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High-order skew-symmetric differentiation matrix on symmetric grid[☆]

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ABSTRACT

Hairer and Iserles (2016) presented a detailed study of skew-symmetric matrix approximation to a first derivative which is proved to be fundamental in ensuring stability of discretisation for evolutional partial differential equations with variable coefficients. An open problem is proposed in that paper which concerns about the existence and construction of the perturbed grid that supports high-order skew-symmetric differentiation matrix for a given grid and only the case $p = 2$ for this problem have been solved. This paper is an attempt to solve the problem for any $p \geqslant 3$. We focus ourselves on the symmetric grid and prove the existence of the perturbed grid for arbitrarily high order p and give in detail the construction of the perturbed grid. Numerical experiments are carried out to illustrate our theory.

1. Introduction

Stability is a necessary condition to ensure convergence of a numerical method for partial differential equations (PDEs) of evolution. Moreover, in practice, the efficiency of the numerical method is closely related to its stability behaviour. The analysis of numerical stability is normally based on the PDE

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f, \qquad x \in \Omega, \qquad t > 0 \tag{1}$$

with suitably smooth initial and boundary conditions. Here, $u = u(x, t)$, $\Omega \subseteq \mathbb{R}^d$, $\mathcal{L}$ is a linear, time-independent differential operator. Consider the discretisation of the PDE (1) with the form

$$\mathbf{u}_N^{n+1} = \mathcal{A}_N \mathbf{u}_N^n + \mathbf{f}_N^n, \tag{2}$$

where $\mathbf{u}_N^n = (u_{N,1}^n, u_{N,2}^n, \ldots, u_{N,N}^n)$ with $u_{N,m}^n \approx u(x_m, n\Delta t)$, x_m the grid point and $\Delta t > 0$ the time step. $\mathcal{A}_N$ is a finite dimensional approximation of L on some grid. The *stability* of (2) means the uniform well-posedness of the operator $\{\mathcal{A}_N\}$ as

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$N \to \infty$ in the time interval $[0, T]$ (see [1]), which together with *consistency* of (2) guarantees *convergence*, i.e., $\mathbf{u}_N^n$ converges pointwise to the exact solution of (1).

If the coefficients of $\mathcal{L}$ are constant, several powerful techniques can be used to investigate the numerical stability of discretisation of (1), such as Fourier transformation and the corresponding Von Neumann condition for analysing stability, eigenvalue analysis and energy methods. For topics on this subject, we refer the reader to [1–7] and the references therein.

Once the coefficients of $\mathcal{L}$ depend on spatial variable x, the analysis becomes considerably complicated. A natural idea is to freeze the coefficients and check the stability using above-mentioned methods. However, as it turns out, the stability of the method for frozen problem is neither sufficient nor necessary for stability in the variable coefficient case. In [8], E. Hairer and A. Iserles showed that the skew-symmetric differentiation matrix approximation for first derivative is fundamental in ensuring stability of discretisation for evolutional PDEs with variable coefficients (see [8], Theorem 1). A wide range of PDEs such as Liouville equation, Convection–diffusion equation, the Fokker–Planck equation, can be discretized stably once first space derivative are approximated by skew-symmetric matrices. However, it has been proved in [9] that the highest order a skew-symmetric differentiation matrix could get on a uniform grid is just two. With mild perturbation of the uniform grid, the third and fourth-order differentiation matrices for first derivative are constructed on the modified grid. Later, this result is extended to a more general case [8]: given arbitrary grid, not necessarily uniform, the existence of a perturbed grid which supports skew-symmetric differentiation matrices of a given order p is discussed. However, a constructive proof is presented only for $p = 2$ based on a necessary condition given in [8] leaving $p \geqslant 3$ as an open problem. In [10], E. Hairer and A. Iserles derived banded and skew-symmetric differentiation matrices of orders up to 6 based on a perturbation of a uniform grid. In this paper, we will addresses the open problem for general symmetric grid, i.e., given arbitrary symmetric grid, we will discuss in detail the existence and construction of symmetric perturbed grid that supports skew-symmetric differentiation matrix approximation of arbitrary order $p \geqslant 3$ for first derivative.

The outline of this paper is as follows. In Section 2, we introduce some notation and revisit some existing results from [8]. In Section 3, we commence from $p = 3$ and prove the existence of symmetric perturbed grid that supports skew-symmetric differentiation matrix approximation of order 3 for first derivative. Then the analysis is extended to the case of arbitrary order p. An efficient approach to construct the perturbed grid is also provided. Numerical experiments are carried out in Section 4. The last section is devoted to conclusions.

2. Notation and some existing results

Following [8], we assume for simplicity zero Dirichlet boundary conditions and consider the one-dimensional case. Let $\Omega = [0, 1]$ and the grid be $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$. An $N \times N$ differentiation matrix $\mathcal{D}$ defined on this grid is of order p, if

$$u'(x_m) = \sum_{k=1}^{N} \mathcal{D}_{m,k} u(x_k), \quad m = 1, 2, \ldots, N,$$

for every polynomial $u(x)$ of degree p that vanishes at the endpoints.

Theorem 2.1 (*[8]*). *Consider a differentiation matrix $\mathcal{D}$ of order p ($p \geqslant 2$) corresponding to the grid be $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$. The necessary condition for $\mathcal{D}$ to be skew-symmetric is*

$$R_N^{[1]} = R_N^{[2]} = \cdots = R_N^{[2p-3]} = 0 \tag{3}$$

with

$$R_m^{[s]} = \sum_{k=1}^{m} x_k^s (1 - x_k)[(s+1) - (s+3)x_k], \quad s \in \mathbb{N}. \tag{4}$$

Let $q \geqslant 1$ and $p = q + 1$. The order conditions (3) are sufficient for the existence of a skew-symmetric differentiation matrix of order p with bandwidth $2q + 1$.

