Chapter 1
Motivation

This note is about the basic category theory. There have been many books on category theory, almost all of them contains many examples from multiple areas of mathematics. In this note, however, we focus on the pure aspects, trying to understand the core concepts of category theory in an intuitive way. Examples are shown only when it is essential.

Chapter 2

Category, Functor, and Natural Transformation

2.1 Category

2.1.1 Category is about Arrows

Category is the fundamental element of category theory. A category consists of arrows and objects, which are employed to declare arrows: where an arrow emits, and where it ends. Moreover, several properties relating to these components should be satisfied.

Strangely, in category, arrow is called morphism, a word derived from isomorphism. And isomorphism is constructed from iso-morphe-ism, where morphe, a Greek word, means shape or form. So, isomorphism means equal shape or form. This can be easily illustrated in topology, where the two isomorphic tologogical space share the same form (but may not the same shape). But semantically, this is far from what arrow should mean. So, the question is why mathematician use the word morphism for arrow.

Definition 2.1. *[Category] A* ***category*** C *consists of*

- *a collection of* ***objects****,* ob_{C}*,*
- *for each* $A, B \in \mathrm{ob}_{\mathsf{C}}$*, a collection of* ***morphisms*** *from* A *to* B*,* $\mathrm{mor}_{\mathsf{C}}(A, B)$*, where* A *is the* ***domain*** *and* B *the* ***codomain****,*
- *for each* $f \in \mathrm{mor}_{\mathsf{C}}(A, B)$ *and* $g \in \mathrm{mor}_{\mathsf{C}}(B, C)$*, a* ***composition*** *of* f *and* g *that furnishes an arrow* $g \circ f \in \mathrm{mor}_{\mathsf{C}}(A, C)$*, and*
- *for each* $A \in \mathrm{ob}_{\mathsf{C}}$*, an* ***identity*** $1_A \in \mathrm{mor}_{\mathsf{C}}(A, A)$*,*

such that the following axioms are satisified:

- ***associativity****: for each* $f \in \mathrm{mor}_{\mathsf{C}}(A, B)$*,* $g \in \mathrm{mor}_{\mathsf{C}}(B, C)$*, and* $h \in \mathrm{mor}_{\mathsf{C}}(C, D)$*, we have* $(h \circ g) \circ f = h \circ (g \circ f)$*, and*
- ***identity****: for each* $f \in \mathrm{mor}_{\mathsf{C}}(A, B)$*, we have* $f \circ 1_A = 1_B \circ f$*.*

But, this notation system is a little complicated. Usually, we simplify it by employing the notations from set theory.

Notation 2.2. *Given category* C*, we simplify the notation* $A \in \mathrm{ob}_{\mathsf{C}}$ *by* $A \in \mathsf{C}$*, and for each* A*,* $B \in \mathsf{C}$*, denote* $f \in \mathrm{mor}_{\mathsf{C}}(A, B)$ *by* $f\colon A \to B$ *or* $A \xrightarrow{f} B$*.*

Now, category becomes much more familiar to us. We can think the objects of C as sets, and morphism as function, which is the map between sets. Indeed, the collection of functions does form a category: the category of sets.

Definition 2.3. *[Category of Sets] The* ***category of sets****, denoted as* Set*, has the collection of all sets as its objects, and for each* $A, B \in \mathsf{Set}$*, the collection of all functions from* A *to* B *as its morphisms from* A *to* B*.*

It is easy to check that the axioms are satisfied. But, be caution! Set is just a specific category, it helps us understanding what a category might look like. But, Set has much more axioms, or restrictions, than the category itself, thus may blind us to the potential power of arrows.

Indeed, there exists categories whose objects are not sets, or whose morphisms are not maps. So, a better way of thinking category is keeping objects and morphisms abstract. You can think objects as colored dots and morphisms as arrows between the colored dots.

2.1.2 Objects may not be Sets

We know that the symmetry group of rectangle is dihedral group D_2. The group elements are operations: identity, rotation of 180 degrees, and reflections along vertical and horizental directions. The operand is unique: the rectangle. These operations can be viewed as arrows from the rectangle to itself. So, this symmetry group describes a category, called $\mathsf{B}D_2$. The axioms of category are satisfied because of the group properties. In this category, object is not a set, but an rectangle.

In fact, all groups are examples illustrating that objects may not be sets. Here, we need the definition of isomorphism in category.

