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The Cauchy problem for linear inhomogeneous wave equations with variable coefficients☆

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ABSTRACT

In this paper, we present a new analytical formula for the Cauchy problem of the linear inhomogeneous wave equation with variable coefficients. The formula gives a much simpler solution than that given by the classical Poisson formula. The derivation is based on Duhamel's Principle and the theory of pseudodifferential operator. An example is solved by using the formula to illustrate the feasibility.

1. Introduction

The history of partial differential equations can date back to the 18th century when the first one-dimensional wave equation $u_{tt} = u_{xx}$ was introduced by d'Alembert as a model of a vibrating string [1]. Giving its initial displacement $u(x,0) = \varphi(x)$ and velocity $u_t(x,0) = \psi(x)$, it is known that an explicit solution to the Cauchy, or initial value, problem of the wave equation is

$$u(x,t) = \frac{1}{2}(\varphi(x-t) + \varphi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)d\xi.$$

If an external force $f(x,t)$ (per unit mass) is applied on the string, then the equation becomes an inhomogeneous wave equation $u_{tt} = u_{xx} + f(x,t)$. In this case, the solution which can be derived by using Duhamel's Principle is

$$u(x,t) = \frac{1}{2}(\varphi(x-t) + \varphi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)d\xi + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f(\sigma,\tau)d\sigma d\tau.$$

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Various methods have been utilized to solve the Cauchy problem for higher dimensional wave equation such as the Hadamard method of descent for $n = 2$, the method of spherical means for $n = 3$ [2], the analytic continuation of an integral of fractional order [3] and the finite part of a divergent integral [4]. A formal series solution has been derived by using a classical power series in [5] and the same formula is obtained in [6] by using the Adomian decomposition method. A direct derivation of the spherical means solution for arbitrary dimension n is presented using Fourier transform in [7].

Consider the Cauchy problem for the following linear inhomogeneous wave equation with variable coefficients in $\mathbb{R}^n$

$$\begin{cases} u_{tt} + Lu = f(x,t), & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where L is a linear differential operator of order m with variable coefficients and L has the following form

$$L \equiv L(x,D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha. \tag{2}$$

Here, the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is an n-tuple of nonnegative integers. The length $|\alpha|$ of α is $|\alpha| = \sum_{j=1}^n \alpha_j = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, the differential operator D^α is defined as

$$D^\alpha h(x) = \frac{1}{i^{|\alpha|}} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} h(x).$$

It is noted that wave equation (1) has the same form as the following oscillatory ordinary differential equation

$$\begin{cases} q_{tt}(t) + Mq(t) = f(q(t)), \\ q(0) = q_0, q_t(0) = p_0, \end{cases} \tag{3}$$

where $q \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$ and M is a symmetric and positive semi-definite matrix.

In the past few decades, much effort has been spent developing numerical methods to solve oscillatory system (3). In 1961, Gautschi [8] proposed the Gautschi method based on trigonometric polynomials. After that, many researchers have been constructing and analyzing Gautschi-type methods [9,10]. In [11], Gonzalez et al. designed a new family of explicit Runge–Kutta type methods with G-functions for the numerical integration of highly oscillatory system. Recently, Wu et al. [12] formulated a standard form of the multi-frequency and multidimensional ERKN (extended Runge–Kutta–Nyström) integrators in which both the internal stages and updates are incorporated with the structure brought by Mq. The ERKN integrators naturally integrate exactly the unperturbed linear equation $q_{tt} + Mq = 0$. For more details on numerical methods for oscillatory system (3), we refer the reader to [13–16]. It is noted that all the numerical methods mentioned above are essentially related with the following variation-of-constants formula which is established in many literatures with different notations (e.g., [10,11,17]). It gives the exact solution and its derivative for the oscillatory system (3) (following the setup in [17]):

$$\begin{cases} q(t) = \phi_0(t^2 M) q_0 + t\phi_1(t^2 M) p_0 + \displaystyle\int_0^t (t-s)\phi_1((t-s)^2 M) f\big(q(s)\big) ds, \\ q_t(t) = -tM\phi_1(t^2 M) q_0 + \phi_0(t^2 M) p_0 + \displaystyle\int_0^t \phi_0((t-s)^2 M) f\big(q(s)\big) ds, \end{cases} \tag{4}$$

where the analytical functions $\phi_j(\cdot), j = 0, 1, \ldots$ are defined as

$$\phi_j(s) := \sum_{k=0}^{\infty} \frac{(-1)^k s^k}{(2k+j)!}, \quad j = 0, 1, \ldots. \tag{5}$$

In this paper, we will show that the linear inhomogeneous wave equation (1) also admits solution of the form (4). The rest of the paper is organized as follows. Some preliminaries and notation are introduced in Section 2. The new analytical formula for the wave equation is derived in Section 3. In Section 4, the feasibility of the formula is illustrated by an example.

