Chapter 1
Stochastics

1.1 Ordinary Differential Equation

Let $\mathcal{M}$ a smooth manifold, and $f\colon \mathbb{R}\rightarrow\mathcal{M}\rightarrow T\mathcal{M}$ [1.1], we have ordinary differential equation[1.2]

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t)=f^a(t,x). \tag{1.1}$$

This ordinary differential equation induces a push-forward operator, $\hat{T}_{t\rightarrow t'}\colon \mathbb{R}\rightarrow\mathbb{R}\rightarrow\mathcal{M}\rightarrow\mathcal{M}$, which pushes the particle on position on $\mathcal{M}$ at time t to another position on $\mathcal{M}$ at time t'.

Let $\Omega^k(\mathcal{M})$ the space of k-forms on $\mathcal{M}$, where $k\leqslant\dim(\mathcal{M})$. This ordinary differential equation also induces a pull-back operator on k-forms[1.3], $\hat{T}^*_{t\rightarrow t'}\colon \mathbb{R}\rightarrow\mathbb{R}\rightarrow\Omega^k(\mathcal{M})\rightarrow\Omega^k(\mathcal{M})$, which describes the transition of k-form from time t to t'.

Precisely, consider $\forall\psi^{(k)}\in\Omega^k(\mathcal{M})$, we can write it explicitly by indices, as

$$\psi^{(k)}(x)=(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}, \tag{1.2}$$

where the indices in $\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)$ is anti-symmetric. Regardless of the $1/k!$ factor, the $\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)$ can be viewed as the local density at x and the $\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}$ as area or volume unit. Thus, $\psi^{(k)}(x)$, as a whole, is the mass unit. $\hat{T}^*$ is thus is a forward pushing transition of mass unit.

Lemma 1.1. *Explicitly, we have*

$$(\hat{T}^*_{t\rightarrow t'}\,\psi^{(k)})(x')=(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(\hat{T}_{t'\rightarrow t}\,x')\bigwedge_{i=1}^{k}\,(\mathcal{D}\hat{T}_{t'\rightarrow t})^{\alpha_i}_{\beta_i}(x')\,\mathrm{d}x'^{\beta_i}, \tag{1.3}$$

where $\mathcal{D}\hat{T}_{t'\rightarrow t}$ denotes the Jacobian of $\hat{T}_{t'\rightarrow t}$.

Proof. Let $x'=\hat{T}_{t\rightarrow t'}\,x$ and $\psi'^{(k)}=\hat{T}^*_{t\rightarrow t'}\,\psi^{(k)}$, by conservation of mass during the pushing, we have

$$\psi'^{(k)}_{\alpha_1\cdots\alpha_k}(x')\,\mathrm{d}x'^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x'^{\alpha_k}=\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}.$$

Replace x by $x=\hat{T}_{t'\rightarrow t}\,x'$, we get

$$\psi'^{(k)}_{\alpha_1\cdots\alpha_k}(x')\,\mathrm{d}x'^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x'^{\alpha_k}=\psi^{(k)}_{\alpha_1\cdots\alpha_k}(\hat{T}_{t'\rightarrow t}\,x')\bigwedge_{i=1}^{k}\,(\mathcal{D}\hat{T}_{t'\rightarrow t})^{\alpha_i}_{\beta_i}(x')\,\mathrm{d}x'^{\beta_i}.$$

Inserting back the $(1/k!)$ factor, we arrive at

$$(\hat{T}^*_{t\rightarrow t'}\,\psi^{(k)})(x')=(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(\hat{T}_{t'\rightarrow t}\,x')\bigwedge_{i=1}^{k}\,(\mathcal{D}\hat{T}_{t'\rightarrow t})^{\alpha_i}_{\beta_i}(x')\,\mathrm{d}x'^{\beta_i}.$$

□

1.1.1 Infinitesimal Pull-back

Now, we try to derive the explicit expression of $\hat{T}^*_{t\rightarrow t'}$ depending on f^a in the limit $t'\rightarrow t$. This infinitesimal version of pull-back can be described by Lie derivative.

Definition 1.2. *[Lie Derivative] Given $f\colon \mathbb{R}\rightarrow\mathcal{M}\rightarrow T\mathcal{M}$, Lie derivative $\hat{L}_f\colon \mathbb{R}\rightarrow\Omega^k(\mathcal{M})\rightarrow\Omega^k(\mathcal{M})$ is defined as*

$$\hat{L}_f(t):=\lim_{\Delta t\rightarrow 0}\frac{1-\hat{T}^*_{t\rightarrow t+\Delta t}}{\Delta t}, \tag{1.4}$$

where $\hat{T}^*$ is the pull-back induced by f.

