Chapter 1
Stochastics

1.1 Ordinary Differential Equation

Let $\mathcal{M}$ a smooth manifold, and $f\colon \mathbb{R} \to \mathcal{M} \to T\mathcal{M}$ [1.1], we have ordinary differential equation[1.2]

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(t,x). \tag{1.1}$$

This ordinary differential equation induces a push-forward operator, $\hat{T}_{t \to t'}\colon \mathbb{R} \to \mathbb{R} \to \mathcal{M} \to \mathcal{M}$, which pushes the particle on position on $\mathcal{M}$ at time t to another position on $\mathcal{M}$ at time t'.

Let $\Omega^k(\mathcal{M})$ the space of k-forms on $\mathcal{M}$, where $k \leqslant \dim(\mathcal{M})$. This ordinary differential equation also induces a pull-back operator on k-forms[1.3], $\hat{T}^*_{t \to t'}\colon \mathbb{R} \to \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$, which describes the transition of k-form from time t to t'.

Precisely, consider $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$, we can write it explicitly by indices, as

$$\psi^{(k)}(x) = (1/k!)\,\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \tag{1.2}$$

where the indices in $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ is anti-symmetric. Regardless of the $1/k!$ factor, the $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ can be viewed as the local density at x and the $\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}$ as area or volume unit. Thus, $\psi^{(k)}(x)$, as a whole, is the mass unit. $\hat{T}^*$ is thus is a forward pushing transition of mass unit.

Lemma 1.1. *Explicitly, we have*

$$(\hat{T}^*_{t \to t'}\,\psi^{(k)})(x') = (1/k!)\,\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\,x') \bigwedge_{i=1}^{k}\,(\mathcal{D}\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\,\mathrm{d}x'^{\beta_i}, \tag{1.3}$$

where $\mathcal{D}\hat{T}_{t' \to t}$ denotes the Jacobian of $\hat{T}_{t' \to t}$.

Proof. Let $x' = \hat{T}_{t \to t'}\,x$ and $\psi'^{(k)} = \hat{T}^*_{t' \to t'}\,\psi^{(k)}$, by conservation of mass during the pushing, we have

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\,\mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}.$$

Replace x by $x = \hat{T}_{t' \to t}\,x'$, we get

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\,\mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\,x') \bigwedge_{i=1}^{k}\,(\mathcal{D}\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\,\mathrm{d}x'^{\beta_i}.$$

Inserting back the $(1/k!)$ factor, we arrive at

$$(\hat{T}^*_{t \to t'}\,\psi^{(k)})(x') = (1/k!)\,\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\,x') \bigwedge_{i=1}^{k}\,(\mathcal{D}\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\,\mathrm{d}x'^{\beta_i}. \qquad \square$$

1.1.1 Infinitesimal Pull-back

Now, we try to derive the explicit expression of $\hat{T}^*_{t \to t'}$ depending on f^a in the limit $t' \to t$. This infinitesimal version of pull-back can be described by Lie derivative.

Definition 1.2. *[Lie Derivative] Given $f\colon \mathbb{R} \to \mathcal{M} \to T\mathcal{M}$, Lie derivative $\hat{L}_f\colon \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$ is defined as*

$$\hat{L}_f(t) := \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t \to t + \Delta t}}{\Delta t}, \tag{1.4}$$

where $\hat{T}^*$ is the pull-back induced by f.

Some useful definitions in exterior algebra are recalled. Operators $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$ [1.4] and, for $\forall f$, $\hat{i}_f := f^\alpha\,i_\alpha$, where i_α is the interior product[1.5]. Let A and B compositions of $\mathrm{d}x^a \wedge$ and i_a, then $[A, B]$ is commutator if both A and B have balanced $\mathrm{d}x^a \wedge$ and i_a, otherwise anti-commutator.
With these definitions, we conclude the explicit relatoin between f and $\hat{L}_f$, as follow.

[1.1]. The notation $A \to B \to C$ in declarations always means $A \to (B \to C)$. Further, $A \to B \to \cdots$ means $A \to (B \to (\cdots))$. This is an useful convension from Haskell.

[1.2]. We employ Einstein's convension of summation thoroughly.

[1.3]. Even though we call it something-back, but it pushes forward the k-forms. The name comes from the fact that forward pushing of k-forms is equivalent to backward pushing the mass unit, as the following discussion shows.

[1.4]. Operator ∂_α is short for $\partial/\partial x^\alpha$.