Definition 2.4. *[Isomorphism in Category] In a category C, for any objects $A, B \in \mathsf{C}$, a morphism $f: A \to B$ is an **isomorphism** if there exists $g: B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Denote $A \cong B$ if A and B are isomorphic.*

With the aid of isomorphism, we definition the groupid.

Definition 2.5. *[Groupid] A **groupid** is a category in which all morphisms are isomorphisms.*

Now, we come to the big step: define the group! But, wait a minute. We have learned abstract algebra and known what a group is. The point here is that category theory provide a new, but equivalent, way of defining group, using arrows!

Definition 2.6. *[Group as Category] A group is a groupid in which there is only one object.*

Notation 2.7. *Because there is only one object, we can simplify the notation of morphism like $f: A \to B$ to f, and denote $f \in \mathsf{G}$ for that f is a morphism of a group G.*

If we start at defining group by arrows, we have to declare that the properties (axioms) of group studied in algebra are satisfied.

Theorem 2.8. *Let G a group, then we have*

- ***associativity**: for $\forall f, g, h \in \mathsf{G}$, $(f \circ g) \circ h = f \circ (g \circ h)$,*
- ***identity element**: there exists $1 \in \mathsf{G}$ such that for $\forall f \in \mathsf{G}$, $f \circ 1 = 1 \circ f = f$.*
- ***inverse element**: for $\forall f \in \mathsf{G}$, there exists $g \in \mathsf{G}$ such that $f \circ g = g \circ f = 1$.*

Proof. The associativity of group is identified as the associativity of category. The same for identity element. The inverse element comes from the fact that all morphisms are isomorphisms. □

So, a group is a category. And as compared with the group defined in abstract algebra, we find that the unique object in this category is not set, and that discussing what the object should be is meaningless.

2.1.3 Morphisms may not be Maps

To illustrate that morphisms may not be maps, we need to define preorder and poset.

Definition 2.9. *[Preorder] Given a set S, a preorder P on S is a subset of $S \times S$ such that the following axioms are satisfied:*

- ***reflexivity**: for $\forall a \in S$, $(a, a) \in P$, and*
- ***tansitivity** for $\forall a, b, c \in S$, if $(a, b) \in P$ and $(b, c) \in P$, then $(a, c) \in P$.*

For example, "no greater than" is a preorder, where S is the set of real numbers and $(a,b)\in P$ means $a\leqslant b$. "Is subset of" is another example of preorder, where S is the set of sets and $(a,b)\in P$ means $a\subseteq b$.

Definition 2.10. *[Poset] A preordered set, or* ***poset****, (S,P) is a set S equiped with a preorder P on S.*

With these prelimiaries, we claim that a poset is a category.

Definition 2.11. *[Poset as Category] Given poset (S,P), a category* Poset *can be constructed by regarding the elements in S as objects and regarding $(a,b)\in P$ as $a\to b$.*

Because of the axioms of preorder, the axioms of category are satisfied. The category Poset illustrates that morphisms are not maps. In Poset, morphisms are "no greater than"s or "is subset of"s.

2.1.4 Isomorphic Objects should be Viewed as One

Isomorphic topological spaces are the same. The same is for the isomorphic groups, isomorphic vector spaces, and so on. So, we should view isomorphic objects are one object. If visualizing a category as diagrams of dots and arrows between dots, then we should pinch two isomorphic objects together. This leads to equivalent, but simplified, diagrams. The category obtained by pinching isomorphic objects as one in category C is called the **skeleton** of C.

2.1.5 Supremum and Infimum are Dual

Arrows can represent many mathematical objects. For example, in Poset with set $\mathbb{R}$ and preorder $\leqslant$, we can describe supremum as follow.

Definition 2.12. *[Supremum in Category] Given a subset $A\subset\mathbb{R}$. An $x\in\mathbb{R}$ is the* ***supermum*** *of A if it satisfies:*

- *for $\forall a\in A$, $a\to x$ and,*
- *for $\forall y\in\mathbb{R}$ and $\forall a\in A$, if $y\to a$, then $y\to x$.*

This is, again, a weired definition on supremum. But, if you check carefully, you can see that this definition is equivalent to that studied in analysis. Also, we can define the infimum in the same fashion.

