Chapter 1
Stochastics

1.1 Ordinary Differential Equation

1.1.1 Push-forward

Let $\mathcal{M}$ a smooth manifold, and $f: \mathbb{R} \to \mathcal{M} \to T\mathcal{M}$ [1.1], we have ordinary differential equation[1.2]

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(t, x). \tag{1.1}$$

This ordinary differential equation induces a push-forward operator, $\hat{T}_{t \to t'}: \mathbb{R} \to \mathbb{R} \to \mathcal{M} \to \mathcal{M}$, which pushes the particle on position on $\mathcal{M}$ at time t to another position on $\mathcal{M}$ at time t'.

1.1.2 Pull-back

Let $\Omega^k(\mathcal{M})$ the space of k-forms on $\mathcal{M}$, where $k \leqslant \dim(\mathcal{M})$.[1.3] This ordinary differential equation also induces a pull-back operator on k-forms[1.4], $\hat{T}^*_{t \to t'}: \mathbb{R} \to \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$, which describes the transition of k-form from time t to t'.

Precisely, consider $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$, we can write it explicitly by indices, as

$$\psi^{(k)}(x) = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \tag{1.2}$$

where the indices in $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ is anti-symmetric. Regardless of the $1/k!$ factor, the $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ can be viewed as the local density at x and the $\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}$ as area or volume unit. Thus, $\psi^{(k)}(x)$, as a whole, is the mass unit. $\hat{T}^*$ is thus is a forward pushing transition of mass unit.

Lemma 1.1. *Explicitly, we have*

$$(\hat{T}^*_{t \to t'}\, \psi^{(k)})(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\, x') \bigwedge_{i=1}^{k} (\mathcal{D}\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}, \tag{1.3}$$

where $\mathcal{D}\hat{T}_{t' \to t}$ denotes the Jacobian of $\hat{T}_{t' \to t}$.

Proof. Let $x' = \hat{T}_{t \to t'}\, x$ and $\psi'^{(k)} = \hat{T}^*_{t \to t'}\, \psi^{(k)}$, by conservation of mass during the pushing, we have

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, \mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x(x'))\, \mathrm{d}x^{\alpha_1}(x') \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}(x'),$$

where $x(x') = \hat{T}_{t' \to t}\, x'$. With direct exterior algebra calculus, we get

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, \mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\, x') \bigwedge_{i=1}^{k} (\mathcal{D}\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}.$$

Inserting back the $(1/k!)$ factor, we arrive at

$$(\hat{T}^*_{t \to t'}\, \psi^{(k)})(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\, x') \bigwedge_{i=1}^{k} (\mathcal{D}\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}.$$

□

Lemma 1.2. *$\hat{T}^*_{t \to t'}$ forms a group. That is*

$$\hat{T}^*_{t \to t'}\, \hat{T}^*_{t' \to t''} = \hat{T}^*_{t \to t''}. \tag{1.4}$$

Proof. TODO □

1.1.3 Lie Derivative

Now, we try to derive the explicit expression of $\hat{T}^*_{t \to t'}$ depending on f^a in the limit $t' \to t$. This infinitesimal version of pull-back can be described by Lie derivative.

Definition 1.3. *[Lie Derivative] Given $f: \mathbb{R} \to \mathcal{M} \to T\mathcal{M}$, Lie derivative $\hat{L}_f: \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$ is defined as*

$$\hat{L}_f(t) := \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t \to t + \Delta t}}{\Delta t}, \tag{1.5}$$

where $\hat{T}^*$ is the pull-back induced by f.

1.1. The notation $A \to B \to C$ in declarations always means $A \to (B \to C)$. Further, $A \to B \to \cdots$ means $A \to (B \to (\cdots))$. This is an useful convension from Haskell.

1.2. We employ Einstein's convension of summation thoroughly.

1.3. The basic knowledge of forms is contained in Tao's very intuitive lecture 1.10.

1.4. Even though we call it something-back, but it pushes forward the k-forms. The name comes from the fact that forward pushing of k-forms is equivalent to backward pushing the mass unit, as the following discussion shows.

