Chapter 1
Stochastics

1.1 Ordinary Differential Equation

Let $\mathcal{M}$ a smooth manifold, and $f\colon \mathbb{R} \to \mathcal{M} \to T\mathcal{M}$ [1.1], we have ordinary differential equation[1.2]

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(t, x). \tag{1.1}$$

This ordinary differential equation induces a push-forward operator, $\hat{T}_{t\to t'}\colon \mathbb{R} \to \mathbb{R} \to \mathcal{M} \to \mathcal{M}$, which pushes the particle on position on $\mathcal{M}$ at time t to another position on $\mathcal{M}$ at time t'.

Let $\Omega^k(\mathcal{M})$ the space of k-forms on $\mathcal{M}$, where $k \leqslant \dim(\mathcal{M})$. This ordinary differential equation also induces a pull-back operator on k-forms[1.3], $\hat{T}^*_{t\to t'}\colon \mathbb{R} \to \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$, which describes the transition of k-form from time t to t'.

Precisely, consider $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$, we can write it explicitly by indices, as

$$\psi^{(k)}(x) = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \tag{1.2}$$

where the indices in $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ is anti-symmetric. Regardless of the $1/k!$ factor, the $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ can be viewed as the local density at x and the $\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}$ as area or volume unit. Thus, $\psi^{(k)}(x)$, as a whole, is the mass unit. $\hat{T}^*$ is thus is a forward pushing transition of mass unit.

Lemma 1.1. *Explicitly, we have*

$$(\hat{T}^*_{t\to t'}\, \psi^{(k)})(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t'\to t}\, x') \bigwedge_{i=1}^{k} (\mathcal{D}\hat{T}_{t'\to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}, \tag{1.3}$$

where $\mathcal{D}\hat{T}_{t'\to t}$ denotes the Jacobian of $\hat{T}_{t'\to t}$.

Proof. Let $x' = \hat{T}_{t\to t'}\, x$ and $\psi'^{(k)} = \hat{T}^*_{t\to t'}\, \psi^{(k)}$, by conservation of mass during the pushing, we have

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, \mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}.$$

Replace x by $x = \hat{T}_{t'\to t}\, x'$, we get

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, \mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t'\to t}\, x') \bigwedge_{i=1}^{k} (\mathcal{D}\hat{T}_{t'\to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}.$$

Inserting back the $(1/k!)$ factor, we arrive at

$$(\hat{T}^*_{t\to t'}\, \psi^{(k)})(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t'\to t}\, x') \bigwedge_{i=1}^{k} (\mathcal{D}\hat{T}_{t'\to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}.$$ □

Lemma 1.2. *$\hat{T}^*_{t\to t'}$ forms a group. That is*

$$\hat{T}^*_{t\to t'}\, \hat{T}^*_{t'\to t''} = \hat{T}^*_{t\to t''}. \tag{1.4}$$

Proof. TODO □

1.1.1 Infinitesimal Pull-back

Now, we try to derive the explicit expression of $\hat{T}^*_{t\to t'}$ depending on f^a in the limit $t' \to t$. This infinitesimal version of pull-back can be described by Lie derivative.

Definition 1.3. *[Lie Derivative] Given $f\colon \mathbb{R} \to \mathcal{M} \to T\mathcal{M}$, Lie derivative $\hat{L}_f\colon \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$ is defined as*

$$\hat{L}_f(t) := \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t\to t+\Delta t}}{\Delta t}, \tag{1.5}$$

where $\hat{T}^*$ is the pull-back induced by f.

1.1. The notation $A \to B \to C$ in declarations always means $A \to (B \to C)$. Further, $A \to B \to \cdots$ means $A \to (B \to (\cdots))$. This is an useful convension from Haskell.

1.2. We employ Einstein's convension of summation thoroughly.

1.3. Even though we call it something-back, but it pushes forward the k-forms. The name comes from the fact that forward pushing of k-forms is equivalent to backward pushing the mass unit, as the following discussion shows.

Some useful definitions in exterior algebra are recalled. Operators $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$ [1.4] and, for $\forall f$, $\hat{i}_f := f^\alpha i_\alpha$, where i_α is the interior product[1.5]. Let operators A and B compositions of $\mathrm{d}x^a\wedge$ and i_a, then $[A, B]$ is commutator if both A and B are closed in $\Omega^k(\mathcal{M})$ for $\forall k$ [1.6], otherwise anti-commutator.

