1 Generalized Gaussian Integral

1 Basic Idea

1.1 Representation Theory May Generalize Gaussian Integral

Originally, the multi-dimensional Gaussian integral is, for any positive definite real symmetric matrix A and vector b,

$$\int_{\mathbb{R}^n} \mathrm{d}\varphi \exp\left(-\frac{1}{2}\varphi^t A \varphi + b^t \varphi\right) = \sqrt{\frac{(2\pi)^n}{\det(A)}} \exp\left(\frac{1}{2} b^t A^{-1} b\right), \tag{1}$$

where φ^t denotes the transpose of vector φ. With $\det(A) = \exp(\operatorname{tr}\ln(A))$, we can absorb the $(\det(A))^{-1/2}$ factor into the exponential, as

$$\ln\left[\int_{\mathbb{R}^n} \mathrm{d}\varphi \exp\left(-\frac{1}{2}\varphi^t A \varphi + b^t \varphi\right)\right] = \frac{1}{2} b^t A^{-1} b - \frac{1}{2}\operatorname{tr}\ln(A) + \frac{n}{2}\ln(2\pi). \tag{2}$$

There is, however, a functional version, which is very useful in quantum field theory. Let A a positive definite symmetric kernel and b a function,

$$\begin{aligned}
&\ln\left[\left(\prod_{x\in V}\int_{\mathbb{R}} \mathrm{d}[\varphi(x)]\right)\exp\left(-\frac{1}{2}\int_V \mathrm{d}x \int_V \mathrm{d}y \varphi(x) A(x,y) \varphi(y) + \int_V \mathrm{d}x b(x) \varphi(x)\right)\right] \\
&= \frac{1}{2}\int_V \mathrm{d}x \int_V \mathrm{d}y b(x)\,(A^{-1})(x,y)\, b(y) - \frac{1}{2}\int_V \mathrm{d}x \ln(A(x,x)) + \mathrm{Const},
\end{aligned}$$

where A^{-1} is the functional inverse of A, defined as $\int_V \mathrm{d}y A(x,y)\,(A^{-1})(y,z) = \delta(x-z)$. It must be noted that, in functional integral, product like $\prod_x \mathrm{d}[\varphi(x)]$ shall be realized as wedged product. This, however, is not the end. There is also functional version in momentum space, and so on, and so on.

This motives us to seek for a generalized version of Gaussian integral, so that all these formulae are nothing but viewing the same result from different perspectives.

An educated guess is using Dirac's representation theory. For instance, letting A an Hermitian operator and $|b\rangle$ a general ket, we may have

$$\ln\left[\int \mathrm{d}\mu(|\varphi\rangle) \exp(-\langle\varphi|A|\varphi\rangle + \langle b|\varphi\rangle)\right] = \frac{1}{2}\langle b|A^{-1}|b\rangle - \frac{1}{2}\operatorname{tr}\ln(A) + \mathrm{Const}. \tag{3}$$

With this expression, by inserting $\sum_\alpha |\alpha\rangle\langle\alpha| = 1$, we recover the multi-dimensional version; and by inserting $\int_V \mathrm{d}x |x\rangle\langle x| = 1$, we recover the functional version. This provides an abstract expression by using Dirac's bracket notation.

The problems left are:

- How is the measurement $\mathrm{d}\mu(|\varphi\rangle)$ defined?
- Is $\operatorname{tr}\ln(A)$ independent of representation?

Now, we are to deal with the first problem. Recall that bracket notation deals with wave-function of quantum state, which is complex. So, $|\varphi\rangle$ is intrinsically complex. This hints us that we shall consider the Gaussian integral on complex plane.

2 Complex Gaussian Integral

2.1 One-Dimensional Complex Gaussian Integral

In complex plane, we have to ensure that, the measurement is real, so is the integrand. So, the only way of writting a complex Gaussian-like integral is

$$\int_{\mathbb{C}} \mathrm{d}z \mathrm{d}\bar{z} \exp(-\bar{z}\, A\, z + \bar{b}\, z + b\, \bar{z}),$$

for a real positive number A and a complex number b. Next, we are to integrate this by first converting it to real plane. To do this, let $z = x + \mathrm{i}\, y$ and $b = p + \mathrm{i}\, q$. Recall that A is itself real.

