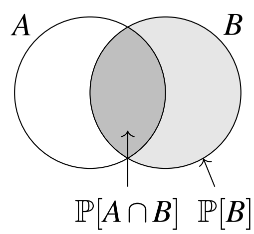
Conditional Probability Intro

Note 14

Conditional Probability: Probability of event A, *given* that event B has happened;

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Think of like restricting our sample space:



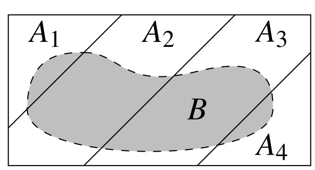
Bayes Rule: A consequence of conditional probability - notice $\mathbb{P}[A \cap B] = \mathbb{P}[A \mid B]\,\mathbb{P}[B] = \mathbb{P}[B \mid A]\,\mathbb{P}[A]$, so

$$\mathbb{P}[B \mid A] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[A]} = \frac{\mathbb{P}[A \mid B]\,\mathbb{P}[B]}{\mathbb{P}[A]}.$$

Total Probability Rule: If disjoint events $A_1, \ldots, A_n$ form a partition on the sample space Ω, we then have

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \cap A_i] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i]\,\mathbb{P}[A_i].$$

Visually, we're splitting an event into partitions and looking at each intersection individually:



Independence: Two events are independent if the following (equivalent) conditions are satisfied. The second definition is probably more intuitive - B happening does not affect the probability of A happening.

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$
$$\mathbb{P}[A \mid B] = \mathbb{P}[A]$$

1 Poisoned Smarties

Note 14

Supposed there are 3 people who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the owners, produces considerably more Smarties than her competitors and has a commanding 50% of the market share. Yousef See, who inherited her riches, lags behind Burr and produces 40% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 10%. However, a recent string of Smarties related food poisoning has forced the FDA to investigate these factories to find the root of the problem. Through her investigations, the inspector found that 2 Smarties out of every 100 at Kelly's factory was poisonous. At See's factory, 5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.1.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) If a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

Solution:

(a) Let S be the event that a smarty is safe to eat. Let BK be the event that a smarty is from Burr Kelly's factory. Let YS be the event that a smarty is from Yousef See's factory. Finally, let SF be the event that a smarty is from Stan Furd's factory.

By total probability, we have

$$\begin{aligned}\mathbb{P}[S] &= \mathbb{P}[BK]\,\mathbb{P}[S \mid BK] + \mathbb{P}[YS]\,\mathbb{P}[S \mid YS] + \mathbb{P}[SF]\,\mathbb{P}[S \mid SF] \\ &= \frac{1}{2}\cdot\frac{49}{50} + \frac{2}{5}\cdot\frac{19}{20} + \frac{1}{10}\cdot\frac{9}{10} \\ &= \frac{49}{100} + \frac{38}{100} + \frac{9}{100} \\ &= \frac{96}{100} = \frac{24}{25} = 0.96\end{aligned}$$

Therefore the probability that a Smarty is safe to eat is 0.96.

(b) Let P be the event that a smarty is poisonous.

$$\mathbb{P}[P \mid \overline{BK}] = \frac{\mathbb{P}[\overline{BK} \cap P]}{\mathbb{P}[\overline{BK}]}$$

Since BK, YS, SF are a partition of the entire sample space, we know that if BK did not occur, then either YS occurred, or SF occurred:

$$\begin{aligned}
&= \frac{\mathbb{P}[YS \cap P]}{\mathbb{P}[\overline{BK}]} + \frac{\mathbb{P}[SF \cap P]}{\mathbb{P}[\overline{BK}]} \\
&= \frac{\mathbb{P}[P \mid YS]\,\mathbb{P}[YS]}{1 - \mathbb{P}[BK]} + \frac{\mathbb{P}[P \mid SF]\,\mathbb{P}[SF]}{1 - \mathbb{P}[BK]} \\
&= \frac{\frac{1}{20} \cdot \frac{2}{5}}{\frac{1}{2}} + \frac{\frac{1}{10} \cdot \frac{1}{10}}{\frac{1}{2}} = 2 \cdot \frac{2}{100} + 2 \cdot \frac{1}{100} \\
&= \frac{6}{100} = \frac{3}{50} = 0.06
\end{aligned}$$