1.1.4 Cartan Magic Formula

Some useful definitions in exterior algebra are recalled. Operators $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$ [1.5] and, for $\forall f$, $\hat{i}_f := f^\alpha\, i_\alpha$, where i_α is the interior product[1.6]. Let operators A and B compositions of $\mathrm{d}x^a \wedge$ and i_a, then $[A, B]$ is commutator if both A and B are closed in $\Omega^k(\mathcal{M})$ for $\forall k$ [1.7], otherwise anti-commutator.

With these definitions, we conclude the explicit relatoin between f and $\hat{L}_f$, as follow.

Theorem 1.4. *[Cartan Magic Formula] We have*

$$\hat{L}_f = [\hat{\mathrm{d}}, \hat{i}_f]. \tag{1.7}$$

Proof. As $t' = t + \Delta t$ with Δt tiny, we have $\hat{T}_{t' \to t}\, x' = x' - f(t', x')\, \Delta t$. Then, $\mathcal{D}\hat{T}_{t' \to t} = 1 - \mathcal{D}f\, \Delta t$, where $\mathcal{D}f$ denotes the Jacobian of f. Now, insert this two expressions into the definition of $\hat{T}^*_{t \to t'}\, \psi^{(k)}$, we find

$$(\hat{T}^*_{t \to t'}\, \psi^{(k)})(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x' - f(t', x')\, \Delta t) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \Delta t) \mathrm{d}x'^{\beta_i}.$$

First, consider the expansion of $(1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \Delta t) \mathrm{d}x'^{\beta_i}$, up to $\mathcal{O}(\Delta t)$,

$$\begin{aligned}
&(1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \Delta t) \mathrm{d}x'^{\beta_i} \\
&= (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&- \Delta t \sum_{i=1}^{k} (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},
\end{aligned}$$

where $A \to B$ means that the original A is replaced by B. Now, we show that summation in the last line equals to $\partial_\beta f^\alpha(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha\, \psi^{(k)}(x)$. Recall that

$$i_\alpha\, \psi^{(k)}(x) := (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_i} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},$$

where A means the original A is deleted. Indeed,

$$\begin{aligned}
&(\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha\, \psi^{(k)}(x) \\
&= (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_i} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&= (\partial_\beta f^\alpha)(t', x') \sum_{i=1}^{k} (1/k!)\, \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^\beta) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&= \sum_{i=1}^{k} (\partial_{\beta_i} f^{\alpha_i})(t', x')\, (1/k!)\, \psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&= \sum_{i=1}^{k} (\partial_{\beta_i} f^{\alpha_i})(t', x')\, (1/k!)\, \psi_{\alpha_1 \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},
\end{aligned}$$

where in the last two lines, we replaced the dummy indices $\alpha \to \alpha_i$ and $\beta \to \beta_i$, and then found that $\psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}$ can be written back to $\psi_{\alpha_1 \cdots \alpha_k}$. Thus,

$$(1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\, \Delta t) \mathrm{d}x'^{\beta_i} = (1 - \Delta t\, (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}(x).$$

So, we find,

$$\begin{aligned}
(\hat{T}^*_{t \to t+\Delta t}\, \psi^{(k)})(x') &= (1 - \Delta t\, (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}(x) \\
\{x = x' - f(t', x')\, \Delta t\} &= (1 - \Delta t\, (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}(x' - f(t', x')\, \Delta t) \\
&= (1 - \Delta t\, (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, (1 - \Delta t\, f^\alpha(t', x')\, \partial_\alpha) \psi^{(k)}(x') \\
&= \psi^{(k)}(x') - (f^a(t', x') \partial_a + (\partial_\beta f^\alpha)(t', x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}(x')\, \Delta t + \mathcal{O}(\Delta t^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{L}_f &:= \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t \to t+\Delta t}}{\Delta t} \\
&= f^a \partial_a + (\partial_\beta f^\alpha)\, \mathrm{d}x^\beta \wedge i_\alpha.
\end{aligned}$$