First, we deal with the measurement. It is $\mathrm{d}z \wedge \mathrm{d}\bar{z}$ in full form. We have

$$\begin{aligned} & \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ \{z = x + \mathrm{i}\, y\} &= (\mathrm{d}x + \mathrm{i}\mathrm{d}y) \wedge (\mathrm{d}x - \mathrm{i}\mathrm{d}y) \\ \{\text{expand}\} &= -\mathrm{i}\mathrm{d}x \wedge \mathrm{d}y + \mathrm{i}\mathrm{d}y \wedge \mathrm{d}x \\ \{\mathrm{d}y \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}y\} &= (-2\mathrm{i})\, \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Next, we deal with the integrand. We have

$$\begin{aligned} & -\bar{z}\, A\, z + \bar{b}\, z + b\, \bar{z} \\ \{z, b = \cdots\} &= -(x - \mathrm{i}\, y)\, A\, (x + \mathrm{i}\, y) + (p - \mathrm{i}\, q)(x + \mathrm{i}\, y) + (p + \mathrm{i}\, q)(x - \mathrm{i}\, y) \\ \{\text{expand}\} &= -A x^2 - A y^2 + 2 p x + 2 q y. \end{aligned}$$

So, we get

$$\int_{\mathbb{C}} \mathrm{d}z \mathrm{d}\bar{z} \exp(-\bar{z}\, A\, z + \bar{b}\, z + b\, \bar{z}) = (-2\mathrm{i}) \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \exp(-A x^2 - A y^2 + 2 p x + 2 q y).$$

The right hand side can be integrated as

$$\begin{aligned} & (-2\mathrm{i}) \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \exp(-A x^2 - A y^2 + 2 p x + 2 q y) \\ = & (-2\mathrm{i}) \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \exp\left(-A \left(x - \frac{p}{A} \right)^2 - A \left(y - \frac{q}{A} \right)^2 + \frac{p^2 + q^2}{A} \right) \\ = & (-2\mathrm{i}) \exp\left(\frac{p^2 + q^2}{A} \right) \times \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}y \exp\left(-A \left(x - \frac{p}{A} \right)^2 - A \left(y - \frac{q}{A} \right)^2 \right). \end{aligned}$$

Noticing that $p^2 + q^2 = b\, \bar{b}$, and defining $u := \sqrt{A}(x - p / A)$ and $v := \sqrt{A}(y - q / A)$, we get

$$\frac{-2\mathrm{i}}{A} \exp\left(\frac{b\, \bar{b}}{A} \right) \times \int_{\mathbb{R}^2} \mathrm{d}u \mathrm{d}v\, \mathrm{e}^{-(u^2 + v^2)}.$$

Now, by converting to polar coordinates,

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \mathrm{d}u\mathrm{d}v\, \mathrm{e}^{-(u^2+v^2)}\\
\{\text{polar coordinates}\} &= \int_0^{2\pi}\mathrm{d}\theta \int_0^{+\infty} \mathrm{d}r\, r\, \mathrm{e}^{-r^2}\\
\left\{\int_0^{+\infty}\mathrm{d}r\,(2r) = \int_0^{+\infty}\mathrm{d}r^2\right\} &= \frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta \int_0^{+\infty}\mathrm{d}r^2\, \mathrm{e}^{-r^2}\\
\left\{\int_0^{+\infty}\mathrm{d}r^2\, \mathrm{e}^{-r^2} = 1\right\} &= \pi.
\end{aligned}
$$

So, we arrive at

$$
\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, \exp(-\bar{z}\,A\,z + \bar{b}\,z + b\,\bar{z}) = \frac{-2\pi\mathrm{i}}{A}\exp\!\left(\frac{b\,\bar{b}}{A}\right). \tag{4}
$$

Surprisingly, during this derivation, we did not employ the famous Gauss's trick of self-multiplication. That is, to evaluate $I := \int_{\mathbb{R}}\mathrm{d}x\, \mathrm{e}^{-x^2}$, Gauss instead evaluated $I^2 = \int_{\mathbb{R}^2}\mathrm{d}x\mathrm{d}y\, \mathrm{e}^{-(x^2+y^2)}$ by which polar coordinates can then intervene. As what we have calculated, it results in $I^2 = \pi$. So, Gauss concluded that $I = \sqrt{\pi}$. Contrarily, in our derivation, all is natural. It is suspected that Gaussian integral should be expressed in complex plane!