If the grid is symmetric, i.e., $x_m = 1 - x_{N+1-m}$ for $m = 1, \ldots, N$, then $R_N^{[1]} = R_N^{[3]} = \cdots = R_N^{[2p-3]} = 0$ automatically, therefore the order conditions (3) reduce to

$$R_N^{[2]} = R_N^{[4]} = \cdots = R_N^{[2p-4]} = 0. \tag{5}$$

In practice, the nature of grid is determined by the coefficients of $\mathcal{L}$. That means we cannot expect order conditions (3) (or (5) if the grid is symmetric) hold in the first place. Our concern is that given a grid $\{x_m\}_{m=0}^{N+1}$ to find a perturbed grid $\{\tilde{x}_m\}_{m=0}^{N+1}$ as close to the given grid as possible (e.g., $\tilde{x}_m = x_m + O(N^{-\alpha})$ for $m = 1, \ldots, N$ and $\alpha \geqslant 1$) such that the order conditions (3) hold. Suppose that the grid $\{x_m\}_{m=0}^{N+1}$ can be expressed as the form $x_m = g(m/(N+1))$, $m = 0, 1, \ldots, N+1$, where g is a strictly monotonically increasing, sufficiently smooth function that maps $[0, 1]$ to itself, the following lemma gives a necessary condition for the existence of the perturbed grid.

Lemma 2.1 ([8]). *A necessary condition for the existence of the perturbed grid $\tilde{x}_m = x_m + \mathcal{O}(N^{-\alpha}), \alpha \geqslant 1$ for the grid $x_m = g(m/(N+1)), m = 0, 1, \ldots, N+1$ such that $\tilde{R}_N^{[s]} = 0, s = 1, \ldots, 2p-3$ is*

$$I_1[g] = I_2[g] = \cdots = I_{2p-3}[g] = 0, \tag{6}$$

where

$$I_s[g] = \int_0^1 g^s(\tau)[1-g(\tau)][(s+1)-(s+3)g(\tau)]d\tau. \tag{7}$$

If the grid is symmetric, the necessary conditions (6) reduce to

$$I_2[g] = I_4[g] = \cdots = I_{2p-4}[g] = 0. \tag{8}$$

3. The problem

In [8], the authors left an open problem stated as follows.

Open Problem ([8]). *Give $p \geqslant 3$ and a grid $\{x_m^{\text{old}}\}_{m=0}^{N+1}$ in the interval $[0, 1]$, does there exist another grid $\{x_m^{\text{new}}\}_{m=0}^{N+1}$, sufficiently near to the old grid (e.g. such that $x_m^{\text{new}} = x_m^{\text{old}} + \mathcal{O}(N^{-2}), m = 1, 2, \ldots, N$) which obeys the order conditions (3).*

Lemma 2.1 gives a necessary condition for the existence of the perturbed grid. In fact, the open problem raised such a question that if the necessary condition is sufficient for the existence. In this paper, we focus on the case of symmetric grid and our problem can be stated as follows.

Problem. *Given a symmetric grid function $g \in C^2[0, 1]$ satisfying*

$$g(0) = 0,\ g(1) = 1,\ g'(x) > 0 \text{ and } g(x) + g(1-x) = 1 \text{ for } x \in [0, 1].$$

Assume that the necessary conditions (8) hold, then give $p \geqslant 3$ and a grid $\{x_m\}_{m=0}^{N+1}$ with $x_m = g(m/(N+1))$, find a perturbed grid $\tilde{x}_m = x_m + \mathcal{O}(N^{-2})$ which obeys the order conditions (5).

Before addressing this problem, we give the following lemma which plays an important role throughout the paper.

Lemma 3.1. *Let $x_1^* = \frac{m_1}{N+1}, x_2^* = \frac{m_2}{N+1}$ with $m_1, m_2 \in \mathbb{Z}_+$ satisfying $0 \leqslant m_1 < m_2 \leqslant N+1$. Then for every $f \in C^2[0, 1]$, we have*

$$\sum_{i=m_1}^{m_2} f(g(x_i)) = (N+1)\int_{x_1^*}^{x_2^*} f(g(\tau))d\tau + \frac{1}{2}[f(g(x_1^*)) + f(g(x_2^*))] - \frac{1}{12}\frac{1}{N+1}[f'(g(x_1^*))g'(x_1^*) - f'(g(x_2^*))g'(x_2^*)] + \mathcal{O}(N^{-2}).$$

Here, $\mathbb{Z}_+$ is the set of nonnegative integers.

Proof. Using the Euler–Maclaurin formula [2],

$$\sum_{i=0}^{N} f(i) = \int_0^N f(\tau)d\tau + \frac{1}{2}[f(0) + f(N)] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(N) - f^{(2k-1)}(0)),$$

where B_{2k} is the $2k$th Bernoulli number, we have

$$\begin{aligned}
\sum_{i=m_1}^{m_2} f(g(x_i)) &= \sum_{i=0}^{m_2-m_1} f(g(\frac{m_1+i}{N+1})) \\
&= \int_0^{m_2-m_1} f(g(\frac{m_1+\tau}{N+1}))d\tau + \frac{1}{2}[f(g(x_1^*)) + f(g(x_2^*))] \\
&\quad + \frac{1}{12}\left[\left(f(g(\frac{m_1+\tau}{N+1}))\right)'_\tau\Big|_{\tau=m_2-m_1} - \left(f(g(\frac{m_1+\tau}{N+1}))\right)'_\tau\Big|_{\tau=0}\right] \\
&\quad + \sum_{k=2}^{\infty}\frac{B_{2k}}{(2k)!}\left[\left(f(g(\frac{m_1+\tau}{N+1}))\right)^{(2k-1)}_{\tau\cdots\tau}\Big|_{\tau=m_2-m_1} - \left(f(g(\frac{m_1+\tau}{N+1}))\right)^{(2k-1)}_{\tau\cdots\tau}\Big|_{\tau=0}\right] \\
&= (N+1)\int_{x_1^*}^{x_2^*} f(g(\tau))d\tau + \frac{1}{2}[f(g(x_1^*)) + f(g(x_2^*))] \\
&\quad + \frac{1}{12}\frac{1}{N+1}[f'(g(x_2^*))g'(x_2^*) - f'(g(x_1^*))g'(x_1^*)] + \mathcal{O}(N^{-2}). \qquad \square
\end{aligned}$$

Corollary 3.1. *Under the assumption $I_s[g] = 0$, we have $R_N^{[s]} = \mathcal{O}(N^{-1})$.*

Proof. Let $m_1 = 0$, $m_2 = N + 1$ and $f(x) = x^s(1 - x)[(s + 1) - (s + 3)x]$ in Lemma 3.1 and notice that $g(0) = 0$, $g(1) = 1$. $\square$

