1 Generalized Gaussian Integral

1.1 Introduction

In this note, we explore the Gaussian integral in complex space.

To define and evaluate Gaussian integral in complex space, we first consider the one-dimensional case in section 1.2, And then generalize the result to multi-dimensional case in section 1.4. To check the result, we reduce it to real Gaussian integral in section 1.5.

Importantly, in section 1.3, 1.6, and 1.7, we find that the result of complex Gaussian integral is invariant for unitary transformation. This enable us to express it in Dirac's representation theory, in section 1.8. This expression is abstract. Explicit expression can be obtained by simply inserting a complete relation. It can be seen as a generalization of Gaussian integral.

1.2 One-Dimensional Complex Gaussian Integral

In complex plane, we have to ensure that, the measurement is real, so is the integrand. So, the only way of writing a complex Gaussian-like integral is

$$\int_{\mathbb{C}} \mathrm{d}z \mathrm{d}\bar{z} \exp(-\bar{z}\,A\,z + \bar{b}\,z + b\,\bar{z}),$$

for a real positive number A and a complex number b. Next, we are to integrate this by first converting it to real plane. To do this, let $z = x + \mathrm{i}\,y$ and $b = p + \mathrm{i}\,q$. Recall that A is itself real.

First, we deal with the measurement. It is $\mathrm{d}z \wedge \mathrm{d}\bar{z}$ in full form. We have

$$\begin{aligned} & \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ \{z = x + \mathrm{i}\,y\} = & (\mathrm{d}x + \mathrm{i}\mathrm{d}y) \wedge (\mathrm{d}x - \mathrm{i}\mathrm{d}y) \\ \{\text{expand}\} = & -\mathrm{i}\mathrm{d}x \wedge \mathrm{d}y + \mathrm{i}\mathrm{d}y \wedge \mathrm{d}x \\ \{\mathrm{d}y \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}y\} = & (-2\mathrm{i})\,\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Next, we deal with the integrand. We have

$$\begin{aligned} & -\bar{z}\,A\,z + \bar{b}\,z + b\,\bar{z} \\ \{z, b = \cdots\} = & -(x - \mathrm{i}\,y)\,A\,(x + \mathrm{i}\,y) + (p - \mathrm{i}\,q)(x + \mathrm{i}\,y) + (p + \mathrm{i}\,q)(x - \mathrm{i}\,y) \\ \{\text{expand}\} = & -A x^2 - A y^2 + 2 p x + 2 q y. \end{aligned}$$

So, we get

$$\int_{\mathbb{C}} \mathrm{d}z \mathrm{d}\bar{z} \exp(-\bar{z}\,A\,z + \bar{b}\,z + b\,\bar{z}) = (-2\mathrm{i}) \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \exp(-A x^2 - A y^2 + 2 p x + 2 q y).$$

The right hand side can be integrated as

$$\begin{aligned} & (-2\mathrm{i}) \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \exp(-A x^2 - A y^2 + 2 p x + 2 q y) \\ = & (-2\mathrm{i}) \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \exp\left(-A \left(x - \frac{p}{A} \right)^2 - A \left(y - \frac{q}{A} \right)^2 + \frac{p^2 + q^2}{A} \right) \\ = & (-2\mathrm{i}) \exp\left(\frac{p^2 + q^2}{A} \right) \times \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \exp\left(-A \left(x - \frac{p}{A} \right)^2 - A \left(y - \frac{q}{A} \right)^2 \right). \end{aligned}$$

Noticing that $p^2+q^2=b\,\bar{b}$, and defining $u:=\sqrt{A}(x-p/A)$ and $v:=\sqrt{A}(y-q/A)$, we get

$$\frac{-2\mathrm{i}}{A}\exp\!\left(\frac{b\,\bar{b}}{A}\right)\times\int_{\mathbb{R}^2}\mathrm{d}u\mathrm{d}v\,\mathrm{e}^{-(u^2+v^2)}.$$

Now, by converting to polar coordinates,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\mathrm{d}u\mathrm{d}v\,\mathrm{e}^{-(u^2+v^2)}\\
\{\text{polar coordinates}\}=&\int_0^{2\pi}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}r\,r\,\mathrm{e}^{-r^2}\\
\left\{\int_0^{+\infty}\mathrm{d}r\,(2r)=\int_0^{+\infty}\mathrm{d}r^2\right\}=&\frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{+\infty}\mathrm{d}r^2\,\mathrm{e}^{-r^2}\\
\left\{\int_0^{+\infty}\mathrm{d}r^2\,\mathrm{e}^{-r^2}=1\right\}=&\,\pi.
\end{aligned}$$