Another difference between the real and complex one-dimensional Gaussian integral is that it is $1/A$ instead of $\sqrt{1/A}$. We will make this difference clear in section 2.3.

2.2 Multi-Dimensional Complex Gaussian Integral

We are to generalize the previous result from one-dimensional to multi-dimensional, that is

$$
\int_{\mathbb{C}^{2n}} \mathrm{d}z\mathrm{d}\bar{z}\, \exp(-\bar{z}\,A\,z + \bar{b}\,z + \bar{z}\,b),
$$

where A is a positive definite Hermitian matrix and b a complex vector. It must be noted that b and z are now vectors, $\bar{b}$ or $\bar{z}$ means more than b^* and z^*, but including transpose [1]. For this reason, we write $\bar{z}\,b$ instead of $b\,\bar{z}$. In addition, $\mathrm{d}z\mathrm{d}\bar{z}$ means $\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\bar{z}^1 \wedge \cdots \wedge \mathrm{d}\bar{z}^n$.

Following the same strategy used for real Gaussian integral, we first diagonalize A as $A = U D \bar{U}$ with U unitary and D diagonal. Defining the new coordinate $w := \bar{U}\,z$, we are to change the coordinates from z to w.

First, we have to declare how $\mathrm{d}z\mathrm{d}\bar{z}$ changes with coordinates. Recall the case in real space, we have $\mathrm{d}x = \det(\partial x / \partial y)\,\mathrm{d}y$. The derivation of this formula is purely algebric, thus can be directly generalized to complex space. So, from $z = Uw$, we get $\mathrm{d}z = \det(U)\,\mathrm{d}w$; and from $\bar{z} = \bar{w}\,\bar{U}$, we get $\mathrm{d}\bar{z} = \det(\bar{U})\,\mathrm{d}\bar{w}$. Thus, we have

$$
\begin{aligned}
&\mathrm{d}z\mathrm{d}\bar{z}\\
\{\mathrm{d}z, \mathrm{d}\bar{z} = \cdots\} &= \det(U)\det(\bar{U})\,\mathrm{d}w\mathrm{d}\bar{w}\\
\{\det(A)\det(B) = \det(AB)\} &= \det(U\bar{U})\,\mathrm{d}w\mathrm{d}\bar{w}\\
\{U\bar{U} = 1\} &= \mathrm{d}w\mathrm{d}\bar{w}.
\end{aligned}
$$

Then, defining h by $b =: Uh$, we have $\bar{z}\,A\,z = \bar{w}\,\bar{U}\,A\,U\,w = \bar{w}\,D\,w$, $\bar{b}\,z = \bar{h}\,\bar{U}\,U\,w = \bar{h}\,w$, $b\,\bar{z} = h\,\bar{w}$. Altogether, we find

$$
\int_{\mathbb{C}^{2n}} \mathrm{d}z\mathrm{d}\bar{z}\, \exp(-\bar{z}\,A\,z + \bar{b}\,z + b\,\bar{z}) = \int_{\mathbb{C}^{2n}} \mathrm{d}w\mathrm{d}\bar{w}\, \exp(-\bar{w}\,D\,w + \bar{h}\,w + h\,\bar{w}).
$$

1. For a complex vector z, $\bar{z} := (z^*)^t$.

Now, we can separate each dimension and compute for each dimension individually, as