Some useful definitions in exterior algebra are recalled. Operators $\hat{\mathrm{d}}:=\mathrm{d}x^\alpha\wedge\partial_\alpha$ [1.4] and, for $\forall f$, $\hat{i}_f:=f^\alpha\,i_\alpha$, where i_α is the interior product[1.5]. Let operators A and B compositions of $\mathrm{d}x^a\wedge$ and i_a, then $[A,B]$ is commutator if both A and B are closed in $\Omega^k(\mathcal{M})$ for $\forall k$ [1.6], otherwise anti-commutator.

1.1. The notation $A\rightarrow B\rightarrow C$ in declarations always means $A\rightarrow(B\rightarrow C)$. Further, $A\rightarrow B\rightarrow\cdots$ means $A\rightarrow(B\rightarrow(\cdots))$. This is an useful convension from Haskell.

1.2. We employ Einstein's convension of summation thoroughly.

1.3. Even though we call it something-back, but it pushes forward the k-forms. The name comes from the fact that forward pushing of k-forms is equivalent to backward pushing the mass unit, as the following discussion shows.

1.4. Operator ∂_α is short for $\partial/\partial x^\alpha$.

1.5. Interior product $i_a\colon\Omega^k(\mathcal{M})\rightarrow\Omega^{k-1}(\mathcal{M})$ is defined as, for $\forall\psi^{(k)}\in\Omega^k(\mathcal{M})$,

$$i_\alpha\,\psi^{(k)}:=(1/k!)\sum_{i=1}^{k}\,(-1)^{i-1}\psi_{\alpha_1\cdots(\alpha_i\rightarrow\alpha)\cdots\alpha_k}\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_i}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}, \tag{1.5}$$

where A means that A is deleted, and $A\rightarrow B$ means that the original A is replaced by B. That is, it annihilates a $\mathrm{d}x^a\wedge$. The most useful property of interior product is the anti-commutator $[\mathrm{d}x^\alpha\wedge,i_\beta]_+=\delta^\alpha_\beta$.

1.6. Recall that operator A is closed in space V if $A\colon V\rightarrow V$.

With these definitions, we conclude the explicit relatoin between f and $\hat{L}_f$, as follow.

Theorem 1.3. *[Cartan's Magic Formula] We have*

$$\hat{L}_f = [\hat{\mathrm{d}}, \hat{\imath}_f]. \tag{1.6}$$

Proof. As $t' = t + \Delta t$ with Δt tiny, we have $\hat{T}_{t' \to t}\, x' = x' - f(t', x')\, \Delta t$. Then, $\mathcal{D}\hat{T}_{t' \to t} = 1 - \mathcal{D}f\, \Delta t$, where $\mathcal{D}f$ denotes the Jacobian of f. Now, insert this two expressions into the definition of $\hat{T}^*_{t \to t'}\, \psi^{(k)}$, we find

$$(\hat{T}^*_{t \to t'}\, \psi^{(k)})(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x' - f(t', x')\, \Delta t) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \Delta t) \mathrm{d}x'^{\beta_i}.$$

First, consider the expansion of $(1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \Delta t) \mathrm{d}x'^{\beta_i}$, up to $\mathcal{O}(\Delta t)$,

$$\begin{aligned}
&(1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \Delta t) \mathrm{d}x'^{\beta_i} \\
&= (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&- \Delta t \sum_{i=1}^{k} (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},
\end{aligned}$$

where $A \to B$ means that the original A is replaced by B. Now, we show that summation in the last line equals to $\partial_\beta f^\alpha(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha\, \psi^{(k)}(x)$. Recall that

$$i_\alpha\, \psi^{(k)}(x) := (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_i} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},$$

where A means the original A is deleted. Indeed,

$$\begin{aligned}
&(\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha\, \psi^{(k)}(x) \\
&= (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_i} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&= (\partial_\beta f^\alpha)(t', x') \sum_{i=1}^{k} (1/k!)\, \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^\beta) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&= \sum_{i=1}^{k} (\partial_{\beta_i} f^{\alpha_i})(t', x')\, (1/k!)\, \psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&= \sum_{i=1}^{k} (\partial_{\beta_i} f^{\alpha_i})(t', x')\, (1/k!)\, \psi_{\alpha_1 \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},
\end{aligned}$$

where in the last two lines, we replaced the dummy indices $\alpha \to \alpha_i$ and $\beta \to \beta_i$, and then found that $\psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}$ can be written back to $\psi_{\alpha_1 \cdots \alpha_k}$. Thus,

$$(1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \Delta t) \mathrm{d}x'^{\beta_i} = (1 - \Delta t\, (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}(x).$$

