

1 Generalized Gaussian Integral

1.1 Representation Theory May Generalize Gaussian Integral

Originally, the multi-dimensional Gaussian integral is, for any positive definite real symmetric matrix A and vector b ,

$$\int_{\mathbb{R}^n} d\varphi \exp\left(-\frac{1}{2}\varphi^t A \varphi + b^t \varphi\right) = \sqrt{\frac{(2\pi)^n}{\det(A)}} \exp\left(\frac{1}{2}b^t A^{-1} b\right). \quad (1)$$

With $\det(A) = \exp(\text{tr} \ln(A))$, we can absorb the $(\det(A))^{-1/2}$ factor into the exponential, as

$$\ln \left[\int_{\mathbb{R}^n} d\varphi \exp\left(-\frac{1}{2}\varphi^t A \varphi + b^t \varphi\right) \right] = \frac{1}{2}b^t A^{-1} b - \frac{1}{2}\text{tr} \ln(A) + \frac{n}{2}\ln(2\pi). \quad (2)$$

There is, however, a functional version, which is very useful in quantum field theory. Let A a positive definite symmetric kernel and b a function,

$$\begin{aligned} & \ln \left[\left(\prod_{x \in V} \int_{\mathbb{R}} d[\varphi(x)] \right) \exp\left(-\frac{1}{2} \int_V dx \int_V dy \varphi(x) A(x, y) \varphi(y) + \int_V dx b(x) \varphi(x)\right) \right] \\ &= \frac{1}{2} \int_V dx \int_V dy b(x) (A^{-1})(x, y) b(y) - \frac{1}{2} \int_V dx \ln(A(x, x)) + \text{Const}, \end{aligned}$$

where A^{-1} is the functional inverse of A , defined as $\int_V dy A(x, y) (A^{-1})(y, z) = \delta(x - z)$. It must be noted that, in functional integral, product like $\prod_x d[\varphi(x)]$ shall be realized as wedged product. This, however, is not the end. There is also functional version in momentum space, and so on, and so on.

This motives us to seek for a generalized version of Gaussian integral, which reduces to all these formulae just in one step.

An educated guess is using Dirac's representation theory. Let A a Hermitian operator and $|b\rangle$ a general ket, we may have

$$\ln \left[\int d\mu(|\varphi\rangle) \exp(-\langle \varphi | A | \varphi \rangle + \langle b | \varphi \rangle + \langle \varphi | b \rangle) \right] = \frac{1}{2} \langle b | A^{-1} | b \rangle - \frac{1}{2} \text{tr} \ln(A) + \text{Const}. \quad (3)$$

With this expression, by inserting $\sum_{\alpha} |\alpha\rangle \langle \alpha| = 1$, we recover the multi-dimensional version; and by inserting $\int_V dx |x\rangle \langle x| = 1$, we recover the functional version. This provides an abstract expression by using Dirac's bracket notation.

The problems left are:

- How is the measurement $d\mu(|\varphi\rangle)$ defined?
- Is $\text{tr} \ln(A)$ independent of representation?

1.2 One-Dimensional Complex Gaussian Integral

Now, we are to deal with the first problem. Recall that bracket notation deals with wave-function of quantum state, which is complex. So, $|\varphi\rangle$ is intrinsically complex. This hints us that we shall consider the Gaussian integral on complex plane.

In complex plane, we have to ensure that, the measurement is real, so is the integrand. So, the only way of writing a complex Gaussian-like integral is

$$\int_{\mathbb{C}} dz d\bar{z} \exp(-\bar{z} A z + \bar{b} z + b \bar{z}),$$

for a real positive number A and a complex number b . Next, we are to integrate this by first converting it to real plane. To do this, let $z = x + iy$ and $b = p + iq$. Recall that A is itself real.

First, we deal with the measurement. It is $dz \wedge d\bar{z}$ in full form. We have

$$\begin{aligned} dz \wedge d\bar{z} \\ \{z = x + iy\} &= (dx + i dy) \wedge (dx - i dy) \\ \{\text{expand}\} &= -i dx \wedge dy + i dy \wedge dx \\ \{dy \wedge dx = -dx \wedge dy\} &= (-2i) dx \wedge dy. \end{aligned}$$

Next, we deal with the integrand. We have

$$\begin{aligned} -\bar{z} A z + \bar{b} z + b \bar{z} \\ \{z, b = \dots\} &= -(x - iy) A (x + iy) + (p - iq)(x + iy) + (p + iq)(x - iy) \\ \{\text{expand}\} &= -Ax^2 - Ay^2 + 2px + 2qy. \end{aligned}$$

So, we get

$$\int_{\mathbb{C}} dz d\bar{z} \exp(-\bar{z} A z + \bar{b} z + b \bar{z}) = (-2i) \int_{\mathbb{R}^2} dx dy \exp(-Ax^2 - Ay^2 + 2px + 2qy).$$

