Category Theory in Brief

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Chapter 1
Preface

What I cannot create, I do not understand. — Richard Feynman

1.1 Motivation

This note is about the basic aspects of category theory. There have been many books on category theory, almost all of them contains many examples from multiple areas of mathematics. These examples, however, need mathematical background on many areas, making it hard to read.

In this note we focus on the core concepts of category theory, trying to understand them in an intuitive way, and *to build theory from the ground up*. This endeavor make us think about things in the framework of category theory. Examples are shown only when it is essential.

1.2 Writing Style

The writing style follows the suggestions given by *How to Think Like a Mathematician*. Some conventions make it better:

- Title of each subsection is a sentence that summarizes that section. With this, you can review by simply reading content.
- Definitions are bold.

1.3 Technique Limitation

It is hard to draw diagrams in TeXmacs. So, all the commutative diagrams are drawn by quiver online, and pasted to the text by screenshot. For each commutative diagram, the link to quiver is provided.

1.4 How to Read

While reading this book, it is strongly suggested to draw commutative diagrams whenever it is needed. You will find it quite easy if you do keep drawing commutative diagrams and quite difficult if not.

Chapter 2
Category and Functor

2.1 Category

2.1.1 Category is about arrows

Category is the fundamental element of category theory. A category consists of arrows and objects, which are employed to declare arrows: where an arrow emits, and where it ends. Moreover, several properties relating to these components should be satisfied.

Strangely, in category, arrow is called morphism, a word derived from isomorphism. And isomorphism is constructed from iso-morphe-ism, where morphe, a Greek word, means shape or form. So, isomorphism means equal shape or form. This can be easily illustrated in topology, where the two isomorphic topological space share the same form (but may not the same shape). But semantically, this is far from what arrow should mean. So, the question is why mathematician use the word morphism for arrow. A guess is that it may come from homomorphism; and in algebra, a homomorphism is an arrow between algebraic structures.

Definition 2.1. *[Category] A **category** C consists of*

- *a collection*[2.1] *of **objects**, denoted by C itself,*
- *for each $A, B \in \mathsf{C}$, a collection of **morphisms** from A to B, denoted by $\mathsf{C}(A, B)$, where A is the **domain** and B the **codomain**,*
- *for each $f \in \mathsf{C}(A, B)$ and $g \in \mathsf{C}(B, C)$, a **composition** of f and g that furnishes a morphism $g \circ f \in \mathsf{C}(A, C)$, and*
- *for each $A \in \mathsf{C}$, an **identity** $1_A \in \mathsf{C}(A, A)$,*

such that the following axioms are satisfied:

- ***associativity**: for each $f \in \mathsf{C}(A, B)$, $g \in \mathsf{C}(B, C)$, and $h \in \mathsf{C}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$, and*
- ***identity**: for each $f \in \mathsf{C}(A, B)$, we have $f \circ 1_A = 1_B \circ f$.*

Notation 2.2. *We also use the notation $f: A \to B$ or $A \xrightarrow{f} B$ for $f \in \mathsf{C}(A, B)$, and $A: \mathsf{C}$ for $A \in \mathsf{C}$.*

Now, category becomes much more familiar to us. We can think the objects of C as sets, and morphism as function, which is the map between sets. Indeed, the collection of sets and functions does form a category: the category of sets.

Definition 2.3. *[Category of Sets] The **category of sets**, denoted as Set, has the collection of all sets as its objects, and for each $A, B \in \mathsf{Set}$, the collection of all functions from A to B as its morphisms from A to B.*

2.1. Usually in set theory, a collection of something is a set. But in fact, not every collection is a set, such as the collection of all sets that do not contain itself, which is Russell's paradox. In this situation, we call the collection a **class**. It looks like a "super" set, beyond the axioms of set theory. Throughout this note, we do not distinguish set from class, just using the word "collection" for general purpose. In many textures, a class that is in fact a set is said to be **small**. So, we do not distinguish small from large too. In addition, we employ the notation $\in$ of set theory for indicating that something is in some collection.

It is easy to check that the axioms are satisfied. But, be caution! Set is just a specific category, it helps us understanding what a category might look like. But, Set has much more axioms, or restrictions, than the category itself, thus may blind us to the potential power of arrows.

Indeed, there exist categories whose objects are not sets, or whose morphisms are not maps. So, a better way of thinking category is keeping objects and morphisms abstract. You can think objects as dots and morphisms as arrows between the dots.

2.1.2 Objects may not be sets

We know that the symmetry group of rectangle is dihedral group D_2. The group elements are operations: identity, rotation of 180 degrees, and reflections along vertical and horizontal directions. The operand is unique: the rectangle. These operations can be viewed as arrows from the rectangle to itself. So, this symmetry group describes a category, called $\mathsf{B}D_2$. The axioms of category are satisfied because of the group properties. In this category, object is not a set, but an rectangle.

In fact, all groups are examples illustrating that objects may not be sets. Here, we need the definition of isomorphism in category.

Definition 2.4. *[Isomorphism in Category] In a category C, for any objects $A, B \in \mathsf{C}$, a morphism $f: A \to B$ is an **isomorphism** if there exists $g: B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Denote $A \cong B$ if A and B are isomorphic.*

With the aid of isomorphism, we definition the groupid.

Definition 2.5. *[Groupid] A **groupid** is a category in which all morphisms are isomorphisms.*

Now, we come to the big step: define the group! But, wait a minute. We have learned abstract algebra and known what a group is. The point here is that category theory provide a new, but equivalent, way of defining group, using arrows!

Definition 2.6. *[Group as Category] A **group** is a groupid in which there is only one object.*

Notation 2.7. *Because there is only one object, we can simplify the notation of morphism like $f: A \to B$ to f, and denote $f \in \mathsf{G}$ for that f is a morphism of a group G.*

If we start at defining group by arrows, we have to declare that the properties (axioms) of group studied in algebra are satisfied.

Theorem 2.8. *Let G a group (defined in 2.6). We have*

- ***associativity**: for $\forall f, g, h \in \mathsf{G}$, $(f \circ g) \circ h = f \circ (g \circ h)$,*
- ***identity element**: there exists $1 \in \mathsf{G}$ such that for $\forall f \in \mathsf{G}$, $f \circ 1 = 1 \circ f = f$.*
- ***inverse element**: for $\forall f \in \mathsf{G}$, there exists $g \in \mathsf{G}$ such that $f \circ g = g \circ f = 1$.*

Proof. The associativity of group is identified as the associativity of category. The same for identity element. The inverse element comes from the fact that all morphisms are isomorphisms. □

So, a group is a category. And as compared with the group defined in abstract algebra, we find that the unique object in this category is not set, and that discussing what the object should be is meaningless.

