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An integral evolution formula of boundary value problem for wave equations☆

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ABSTRACT

This article proposes a new integral evolution formula of boundary value problem for wave equations of the form $u_{tt}(x,t)+\mathcal{L}(x,D)u(x,t)=f(x,t)$. By introducing the operator functions, e.g., ϕ-functions, and using the Duhamel's principle, a compact integral evolution formula is established for inhomogeneous wave equations. The derivation is based on Duhamel's principle and the theory of operational calculus.

1. Introduction

The history of partial differential equations can date back to the 18th century when the first wave equation $u_{tt}-u_{xx}=0$ was firstly introduced to modelling a vibrating string by d'Alembert [1]. Many phenomena in physics, engineering, chemistry, other sciences can be described by partial differential equations. The investigation on analytical solutions of partial differential equations is an important subject since the exact analytical solutions may help physicists and engineers to examine the sensitivity of the model by adjusting some physical parameters, and give good enough support to numerical simulation. Many research has been done in this developing field [2–5]. In the past few years, significant progression has been made in the development of powerful methods, such as truncated expansion, inverse scattering transformation, Jacobi elliptic function expansion [5–8], etc.

In this article, we will present a new integral evolution formula of boundary value problem for wave equations with the form $u_{tt}(x,t)+\mathcal{L}(x,D)u(x,t)=f(x,t)$. The techniques we use in this paper are based on the ϕ-functions arising from the development of numerical integration of highly oscillatory systems $q_{tt}(t)+Mq(t)=f(q(t))$. In recent years, much attention has been paid on the numerical simulation of

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highly oscillatory problems, which is a main topic in geometric numerical integration. We refer the reader to [9–15] for recent surveys of this research.

The key point is that the form of wave equations we investigated is practically the same as the highly oscillatory system $q_{tt}(t) + Mq(t) = f(q(t))$. In this paper, under suitable assumption on the linear differential operator $\mathcal{L}$, a new integral evolution formula of boundary value problem for wave equations $u_{tt}(x,t) + \mathcal{L}(x,D)u(x,t) = f(x,t)$ will be established. It can be viewed as a generalization of the integral formula for finite-dimensional oscillatory system to the infinite-dimensional case. The rest of the paper is organized as follows. Some preliminaries and notations are introduced in Section 2. The new integral evolution formula of boundary value problem for wave equations is derived in Section 3. The last section is devoted to conclusions and remarks.

2. Preliminaries and notations

We consider the following linear inhomogeneous wave equation

$$\begin{cases} u_{tt}(x,t) + \mathcal{L}(x,D)u(x,t) = f(x,t), \qquad x \in \Omega, \quad 0 \le t \le T, \\ u(x,0) = \varphi(x), \qquad x \in \Omega, \\ u_t(x,0) = \psi(x), \qquad x \in \Omega \end{cases} \tag{1}$$

with suitable boundary conditions $\mathcal{B}\, u = 0$ on $\partial\Omega$. Here, Ω is an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, and $\mathcal{L}(x,D)$ denotes a linear differential operator, densely defined in $L^2(\Omega)$. The linear differential operator $\mathcal{L}(x,D)$ has the following form

$$\mathcal{L} \equiv \mathcal{L}(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha. \tag{2}$$

The multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is an n-tuple of nonnegative integers. The length $|\alpha|$ of α is $|\alpha| = \sum_{j=1}^n \alpha_j = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, the differential operator D^α is defined as

$$D^\alpha h(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} h(x).$$

Our point of view is a typical assumption in spectral theory, namely $\mathcal{L}$ is self-adjoint, positive definite and has a compact inverse. We therefore assume that the linear differential operator $\mathcal{L}(x,D)$ has the following properties:

(i) Each eigenvalue of $\mathcal{L}$ is real and of finite multiplicity. Furthermore, if we repeat each eigenvalue according to its multiplicity, we have

$$\sum = \{\lambda_k\}_{k=1}^\infty,$$

where

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots,$$

and

$$\lambda_k \to \infty \quad \text{as} \quad k \to \infty.$$

(ii) The eigenfunctions φ_k (corresponding to λ_k), $k = 1, 2, \ldots$ form an orthonormal basis of $L^2(\Omega)$, so that any $\psi \in L^2(\Omega)$ can be expressed by the (generalized) Fourier series

$$\psi = \sum_{k=1}^\infty \psi_k \varphi_k \qquad \text{in } L^2(\Omega). \tag{3}$$

(iii) The mapping
$$\begin{cases} L^2(\Omega) \to l^2 \\ \psi = \sum \psi_k \varphi_k \mapsto (\psi_k) \end{cases} \quad \text{is a homeomorphism} \tag{4}$$
(i.e., one-to-one and continuous in both directions). Where l^2 denotes, as usual, the Hilbert space of sequences $(\psi_k)_{k\geq 1}$ for which $\sum |\psi_k|^2 < \infty$.

