y_n + c\otimes hy'_n \\ e\otimes hy'_n \end{pmatrix}. \tag{24}$$

Substituting (24) into the last two equation in (21) yields

$$\begin{aligned}\begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_n \\ hy'_n \end{pmatrix} - \begin{pmatrix} \nu^2\bar{b}^\intercal Y + \sigma\bar{b}^\intercal hY' \\ \nu^2 b^\intercal Y + \sigma b^\intercal hY' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_n \\ hy'_n \end{pmatrix} - \begin{pmatrix} \nu^2\bar{b}^\intercal & \sigma\bar{b}^\intercal \\ \nu^2 b^\intercal & \sigma b^\intercal \end{pmatrix}\begin{pmatrix} Y \\ hY' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_n \\ hy'_n \end{pmatrix} - \begin{pmatrix} \nu^2\bar{b}^\intercal & \sigma\bar{b}^\intercal \\ \nu^2 b^\intercal & \sigma b^\intercal \end{pmatrix} \cdot \\ &\quad \begin{pmatrix} (I+\sigma A)C^{-1} & -\sigma/\nu^2 I + (\sigma/\nu^2 I + \sigma^2/\nu^2 A)C^{-1} \\ -\nu^2 AC^{-1} & I - \sigma AC^{-1} \end{pmatrix}\begin{pmatrix} e\otimes y_n + c\otimes hy'_n \\ e\otimes hy'_n \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_n \\ hy'_n \end{pmatrix} - \begin{pmatrix} \nu^2\bar{b}^\intercal & \sigma\bar{b}^\intercal \\ \nu^2 b^\intercal & \sigma b^\intercal \end{pmatrix} \cdot \\ &\quad \begin{pmatrix} (I+\sigma A)C^{-1} & -\sigma/\nu^2 I + (\sigma/\nu^2 I + \sigma^2/\nu^2 A)C^{-1} \\ -\nu^2 AC^{-1} & I - \sigma AC^{-1} \end{pmatrix}\begin{pmatrix} e & c \\ 0 & e \end{pmatrix}\begin{pmatrix} y_n \\ hy'_n \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_n \\ hy'_n \end{pmatrix} - \begin{pmatrix} \nu^2\bar{b}^\intercal C^{-1}e & \nu^2\bar{b}^\intercal C^{-1}c + \sigma\bar{b}^\intercal C^{-1}e \\ \nu^2 b^\intercal C^{-1}e & \nu^2 b^\intercal C^{-1}c + \sigma b^\intercal C^{-1}e \end{pmatrix}\begin{pmatrix} y_n \\ hy'_n \end{pmatrix} \\ &= \begin{pmatrix} 1-\nu^2\bar{b}^\intercal C^{-1}e & 1-\nu^2\bar{b}^\intercal C^{-1}c - \sigma\bar{b}^\intercal C^{-1}e \\ -\nu^2 b^\intercal C^{-1}e & 1-\nu^2 b^\intercal C^{-1}c - \sigma b^\intercal C^{-1}e \end{pmatrix}\begin{pmatrix} y_n \\ hy'_n \end{pmatrix},\end{aligned} \tag{25}$$

namely, the characteristic matrix of the RKN methods is given by

$$M_{RKN} = \begin{pmatrix} 1-\nu^2\bar{b}^\intercal C^{-1}e & 1-\nu^2\bar{b}^\intercal C^{-1}c - \sigma\bar{b}^\intercal C^{-1}e \\ -\nu^2 b^\intercal C^{-1}e & 1-\nu^2 b^\intercal C^{-1}c - \sigma b^\intercal C^{-1}e \end{pmatrix}.$$

3.2 ARKN methods

If $g(y, y')$ in (18) has the form $g(y, y') = -Ky + f(y, y')$, the oscillatory system can be written in the form

$$\begin{cases} y''(t) + Ky(t) = f(y(t), y'(t)), \\ y(t_0) = y_0, \quad y'(t_0) = y'_0, \end{cases} \tag{26}$$

where K is a $d \times d$ positive semi-definite matrix that implicitly contains the dominant frequencies of the oscillatory problem and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$.

We first introduce matrix-valued functions [25]

$$\phi_j(K) := \sum_{i=0}^{\infty} \frac{(-1)^i K^i}{(2i+j)!}, \quad j = 0, 1, 2, \cdots, \tag{27}$$

where K is a $d \times d$ matrix.