[1.5]. Interior product $i_a\colon \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ is defined as, for $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$,

$$i_\alpha\,\psi^{(k)} := (1/k!)\sum_{i=1}^{k}\,(-1)^{i-1}\psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_i} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \tag{1.5}$$

where A means that A is deleted, and $A \to B$ means that the original A is replaced by B. That is, it annihilates a $\mathrm{d}x^a \wedge$. The most useful property of interior product is the anti-commutator $[\mathrm{d}x^\alpha \wedge, i_\beta]_+ = \delta^\alpha_\beta$.

Theorem 1.3. *[Cartan's Magic Formula] We have*

$$\hat{L}_f = [\hat{d}, \hat{i}_f]. \tag{1.6}$$

Proof. As $t' = t + \Delta t$ with Δt tiny, we have $\hat{T}_{t' \to t} x' = x' - f(t', x') \Delta t$. Then, $\mathcal{D}\hat{T}_{t' \to t} = 1 - \mathcal{D}f \Delta t$, where $\mathcal{D}f$ denotes the Jacobian of f. Now, insert this two expressions into the definition of $\hat{T}^*_{t \to t'} \psi^{(k)}$, we find

$$(\hat{T}^*_{t \to t'} \psi^{(k)})(x') = (1/k!) \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x' - f(t', x') \Delta t) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x') \Delta t) dx'^{\beta_i}.$$

First, consider the expansion of $(1/k!) \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x') \Delta t) dx'^{\beta_i}$, up to $\mathcal{O}(\Delta t)$,

$$(1/k!) \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x') \Delta t) dx'^{\beta_i}$$
$$= (1/k!) \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$$
$$- \Delta t \sum_{i=1}^{k} (1/k!) \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) (\partial_{\beta_i} f^{\alpha_i})(t', x') dx^{\alpha_1} \wedge \cdots \wedge (dx^{\alpha_i} \to dx^{\beta_i}) \wedge \cdots \wedge dx^{\alpha_k},$$

where $A \to B$ means that the original A is replaced by B. Now, we show that summation in the last line equals to $\partial_\beta f^\alpha(t', x') dx^\beta \wedge i_\alpha \psi^{(k)}(x)$. Recall that

$$i_\alpha \psi^{(k)}(x) := (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}(x) dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_i} \wedge \cdots \wedge dx^{\alpha_k},$$

where A means the original A is deleted. Indeed,

$$(\partial_\beta f^\alpha)(t', x') dx^\beta \wedge i_\alpha \psi^{(k)}(x)$$
$$= (\partial_\beta f^\alpha)(t', x') dx^\beta \wedge (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}(x) dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_i} \wedge \cdots \wedge dx^{\alpha_k}$$
$$= (\partial_\beta f^\alpha)(t', x') \sum_{i=1}^{k} (1/k!) \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}(x) dx^{\alpha_1} \wedge \cdots \wedge (dx^{\alpha_i} \to dx^\beta) \wedge \cdots \wedge dx^{\alpha_k}$$
$$= \sum_{i=1}^{k} (\partial_{\beta_i} f^{\alpha_i})(t', x') (1/k!) \psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}(x) dx^{\alpha_1} \wedge \cdots \wedge (dx^{\alpha_i} \to dx^{\beta_i}) \wedge \cdots \wedge dx^{\alpha_k}$$
$$= \sum_{i=1}^{k} (\partial_{\beta_i} f^{\alpha_i})(t', x') (1/k!) \psi_{\alpha_1 \cdots \alpha_k}(x) dx^{\alpha_1} \wedge \cdots \wedge (dx^{\alpha_i} \to dx^{\beta_i}) \wedge \cdots \wedge dx^{\alpha_k},$$

where in the last two lines, we replaced the dummy indices $\alpha \to \alpha_i$ and $\beta \to \beta_i$, and then found that $\psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}$ can be written back to $\psi_{\alpha_1 \cdots \alpha_k}$. Thus,

$$(1/k!) \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x') \Delta t) dx'^{\beta_i} = (1 - \Delta t (\partial_\beta f^\alpha)(t', x') dx^\beta \wedge i_\alpha) \psi^{(k)}(x).$$