We commence the problem from the case $p = 3$. For a symmetric grid, the order condition for $p = 3$ becomes

$$R_N^{[2]} = \sum_{k=1}^{N} x_k^2(1 - x_k)(3 - 5x_k) = 0. \tag{9}$$

Since the grid is symmetric, (9) can be rewritten as

$$\begin{aligned} R_N^{[2]} &= \sum_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} [x_k^2(1 - x_k)(3 - 5x_k) + (1 - x_k)^2 x_k(3 - 5(1 - x_k))] \\ &= \sum_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} x_k(1 - x_k)(-10x_k^2 + 10x_k - 2) = 0, \end{aligned} \tag{10}$$

where $\lfloor x \rfloor = \max\{m \in \mathbb{Z} | m \leqslant x\}$. Here and in the sequel we always assume N is even. For the case N is odd, the discussion is the same. Let

$$F(x) = \int_0^x g(\tau)(2g(\tau) - 1)(1 - 10g(\tau) + 10g^2(\tau))d\tau.$$

Choose $x^* \in (0, \frac{2}{5})$ such that $F(x^*) \neq 0$ and set $K = \lfloor (N + 1)x^* \rfloor$. To keep the symmetry of the grid, we perturb the old grid to the following form.

$$\tilde{x}_m = \begin{cases} (1 - \gamma)x_m, & m = 1, 2, \ldots, K, \\ x_m, & m = K + 1, \ldots, \lfloor \dfrac{N+1}{2} \rfloor, \\ 1 - \tilde{x}_{N+1-m}, & m = \lfloor \dfrac{N+1}{2} \rfloor + 1, \ldots, N, \end{cases} \tag{11}$$

where a parameter γ is introduced and would be determined such that the order condition $\tilde{R}_N^{[2]} = 0$ holds. Substituting the new grid $\{\tilde{x}_m\}_{m=1}^N$ into the order condition gives

$$\begin{aligned} \tilde{R}_N^{[2]} =& R_N^{[2]} - 2\gamma \sum_{k=1}^{K} x_k(2x_k - 1)(1 - 10x_k + 10x_k^2) + 12\gamma^2 \sum_{k=1}^{K} x_k^2(1 - 5x_k + 5x_k^2) \\ &- 20\gamma^3 \sum_{k=1}^{K} x_k^3(2x_k - 1) + 10\gamma^4 \sum_{k=1}^{K} x_k^4 = 0. \end{aligned} \tag{12}$$

By Lemma 3.1, we have

$$\begin{aligned} \tilde{R}_N^{[2]} =& R_N^{[2]} - 2\gamma[(N + 1)F(x^*) + \mathcal{O}(1)] \\ &+ 12\gamma^2[(N + 1)\int_0^{x^*} g^2(\tau)(1 - 5g(\tau) + 5g^2(\tau))d\tau + \mathcal{O}(1)] \\ &- 20\gamma^3[(N + 1)\int_0^{x^*} g^3(\tau)(2g(\tau) - 1)d\tau + \mathcal{O}(1)] \\ &+ 10\gamma^4[(N + 1)\int_0^{x^*} g^4(\tau)d\tau + \mathcal{O}(1)] = 0. \end{aligned} \tag{13}$$

In the following, we will prove that (12) has a zero. Rearranging (13) as

$$\begin{aligned} \gamma =& \frac{R_N^{[2]}}{2[(N + 1)F(x^*) + \mathcal{O}(1)]} + \gamma^2 \frac{12[(N + 1)\int_0^{x^*} g^2(\tau)(1 - 5g(\tau) + 5g^2(\tau))d\tau + \mathcal{O}(1)]}{2[(N + 1)F(x^*) + \mathcal{O}(1)]} \\ &- \gamma^3 \frac{20[(N + 1)\int_0^{x^*} g^3(\tau)(2g(\tau) - 1)d\tau + \mathcal{O}(1)]}{2[(N + 1)F(x^*) + \mathcal{O}(1)]} \\ &+ \gamma^4 \frac{10[(N + 1)\int_0^{x^*} g^4(\tau)d\tau + \mathcal{O}(1)]}{2[(N + 1)F(x^*) + \mathcal{O}(1)]} := G(\gamma). \end{aligned} \tag{14}$$

To find the zero of (12) is equivalent to find the fixed point of $G(\gamma)$. Bearing in mind that $F(x^*) \neq 0$ and $R_N^{[2]} = \mathcal{O}(N^{-1})$ by Corollary 3.1, for $\forall \gamma \in [-\frac{R_N^{[2]}}{2NF(x^*)}, \frac{R_N^{[2]}}{2NF(x^*)}]$ (or $[\frac{R_N^{[2]}}{2NF(x^*)}, -\frac{R_N^{[2]}}{2NF(x^*)}]$ if $\frac{R_N^{[2]}}{2NF(x^*)} < 0$), we have

$$
\begin{aligned}
|G(\gamma)| \leqslant & \left|\frac{R_N^{[2]}}{2[(N+1)F(x^*)+O(1)]}\right| \\
& + \left(\frac{R_N^{[2]}}{2NF(x^*)}\right)^2 \left|\frac{12[(N+1)\int_0^{x^*} g^2(\tau)(1-5g(\tau)+5g^2(\tau))d\tau + \mathcal{O}(1)]}{2[(N+1)F(x^*)+\mathcal{O}(1)]}\right| \\
& + \left(\frac{R_N^{[2]}}{2NF(x^*)}\right)^3 \left|\frac{20[(N+1)\int_0^{x^*} g^3(\tau)(2g(\tau)-1)d\tau + \mathcal{O}(1)]}{2[(N+1)F(x^*)+\mathcal{O}(1)]}\right| \\
& + \left(\frac{R_N^{[2]}}{2NF(x^*)}\right)^4 \left|\frac{10[(N+1)\int_0^{x^*} g^4(\tau)d\tau + \mathcal{O}(1)]}{2[(N+1)F(x^*)+\mathcal{O}(1)]}\right| \\
= & \left|\frac{R_N^{[2]}}{2[(N+1)F(x^*)+O(1)]}\right| + \mathcal{O}(N^{-4}) \leqslant \left|\frac{R_N^{[2]}}{2NF(x^*)}\right|.
\end{aligned}
\tag{15}
$$