In the recent paper [25], we presented the following matrix-variation-of-constants formula for the exact solution and its derivative for the oscillatory system (26).

Theorem 3.1 *If $K \in \mathbb{R}^{d\times d}$ is a positive semi-definite matrix and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous in (26), then the solution of (26) and its derivative satisfy the following equations*

$$\begin{cases} y(t) = \phi_0((t-t_0)^2K)y_0 + (t-t_0)\phi_1((t-t_0)^2K)y'_0 + \int_{t_0}^t (t-\xi)\phi_1((t-\xi)^2K)\hat{f}(\xi)d\xi, \\ y'(t) = -(t-t_0)K\phi_1((t-t_0)^2K)y_0 + \phi_0((t-t_0)^2K)y'_0 + \int_{t_0}^t \phi_0((t-\xi)^2K)\hat{f}(\xi)d\xi \end{cases} \tag{28}$$

for any real number t_0, t, where $\hat{f}(\xi) = f(y(\xi), y'(\xi))$.

It follows from (28) that

$$\begin{cases} y(t_n + h) = \phi_0\left(h^2K\right)y_n + h\phi_1\left(h^2K\right)y'_n \\ \qquad\qquad + h^2\int_0^1 (1-\gamma)\phi_1\left((1-\gamma)^2h^2K\right)\hat{f}(t_n + h\gamma)d\gamma, \\ y'(t_n + h) = -hK\phi_1\left(h^2K\right)y_n + \phi_0\left(h^2K\right)y'_n \\ \qquad\qquad + h\int_0^1 \phi_0\left((1-\gamma)^2h^2K\right)\hat{f}(t_n + h\gamma)d\gamma. \end{cases} \tag{29}$$

It is clear that (29) contains the structure of the updates for an RKN-type method. Then, revising the updates of RKN methods gives the following multi-frequency and multidimensional adapted RKN (ARKN) methods.

Definition 3.2 An s-stage multidimensional ARKN method for the numerical integration of the initial value problem (26) is defined as the following scheme

$$\begin{cases} Y_i &= y_n + hc_i y_n' + h^2 \sum_{j=1}^{s} \bar{a}_{ij}\big(f(Y_j, Y_j') - KY_j\big), \qquad i = 1, \cdots, s, \\ Y_i' &= y_n' + h \sum_{j=1}^{s} a_{ij}\big(f(Y_j, Y_j') - KY_j\big), \qquad i = 1, \cdots, s, \\ y_{n+1} &= \phi_0(V) y_n + h\phi_1(V) y_n' + h^2 \sum_{i=1}^{s} \bar{b}_i(V) f(Y_i, Y_i'), \\ y_{n+1}' &= \phi_0(V) y_n' - hK\phi_1(V) y_n + h \sum_{i=1}^{s} b_i(V) f(Y_i, Y_i'), \end{cases} \tag{30}$$

where $a_{i,j}, \bar{a}_{i,j}, i, j = 1, \ldots, s \in \mathbb{R}$ and the weights $b_i : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ and $\bar{b}_i : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}, i = 1, \cdots, s$ in the updates are matrix-valued functions of $V = h^2K$, which can be expanded into series in powers of V with real coefficients. The scheme (30) can also be denoted by the Butcher tableau

$$\begin{array}{c|cc} c & A & \bar{A} \\ \hline & b^{\intercal}(V) & \bar{b}^{\intercal}(V) \end{array} = \begin{array}{c|ccc|ccc} c_1 & a_{11} & \cdots & a_{1s} & \bar{a}_{11} & \cdots & \bar{a}_{1s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} & \bar{a}_{s1} & \cdots & \bar{a}_{ss} \\ \hline & b_1(V) & \cdots & b_s(V) & \bar{b}_1(V) & \cdots & \bar{b}_s(V) \end{array}$$

The order conditions for the multidimensional ARKN methods (30) are given in [25].