$$\begin{aligned}
&\int_{\mathbb{C}^{2n}} \mathrm{d}z\mathrm{d}\bar{z}\,\exp(-\bar{z}\,A\,z+\bar{b}\,z+b\,\bar{z})\\
&= \int_{\mathbb{C}^{2n}} \mathrm{d}w\mathrm{d}\bar{w} \prod_{\alpha=1}^{n} \exp(-\bar{w}^{\alpha}\,D_{\alpha\alpha}w^{\alpha}+\bar{h}^{\alpha}\,w^{\alpha}+h_{\alpha}\,\bar{w}^{\alpha})\\
\{\text{move } \mathrm{d}w^{\alpha} \text{ to } \mathrm{d}\bar{w}^{\alpha}\} &= (-1)^{n(n-1)/2} \prod_{\alpha=1}^{n} \left[\int_{\mathbb{C}^{2}} \mathrm{d}w^{\alpha}\mathrm{d}\bar{w}^{\alpha}\exp(-\bar{w}^{\alpha}\,D_{\alpha\alpha}w^{\alpha}+\bar{h}^{\alpha}\,w^{\alpha}+h_{\alpha}\,\bar{w}^{\alpha})\right]\\
\{\text{1-dimensional result}\} &= (-1)^{n(n-1)/2} \prod_{\alpha=1}^{n} \left[\frac{-2\pi\mathrm{i}}{D_{\alpha\alpha}}\exp\!\left(\frac{h_{\alpha}\,\bar{h}_{\alpha}}{D_{\alpha\alpha}}\right)\right]\\
&= \frac{\mathrm{i}^{n(n+2)}\,(2\pi)^{n}}{\prod_{\alpha=1}^{n} D_{\alpha\alpha}} \exp\!\left(\sum_{\alpha=1}^{n} \frac{h_{\alpha}\,\bar{h}_{\alpha}}{D_{\alpha\alpha}}\right).
\end{aligned}$$

The final step is changing coordinates back to z from w. We have $D=\bar{U}\,A\,U$ and $h=\bar{U}\,b$. We have

$$\begin{aligned}
&\prod_{\alpha=1}^{n} D_{\alpha\alpha}\\
&= \det(D)\\
\{D=\cdots\} &= \det(\bar{U}\,A\,U)\\
&= \det(\bar{U})\,\det(A)\,\det(U)\\
&= \det(U\,\bar{U})\,\det(A)\\
\{U\,\bar{U}=1\} &= \det(A).
\end{aligned}$$

And since

$$\begin{aligned}
&(\bar{U}\,A\,U)^{-1}\\
\{\text{property of inverse}\} &= U^{-1}\,A^{-1}\,(\bar{U})^{-1}\\
\{U\,\bar{U}=1\} &= \bar{U}\,A^{-1}\,U,
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{\alpha=1}^{n} \frac{h_{\alpha}\,\bar{h}_{\alpha}}{D_{\alpha\alpha}}\\
&= \bar{h}\,D^{-1}\,h\\
\{\text{previous conclusion}\} &= \bar{b}\,U\,(\bar{U}\,A\,U)^{-1}\,\bar{U}\,b\\
\{U\,\bar{U}=1\} &= \bar{b}\,U\,\bar{U}A^{-1}\,U\,\bar{U}\,b\\
&= \bar{b}\,A^{-1}\,b.
\end{aligned}$$

Altogether, we find

$$\int_{\mathbb{C}^{2n}} \mathrm{d}z\mathrm{d}\bar{z}\,\exp(-\bar{z}\,A\,z+\bar{b}\,z+b\,\bar{z}) = \frac{\mathrm{i}^{n(n+2)}(2\pi)^{n}}{\det(A)}\exp(\bar{b}\,A^{-1}\,b), \tag{5}$$

or, since $1/\det(A)=\det(A^{-1})$,

$$\int_{\mathbb{C}^{2n}} \mathrm{d}z\mathrm{d}\bar{z}\,\exp(-\bar{z}\,A\,z+\bar{b}\,z+b\,\bar{z}) = \mathrm{i}^{n(n+2)}(2\pi)^{n}\det(A^{-1})\exp(\bar{b}\,A^{-1}\,b). \tag{6}$$

Go back to the first problem: what is the measurement $\mu(|\varphi\rangle)$? Now, we can say, given a complete orthogonal basis $|x\rangle$, $\mu(|\varphi\rangle)=\prod_{x}\mathrm{d}[\varphi(x)]\mathrm{d}[\overline{\varphi(x)}]$ which is invariant when changing from $|x\rangle$ to another complete orthogonal basis. Indeed, when $|x\rangle\to|k\rangle$ where $|k\rangle$ is another complete orthogonal basis, we have $\prod_{x}\mathrm{d}[\varphi(x)]\mathrm{d}[\overline{\varphi(x)}]=\det(U)\,\det(\bar{U})\,\prod_{k}\mathrm{d}[\varphi(k)]\mathrm{d}[\overline{\varphi(k)}]=\prod_{k}\mathrm{d}[\varphi(k)]\mathrm{d}[\overline{\varphi(k)}]$, where $U_{x,k}:=\langle x\,|k\rangle$ is unitary.