(c) From Bayes' Rule, we know that:

$$\mathbb{P}[SF \mid P] = \frac{\mathbb{P}[P \mid SF]\,\mathbb{P}[SF]}{\mathbb{P}[P]}.$$

In part (a), we calculated the probability that any random Smarty was safe to eat; here, notice that $\mathbb{P}[P] = 1 - \mathbb{P}[S]$. This means we have

$$\begin{aligned}
\mathbb{P}[SF \mid P] &= \frac{\mathbb{P}[P \mid SF]\,\mathbb{P}[SF]}{1 - \mathbb{P}[S]} \\
&= \frac{\frac{1}{10} \cdot \frac{1}{10}}{1 - \frac{24}{25}} = \frac{\frac{1}{100}}{\frac{1}{25}} \\
&= \frac{25}{100} = \frac{1}{4} = 0.25
\end{aligned}$$

2 Symmetric Marbles

Note 14

A bag contains 4 red marbles and 4 blue marbles. Rachel and Brooke play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Rachel wins if there are more red than blue marbles, and Brooke wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

(a) Let A_1 be the event that the first marble is red and let A_2 be the event that the second marble is red. Are A_1 and A_2 independent?

(b) What is the probability that Rachel wins the game?

(c) Given that Rachel wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles and we add a tiebreaker to the game: if there are an equal number of red and blue marbles among the four drawn, Rachel wins if the third marble is red, and Brooke wins if the third marble is blue.

(d) What is the probability that the third marble is red?

(e) Given that there are k red marbles among the four drawn, where $0 \le k \le 4$, what is the probability that the third marble is red? Answer in terms of k.

(f) Given that the third marble is red, what is the probability that Rachel wins the game?

Solution:

(a) They are not independent; removing one red marble lowers the probability of the next marble being red.

(b) Let p be the probability that Rachel wins. Since there are an equal number of red and blue marbles, by symmetry, the probability that Rachel wins and the probability that Brooke wins is the same. Thus, the probability that there is a tie is $1 - p - p = 1 - 2p$.

We now compute the probability that there is a tie. For there to be a tie, two of the four marbles need to be red. There are $\binom{8}{4}$ ways to pick 4 marbles, and $\binom{4}{2}\binom{4}{2}$ to pick 2 red and blue marbles, respectively, giving a probability of

$$\frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{36}{70} = \boxed{\frac{18}{35}}.$$

We conclude that $1 - 2p = \frac{18}{35}$. Solving for p gives $p = \boxed{\frac{17}{70}}$.

(c) Let A be the event that there are 3 red marbles drawn, and let B be the event that there are 4 red marbles drawn. We wish to compute

$$\mathbb{P}[B \mid (A \cup B)] = \frac{\mathbb{P}[B \cap (A \cup B)]}{\mathbb{P}[A \cup B]} = \frac{\mathbb{P}[B]}{\mathbb{P}[A] + \mathbb{P}[B]}.$$

Similar to the calculation in part (b), the probability that there are 3 red marbles drawn is $\frac{\binom{4}{3}\binom{4}{1}}{\binom{8}{4}} = \frac{16}{70}$, and the probability that there are 4 red marbles drawn is $\frac{\binom{4}{4}\binom{4}{0}}{\binom{8}{4}} = \frac{1}{70}$, giving a final answer of $\frac{\frac{1}{70}}{\frac{16}{70} + \frac{1}{70}} = \boxed{\frac{1}{17}}$.

(d) By symmetry, the probability that the third marble is red is the same as the probability that the first marble is red, or the same as any marble being red. One way to see this is to imagine drawing the four marbles in order, then moving the first marble drawn to the third position. This is another way to draw four marbles that yields the same distribution.

There are 8 red marbles, and 12 marbles in total. Thus, the probability that the third marble is red is $\frac{8}{12} = \boxed{\frac{2}{3}}$.

(e) We are given that there are k red marbles among the 4 drawn. By symmetry, each marble has the same probability of being red, so the probability that the third marble is red is $\boxed{\frac{k}{4}}$.