So, we find,

$$(\hat{T}^*_{t \to t+\Delta t} \psi^{(k)})(x') = (1 - \Delta t (\partial_\beta f^\alpha)(t', x') dx^\beta \wedge i_\alpha) \psi^{(k)}(x)$$
$$\{x = x' - f(t', x') \Delta t\} = (1 - \Delta t (\partial_\beta f^\alpha)(t', x') dx^\beta \wedge i_\alpha) \psi^{(k)}(x' - f(t', x') \Delta t)$$
$$= (1 - \Delta t (\partial_\beta f^\alpha)(t', x') dx^\beta \wedge i_\alpha) (1 - \Delta t f^\alpha(t', x') \partial_\alpha) \psi^{(k)}(x')$$
$$= \psi^{(k)}(x') - (f^a(t', x') \partial_a + (\partial_\beta f^\alpha)(t', x') dx^\beta \wedge i_\alpha) \psi^{(k)}(x') \Delta t + \mathcal{O}(\Delta t^2).$$

Thus,

$$\hat{L}_f := \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t \to t+\Delta t}}{\Delta t}$$
$$= f^a \partial_a + (\partial_\beta f^\alpha) dx^\beta \wedge i_\alpha.$$

Since $\hat{d} := dx^\alpha \wedge \partial_\alpha$ and $\hat{i}_f := f^\alpha i_\alpha$, we have

$$[\hat{d}, \hat{i}_f] = dx^\alpha \wedge \partial_\alpha f^\beta i_\beta + f^\beta i_\beta dx^\alpha \wedge \partial_\alpha$$
$$= (\partial_\alpha f^\beta) dx^\alpha \wedge i_\beta + f^\beta dx^\alpha \wedge i_\beta \partial_\alpha$$
$$\{[dx^a \wedge, i_\beta] = \delta^a_\beta\} + f^\beta \delta^\alpha_\beta \partial_\alpha - f^\beta dx^\alpha \wedge i_\beta \partial_\alpha$$
$$= (\partial_\alpha f^\beta) dx^\alpha \wedge i_\beta + f^a \partial_a,$$

which is $\hat{L}_f$. $\qquad \square$

1.2 Stochastic Differential Equation

A direct generalization of ordinary differential equatoin is adding a Gaussian noise, as stochastic differential equation

$$\frac{dx^a}{dt}(t) = f^a(t, x) + g^a_\beta(t, x) \eta^\beta(t), \tag{1.7}$$

where, for $\forall t$ and α, $\eta^\alpha(t) \sim \mathcal{N}(0, 1/dt)$. Thus, $\eta: \mathbb{R} \to V$ with V an Euclidean space, and $g: \mathbb{R} \to \mathcal{M} \to V \to T\mathcal{M}$.

For any functional $F[\eta]$, split the time interval $[t, t']$ by $t = t_1 < t_2 < \cdots < t_N = t'$, with $t_{i+1} - t_i \equiv \Delta t$, then define the expectation as

$$\langle F \rangle := \int D[\eta] \exp\left(-\frac{1}{2} \int dt \, \delta_{\alpha\beta} \eta^a(t) \, \eta^\beta(t) \right) F[\eta]$$

$$:= \lim_{\Delta t \to 0} Z^{-1} \int d\eta(t_1) \cdots d\eta(t_N) \exp\left(-\frac{1}{2} \sum_i \Delta t \, \delta_{\alpha\beta} \eta^a(t_i) \, \eta^\beta(t_i) \right) F[\eta],$$

where Z the normalization factor so that $\langle 1 \rangle = 1$. Notice that the dimension of η and that of f may not equal.

1.2.1 Infinitesimal Pull-back Expectation

For any configuration η given, the stochastic differential equation is reduced to an ordinary differential equation. In this case, the pull-back depends on η, that is, $\hat{T}^*_{t \to t'}[\eta]$. We care about the expectation $\langle \hat{T}^*_{t \to t'}[\eta] \rangle$ over all possible configuration of η, especially it's infinitesimal version.

As before, define the stochastic version of Lie derivative, as

$$\hat{H}_{(f,g)}(t) := \lim_{\Delta t \to 0} \frac{1 - \langle \hat{T}^*_{t \to t+\Delta t}[\cdot]\rangle}{\Delta t}, \tag{1.8}$$

where $\hat{T}^*_{t \to t+\Delta t}$ depends on the configuration η.

Theorem 1.4. *We have*

$$\hat{H}_{(f,g)} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2, \tag{1.9}$$

where $\hat{L}_g^2 := \delta^{\alpha\beta}\hat{L}_{g_\alpha}\hat{L}_{g_\beta}$.