2. Preliminaries and some notation

First we will recall some basic definitions. Denote the Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^n$ as

$$\mathcal{S}(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty, \forall \alpha, \beta \in \mathbb{Z}_+^n\}$$

with $\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n}|x^\alpha D^\beta f|$. The Fourier transform of a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ [18,19] is given by

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} \varphi(x)e^{-ix\cdot\xi}dx, \tag{6}$$

where $x \cdot \xi$ is the inner product of x and ξ in $\mathbb{R}^n$. It follows immediately from integral by parts that

$$\widehat{D^\alpha\varphi}(\xi) = \xi^\alpha\widehat{\varphi}(\xi) \quad \text{and} \quad D^\alpha\widehat{\varphi}(\xi) = \widehat{x^\alpha\varphi}(\xi). \tag{7}$$

Therefore, if $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{\varphi}(\xi) \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, the Fourier inversion formula holds

$$\varphi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{\varphi}(\xi)e^{i\xi\cdot x}d\xi. \tag{8}$$

And we have

$$\widehat{(\varphi * \psi)}(\xi) = \widehat{\varphi}(\xi)\widehat{\psi}(\xi), \tag{9}$$

where $*$ denotes the convolution operation defined as

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x-y)\psi(y)dy.$$

According to Eqs. (7) and (9), we have, for $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$

$$L\varphi(x) = L(x,D)\varphi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} L(x,\xi)\hat{\varphi}(\xi)e^{i\xi\cdot x}d\xi, \tag{10}$$

where $L(x,\xi) = \sum_{|\alpha|\leqslant m} a_\alpha(x)\xi^\alpha$ is a multivariate polynomial of degree m with respect to ξ. Not only for polynomials, the right-hand side of Eq. (10) is also well defined for general functions $\phi(x,\xi)$. To be more precise, if $\phi(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies

$$\left|D_\xi^\alpha D_x^\beta\phi(x,\xi)\right| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}, \quad \forall \alpha,\beta \in \mathbb{Z}_+^n \tag{11}$$

for some $m \in \mathbb{R}_+$, then we have the following definition .

Definition 2.1 (*[20]*).

$$(\Phi\varphi)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \phi(x,\xi)\hat{\varphi}(\xi)e^{i\xi\cdot x}d\xi. \tag{12}$$

is said to be pseudodifferential operator. The function $\phi(x,\xi)$ is called symbol of the pseudodifferential operator Φ.

3. Solution to the wave equation

It can be verified that the first three ϕ-functions (5) are

$$\phi_0(s) = \cos(\sqrt{s}), \quad \phi_1(s) = \frac{\sin(\sqrt{s})}{\sqrt{s}}, \quad \phi_2(s) = \frac{\sin^2(\sqrt{s}/2)}{s/2}.$$

Since we have $\lim_{s\to 0^+}\phi_j(s) = \frac{1}{j!}, j = 0, 1, \ldots$, the function $\phi_j(s), j = 0, 1, \ldots$ is well defined at $s = 0$.

Now we are in a position to present our main results. Assume that for the linear differential operator L in wave equation (1), the symbol $L(x, \xi) \geqslant 0$, for $\forall x, \xi \in \mathbb{R}^n$. With reasonable restrictions on the smoothness and growth of coefficients $a_\alpha(x), |\alpha| \leqslant m$ such that (11) holds for $L(x, \xi)$, the following two pseudodifferential operators, with symbols $\phi_0(t^2L(x, \xi))$ and $\phi_1(t^2L(x, \xi))$, respectively, are well defined for $\forall t \in \mathbb{R}_+$.

$$\begin{aligned} (\phi_0(t^2L)\varphi)(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \phi_0(t^2L(x,\xi))\hat{\varphi}(\xi)e^{i\xi\cdot x}d\xi, \\ (\phi_1(t^2L)\varphi)(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \phi_1(t^2L(x,\xi))\hat{\varphi}(\xi)e^{i\xi\cdot x}d\xi \end{aligned} \tag{13}$$

First, we consider the corresponding homogeneous wave equation of (1)

$$\begin{cases} u_{tt} + Lu = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}^n, \\ u_t(x,0) = \psi(x), & x \in \mathbb{R}^n. \end{cases} \tag{14}$$

Based on the two pseudodifferential operators defined by (13), the following theorem for the homogeneous wave equation (14) can be obtained.