So, we arrive at

$$\int_{\mathbb{C}}\mathrm{d}z\mathrm{d}\bar{z}\exp(-\bar{z}\,A\,z+\bar{b}\,z+b\,\bar{z})=\frac{-2\pi\mathrm{i}}{A}\exp\!\left(\frac{b\,\bar{b}}{A}\right). \tag{1}$$

Surprisingly, during this derivation, we did not employ the famous Gauss's trick of self-multiplication. That is, to evaluate $I:=\int_{\mathbb{R}}\mathrm{d}x\mathrm{e}^{-x^2}$, Gauss instead evaluated $I^2=\int_{\mathbb{R}^2}\mathrm{d}x\mathrm{d}y\mathrm{e}^{-(x^2+y^2)}$ by which polar coordinates can then intervene. As what we have calculated, it results in $I^2=\pi$. So, Gauss concluded that $I=\sqrt{\pi}$. Contrarily, in our derivation, all is natural. It is suspected that Gaussian integral should be expressed in complex plane!

Another difference between the real and complex one-dimensional Gaussian integral is that it is $1/A$ instead of $\sqrt{1/A}$. We will make this difference clear in section 2.5.

1.3 Differential Form Is Invariant for Unitary Transformation

To generalize the previous result multi-dimension, we have to declare the complex differential form used for writing down the multi-dimensional complex Gaussian integral. A general complex integral has the form

$$\int_{\mathbb{C}^{2n}}\mathrm{d}z\mathrm{d}\bar{z}\;\{\text{integrand}\},$$

where $z\in\mathbb{C}^n$, and the differential form $\mathrm{d}z\mathrm{d}\bar{z}:=\mathrm{d}z^1\wedge\cdots\wedge\mathrm{d}z^n\wedge\mathrm{d}\bar{z}^1\wedge\cdots\wedge\mathrm{d}\bar{z}^n$.

Next, we wonder how the $\mathrm{d}z\mathrm{d}\bar{z}$ changes with unitary transformation. Let U an $n\times n$ unitary matrix, and $w:=\bar{U}\,z$ where $\bar{U}$ denotes the transjugate of U [1]. Recall the case in real space, we have $\mathrm{d}x=\det(\partial x/\partial y)\,\mathrm{d}y$. The derivation of this formula is purely algebraic, thus can be directly generalized to complex space. So, from $z=Uw$, we get $\mathrm{d}z=\det(U)\,\mathrm{d}w$; and from $\bar{z}=\bar{w}\,\bar{U}$, we get $\mathrm{d}\bar{z}=\det(\bar{U})\,\mathrm{d}\bar{w}$. Thus, we have

$$\begin{aligned}
&\mathrm{d}z\mathrm{d}\bar{z}\\
\{\mathrm{d}z,\mathrm{d}\bar{z}=\cdots\}=&\det(U)\det(\bar{U})\,\mathrm{d}w\mathrm{d}\bar{w}\\
\{\det(A)\det(B)=\det(AB)\}=&\det(U\bar{U})\,\mathrm{d}w\mathrm{d}\bar{w}\\
\{U\bar{U}=1\}=&\,\mathrm{d}w\mathrm{d}\bar{w}.
\end{aligned}$$

1. Generally, we should use dagger-notation, instead of bar-notation, for transjugate of vector or matrix. Here, it should be $\mathrm{d}z^\dagger$ instead of $\mathrm{d}\bar{z}$, and $U^\dagger$ instead of $\bar{U}$. And usually, the bar-notation is left to complex conjugate. But for the fact that complex conjugate is absent in this note, and that notations shall be consistent throughout the note, we employ bar-notation for representing transjugate.

So, we find that the differential form is invariant for unitary transformation.

