CS 70 Discrete Mathematics and Probability Theory
Summer 2025 Tate

HW 04

1 Is This EECS 126?

Note 13
Note 14

Youngmin loves birdwatching. There are three sites he birdwatches at: Site A, B, and C. He goes to Site A 60% of the time, Site B 35% of the time, and Site C 5% of the time. The probability of seeing a nightingale at each site is $\frac{1}{10}$, $\frac{3}{10}$, and $\frac{2}{5}$, respectively. Using the information above, answer the following questions.

(a) What is the total probability that Youngmin sees a nightingale?

(b) Given that Youngmin sees a nightingale, what is the probability that Youngmin went to site B?

(c) Given that Youngmin didn't go to site B, what is the probability that Youngmin doesn't see a nightingale?

Solution:

(a) Let E be the event Youngmin sees a nightingale, A be the event that he goes to site A, B be the event that he goes to site B, and C be the event that he goes to site C. Then, we can calculate $\mathbb{P}[E]$ by total probability:

$$\begin{aligned}\mathbb{P}[E] &= \mathbb{P}[E|A]\,\mathbb{P}[A] + \mathbb{P}[E|B]\,\mathbb{P}[B] + \mathbb{P}[E|C]\,\mathbb{P}[C] \\ &= \frac{1}{10}\cdot\frac{3}{5} + \frac{3}{10}\cdot\frac{7}{20} + \frac{2}{5}\cdot\frac{1}{20} \\ &= \frac{3}{50} + \frac{21}{200} + \frac{1}{50} \\ &= \frac{37}{200}\end{aligned}$$

(b) Using Bayes' Rule,

$$\mathbb{P}[B|E] = \frac{\mathbb{P}[E|B]\,\mathbb{P}[B]}{\mathbb{P}[E]} = \frac{\frac{3}{10}\cdot\frac{7}{20}}{\frac{37}{200}} = \frac{21}{37}$$

(c) Noting that $\overline{B} = A \cup C$, we can write $\mathbb{P}[\overline{E}|\overline{B}]$ as

$$\mathbb{P}[\overline{E}|\overline{B}] = \mathbb{P}[\overline{E}|A\cup C] = 1 - \mathbb{P}[E|A\cup C]$$

Calculating $\mathbb{P}[E|A\cup C]$ can be done as the following:

$$\begin{aligned}\mathbb{P}[E|A\cup C] &= \frac{\mathbb{P}[E\cap A]}{\mathbb{P}[A\cup C]}+\frac{\mathbb{P}[E\cap C]}{\mathbb{P}[A\cup C]}\\ &= \frac{\mathbb{P}[E|A]\,\mathbb{P}[A]}{\mathbb{P}[A\cup C]}+\frac{\mathbb{P}[E|C]\,\mathbb{P}[C]}{\mathbb{P}[A\cup C]}\\ &= \frac{\frac{1}{10}\cdot\frac{3}{5}}{\frac{3}{5}+\frac{1}{20}}+\frac{\frac{2}{5}\cdot\frac{1}{20}}{\frac{3}{5}+\frac{1}{20}}\\ &= \frac{6}{65}+\frac{2}{65}=\frac{8}{65}\end{aligned}$$

2 Presidential Election

Note 13

We traveled back in time to 1960 and want to determine which presidential candidate will win this coming election. There are two candidates, John F. Kennedy and Richard Nixon.

1. For each state, Kennedy has probability p of winning. What is the probability that Kennedy will win exactly half the states (there are 50 states)?
2. What is the probability that Kennedy will win at least one state?
3. Say that $p = 0.25$ or $p = 0.75$ with equal chance. If Kennedy wins every single state, what is the probability that $p = 0.75$?

Solution:

1. There are $\binom{50}{25}$ ways to choose 25 out of 50 states for Kennedy to win. In this situation, there is probability p that Kennedy will win each state, so the total probability he will win all 25 states is p^{25}. There is $(1-p)$ chance that Nixon will win the other 25 states, so the total probability he will win the other 25 states is $(1-p)^{25}$. Combining these probabilities with the ways to choose 25 states yields $\binom{50}{25}p^{25}(1-p)^{25}$.
2. We will compute the complement probability and then subtract from 1, since the complement is much easier to compute. The complement of Kennedy winning at least one state is that he wins no states. This is equivalent to Nixon winning all states, or $(1-p)^{50}$. Subtracting this from one yields $1-(1-p)^{50}$.