Applying an ARKN method (30) to the test equation (3) with $\nu = h\omega, \sigma = h\mu$ gives

$$\begin{cases} Y_i &= y_n + hc_i y_n' + h^2 \sum_{j=1}^{s} \bar{a}_{ij}\big(-\omega^2 Y_j - \mu Y_j'\big), \qquad i = 1, \cdots, s, \\ hY_i' &= hy_n' + h^2 \sum_{j=1}^{s} a_{ij}\big(-\omega^2 Y_j - \mu Y_j'\big), \qquad i = 1, \cdots, s, \\ y_{n+1} &= \phi_0(\nu^2) y_n + h\phi_1(\nu^2) y_n' + h^2 \sum_{i=1}^{s} \bar{b}_i(\nu^2)\big(-\mu Y_i'\big), \\ hy_{n+1}' &= \phi_0(\nu^2) hy_n' - \nu^2\phi_1(\nu^2) y_n + h^2 \sum_{i=1}^{s} b_i(\nu^2)\big(-\mu Y_i'\big). \end{cases} \tag{31}$$

or in the compact form

$$\begin{cases} Y &= e \otimes y_n + c \otimes hy_n' + \bar{A}\big(-\nu^2 Y - \sigma hY'\big), \\ hY' &= e \otimes hy_n' + A\big(-\nu^2 Y - \sigma hY'\big), \\ y_{n+1} &= \phi_0 y_n + \phi_1 hy_n' - \sigma\bar{b}^{\intercal} hY', \\ hy_{n+1}' &= \phi_0 hy_n' - \nu^2\phi_1 y_n - \sigma b^{\intercal} hY', \end{cases} \tag{32}$$

where the arguments (ν^2) are suppressed.

The first two equations in (32) are the same as in (21). So substituting (24) into the last two equations in (32) yields

$$\begin{aligned}
\begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix} &= \begin{pmatrix} \phi_0(\nu^2) & \phi_1(\nu^2) \\ -\nu^2\phi_1(\nu^2) & \phi_0(\nu^2) \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix} - \begin{pmatrix} \sigma\bar{b}^{\intercal} hY' \\ \sigma b^{\intercal} hY' \end{pmatrix} \\
&= \begin{pmatrix} \phi_0 & \phi_1 \\ -\nu^2\phi_1 & \phi_0 \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix} - \begin{pmatrix} 0 & \sigma\bar{b}^{\intercal} \\ 0 & \sigma b^{\intercal} \end{pmatrix} \begin{pmatrix} Y \\ hY' \end{pmatrix} \\
&= \begin{pmatrix} \phi_0 & \phi_1 \\ -\nu^2\phi_1 & \phi_0 \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix} - \begin{pmatrix} 0 & \sigma\bar{b}^{\intercal} \\ 0 & \sigma b^{\intercal} \end{pmatrix} \cdot \\
&\quad \begin{pmatrix} (I+\sigma A)C^{-1} & -\sigma/\nu^2 I + (\sigma/\nu^2 I + \sigma^2/\nu^2 A)C^{-1} \\ -\nu^2 AC^{-1} & I - \sigma AC^{-1} \end{pmatrix} \begin{pmatrix} e \otimes y_n + c \otimes hy'_n \\ e \otimes hy'_n \end{pmatrix} \\
&= \begin{pmatrix} \phi_0 & \phi_1 \\ -\nu^2\phi_1 & \phi_0 \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix} - \begin{pmatrix} 0 & \sigma\bar{b}^{\intercal} \\ 0 & \sigma b^{\intercal} \end{pmatrix} \cdot \\
&\quad \begin{pmatrix} (I+\sigma A)C^{-1} & -\sigma/\nu^2 I + (\sigma/\nu^2 I + \sigma^2/\nu^2 A)C^{-1} \\ -\nu^2 AC^{-1} & I - \sigma AC^{-1} \end{pmatrix} \begin{pmatrix} e & c \\ 0 & e \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix} \\
&= \begin{pmatrix} \phi_0 & \phi_1 \\ -\nu^2\phi_1 & \phi_0 \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix} \\
&\quad + \begin{pmatrix} \sigma\nu^2\bar{b}^{\intercal}AC^{-1}e & \sigma\nu^2\bar{b}^{\intercal}AC^{-1}c + \sigma^2\bar{b}^{\intercal}AC^{-1}e - \sigma\bar{b}^{\intercal}e \\ \sigma\nu^2 b^{\intercal}AC^{-1}e & \sigma\nu^2 b^{\intercal}AC^{-1}c + \sigma^2 b^{\intercal}AC^{-1}e - \sigma b^{\intercal}e \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix} \\
&= \begin{pmatrix} \phi_0 + \sigma\nu^2\bar{b}^{\intercal}AC^{-1}e & \phi_1 + \sigma\nu^2\bar{b}^{\intercal}AC^{-1}c + \sigma^2\bar{b}^{\intercal}AC^{-1}e - \sigma\bar{b}^{\intercal}e \\ -\nu^2\phi_1 + \sigma\nu^2 b^{\intercal}AC^{-1}e & \phi_0 + \sigma\nu^2 b^{\intercal}AC^{-1}c + \sigma^2 b^{\intercal}AC^{-1}e - \sigma b^{\intercal}e \end{pmatrix} \begin{pmatrix} y_n \\ hy'_n \end{pmatrix}.
\end{aligned} \tag{33}$$