So, we find,

$$\begin{aligned}
(\hat{T}^*_{t \to t+\Delta t}\, \psi^{(k)})(x') &= (1 - \Delta t\, (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}(x) \\
\{x = x' - f(t', x')\, \Delta t\} &= (1 - \Delta t\, (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}(x' - f(t', x')\, \Delta t) \\
&= (1 - \Delta t\, (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, (1 - \Delta t\, f^\alpha(t', x')\, \partial_\alpha)\, \psi^{(k)}(x') \\
&= \psi^{(k)}(x') - (f^a(t', x') \partial_a + (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}(x')\, \Delta t + \mathcal{O}(\Delta t^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{L}_f &:= \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t \to t+\Delta t}}{\Delta t} \\
&= f^a \partial_a + (\partial_\beta f^\alpha)\, \mathrm{d}x^\beta \wedge i_\alpha.
\end{aligned}$$

Since $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$ and $\hat{\imath}_f := f^\alpha\, i_\alpha$, we have

$$\begin{aligned}
[\hat{\mathrm{d}}, \hat{\imath}_f] &= \mathrm{d}x^\alpha \wedge \partial_\alpha\, f^\beta\, i_\beta + f^\beta\, i_\beta \mathrm{d}x^\alpha \wedge \partial_\alpha \\
&= (\partial_\alpha\, f^\beta) \mathrm{d}x^\alpha \wedge i_\beta + f^\beta\, \mathrm{d}x^\alpha \wedge i_\beta\, \partial_\alpha \\
\{[\mathrm{d}x^a \wedge, i_\beta] = \delta^a_\beta\} &+ f^\beta\, \delta^a_\beta \partial_\alpha - f^\beta\, \mathrm{d}x^\alpha \wedge i_\beta\, \partial_\alpha \\
&= (\partial_\alpha\, f^\beta) \mathrm{d}x^\alpha \wedge i_\beta + f^a \partial_a,
\end{aligned}$$

which is $\hat{L}_f$. □

1.2 Stochastic Differential Equation

A direct generalization of ordinary differential equatoin is adding a Gaussian noise, as stochastic differential equation

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(t, x) + g^a_\beta(t, x)\, \eta^\beta(t), \tag{1.7}$$

where, for $\forall t$ and α, $\eta^\alpha(t) \sim \mathcal{N}(0, 1/\mathrm{d}t)$. Thus, $\eta: \mathbb{R} \to V$ with V an Euclidean space, and $g: \mathbb{R} \to \mathcal{M} \to V \to T\mathcal{M}$.

For any functional $F[\eta]$, split the time interval $[t, t']$ by $t = t_1 < t_2 < \cdots < t_N = t'$, with $t_{i+1} - t_i \equiv \Delta t$, then define the expectation as

$$\begin{aligned}
\langle F \rangle &:= \int D[\eta] \exp\left(-\frac{1}{2} \int \mathrm{d}t\, \delta_{\alpha\beta}\, \eta^a(t)\, \eta^\beta(t)\right) F[\eta] \\
&:= \lim_{\Delta t \to 0} Z^{-1} \int \mathrm{d}\eta(t_1) \cdots \mathrm{d}\eta(t_N) \exp\left(-\frac{1}{2} \sum_i \Delta t\, \delta_{\alpha\beta}\, \eta^a(t_i)\, \eta^\beta(t_i)\right) F[\eta],
\end{aligned}$$

where Z the normalization factor so that $\langle 1 \rangle = 1$. Notice that the dimension of η and that of f may not equal.

1.2.1 Infinitesimal Pull-back Expectation

For any configuration η given, the stochastic differential equation is reduced to an ordinary differential equation. In this case, the pull-back depends on η, that is, $\hat{T}^*_{t\to t'}[\eta]$. We care about the expectation $\langle \hat{T}^*_{t\to t'}[\eta]\rangle$ over all possible configuration of η, especially it's infinitesimal version.

As before, define the stochastic version of Lie derivative, as

$$\hat{H}_{(f,g)}(t) := \lim_{\Delta t\to 0} \frac{1 - \langle \hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle}{\Delta t}, \tag{1.8}$$

where $\hat{T}^*_{t\to t+\Delta t}$ depends on the configuration η.

Theorem 1.4. *We have*

$$\hat{H}_{(f,g)} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2, \tag{1.9}$$

where $\hat{L}_g^2 := \delta^{\alpha\beta}\,\hat{L}_{g_\alpha}\hat{L}_{g_\beta}$.