Since $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$ and $\hat{i}_f := f^\alpha\, i_\alpha$, we have

$$\begin{aligned}
[\hat{\mathrm{d}}, \hat{i}_f] &= \mathrm{d}x^\alpha \wedge \partial_\alpha\, f^\beta\, i_\beta + f^\beta\, i_\beta \mathrm{d}x^\alpha \wedge \partial_\alpha \\
&= (\partial_\alpha\, f^\beta) \mathrm{d}x^\alpha \wedge i_\beta + f^\beta\, \mathrm{d}x^\alpha \wedge i_\beta\, \partial_\alpha \\
\{[\mathrm{d}x^a \wedge, i_\beta] = \delta^a_\beta\} &+ f^\beta\, \delta^\alpha_\beta \partial_\alpha - f^\beta\, \mathrm{d}x^\alpha \wedge i_\beta\, \partial_\alpha \\
&= (\partial_\alpha\, f^\beta) \mathrm{d}x^\alpha \wedge i_\beta + f^a \partial_a,
\end{aligned}$$

which is $\hat{L}_f$. □

1.1.5 Dyson Series

From Lie derivative $\hat{L}_f$, we can go back to $\hat{T}^*_{t \to t'}$ via the Dyson series.

Lemma 1.5. *[Dyson Series] If $\hat{L}_f$ the Lie derivative of the pull-back $\hat{T}^*_{t \to t'}$, then*

$$\hat{T}^*_{t \to t'} = 1 - \int_t^{t'} \mathrm{d}\tau_1\, \hat{L}_f(\tau_1) + \int_t^{t'} \mathrm{d}\tau_1\, \hat{L}_f(\tau_1) \int_t^{\tau_1} \mathrm{d}\tau_2\, \hat{L}_f(\tau_2) - \cdots. \tag{1.8}$$

1.5. Operator ∂_α is short for $\partial / \partial x^\alpha$.

1.6. Interior product $i_a \colon \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ is defined as, for $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$,

$$i_\alpha\, \psi^{(k)} := (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k} \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_i} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \tag{1.6}$$

where A means that A is deleted, and $A \to B$ means that the original A is replaced by B. That is, it annihilates a $\mathrm{d}x^a \wedge$. The most useful property of interior product is the anti-commutator $[\mathrm{d}x^\alpha \wedge, i_\beta]_+ = \delta^\alpha_\beta$.

1.7. Recall that operator A is closed in space V if $A \colon V \to V$.

Proof. By definition of $\hat{L}_f$, we have

$$\begin{aligned}\frac{\partial \hat{T}^*_{t\to t'}}{\partial t'} &:= \lim_{\Delta t\to 0}\frac{\hat{T}^*_{t\to t'+\Delta t}-\hat{T}^*_{t\to t'}}{\Delta t}\\ \{\hat{T}^*_{t\to t'}\text{ forms a group}\} &= \lim_{\Delta t\to 0}\frac{\hat{T}^*_{t'\to t'+\Delta t}\,\hat{T}^*_{t\to t'}-\hat{T}^*_{t\to t'}}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{\hat{T}^*_{t'\to t'+\Delta t}-1}{\Delta t}\,\hat{T}^*_{t\to t'}\\ \{\hat{L}_f:=\cdots\} &= -\hat{L}_f(t')\,\hat{T}^*_{t\to t'},\end{aligned}$$

where we employed the fact that $\hat{T}^*_{t\to t'}$ forms a group.

It can be checked by direct calculus that the Dyson series satisfies this differential equation. Thus, the $\hat{T}^*_{t\to t'}$ can be expressed so. □

1.2 Stochastic Differential Equation

1.2.1 Definitions

A direct generalization of ordinary differential equatoin is adding a Gaussian noise, as stochastic differential equation

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t)=f^a(t,x)+g^a_\beta(t,x)\,\eta^\beta(t). \tag{1.9}$$

Thus, $\eta\colon\mathbb{R}\to V$ with V an Euclidean space, and $g\colon\mathbb{R}\to\mathcal{M}\to V\to T\mathcal{M}$. Notice that the dimension of η and that of f may not equal.