2.3 Reducing to Real Gaussian Integral (TODO)

To reduce to real Gaussian integral, say multi-dimensional version, we simply let the A and b completely real. Then, by changing coordinates from $\mathrm{d}z\mathrm{d}\bar{z}$ to $\mathrm{d}x\mathrm{d}y$, where $z = x + \mathrm{i}\,y$, we will arrive at two individual real Gaussian integrals. So, the result is a multiplication of two real Gaussian integral. It is for this reason, it is the sqrt of $\det(A)$ in real Gaussian integral.

Explicitly, we change coordinates from $\mathrm{d}z\mathrm{d}\bar{z}$ to $\mathrm{d}x\mathrm{d}y$. As it has been derived,

$$\begin{aligned}
&\int_{\mathbb{C}^{2n}} \mathrm{d}z\mathrm{d}\bar{z}\exp(-\bar{z}\,A\,z + \bar{b}\,z + \bar{z}\,b)\\
\{z = x + \mathrm{i}\,y\} &= (-1)^{n(n-1)/2}\,(-2\mathrm{i})^n \int_{\mathbb{R}^{2n}} \mathrm{d}x\mathrm{d}y\exp(-x^t\,A x - y^t\,A y + 2 b^t\,x)\\
&= \mathrm{i}^{n\,(n+2)}\,2^n \int_{\mathbb{R}^n} \mathrm{d}x\exp(-x^t\,A x + 2 b^t\,x)\int_{\mathbb{R}^n} \mathrm{d}y\exp(-y^t\,A y)\\
\{\text{previous result}\} &= \frac{\mathrm{i}^{n\,(n+2)}(2\pi)^n}{\det(A)}\exp(\bar{b}\,A^{-1}\,b).
\end{aligned}$$

So, we have

$$\frac{\pi^n}{\det(A)}\exp(\bar{b}\,A^{-1}\,b) = \int_{\mathbb{R}^n} \mathrm{d}x\exp(-x^t\,A x + 2 b^t\,x)\int_{\mathbb{R}^n} \mathrm{d}y\exp(-y^t\,A y)$$

First letting $b = 0$, since both x and y are dummy variables, we find

$$\int_{\mathbb{R}^n} \mathrm{d}y\exp(-y^t\,A y) = \sqrt{\frac{\pi^n}{\det(A)}}.$$

Plugging back, we find

$$\int_{\mathbb{R}^n} \mathrm{d}x\exp(-x^t\,A x + 2 b^t\,x) = \sqrt{\frac{\pi^n}{\det(A)}}\exp(\bar{b}\,A^{-1}\,b),$$

which is exactly the formula of real multi-dimensional Gaussian integral.

3 Trace (TODO)

3.1 Trace in Continuous Representation

We have known what trace means for finite-dimensional matrix, and even for representation with discrete spectrum. We are to determine how trace is defined in representation with continuous spectrum. To do so, we convert from a discrete representation $|\alpha\rangle$ to a continuous one $|x\rangle$.

Let A an operator. We have known that $\mathrm{tr}\,(A)$ in representation $|\alpha\rangle$ is defined by $\sum_\alpha \langle\alpha|A|\alpha\rangle$. By inserting the complete relation $\int \mathrm{d}x |x\rangle\langle x| = 1$, we have

$$\begin{aligned}
&\sum_\alpha \langle\alpha|A|\alpha\rangle\\
&= \sum_\alpha \int \mathrm{d}x \int \mathrm{d}x'\,\langle\alpha|x\rangle\langle x|A|x'\rangle\langle x'|\alpha\rangle\\
\left\{\sum_\alpha |\alpha\rangle\langle\alpha| = 1\right\} &= \int \mathrm{d}x \int \mathrm{d}x'\,\langle x|A|x'\rangle\left(\sum_\alpha \langle x'|\alpha\rangle\langle\alpha|x\rangle\right)\\
&= \int \mathrm{d}x \int \mathrm{d}x'\,\langle x|A|x'\rangle\,\delta(x - x')\\
&= \int \mathrm{d}x\,\langle x|A|x\rangle.
\end{aligned}$$