This means that the characteristic matrix of the ARKN methods is given by

$$M_{ARKN} = \begin{pmatrix} \phi_0 + \sigma\nu^2\bar{b}^{\intercal}AC^{-1}e & \phi_1 + \sigma\nu^2\bar{b}^{\intercal}AC^{-1}c + \sigma^2\bar{b}^{\intercal}AC^{-1}e - \sigma\bar{b}^{\intercal}e \\ -\nu^2\phi_1 + \sigma\nu^2 b^{\intercal}AC^{-1}e & \phi_0 + \sigma\nu^2 b^{\intercal}AC^{-1}c + \sigma^2 b^{\intercal}AC^{-1}e - \sigma b^{\intercal}e \end{pmatrix}.$$

4 Numerical evidences

In this section, we present some numerical experiments to verify our theory of dissipation and dissipation based on the new test model (3). Given the the corresponding characteristic matrices of two different kinds of numerical methods for general oscillatory systems, we calculate the orders of dissipation and dispersion and plot the stability regions of the underlying numerical methods.

To show the effect of phase properties on the behavior of the numerical methods, and the priority of the new theory based on the new test model (3) in comparison with the traditional one on the basis of (2), we present two RKN methods and an ARKN method with the same algebraic order from the order conditions for RKN methods given in [8] and the order conditions for ARKN methods derived by [25]. The three methods are chosen in a such way that they are all explicit and of order three but with different dispersions and dissipations under both the new and the traditional definitions. The methods and their the dispersion errors and dissipation errors are listed one by one as follows.

RKN methods

1. The third-order RKN method denoted by RKN1 with the Butcher tableau

$$\begin{array}{c|cc} c & A & \bar{A} \\ \hline & b^{\mathsf{T}} & \bar{b}^{\mathsf{T}} \end{array} = \begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 1 & -1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{6} & \frac{1}{12} \end{array}$$

By the new definitions based on the model (3), the dispersion error and the dissipation error are

$$\phi_{new}(\nu) = \frac{\left(\zeta - 24\zeta^3 + 32\,\zeta^5\right)\nu^4}{96\sqrt{1-\zeta^2}} + \mathcal{O}(\nu^5), \quad \alpha_{new}(\nu) = \frac{\left(1 - 8\zeta^2 + 32\,\zeta^4\right)\nu^4}{96} + \mathcal{O}(\nu^5),$$

respectively, where $\zeta = \frac{\mu}{2\omega}$ as given in (14). When $\zeta \to 0$, the dispersion error and the dissipation error reduce to the traditional ones based on the traditional model (2)

$$\phi_{old}(\nu) = -\frac{1}{480}\nu^5 + \mathcal{O}(\nu^7), \qquad \alpha_{old}(\nu) = \frac{1}{96}\nu^4 + \mathcal{O}(\nu^6).$$

The third-order RKN method denoted by RKN2 with the Butcher tableau

$$\begin{array}{c|cc} c & A & \bar{A} \\ \hline & b^{\mathsf{T}} & \bar{b}^{\mathsf{T}} \end{array} = \begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{775-6\sqrt{3710}}{3270} & 0 & 0 \\ 1 & -1 & 2 & 0 & \frac{-11\left(-2135+32\sqrt{3710}\right)}{11445} & \frac{2\left(-105+2\sqrt{3710}\right)}{105} & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \frac{2}{9} & \frac{2}{9} & \frac{1}{18} \end{array}$$