To declare the distribution of η, for any $F[\eta]$ as test functional, split the time interval $[t,t']$ by $t=t_1<t_2<\cdots<t_N=t'$, with $t_{i+1}-t_i\equiv\Delta t$, then define the expectation as

$$\begin{aligned}\langle F\rangle &:= \int D[\eta]\exp\!\left(-\frac{1}{2}\int \mathrm{d}t\,\delta_{\alpha\beta}\,\eta^a(t)\,\eta^\beta(t)\right)F[\eta]\\ &:= \lim_{\Delta t\to 0} Z^{-1}\int \mathrm{d}\eta(t_1)\cdots\mathrm{d}\eta(t_N)\exp\!\left(-\frac{1}{2}\sum_i \Delta t\,\delta_{\alpha\beta}\,\eta^a(t_i)\,\eta^\beta(t_i)\right)F[\eta],\end{aligned}$$

where Z is the normalization factor so that $\langle 1\rangle=1$. So, roughly speaking, $\eta^\alpha(t)\sim\mathcal{N}(0,1/\mathrm{d}t)$ for $\forall\alpha,t$. With this, we find

$$\begin{aligned}\langle\eta^\alpha(t)\rangle &= 0;\\ \langle\eta^\alpha(t)\,\eta^\beta(t')\rangle &= \delta^{\alpha\beta}\,\delta(t-t').\end{aligned}$$

Higher order expectations can be obtained directly by Wick theorem.

1.2.2 Pull-back

For any configuration η given, the stochastic differential equation is reduced to an ordinary differential equation. In this case, the pull-back depends on η, that is, $\hat{T}^*_{t\to t'}[\eta]$. We care about the expectation $\langle\hat{T}^*_{t\to t'}[\eta]\rangle$ over all possible configuration of η, especially it's infinitesimal version.

1.2.3 Lie Derivative

As before, define the stochastic version of Lie derivative, as

$$\hat{H}_{(f,g)}(t):=\lim_{\Delta t\to 0}\frac{1-\langle\hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle}{\Delta t}, \tag{1.10}$$

where $\hat{T}^*_{t\to t+\Delta t}$ depends on the configuration η.

Theorem 1.6. *We have*

$$\hat{H}_{(f,g)}=\hat{L}_f-\frac{1}{2}\hat{L}^2_g, \tag{1.11}$$

where $\hat{L}^2_g:=\delta^{\alpha\beta}\,\hat{L}_{g_\alpha}\hat{L}_{g_\beta}$.

Proof. Given configuration of η, let $F^\alpha_\eta(t,x):=f^\alpha(t,x)+g^\alpha_\beta(t,x)\eta^\beta(t)$. Directly, we have

$$\hat{L}_{F_\eta}=\hat{L}_f+\hat{L}_{g_\beta}\eta^\beta.$$

Since $\hat{T}^*_{t\to t'}=1-\int_t^{t'}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)+\int_t^{t'}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)\int_t^{\tau_1}\mathrm{d}\tau_2\,\hat{L}_{F_\eta}(\tau_2)-\cdots$, we have

$$\langle\hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle=1-\left\langle\int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)\right\rangle+\left\langle\int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)\int_t^{\tau_1}\mathrm{d}\tau_2\,\hat{L}_{F_\eta}(\tau_2)\right\rangle-\cdots.$$

Since $\langle\eta\rangle=0$,

$$\begin{aligned}&\left\langle\int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_{F_\eta}(\tau_1)\right\rangle\\ &= \int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_f(\tau_1)+\int_t^{t+\Delta t}\mathrm{d}\tau_1\hat{L}_{g_\beta}(\tau_1)\,\langle\eta^\beta(\tau_1)\rangle\\ \{\langle\eta^\beta\rangle=0\} &= \int_t^{t+\Delta t}\mathrm{d}\tau_1\,\hat{L}_f(\tau_1)\end{aligned}$$

And since $\langle \eta^\alpha(t)\, \eta^\beta(t') \rangle = \delta^{\alpha\beta}\, \delta(t-t')$,

