CS 70 Discrete Mathematics and Probability Theory
Summer 2025 Tate

DIS 4A

Discrete Probability Intro

Note 13

Probability Space: A probability space is a tuple $(\Omega, \mathbb{P})$, where Ω is the *sample space* and $\mathbb{P}$ is the *probability function* on the sample space.

Specifically, Ω is the set of all outcomes ω, and $\mathbb{P}$ is a function $\mathbb{P}\colon \Omega \to [0,1]$, assigning a probability to each outcome, satisfying the following conditions:

$$0 \leq \mathbb{P}[\omega] \leq 1 \quad \text{and} \quad \sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1.$$

Event: an event A is a subset of Ω, i.e. a collection of some outcomes in the sample space. We define

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega].$$

Uniform Probability Space: all outcomes are assigned the same probability, i.e. $\mathbb{P}[\omega] = \frac{1}{|\Omega|}$; this is just counting!

With an event A in a uniform probability space, $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$, which is again more counting!

1 Symmetry

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In this problem, we will walk you through the idea of *symmetry* and its formal justification. Consider an experiment where you have a bag with m red marbles and $n-m$ blue marbles. You draw marbles from the bag, one at a time without replacement until the bag is empty.

(a) Define the sample space Ω. (No need to write out every element, a brief description is fine). Is this a uniform probability space?

(b) What is the probability that the first marble you draw is red?

(c) Suppose you've drawn all but the final marble, setting each marble aside as you draw it *without looking at it*. We want to find the probability that the final marble left in the bag will be red.

Let A be the event containing outcomes where the first marble is red, and let B be the event containing outcomes where the final marble is red. Provide a bijective function $f : A \to B$ mapping outcomes in A to outcomes in B, and explain why it is a bijection. Note that there can be multiple valid bijections.

(d) Use the previous parts to find the probability that the final marble will be red.

(e) You repeat the experiment. Find the probability that the last two marbles you draw will be red.

(f) You repeat the experiment again, but this time you see that the first marble you draw is red. Find the probability that the second-to-last marble you draw will also be red.

Solution:

(a) The sample space is the set of all length n sequences with m reds and $n-m$ blues. This is a uniform probability space; there are a total of $\binom{n}{m}$ outcomes in the sample space, and each outcome has probability $\frac{1}{\binom{n}{m}}$.

(b) Of the n marbles, m are red, giving a probability of $\frac{m}{n}$.

(c) The inputs to f will be sequences of length n with m red draws and $n-m$ blue draws, and the output will be the same sequence except with the first and last draws swapped. This uniquely transforms each sequence of draws with a red marble first into a sequence with a red marble last, and vice versa (f is its own inverse).

(d) Since we have a bijection between the events A and B, they have the same number of outcomes. Additionally, we have a uniform probability space, so $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$ and $\mathbb{P}[B] = \frac{|B|}{|\Omega|}$. But our bijection showed that $|A| = |B|$, so $\mathbb{P}[A] = \mathbb{P}[B]$, which means the probability of drawing a red marble last is the same as the probability of drawing a red marble first, which is $\frac{m}{n}$.

Note: We don't require a uniform probability space in order to apply the idea of symmetry. The mapping f only needs to map outcomes in A to outcomes in B with the same probability. Mathematically, we require that for every $\omega \in A$, we have $\mathbb{P}[\omega] = \mathbb{P}[f(\omega)]$. Then we'd have

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega] = \sum_{\omega \in A} \mathbb{P}[f(\omega)] = \sum_{\omega \in B} \mathbb{P}[\omega] = \mathbb{P}[B]$$

as desired.

(e) By the same logic as before, the probability that the last two marbles are red is the same as the probability that the first two marbles are red, which is $\frac{m}{n} \times \frac{m-1}{n-1} = \frac{m(m-1)}{n(n-1)}$. The explicit bijection swaps the first two marbles with the last two marbles.

(f) After seeing the first marble is red, there are $m-1$ red marbles left and $n-1$ total marbles. By symmetry, the probability that the second-to-last marble will be red is the same as the probability that the second marble will be red, which is $\frac{m-1}{n-1}$.

2 Flippin' Coins

Note 13

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

(g) What is the probability of the event that our outcome has at least one head?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) An event must be a subset of Ω, meaning that it must consist of possible outcomes.

- No
- Yes
- Yes
- Yes
- Yes

(c) $\{(T,H,H),(T,H,T),(T,T,H)\}$

(d) $\{(T,H,H),(H,H,T),(H,T,H),(T,T,T)\}$

(e) Since $|\Omega| = 2^3 = 8$ and every outcome has equal probability, $\mathbb{P}[(H,H,T)] = 1/8$.