Therefore, G is a map from $[-\frac{R_N^{[2]}}{2NF(x^*)}, \frac{R_N^{[2]}}{2NF(x^*)}]$ to itself. Furthermore, it is easy to verify that $G'(\gamma) = \mathcal{O}(N^{-2})$ for $\gamma \in [-\frac{R_N^{[2]}}{2NF(x^*)}, \frac{R_N^{[2]}}{2NF(x^*)}]$. Thus, for $\forall \gamma_1, \gamma_2 \in [-\frac{R_N^{[2]}}{2NF(x^*)}, \frac{R_N^{[2]}}{2NF(x^*)}]$

$$
|G(\gamma_1) - G(\gamma_2)| = |G'(\xi)(\gamma_1 - \gamma_2)| < m_0|\gamma_1 - \gamma_2|, \tag{16}
$$

where $m_0 < 1$. By Banach fixed point theorem [11], G has a unique fixed point in $[-\frac{R_N^{[2]}}{2NF(x^*)}, \frac{R_N^{[2]}}{2NF(x^*)}]$, or equivalently, (12) has a unique zero $\gamma = \gamma^* \in [-\frac{R_N^{[2]}}{2NF(x^*)}, \frac{R_N^{[2]}}{2NF(x^*)}]$. This implies that $\gamma^* = \mathcal{O}(N^{-2})$ and

$$
\tilde{x}_m - x_m = \begin{cases} -\gamma^* x_m = \mathcal{O}(N^{-2}), & m = 1, 2, \ldots, K, \\ 0, & m = K+1, \ldots, N+1-K \\ \gamma^* x_m = \mathcal{O}(N^{-2}), & m = N-K, \ldots, N, \end{cases}
$$

which assures that $\tilde{x}_{m+1} - \tilde{x}_m > 0,\ m = 0, 1, \ldots, N$ for sufficiently large N. Therefore, the existence of the perturbed grid $\tilde{x}_m = x_m + \mathcal{O}(N^{-2})$ which obeys the order conditions is proved.

Remark 3.1. Here and in the sequel, we assume that all the integrals arising in the proof are $\mathcal{O}(1)$. This can be assured by choosing $x_1^*, x_2^*, \ldots$ such that the intervals of the integrals are $\mathcal{O}(1)$. The demand is easy to meet since our interest is in the stability for $N \gg p$.

Now that we have proved there exists a zero $\gamma = \mathcal{O}(N^{-2})$ one can always seek γ by fixed point iteration. However, to give an insight of what γ is, in the following, we present γ explicitly in a constructive way. Since $\gamma = \mathcal{O}(N^{-2})$, γ admits the expansion in inverse power of N,

$$
\gamma = \sum_{i=2}^{\infty} \frac{\gamma_i}{N^i}.
$$

For simplicity, we rewrite (12) in a compact form

$$
\tilde{R}_N^{[2]} = \frac{F_0}{N} + NF_1\gamma + NF_2\gamma^2 + NF_3\gamma^3 + NF_4\gamma^4, \tag{17}
$$

where $F_0 = NR_N^{[2]}$, and NF_i is the coefficient in front of γ^i, $i = 1, 2, 3, 4$. Note that $F_i = \mathcal{O}(1)$, $i = 0, 1, 2, 3, 4$ by (13) and $F_1 \neq 0$ under the assumption.

Substituting the expansion of γ into (17) and letting the coefficient of every inverse power of N equal 0 gives the values of γ_i, $i = 2, 3, \ldots$. The first few terms of the expansion of γ are as follows.

$$
\begin{aligned}
&\gamma_2 = -\frac{F_0}{F_1},\ \gamma_3 = 0,\ \gamma_4 = -\frac{F_2\gamma_2^2}{F_1}, \\
&\gamma_5 = 0,\ \gamma_6 = \frac{F_2(2\gamma_2\gamma_4 + \gamma_3^2) + F_3\gamma_2^3}{F_1}, \ldots.
\end{aligned}
\tag{18}
$$

From this pattern, we guess that $\gamma_{2k+1} = 0, k = 1, 2, \ldots,$ which can be proven by induction and we leave it to the reader. Thus, γ has the expansion

$$\gamma = \sum_{i=1}^{\infty} \frac{\gamma_{2i}}{N^{2i}}.$$

Accordingly, (17) becomes

$$\frac{F_0}{N^2} + F_1 \sum_{i=1}^{\infty} \frac{\gamma_{2i}}{N^{2i}} + F_2\Big(\sum_{i=1}^{\infty} \frac{\gamma_{2i}}{N^{2i}}\Big)^2 + F_3\Big(\sum_{i=1}^{\infty} \frac{\gamma_{2i}}{N^{2i}}\Big)^3 + F_4\Big(\sum_{i=1}^{\infty} \frac{\gamma_{2i}}{N^{2i}}\Big)^4 = 0. \tag{19}$$

Considering the coefficients of N^{-2n} in (19), we have

$$F_0 + F_1\gamma_2 = 0, \quad n = 1, \tag{20}$$

$$F_1\gamma_{2n} + \sum_{l=2}^{4} F_l \sum_{\mathbf{j}\in\mathbb{J}_{n,l}} \binom{l}{j_1, \ldots, j_{n-1}} \gamma_2^{2j_1}\gamma_4^{2j_2}\cdots\gamma_{2n-2}^{2j_{n-1}} = 0, \quad n = 2, 3, \ldots \tag{21}$$

where $\mathbb{J}_{n,l}$ is defined as

$$\mathbb{J}_{n,l} = \{(j_1, \ldots, j_{n-1}) \in \mathbb{Z}_+^{n-1} : j_1 + j_2 + \cdots + j_{n-1} = l,\ j_1 + 2j_2 + \cdots + (n-1)j_{n-1} = n\} \tag{22}$$

and

$$\binom{l}{j_1, j_2, \ldots, j_{n-1}} = \frac{l!}{j_1!j_2!\cdots j_{n-1}!}.$$

Thus solving (20) and (21) for γ_2 and γ_{2n}, $n = 2, 3, \ldots$ gives

$$\gamma_2 = -\frac{F_0}{F_1}, \quad n = 1, \tag{23}$$

$$\gamma_{2n} = -F_1^{-1} \sum_{l=2}^{4} F_l \sum_{\mathbf{j}\in\mathbb{J}_{n,l}} \binom{l}{j_1, \ldots, j_{n-1}} \gamma_2^{2j_1}\gamma_4^{2j_2}\cdots\gamma_{2n-2}^{2j_{n-1}}, \quad n = 2, 3, \ldots. \tag{24}$$

In practice, γ could be calculated by truncating the expansion to certain power of N^{-2} where the desired accuracy is achieved.