3. Let K be the event that Kennedy wins every single state. Using Bayes Rule,

$$\begin{aligned}
\mathbb{P}(p=0.75 \mid K) &= \frac{\mathbb{P}(K \mid p=0.75)\,\mathbb{P}(0.75)}{\mathbb{P}(K)} \\
&= \frac{\mathbb{P}(K \mid p=0.75)\,\mathbb{P}(0.75)}{\mathbb{P}(K \mid p=0.75)\,\mathbb{P}(0.75)+\mathbb{P}(K \mid p=0.25)\,\mathbb{P}(0.25)} \\
&= \frac{(0.75^{50})(0.5)}{(0.75^{50})(0.5)+(0.25^{50})(0.5)} \\
&\approx 1.
\end{aligned}$$

3 Solve the Rainbow

Note 14

Your roommate was having Skittles for lunch and they offer you some. There are five different colors in a bag of Skittles: red, orange, yellow, green, and purple, and there are 20 of each color. You know your roommate is a huge fan of the green Skittles. With probability $1/2$ they ate all of the green ones, with probability $1/4$ they ate half of them, and with probability $1/4$ they only ate 5 green ones.

(a) If you take a Skittle from the bag, what is the probability that it is green?

(b) If you take two Skittles from the bag, what is the probability that at least one is green?

(c) If you take three Skittles from the bag, what is the probability that they are all green?

(d) If all three Skittles you took from the bag are green, what are the probabilities that your roommate had all of the green ones, half of the green ones, or only 5 green ones?

(e) If you take three Skittles from the bag, what is the probability that they are all the same color?

Solution:

(a) We will use the law of total probability. Let G be the event that you take a green Skittles from the bag, A be the event that your roommate ate all of the green Skittles, H be the event that your roommate ate half the green Skittles, and F be the event that your roommate ate five green Skittles. Then, we get the total probability as following:

$$\begin{aligned}
\mathbb{P}(G) &= \mathbb{P}(G \cap A)+\mathbb{P}(G \cap H)+\mathbb{P}(G \cap F) && (1)\\
&= \mathbb{P}(G \mid A)\,\mathbb{P}(A)+\mathbb{P}(G \mid H)\,\mathbb{P}(H)+\mathbb{P}(G \mid F)\,\mathbb{P}(F) && (2)\\
&= 0 \cdot \frac{1}{2}+\frac{10}{90}\cdot\frac{1}{4}+\frac{15}{95}\cdot\frac{1}{4} \approx 0.0673 && (3)
\end{aligned}$$

(b) We will consider the complement event, that neither of them are green. Let's call the event that at least one of them is green B, this makes the complement $\overline{B}$ the event that neither

Skittles are green. Using the same approach as the previous part, we will get the following:

$$\begin{aligned}
\mathbb{P}(\overline{B}) &= \mathbb{P}(\overline{B} \cap A) + \mathbb{P}(\overline{B} \cap H) + \mathbb{P}(\overline{B} \cap F) && (4)\\
&= \mathbb{P}(\overline{B} \mid A)\,\mathbb{P}(A) + \mathbb{P}(\overline{B} \mid H)\,\mathbb{P}(H) + \mathbb{P}(\overline{B} \mid F)\,\mathbb{P}(F) && (5)\\
&= 1 \cdot \frac{1}{2} + \frac{80}{90} \cdot \frac{79}{89} \cdot \frac{1}{4} + \frac{80}{95} \cdot \frac{79}{94} \cdot \frac{1}{4} \approx 0.874 && (6)
\end{aligned}$$

This makes our final answer the following:

$$\mathbb{P}(B) = 1 - \mathbb{P}(\overline{B}) \approx 0.126$$

(c) Let's call the event of having 3 green Skittles G_3. This event is impossible if our roommate ate all the green Skittles.