The right hand side can be integrated as

$$\begin{aligned} &(-2i) \int_{\mathbb{R}^2} dx dy \exp(-Ax^2 - Ay^2 + 2px + 2qy) \\ &= (-2i) \int_{\mathbb{R}^2} dx dy \exp\left(-A\left(x - \frac{p}{A}\right)^2 - A\left(y - \frac{q}{A}\right)^2 + \frac{p^2 + q^2}{A}\right) \\ &= (-2i) \exp\left(\frac{p^2 + q^2}{A}\right) \times \int_{\mathbb{R}^2} dx dy \exp\left(-A\left(x - \frac{p}{A}\right)^2 - A\left(y - \frac{q}{A}\right)^2\right). \end{aligned}$$

Noticing that $p^2 + q^2 = b\bar{b}$, and defining $u := \sqrt{A}(x - p/A)$ and $v := \sqrt{A}(y - q/A)$, we get

$$\frac{-2i}{A} \exp\left(\frac{b\bar{b}}{A}\right) \times \int_{\mathbb{R}^2} du dv e^{-(u^2 + v^2)}.$$

Now, by converting to polar coordinates,

$$\begin{aligned} &\int_{\mathbb{R}^2} du dv e^{-(u^2 + v^2)} \\ \{\text{polar coordinates}\} &= \int_0^{2\pi} d\theta \int_0^{+\infty} dr r e^{-r^2} \\ \left\{ \int_0^{+\infty} dr (2r) = \int_0^{+\infty} dr^2 \right\} &= \frac{1}{2} \int_0^{2\pi} d\theta \int_0^{+\infty} dr^2 e^{-r^2} \\ \left\{ \int_0^{+\infty} dr^2 e^{-r^2} = 1 \right\} &= \pi. \end{aligned}$$

So, we arrive at

$$\int_{\mathbb{C}} dz d\bar{z} \exp(-\bar{z} A z + \bar{b} z + b \bar{z}) = \frac{-2\pi i}{A} \exp\left(\frac{b \bar{b}}{A}\right). \quad (4)$$

Surprisingly, during this derivation, we did not employ the famous Gauss's trick of self-multiplication. That is, to evaluate $I := \int_{\mathbb{R}} dx e^{-x^2}$, Gauss instead evaluated $I^2 = \int_{\mathbb{R}^2} dx dy e^{-(x^2+y^2)}$ by which polar coordinates can then intervene. As what we have calculated, it results in $I^2 = \pi$. So, Gauss concluded that $I = \sqrt{\pi}$. Contrarily, in our derivation, all is natural. It is suspected that Gaussian integral should be expressed in complex plane!

Another difference between the real and complex one-dimensional Gaussian integral is that it is $1/A$ instead of $\sqrt{1/A}$. We will make this difference clear in section 1.4.

1.3 Multi-Dimensional Complex Gaussian Integral

We are to generalize the previous result from one-dimensional to multi-dimensional, that is

$$\int_{\mathbb{C}^{2n}} dz d\bar{z} \exp(-\bar{z} A z + \bar{b} z + \bar{z} b),$$

where A is a positive definite Hermitian matrix and b a complex vector. Notice that b and z are now vectors, \bar{b} or \bar{z} means more than b^* and z^* , but including transpose¹. For this reason, we write $\bar{z} b$ instead of $b \bar{z}$ which is meaningless. In addition, $dz d\bar{z}$ means $dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^n$.

Following the same strategy used for real Gaussian integral, we first diagonalize A as $A = U D \bar{U}$ with U unitary and D diagonal. Defining the new coordinate $w := \bar{U} z$, we are to change the coordinates from z to w .

First, we have to declare how $dz d\bar{z}$ changes with coordinates. **Since that $z = U w$ and that dz changes algebraically, we can employ the result in real forms, as $dz = \det(U) dw$.** The same, $d\bar{z} = \det(\bar{U}) d\bar{w}$. So,

$$\begin{aligned} dz d\bar{z} \\ \{dz, d\bar{z} = \dots\} &= \det(U) \det(\bar{U}) dw d\bar{w} \\ \{\det(A) \det(B) = \det(AB)\} &= \det(U \bar{U}) dw d\bar{w} \\ \{U \bar{U} = 1\} &= dw d\bar{w}. \end{aligned}$$

Then, letting $b =: U h$, we have $\bar{z} A z = \bar{w} \bar{U} A U w = \bar{w} D w$, $\bar{b} z = \bar{h} \bar{U} U w = \bar{h} w$, $b \bar{z} = h \bar{w}$. Altogether, we find

$$\int_{\mathbb{C}^{2n}} dz d\bar{z} \exp(-\bar{z} A z + \bar{b} z + b \bar{z}) = \int_{\mathbb{C}^{2n}} dw d\bar{w} \exp(-\bar{w} D w + \bar{h} w + h \bar{w}).$$