The whole idea above can be extended to any order $p \geqslant 3$. As a matter of fact, order conditions of symmetric grid for $p \geqslant 3$ read

$$R_N^{[s]} = \sum_{k=1}^{\lfloor \frac{N+1}{2} \rfloor} x_k(1 - x_k)f_s(x_k) = 0, s = 2, 4, \ldots, 2p - 4, \tag{25}$$

where $f_s(x) = x(1 - x)[(s + 1)(x^{s-1} + (1 - x)^{s-1}) - (s + 3)(x^s + (1 - x)^s)]$. The number of the equations is $p - 2$, which motivates us to consider the following perturbed grid

$$\tilde{x}_m = \begin{cases} (1-\gamma_1)x_m, & m = 1, 2, \ldots, K_1, \\ x_m, & m = K_1 + 1, \ldots, \left\lfloor \frac{2}{2p-1}(N+1) \right\rfloor - 1, \\ (1-\gamma_2)x_m, & m = \left\lfloor \frac{2}{2p-1}(N+1) \right\rfloor, \ldots, K_2, \\ \vdots & \\ (1-\gamma_{p-2})x_m, & m = \left\lfloor \frac{2}{7}(N+1) \right\rfloor, \ldots, K_{p-2}, \\ x_m, & m = K_{p-2} + 1, \ldots, \lfloor \frac{1}{2}(N+1) \rfloor - 1, \\ 1 - \tilde{x}_{N+1-m}, & m = \lfloor \frac{1}{2}(N+1) \rfloor, \ldots, N. \end{cases} \tag{26}$$

where $K_i = \lfloor (N+1)x_i^* \rfloor$, $i = 1, 2, \ldots, p-2$ and

$$x_1^* \in (0, \frac{2}{2p-1}),\ x_i^* \in (\frac{2}{2p-2i+3}, \frac{2}{2p-2i+1}),\ i = 2, 3, \ldots, p-2$$

are chosen such that

$$\det(F(x_1^*, x_2^*, \ldots, x_{p-2}^*)) \neq 0 \tag{27}$$

with

$$F(x_1^*, x_2^*, \ldots, x_{p-2}^*) = \begin{pmatrix} F_{11}(x_1^*) & F_{12}(x_2^*) & \cdots & F_{1,p-2}(x_{p-2}^*) \\ F_{21}(x_1^*) & F_{22}(x_2^*) & \cdots & F_{2,p-2}(x_{p-2}^*) \\ \vdots & \vdots & \vdots & \vdots \\ F_{p-2,1}(x_1^*) & F_{p-2,2}(x_2^*) & \cdots & F_{p-2,p-2}(x_{p-2}^*) \end{pmatrix}. \tag{28}$$

$F_{ij}(x)$, $i, j = 1, \ldots, p-2$ are defined as follows.

Substituting $\tilde{x} = (1-\gamma)x$ for x into $f_s(x)$ gives

$$f_s(\tilde{x}) = f_s(x) - \gamma \tilde{f}_s(x) + \sum_{j=2}^{s+2} \gamma^j c_j^{[s]}(x),$$

where

$$\tilde{f}_s(x) = x^s(s(s+1) - 2(s+1)(s+2)x + (s+2)(s+3)x^2) - (1-x)^{s-1}x(2 - 4(s+2)x + (s+2)(s+3)x^2) \tag{29}$$

and $C_k^{[s]}(x)$ represents the coefficient of γ^k, $k = 2, \ldots, s+2$ in $f_s(\tilde{x})$.

Then substituting (26) into (25) gives

$$\begin{aligned} \tilde{R}_N^{[s]} &= R_N^{[s]} - \gamma_1 \sum_{k=1}^{K_1} \tilde{f}_s(x_k) - \sum_{i=2}^{p-2} \gamma_i \sum_{k=\left\lfloor \frac{2(N+1)}{2p-2i+3} \right\rfloor}^{K_i} \tilde{f}_s(x_k) \\ &\quad + \sum_{j=2}^{s+2} [\gamma_1^j \sum_{k=1}^{K_1} c_j^{[s]}(x_k) + \sum_{i=2}^{p-2} \gamma_i^j \sum_{k=\left\lfloor \frac{2(N+1)}{2p-2i+3} \right\rfloor}^{K_i} c_j^{[s]}(x_k)] = 0 \\ &\quad s = 2, 4, \ldots, 2p-4. \end{aligned} \tag{30}$$

Let

$$\begin{aligned} F_{i1}(x) &= \int_0^x \tilde{f}_{2i}(g(\tau))d\tau, \qquad i = 1, \ldots, p-2, \\ F_{i,j}(x) &= \int_{2/(2p-2j+3)}^x \tilde{f}_{2i}(g(\tau))d\tau, \qquad i = 1, \ldots, p-2, \quad j = 2, 4, \ldots, 2p-2, \end{aligned} \tag{31}$$

and

$$\begin{aligned} C_{i1}^{[j]}(x) &= \int_0^x c_j^{[2i]}(g(\tau))d\tau, \qquad i = 1, \ldots, p-2, \quad j = 2, 4, \ldots, s+2, \\ C_{i,k}^{[j]}(x) &= \int_{2/(2p-2k+3)}^x c_j^{[2i]}(g(\tau))d\tau, \qquad i, k = 1, \ldots, p-2, \quad j = 2, 4, \ldots, s+2. \end{aligned} \tag{32}$$

By Lemma 3.1, (30) can be rewritten as

$$\begin{aligned} \tilde{R}_N^{[2s]} &= R_N^{[2s]} - \sum_{i=1}^{p-2} \gamma_i[(N+1)F_{si}(x_i^*) + \mathcal{O}(1)] + T_s(\gamma_1, \ldots, \gamma_{p-2}) = 0, \\ &\quad s = 1, 2, \ldots, p-2, \end{aligned} \tag{33}$$

where

$$\begin{aligned} T_s(\gamma_1, \ldots, \gamma_{p-2}) &= \sum_{j=2}^{s+2} \sum_{i=1}^{p-2} \gamma_i^j [(N+1)C_{si}^{[j]}(x_i^*) + \mathcal{O}(1)], \\ &\quad s = 1, 2, \ldots, p-2. \end{aligned} \tag{34}$$

With a little different setup, the existence of zero for (33) could be proved in a similar way as the case $p=3$. Letting $R=(R_N^{[2]},R_N^{[4]},\dots,R_N^{[2p-4]})^T$, $\gamma=(\gamma_1,\dots,\gamma_{p-2})^T$ and $T(\gamma)=(T_1(\gamma_1,\dots,\gamma_{p-2}),T_2(\gamma_1,\dots,\gamma_{p-2}),\dots,T_{p-2}(\gamma_1,\dots,\gamma_{p-2}))^T$, (33) can be written compactly as

$$R-\left((N+1)F(x_1^*,x_2^*,\dots,x_{p-2}^*)+\mathcal{O}(1)\right)\gamma+T(\gamma)=0. \tag{35}$$