With these definitions, we conclude the explicit relatoin between f and $\hat{L}_f$, as follow.

Theorem 1.4. *[Cartan's Magic Formula] We have*

$$\hat{L}_f = [\hat{\mathrm{d}}, \hat{i}_f]. \tag{1.7}$$

Proof. As $t' = t + \Delta t$ with Δt tiny, we have $\hat{T}_{t'\to t}\, x' = x' - f(t', x')\,\Delta t$. Then, $\mathcal{D}\hat{T}_{t'\to t} = 1 - \mathcal{D}f\,\Delta t$, where $\mathcal{D}f$ denotes the Jacobian of f. Now, insert this two expressions into the definition of $\hat{T}^*_{t\to t'}\,\psi^{(k)}$, we find

$$(\hat{T}^*_{t\to t'}\,\psi^{(k)})(x') = (1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x' - f(t', x')\,\Delta t)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\,\Delta t)\mathrm{d}x'^{\beta_i}.$$

First, consider the expansion of $(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\,\Delta t)\mathrm{d}x'^{\beta_i}$, up to $\mathcal{O}(\Delta t)$,

$$\begin{aligned}
&(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\,\Delta t)\mathrm{d}x'^{\beta_i}\\
&= (1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}\\
&\quad - \Delta t\sum_{i=1}^{k}(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\,(\partial_{\beta_i} f^{\alpha_i})(t', x')\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge(\mathrm{d}x^{\alpha_i}\to\mathrm{d}x^{\beta_i})\wedge\cdots\wedge\mathrm{d}x^{\alpha_k},
\end{aligned}$$

where $A\to B$ means that the original A is replaced by B. Now, we show that summation in the last line equals to $\partial_\beta f^\alpha(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha\,\psi^{(k)}(x)$. Recall that

$$i_\alpha\,\psi^{(k)}(x) := (1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_i}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k},$$

where A means the original A is deleted. Indeed,

$$\begin{aligned}
&(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha\,\psi^{(k)}(x)\\
&= (\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge(1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_i}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}\\
&= (\partial_\beta f^\alpha)(t', x')\sum_{i=1}^{k}(1/k!)\,\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge(\mathrm{d}x^{\alpha_i}\to\mathrm{d}x^{\beta})\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}\\
&= \sum_{i=1}^{k}(\partial_{\beta_i} f^{\alpha_i})(t', x')\,(1/k!)\,\psi_{\alpha_1\cdots\alpha_i\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge(\mathrm{d}x^{\alpha_i}\to\mathrm{d}x^{\beta_i})\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}\\
&= \sum_{i=1}^{k}(\partial_{\beta_i} f^{\alpha_i})(t', x')\,(1/k!)\,\psi_{\alpha_1\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge(\mathrm{d}x^{\alpha_i}\to\mathrm{d}x^{\beta_i})\wedge\cdots\wedge\mathrm{d}x^{\alpha_k},
\end{aligned}$$

where in the last two lines, we replaced the dummy indices $\alpha\to\alpha_i$ and $\beta\to\beta_i$, and then found that $\psi_{\alpha_1\cdots\alpha_i\cdots\alpha_k}$ can be written back to $\psi_{\alpha_1\cdots\alpha_k}$. Thus,

$$(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\,\Delta t)\mathrm{d}x'^{\beta_i} = (1 - \Delta t\,(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}(x).$$

So, we find,

$$\begin{aligned}
(\hat{T}^*_{t\to t+\Delta t}\,\psi^{(k)})(x') &= (1 - \Delta t\,(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}(x)\\
\{x = x' - f(t', x')\,\Delta t\} &= (1 - \Delta t\,(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}(x' - f(t', x')\,\Delta t)\\
&= (1 - \Delta t\,(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,(1 - \Delta t\, f^\alpha(t', x')\,\partial_\alpha)\,\psi^{(k)}(x')\\
&= \psi^{(k)}(x') - (f^a(t', x')\partial_a + (\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}(x')\,\Delta t + \mathcal{O}(\Delta t^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{L}_f &:= \lim_{\Delta t\to 0}\frac{1 - \hat{T}^*_{t\to t+\Delta t}}{\Delta t}\\
&= f^a\partial_a + (\partial_\beta f^\alpha)\,\mathrm{d}x^\beta\wedge i_\alpha.
\end{aligned}$$