1.4 Multi-Dimensional Complex Gaussian Integral

In this section, we are to generalize the complex Gaussian integral from one-dimensional to multi-dimensional, that is

$$\int_{\mathbb{C}^{2n}} \mathrm{d}z\mathrm{d}\bar{z}\,\exp(-\bar{z}\,A\,z+\bar{b}\,z+\bar{z}\,b),$$

where A is a positive definite Hermitian matrix and b a complex vector. It must be noted that b and z are now vectors, $\bar{b}$ or $\bar{z}$ means more than complex conjugate, but including transpose. For this reason, we write $\bar{z}\,b$ instead of $b\,\bar{z}$.

Following the same strategy used for real Gaussian integral, we first diagonalize A as $A=UD\bar{U}$ with U unitary and D diagonal. Defining the new coordinate $\zeta:=\bar{U}\,z$, we are to change the coordinates from z to ζ. We have, $\bar{z}\,Az=\bar{\zeta}\,\bar{U}\,AU\zeta=\bar{\zeta}\,D\zeta$. As discussed in section 1.3, we have $\mathrm{d}z\mathrm{d}\bar{z}=\mathrm{d}\zeta\mathrm{d}\bar{\zeta}$.

Then, defining $h:=\bar{U}\,b$, we have $\bar{b}\,z=\bar{h}\,\bar{U}\,U\zeta=\bar{h}\,\zeta$ and $\bar{z}\,b=\bar{\zeta}\,h$. Altogether, we find

$$\int_{\mathbb{C}^{2n}} \mathrm{d}z\mathrm{d}\bar{z}\,\exp(-\bar{z}\,A\,z+\bar{b}\,z+\bar{z}\,b)=\int_{\mathbb{C}^{2n}} \mathrm{d}\zeta\mathrm{d}\bar{\zeta}\,\exp(-\bar{\zeta}\,D\,\zeta+\bar{h}\,\zeta+\bar{\zeta}\,h).$$

Now, we can separate each dimension and compute for each dimension individually, as

$$\begin{aligned}
&\int_{\mathbb{C}^{2n}} \mathrm{d}\zeta\mathrm{d}\bar{\zeta}\prod_{\alpha=1}^{n}\exp(-\bar{\zeta}\,D\zeta+\bar{h}\,\zeta+\bar{\zeta}\,h)\\
\{\text{move } \mathrm{d}\zeta^{\alpha} \text{ to } \mathrm{d}\bar{\zeta}^{\alpha}\} &= (-1)^{n\,(n-1)/2}\prod_{\alpha=1}^{n}\left[\int_{\mathbb{C}^{2}} \mathrm{d}\zeta^{\alpha}\mathrm{d}\bar{\zeta}^{\alpha}\exp(-\bar{\zeta}^{\alpha}\,D_{\alpha\alpha}\zeta^{\alpha}+\bar{h}^{\alpha}\,\zeta^{\alpha}+\bar{\zeta}^{\alpha}\,h_{\alpha})\right]\\
\{\text{1-dimensional result}\} &= (-1)^{n\,(n-1)/2}\prod_{\alpha=1}^{n}\left[\frac{-2\pi\mathrm{i}}{D_{\alpha\alpha}}\exp\!\left(\frac{h_{\alpha}\,\bar{h}_{\alpha}}{D_{\alpha\alpha}}\right)\right]\\
&= \frac{\mathrm{i}^{n\,(n+2)}\,(2\pi)^{n}}{\prod_{\alpha=1}^{n}D_{\alpha\alpha}}\exp\!\left(\sum_{\alpha=1}^{n}\frac{h_{\alpha}\,\bar{h}_{\alpha}}{D_{\alpha\alpha}}\right).
\end{aligned}$$

The final step is changing coordinates back to z from ζ. Since $D=\bar{U}\,AU$ and $h=\bar{U}\,b$, we have

$$\begin{aligned}
&\prod_{\alpha=1}^{n}D_{\alpha\alpha}\\
&= \det(D)\\
&= \det(\bar{U}\,AU)\\
&= \det(\bar{U})\,\det(A)\,\det(U)\\
&= \det(U\bar{U})\,\det(A)\\
\{U\bar{U}=1\} &= \det(A).
\end{aligned}$$