Theorem 3.1. *Assume that the initial data $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Then the solution to the Cauchy problem for the wave equation (14) and its derivative with respect to t are given as*

$$\begin{aligned} u(x,t) &= (\phi_0(t^2L)\varphi)(x) + t(\phi_1(t^2L)\psi)(x), \\ u_t(x,t) &= -t(L\phi_1(t^2L)\varphi)(x) + (\phi_0(t^2L)\psi)(x), \end{aligned} \tag{15}$$

Proof. Differentiating both sides of the first equation in (15) with respect to t twice and noticing that

$$\frac{d}{dt}\phi_0(t^2L) = -tL\phi_1(t^2L), \quad \frac{d}{dt}\left(t\phi_1(t^2L)\right) = \phi_0(t^2L),$$

we have

$$u_t(x,t) = -t(L\phi_1(t^2L)\varphi)(x) + (\phi_0(t^2L)\psi)(x)$$

and

$$u_{tt}(x,t) = -(L\phi_0(t^2L)\varphi)(x) - t(L\phi_1(t^2L)\psi)(x) = -Lu(x,t).$$

The proof is completed. □

Remark 3.1. Substituting ϕ_0 and ϕ_1 by their series expansions in (15) gives a formal series solution to the Cauchy problem of homogeneous wave equation (14)

$$u(x,t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}(-1)^k L^k\varphi(x) + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(-1)^k L^k\psi(x), \tag{16}$$

where $L^k\varphi$ is interpreted as $L(L^{k-1}\varphi)$. If L is the negative Laplacian operator $-\Delta$, this solution is coincided with the formula obtained previously in [5,6].

Generally speaking, the closed form solutions for concrete problems are not easily obtainable. In this case, truncated series in the form of the sequence of partial sum

$$u_K(x,t) = \sum_{k=0}^{K} \frac{t^{2k}}{(2k)!}(-1)^k L^k \varphi(x) + \sum_{k=0}^{K} \frac{t^{2k+1}}{(2k+1)!}(-1)^k L^k \psi(x)$$

can be used to approximate the solution.

Now we consider the inhomogeneous wave equation (1). Duhamel's Principle states that if $S(t)$ is the solution operator for the first-order initial-value problem $Ut + AU = 0, U(0) = \Phi$, then the solution of the inhomogeneous problem $U_t + AU = F, U(0) = \Phi$ should be given by $U(t) = S(t)\Phi + \int_0^t S(t-s)F(s)ds$. Introducing a new function $v(x,t) = u_t(x,t)$, (1) can be rewritten as

$$\begin{cases} \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}_t' + \begin{pmatrix} 0 & -1 \\ L & 0 \end{pmatrix} \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix}, \\ \begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} \end{cases} \tag{17}$$

Theorem 3.1 implies that the solution operator $S(t)$ associated with (17) is given by

$$S(t)\begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} \phi_0(t^2L) & t\phi_1(t^2L) \\ -tL\phi_1(t^2L) & \phi_0(t^2L) \end{pmatrix} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} \tag{18}$$

Applying Duhamel's Principle to (17), the solution is given by

$$\begin{aligned} \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} &= \begin{pmatrix} \phi_0(t^2L) & t\phi_1(t^2L) \\ -tL\phi_1(t^2L) & \phi_0(t^2L) \end{pmatrix} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} \\ &\quad + \int_0^t \begin{pmatrix} \phi_0((t-s)^2L) & (t-s)\phi_1((t-s)^2L) \\ -(t-s)L\phi_1((t-s)^2L) & \phi_0((t-s)^2L) \end{pmatrix} \begin{pmatrix} 0 \\ f(x,s) \end{pmatrix} ds \end{aligned}$$

Noticing that the solution of (1) is given by the first component of the solution of (17), we have the following theorem.

Theorem 3.2. *Assume that the initial data $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and the source term $f(x,t) \in \mathcal{S}(\mathbb{R}^n)$ for all $t \geqslant 0$. Then the solution to the Cauchy problem for the wave equation (1) is given by*

$$u(x,t) = (\phi_0(t^2L)\varphi)(x) + t(\phi_1(t^2L)\psi)(x) + \int_0^t (t-s)(\phi_1((t-s)^2L)f)(x,s)ds. \tag{19}$$

If the right-hand side of (1) $f(x,t) \equiv f(x)$ is independent of time variable t, then the solution (20) can be reduced to

$$u(x,t) = (\phi_0(t^2L)\varphi)(x) + t(\phi_1(t^2L)\psi)(x) + t^2(\phi_2(t^2L)f)(x). \tag{20}$$