So, in continuous representation like $|x\rangle$, the $\mathrm{tr}(A)$ is defined as $\int \mathrm{d}x \, \langle x|A|x\rangle$.

3.2 Trace of Logorithm is Representation Independent

Now, we have solved the first problem and come to the second. That is, how $\mathrm{tr}\ln(A)$ changes with representation.

But, first of all, we have to check what happens when changing representation. We start at a known formula

$$\mathrm{e}^{UX\bar{U}} = U\,\mathrm{e}^X\,\bar{U},$$

where U is a unitary operator and X is an arbitrary operator. It can be proven by simply noticing $U\bar{U}=1$. Taking logorithm on both sides, we find

$$UX\bar{U} = \ln(U\,\mathrm{e}^X\,\bar{U}).$$

And letting $A := \mathrm{e}^X$, we arrive at

$$U\ln(A)\,\bar{U} = \ln(UA\bar{U}). \tag{7}$$

When changing representation from $|\alpha\rangle$ to $|\beta\rangle$, unitary operator U represents $\langle\alpha|\beta\rangle$. And this formula states that

$$\sum_\beta \sum_{\beta'} \langle\alpha|\beta\rangle \ln(\langle\beta|A|\beta'\rangle)\,\langle\beta'|\alpha'\rangle = \ln\left(\sum_\beta \sum_{\beta'} \langle\alpha|\beta\rangle\langle\beta|A|\beta'\rangle\langle\beta'|\alpha'\rangle\right).$$

Notice that $\sum_\beta \sum_{\beta'} \langle\alpha|\beta\rangle\langle\beta|A|\beta'\rangle\langle\beta'|\alpha'\rangle = \langle\alpha|A|\alpha'\rangle$, we then have

$$\sum_\beta \sum_{\beta'} \langle\alpha|\beta\rangle \ln(\langle\beta|A|\beta'\rangle)\,\langle\beta'|\alpha'\rangle = \ln(\langle\alpha|A|\alpha'\rangle).$$

Now, we take trace on both sides. The right hand side comes to be the $\mathrm{tr}\ln(A)$ under the $|\alpha\rangle$ representation. While the left hand side comes to be

$$\begin{aligned} & \sum_\alpha \sum_\beta \sum_{\beta'} \langle\alpha|\beta\rangle \ln(\langle\beta|A|\beta'\rangle)\,\langle\beta'|\alpha\rangle \\ &= \sum_\beta \sum_{\beta'} \ln(\langle\beta|A|\beta'\rangle)\left(\sum_\alpha \langle\beta'|\alpha'\rangle\langle\alpha|\beta\rangle\right) \\ \left\{\sum_\alpha |\alpha\rangle\langle\alpha| = 1\right\} &= \sum_\beta \sum_{\beta'} \ln(\langle\beta|A|\beta'\rangle)\,\langle\beta'|\beta\rangle \\ &= \sum_\beta \ln(\langle\beta|A|\beta\rangle), \end{aligned}$$

which is the $\mathrm{tr}\ln(A)$ under the $|\beta\rangle$ representation. So, $\mathrm{tr}\ln(A)$ in different representations are equal.

Now, we can answer the second problem: $\mathrm{tr}\ln(A)$ is independent of representation.

A Berezin Integral (TODO)

C.f. wikipedia.

We have,

$$\mathrm{i}^{n(n+2)}(2\pi)^n \exp(\bar{b}\,A^{-1}\,b) = \int \mathrm{d}z\,\mathrm{d}\bar{z}\,\mathrm{d}\theta\mathrm{d}\eta\,\exp(-\bar{z}\,A\,z + \bar{b}\,z + b\,\bar{z} - \theta^t A\eta) \tag{8}$$