Example 2.1. A typical linear differential operator satisfying assumptions (i)–(iii) is the self-adjoint uniformly elliptical operator of order $m = 2s$ associated with homogeneous Dirichlet boundary conditions. The linear differential operator has the form
$$\mathcal{L}(x, D)u = \sum_{|\alpha|,|\beta|\leq s} (-1)^{|\alpha|} D^{\alpha} \left(a_{\alpha\beta}(x) D^{\beta} u\right), \tag{5}$$
with $a_{\alpha\beta}(x) = \overline{a_{\beta\alpha}}(x)$ for $|\alpha| \leq s$ and $|\beta| \leq s$. And the principal symbol of $\mathcal{L}(x, D)$ satisfies that there exists $c > 0$
$$\sum_{|\alpha|=|\beta|=s} a_{\alpha\beta}(x)\xi^{\alpha}\xi^{\beta} \geq c|\xi|^{2s},$$
for all $\xi \in \mathbb{R}^n$ and $x \in \overline{\Omega}$. The homogeneous boundary condition is expressed as
$$B_j u(x)|_{x\in\partial\Omega} = \sum_{|l|=r_j} b^{(l_1,\ldots,l_n)} \left.\frac{\partial^{r_j} u(x)}{\partial x_1^{l_1} \ldots \partial x_n^{l_n}}\right|_{x\in\partial\Omega} = 0, j = 1, \ldots, s, 0 \leqslant r_j \leqslant 2s - 1. \tag{6}$$
The domain of the linear operator $\mathcal{L}(x, D)$ is
$$\mathrm{Dom}(\mathcal{L}) = H^{2s}(\Omega) \cap H_0^s(\Omega),$$
where $H^s(\Omega)$ denotes the Sobolev space of functions $u \in L^2(\Omega)$ such that the distributional derivatives $D^{\alpha}u$ are in $L^2(\Omega)$ for $|\alpha| \leq s$. And $H_0^s(\Omega)$ is the subspace of $H^s(\Omega)$ obtained by completing $C_0^{\infty}(\Omega)$ with respect to the norm of $H^s(\Omega)$; here, $C_0^{\infty}(\Omega)$ denotes the space of continuously differentiable functions with compact support contained in Ω.

One of the most important concrete examples of the above mentioned linear differential operator is the second-order elliptic operator
$$\mathcal{L}(x, D)u = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u,$$
or equivalently
$$\mathcal{L}(x, D)u = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i(x)u_{x_i} + c(x)u$$
with $\tilde{b}^i := b^i - \sum_{j=1}^{n} a^{ij}_{x_j} (i = 1, \ldots, n)$. The ellipticity means that for each point $x \in \overline{\Omega}$, the symmetric $n \times n$ matrix $\mathbf{A}(x) = \left(a^{ij}(x)\right)$ is positive definite, with smallest eigenvalue greater than or equal to a positive number c [16].

3. A new integral evolution formula for wave equations

In this section, we will derive the new integral evolution formula for the linear differential equation (1). To this end, we first define some analytical functions $\phi_j(\cdot),\ j = 0, 1, \ldots$ as
$$\phi_j(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(2k + j)!},\ j = 0, 1, \ldots. \tag{7}$$

Since we have $\lim_{x\to 0^+}\phi_j(x)=\frac{1}{j!},\ j=0,1,\ldots$, the functions $\phi_j(x),\ j=0,1,\ldots$ are well defined at $x=0$. It can be verified that $|\phi_j(x)|\le\frac{1}{j!}, j=0,1,\ldots$ for all $x\ge 0$. The first three ϕ-functions are

$$\phi_0(x)=\cos(\sqrt{x}),\qquad \phi_1(x)=\frac{\sin(\sqrt{x})}{\sqrt{x}},\qquad \phi_2(x)=\frac{\sin^2(\sqrt{x}/2)}{x/2}.$$

The ϕ-functions are firstly introduced by Wu et al. [17], where a standard form of the multi-frequency and multidimensional ERKN (extended Runge–Kutta–Nyström) integrators are formulated to solve oscillatory system

$$q_{tt}(t)+Mq(t)=f(q(t)),\quad q(0)=q_0,\quad q_t(0)=p_0, \tag{8}$$

where $q:\mathbb{R}\to\mathbb{R}^d$ and $M\in\mathbb{R}^{d\times d}$ is a symmetric and positive definite matrix governing the main oscillation of the system.