And since U and $\bar{U}$ are mutually inverse, we have

$$\begin{aligned}
&(\bar{U}\,AU)^{-1}\\
\{\text{property of inverse}\} &= U^{-1}\,A^{-1}\,(\bar{U})^{-1}\\
\{U\bar{U}=1\} &= \bar{U}\,A^{-1}\,U,
\end{aligned}$$

thus

$$\begin{aligned}
&\sum_{\alpha=1}^{n} \frac{h_\alpha \bar{h}_\alpha}{D_{\alpha\alpha}} \\
&= \bar{h}\, D^{-1}\, h \\
\{\text{previous conclusion}\} &= \bar{b}\, U\, (\bar{U} A U)^{-1}\, \bar{U} b \\
\{U\bar{U}=1\} &= \bar{b}\, U \bar{U} A^{-1}\, U \bar{U} b \\
&= \bar{b}\, A^{-1}\, b.
\end{aligned}$$

Altogether, we find

$$\int_{\mathbb{C}^{2n}} \mathrm{d}z \mathrm{d}\bar{z} \exp(-\bar{z}\, A\, z + \bar{b}\, z + \bar{z}\, b) = \frac{\mathrm{i}^{n\,(n+2)} (2\pi)^n}{\det(A)} \exp(\bar{b}\, A^{-1}\, b).$$

By the formula $\det(A) = \exp(\mathrm{tr}\ln(A))$, we can convert $\det(A)$ into exponential. Finally, we arrive at

$$\ln\!\left[\int_{\mathbb{C}^{2n}} \mathrm{d}z \mathrm{d}\bar{z} \exp(-\bar{z}\, A\, z + \bar{b}\, z + \bar{z}\, b)\right] = \bar{b}\, A^{-1}\, b - \mathrm{tr}\ln(A) + \mathrm{Const}, \tag{2}$$

where the constant $\mathrm{Const} = \ln(\mathrm{i}^{n\,(n+2)} (2\pi)^n) = n\,(n+2)\,(\pi \mathrm{i}/2) + n \ln(2\pi)$.

1.5 From Complex Gaussian Integral to Real Gaussian Integral

To reduce to multi-dimensional real Gaussian integral, we simply let the A and b completely real. Then, by changing coordinates from $\mathrm{d}z\mathrm{d}\bar{z}$ to $\mathrm{d}x\mathrm{d}y$, where $z = x + \mathrm{i}\, y$, we will arrive at two individual real Gaussian integrals. So, the result is a multiplication of two real Gaussian integral. It is for this reason, it is the square root of $\det(A)$ in real Gaussian integral.

Explicitly, we change coordinates from $\mathrm{d}z\mathrm{d}\bar{z}$ to $\mathrm{d}x\mathrm{d}y$. As it has been derived,

$$\begin{aligned}
&\int_{\mathbb{C}^{2n}} \mathrm{d}z \mathrm{d}\bar{z} \exp(-\bar{z}\, A\, z + \bar{b}\, z + \bar{z}\, b) \\
\{z = x + \mathrm{i}\, y\} &= (-1)^{n(n-1)/2}\, (-2\mathrm{i})^n \int_{\mathbb{R}^{2n}} \mathrm{d}x \mathrm{d}y \exp(-x^t\, A x - y^t\, A y + 2 b^t\, x) \\
&= \mathrm{i}^{n\,(n+2)}\, 2^n \int_{\mathbb{R}^n} \mathrm{d}x \exp(-x^t\, A x + 2 b^t\, x) \int_{\mathbb{R}^n} \mathrm{d}y \exp(-y^t\, A y).
\end{aligned}$$

By plugging in the result of complex Gaussian integral, we have

$$\mathrm{i}^{n\,(n+2)}\, 2^n \int_{\mathbb{R}^n} \mathrm{d}x \exp(-x^t\, A x + 2 b^t\, x) \int_{\mathbb{R}^n} \mathrm{d}y \exp(-y^t\, A y) = \frac{\mathrm{i}^{n\,(n+2)} (2\pi)^n}{\det(A)} \exp(\bar{b}\, A^{-1}\, b).$$

So, we get

$$\frac{\pi^n}{\det(A)} \exp(\bar{b}\, A^{-1}\, b) = \int_{\mathbb{R}^n} \mathrm{d}x \exp(-x^t\, A x + 2 b^t\, x) \int_{\mathbb{R}^n} \mathrm{d}y \exp(-y^t\, A y)$$

First letting $b = 0$, since both x and y are dummy variables, we find

$$\int_{\mathbb{R}^n} \mathrm{d}y \exp(-y^t\, A y) = \sqrt{\frac{\pi^n}{\det(A)}}.$$