By the new definitions, the dispersion error and the dissipation error are

$$\phi_{new}(\nu) = \frac{\zeta\left(17465 - 285\sqrt{3710} + \left(-38745 + 352\sqrt{3710}\right)\zeta^2 + 22890\zeta^4\right)\nu^4}{68670\sqrt{1-\zeta^2}} + \mathcal{O}(\nu^5)$$

$$\alpha_{new}(\nu) = \left(\frac{7}{72} - \frac{\sqrt{\frac{53}{70}}}{9} + \frac{2\left(-6825 + 88\sqrt{3710}\right)\zeta^2}{34335} + \frac{\zeta^4}{3}\right)\nu^4 + \mathcal{O}(\nu^5),$$

respectively, and when $\zeta \to 0$, the dispersion error and the dissipation error reduce to the traditional ones

$$\phi_{old}(\nu) = \frac{\left(1522 - 25\sqrt{3710}\right)\nu^9}{1020600} + \mathcal{O}(\nu^{11}), \quad \alpha_{old}(\nu) = \left(\frac{7}{72} - \frac{\sqrt{\frac{53}{70}}}{9}\right)\nu^4 + \mathcal{O}(\nu^6).$$

It can be observed that the two RKN methods have the same order of dispersion and dissipation by the new definition. However, the method RKN2 has a higher dispersion order than the method RKN1 in the sense of the traditional analysis based on the test model (2).

ARKN methods

1. The third-order ARKN method denoted by ARKN1 with Butcher tableau

$$\begin{array}{c|cc} c & A & \bar{A} \\ \hline & b^{\intercal}(V) & \bar{b}^{\intercal}(V) \end{array} = \begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 1 & -1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ \hline & b_1(V) & b_2(V) & b_3(V) & \bar{b}_1(V) & \bar{b}_2(V) & \bar{b}_3(V) \end{array}$$

with

$$\begin{aligned} b_1(V) &= \phi_1(V) - 3\phi_2(V) + 4\phi_3(V), & \bar{b}_1(V) &= \phi_2(V) - \frac{3}{2}\phi_3(V), \\ b_2(V) &= 4\phi_2(V) - 8\phi_3(V), & \bar{b}_2(V) &= \phi_3(V), \\ b_3(V) &= -\phi_2(V) + 4\phi_3(V), & \bar{b}_3(V) &= \frac{1}{2}\phi_3(V). \end{aligned}$$

By the new definitions, the dispersion error and the dissipation error are given by

$$\phi_{new}(\nu) = \frac{-\zeta^2\nu^3}{3\sqrt{1-\zeta^2}} + \mathcal{O}(\nu^4), \quad \alpha_{new}(\nu) = \frac{\zeta\nu^3}{3} + \mathcal{O}(\nu^4).$$

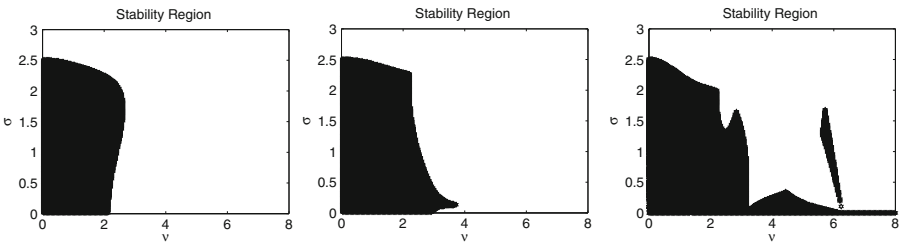
As we have known that ARKN methods exactly integrate the homogeneous oscillatory system $y''+Ky = 0$, therefore, the method ARKN1 is zero-dispersive and zero-dissipative in the light of the traditional stability definition based on (2), i.e.,

$$\phi_{old}(\nu) = 0, \quad \alpha_{old}(\nu) = 0.$$

The stability regions of RKN1, RKN2, and ARKN1 are shown in Fig. 1. We also plot the α-stability regions of the three methods when $\alpha = 0.1$ in Fig. 2.