$$\begin{aligned}
& \left\langle \int_t^{t+\Delta t} \mathrm{d}\tau_1\, \hat{L}_{F_\eta}(\tau_1) \int_t^{\tau_1} \mathrm{d}\tau_2\, \hat{L}_{F_\eta}(\tau_2) \right\rangle \\
\{\hat{L}_{F_\eta} := \cdots\} = & \left\langle \int_t^{t+\Delta t} \mathrm{d}\tau_1\, [\hat{L}_f(\tau_1) + \hat{L}_{g_\alpha}(\tau_1)\, \eta^\alpha(\tau_1)] \int_t^{\tau_1} \mathrm{d}\tau_2\, [\hat{L}_f(\tau_2) + \hat{L}_{g_\beta}(\tau_2)\, \eta^\beta(\tau_2)] \right\rangle \\
\{\text{Expand}\} = & \int_t^{t+\Delta t} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2\, \hat{L}_f(\tau_1) \hat{L}_f(\tau_2) \\
& + \int_t^{t+\Delta t} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2 \hat{L}_f(\tau_1) \hat{L}_{g_\beta}(\tau_2)\, \langle \eta^\beta(\tau_2) \rangle + \int_t^{t+\Delta t} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2 \hat{L}_{g_\alpha}(\tau_1)\, \hat{L}_f(\tau_2)\, \langle \eta^\alpha(\tau_1) \rangle \\
& + \int_t^{t+\Delta t} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2\, \hat{L}_{g_\alpha}(\tau_1) \hat{L}_{g_\beta}(\tau_2)\, \langle \eta^\alpha(\tau_1)\, \eta^\beta(\tau_2) \rangle \\
& = \mathcal{O}(\Delta t^2) \\
\{\langle \eta \rangle = 0\} & + 0 \\
\{\langle \eta^\alpha(t)\, \eta^\beta(t') \rangle = \delta^{\alpha\beta}\, \delta(t-t')\} & + \int_t^{t+\Delta t} \mathrm{d}\tau_1 \int_t^{\tau_1} \mathrm{d}\tau_2\, \hat{L}_{g_\alpha}(\tau_1) \hat{L}_{g_\beta}(\tau_2)\, \delta^{\alpha\beta}\, \delta(\tau_1 - \tau_2) \\
& = \frac{1}{2} \int_t^{t+\Delta t} \mathrm{d}\tau_1\, \hat{L}_{g_\alpha}(\tau_1) \hat{L}_{g_\beta}(\tau_1)\, \delta^{\alpha\beta},
\end{aligned}$$

where in the last line, we employed $\int_t^{\tau_1} \mathrm{d}\tau_2\, \delta(\tau_1 - \tau_2) = 1/2$.[1.8] Thus,

$$\langle \hat{T}^*_{t \to t+\Delta t}[\cdot] \rangle = 1 - \hat{L}_f\, \Delta t + \frac{1}{2} \langle \hat{L}_g^2 \rangle \Delta t + o(\Delta t).$$

So, finally,

$$\hat{H}_{(f,g)} := \lim_{\Delta t \to 0} \frac{\hat{1} - \langle \hat{T}^*_{t \to t+\Delta t}[\cdot] \rangle}{\Delta t} = \hat{L}_f - \frac{1}{2} \hat{L}_g^2. \qquad \square$$

Example 1.7. [Fokker-Planck Equation] In the case $g^\alpha_\beta \equiv \sqrt{2T}\, \delta^\alpha_\beta$, we have $\hat{L}_{g_\alpha} = \sqrt{2T}\, \partial_\alpha$. Thus,

$$\hat{H} = (\partial_\alpha f^\beta)\, \mathrm{d}x^\alpha \wedge i_\beta + f^\alpha \partial_\alpha - T \partial^2, \tag{1.12}$$

where $\partial^2 := \delta^{\alpha\beta} \partial_\alpha \partial_\beta$. Applying on $\psi^{(D)}$ with $D = \dim(\mathcal{M})$, since $\mathrm{d}x^\alpha \wedge \psi^{(D)} = 0$, and thus $\mathrm{d}x^\alpha \wedge i_\beta\, \psi^{(D)} = \delta^\alpha_\beta\, \psi^{(D)}$, we find

$$\hat{H} \psi^{(D)} = (\partial_\alpha f^\alpha - T \partial^2) \psi^{(D)}, \tag{1.13}$$

which is the Fokker-Planck equation.

1.3 Topological Symmetry

1.3.1 Closed & Exact

Definition 1.8. *Let $\psi^{(k)} \in \Omega^k(\mathcal{M})$. Then,*

- *$\psi^{(k)}$ is called $\hat{\mathrm{d}}$-closed, if $\hat{\mathrm{d}} \psi^{(k)} = 0$; and*
- *$\psi^{(k)}$ is called $\hat{\mathrm{d}}$-exact, if $\exists \psi^{(k-1)} \in \Omega^{k-1}(\mathcal{M})$ such that $\psi^{(k)} = \hat{\mathrm{d}} \psi^{(k-1)}$.*