Since $\hat{\mathrm{d}} := \mathrm{d}x^\alpha\wedge\partial_\alpha$ and $\hat{i}_f := f^\alpha\, i_\alpha$, we have

$$\begin{aligned}
[\hat{\mathrm{d}}, \hat{i}_f] &= \mathrm{d}x^\alpha\wedge\partial_\alpha\, f^\beta\, i_\beta + f^\beta\, i_\beta\mathrm{d}x^\alpha\wedge\partial_\alpha\\
&= (\partial_\alpha\, f^\beta)\mathrm{d}x^\alpha\wedge i_\beta + f^\beta\,\mathrm{d}x^\alpha\wedge i_\beta\,\partial_\alpha\\
\{[\mathrm{d}x^a\wedge, i_\beta] = \delta^a_\beta\} &+ f^\beta\,\delta^\alpha_\beta\partial_\alpha - f^\beta\,\mathrm{d}x^\alpha\wedge i_\beta\,\partial_\alpha\\
&= (\partial_\alpha\, f^\beta)\mathrm{d}x^\alpha\wedge i_\beta + f^a\partial_a,
\end{aligned}$$

which is $\hat{L}_f$. □

From Lie derivative $\hat{L}_f$, we can go back to $\hat{T}^*_{t\to t'}$ via the Dyson series.

Lemma 1.5. *[Dyson Series] If $\hat{L}_f$ the Lie derivative of the pull-back $\hat{T}^*_{t\to t'}$, then*

$$\hat{T}^*_{t\to t'} = 1 - \int_t^{t'}\mathrm{d}\tau_1\,\hat{L}_f(\tau_1) + \int_t^{t'}\mathrm{d}\tau_1\,\hat{L}_f(\tau_1)\int_t^{\tau_1}\mathrm{d}\tau_2\,\hat{L}_f(\tau_2) - \cdots. \tag{1.8}$$

Proof. By definition of $\hat{L}_f$, we have

$$\begin{aligned}
\frac{\partial\hat{T}^*_{t\to t'}}{\partial t'} &:= \lim_{\Delta t\to 0}\frac{\hat{T}^*_{t\to t'+\Delta t} - \hat{T}^*_{t\to t'}}{\Delta t}\\
\{\hat{T}^*_{t\to t'}\text{ forms a group}\} &= \lim_{\Delta t\to 0}\frac{\hat{T}^*_{t'\to t'+\Delta t}\,\hat{T}^*_{t\to t'} - \hat{T}^*_{t\to t'}}{\Delta t}\\
&= \lim_{\Delta t\to 0}\frac{\hat{T}^*_{t'\to t'+\Delta t} - 1}{\Delta t}\,\hat{T}^*_{t\to t'}\\
\{\hat{L}_f := \cdots\} &= -\hat{L}_f(t')\,\hat{T}^*_{t\to t'},
\end{aligned}$$

1.4. Operator ∂_α is short for $\partial/\partial x^\alpha$.

1.5. Interior product $i_a\colon\Omega^k(\mathcal{M})\to\Omega^{k-1}(\mathcal{M})$ is defined as, for $\forall\psi^{(k)}\in\Omega^k(\mathcal{M})$,

$$i_\alpha\,\psi^{(k)} := (1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_i}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}, \tag{1.6}$$

where A means that A is deleted, and $A\to B$ means that the original A is replaced by B. That is, it annihilates a $\mathrm{d}x^a\wedge$. The most useful property of interior product is the anti-commutator $[\mathrm{d}x^\alpha\wedge, i_\beta]_+ = \delta^\alpha_\beta$.

1.6. Recall that operator A is closed in space V if $A\colon V\to V$.

where we employed the fact that $\hat{T}^*_{t\to t'}$ forms a group.

It can be checked by direct calculus that the Dyson series satisfies this differential equation. Thus, the $\hat{T}^*_{t\to t'}$ can be expressed so. □

1.2 Stochastic Differential Equation

A direct generalization of ordinary differential equatoin is adding a Gaussian noise, as stochastic differential equation

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(t,x) + g^a_\beta(t,x)\,\eta^\beta(t). \tag{1.9}$$

Thus, $\eta\colon \mathbb{R}\to V$ with V an Euclidean space, and $g\colon \mathbb{R}\to\mathcal{M}\to V\to T\mathcal{M}$. Notice that the dimension of η and that of f may not equal.