Proof. Given configuration of η, let $F^\alpha_\eta(t,x) := f^\alpha(t,x) + g^\alpha_\beta(t,x)\eta^\beta(t)$. Directly, we have

$$\hat{L}_{F_\eta} = \hat{L}_f + \hat{L}_{g_\beta}\eta^\beta.$$

Then, we have $\hat{T}^*_{t\to t+\Delta t}[\eta] = \exp(-\hat{L}_{F_\eta}\Delta t)$.[1.7] Then,

$$\begin{aligned}\langle \hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle &= \int \mathrm{d}\eta(t)\exp\left(-\frac{1}{2}\Delta t\,\delta_{\alpha\beta}\,\eta^\alpha(t)\,\eta^\beta(t)\right)\exp(-\hat{L}_{F_\eta}\Delta t)\\ &= \int \mathrm{d}\eta(t)\exp\left(-\frac{1}{2}\Delta t\,\delta_{\alpha\beta}\,\eta^\alpha(t)\,\eta^\beta(t)\right)\left(1 - \hat{L}_{F_\eta}\Delta t + \frac{1}{2}(\hat{L}_{F_\eta}\Delta t)^2 + \cdots\right)\\ &= 1 - \langle \hat{L}_{F_\eta}\Delta t\rangle + \frac{1}{2}\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle + \cdots\end{aligned}$$

Since $\langle\eta^\beta\rangle = 0$, $\langle \hat{L}_{F_\eta}\Delta t\rangle = \langle \hat{L}_f + \hat{L}_{g_\beta}\eta^\beta\rangle\Delta t = \hat{L}_f\,\Delta t$. And since $\langle\eta^\alpha\,\eta^\beta\rangle = \delta^{\alpha\beta}/\Delta t$,

$$\begin{aligned} &\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle\\ &= \langle(\hat{L}_f + \hat{L}_{g_\alpha}\eta^\alpha)\,(\hat{L}_f + \hat{L}_{g_\beta}\eta^\beta)\rangle\,\Delta t^2\\ \{\langle\eta^\beta\rangle = 0\} &= \hat{L}_f^2\,\Delta t^2 + \hat{L}_{g_\alpha}\hat{L}_{g_\beta}\langle\eta^\alpha\,\eta^\beta\rangle\,\Delta t^2\\ \{\langle\eta^\alpha\,\eta^\beta\rangle = \delta^{\alpha\beta}/\Delta t\} &= \hat{L}_f^2\,\Delta t^2 + \hat{L}_{g_\alpha}\hat{L}_{g_\beta}\,\delta^{\alpha\beta}\,\Delta t\\ \{\hat{L}_g^2 := \cdots\} &= \hat{L}_f^2\,\Delta t^2 + \hat{L}_g^2\,\Delta t.\end{aligned}$$

Thus,

$$\begin{aligned}\langle \hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle &= 1 - \langle\hat{L}_{F_\eta}\rangle\Delta t + \frac{1}{2}\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle + \cdots\\ &= 1 - \hat{L}_f\,\Delta t + \frac{1}{2}\langle\hat{L}_g^2\rangle\Delta t + o(\Delta t).\end{aligned}$$

So, finally,

$$\hat{H}_{(f,g)} := \lim_{\Delta t\to 0}\frac{\hat{1} - \langle \hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle}{\Delta t} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2.$$ □

Example 1.5. [Fokker-Planck Equation] In the case $g^\alpha_\beta \equiv \sqrt{2T}\,\delta^\alpha_\beta$,

$$\hat{H} = (\partial_\alpha f^\beta)\,\mathrm{d}x^\alpha \wedge i_\beta + f^\alpha\partial_\alpha - T\partial^2, \tag{1.10}$$

where $\partial^2 := \delta^{\alpha\beta}\partial_\alpha\partial_\beta$. Applying on $\psi^{(D)}$ with $D = \dim(\mathcal{M})$, since $\mathrm{d}x^\alpha \wedge \psi^{(D)} = 0$, and thus $\mathrm{d}x^\alpha \wedge i_\beta\,\psi^{(D)} = \delta^\alpha_\beta\,\psi^{(D)}$, we find

$$\hat{H}\psi^{(D)} = (\partial_\alpha f^\alpha - T\partial^2)\psi^{(D)}, \tag{1.11}$$

which is the Fokker-Planck equation.

1.2.2 Symmetry (TODO)

1.7. TODO: needs some proof.

Bibliography

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