Definition 2.13. *[Infimum in Category] Given a subset $A\subset\mathbb{R}$. An $x\in\mathbb{R}$ is the* ***infimum*** *of A if it satisfies:*

- *for $\forall a\in A$, $x\to a$ and,*
- *for $\forall y\in\mathbb{R}$ and $\forall a\in A$, if $a\to y$, then $x\to y$.*

Weired again, but now you may have been familiar with the weired. Hint: the word weired also has the meaning of fate. Indeed, you are on the load to wonderland. By comparing the definition of infimum to that of supremum, we find all statements are the same except that we replaced supremum by infimum and domain by codomain (for instance, replaced $y\to x$ by $x\to y$). Two statements are **dual** if you can get one from the other by simply flipping all the arrows, or, equivalently, by exchanging the domain and codomain for each morphism in the statement. We say that supremum and infimum are dual.

2.1.6 Morphisms in the Dual Category of Set are Not Maps

There are also dual categories. Given a category C, its **dual category**, denoted by C^{op}, is obtained from C by exchanging the domain and codomain for each morphism in C.

So, in the dual category of Set, i.e. $\mathsf{Set}^{\mathrm{op}}$, we find that arrows are not functions, not even maps! Yet another example whose morphisms are not maps.

2.1.7 Arrows Generalize Concepts and Theorems from One Area to Every Area in Mathemtatics

Why category theory? Or say, why arrows? One benefit of re-claim everything in arrows is the ability of generalizing a concept in one area to area domain in mathematics. An example comes from generalizing the Cartesian product, also called direct product, in the set theory.

You have been familiar with the direct product of two sets. Given two sets A and B, recall that the direct product $A \times B := \{(a, b) | a \in A, b \in B\}$. Again, for generalizing the concepts using category theory, we have to re-write the concepts using arrows. And again, this re-writing looks weired at the first sight.

Definition 2.14. *[Direct Product of Two Objects] Given a category* C*. For any* $A, B \in \mathsf{C}$*, the* ***direct product*** *of* A *and* B *is another object* $C \in \mathsf{C}$ *together with two morphisms* $\alpha: C \to A$ *and* $\beta: C \to B$ *such that, for any* $C' \in \mathsf{C}$*, any* $\alpha': C' \to A$ *and* $\beta': C' \to B$*, there exists an unique morphism* $\gamma: C' \to C$ *so that* $\alpha' = \alpha \circ \gamma$ *and* $\beta' = \beta \circ \gamma$*.*

So, a direct product of objects A and B and a triplet (C, α, β). Applying to Set, as you can check directly, it goes back to the Cartesian product of two sets. We can also apply it to Grp, which furnishes the group direct product:

Definition 2.15. *[Group Direct Product] Given two groups* G *and* H*, the* ***group direct product*** *of* G *and* H *is defined as* $\{(g, h) | g \in G, h \in H\}$ *equipped with group multiplication* $(g, h) \times (g', h') := (g \circ g', h \cdot h')$ *where* $\circ$ *is the muliplication of* G *and* $\cdot$ *of* H*.*

It is like the Cartesian product, but extra structure are implied.

Also, all specific categories would be benefited from a theorem claimed in category theory. Such as the uniqueness of direct product in the sense of isomorphism.

Theorem 2.16. *[Uniqueness of Direct Product] Given a category* C*. For any* $A, B \in \mathsf{C}$ *and any two direct products* (C, α, β) *and* (C', α', β')*. Then, there exists an unique isomorphism* $\gamma: C' \to C$ *such that* $\alpha' = \alpha \circ \gamma$ *and* $\beta' = \beta \circ \gamma$*.*

That is, C and C', α and α', β and β' are equivalent in the sense of isomorphism.

Proof. □

This theorem holds not only for Cartesian product of sets, but also, for instance, for the group direct product.

2.1.8 Duality is Free Lunch

Another benefit of viewing everything in arrows is duality. In category theory, it is natural to think what would happen if we exchange domain and codomain for all the arrows. Just like the relation between supremum and infimum, it is natural to ask what if we exchange domain and codomain for all the arrows in the definition of direct product. This furnishes a new concept we called direct sum.

Definition 2.17. *[Direct Sum of Two Objects] Given a category* C*. For any* $A, B \in \mathsf{C}$*, the* ***direct product*** *of* A *and* B *is another object* $C \in \mathsf{C}$ *together with two morphisms* $\alpha: A \to C$ *and* $\beta: B \to C$ *such that, for any* $C' \in \mathsf{C}$*, any* $\alpha': A \to C'$ *and* $\beta': B \to C'$*, there exists an unique morphism* $\gamma: C \to C'$ *so that* $\alpha' = \gamma \circ \alpha$ *and* $\beta' = \gamma \circ \beta$*.*

Again, a direct sum of objects A and B and a triplet (C, α, β). Comparing with direct product, direct sum is nothing but exchanging domain and codomain for all the arrows in the statement of direct product.