If they ate half, we have the probability of G_3 as

$$\frac{10 \times 9 \times 8}{90 \times 89 \times 88}.$$

We can see this by noticing that given our roommate ate half the green Skittles, there will be 10 green Skittles left out of the 90 that are still in the bag. After the first one is removed, there will be 9 out of 89 that are green, and so on.

Similarly, if they ate only five green Skittles, we have the probability of G_3 as

$$\frac{15 \times 14 \times 13}{95 \times 94 \times 93},$$

giving us the final result as:

$$\begin{aligned}
\mathbb{P}(G_3) &= \mathbb{P}(G_3 \mid A)\,\mathbb{P}(A) + \mathbb{P}(G_3 \mid H)\,\mathbb{P}(H) + \mathbb{P}(G_3 \mid F)\,\mathbb{P}(F) && (7)\\
&= 0 \cdot \frac{1}{2} + \frac{10 \times 9 \times 8}{90 \times 89 \times 88} \cdot \frac{1}{4} + \frac{15 \times 14 \times 13}{95 \times 94 \times 93} \cdot \frac{1}{4} && (8)\\
&\approx 0.00108 && (9)
\end{aligned}$$

(d) We can use the Bayes Rule to solve this.

$$\mathbb{P}(A \mid G_3) = \frac{\mathbb{P}(G_3 \cap A)}{\mathbb{P}(G_3)} = \frac{\mathbb{P}(G_3 \mid A)\,\mathbb{P}(A)}{\mathbb{P}(G_3)} = \frac{0 \times 1/2}{0.00108} = 0$$

This makes intuitive sense, since if you took three green Skittles out of the bag, it is impossible that your roommate ate all of them. Using it for the two other conditions, we get:

$$\mathbb{P}(H \mid G_3) = \frac{\mathbb{P}(G_3 \cap H)}{\mathbb{P}(G_3)} = \frac{\mathbb{P}(G_3 \mid H)\,\mathbb{P}(H)}{\mathbb{P}(G_3)} = \frac{10 \times 9 \times 8}{90 \times 89 \times 88} \cdot \frac{1}{4} \cdot \frac{1}{0.00108} \approx 0.237$$

$$\mathbb{P}(F \mid G_3) = \frac{\mathbb{P}(G_3 \cap F)}{\mathbb{P}(G_3)} = \frac{\mathbb{P}(G_3 \mid F)\,\mathbb{P}(F)}{\mathbb{P}(G_3)} = \frac{15 \times 14 \times 13}{95 \times 94 \times 93} \cdot \frac{1}{4} \cdot \frac{1}{0.00108} \approx 0.763$$

Note that the sum of these probabilities add up to 1.

(e) We can divide this into two cases. If the color of all the Skittles is green, we have already calculated the probability in the previous part.

For all other colors, we can notice that the probabilities will have the same structure, and since these are disjoint events, we can add them to get our final result. Let's find the probability for the case of getting three red Skittles, let's call this event R_3. We find this probability as follows:

$$\begin{aligned}
\mathbb{P}(R_3) &= \mathbb{P}(R_3 \mid A)\,\mathbb{P}(A) + \mathbb{P}(R_3 \mid H)\,\mathbb{P}(H) + \mathbb{P}(R_3 \mid F)\,\mathbb{P}(F) && (10)\\
&= \frac{20\times 19\times 18}{80\times 79\times 78}\cdot\frac{1}{2} + \frac{20\times 19\times 18}{90\times 89\times 88}\cdot\frac{1}{4} + \frac{20\times 19\times 18}{95\times 94\times 93}\cdot\frac{1}{4} && (11)\\
&\approx 0.0114 && (12)
\end{aligned}$$

If we call the probability of getting three Skittles of the same color X_3, we can find it by adding the probability for the events for different colors such as G_3, and R_3. The probability for getting 3 of the same color for yellow, orange, and purple will be the same as it was for red. Using the same name convention for red and green for the other colors, this can be summed up as:

$$\begin{aligned}
\mathbb{P}(X_3) &= \mathbb{P}(G_3) + \mathbb{P}(R_3) + \mathbb{P}(Y_3) + \mathbb{P}(O_3) + \mathbb{P}(P_3) && (13)\\
&= \mathbb{P}(G_3) + 4\,\mathbb{P}(R_3) && (14)
\end{aligned}$$