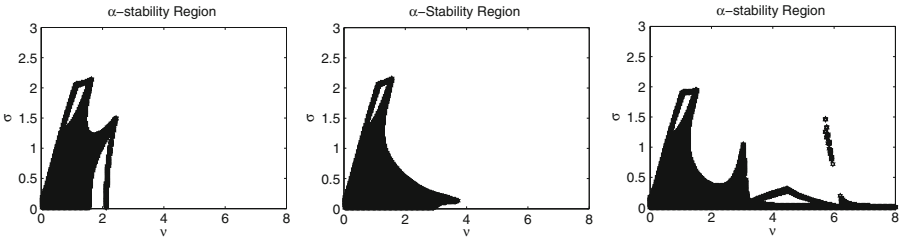
It can be seen clearly from Figs. 1 and 2 that the ARKN method, ARKN1, has larger stability region than those of the two classical RKN methods, RKN1 and RKN2.

To show the significance of the new definition of stability region, we apply the three numerical methods to the following Problem 1. We choose $\omega = 1, 2$, and



(a) $RKN1$ (b) $RKN2$ (c) $ARKN1$

Fig. 1 Stability regions of the three methods RKN1, RKN2, and ARKN1



(a) $RKN1$ (b) $RKN2$ (c) $ARKN1$

Fig. 2 α-stability regions of the three methods RKN1, RKN2, and ARKN1 for $\alpha = 0.1$

$h = 1$ for the three methods so that they lie in their stability intervals in traditional sense, which are exactly the intersection of the stability regions with ν-axis, respectively. We then vary the values of μ and calculate the relative errors of the methods for different values of μ at $t_{end} = 10$, the results are shown in Tables 1 and 2, respectively.

It is observed from Table 1 that for $\omega = 1, h = 1$, since $(h\omega, h\mu)$s for all the μs listed in Table 1 lie in both the stability regions and the 0.1-stability regions of the three numerical methods (see Figs. 1 and 2), the methods behave well. And as $(h\omega, h\mu)$ approaches the boundary of the 0.1-stability region, the methods become worse.

However, as can be seen in Table 2 with $\omega = 2, h = 1$, the behaviors of method RKN1 for different values of μ do not differ much from each other. This is in accordance with the fact that all the $(h\omega, h\mu)$s are near the boundary of the 0.1-stability region of RKN1 (Fig. 2a). For the methods RKN2 and ARKN1, roughly speaking, $\mu = 0.5$ is the critical point that decides whether $(h\omega, h\mu)$ lies in or is beyond the 0.1-stability regions of the two methods. And the numerical results match the stability properties of the methods.

Thus, even though the three numerical methods are stable in the traditional sense, when they are applied to the general problems involving both y and y', the effect of the damping term should also be considered for the stability of the methods. In fact, the damping term plays an important role in the study of numerical stability.

The stability property of a numerical method gives a hint on choosing the stepsize h. That is to choose the stepsize h moderately so that $(h\omega, h\mu)$ lies in the α-stability region with a relatively small α. And when h is fixed, the intrinsic phase properties of

Table 1 Problem 1: The relative errors $\left|\frac{y(t_{end})-y_n}{y(t_{end})}\right|$ of the three methods with $\omega = 1, h = 1$

Method	$\mu = 0$	$\mu = 0.01$	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$
RKN1	0.1105	0.1101	0.1057	0.0977	0.0516	0.3918	0.9954
RKN2	0.0049	0.0055	0.0137	0.0271	0.1007	0.6961	1.0428
ARKN1	6.6158e-16	0.0016	0.0172	0.0367	0.1180	0.6369	1.0374

Table 2 Problem 1: The relative errors $\left|\frac{y(t_{end})-y_n}{y(t_{end})}\right|$ of the three methods with $\omega = 2, h = 1$

Method	$\mu = 0$	$\mu = 0.01$	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$
RKN1	1.0141	1.0134	1.0100	1.0110	1.0302	1.0396	7.9785
RKN2	0.2692	0.2961	0.5757	0.9743	3.0641	21.6228	2.3686e2
ARKN1	1.2243e-15	0.0465	0.5456	1.2884	5.6397	46.7869	4.4647e2

the numerical method describe the qualitative behavior of the method to some extent when applied to oscillatory systems.

In what follows, we apply the three numerical methods to two test problems, the linear test Eq. 3 and a nonlinear wave equation. For each experiment, we will display the efficiency curves: accuracy versus the computational cost measured by the number of function evaluations required by each method.