(f) The only way for Rachel to lose the game given that the third marble is red is if all the other marbles are blue. The probability that the third marble is red and all the other marbles are blue is $\frac{4}{12} \cdot \frac{3}{11} \cdot \frac{8}{10} \cdot \frac{2}{9} = \frac{8}{495}$, and the probability that the third marble is red is $\frac{8}{12} = \frac{2}{3}$, so the probability that Rachel loses given that the third marble is red is $\frac{\frac{8}{495}}{\frac{2}{3}} = \frac{4}{165}$, and the probability that Rachel wins given that the third marble is red is $\boxed{\frac{161}{165}}$.

3 Pairwise Independence

Note 14

Recall that the events A_1, A_2, and A_3 are *pairwise independent* if for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3]$.

Suppose you roll two fair six-sided dice. Let A_1 be the event that the first die lands on 1, let A_2 be the event that the second die lands on 6, and let A_3 be the event that the two dice sum to 7.

(a) Compute $\mathbb{P}[A_1]$, $\mathbb{P}[A_2]$, and $\mathbb{P}[A_3]$.

(b) Are A_1 and A_2 independent?

(c) Are A_2 and A_3 independent?

(d) Are A_1, A_2, and A_3 pairwise independent?

(e) Are A_1, A_2, and A_3 mutually independent?

Solution:

(a) We have that $\mathbb{P}[A_1] = \mathbb{P}[A_2] = \frac{1}{6}$, since we have a $\frac{1}{6}$ probability of getting a particular number on a fair die.

Since there are 6 ways in which the two dice can sum to 7 (i.e. $\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$), we have $\mathbb{P}[A_3] = \frac{1}{6}$ as well.

(b) We want to determine whether $\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]$. We already found the probabilities of A_1 and A_2 from part (a), so let's look at $\mathbb{P}[A_1 \cap A_2]$. There's only one possible outcome where the first die is a 1 and the second die is a 6, so this gives a probability of $\mathbb{P}[A_1 \cap A_2] = \frac{1}{36}$.

Since $\mathbb{P}[A_1]\,\mathbb{P}[A_2] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_1 \cap A_2]$, these two events are independent.

(c) We want to determine whether $\mathbb{P}[A_2 \cap A_3] = \mathbb{P}[A_2]\,\mathbb{P}[A_3]$. We already found the probabilities of A_2 and A_3 from part (a), so let's look at $\mathbb{P}[A_2 \cap A_3]$. These two events both occur if the second die lands on a 6, and the two dice sum to 7. There's only one way that this can happen, i.e. the first die must be a 1, so the intersection has probability $\mathbb{P}[A_2 \cap A_3] = \frac{1}{36}$.

Since $\mathbb{P}[A_2]\,\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_2 \cap A_3]$, these two events are independent.

(d) To see whether the three events are pairwise independent, we need to ensure that all pairs of events are independent. We've already checked that A_1 and A_2 are independent, and that A_2

and A_3 are independent, so it suffices to check whether A_1 and A_3 are independent.

Similar to the previous two parts, the intersection $A_1 \cap A_3$ means that the first die must land on a 1, and the two dice sum to 7. There's only one way for this to happen, i.e. the second die must land on a 6, so the probability is $\mathbb{P}[A_1 \cap A_3] = \frac{1}{36}$.

Since $\mathbb{P}[A_1]\,\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_1 \cap A_3]$, these two events are also independent. Since we've now shown that all possible pairs of events are independent, A_1, A_2, and A_3 are indeed pairwise independent.

(e) Mutual independence requires the additional constraint that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3]$. We've found the individual probabilities of these events in part (a), so we only need to compute $\mathbb{P}[A_1 \cap A_2 \cap A_3]$.

Here, we must have that the first die lands on 1, the second die lands on 6, and the sum of the two dice is equal to 7. There's only one way for this to happen, i.e. the first die is a 1 and the second die is a 6, so the probability of the intersection of all three events is $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \frac{1}{36}$.

However, since $\mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216} \neq \frac{1}{36} = \mathbb{P}[A_1 \cap A_2 \cap A_3]$, these three events are not mutually independent.