2.1.3 Morphisms may not be maps

To illustrate that morphisms may not be maps, we need to define preorder and poset.

Definition 2.9. *[Preorder] Given a set S, a **preorder** P on S is a subset of $S \times S$ such that the following axioms are satisfied:*

- ***reflexivity**: for $\forall a \in S$, $(a, a) \in P$, and*
- ***transitivity** for $\forall a, b, c \in S$, if $(a, b) \in P$ and $(b, c) \in P$, then $(a, c) \in P$.*

For example, “no greater than” is a preorder, where S is the set of real numbers and $(a,b)\in P$ means $a\leqslant b$. “Is subset of” is another example of preorder, where S is the set of sets and $(a,b)\in P$ means $a\subseteq b$.

Definition 2.10. *[Poset] A preordered set, or **poset**, (S,P) is a set S equipped with a preorder P on S.*

With these preliminaries, we claim that a poset is a category.

Definition 2.11. *[Poset as Category] Given poset (S,P), a category $\mathsf{Poset}(S,P)$ can be constructed by regarding the elements in S as objects and regarding $(a,b)\in P$ as $a\rightarrow b$.*

Because of the axioms of preorder, the axioms of category are satisfied. The category Poset illustrates that morphisms are not maps. In $\mathsf{Poset}(\mathbb{R},\leqslant)$, morphisms are “no greater than”s. And in $\mathsf{Poset}(\mathrm{pow}(\mathbb{R}),\subseteq)$ where pow represents power set[2.2], they are “being subset of”.

2.1.4 Supremum and infimum are dual

Arrows can represent many mathematical objects. For example, in $\mathsf{Poset}(\mathbb{R},\leqslant)$, we can describe supremum as follow.

Definition 2.12. *[Supremum in Category] Given a subset $A\subset\mathbb{R}$. An $x\in\mathbb{R}$ is the **supremum** of A if it satisfies:*

- *for $\forall a\in A$, $a\rightarrow x$ and,*
- *for $\forall y\in\mathbb{R}$, if $a\rightarrow y$ for $\forall a\in A$, then $x\rightarrow y$.*

This is, again, a weird definition on supremum. But, if you check carefully, you can see that this definition is equivalent to that studied in analysis. Also, we can define the infimum in the same fashion.

Definition 2.13. *[Infimum in Category] Given a subset $A\subset\mathbb{R}$. An $x\in\mathbb{R}$ is the **infimum** of A if it satisfies:*

- *for $\forall a\in A$, $x\rightarrow a$ and,*
- *for $\forall y\in\mathbb{R}$, if $y\rightarrow a$ for $\forall a\in A$, then $y\rightarrow x$.*

Weird again, but now you are on the load to wonderland. By comparing the definition of infimum to that of supremum, we find all statements are the same except that we replaced supremum by infimum and domain by codomain (for instance, replaced $y\rightarrow x$ by $x\rightarrow y$). Two statements are **dual** if you can get one from the other by simply flipping all the arrows, or, equivalently, by exchanging the domain and codomain for each morphism in the statement. We say that supremum and infimum are dual.

Applying this to category, we find the dual category.

Definition 2.14. *[Dual Category] Given a category C, its **dual category**, denoted by C^{op}, is obtained from C by exchanging the domain and codomain for each morphism in C.*

2.2 Why Category?

2.2.1 Arrows generalize concepts and theorems from one area to every area in mathematics

Why category theory? Or say, why arrows? One benefit of re-claim everything in arrows is the ability of generalizing a concept in one area to area domain in mathematics. An example comes from generalizing the Cartesian product in the set theory.

2.2. Let A be a set. The **power set** of A, denoted by $\mathrm{pow}(A)$, is the set of all subsets of A.

You have been familiar with the Cartesian product of two sets. Given two sets A and B, recall that the Cartesian product $A \times B := \{(a, b) | a \in A, b \in B\}$. Again, for generalizing the concepts using category theory, we have to re-write the concepts using arrows (in fact, re-writing concepts using arrows is the hardest part in category theory). And again, this re-writing looks weird at the first sight.

Definition 2.15. *[Product of Two Objects] Given a category C. For any $A, B \in \mathsf{C}$, the **product** of A and B is another object $C \in \mathsf{C}$ together with two morphisms $\alpha: C \to A$ and $\beta: C \to B$ such that, for any $C' \in \mathsf{C}$, any $\alpha': C' \to A$ and $\beta': C' \to B$, there exists a unique morphism $\gamma: C' \to C$ so that $\alpha' = \alpha \circ \gamma$ and $\beta' = \beta \circ \gamma$.*

So, a product of objects A and B and a triplet (C, α, β). Applying to Set, as you can check directly, it goes back to the Cartesian product of two sets. We can also apply it to Grp, which furnishes the group direct product:

Definition 2.16. *[Group Direct Product] Given two groups G and H, the **group direct product** of G and H is defined as $\{(g, h) | g \in G, h \in H\}$ equipped with group multiplication $(g, h) \times (g', h') := (g \circ g', h \cdot h')$ where $\circ$ is the multiplication of G and $\cdot$ of H.*

It is like the Cartesian product, but extra structure are *implied*.

Also, all specific categories would be benefited from a theorem claimed in category theory. Such as the uniqueness of product in the sense of isomorphism.

Theorem 2.17. *[Uniqueness of Product] Given a category C. For any $A, B \in \mathsf{C}$ and any two products (C, α, β) and (C', α', β'). Then, there exists a unique isomorphism $\gamma: C' \to C$ such that $\alpha' = \alpha \circ \gamma$ and $\beta' = \beta \circ \gamma$.*

That is, C and C', α and α', β and β' are equivalent in the sense of isomorphism.

Proof. Left to reader. □

This theorem holds not only for Cartesian product of sets, but also, for instance, for the group direct product.

2.2.2 Duality helps create new concepts and theorems, freely!

Another benefit of viewing everything in arrows is duality. In category theory, it is natural to think what would happen if we exchange domain and codomain for all the arrows. Just like the relation between supremum and infimum, it is natural to ask what if we exchange domain and codomain for all the arrows in the definition of product. This furnishes a new concept we called coproduct.