Let

$$G(\gamma)=\left((N+1)F(x_1^*,x_2^*,\dots,x_{p-2}^*)+\mathcal{O}(1)\right)^{-1}(R+T(\gamma)),$$

then to find the solution of (30) is equivalent to find the fixed point of G. Denote $\|\cdot\|_d$ be the l^∞ norm on $\mathbb{R}^d$ and its induced matrix norm on $\mathbb{R}^{d\times d}$ and let the domain

$$\mathcal{D}=\left\{\gamma\in\mathbb{R}^{p-2}\,\middle|\,\;\|\gamma\|_{p-2}\leqslant\frac{2}{N+1}\|F^{-1}(x_1^*,x_2^*,\dots,x_{p-2}^*)R\|_{p-2}\right\}.$$

In the following, we will show that G is a contraction mapping from $\mathcal{D}$ to $\mathcal{D}$. Since $R_N^{[2s]}=\mathcal{O}(N^{-1})$, $s=1,2,\dots,p-2$ by Corollary 3.1, we have $\|\gamma\|_{p-2}=\mathcal{O}(N^{-2})$ and $T(\gamma)=\mathcal{O}(N^{-4})$ for all $\gamma\in\mathcal{D}$. For sufficiently large N,

$$\begin{aligned}\|G(\gamma)\|_{p-2}&=\|\left((N+1)F(x_1^*,x_2^*,\dots,x_{p-2}^*)+\mathcal{O}(1)\right)^{-1}(R+T(\gamma))\|_{p-2}\\&=\frac{\|\left(F(x_1^*,x_2^*,\dots,x_{p-2}^*)+\mathcal{O}((N+1)^{-1})\right)^{-1}(R+T(\gamma))\|_{p-2}}{N+1}\\&\leqslant\frac{\|\left(F(x_1^*,x_2^*,\dots,x_{p-2}^*)+\mathcal{O}((N+1)^{-1})\right)^{-1}R\|_{p-2}}{N+1}+\mathcal{O}(N^{-4})\\&\leqslant\frac{2}{N+1}\|F^{-1}(x_1^*,x_2^*,\dots,x_{p-2}^*)R\|_{p-2}.\end{aligned} \tag{36}$$

Thus, G maps $\mathcal{D}$ to $\mathcal{D}$. To see that G is a contraction mapping, one only needs to notice that

$$\begin{aligned}&\|G(\alpha)-G(\beta)\|_{p-2}\\&=\|\left((N+1)F(x_1^*,x_2^*,\dots,x_{p-2}^*)+\mathcal{O}(1)\right)^{-1}(T(\alpha)-T(\beta))\|_{p-2}\\&\leqslant\frac{1}{(N+1)}\|\left(F(x_1^*,x_2^*,\dots,x_{p-2}^*)+\mathcal{O}((N+1)^{-1})\right)^{-1}\|_{p-2}\cdot\|T(\alpha)-T(\beta)\|_{p-2}\end{aligned} \tag{37}$$

and the fact for all $\gamma\in\mathcal{D}$, $\|T'(\gamma)\|_{p-2}=\mathcal{O}(N^{-1})$, where $T'(\gamma)$ is the Jacobian matrix of T with respect to γ. By Banach fixed point theorem, G has a unique fixed point $\gamma=\gamma^*=(\gamma_1^*,\dots,\gamma_{p-2}^*)^T$ in $\mathcal{D}$ which means that (30) has a solution satisfying $\|\gamma\|_{p-2}=\mathcal{O}(N^{-2})$.

Following the line of the case $p=3$, we assume

$$\gamma_1=\sum_{i=1}^{\infty}\frac{\gamma_{1,2i}}{N^{2i}},\gamma_2=\sum_{i=1}^{\infty}\frac{\gamma_{2,2i}}{N^{2i}},\dots,\gamma_{p-2}=\sum_{i=1}^{\infty}\frac{\gamma_{p-2,2i}}{N^{2i}}, \tag{38}$$

and rewrite Eq. (30) in the form

$$\tilde{R}_N^{[s]}=\frac{F_0^{[s]}}{N}+N\sum_{i=1}^{p-2}F_{1,i}^{[s]}\gamma_i+N\sum_{i=1}^{p-2}F_{2,i}^{[s]}\gamma_i^2+\dots+N\sum_{i=1}^{p-2}F_{s+2,i}^{[s]}\gamma_i^{s+2}=0,\quad s=2,4,\dots,2p-4, \tag{39}$$

or

$$\tilde{R}_N^{[s]}=\frac{F_0^{[s]}}{N^2}+\sum_{i=1}^{p-2}F_{1,i}^{[s]}\gamma_i+\sum_{i=1}^{p-2}F_{2,i}^{[s]}\gamma_i^2+\dots+\sum_{i=1}^{p-2}F_{s+2,i}^{[s]}\gamma_i^{s+2}=0,\quad s=2,4,\dots,2p-4, \tag{40}$$

where $F_0^{[s]}=NR_N^{[s]}$ and $F_{k,i}^{[s]}=0$, $i=1,2,\dots,p-2$ for $k>s+2$. Let

$$F_k=\begin{pmatrix}F_{k,1}^{[2]}&F_{k,2}^{[2]}&\cdots&F_{k,p-2}^{[2]}\\F_{k,1}^{[4]}&F_{k,2}^{[4]}&\cdots&F_{k,p-2}^{[4]}\\\vdots&\vdots&\vdots&\vdots\\F_{k,1}^{[2p-4]}&F_{k,2}^{[2p-4]}&\cdots&F_{k,p-2}^{[2p-4]}\end{pmatrix},\ k=1,2,\dots,p-2,$$

$$F_0=(F_0^{[2]},\dots,F_0^{[2p-4]})^T\quad\text{and}\quad\Gamma_{2i}=(\gamma_{1,2i},\dots,\gamma_{p-2,2i})^T,\quad i=1,2,\dots,$$

Substituting (38) into (40), we have

$$\frac{F_0}{N^2} + F_1 \sum_{i=1}^{\infty} \frac{\Gamma_{2i}}{N^{2i}} + F_2 \left(\sum_{i=1}^{\infty} \frac{\Gamma_{2i}}{N^{2i}} \right)^2 + \cdots + F_{p-2} \left(\sum_{i=1}^{\infty} \frac{\Gamma_{2i}}{N^{2i}} \right)^{2p-2} = 0, \tag{41}$$

where the multiplication among $\Gamma_{2i}, i = 1, 2, \dots$ is interpreted as componentwise.