Applying to Set, as it can be directly checked, we get the disjoint union of two sets. Given two sets A and B, recall that the disjoint union $A \cup_d B := \{(a, 1) | a \in A\} \cup \{(b, 2) | b \in B\}$. This is a surprise, since, unlike the duality between supremum and infimum, Cartesian product and disjoint union do not look like a pair at the first sight!

Recall the theorem that direct product is unique in the sense of isomorphism. If we also exchange domain and codomain for all the arrows in the statement of the theorem, as well as in the statement of its proof, then we get another theorem: direct sum is unique in the sense of isomorphism, without re-do the proof!

As a summary, the duality in category theory furnishes free lunch, which include not only the dual concepts that are very generic, but also the dual theorems that need no proof. All about is flipping arrows.

2.2 Functor

2.2.1 Functor is the Morphism of the Category of Categories

We need some examples of category to introduce the next core concept of category theory: functor. The first example is the category of topological spaces.

Definition 2.18. *[Category of Topological Spaces] The **category of topological spaces**, denoted as* Top*, has the collection of all topological spaces as its objects, and for each* $A, B \in \mathsf{Top}$*, the collection of all continuous maps from* A *to* B *as its morphisms from* A *to* B*.*

The next example is the categroy of groups.

Definition 2.19. *[Category of Groups] The **category of groups**, denoted as* Grp*, has the collection of all groups as its objects, and for each* $A, B \in \mathsf{Top}$*, the collection of all homomorphisms from* A *to* B *as its morphisms from* A *to* B*.*

From these two examples, we find an almost free method to construct a category out of objects. That is, a method to assign the morphisms. This method employs the maps that preserve the structure of object as the morphisms. For example, in Top, the preserved structure is continuity, and in Grp, it is the group structure.

Notice that the objects of a category can be anything. So, it can also be categories! To construct a category out of categories, the morphisms between two categories can be the maps that preserve the structure of category. These structure preserving maps in the category of categories are functors.

Definition 2.20. *[Functor] Given two categories* C *and* D*, a **functor*** $F\colon \mathsf{C} \to \mathsf{D}$ *maps*

- *for each* $A \in \mathsf{C}$*,* $F(A) \in \mathsf{D}$*, and*
- *for each* $A, B \in \mathsf{C}$ *and each* $f\colon A \to B$*,* $F(f)\colon F(A) \to F(B)$*,*

such that the structure of category is preserved, that is

- ***composition**: for* $\forall A, B, C \in \mathsf{C}$ *and* $f\colon A \to B, g\colon B \to C$*,* $F(f \circ g) = F(f) \circ F(g)$*,*
- ***identity**: for* $\forall A \in \mathsf{C}$*,* $F(1_A) = 1_{F(A)}$*.*

Imagine a category as a series of diagrams with colored dots and arrows between dots. The apply of a functor changes the shape of the dots and arrows, for instance from simple dots to stars, or from straight arrows to curved arrows. This means it has become another category, but the structure, or form, of the diagrams are invariant.

2.3 Natural Transformation

2.3.1 Natural Transformation is the Morphism of the Category of Functors

As we have defined category, and as we have built a category out of categories by defining functor, we can also build a category out of functors by defining natural transformation. Precisely, given two categories C and D, a category of functors from C to D, denoted by $[\mathsf{C}, \mathsf{D}]$, has the collection of all functors from C to D as its objects.

But, given two functor $F, G\colon \mathsf{C} \to \mathsf{D}$, what is the morphism between F and G? Remember Top and Grp, in which we constructed the category out of the objects; morphism between two objects are structure preserving map. The same goes for the category of functors, in which we know what the objects are and have to define the morphisms. The structure preserving map is natural transformation.

Definition 2.21. *[Natural Transformation] Given two categories* C *and* D*, and two functors* F*,* $G\colon \mathsf{C} \to \mathsf{D}$*, a* ***natural transformation*** $\alpha\colon F \to G$ *is a family of morphisms in* D*,* $\{\alpha_A\colon F(A) \to G(A) | \forall A \in \mathsf{C}\}$*, such that for each* $A, B \in \mathsf{C}$ *and each* $f\colon A \to B$*, we have* $G(f) \circ \alpha_A = \alpha_B \circ F(f)$*. The* α_A *is called a* ***component*** *of* α*.*

Remember the metaphor for functor, wherein the action of a functor is like applying a style to the dots and arrows of the diagrams, the action of α is like changing the style of dots and arrows from style F to style G. So, with natural transformation, the structure of functor is preserved.