Definition 2.18. *[Coproduct of Two Objects] Given a category C. For any $A, B \in \mathsf{C}$, the **coproduct** of A and B is another object $C \in \mathsf{C}$ together with two morphisms $\alpha: A \to C$ and $\beta: B \to C$ such that, for any $C' \in \mathsf{C}$, any $\alpha': A \to C'$ and $\beta': B \to C'$, there exists a unique morphism $\gamma: C \to C'$ so that $\alpha' = \gamma \circ \alpha$ and $\beta' = \gamma \circ \beta$.*

Again, a coproduct of objects A and B and a triplet (C, α, β). Comparing with product, coproduct is nothing but exchanging domain and codomain for all the arrows in the statement of product.

Applying to Set, as it can be directly checked, we get the disjoint union of two sets. Given two sets A and B, recall that the disjoint union $A \cup_d B := \{(a, 1) | a \in A\} \cup \{(b, 2) | b \in B\}$. This is a surprise, since, unlike the duality between supremum and infimum, Cartesian product and disjoint union do not look like a pair at the first sight!

Recall the theorem that product is unique in the sense of isomorphism. If we also exchange domain and codomain for all the arrows in the statement of the theorem, as well as in the statement of its proof, then we get another theorem: coproduct is unique in the sense of isomorphism, without re-do the proof!

As a summary, the duality in category theory furnishes free lunch, which include not only the dual concepts that are very generic, but also the dual theorems that need no proof. All about is flipping arrows.

2.3 Functor

2.3.1 Structure preserving map builds category out of objects

We need some examples of category to introduce the next core concept of category theory: functor. The first example is the category of topological spaces.

Definition 2.19. *[Category of Topological Spaces] The **category of topological spaces**, denoted as* Top*, has the collection of all topological spaces as its objects, and for each* $A, B \in \mathsf{Top}$*, the collection of all continuous maps from* A *to* B *as its morphisms from* A *to* B*.*

The next example is the category of groups.

Definition 2.20. *[Category of Groups] The **category of groups**, denoted as* Grp*, has the collection of all groups as its objects, and for each* $A, B \in \mathsf{Grp}$*, the collection of all homomorphisms from* A *to* B *as its morphisms from* A *to* B*.*

From these two examples, we find an almost free method to construct a category out of objects. That is, a method to assign the morphisms. This method employs the maps that preserve the structure of object as the morphisms. For example, in Top, the preserved structure is continuity, and in Grp, it is the group structure.

2.3.2 Functor is the morphism of the category of categories

Notice that the objects of a category can be anything. So, it can also be categories! To construct a category out of categories, the morphisms between two categories can be the maps that preserve the structure of category. These structure preserving maps in the category of categories are functors.

Definition 2.21. *[Functor] Given two categories* C *and* D*, a **functor*** $F: \mathsf{C} \to \mathsf{D}$ *is a map that takes*

- *for each* $A \in \mathsf{C}$*,* $F(A) \in \mathsf{D}$*, and*
- *for each* $A, B \in \mathsf{C}$ *and each* $f \in \mathsf{C}(A, B)$*,* $F(f) \in \mathsf{D}(F(A), F(B))$*,*

such that the structure of category is preserved, that is

- ***composition**: for* $\forall A, B, C \in \mathsf{C}$ *and* $f: A \to B, g: B \to C$*,* $F(f \circ g) = F(f) \circ F(g)$*,*
- ***identity**: for* $\forall A \in \mathsf{C}$*,* $F(1_A) = 1_{F(A)}$*.*

$$
\begin{tikzcd}
X \arrow[rr] \arrow[dr, "f"] \arrow[dd, "h"'] & & F(X) \arrow[dr, "F(f)"] \arrow[dd, "F(h)"' near start] & \\
& Y \arrow[rr] \arrow[dl, "g"] & & F(Y) \arrow[dl, "F(g)"] \\
Z \arrow[rr] & & F(Z) &
\end{tikzcd}
$$

Figure 2.1. Indicates the functor $F: \mathsf{C} \to \mathsf{D}$. This diagram commutes.

Functor as morphism does build a category of categories. Indeed, functors can be composed by extending the commutative diagram 2.1 to the right (figure 2.2). Identity functor is the one that maps everything in a category to itself. Finally, associativity and identity axioms can be checked directly.

$$
\begin{tikzcd}
X \arrow[rr] \arrow[dr, "f"] \arrow[dd, "h"'] & & F(X) \arrow[rr] \arrow[dr, "F(f)"] \arrow[dd, "F(h)"' near start] & & G(X) \arrow[dr, "G(f)"] \arrow[dd, "G(h)"' near start] & \\
& Y \arrow[rr] \arrow[dl, "g"] & & F(Y) \arrow[rr] \arrow[dl, "F(g)"] & & G(Y) \arrow[dl, "G(g)"] \\
Z \arrow[rr] & & F(Z) \arrow[rr] & & G(Z) &
\end{tikzcd}
$$

Figure 2.2. Extending the commutative diagram 2.1 to the right. This diagram commutes.

2.3.3 Functor preserves the structure of category

It should be checked that functor preserves the structure of category. Comparing with the definition of category, functor has preserved composition and identity. So, we just have to prove that the axioms of category are also preserved.

Directly, for each $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in C, we have

$$
\begin{aligned}
& (F(f) \circ F(g)) \circ F(h) \\
\{\text{composition of } F\} = {} & F(f \circ g) \circ F(h) \\
= {} & F((f \circ g) \circ h) \\
\{\text{associativity of } \mathsf{C}\} = {} & F(f \circ (g \circ h)) \\
\{\text{composition of } F\} = {} & F(f) \circ F(g \circ h) \\
= {} & F(f) \circ (F(g) \circ F(h)),
\end{aligned}
$$

so the associativity axiom is preserved. And, for each $f: A \to B$ in C, since

$$
\begin{aligned}
& F(f) \circ F(1_A) \\
\{\text{identity of } F\} = {} & F(f) \circ 1_{F(A)} \\
\{\text{identity of } \mathsf{D}\} = {} & 1_{F(B)} \circ F(f) \\
\{\text{identity of } F\} = {} & F(1_B) \circ F(f),
\end{aligned}
$$

the identity axiom is preserved.

2.3.4 Surjective functor may not be full

Since functor preserves the structure of category, the image of a functor should be a subcategory of the codomain of the functor. Explicitly, given a functor $F: \mathsf{C} \to \mathsf{D}$, the image $F(\mathsf{C})$ is a subcategory of D. Naturally, we can consider if the subcategory $F(\mathsf{C})$ is D itself. Naturally, you may think that $F(\mathsf{C}) = \mathsf{D}$ if for each object $A' \in \mathsf{D}$, there exists an object $A \in \mathsf{C}$ such that $F(A) = A'$. This, however, is not true.