(41) is exactly of the form (19). In analogy, we have

$$\Gamma_2 = -F_1^{-1} F_0, \quad n = 1, \tag{42}$$

$$\Gamma_{2n} = -F_1^{-1} \sum_{l=2}^{2p-2} F_l \sum_{\mathbf{j} \in \mathbb{J}_{n,l}} \binom{l}{j_1, \dots, j_{n-1}} \Gamma_2^{2j_1} \Gamma_4^{2j_2} \cdots \Gamma_{2n-2}^{2j_{n-1}}, \quad n = 2, 3, \dots. \tag{43}$$

Here, the reversibility of F_1 is guaranteed by the assumption $\det(F(x_1^*, x_2^*, \dots, x_{p-2}^*)) \neq 0$.

4. Numerical experiments

In this section, we illustrate our theory by an example. First of all, we need to construct an symmetric grid function that satisfies the necessary conditions (8) for the existence of the perturbed grid. Before that we state an important lemma given in [8].

Lemma 4.1. *The identity $I_s[g] = 0$ holds for some $s \in \mathbb{N}$ if the inverse function $h = g^{-1}$ is orthogonal to $\tilde{P}_2, \tilde{P}_3, \dots, \tilde{P}_{s+1}$, where $\tilde{P}_n$ is the nth degree Legendre polynomial shifted to the interval $[0, 1]$, $\tilde{P}_n(t) = P_n(2t - 1)$.*

Here, the Legendre polynomials can be represented by *Rodrigues' formula*

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n,$$

which takes the value 1 at $x = 1$, and if n is odd/even then the polynomial is odd/even.

Let $g(x)$ be the symmetric grid function we seek and the expansion of its inverse function $h(x)$ in shifted Legendre polynomials is

$$h(x) = \sum_{m=0}^{\infty} h_m \tilde{P}_m(x).$$

Since $g(x)$ is a symmetric grid function, we have $g(x) + g(1 - x) = 1$ which means that $h(x) + h(1 - x) = 1$. Therefore h is spanned only by odd shifted Legendre polynomials and $\tilde{P}_0$, i.e., $h_{2m} = 0$ for all $m \in \mathbb{N}$. The expansion of $h(x)$ then becomes

$$h(x) = h_0 + \sum_{m=1}^{\infty} h_{2m-1} \tilde{P}_{2m-1}(x) = h_0 + \sum_{m=1}^{\infty} h_{2m-1} P_{2m-1}(2x - 1). \tag{44}$$

Since $h(x) = g^{-1}(x)$ and $g(0) = 0, g(1) = 1$, we have $h(0) = 0$, $h(1) = 1$. Substituting $x = 0, x = 1$ into (44) gives

$$\begin{aligned} h(0) &= h_0 - \sum_{m=1}^{\infty} h_{2m-1} = 0, \\ h(1) &= h_0 + \sum_{m=1}^{\infty} h_{2m-1} = 1, \end{aligned} \tag{45}$$

or

$$h_0 = \frac{1}{2}, \quad \sum_{m=1}^{\infty} h_{2m-1} = \frac{1}{2}. \tag{46}$$

Combining (44) and (46), we have, the inverse function h of the symmetric grid function g admits the expansion in shifted Legendre polynomials

$$h(x) = \frac{1}{2} + \sum_{m=1}^{\infty} h_{2m-1} \tilde{P}_{2m-1}(x), \quad \sum_{m=1}^{\infty} h_{2m-1} = \frac{1}{2}.$$

By Lemma 4.1 and Lemma 2.1, we have the following lemma.

Table 1
Results for order $p = 4$.

x_1^*, x_2^*	γ_1, γ_2 ($\times 10^{-4}$)	$\max_{s=2,4} \lvert\tilde{R}_N^{[s]}\rvert$ ($\times 10^{-15}$)	$\lVert\tilde{x} - x\rVert$ ($\times 10^{-5}$)
$\frac{12}{70}, \frac{34}{70}$	4.74829, 2.14438	2.02616	8.03350
$\frac{13}{70}, \frac{33}{70}$	4.37267, 1.94442	1.58207	8.05035
$\frac{14}{70}, \frac{30}{70}$	4.09013, 1.71268	1.13798	8.14048
$\frac{15}{70}, \frac{30}{70}$	3.87178, 1.41551	0.63838	8.28369

Lemma 4.2. *A symmetric grid function g obeys necessary conditions (8) of order p, if its inverse function h has the expansion in shifted Legendre polynomials*

$$h(x) = \frac{1}{2} + h_1\tilde{P}_1(x) + \sum_{m=p}^{\infty} h_{2m-1}\tilde{P}_{2m-1}(x), \quad h_1 + \sum_{m=p}^{\infty} h_{2m-1} = \frac{1}{2}. \tag{47}$$

Now we are in a position to construct the required grid function. To do that, we commence from a strictly monotone function $\tilde{h}(x) \in C^2[0, 1]$ satisfying $\tilde{h}(0) = 0$, $\tilde{h}(1) = 1$ and $\tilde{h}(x) + \tilde{h}(1-x) = 1$. In the spirit of Lemma 4.2 and the discussion above, firstly we orthogonalise $\tilde{h}(x)$ with respect to $\tilde{P}_3, \tilde{P}_5, \ldots, \tilde{P}_{2p-3}$. Take $p = 4$ for example, it gives

$$\bar{h}(x) = \tilde{h}(x) - \frac{\int_0^1 \tilde{h}(\tau)\tilde{P}_3(\tau)d\tau}{\int_0^1 \tilde{P}_3^2(\tau)d\tau}\tilde{P}_3(x) - \frac{\int_0^1 \tilde{h}(\tau)\tilde{P}_5(\tau)d\tau}{\int_0^1 \tilde{P}_5^2(\tau)d\tau}\tilde{P}_5(x).$$

It cannot be guaranteed that $\bar{h}(0) = 0$, $\bar{h}(1) = 1$ holds after orthogonalisation. Therefore, the second step is to tune the coefficients of the expansion of $\bar{h}$ such that (47) is satisfied without destroying the orthogonality. An easy way is to introduce two parameters a and b and let

$$h(x) = \tilde{h}(x) - \frac{\int_0^1 \tilde{h}(\tau)\tilde{P}_3(\tau)d\tau}{\int_0^1 \tilde{P}_3^2(\tau)d\tau}\tilde{P}_3(x) - \frac{\int_0^1 \tilde{h}(\tau)\tilde{P}_5(\tau)d\tau}{\int_0^1 \tilde{P}_5^2(\tau)d\tau}\tilde{P}_5(x) + a\tilde{P}_0(x) + b\tilde{P}_1(x).$$