Plugging back, we get

$$\int_{\mathbb{R}^n} \mathrm{d}x \exp(-x^t\, A x + 2 b^t\, x) = \sqrt{\frac{\pi^n}{\det(A)}} \exp(\bar{b}\, A^{-1}\, b).$$

By re-defining $A \to A/2$ and $b \to b/2$, we finally arrive at

$$\int_{\mathbb{R}^n} \mathrm{d}x \exp\left(-\frac{1}{2} x^t A x + b^t x\right) = \sqrt{\frac{(2\pi)^n}{\det(A)}} \exp\left(\frac{1}{2} \bar{b} A^{-1} b\right),$$

which is exactly the formula of real multi-dimensional Gaussian integral.

1.6 Trace of Logarithm Is Invariant for Unitary Transformation

In this section, we are to exam how the $\operatorname{tr}\ln(A)$ changes with unitary transformation. We start at the well-known formula

$$\mathrm{e}^{\bar{U} X U} = \bar{U} \mathrm{e}^X U,$$

where U is a unitary operator and X is an arbitrary operator. It can be proven by simply expanding exponential as series while noticing $U\bar{U} = 1$. Taking logarithm on both sides, we find

$$\bar{U} X U = \ln(\bar{U} \mathrm{e}^X U).$$

And letting $A := \mathrm{e}^X$, we arrive at

$$\bar{U} \ln(A) U = \ln(\bar{U} A U). \tag{3}$$

Taking tace on both side, we get $\operatorname{tr}(\bar{U} \ln(A) U) = \operatorname{tr}\ln(A)$.[2] Thus,

$$\operatorname{tr}\ln(A) = \operatorname{tr}\ln(\bar{U} A U).$$

So, trace of logarithm is invariant for unitary transformation.

1.7 Complex Gaussian Integral Is Invariant for Unitary Transformation

Now, we can exam how the result of complex Gaussian integral changes with unitary transformation. We restate the result as

$$\ln\left[\int_{\mathbb{C}^{2n}} \mathrm{d}z \mathrm{d}\bar{z} \exp(-\bar{z} A z + \bar{b} z + \bar{z} b)\right] = \bar{b} A^{-1} b - \operatorname{tr}\ln(A) + \text{Const}.$$

Taking unitary transformation $w := \bar{U} z$, the matrix or operator A transforms as $B := \bar{U} A U$, and the b as $c := \bar{U} b$. Because of $U\bar{U} = 1$, we have $A^{-1} = U B \bar{U}$. As discussed in section 1.3 and section 1.6, the differential form and the trace of logarithm are invariant for unitary transformation. So, the result of complex Gaussian integral is transformed to

$$\ln\left[\int_{\mathbb{C}^{2n}} \mathrm{d}w \mathrm{d}\bar{w} \exp(-\bar{w} B w + \bar{c} w + \bar{w} c)\right] = \bar{c} B^{-1} c - \operatorname{tr}\ln(B) + \text{Const}.$$

2. This is apparent when we write it in component. That is,

$$\begin{aligned}
& \operatorname{tr}(\bar{U} \ln(A) U) \\
\{\text{in component}\} = & \sum_{\alpha,\beta,\gamma} \bar{U}_{\alpha\beta} \ln(A_{\beta\gamma}) U_{\gamma\alpha} \\
= & \sum_{\beta,\gamma} \left(\sum_{\alpha} U_{\gamma\alpha} \bar{U}_{\alpha\beta}\right) \ln(A_{\beta\gamma}) \\
\{U\bar{U} = 1\} = & \sum_{\beta,\gamma} \delta_{\beta\gamma} \ln(A_{\beta\gamma}) \\
= & \sum_{\beta} \ln(A_{\beta\beta}),
\end{aligned}$$

which is the $\operatorname{tr}\ln(A)$ written in component.

It is seen that the result of complex Gaussian integral is invariant for unitary transformation.