A functor is not a map. In fact, it has two maps as components: the map on objects and that on morphisms. So, $F(\mathsf{C})$ can still be a proper subcategory of D even though the map $F: \mathsf{C} \to \mathsf{D}$ on objects is surjective, as long as for some objects $A, B \in \mathsf{C}$, the map $F: \mathsf{C}(A, B) \to \mathsf{D}(F(A), F(B))$ is not surjective. So, for functor, we have to distinguish two kinds of surjection. One is that the map $F: \mathsf{C} \to \mathsf{D}$ on objects is surjective, for which we say F is **surjective on objects**. The other is that, for every $A, B \in \mathsf{C}$, the map $F: \mathsf{C}(A, B) \to \mathsf{D}(F(A), F(B))$ is surjective, for which we say F is **full**.

Surjection on objects and fullness are two independent properties of functor. We may find a functor that is not surjective on objects, but is full. That is, there is object in D that is out of the image of F, but for every two objects in the image of F, say $F(A)$ and $F(B)$, the map on morphisms $F\colon \mathsf{C}(A,B) \to \mathsf{D}(F(A),F(B))$ is always surjective.

2.3.5 Injective functor may not be faithful

The same discussion applies for the injection of functor. Again, you may think that $F(\mathsf{C})$ can still be multiple-to-one correspondent to C even though the map $F\colon \mathsf{C} \to \mathsf{D}$ on objects is injective, as long as for some objects $A, B \in \mathsf{C}$, the map $F\colon \mathsf{C}(A,B) \to \mathsf{D}(F(A),F(B))$ is not injective. So, again, there are two kinds of injection. One is that map $F\colon \mathsf{C} \to \mathsf{D}$ on objects is injective, for which we say F is **injective on objects**. The other is that, for every $A, B \in \mathsf{C}$, the map $F\colon \mathsf{C}(A,B) \to \mathsf{D}(F(A),F(B))$ is injective, for which we say F is **faithful**.

For the same reason, injection on objects and faithfulness are two independent properties of functor.

2.3.6 Fully faithful functor preserves isomorphisms

Fully faithful functor is found to be important because it preserves isomorphisms. To be clear, recall that notation $X \cong Y$ means there is an isomorphism between objects X and Y, we have the following lemma.

Lemma 2.22. *[Fully Faithful Functor] Given a fully faithful functor $F\colon \mathsf{C} \to \mathsf{D}$ and some $X, Y \in \mathsf{C}$, we have*

$$X \cong Y \Leftrightarrow F(X) \cong F(Y).$$

Proof. First, we are to prove that $X \cong Y \Rightarrow F(X) \cong F(Y)$. The $X \cong Y$ means there exists $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ such that $\beta \circ \alpha = 1_X$ and $\alpha \circ \beta = 1_Y$. By the composition axiom, we have $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$. And by the identity axiom, we have $F(1_X) = 1_{F(X)}$. So, we have $F(\beta) \circ F(\alpha) = 1_{F(X)}$. The same, we get $F(\alpha) \circ F(\beta) = 1_{F(Y)}$. This simply means $F(X) \cong F(Y)$.

Then, we are to prove that $F(X) \cong F(Y) \Rightarrow X \cong Y$. The $F(X) \cong F(Y)$ means there exists ω: $F(X) \to F(Y)$ and $\zeta\colon F(Y) \to F(X)$ such that $\zeta \circ \omega = 1_{F(X)}$ and $\omega \circ \zeta = 1_{F(Y)}$. Since F is full, we must have $\alpha\colon X \to Y$ and $\beta\colon Y \to X$ such that $\omega = F(\alpha)$ and $\zeta = F(\beta)$. By the composition axiom, we have $\zeta \circ \omega = F(\beta) \circ F(\alpha) = F(\beta \circ \alpha)$. By the identity axiom, $1_{F(X)} = F(1_X)$. So, we have $F(\beta \circ \alpha) = F(1_X)$. Since F is faithful, this implies $\beta \circ \alpha = 1_X$. The same, we get $\alpha \circ \beta = 1_Y$. This simply means $X \cong Y$. □

2.3.7 Image of functor may not be a category

The image of a group homomorphism is a subgroup of the codomain. But, this is not generally true for category. (Remind that group is the category with single object.)

What is the problem? Since functor preserves the structure of category, the composition, identity, and the axioms of associativity and identity of category are automatically fulfilled. All left is the closure of composition.

Consider a functor $F\colon \mathsf{C} \to \mathsf{D}$, the generic pattern to be examined will be $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ in the image $F(\mathsf{C})$. The problem is if $F(g) \circ F(f)$ still in the $F(\mathsf{C})$? By the axiom of composition, $F(g) \circ F(f) = F(g \circ f)$. So, that $F(g) \circ F(f)$ is in $F(\mathsf{C})$ is equivalent to that $g \circ f$ is in C. The later is only possible when the domain of g is exactly the codomain of f, but this may not be true. For instance, let $A \xrightarrow{f} B$ and $B' \xrightarrow{g} C$ with $F(B) = F(B')$, then $g \circ f$ does not exist at all, neither does $F(g \circ f)$. So, $F(g) \circ F(f)$ is not in $F(\mathsf{C})$, and $F(\mathsf{C})$ is not a category.

Why is it so? The key point in this analysis is $F(B) = F(B')$. It is this construction that destroyed the $g \circ f$. In other words, F is not injective on objects.

Contrarily, when F does be injective on objects, then there must be a unique B that is mapped to $F(B)$. In this case, there is no choice but $A \xrightarrow{f} B \xrightarrow{g} C$. This means $g \circ f$ exists in C, so does the $F(g \circ f)$, or $F(g) \circ F(f)$, in $F(\mathsf{C})$. So, the closure of $F(\mathsf{C})$ is satisfied. Now, $F(\mathsf{C})$ becomes a category.

We conclude this section by a theorem.

Theorem 2.23. *[Functorial Image as Category] Given a functor $F\colon \mathsf{C} \to \mathsf{D}$, the image $F(\mathsf{C})$ is a category if and only if F is injective on objects.*

In the case of group, there is only one object, so a functor, or group homomorphism, is always injective on objects.