Remark 3.2. Substituting ϕ_0, ϕ_1 and ϕ_2 by their series expansions in (20) gives a formal series solution to the Cauchy problem of inhomogeneous wave equation (1)

$$u(x,t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}(-1)^k L^k \varphi(x) + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(-1)^k L^k \psi(x) + \sum_{k=0}^{\infty} \frac{t^{2k+2}}{(2k+2)!}(-1)^k L^k f(x). \tag{21}$$

If the closed form solutions for concrete problems are not obtainable. Truncated series in the form of the sequence of partial sum

$$u_K(x,t) = \sum_{k=0}^{K} \frac{t^{2k}}{(2k)!}(-1)^k L^k \varphi(x) + \sum_{k=0}^{K} \frac{t^{2k+1}}{(2k+1)!}(-1)^k L^k \psi(x) + \sum_{k=0}^{K} \frac{t^{2k+2}}{(2k+2)!}(-1)^k L^k f(x)$$

can be used to approximate the solution.

4. Example

It is worth noting that the formulas (20) and (21) are independent of the space dimension. For computation, only differentiation instead of surface integration as in classical Poisson formula is involved, which makes the calculation much simpler. Now, we illustrate our formula through an example.

Example. Consider the linear inhomogeneous wave equation with $N = 3$

$$\begin{cases} u_{tt}(x,t) - \dfrac{\partial^2 u}{\partial x_1 \partial x_1}(x,t) - \sin^2(x_1+x_2+x_3)\dfrac{\partial^2 u}{\partial x_1 \partial x_2}(x,t) - \dfrac{\partial^2 u}{\partial x_2 \partial x_2}(x,t) \\ \quad - \cos^2(x_1+x_2+x_3)\dfrac{\partial^2 u}{\partial x_3 \partial x_3}(x,t) = e^{-2x_1-8x_2-4x_3}, \\ u(x,0) = e^{x_1+4x_2+2x_3}, \\ u_t(x,0) = (x_1-x_2)^2 - x_3^2. \end{cases} \tag{22}$$

The linear differential operator in space is

$$L = \frac{1}{i^2}\frac{\partial^2}{\partial x_1 \partial x_1} + \sin^2(x_1+x_2+x_3)\frac{1}{i^2}\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{i^2}\frac{\partial^2}{\partial x_2 \partial x_2} + \cos^2(x_1+x_2+x_3)\frac{1}{i^2}\frac{\partial^2}{\partial x_3 \partial x_3}.$$

Its symbol is given by

$$L(x,\xi) = \xi_1^2 + \sin^2(x_1+x_2+x_3)\xi_1\xi_2 + \xi_2^2 + \cos^2(x_1+x_2+x_3)\xi_3^2,$$

which satisfies that $L(x,\xi) \geqslant 0, \forall x, \xi \in \mathbb{R}^3$. Therefore, formula (20) is applicable. It can be seen that

$$\begin{aligned} &L\left(e^{x_1+4x_2+2x_3}\right) = -21e^{x_1+4x_2+2x_3}, && L^k\left(e^{x_1+4x_2+2x_3}\right) = (-21)^k e^{x_1+4x_2+2x_3}, && k = 2,3,\ldots, \\ &L\left((x_1-x_2)^2 - x_3^2\right) = -2, && L^k\left((x_1-x_2)^2 - x_3^2\right) = 0, && k = 2,3,\ldots, \\ &L\left(e^{-2x_1-8x_2-4x_3}\right) = -86e^{-2x_1-8x_2-4x_3}, && L^k\left(e^{-2x_1-8x_2-4x_3}\right) = (-86)^k e^{-2x_1-8x_2-4x_3}, && k = 2,3,\ldots. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi_0(t^2L)\left(e^{x_1+4x_2+2x_3}\right) &= \cosh(\sqrt{21}t)e^{x_1+4x_2+2x_3}, \\ \phi_1(t^2L)\left((x_1-x_2)^2 - x_3^2\right) &= \left(1 + \frac{t^2}{6}\right)\left((x_1-x_2)^2 - x_3^2\right), \\ \phi_2(t^2L)\left(e^{-2x_1-8x_2-4x_3}\right) &= \frac{\cosh(\sqrt{86}t) - 1}{86t^2}e^{-2x_1-8x_2-4x_3}, \end{aligned}$$

The solution to the Cauchy problem for the wave equation (22) is given by

$$u(x,t) = \cosh(\sqrt{21}t)e^{x_1+4x_2+2x_3} + \left(t + \frac{t^3}{6}\right)\left((x_1-x_2)^2 - x_3^2\right) + \frac{\cosh(\sqrt{86}t) - 1}{86}e^{-2x_1-8x_2-4x_3}.$$

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