2.3.2 Natural Isomorphism is Equivalent to Isomorphisms of Category.

With the category of functors, we can discuss whether two functors are equivalent or not. This relates to the isomorphism between functors. Since a morphisim in this category is called a natural transformation, an isomorphism is called a natural isomorphism. Given the general definition of isomorphism, a natural transformation $\alpha\colon F \to G$ is **natural isomorphic** between functors F, G: $\mathsf{C} \to \mathsf{D}$, if there exists a natural transformation $\beta\colon G \to F$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$. As usual, if natural isomorphism exists between F and G, then denote $F \cong G$.

This definition is quite complicated, since it is an isomorphism on $[\mathsf{C}, \mathsf{D}]$, which we are not familiar with. But, because α is a family of morphisms on category D, we can first consider a much simplier case, that is, $\alpha_A\colon F(A) \to G(A)$ is isomorphic. This isomorphism is of category D, instead of $[\mathsf{C}, \mathsf{D}]$. So, we guess, or hope, that, if for each $A \in \mathsf{C}$, there exists a morphism on D, β_A: $G(A) \to F(A)$, such that $\alpha_A \circ \beta_A = 1_{G(A)}$ and $\beta_A \circ a_A = 1_{F(A)}$, then the family of β_A might be the correct natural transformation β we need.

Lemma 2.22. *[Natural Isomorphism] A natural transformation* α *is a natural isomorphism if and only if each component of* α *is an isomorphism.*

Proof. The relation $\beta \circ \alpha = 1_F$ means that $\beta \circ \alpha$ acts as 1_F. What does the natural transformation 1_F acts? For each $A \in \mathsf{C}$, $(1_F)_A = 1_{F(A)}$; and the relation $F(f) \circ (1_F)_A = (1_F)_A \circ F(f)$ should hold. But, if $(1_F)_A = 1_{F(A)}$, then the relation becomes $F(f) \circ 1_{F(A)} = 1_{F(B)} \circ F(f)$, which is fulfilled on its own. So, the relation $\beta \circ \alpha = 1_F$ simply means, for each $A \in \mathsf{C}$, $\beta_A \circ \alpha_A = 1_{F(A)}$. The same statement holds for $\alpha \circ \beta = 1_G$, that is, $\alpha_A \circ \beta_A = 1_{G(A)}$. So, we find the conclusion that α is a natural isomorphism on $[\mathsf{C}, \mathsf{D}]$ if and only if for each $A \in \mathsf{C}$, α_A is an isomorphism on D. □

With this lemma, a isomorphism on $[\mathsf{C}, \mathsf{D}]$ now reduces to a family of isomorphisms on D, which is quite familiar to us.

2.3.3 Natural Isomorphism Describes Equivalence between Categories

Given two categories C and D, how can we say they are equivalent? A natural possibility is using isomorphic functor. Precisely, there exist functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$, such that $G \circ F = 1_{\mathsf{C}}$ and $F \circ G = 1_{\mathsf{D}}$. Even though this definition is quite natural, however, it is not true. The reason is that there exist isomorphic objects. For instance, if $G \circ F(A) = B$, which is not equal, but isomorphic, to A, then the categories can still be equivalent. This reflects our previous idea that isomorphic objects should be pinched together as one. So, instead of $G \circ F(A) = A$, as $G \circ F = 1_{\mathsf{C}}$ indicates, we should say $G \circ F(A) \cong A$. By lemma 2.22, $G \circ F \cong 1_{\mathsf{C}}$ means, for each $A \in \mathsf{C}$, there exists an isomorphism $\alpha_A\colon (G \circ F)(A) \to A$, that is $(G \circ F)(A) \cong A$. This implies that, instead of $G \circ F = 1_{\mathsf{C}}$, the correct condition for equivalence should be $G \circ F \cong 1_{\mathsf{C}}$. The same, $F \circ G \cong 1_{\mathsf{D}}$.

Definition 2.23. *[Equivalent Categories] Categories* C *and* D *are* ***equivalent*** *if there exist functors* $F\colon \mathsf{C} \to \mathsf{D}$ *and* $G\colon \mathsf{D} \to \mathsf{C}$ *such that* $G \circ F \cong 1_{\mathsf{C}}$ *and* $F \circ G \cong 1_{\mathsf{D}}$*.*

Historically, functor is defined for describing natural transformation, and natural transformation, or natural isomorphism, is defined for describing equivalence between categories.