2.4 Equivalences

2.4.1 Isomorphic objects should be viewed as one

Isomorphic topological spaces are the same. So it is for the isomorphic groups, isomorphic vector spaces, and so on. Generally, given an object in a category, we can construct many objects that are isomorphic to each other just by copying the object. In this way, we can extend a category to a larger one. But, we shall view the extended category as the same as the original, since this extension does not add new information to the original category. This means we shall view isomorphic objects as the same object. Regarding to morphisms, consider isomorphic objects X and Y, and another object Z of category C. If X and Y are viewed as one, then there will be bijections between $\mathsf{C}(X, Z)$ and $\mathsf{C}(Y, Z)$, and between $\mathsf{C}(Z, X)$ and $\mathsf{C}(Z, Y)$.

If visualizing a category as diagrams of dots and arrows between dots, then we shall pinch two isomorphic objects together. This leads to equivalent, but simplified, diagrams. The category obtained by pinching isomorphic objects as one in category C is called the **skeleton** of C.

2.4.2 Natural isomorphism describes equivalence between categories

Given two categories C and D, how can we say they are equivalent? A natural possibility is using identity functor, which maps identically on object and morphism. Precisely, there exist functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$, such that $G \circ F = 1_{\mathsf{C}}$ and $F \circ G = 1_{\mathsf{D}}$, where 1_{C} and 1_{D} are identity functors. Even though this definition is quite natural, however, it is not true. The reason is that there exist isomorphic objects. For instance, if $(G \circ F)(A) = A'$, which is not identical but isomorphic to A, then the categories can still be equivalent. This reflects our previous idea that isomorphic objects shall be viewed as one. So, instead of $(G \circ F)(A) = A$, as $G \circ F = 1_{\mathsf{C}}$ indicates, we shall demand $(G \circ F)(A) \cong A$ (recall that the $\cong$ between objects means that there is an isomorphism between them in the category they belong to).

We have discussed about how $G \circ F$ acts on objects. Since F and G are functors, $G \circ F$ also acts on morphisms. For any $A, B \in \mathsf{C}$, we have $(G \circ F)(A) = A' \cong A$ and $(G \circ F)(B) = B' \cong B$. Then for any $f\colon A \to B$, it is mapped to $(G \circ F)(f)\colon A' \to B'$. Since A and A' are viewed as one, so are B and B', there has to be a bijection between $\mathsf{C}(A, B)$ and $\mathsf{C}(A', B')$, since they should be viewed as one too. To find out this bijection, we first draw the commutative diagram in figure 2.3. If this diagram commutes, then we have $(G \circ F)(f) = \alpha_B \circ f \circ \alpha_A^{-1}$. So, the map

$$\varphi(f) := \alpha_B \circ f \circ \alpha_A^{-1}$$

takes f to $(G \circ F)(f)$. We are to show that φ is bijective between $\mathsf{C}(A, B)$ and $\mathsf{C}(A', B')$. To do this, we have to prove that φ is one-to-one (injective) and onto (surjective). For injection, suppose $\varphi(f) = \varphi(f')$, then $\alpha_B \circ f \circ \alpha_A^{-1} = \alpha_B \circ f' \circ \alpha_A^{-1}$, which implies $f = f'$. Then for surjection, let g: $A' \to B'$, we construct $f := \alpha_B^{-1} \circ g \circ \alpha_A$, which has $\varphi(f) = \alpha_B \circ (\alpha_B^{-1} \circ g \circ \alpha_A) \circ \alpha_A^{-1} = g$. So, we have shown that φ is bijective.

We summarize this section in the following definition.

Definition 2.24. *[Equivalent Categories] Categories* C *and* D *are said* ***equivalent*** *if there are functors* $F\colon\mathsf{C}\!\longrightarrow\!\mathsf{D}$ *and* $G\colon\mathsf{D}\!\longrightarrow\!\mathsf{C}$ *as well as a set of isomorphisms* $\alpha:=\{\alpha_X:|\forall X\in\mathsf{C}\}$ *in* C*, such that figure 2.3 commutes.*

$$\begin{tikzcd}
A \arrow[r, "\alpha_A"] \arrow[d, "f"] & (G\circ F)(A) \arrow[d, "(G\circ F)(f)"] \\
B \arrow[r, "\alpha_B"] & (G\circ F)(B)
\end{tikzcd}$$

Figure 2.3. In this figure, the α_A and α_B are isomorphisms.

The set α is called a natural isomorphism. Because isomorphism is "self-dual" (being invariant by duality), we can equivalently define natural isomorphism by "*... as well as a set of isomorphisms* $\beta:=\{\beta_X|\forall X\in\mathsf{D}\}$ *in* D*, such that figure 2.4 commutes, where the set* β *is called a natural isomorphism.*" Thus, natural isomorphism is "self-dual" too.

$$\begin{tikzcd}
A \arrow[r, "\beta_A"] \arrow[d, "f"] & (F\circ G)(A) \arrow[d, "(F\circ G)(f)"] \\
B \arrow[r, "\beta_B"] & (F\circ G)(B)
\end{tikzcd}$$

Figure 2.4. In this figure, the β_A and β_B are isomorphisms.

Not only the equivalence of categories can natural isomorphism be used for, but also the equivalence of functors. Given two functors $F\colon\mathsf{C}\!\longrightarrow\!\mathsf{D}$ and $G\colon\mathsf{C}\!\longrightarrow\!\mathsf{D}$, they are equivalent if there is a set, or natural isomorphism, $\{\alpha_X|\forall X\in\mathsf{D}\}$ such that figure 2.5 commutes. We find figure 2.5 is much generic than figure 2.3. Indeed, in figure 2.5, if we replace F by identity functor and G by $(G\circ F)$, we get figure 2.3 (the same for figure 2.4). So, we shall use figure 2.5 as the definition of natural isomorphism.

Definition 2.25. *[Natural Isomorphism] Given two functors* $F\colon\mathsf{C}\!\longrightarrow\!\mathsf{D}$ *and* $G\colon\mathsf{C}\!\longrightarrow\!\mathsf{D}$*, a* ***natural isomorphism*** *between* F *and* G *is a set of isomorphisms* $\{\alpha_X|\forall X\in\mathsf{D}\}$ *such that figure 2.5 commutes. The element* α_X *is called a* ***component*** *of natural isomorphism. We also say "*$F(X)\cong G(X)$ ***natural in*** X*".*

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\alpha_B"] & G(B)
\end{tikzcd}$$

Figure 2.5. In this figure, the α_A and α_B are isomorphisms.