To declare the distribution of η, for any $F[\eta]$ as test functional, split the time interval $[t,t']$ by $t=t_1<t_2<\cdots<t_N=t'$, with $t_{i+1}-t_i\equiv\Delta t$, then define the expectation as

$$\begin{aligned}\langle F\rangle &:= \int D[\eta]\exp\biggl(-\frac{1}{2}\int \mathrm{d}t\,\delta_{\alpha\beta}\,\eta^a(t)\,\eta^\beta(t)\biggr)F[\eta]\\ &:= \lim_{\Delta t\to 0} Z^{-1}\int \mathrm{d}\eta(t_1)\cdots\mathrm{d}\eta(t_N)\exp\biggl(-\frac{1}{2}\sum_i \Delta t\,\delta_{\alpha\beta}\,\eta^a(t_i)\,\eta^\beta(t_i)\biggr)F[\eta],\end{aligned}$$

where Z is the normalization factor so that $\langle 1\rangle = 1$. So, roughly speaking, $\eta^\alpha(t)\sim\mathcal{N}(0,1/\mathrm{d}t)$ for $\forall\alpha,t$. With this, we find

$$\begin{aligned}\langle\eta^\alpha(t)\rangle &= 0;\\ \langle\eta^\alpha(t)\,\eta^\beta(t')\rangle &= \delta^{\alpha\beta}\,\delta(t-t').\end{aligned}$$

Higher order expectations can be obtained directly by Wick theorem.

1.2.1 Infinitesimal Pull-back Expectation

For any configuration η given, the stochastic differential equation is reduced to an ordinary differential equation. In this case, the pull-back depends on η, that is, $\hat{T}^*_{t\to t'}[\eta]$. We care about the expectation $\langle\hat{T}^*_{t\to t'}[\eta]\rangle$ over all possible configuration of η, especially it's infinitesimal version.

As before, define the stochastic version of Lie derivative, as

$$\hat{H}_{(f,g)}(t) := \lim_{\Delta t\to 0}\frac{1-\langle\hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle}{\Delta t}, \tag{1.10}$$

where $\hat{T}^*_{t\to t+\Delta t}$ depends on the configuration η.

Theorem 1.6. *We have*

$$\hat{H}_{(f,g)} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2, \tag{1.11}$$

where $\hat{L}_g^2 := \delta^{\alpha\beta}\,\hat{L}_{g_\alpha}\hat{L}_{g_\beta}$.

Proof. Given configuration of η, let $F^\alpha_\eta(t,x) := f^\alpha(t,x) + g^\alpha_\beta(t,x)\eta^\beta(t)$. Directly, we have

$$\hat{L}_{F_\eta} = \hat{L}_f + \hat{L}_{g_\beta}\eta^\beta.$$

Since $\hat{T}^*_{t\to t'} = 1 - \int_t^{t'}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1) + \int_t^{t'}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)\int_t^{\tau_1}\mathrm{d}\tau_2\,\hat{L}_{F_\eta}(\tau_2) - \cdots$, we have

$$\langle\hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle = 1 - \left\langle\int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)\right\rangle + \left\langle\int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)\int_t^{\tau_1}\mathrm{d}\tau_2\,\hat{L}_{F_\eta}(\tau_2)\right\rangle - \cdots.$$

Since $\langle\eta\rangle = 0$,

$$\begin{aligned}&\left\langle\int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)\right\rangle\\ &= \int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_f(\tau_1) + \int_t^{t+\Delta t}\mathrm{d}\tau_1\hat{L}_{g_\beta}(\tau_1)\,\langle\eta^\beta(\tau_1)\rangle\\ \{\langle\eta^\beta\rangle = 0\} &= \int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_f(\tau_1)\end{aligned}$$