The above holds since these are all disjoint events, we can't get all three Skittles to be the same color for different colors at the same time. Overall, getting this means we are adding these probabilities, giving us:

$$\mathbb{P}(X_3) = \mathbb{P}(G_3) + 4\cdot\mathbb{P}(R_3) \approx 0.0468$$

4 Cliques in Random Graphs

Note 13
Note 14

Consider the graph $G = (V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in a graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example, a 3-clique is a triangle. Let E_S be the event that a set S forms a clique. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1,\ldots,v_\ell\}$ and $V_2 = \{w_1,\ldots,w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \leq n^k$. (You might find this useful in part (e)).

(e) Prove that the probability that the graph contains a k-clique, for $k \geq 4\log_2 n + 1$, is at most $1/n$. *Hint:* Use the union bound.

Solution:

(a) Between every pair of vertices, there is either an edge or there isn't. Since there are two choices for each of the $\binom{n}{2}$ pairs of vertices, the size of the sample space is $2^{\binom{n}{2}}$.

(b) For a fixed set of k vertices to be a k-clique, all of the $\binom{k}{2}$ pairs of those vertices have to be connected by an edge. The probability of this event is $1/2^{\binom{k}{2}}$.

(c) E_{V_1} and E_{V_2} are independent if and only if V_1 and V_2 share at most one vertex: If V_1 and V_2 share at most one vertex, then since edges are added independently of each other, we have

$$\begin{aligned}
\mathbb{P}[E_{V_1} \cap E_{V_2}] &= \mathbb{P}[\text{all edges in } V_1 \text{ and all edges in } V_2 \text{ are present}] \\
&= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2}} \\
&= \mathbb{P}[E_{V_1}] \cdot \mathbb{P}[E_{V_2}].
\end{aligned}$$

Conversely, if V_1 and V_2 share at least two vertices, then their intersection $V_3 = V_1 \cap V_2$ has at least 2 elements, so we have

$$\begin{aligned}
\mathbb{P}[E_{V_1} \cap E_{V_2}] &= \left(\frac{1}{2}\right)^{\binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} - \binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2} - \binom{|V_3|}{2}} \\
&= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} + \binom{|V_2|}{2} - \binom{|V_3|}{2}} \neq \mathbb{P}[E_{V_1}] \cdot \mathbb{P}[E_{V_2}].
\end{aligned}$$

(d) The algebraic solution is an application of the definition of $\binom{n}{k}$:

$$\begin{aligned}
\binom{n}{k} = \frac{n!}{(n-k)!k!} &= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \\
&\leq n \cdot (n-1) \cdots (n-k+1) \\
&\leq n^k
\end{aligned}$$

(e) Let A_S denote the event that S is a k-clique, where $S \subseteq V$ is of size k. Then, the event that the graph contains a k-clique can be described as the union of A_S's over all $S \subseteq V$ of size k. Using the union bound,

$$\mathbb{P}\left[\bigcup_{S \subseteq V, |S|=k} A_S\right] \leq \sum_{S \subseteq V, |S|=k} \mathbb{P}[A_S] = \sum_{S \subseteq V, |S|=k} \frac{1}{2^{\binom{k}{2}}}.$$

Now, since there are $\binom{n}{k}$ ways of choosing a subset $S \subseteq V$ of size k, the right-hand side of the above equality is

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{\binom{n}{k}}{2^{k(k-1)/2}} \leq \frac{n^k}{\left(2^{(k-1)/2}\right)^k} \leq \frac{n^k}{\left(2^{(4\log n + 1 - 1)/2}\right)^k} = \frac{n^k}{\left(2^{2\log n}\right)^k} = \frac{n^k}{n^{2k}} = \frac{1}{n^k} \leq \frac{1}{n}.$$

5 Pairwise Independence

Note 14 The events A_1, A_2, A_3 are *pairwise independent* if, for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}(A_1, A_2, A_3) = \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3)$.

Try to construct an example where three events are pairwise independent but not mutually independent.

Here is one potential starting point: Let A_1, A_2 be the respective results of flipping two fair coins. Can you come up with an event A_3 that works?

Solution:

A_1: the first result is Head; A_2: the second result is Head; A_3: both results are the same.