Remark 2.26. [Naturality] The word "nature" indicates figure 2.5. This is the natural way of extending the isomorphism of objects to the bijection of morphisms. Indeed, the bijection can be read out directly from figure 2.5 as

$$\psi\colon \mathsf{D}(F(A), F(B)) \to \mathsf{D}(G(A), G(B)) \text{ and } \psi(g) := \alpha_B \circ g \circ \alpha_A^{-1}.$$

2.5 Summary

2.5.1 Category theory is built by recursion

In this chapter we first defined category. This was the unique starting point; and all the left were built by recursion. When the category was defined, the object was quite abstract and generic. It can be anything. So, it can be category itself! This implied a category of categories. Therein, the morphism, or functor, was defined as the structure preserving map.

So, the basic conceptions, which are category and functor, were defined recursively.

2.5.2 Proof in category theory is easy

As you might noticed, the proof in category theory is almost nothing but expanding definitions. Once you have clearly realized what the concepts mean, you get the proof.

Why is proof in category theory so easy? An educated guess is that category theory is quite fundamental. In analysis, almost for every critical concept, there are a plenty of lemmas, theorems, and corollaries related to this concept. This is because analysis is not fundamental, and is supported by other mathematical areas, such as set theory, topology, and linear algebra (for higher dimension). So, to prove a theorem, there will be a large amount of combinations of the more fundamental lemmas, theorems, and corollaries. The proof, thus, cannot be generally easy. But, for category theory, there is no other mathematical area that supports, and the combination is quite limited. Even though most proofs in category theory are trivial, the theorems to be proven are generally quite far-reaching.

Chapter 3
Representation

3.1 Hom-Functor and Yoneda Lemma

3.1.1 Object equals to its relations with others and with itself

If you want to know someone, what should you do? You can talk with this person, but this is not enough. You shall also talk with everyone who knows this person, realizing their impressions about him. Only then can you say you have known this person. Namely, the information of an object (the person) is encoded in the morphisms (impressions) toward the object.

3.1.2 Morphisms with fixed codomain can be represented by hom-functor

Given the object, there will be many morphisms with this object as codomain (or domain). But, for the convenience of discussion, it will be better to use one morphism to represent them all. Precisely, consider a category C. For each $X \in \mathsf{C}$, we are to represent the collection $\{\mathsf{C}(Y, X) | \forall Y \in \mathsf{C}\}$ by a map $Y \to \mathsf{C}(Y, X)$. Say, a map $\mathsf{C}(-, X): Y \to \mathsf{C}(Y, X)$. In addition, we hope that this map preserves the structure of category, which is important when we are discussing category theory. That is, we are to define how the $\mathsf{C}(-, X)$ acts on morphisms of C, so that it can be a functor.

To figure this out, we have to claim the problem explicitly. We want to find a map from a morphism $f: Y \to Z$ in C to a map from set $\mathsf{C}(Y, X)$ to set $\mathsf{C}(Z, X)$. The later maps a morphism φ: $Y \to X$ to a morphism $\psi: Z \to X$. How can it be? By chaining the morphisms f and φ (these are all what we temporally have), we find it impossible. So, we conclude that there is no such functor map from C. One possibility to solve this problem is consider the dual of C, the category C^{op}, where the arrow of f is flipped to $f: Z \to Y$. Now, we find an arrow $\varphi \circ f: Z \to X$! By denoting $f^*(\varphi) := \varphi \circ f$, we have $\psi = f^*(\varphi)$. So, we guess that, for each morphism $f: Z \to Y$ in C^{op}, $\mathsf{C}(-, X)(f) := f^*$.

Remark 3.1. In the course of this reasoning, we find that making an educated guess in category theory is quite easy, since with the restriction of "types" (in programming language, a function f: $A \to B$ has types A and B), only a few possibilities are left to examine. So, we can quick reach the destination, no matter whether the ending is positive (constructed what we want) or not (found it impossible to construct). The types is extremely helpful in computer programming, so is it in category theory!

Next is to check if $\mathsf{C}(-, X)$ constructed as such is a functor. We need to check the composition and identity axioms of functor. Indeed, for each $C \xrightarrow{g} B \xrightarrow{f} A$ in C and each $\varphi \in \mathsf{C}(A, X)$,

$$
\begin{aligned}
& \mathsf{C}(-, X)(f \circ g)(\varphi) \\
\{\text{definition of } \mathsf{C}(-, X)\} = {} & (f \circ g)^*(\varphi) \\
\{\text{definition of } (f \circ g)^*\} = {} & \varphi \circ (f \circ g) \\
\{\text{associativity}\} = {} & (\varphi \circ f) \circ g \\
\{\text{definition of } f^*\} = {} & f^*(\varphi) \circ g \\
\{\text{definition of } g^*\} = {} & g^*(f^*(\varphi)) \\
\{\text{rewrite}\} = {} & (g^* \circ f^*)(\varphi) \\
\{\text{definition of } \mathsf{C}(-, X)\} = {} & [\mathsf{C}(-, X)(g) \circ \mathsf{C}(-, X)(f)](\varphi)
\end{aligned}
$$

so the composition axiom, $\mathsf{C}(-,X)(f\circ g)=\mathsf{C}(-,X)(g)\circ\mathsf{C}(-,X)(f)$, is satisfied. (Recall that the domain of $\mathsf{C}(-,X)$ is the dual category of C. So, as figure ? indicates, applying $\mathsf{C}(-,X)$ flips the direction of morphism, thus the direction of morphic composition.) And since

$$\begin{aligned} & \mathsf{C}(-,X)(1_A)(\varphi) \\ \{\text{definition of } \mathsf{C}(-,X)\} = {} & (1_A)^*(\varphi) \\ \{\text{definition of } (1_A)^*\} = {} & \varphi\circ 1_A \\ \{\text{identity}\} = {} & \varphi \\ \{\text{definition of identity}\} = {} & 1_{\mathsf{C}(A,X)}(\varphi), \end{aligned}$$

the identity axiom, $\mathsf{C}(-,X)(1_A)=1_{\mathsf{C}(A,X)}$ is satisfied. So, the $\mathsf{C}(-,X)\colon \mathsf{C}^{\mathrm{op}}\to\mathsf{Set}$ does be a functor, which is called the hom-functor of X.

Definition 3.2. *[Hom-Functor] Let* C *a category. For any object* $X\in\mathsf{C}$*, functor* $\mathsf{C}(-,X)\colon \mathsf{C}^{\mathrm{op}}\to\mathsf{Set}$ *is defined by*

- *for each* $Y\in\mathsf{C}$*,* $\mathsf{C}(-,X)(Y):=\mathsf{C}(Y,X)$*, and*
- *for each* $Y,Z\in\mathsf{C}$ *and each* $f\colon Z\to Y$*,* $\mathsf{C}(-,X)(f):=f^*$*, where* $f^*(\varphi):=\varphi\circ f$*.*

This $\mathsf{C}(-,X)$ *is called the* ***hom-functor*** *of* X *in* C*.*

$$\begin{tikzcd} Y \arrow[r] & C(Y,X) \arrow[d, "f^*"] \\ Z \arrow[u, "f"] \arrow[r] & C(Z,X) \end{tikzcd}$$

Figure 3.1. Indicates $\mathsf{C}(-,X)\colon \mathsf{C}^{\mathrm{op}}\to\mathsf{Set}$.

