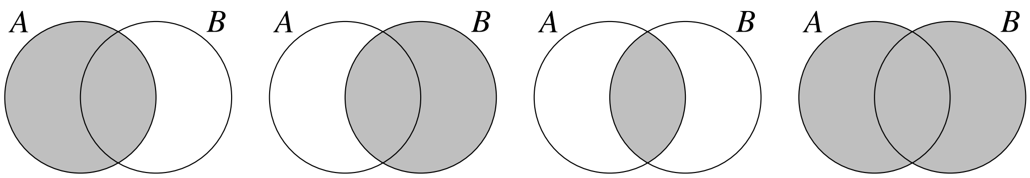
Counting Intro II

Note 10

Inclusion-exclusion: With two sets,



$$|A| + |B| - |A \cap B| = |A \cup B|$$

With more sets,

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = &|A_1| + |A_2| + \cdots + |A_n| \\ &- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - \left|A_i \cap A_j\right| - \cdots - |A_{n-1} \cap A_n| \\ &+ |A_1 \cap A_2 \cap A_3| + \cdots + \left|A_i \cap A_j \cap A_k\right| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\ &\cdots\end{aligned}$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} \left|\bigcap_{i \in S} A_i\right|$$

That is, for each size k, iterate through all ways of picking k sets from $\{A_1, \ldots, A_n\}$, and alternate between adding and subtracting the sizes of their intersection.

Combinatorial proofs: A technique for proving combinatorial identities. There should be very little math involved (usually none): use two different ways of counting the same scenario. One way should correspond to the left-hand side of the equality, and the other way should correspond to the right-hand side of the equality. The fact that we're counting the same scenario means that the two sides are equal.

1 Inclusion and Exclusion

Note 10

What is the total number of positive integers strictly less than 60 that are also coprime to 60?

Solution: It is sufficient to count the opposite: what is the total number of positive integers strictly less than 60 and *not* coprime to 60?

If a number is not coprime to 60, this means that the number is a multiple of 2, multiple of 3, or multiple of 5. In this case, we have:

- 29 multiples of 2
- 19 multiples of 3
- 11 multiples of 5
- 9 multiples of both 2 and 3
- 5 multiples of both 2 and 5
- 3 multiples of both 3 and 5
- 1 multiple of all three

By inclusion-exclusion, the total number of positive integers not coprime to 60 is $29+19+11-9-5-3+1=43$, and there are 59 positive integers strictly less than 60.

As such, in total there are $59-43=16$ different positive integers strictly less than 60 that are coprime to 60.

2 Fibonacci Fashion

Note 10

You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \ldots, S_t$, where $S_1 \subseteq \{1,2,\ldots,n\}$ and for $2 \le i \le t$, $S_i \subseteq \{1,2,\ldots,n\}$ and S_i is disjoint from S_{i-1}.

(a) For $t \ge 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \ge 1$ and $n \ge 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1 \ge 0} \sum_{x_2 \ge 0} \cdots \sum_{x_t \ge 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)

Solution:

(a) We will prove this by strong induction.

Base cases: For $k=1$, the only binary strings possible are 0 and 1. Therefore, there are two possible binary strings, and $F_{k+2} = F_3 = 2$. For $k=2$, the binary strings possible are 11, 01, and 10, and we have $F_{k+2} = F_4 = 3$, so the identity holds.

Inductive hypothesis: For $k \geq 2$, assume that for all $1 \leq x \leq k$, there are F_{x+2} binary strings of length x with no consecutive zeros.

Inductive step: Consider the set of binary strings of length $k+1$ with no consecutive zeros. We can group these into two sets: those which end with 0, and those which end with 1. For those that end with a 0, these can be constructed by taking the set of binary strings of length $k-1$ with no consecutive zeros and appending 10 to the end of them. Then by the inductive hypothesis, this set is of size F_{k+1}. For those that end with a 1, these can be constructed by taking the set of binary strings of length k with no consecutive zeros and appending a 1 to the end of them. Then by the inductive hypothesis, this set is of size F_{k+2}. Since the union of these two subsets (those which end with 0 and those which end with 1) cover all possible elements in the set of binary strings of length $k+1$ with no consecutive zeros, the size of this set will be $F_{k+1}+F_{k+2}=F_{k+3}$. This thus proves the inductive hypothesis.