Now, we can separate each dimension and compute for each dimension individually, as

$$\begin{aligned} \int_{\mathbb{C}^{2n}} dz d\bar{z} \exp(-\bar{z} A z + \bar{b} z + b \bar{z}) \\ \{\text{separate dimensions}\} &= \prod_{\alpha=1}^n \left[\int_{\mathbb{C}^2} dw^\alpha d\bar{w}^\alpha \exp(-\bar{w}^\alpha D_{\alpha\alpha} w^\alpha + \bar{h}_\alpha w^\alpha + h_\alpha \bar{w}^\alpha) \right] \\ \{1\text{-dimensional result}\} &= \prod_{\alpha=1}^n \left[\frac{-2\pi i}{D_{\alpha\alpha}} \exp\left(\frac{h_\alpha \bar{h}_\alpha}{D_{\alpha\alpha}}\right) \right] \\ &= \frac{(-2\pi i)^n}{\prod_{\alpha=1}^n D_{\alpha\alpha}} \exp\left(\sum_{\alpha=1}^n \frac{h_\alpha \bar{h}_\alpha}{D_{\alpha\alpha}}\right). \end{aligned}$$

1. For a complex vector z , $\bar{z} := (z^*)^t$.

The final step is changing coordinates back to z from w . We have $D = \bar{U} A U$ and $h = \bar{U} b$. We have

$$\begin{aligned}
& \prod_{\alpha=1}^n D_{\alpha\alpha} \\
&= \det(D) \\
\{D = \dots\} &= \det(\bar{U} A U) \\
&= \det(\bar{U}) \det(A) \det(U) \\
&= \det(U \bar{U}) \det(A) \\
\{U \bar{U} = 1\} &= \det(A).
\end{aligned}$$

And since

$$\begin{aligned}
& (\bar{U} A U)^{-1} \\
\{\text{property of inverse}\} &= U^{-1} A^{-1} (\bar{U})^{-1} \\
\{U \bar{U} = 1\} &= \bar{U} A^{-1} U,
\end{aligned}$$

we have

$$\begin{aligned}
& \sum_{\alpha=1}^n \frac{h_{\alpha} \bar{h}_{\alpha}}{D_{\alpha\alpha}} \\
&= \bar{h} D^{-1} h \\
\{\text{previous conclusion}\} &= \bar{b} U (\bar{U} A U)^{-1} \bar{U} b \\
\{U \bar{U} = 1\} &= \bar{b} U \bar{U} A^{-1} U \bar{U} b \\
&= \bar{b} A^{-1} b.
\end{aligned}$$

Altogether, we find

$$\int_{\mathbb{C}^{2n}} dz d\bar{z} \exp(-\bar{z} A z + \bar{b} z + b \bar{z}) = \frac{(-2\pi i)^n}{\det(A)} \exp(\bar{b} A^{-1} b), \quad (5)$$

or, since $1/\det(A) = \det(A^{-1})$,

$$\int_{\mathbb{C}^{2n}} dz d\bar{z} \exp(-\bar{z} A z + \bar{b} z + b \bar{z}) = (-2\pi i)^n \det(A^{-1}) \exp(\bar{b} A^{-1} b). \quad (6)$$

Go back to the first problem: what is the measurement $\mu(|\varphi\rangle)$? Now, we can say, given a complete orthogonal basis $|x\rangle$, $\mu(|\varphi\rangle) = \prod_x d[\varphi(x)] d[\overline{\varphi(x)}]$ which is invariant when changing from $|x\rangle$ to another complete orthogonal basis. Indeed, when $|x\rangle \rightarrow |k\rangle$ where $|k\rangle$ is another complete orthogonal basis, we have $\prod_x d[\varphi(x)] d[\overline{\varphi(x)}] = \det(U) \det(\bar{U}) \prod_k d[\varphi(k)] d[\overline{\varphi(k)}] = \prod_k d[\varphi(k)] d[\overline{\varphi(k)}]$, where $U_{x,k} := \langle x | k \rangle$ is unitary.

1.4 Reducing to Real Gaussian Integral (TODO)

To reduce to real Gaussian integral, say multi-dimensional version, we simply let the A and b completely real. Then, by changing coordinates from $dz d\bar{z}$ to $dx dy$, where $z = x + iy$, we will arrive at two individual real Gaussian integrals. So, the result is a multiplication of two real Gaussian integral. It is for this reason, it is the sqrt of $\det(A)$ in real Gaussian integral.

Explicitly, TODO

A Berezin Integral (TODO)

C.f. [wikipedia](#).

We have,

$$(-2\pi i)^n \exp(\bar{b} A^{-1} b) = \int dz d\bar{z} d\theta d\eta \exp(-\bar{z} A z + \bar{b} z + b \bar{z} - \theta^t A \eta) \quad (7)$$