Much effort has been spent in developing numerical integration of oscillatory system (8). In 1961, Gautschi [18] proposed the Gautschi's method based on trigonometric polynomials. In [19], Gonzalez et al. designed a new family of explicit Runge–Kutta type methods with G-functions for the numerical integration of highly oscillatory system. The ϕ-functions are related to trigonometric polynomials and the G-functions. We refer the reader to [20–24] for more details on the properties of the ϕ-functions.

Since all the ϕ-functions are bounded in the interval $[0,+\infty)$ and the eigenvalues of the linear differential operator $\mathcal{L}(x,D)$ are positive, we may define the operators $\phi_j(t^2\mathcal{L}), j=0,1,\ldots$ by defining them on the spectrum, i.e.,

$$\phi_j(t^2\mathcal{L})\psi=\sum_{k=1}^{\infty}\phi_j(t^2\lambda_k)\psi_k\varphi_k,\quad \textit{for all}\ \psi\in\mathrm{Dom}(\phi_j(t^2\mathcal{L})),\quad j=0,1,\ldots \tag{9}$$

with their corresponding domains

$$\mathrm{Dom}(\phi_j(t^2\mathcal{L}))=\left\{\psi=\sum_{k=1}^{\infty}\psi_k\varphi_k\in L^2(\Omega);\ (\phi_j(t^2\lambda_k)\psi_k)\in l^2\right\},\quad j=0,1,\ldots. \tag{10}$$

Note the assumption (iii) implies that there exist two positive constants c_1 and c_2 such that every $\psi=\sum_{k=1}^{\infty}\psi_k\varphi_k$ in $L^2(\Omega)$ satisfies

$$c_1\left(\sum_{k=1}^{\infty}|\psi_k|^2\right)^{1/2}\le\|\psi\|_{L^2(\Omega)}\le c_2\left(\sum_{k=1}^{\infty}|\psi_k|^2\right)^{1/2}. \tag{11}$$

Therefore, we have

$$\|\phi_j(t^2\mathcal{L})\psi\|=\left\|\sum_{k=1}^{\infty}\phi_j(t^2\lambda_k)\psi_k\varphi_k\right\|\le\frac{c_2}{j!c_1}\|\psi\|$$

and $\mathrm{Dom}(\phi_j(t^2\mathcal{L}))=L^2(\Omega), j=0,1,\ldots$. Thus, we have proved the following lemma:

Lemma 3.1. *Under the assumptions (i)–(iii), the linear differential operators $\phi_j(t^2\mathcal{L}):L^2(\Omega)\to L^2(\Omega)$ are bounded operators with $\mathrm{Dom}\left(\phi_j(t^2\mathcal{L})\right)=L^2(\Omega), j=0,1,\ldots$ for all $t\in\mathbb{R}$.*

Now, we are in the position to propose our main result.

Theorem 3.1. *Assume that the initial data $\varphi,\psi\in L^2(\Omega)$ and the source term $f\in C\left([0,T],L^2(\Omega)\right)$. Then the solution to the problem (1) is given by*

$$u(x,t)=(\phi_0(t^2\mathcal{L})\varphi)(x)+t(\phi_1(t^2\mathcal{L})\psi)(x)+\int_0^t(t-\tau)(\phi_1((t-s)^2\mathcal{L})f)(x,\tau)d\tau. \tag{12}$$

Proof. First of all, we consider the linear homogeneous differential equation associated to the linear partial differential equation (1)

$$\begin{cases} u_{tt}(x,t) + \mathcal{L}(x,D)u(x,t) = 0, \qquad x \in \Omega, \quad 0 \le t \le T, \\ u(x,0) = \varphi(x), \qquad x \in \Omega, \\ u_t(x,0) = \psi(x), \qquad x \in \Omega, \end{cases} \tag{13}$$

with boundary conditions $\mathcal{B}\ u = 0$. Introducing the new variable $v(x,t) = u_t(x,t)$ and setting $w = (u,v)^T$ and $\mathcal{A}w = (v, -\mathcal{L}u)^T$. Eq. (13) is equivalent to the following first-order system:

$$w_t = \mathcal{A}w, \quad w(0) = w_0, \tag{14}$$

where $w_0(x) = (\varphi(x), \psi(x))^T$. The space $X = \mathrm{Dom}(\mathcal{L}) \times L^2(\Omega)$ with the inner product

$$\langle w_1, w_2 \rangle_X := \langle \mathcal{L}u_1, u_2 \rangle_{L^2(\Omega)} + \langle v_1, v_2 \rangle_{L^2(\Omega)}$$

becomes a Hilbert space, and the operator $\mathcal{A} : X \to X$ is linear, continuous, and skew-symmetric, i.e.,

$$\langle \mathcal{A}w_1, w_2 \rangle_X = -\langle w_1, \mathcal{A}w_2 \rangle_X \qquad \text{for all } w_1, w_2 \in X.$$