And since $\langle\eta^\alpha(t)\,\eta^\beta(t')\rangle = \delta^{\alpha\beta}\,\delta(t-t')$,

$$\begin{aligned}&\left\langle\int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)\int_t^{\tau_1}\mathrm{d}\tau_2\,\hat{L}_{F_\eta}(\tau_2)\right\rangle\\ \{\hat{L}_{F_\eta} := \cdots\} &= \left\langle\int_t^{t+\Delta t}\mathrm{d}\tau_1\,[\hat{L}_f(\tau_1) + \hat{L}_{g_\alpha}(\tau_1)\,\eta^\alpha(\tau_1)]\int_t^{\tau_1}\mathrm{d}\tau_2\,[\hat{L}_f(\tau_2) + \hat{L}_{g_\beta}(\tau_2)\,\eta^\beta(\tau_2)]\right\rangle\\ \{\text{Expand}\} &= \int_t^{t+\Delta t}\mathrm{d}\tau_1\int_t^{\tau_1}\mathrm{d}\tau_2\,\hat{L}_f(\tau_1)\hat{L}_f(\tau_2)\\ &+ \int_t^{t+\Delta t}\mathrm{d}\tau_1\int_t^{\tau_1}\mathrm{d}\tau_2\hat{L}_f(\tau_1)\hat{L}_{g_\beta}(\tau_2)\,\langle\eta^\beta(\tau_2)\rangle + \int_t^{t+\Delta t}\mathrm{d}\tau_1\int_t^{\tau_1}\mathrm{d}\tau_2\hat{L}_{g_\alpha}(\tau_1)\,\hat{L}_f(\tau_2)\,\langle\eta^\alpha(\tau_1)\rangle\\ &+ \int_t^{t+\Delta t}\mathrm{d}\tau_1\int_t^{\tau_1}\mathrm{d}\tau_2\,\hat{L}_{g_\alpha}(\tau_1)\hat{L}_{g_\beta}(\tau_2)\,\langle\eta^\alpha(\tau_1)\,\eta^\beta(\tau_2)\rangle\\ &= \mathcal{O}(\Delta t^2)\\ \{\langle\eta\rangle = 0\} &+ 0\\ \{\langle\eta^\alpha(t)\,\eta^\beta(t')\rangle = \delta^{\alpha\beta}\,\delta(t-t')\} &+ \int_t^{t+\Delta t}\mathrm{d}\tau_1\int_t^{\tau_1}\mathrm{d}\tau_2\,\hat{L}_{g_\alpha}(\tau_1)\hat{L}_{g_\beta}(\tau_2)\,\delta^{\alpha\beta}\,\delta(\tau_1-\tau_2)\\ &= \frac{1}{2}\int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_{g_\alpha}(\tau_1)\hat{L}_{g_\beta}(\tau_1)\,\delta^{\alpha\beta},\end{aligned}$$

where in the last line, we employed $\int_t^{\tau_1} \mathrm{d}\tau_2\, \delta(\tau_1 - \tau_2) = 1/2$.[1.7] Thus,

$$\langle \hat{T}^*_{t\to t+\Delta t}[\cdot] \rangle = 1 - \hat{L}_f\, \Delta t + \frac{1}{2} \langle \hat{L}_g^2 \rangle \Delta t + o(\Delta t).$$

So, finally,

$$\hat{H}_{(f,g)} := \lim_{\Delta t\to 0} \frac{\hat{1} - \langle \hat{T}^*_{t\to t+\Delta t}[\cdot] \rangle}{\Delta t} = \hat{L}_f - \frac{1}{2} \hat{L}_g^2. \qquad \square$$

Example 1.7. [Fokker-Planck Equation] In the case $g^{\alpha}_{\beta} \equiv \sqrt{2T}\, \delta^{\alpha}_{\beta}$,

$$\hat{H} = (\partial_\alpha f^\beta)\, \mathrm{d}x^\alpha \wedge i_\beta + f^\alpha \partial_\alpha - T\partial^2, \tag{1.12}$$

where $\partial^2 := \delta^{\alpha\beta}\partial_\alpha\partial_\beta$. Applying on $\psi^{(D)}$ with $D = \dim(\mathcal{M})$, since $\mathrm{d}x^\alpha \wedge \psi^{(D)} = 0$, and thus $\mathrm{d}x^\alpha \wedge i_\beta\, \psi^{(D)} = \delta^\alpha_\beta\, \psi^{(D)}$, we find

$$\hat{H}\psi^{(D)} = (\partial_\alpha f^\alpha - T\partial^2)\, \psi^{(D)}, \tag{1.13}$$

which is the Fokker-Planck equation.

1.2.2 Symmetry (TODO)

1.7. TODO: explain this.

Bibliography

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