Remark 3.3. [Hom-] In the word "hom-functor", we would better to use "mor", which means "morphic", instead of "hom", which means "homomorphic". But, historically, mathematicians employed homomorphism for indicating morphism. So now, the morphic functor, like $\mathsf{C}(-,X)$, is named by homomorphic functor.

3.1.3 Yoneda lemma relates objects and hom-functors

Our aim is to study the relation between an object and its hom-functor, say between X and $\mathsf{C}(-,X)$. In category theory, object and functor locate in different level of concepts, hence we cannot say "the object X is equivalent to the functor $\mathsf{C}(-,X)$". What really interests us is the equivalence of things, or in the language of category theory, isomorphism. Namely, if $X\cong Y$ for some $Y\in\mathsf{C}$, then will $\mathsf{C}(-,X)\cong\mathsf{C}(-,Y)$? And conversely, if $\mathsf{C}(-,X)\cong\mathsf{C}(-,Y)$, then will $X\cong Y$? Recall that isomorphism between functors are natural isomorphism.

Suppose that $X\cong Y$, namely there is a isomorphism $f\colon X\to Y$. We are to show that there is a set $\eta=\{\eta_A|\forall A\in\mathsf{C}\}$ in which η_A is isomorphic and, for any $g\colon B\to A$ and any $\phi\colon A\to X$, figure 3.2 commutes (for naturality). Let $\eta_A(\varphi):=f\circ\varphi$ for any $A\in\mathsf{C}$ and $\varphi\colon A\to X$. We first show that η_A is isomorphic. Since f is isomorphic, we have $f^{-1}\colon Y\to X$ such that $f^{-1}\circ f=1_X$. Let $\eta_A^{-1}(\varphi):=f^{-1}\circ\varphi$, we then have $(\eta_A^{-1}\circ\eta_A)(\varphi)=f^{-1}\circ f\circ\varphi=\varphi=1_{\mathsf{C}(A,X)}(\varphi)$, namely $\eta_A^{-1}\circ\eta_A=1_{\mathsf{C}(A,X)}$ and η_A is isomorphic. Next, we prove that figure 3.2 commutes, namely $\eta_B(\varphi\circ g)=\eta_A(\varphi)\circ g$ for any $g\colon B\to A$. Directly, the left hand side is $f\circ(\varphi\circ g)$ and the right hand side is $(f\circ\varphi)\circ g$. By associtivity, they are equal. So, we have shown that $\mathsf{C}(-,X)\cong\mathsf{C}(-,Y)$. And for convenience, we denote $f_*(\varphi):=f\circ\varphi$, thus $\eta_A=f_*$.

$$\begin{tikzcd}
A & \phi : A \to X \arrow[r, mapsto, "\eta_A"] \arrow[d, mapsto, "g^*"] & \eta_A(\phi) : A \to Y \arrow[d, mapsto, "g^*"] \\
B \arrow[u, "g"] & \phi \circ g : B \to X \arrow[r, mapsto, "\eta_B"] & \eta_B(\phi \circ g) = \eta_A(\phi) \circ g : B \to Y
\end{tikzcd}$$

Figure 3.2. This internal diagram TODO

Conversely, suppose $\mathsf{C}(-,X) \cong \mathsf{C}(-,Y)$. That is, there is a set $\eta = \{\eta_A | \forall A \in \mathsf{C}\}$ in which η_A: $\mathsf{C}(A,X) \to \mathsf{C}(A,Y)$ is isomorphic and figure 3.3 commutes. We are to show that there is an isomorphism $f\colon X \to Y$ in C. Figure 3.3 says, for any $g\colon X \to Y$, that $\eta_X(\eta_Y^{-1}(1_Y) \circ g) = 1_Y \circ g$. Define $f := \eta_X(1_X)$, we have $\eta_X(\eta_Y^{-1}(1_Y) \circ f) = f$ Since η_X is bijection (namely, isomorphism) and $\eta_X(1_X) = f$, we have $\eta_Y^{-1}(1_Y) \circ f = 1_X$, thus $f^{-1} = \eta_Y^{-1}(1_Y)$. So, f is an isomorphism in C.

$$\begin{tikzcd}
X \arrow[d, "g"] & C(X,X) \arrow[r, "\eta_X"] & C(X,Y) \\
Y & C(Y,X) \arrow[r, "\eta_Y"] \arrow[u, "g^*"] & C(Y,Y) \arrow[u, "g^*"]
\end{tikzcd}$$

Figure 3.3. In this figure, η_X and η_Y are isomorphisms.

Now, given an isomorphism $f\colon X \to Y$, we can construct a natural isormorphism $\eta\colon \mathsf{C}(-,X) \to \mathsf{C}(-,Y)$ via $\eta_A := f_*$. Conversely, given a natural isormorphism $\eta\colon \mathsf{C}(-,X) \to \mathsf{C}(-,Y)$, we can construct an isomorphsim $f\colon X \to Y$ via $f := \eta_X(1_X)$. We have to prove that these two constructions are consistent. On one hand, by replacing $\eta_X = f_*$ in $f = \eta_X(1_X)$, we are going to check if $f = (f_*)(1_X)$. By the definition of f_*, we get $(f_*)(1_X) = f \circ 1_X = f$. On the other hand, by replacing $f = \eta_X(1_X)$ in $\eta_X = f_*$, we are going to check if $\eta_X = [\eta_X(1_X)]_*$. By applying on an arbitrary morphism $\varphi\colon X \to X$, we have $[\eta_X(1_X)]_*(\varphi) = \eta_X(1_X) \circ \varphi = \varphi^*(\eta_X(1_X))$. And by naturality (figure 3.4), $\varphi^*(\eta_X(1_X)) = \eta_X(\varphi^*(1_X)) = \eta_X(\varphi)$. Altogether, we find $[\eta_X(1_X)]_*(\varphi) = \eta_X(\varphi)$.

$$\begin{tikzcd}
X & 1_X : X \to X \arrow[r, mapsto, "\eta_X"] \arrow[d, mapsto, "\phi^*"] & \eta_X(1_X) : X \to Y \arrow[d, mapsto, "\phi^*"] \\
X \arrow[u, "\phi"] & \phi : X \to X \arrow[r, mapsto, "\eta_X"] & \eta_X(\phi) = \eta_X(1_X) \circ \phi : X \to Y
\end{tikzcd}$$

Figure 3.4. This internal diagram TODO

This isomorphic relationship between objects and hom-functors was first revealed by the Japanese mathematician Yoneda Nobuno. Interestingly, Yoneda is also a computer scientist, supported the programming language ALGOL. We summarize this section by the following lemma named by Yoneda.