(b) We first consider the left-hand-side of the identity. To create subsets of accessories that are consecutively disjoint with sizes $x_i = |S_i|$, $1 \leq i \leq n$, there are $\binom{n}{x_1}$ ways to create S_1, the subset of accessories you will wear on the first day. Then since S_2 must be disjoint from S_1, there are $\binom{n-x_1}{x_2}$ ways choose accessories to create S_2. Since S_3 must be disjoint from S_2, there are $\binom{n-x_2}{x_3}$ ways choose accessories to create S_3, and so on. Thus there are $\binom{n}{x_1}\binom{n-x_1}{x_2}\cdots\binom{n-x_{t-1}}{x_t}$ ways to create subsets of accessories $S_1,\ldots,S_t$ with respective sizes $x_1,\ldots,x_t$. Then altogether, $S_1,\ldots,S_t$ can be created in

$$\sum_{x_1 \geq 0}\sum_{x_2 \geq 0}\cdots\sum_{x_t \geq 0}\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t}$$

ways.

Now, consider the right-hand-side of the identity. Now for each accessory $i \in \{1,\ldots,n\}$, we will first decide which subsets $S_1,\ldots,S_t$ will contain accessory i, where we can't assign item i to consecutive subsets. For each accessory, we create a binary string of length t, where the leading digit represents S_1, the next digit represents S_2, and so on. We will say that a 0 in digit k means that we will wear the accessory on day k. Therefore, the number of ways we can assign accessory i to subsets $S_1,\ldots,S_t$ such that no two consecutive subsets both have accessory i is the same as the number of binary strings of length t with no consecutive zeros. Thus using the result in part (a), there are F_{t+2} ways to select the nonconsecutive subsets containing i among $S_1,\ldots,S_t$. Since we have n accessories, accessories $1,\ldots,n$ can be placed into subsets $S_1,\ldots,S_t$ in $(F_{t+2})^n$ ways.

This thus proves the identity.

3 CS70: The Musical

Note 10

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$

Solution:

(a) Say that we would like to select 2 directors.

LHS: This is the number of ways to choose 2 directors out of the $2n$ candidates.

RHS: Split the $2n$ directors into two groups of n; one group consisting of experienced directors, or inexperienced directors (you can split arbitrarily). Then, we consider three cases: either we choose:

1. Both directors from the group of experienced directors,
2. Both directors from the group of inexperienced directors, or
3. One experienced director and one inexperienced director.

The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. Since these cases are mutually exclusive and cover all possibilities, it must also count the total number of ways to choose 2 directors out of the $2n$ candidates. This completes the proof.

(b) Say that we would like to select k crew members.

LHS: This is simply the number of ways to choose k crew members out of n candidates.

RHS: We select the k crew members in a different way. First, Edward looks at the first candidate he sees and decides whether or not he would like to choose the candidate. If he selects the first candidate, then Edward needs to choose $k-1$ more crew members from the remaining $n-1$ candidates. Otherwise, he needs to select all k crew members from the remaining $n-1$ candidates.

We are not double counting here - since in the first case, Edward takes the first candidate he encounters, and in the other case, we do not.

(c) In this part, Edward selects a subset of the n actors to be in his musical. Additionally, assume that he must select one individual as the lead for his musical.

LHS: Edward casts k actors in his musical, and then selects one lead among them (note that $k = \binom{k}{1}$). The summation is over all possible sizes for the cast - thus, the expression accounts for all subsets of the n actors.

RHS: From the n people, Edward selects one lead for his musical. Then, for the remaining $n-1$ actors, he decides whether or not he would like to include them in the cast. 2^{n-1} represents the amount of (possibly empty) subsets of the remaining actors. *(Note that for each actor, Edward has 2 choices: to include them, or to exclude them.)*

(d) In this part, Edward selects a subset of the n actors to be in the musical; additionally he must select j lead actors (instead of only 1 in the previous part).

LHS: Edward casts $k \geq j$ actors in his musical, then selects the j leads among them. Again, the summation is over all possible sizes for the cast (note that any cast that has $< j$ members is invalid, since Edward would be unable to select j lead actors) - thus, the expression accounts for all valid subsets of the n actors.

RHS: From the n people, Edward selects j leads for his musical. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. 2^{n-j} represents the amount of ways that Edward can do this.

4 August Absurdity

Since March Madness was cancelled, the council unanimously decided to have August Absurdity instead - an online Discrete Mathematics tournament! There are 64 teams (including Cal) in the single-elimination tournament - that means, every match is between two teams and will decide a winner who moves on to the next round and a loser who is eliminated from the tournament. Thus the first round will have 64 teams, the next will have 32, and so on until 1 remains. There is a single, randomly initialized, starting bracket.

(a) How many tournament outcomes exist such that Cal wins the entire tournament?