(f) The event of interest is $E = \{(H,H,T),(H,T,H),(T,H,H)\}$, which has size 3. Whence $\mathbb{P}[E] = 3/8$.

(g) If we do not see at least one head, then we must see at exactly three tails. The event $\overline{E} = \{(T,T,T)\}$ of seeing exactly three tails is thus the complement of the event E that we see at least one head. $\overline{E}$ occurs with probability $(1/2)^3 = 1/8$, so its complement E must occur with probability $1 - 1/8 = 7/8$.

3 Sampling

Note 13 Suppose you have balls numbered $1, \dots, n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

Solution:

(a) Out of n^2 pairs of balls that you could have chosen, only one pair $(1,2)$ corresponds to the event we are interested in, so the probability is $1/n^2$.

(b) Again, there are n^2 total outcomes. Now, we want to count the number of outcomes where the second ball's number is strictly less than the first ball's number. Similarly to the last part, we can view any outcome as an ordered pair (n_1, n_2), where n_1 is the number on the first ball, and n_2 is the number on the second ball. There are $\binom{n}{2}$ outcomes where $n_1 > n_2$; select two distinct numbers from $[1,n]$, and assign the higher number to n_1. Thus, the probability is $\frac{\binom{n}{2}}{n^2} = \frac{n-1}{2n}$.

Alternate Solution: The probability that the two balls have the same number is $n/n^2 = 1/n$, so the probability that the balls have different numbers is $1 - 1/n = (n-1)/n$. By symmetry, it is equally likely for the first ball to have a greater number and for the second ball to have a greater number, so we take the probability $(n-1)/n$ and divide it by two to obtain $(n-1)/(2n)$.

(c) Again, there are n^2 pairs of balls that we could have drawn, but there are $n-1$ pairs of balls which correspond to the event we are interested in: $\{(1,2), (2,3), \dots, (n-1,n)\}$. So, the probability is $(n-1)/n^2$.

(d) There are a total of $n(n-1)$ pairs of balls that we could have drawn, and only the pair $(1,2)$ corresponds to the event that we are interested in, so the probability is $1/(n(n-1))$.

The probability that the two balls are the same is now 0, but the symmetry described earlier still applies, so the probability that the second ball has a smaller number is $1/2$.

There are a total of $n(n-1)$ pairs of balls that we could have drawn, and we are interested in the $n-1$ pairs $(1,2), (2,3), \dots, (n-1,n)$ as before. Thus, the probability that the second ball is one greater than the first ball is $1/n$.

4 Intransitive Dice

Note 13

You're playing a game with your friend Bob, who has a set of three dice. You'll each choose a different die, roll it, and whoever had the higher result wins. The dice have sides as follows:

- Die A has sides 2, 2, 4, 4, 9, and 9.
- Die B has sides 1, 1, 6, 6, 8, and 8.
- Die C has sides 3, 3, 5, 5, 7, and 7.

(a) Suppose you have chosen die A and Bob has chosen die B. What is the probability that you win?
Hint: It may be easier to work with a sample space smaller than 6×6.

(b) Suppose you have chosen die B and Bob has chosen die C. What is the probability that you win?

(c) Suppose you have chosen die C and Bob has chosen die A. What is the probability that you win?

(d) Bob offers to let you choose your die first so that you can choose the best one. Is this an offer your should accept? Why or why not?

Solution: Each die has 6 sides, giving us $6 \times 6 = 36$ sample points. However, since each die has three sides repeated twice, we can treat the dice as 3 sided dice with each of their unique sides once. With this simplication, each die has 3 possible outcomes, so for each pair of dice, there are $3 \times 3 = 9$ possible outcomes. Each outcome occurs $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ of the time.

(a) The event that Die A beats die B consists of the following sample points: (2,1), (4,1), (9,1), (9,6), (9,8). This gives us a total of 5 outcomes. The probability that you win is the sum of the probabilities of the outcomes, which is $5 \times \frac{1}{9} = \frac{5}{9}$.

(b) The event that Die B beats die C consists of the following sample points: (6,3), (6,5), (8,3), (8,5), (8,7). This gives us a total of 5 outcomes. The probability that you win is the sum of the probabilities of the outcomes, which is $5 \times \frac{1}{9} = \frac{5}{9}$.

(c) The event that Die C beats die A consists of the following sample points: (3,2), (5,2), (7,2), (5,4), (7,4). This gives us a total of 5 outcomes. The probability that you win is the sum of the probabilities of the outcomes, which is $5 \times \frac{1}{9} = \frac{5}{9}$.

(d) You should not accept! Bob is trying to trick you into choosing first. Notice that for each die, there is a different die that has an advantage against it. This means that if you choose first, Bob can always choose the die that beats your die. Therefore, you should not accept Bob's offer. This is due to the *intransitive* nature of the dice, just like rock-paper-scissors.