Solving the equations $h(0) = 0$, $h(1) = 1$ for a and b yields

$$a = 0, \quad b = \frac{\int_0^1 \tilde{h}(\tau)\tilde{P}_3(\tau)d\tau}{\int_0^1 \tilde{P}_3^2(\tau)d\tau} + \frac{\int_0^1 \tilde{h}(\tau)\tilde{P}_5(\tau)d\tau}{\int_0^1 \tilde{P}_5^2(\tau)d\tau}.$$

There is no guarantee that $h(x)$ is strictly monotone, and this need to be checked from case to case. Thus the last step is to check the monotonicity of the function h. If h is strictly monotone, then $g = h^{-1}$ is exactly the grid function we are seeking for.

In the following, we illustrate the whole procedure with $\tilde{h}(x) = \sin^2\frac{\pi x}{2}$. Note that $\tilde{h}(0) = 0$, $\tilde{h}(1) = 1$ and $\tilde{h}(x)+\tilde{h}(1-x) = 1$.

First consider the case $p = 4$. Following the procedure above we have

$$\begin{aligned} h(x) =& \sin^2\frac{\pi x}{2} - -\frac{84(\pi^2 - 10)}{\pi^4}\tilde{P}_3(x) - \frac{330(\pi^4 - 112\pi^2 + 1008)}{\pi^6}\tilde{P}_5(x) \\ &+ \frac{18(23\pi^4 - 2100\pi^2 + 18480)}{\pi^6}\tilde{P}_1(x). \end{aligned}$$

It is easy to verify that the derivative of h with respect to x is greater than 0 in the interval $[0, 1]$. Then $g(x) = h^{-1}(x)$ is the symmetric grid function that obeys necessary conditions (8) with $p = 4$. To find the perturbed grid that satisfies conditions (5), we choose $x_1^* \in (0, \frac{2}{7})$, $x_2^* \in (\frac{2}{7}, \frac{2}{5})$ randomly, and solve γ_1, γ_2 by the expansion with truncation to N^{-8}, the results for $N = 200$ are shown in Table 1.

Next we consider the case $p = 6$. Similarly, first we derive $h(x)$ from $\tilde{h}(x) = \sin^2\frac{\pi x}{2}$, which gives

$$\begin{aligned} h(x) =& \tilde{h}(x) - \frac{84(\pi^2 - 10)}{\pi^4}\tilde{P}_3(x) - \frac{330(\pi^4 - 112\pi^2 + 1008)}{\pi^6}\tilde{P}_5(x) \\ &- \frac{840(\pi^6 - 450\pi^4 + 35640\pi^2 - 308880)}{\pi^8}\tilde{P}_7(x) \\ &- \frac{1710(\pi^8 - 1232\pi^6 + 336336\pi^4 - 23063040\pi^2 + 196035840)}{\pi^{10}}\tilde{P}_9(x) \\ &+ \frac{156(19\pi^8 - 16170\pi^6 + 3880800\pi^4 - 254469600\pi^2 + 2148854400)}{\pi^{10}}\tilde{P}_1(x). \end{aligned}$$

Table 2
Result for order $p = 6$.

$x_1^*, x_2^*, x_3^*, x_4^*$	$\gamma_1, \gamma_2, \gamma_3, \gamma_4$ ($\times 10^{-3}$)	$\max_{s=2,4,6,8} \lvert \tilde{R}_N^{[s]} \rvert$ ($\times 10^{-15}$)	$\lvert\lvert \tilde{x} - x \rvert\rvert$ ($\times 10^{-4}$)
$\frac{200}{3465}, \frac{680}{3465}, \frac{900}{3465}, \frac{1100}{3465}$	2.97959, 1.45150, −2.07073, 1.31921	7.97278	4.53295
$\frac{300}{3465}, \frac{700}{3465}, \frac{800}{3465}, \frac{1200}{3465}$	2.04141, 1.16980, −1.69376, 1.15317	2.42167	3.70772
$\frac{400}{3465}, \frac{720}{3465}, \frac{850}{3465}, \frac{1300}{3465}$	1.75559, 0.82827, −1.36721, 1.06210	1.87350	3.01192
$\frac{500}{3465}, \frac{700}{3465}, \frac{900}{3465}, \frac{1100}{3465}$	1.62490, −0.15508, −0.88501, 0.98324	1.60982	2.78831

It can also be verified that h is strictly monotone and thus $g = h^{-1}$ satisfies necessary conditions (8). The conditions for order $p = 6$ consist of four equations $R_N^{[s]} = 0$, $s = 2, 4, 6, 8$. We choose $x_1^* \in (0, \frac{2}{11})$, $x_2^* \in (\frac{2}{11}, \frac{2}{9})$, $x_3^* \in (\frac{2}{9}, \frac{2}{7})$, $x_4^* \in (\frac{2}{7}, \frac{2}{5})$ randomly and solve $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ by expansion with truncation to N^{-12}, the result for $N = 200$ is shown in Table 2.

It can be seen from the results that Given an old symmetric grid function that obeys necessary conditions (8), our procedure of perturbation to the old symmetric does give a new symmetric grid which obeys the order conditions (5). As long as the assumption $\det(F(x_1^*, x_2^*, \ldots, x_{p-2}^*)) \neq 0$ holds, the choice of x^* gives little difference on the construction of the perturbed grid.

5. Conclusions

In [8], E, Hairer and A. Iserles investigated in detail the numerical stability of PDEs in the presence of variable coefficients. This paper addresses one of the open problems proposed in [8]. It is concerned about the existence and construction of the perturbed grid that supports high-order skew-symmetric differentiation matrix for a given grid. The discussion in [8] has been restricted to $p = 2$. In this paper, we focus ourselves on symmetric grid and give a detailed study upon the problem for the case of order $p \geqslant 3$. By subdividing the space interval properly into some subintervals and giving independent perturbation to the old symmetric grid in each subinterval, we show the existence of the perturbed grid that obeys order conditions (5) and present an efficient way to construct it for arbitrarily high order p. Numerical experiments are carried out and the results show the feasibility of our method. Last but not least, it should be noted that the procedure given in the paper can be easily extended to the general grids without symmetry.

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