Proof. Given configuration of η, let $F_\eta^\alpha(t,x) := f^\alpha(t,x) + g_\beta^\alpha(t,x)\eta^\beta(t)$. Directly, we have

$$\hat{L}_{F_\eta} = \hat{L}_f + \hat{L}_{g_\beta}\eta^\beta.$$

Then, we have $\hat{T}^*_{t \to t+\Delta t}[\eta] = \exp(-\hat{L}_{F_\eta}\Delta t)$.[1.6] Then,

$$
\begin{aligned}
\langle \hat{T}^*_{t \to t+\Delta t}[\cdot]\rangle &= \int \mathrm{d}\eta(t)\exp\left(-\frac{1}{2}\Delta t\,\delta_{\alpha\beta}\,\eta^\alpha(t)\,\eta^\beta(t)\right)\exp(-\hat{L}_{F_\eta}\Delta t)\\
&= \int \mathrm{d}\eta(t)\exp\left(-\frac{1}{2}\Delta t\,\delta_{\alpha\beta}\,\eta^\alpha(t)\,\eta^\beta(t)\right)\left(1 - \hat{L}_{F_\eta}\Delta t + \frac{1}{2}(\hat{L}_{F_\eta}\Delta t)^2 + \cdots\right)\\
&= 1 - \langle \hat{L}_{F_\eta}\Delta t\rangle + \frac{1}{2}\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle + \cdots
\end{aligned}
$$

Since $\langle\eta^\beta\rangle = 0$, $\langle\hat{L}_{F_\eta}\Delta t\rangle = \langle\hat{L}_f + \hat{L}_{g_\beta}\eta^\beta\rangle\Delta t = \hat{L}_f\,\Delta t$. And since $\langle\eta^\alpha\,\eta^\beta\rangle = \delta^{\alpha\beta}/\Delta t$,

$$
\begin{aligned}
&\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle\\
&= \langle(\hat{L}_f + \hat{L}_{g_\alpha}\eta^\alpha)\,(\hat{L}_f + \hat{L}_{g_\beta}\eta^\beta)\rangle\,\Delta t^2\\
\{\langle\eta^\beta\rangle = 0\} &= \hat{L}_f^2\,\Delta t^2 + \hat{L}_{g_\alpha}\hat{L}_{g_\beta}\langle\eta^\alpha\,\eta^\beta\rangle\,\Delta t^2\\
\{\langle\eta^\alpha\,\eta^\beta\rangle = \delta^{\alpha\beta}/\Delta t\} &= \hat{L}_f^2\,\Delta t^2 + \hat{L}_{g_\alpha}\hat{L}_{g_\beta}\delta^{\alpha\beta}\,\Delta t\\
\{\hat{L}_g^2 := \cdots\} &= \hat{L}_f^2\,\Delta t^2 + \hat{L}_g^2\,\Delta t.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\langle \hat{T}^*_{t \to t+\Delta t}[\cdot]\rangle &= 1 - \langle\hat{L}_{F_\eta}\rangle\Delta t + \frac{1}{2}\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle + \cdots\\
&= 1 - \hat{L}_f\,\Delta t + \frac{1}{2}\langle\hat{L}_g^2\rangle\Delta t + o(\Delta t).
\end{aligned}
$$

So, finally,

$$\hat{H}_{(f,g)} := \lim_{\Delta t \to 0}\frac{\hat{1} - \langle\hat{T}^*_{t \to t+\Delta t}[\cdot]\rangle}{\Delta t} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2. \qquad \square$$

Example 1.5. [Fokker-Planck Equation] In the case $g_\beta^\alpha \equiv \sqrt{2T}\,\delta_\beta^\alpha$,

$$\hat{H} = (\partial_\alpha f^\beta)\,\mathrm{d}x^\alpha \wedge i_\beta + f^\alpha\partial_\alpha - T\partial^2, \tag{1.10}$$

where $\partial^2 := \delta^{\alpha\beta}\partial_\alpha\partial_\beta$. Applying on $\psi^{(D)}$ with $D = \dim(\mathcal{M})$, since $\mathrm{d}x^\alpha \wedge \psi^{(D)} = 0$, and thus $\mathrm{d}x^\alpha \wedge i_\beta\psi^{(D)} = \delta_\beta^\alpha\,\psi^{(D)}$, we find

$$\hat{H}\psi^{(D)} = (\partial_\alpha f^\alpha - T\partial^2)\psi^{(D)}, \tag{1.11}$$

which is the Fokker-Planck equation.

1.2.2 Symmetry (TODO)

1.6. TODO: needs some proof.

Bibliography

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