Problem 1 We consider the linear test (3) with initial values

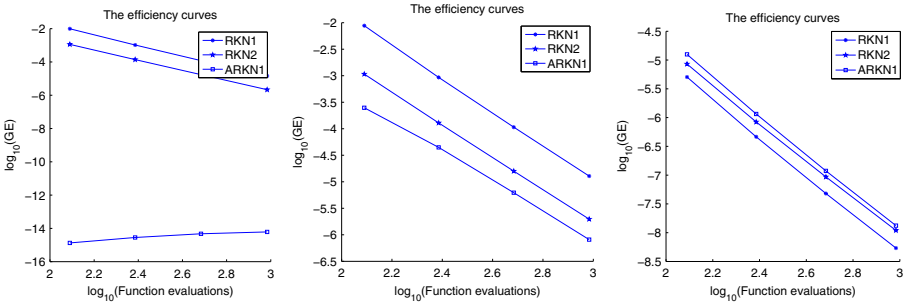
$$y(0) = 1, \quad y'(0) = -\frac{\mu}{2}.$$

The analytic solution of the problem is given by

$$y(t) = \exp(-\frac{\mu}{2}t)\cos(\sqrt{\omega^2 - \frac{\mu^2}{4}}\, t).$$

In this test, we choose the parameter values $\omega = 1,\ \mu = 0, 10^{-2}, 0.9$. We integrate the problem on the interval $[0, 20]$ with the stepsizes $h = 1/2^j,\ j = 1, \ldots, 4$. The numerical results are stated in Fig. 3.

It is observed from Fig. 3 that for $\mu = 0$, the traditional theory of dispersion and dissipation can explain Fig. 3a perfectly, since $\mu = 0$ means that no velocity is



(a) $\mu = 0$ (b) $\mu = 10^{-2}$ (c) $\mu = 0.9$

Fig. 3 Problem 1: The number of function evaluations against $\log_{10}(GE)$, the logarithm of the maximum global error over the integration interval

involved in the equation. As μ increases slightly to 10^{-2}, $\zeta = \frac{\mu}{2\omega} = 5 \times 10^{-3}$, in this case, up to a little change brought by the term $\mu y'$, the efficient curves in Fig. 3b still fit the traditional theory that numerical methods with higher-order dispersion and dissipation in the traditional sense tend to be more accurate for oscillatory problems when they have the same algebraic order. The reason is that when ζ is very small, the leading terms of the new dispersion errors and dissipation errors of the numerical methods are also small and hence are negligible compared with the the leading terms of the traditional ones. The leading terms of traditional dispersion errors and dissipation errors are dominant. However, in Fig. 3c with $\mu = 0.9$ or $\zeta = 0.45$, the traditional theory cannot explain the phenomenon, because the leading terms of the dispersion errors and dissipation errors of the numerical methods under the new definitions are not negligible and play an important role in the dispersion and dissipation relation. Thus, the behaviors of the numerical methods fit the new theory very well.

In nonlinear models, it is not easy to determine ω, μ, especially in high-dimensional case. In what follows, we show that the theory derived on the basis of the linear equation may also be applicable to nonlinear models.

Problem 2 Consider the damped wave equation with periodic boundary conditions [21]

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \delta \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f(u), \quad -1 < x < 1, \ \ t > 0, \\ u(-1, t) = u(1, t). \end{cases}$$

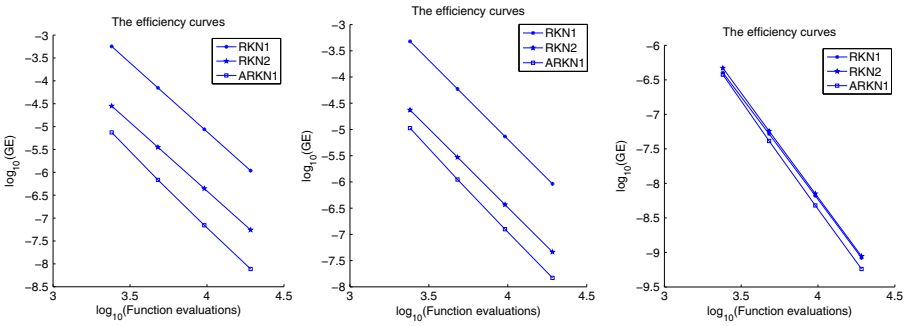
A second-order symmetric differences discretization in the spatial variable leads to the following system of second-order ODEs in time