Therefore, for each given $w_0 \in X$, System (14) has the unique solution

$$w(t) = e^{t\mathcal{A}}w_0 \qquad \text{for all } t \in \mathbb{R},$$

where $\{e^{t\mathcal{A}}\}$ represents a one-parameter group on X [25]. More precisely, we have

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{t\mathcal{A}} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \exp\left(t \begin{pmatrix} 0 & I \\ -\mathcal{L} & 0 \end{pmatrix} \right) \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} \phi_0(t^2\mathcal{L}) & t\phi_1(t^2\mathcal{L}) \\ -t\phi_1(t^2\mathcal{L}) & \phi_0(t^2\mathcal{L}) \end{pmatrix} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix}. \tag{15}$$

The derivation of the third equality in (15) can be found in [26]. From (15), we have that the solution to the homogeneous equation (13) is

$$u(x,t) = (\phi_0(t^2\mathcal{L})\varphi)(x) + t(\phi_1(t^2\mathcal{L})\psi)(x). \tag{16}$$

According to Duhamel's Principle, the solution of the inhomogeneous equation (1) is

$$u(t) = \begin{pmatrix} \text{solution at time } t \text{ to} \\ u_{tt}(x,t) + \mathcal{L}u(x,t) = 0 \text{ with} \\ u(x,0) = \varphi(x) \& u_t(x,t) = \psi(x) \end{pmatrix} + \int_0^t \begin{pmatrix} \text{solution at time } t \text{ to} \\ u_{tt}(x,t) + \mathcal{L}u(x,t) = 0 \text{ with} \\ u(x,\tau) = 0 \& u_t(x,\tau) = f(x,\tau) \end{pmatrix} d\tau. \tag{17}$$

Substituting the homogeneous solution (16) into (17) yields the solution to the linear inhomogeneous partial differential equation (1):

$$u(x,t) = (\phi_0(t^2\mathcal{L})\varphi)(x) + t(\phi_1(t^2\mathcal{L})\psi)(x) + \int_0^t (t-\tau)(\phi_1((t-s)^2\mathcal{L})f)(x,\tau)d\tau. \quad \square$$

Remark 3.1. The exact solution for the oscillatory system (8) has the form (see [26]) :

$$q(t) = \phi_0(t^2M)q_0 + t\phi_1(t^2M)p_0 + \int_0^t (t-\tau)\phi_1((t-\tau)^2M)f\big(q(\tau)\big)d\tau. \tag{18}$$

If we consider $\mathcal{M}q = Mq$ as a linear operator from $\mathbb{R}^d$ to $\mathbb{R}^d$, all the assumptions (i)–(iii) are satisfied by the symmetry and positive definiteness of the matrix M. The formula (18) can be easily recovered by the approach in this section.

Remark 3.2. If the right-hand-side function of (1) $f(x,t) \equiv f(x)$ is independent of time variable t, then the solution (12) can be reduced to

$$u(x,t) = (\phi_0(t^2\mathcal{L})\varphi)(x) + t(\phi_1(t^2\mathcal{L})\psi)(x) + t^2(\phi_2(t^2\mathcal{L})f)(x)$$

by the equation

$$\int_0^t (t-\tau)\phi_1((t-\tau)^2\mathcal{L})d\tau = \phi_2(t^2\mathcal{L}).$$

4. Conclusions and remarks

Under the assumption that the spatial differential operator $\mathcal{L}$ is self-adjoint, positive definite and has a compact inverse, we have derived a new exact representation for solutions of boundary value problem for wave equations $u_{tt}(x,t) + \mathcal{L}(x,D)u(x,t) = f(x,t)$. The formula is of analytical interest in its own right. It avoids the cumbersome work needed by traditional classical techniques, such as, e.g., the method of separating variables, d'Alembert formulae.

Furthermore, the method can be applied to numerically solving nonlinear wave equations [27]. To be more precise, the nonlinear wave equation $u_{tt}(x,t) + \mathcal{L}(x,D)u(x,t) = f(u(x,t))$ has a formal integral formula

$$u(x,t) = (\phi_0(t^2\mathcal{L})\varphi)(x) + t(\phi_1(t^2\mathcal{L})\psi)(x) + \int_0^t (t-\tau)(\phi_1((t-s)^2\mathcal{L})f(u))(x,\tau)d\tau. \tag{19}$$

With a finite dimensional approximation of $\mathcal{L}$ on some grid and suitable quadrature formulae for approximating the integral in (19), efficient schemes for numerically solving the nonlinear wave equation yield. The numerical schemes can deal with the high oscillation brought by the linear differential operator $\mathcal{L}$ effectively (see e.g., [9,12] for more details on this topic).

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