6 Independent Complements

Note 14 Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ must be independent.

(b) Prove or disprove: A and $\overline{B}$ must be independent.

(c) Prove or disprove: A and $\overline{A}$ must be independent.

(d) Prove or disprove: It is possible that $A = B$.

Solution:

(a) True. $\overline{A}$ and $\overline{B}$ must be independent:

$$\begin{aligned}
\mathbb{P}[\overline{A} \cap \overline{B}] &= \mathbb{P}[\overline{A \cup B}] && \text{(by De Morgan's law)} \\
&= 1 - \mathbb{P}[A \cup B] && \text{(since } \mathbb{P}[\overline{E}] = 1 - \mathbb{P}[E] \text{ for all } E\text{)} \\
&= 1 - (\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]) && \text{(union of overlapping events)} \\
&= 1 - \mathbb{P}[A] - \mathbb{P}[B] + \mathbb{P}[A]\,\mathbb{P}[B] && \text{(since } A \text{ and } B \text{ are independent)} \\
&= (1 - \mathbb{P}[A])(1 - \mathbb{P}[B]) \\
&= \mathbb{P}[\overline{A}]\,\mathbb{P}[\overline{B}] && \text{(since } \mathbb{P}[\overline{E}] = 1 - \mathbb{P}[E] \text{ for all } E\text{)}
\end{aligned}$$

(b) True. A and $\overline{B}$ must be independent:

$$\begin{aligned}
\mathbb{P}[A \cap \overline{B}] &= \mathbb{P}[A - (A \cap B)] \\
&= \mathbb{P}[A] - \mathbb{P}[A \cap B] \\
&= \mathbb{P}[A] - \mathbb{P}[A]\,\mathbb{P}[B] \\
&= \mathbb{P}[A](1 - \mathbb{P}[B]) \\
&= \mathbb{P}[A]\,\mathbb{P}[\overline{B}]
\end{aligned}$$

(c) False in general. If $0 < \mathbb{P}[A] < 1$, then $\mathbb{P}[A \cap \overline{A}] = \mathbb{P}[\varnothing] = 0$ but $\mathbb{P}[A]\,\mathbb{P}[\overline{A}] > 0$, so $\mathbb{P}[A \cap \overline{A}] \neq \mathbb{P}[A]\,\mathbb{P}[\overline{A}]$; therefore A and $\overline{A}$ are not independent in this case.

(d) True. To give one example, if $\mathbb{P}[A] = \mathbb{P}[B] = 0$, then $\mathbb{P}[A \cap B] = 0 = 0 \times 0 = \mathbb{P}[A]\,\mathbb{P}[B]$, so A and B are independent in this case. (Another example: If $A = B$ and $\mathbb{P}[A] = 1$, then A and B are independent.)

7 Pairs of Beads

Note 14

Sinho has a set of $2n$ beads ($n \geq 2$) of n different colors, such that there are two beads of each color. He wants to give out pairs of beads as gifts to all the other $n-1$ TAs, and plans on keeping the final pair for himself (since he is, after all, also a TA). To do so, he first chooses two beads at random to give to the first TA he sees. Then he chooses two beads at random from those remaining to give to the second TA he sees. He continues giving each TA he sees two beads chosen at random from his remaining beads until he has seen all $n-1$ TAs, leaving him with just the two beads he plans to keep for himself. Prove that the probability that at least one of the other TAs (*not* including Sinho himself) gets two beads of the same color is at most $\frac{1}{2}$.

Solution:

Denote A_i as the event that the ith TA gets a matching pair. We see that $\mathbb{P}[A_1] = \frac{1}{2n-1}$ since the first bead doesn't matter; what matters is that the second bead (of a possible $2n-1$) matches the first (only 1 corresponding matching bead). By symmetry, we see that $\mathbb{P}[A_i]$ is equivalent across all i. Hence, by union-bound

$$\mathbb{P}[A_1 \cup A_2 \cup \cdots \cup A_{n-1}] \leq \sum_{i=1}^{n-1} \mathbb{P}[A_i] = \frac{n-1}{2n-1} \leq \frac{n-1}{2n-2} = \frac{1}{2}$$

as desired.