1.8 Gaussian Integral in Dirac's Representation Theory

In Dirac's representation theory [3], we can change from one representation to another by changing the complete orthogonal basis. For instance, let $|\varphi\rangle$ an abstract ket. In α-representation with complete orthogonal basis $\{|\alpha\rangle | \alpha \in A\}$, it is $\langle\alpha|\varphi\rangle$. Changing to β-representation with complete orthogonal basis $\{|\beta\rangle | \beta \in B\}$ makes

$$\begin{aligned} & \langle\alpha|\varphi\rangle \\ \left\{\sum_{\beta\in B} |\beta\rangle\langle\beta| = 1\right\} = & \sum_{\beta\in B} \langle\alpha|\beta\rangle\langle\beta|\varphi\rangle. \end{aligned}$$

This change is nothing but a unitary transformation, where the unitary matrix or operator has component $\langle\alpha|\beta\rangle$. It indicates that the result of complex Gaussian integral can be expressed in Dirac's bracket notation, and the invariance for unitary transformation means that the expression is independent of specific representations.

Explicitly, for any positive definite Hermitian operator A and any ket $|b\rangle$ which may not be normalized, we have

$$\int \mathrm{d}|\varphi\rangle \mathrm{d}\langle\varphi| \exp(-\langle\varphi|A|\varphi\rangle + \langle b|\varphi\rangle + \langle\varphi|b\rangle) = \exp(\langle b|A^{-1}|b\rangle - \operatorname{tr}\ln(A) + \mathrm{Const}), \tag{4}$$

where for any complete orthogonal basis $\{|x\rangle | x \in X\}$, $\mathrm{d}|\varphi\rangle \mathrm{d}\langle\varphi| = \prod_{x\in X} \mathrm{d}[\langle x|\varphi\rangle] \mathrm{d}[\langle\varphi|x\rangle]$.

This expression is abstract. We can get an explicit expression by simply inserting a complete relation, like $\sum_\alpha |\alpha\rangle\langle\alpha| = 1$ or $\int_X \mathrm{d}x |x\rangle\langle x| = 1$ [4]. This this reason, it is called the **generalized Gaussian integral**.

A Continuous Representation

A.1 Trace in Continuous Representation

We have known what trace means for finite-dimensional matrix, and even for representation with discrete spectrum. We are to determine how trace is defined in representation with continuous spectrum. To do so, we convert from a discrete representation $|\alpha\rangle$ to a continuous one $|x\rangle$.

Let A an operator. We have known that $\operatorname{tr}(A)$ in representation $|\alpha\rangle$ is defined by $\sum_\alpha \langle\alpha|A|\alpha\rangle$. By inserting the complete relation $\int \mathrm{d}x |x\rangle\langle x| = 1$, we have

$$\begin{aligned} & \sum_\alpha \langle\alpha|A|\alpha\rangle \\ \left\{\text{insert } \int \mathrm{d}x|x\rangle\langle x| = 1\right\} = & \sum_\alpha \int \mathrm{d}x \int \mathrm{d}x' \langle\alpha|x\rangle\langle x|A|x'\rangle\langle x'|\alpha\rangle \\ = & \int \mathrm{d}x \int \mathrm{d}x' \langle x|A|x'\rangle \left(\sum_\alpha \langle x'|\alpha\rangle\langle\alpha|x\rangle\right) \\ \left\{\sum_\alpha |\alpha\rangle\langle\alpha| = 1\right\} = & \int \mathrm{d}x \int \mathrm{d}x' \langle x|A|x'\rangle \langle x|x'\rangle \\ \{\langle x|x'\rangle = \delta(x - x')\} = & \int \mathrm{d}x \int \mathrm{d}x' \langle x|A|x'\rangle \delta(x - x') \\ = & \int \mathrm{d}x \langle x|A|x\rangle. \end{aligned}$$

3. For Dirac's representation theory, we reference to the book *The Principles of Quantum Mechanics* by P. A. M. Dirac.

4. The expression of $\operatorname{tr}\ln(A)$ for continuous representation is discussed in appendix A.1.

So, in continuous representation like $|x\rangle$, the $\mathrm{tr}(A)$ is defined as $\int \mathrm{d}x \, \langle x|A|x\rangle$.

B Berezin Integral (TODO)

C.f. wikipedia.

We have,

$$\mathrm{i}^{n(n+2)}(2\pi)^n \exp(\bar{b}\, A^{-1}\, b) = \int \mathrm{d}z \, \mathrm{d}\bar{z} \, \mathrm{d}\theta \mathrm{d}\eta \exp(-\bar{z}\, A\, z + \bar{b}\, z + \bar{z}\, b - \theta^t\, A \eta) \tag{5}$$