(b) In the first round, Cal will face a no-name school called LJSU (some people call it Stanfurd?). The format of each match is as follows: Each of the two teams have 8 players labelled from 1 to 8. They play a series of games. In the first game, the two 1's play each other. The loser of the game is eliminated and replaced by the next player of the same team until all players from one team are eliminated, ending the match. What is the number of possible sequences of games such that Cal wins the match?

(c) Cal employs a blasphemous strategy that even baffles themselves. They place their players in an order such that each player is either taller than all the preceding players or shorter than all the preceding players. Let 1-8 represent the players' heights. An example of a valid ordering: 4, 5, 6, 3, 2, 7, 1, 8. An example of an invalid ordering: 1, 2, 3, 4, 5, 6, 8, 7. (invalid since 7 is neither taller or shorter than all the preceding players). How many such orderings exist?

(d) To keep viewership up after the tournament finishes, the council plans an All-Star match. The 16 greatest players in the league were chosen, including Oski and a tree..? Oski refuses to play on the same team as the tree. How many ways can the 16 players be distributed into two teams of 8 players such that Oski and the tree are in opposite teams?

(e) Provide an explanation for the following combinatorial identity. Hint: Solve the previous part using another method. Those two methods should correspond to the two sides of the equality.

$$\binom{n}{r} - \binom{n-2}{r-2} - \binom{n-2}{r} = 2\binom{n-2}{r-1}$$

Solution:

(a) In the first round, there are 32 matches, each with 2 possible outcomes. Thus there are 2^{32} possibilities of the first round. In the next round, there are 16 matches, again each with 2 possible outcomes for a total of 2^{16}. Continuing on this pattern, $2^{32}2^{16}2^{8}2^{4}2^{2}2^{1} = 2^{63}$. More succinctly, the tournament has a total of 63 matches, each with 2 options, thus there are 2^{63} outcomes from the first rule of counting. Now, since we are considering only the outcomes where Cal wins, we have overcounted. We only care about $1/2^6$ of the previous outcomes because we already know that Cal wins in each of the 6 rounds. Thus, the total amount of outcomes is 2^{57}.

(b) Let Cal and Stanford be C and S. We can write the sequence of games as a sequence of letters that represent the winner of each game. For example, SSCCCCCC means Stanford won the first two games and then Cal won the next 8, ending the match. Note that for any sequence of games, there is only 1 way to arrange the unplayed games at the end. So we can append those games to the end and that would not make a difference. If we add the rest of Stanford's remaining letters to the sequence, we are counting the number of 16 letter sequences with 8 C's and 8 S's. So in fact, the total number of sequences of games is $\binom{16}{8}$. (Verify this holds for a small example such as n=4 instead of 16). Now, since we are considering the number of sequences of games such that Cal wins, by symmetry, that is exactly half of these. Thus, the answer is $\binom{16}{8}/2$.

(c) Each element in the set is constrained by everything to the left of it. Notably, the leftmost number can be anything, yet the last number has to be either greater than all the previous elements or smaller than them. We want focus on our most constrained variables since that is easiest to count. So, let's start from the end and work backwards. The last number must be the maximum or minimum element of the set. Thus, it can be only 1 or 8. Now, similarly in either case, the second to last number must be the minimum or maximum element of the remaining set. i.e, if the last element was 8, then the second to last must be 7 or 1. If the last element was 1, then the second to last must be 2 or 8. And so on until the very first element, which only has one remaining option. Thus, by the first rule of counting we get 2^7.

(d) **Answer 1: Complementary** We first count the total number of ways to distribute the players into two teams without any restrictions, which is $\binom{16}{8}$ since we choose 8 to be in team A (and the remaining makes up team B). Then, we subtract off the cases we dont want - that Oski and tree are in the same team. We fix the two onto a team and pick the remaining players to make up team A. The two cases are $\binom{14}{6}$ if both on team A, and $\binom{14}{8}$ if both on team B. Thus, we get $\binom{16}{8}$ - $\binom{14}{6}$ - $\binom{14}{8}$

Answer 2: Casework We count the two cases: Oski A/tree B, and Oski B/tree A. So we fix them on their respective teams, and pick the remaining players to make up team A to get $\binom{14}{7}$ in both cases. Thus, we get $2\binom{14}{7}$

(e) Let n represent the number of players and r represent the number of players on team A. Both methods of the previous subparts are ways of calculating the number of ways to distribute n people into two teams of size r and $n-r$ (in this case, $r = n-r$). LHS is answer 1 and RHS is answer 2. Refer to the previous part to understand the details.