$$U'' + KU = F(U, U'), \quad 0 < t \le t_{end},$$

where $U(t) = (u_1(t), \cdots, u_N(t))^\intercal$ with $u_i(t) \approx u(x_i, t), i = 1, \cdots, N$,

$$K = \frac{1}{\Delta x^2} \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

with $\Delta x = 2/N$ and $x_i = -1 + i\Delta x$, and $F(U, U') = \left(f(u_1) - \delta u_1', \cdots, f(u_N) - \delta u_N'\right)^\intercal$.



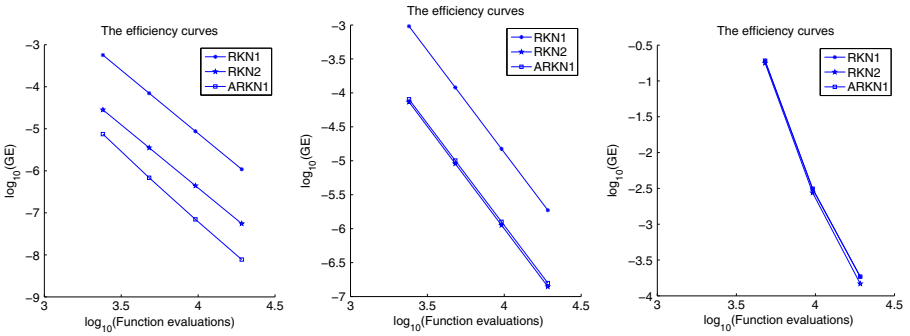
(a) $\delta = 0$ (b) $\delta = 10^{-2}$ (c) $\delta = 0.9$

Fig. 4 Problem 2: The number of function evaluations against $\log_{10}(GE)$, the logarithm of the maximum global error over the integration interval

In this experiment, we take $f(u) = -\frac{1}{10}\sin u$ and the initial conditions:

$$U(0) = (\pi)_{i=1}^{N}, \quad U_t(0) = \sqrt{N}\big(0.01 + \sin(\frac{2\pi i}{N})\big)_{i=1}^{N}.$$

We choose $N = 64$, $\delta = 0, 10^{-2}, 0.9$ and integrate the system on the interval $[0, 20]$ with the stepsizes $h = 0.1/2^j$, $j = 2, \ldots, 5$. The reference numerical solution is obtained by the method given in Hairer et al. [8] (II.14) with a very small stepsize. The numerical results are shown in Fig. 4.



(a) $\delta = 0$ (b) $\delta = 10^{-2}$ (c) $\delta = 0.9$

Fig. 5 Problem 3: The number of function evaluations against $\log_{10}(GE)$, the logarithm of the maximum global error over the integration interval

Problem 3 Consider the following damped wave equation with a nonlinear damping term

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \delta \left(\dfrac{\partial u}{\partial t} \right)^3 - \dfrac{\partial^2 u}{\partial x^2} = f(u), \quad -1 < x < 1, \ \ t > 0, \\ u(-1, t) = u(1, t). \end{cases}$$

With the same setup as in Problem 2, we plot the numerical results are shown in Fig. 5.

As shown in Figs. 4 and 5, the same pattern occurs in the nonlinear models as in the linear test equation, different frequencies and damping ratios lead to different numerical results.

5 Conclusions

It is noted that the phase properties and stability of RKN-type methods designed for general second-order initial value problems whose right-hand side functions *depend on both y and y′* have not been discussed in the existing scientific literature. Therefore, in this paper, by introducing a new linear test model $y''(t)+\omega^2 y(t)+\mu y'(t) = 0$ with $\mu < 2\omega$, we commence to make further discussions and analysis on the phase properties and stability of numerical methods for general second-order initial value problems. The new definitions of dispersion and dissipation are presented. The phase properties of two kinds of numerical methods, RKN methods and ARKN integrators, are analyzed in detail based on the new definitions. Numerical experiments are carried out and the numerical results verify our new theory. It turns out that the analysis of phase property and stability presented based on (3) in this paper is more suitable for numerical methods designed for the general second-order initial value problem involving both y and y' than the conventional one on the basis of the linear test model (2).

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