Lemma 3.4. *[Yoneda] For any $X, Y \in \mathsf{C}$, we have*

$$X \cong Y \Leftrightarrow \mathsf{C}(-,X) \cong \mathsf{C}(-,Y).$$

And the isomorphism $f\colon X\to Y$ on the left hand side and the natural isomorphism $\eta\colon \mathsf{C}(-,X)\to\mathsf{C}(-,Y)$ on the right hand side have the (equivalent) correspondence

$$f=\eta_X(1_X) \quad \text{or} \quad \eta=f_*.$$

3.2 Universal Element

3.2.1 A presheaf is representable if there is universal element

The hom-functor $\mathsf{C}(-,X)$ maps from category C^{op} to category Set. It is natural to ask the inverse problem that, for any functor $F\colon\mathsf{C}^{\mathrm{op}}\to\mathsf{Set}$, whether or not there exists an object $\hat{F}\in\mathsf{C}$ such that F is the hom-functor of $\hat{F}$, namely $F\cong\mathsf{C}(-,\hat{F})$ (recall that isomorphic things should be viewed as one).

A functor of the "type" $\mathsf{C}^{\mathrm{op}}\to\mathsf{Set}$ is called a **presheaf** on C. If a presheaf can be written as the hom-functor of an object, then we say the presheaf is **representable**, or **represented** by the object. And the object is called a **representation** of the presheaf.

Suppose that there exists an object $\hat{F}\in\mathsf{C}$ and natural isomorphism $\eta\colon \mathsf{C}(-,\hat{F})\to F$, then figure 3.5 commutes. Interestingly, figure 3.5 also indicates that, instead of defining the e by $\eta_{\hat{F}}$ as $e:=\eta_{\hat{F}}(1_{\hat{F}})$, you can conversely define the natural transformation η by e! Indeed, its component $\eta_X\colon \mathsf{C}(X,\hat{F})\to F(X)$ can be defined as $\eta_X:=F(-)(e)$, since figure 3.5 implies $\eta_X(\varphi)=F(\varphi)(e)$ for any $\varphi\colon X\to\hat{F}$. The naturality of η is an immediate result of the functoriality of F, as figure 3.6 indicates. And that η_X is isomorphic demands that $F(-)(e)$ shall be isomorphic. The e, as an element in the set $F(\hat{F})$, is called **universal element**, because it is universal for defining all components of η.

$$\begin{tikzcd}
\hat{F} & 1_{\hat{F}} : \hat{F}\to\hat{F} \arrow[r, mapsto, "\eta_{\hat{F}}"] \arrow[d, mapsto, "\phi^*"] & e := \eta_{\hat{F}}(1_{\hat{F}}) \in F(\hat{F}) \arrow[d, mapsto, "F(\phi)"] \\
X \arrow[u, "\phi"] & \phi : X\to\hat{F} \arrow[r, mapsto, "\eta_X"] & \eta_X(\phi) = F(\phi)(e) \in F(X)
\end{tikzcd}$$

Figure 3.5. This internal diagram indicates how the e is defined by η, or conversely how the η is defined by e.

$$\begin{tikzcd}
X & \phi : X\to\hat{F} \arrow[r, "F(-)(e)"] \arrow[d, mapsto, "f^*"] & F(\phi)(e) \in F(X) \arrow[d, mapsto, "F(f)"] \\
Y \arrow[u, "f"] & \phi\circ f : Y\to\hat{F} \arrow[r, "F(-)(e)"] & F(\phi\circ f)(e) = [F(f)\circ F(\phi)](e) \in F(Y)
\end{tikzcd}$$

Figure 3.6. This internal diagram proves that the naturality of $F(-)(e)$ is an immediate result of the functoriality of F. It should be noticed that the domain of F is the dual category of C, so it should be $F(\varphi\circ f)=F(f)\circ F(\varphi)$.

We summarize this section by the following theorem.

Theorem 3.5. *[Universal Element] A presheaf $F\colon\mathsf{C}^{\mathrm{op}}\to\mathsf{Set}$ is representable if and only if there is an object $\hat{F}$ in C together with an element e in the set $F(\hat{F})$ such that, for any $X\in\mathsf{C}$, the function (morphism in Set) $F(-)(e)\colon\mathsf{C}(X,\hat{F})\to F(X)$ is bijective (isomorphic).*

3.2.2 Representation is unique up to isomorphism

Given a presheaf $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, suppose that it has two representations $\hat{F}$ and $\hat{F}'$, then we have

$$F \cong \mathsf{C}(-,\hat{F}) \ \text{ and } \ F \cong \mathsf{C}(-,\hat{F}'),$$

which means $\mathsf{C}(-,\hat{F}) \cong \mathsf{C}(-,\hat{F}')$. By Yoneda lemma (lemma 3.4), we have $\hat{F} \cong \hat{F}'$. Thus, we have the following theorem.

Theorem 3.6. *If a presheaf F is represented by both object A and object B, then $A \cong B$.*

It is in this sense that we say an object $\hat{F}$ is *the* representation of presheaf F.

3.3 Summary

3.3.1 Embedding in the framework of category theory is the right way to extend category theory

The Yoneda functor was given out by embedding the object and its hom-functor in the framework of category theory. The same goes for diagram and cone. So, in category theory, considering a concept in the framework of category is the right way to extend category theory.

3.3.2 "Types" help to restrict the possibility of construction

In programming languages, especially for the strong type languages, types are important. For instance, a function $f\colon A \to B$ has types A and B. The same goes for category theory.

While constructing the hom-functor and Yoneda functor, we find that "types" are extremely helpful for restricting the possibility of construction, almost make it unique. So, when construct something in category theory, types should be considered at the first place.

3.3.3 "Types" help check the correctness of derivation

As in the case of static programming language, "types" provides a direct way of checking the correctness of the final result after a long long derivation. It is simple but very efficient for finding errors. And it is free of charge in category theory.