If a form is $\hat{\mathrm{d}}$-exact, then it must be $\hat{\mathrm{d}}$-closed, since $\hat{\mathrm{d}}^2 = 0$. The inverse, however, is not always true. This can be illustrated with Stokes theorem, that is

$$\int_{\mathcal{S}} \hat{\mathrm{d}} \omega = \int_{\partial \mathcal{S}} \omega, \tag{1.14}$$

for any form $\omega \in \Omega(\mathcal{M})$ and any $\mathcal{S} \subset \mathcal{M}$. If ω is $\hat{\mathrm{d}}$-closed, then for $\forall x \in \mathcal{M}$, let $\mathcal{S} \in \mathcal{M}$, in $\int_{\partial \mathcal{S}} \omega = \int_{\mathcal{S}} \hat{\mathrm{d}} \omega = 0$. TODO.

Example 1.9. [Closed but not Exact] Let $\mathcal{M} = \mathbb{R}^2 \setminus \{(0,0)\}$, and $\omega = f_\alpha \mathrm{d}x^\alpha$ with

$$f(x, y) = \left(-\frac{y}{(x^2 + y^2)}, \frac{x}{(x^2 + y^2)} \right), \tag{1.15}$$

visualized as

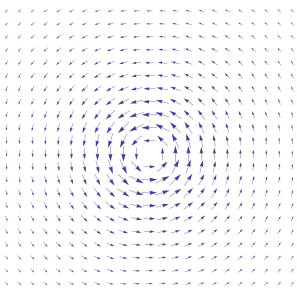

Figure 1.1.

1.8. TODO: explain this.

Direct calculus shows that $\hat{\mathrm{d}}\omega = 0$. But, obviously, the integral on any circle around $(0,0)$ will not vanish.

1.3.2 Flux Operator

Let

$$\hat{j} := \hat{i}_f - \frac{1}{2}\delta^{\alpha\beta}\,\hat{i}_{g_\alpha}\,\hat{L}_{g_\beta}. \tag{1.16}$$

Then, $\hat{H}$ can be re-written as

$$\hat{H} = [\hat{\mathrm{d}}, \hat{j}]. \tag{1.17}$$

The $\hat{j}$ operator is for the flux in the continuity equation. Indeed, $\partial_t \psi^{(D)} = -\hat{H}\psi^{(D)} = -\hat{\mathrm{d}}(\hat{j}\psi^{(D)}) = 0$, indicating that $\hat{j}\psi^{(D)}$ is the mass flux.

1.3.3 Spectrum of Lie Derivative

Example 1.10. [Eigen-value] Let $\mathcal{M} = \mathbb{R}^2$. Thus, let $\psi^{(2)} \in \Omega^2(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \psi^{(2)} = 1$. With indices, $\psi^{(2)}(x) = \psi_{12}(x)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$. Here, the $\psi_{12}(x)$ can be viewed as a p.d.f. If $\psi^{(2)}$ is an eigen-state of $\hat{H}$, then we have $\hat{H}\,\psi^{(2)} = \lambda\psi^{(2)}$, where $\lambda \in \mathbb{C}$. We have

$$\begin{aligned}
\lambda &= \int_{\mathbb{R}^2} \lambda\psi^{(2)} \\
\{\hat{H}\,\psi^{(2)} = \lambda\psi^{(2)}\} &= \int_{\mathbb{R}^2} \hat{H}\psi^{(2)} \\
\{\hat{H} = [\hat{\mathrm{d}}, \hat{j}]\} &= \int_{\mathbb{R}^2} [\hat{\mathrm{d}}, \hat{j}]\,\psi^{(2)} \\
\{\hat{\mathrm{d}}\psi^{(2)} = 0\} &= \int_{\mathbb{R}^2} \hat{\mathrm{d}}\hat{j}\psi^{(2)} \\
\{\text{Stokes theorem}\} &= \int_{\partial\mathbb{R}^2} \hat{j}\psi^{(2)}.
\end{aligned}$$

Explicitly,

$$\begin{aligned}
\hat{j}\psi^{(2)} &= \left(f^\alpha - \sqrt{T/2}\,\partial^\alpha\right) i_\alpha\,\psi_{12}(x)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
\{i_\alpha := \cdots\} &= \left(f^1 - \sqrt{T/2}\,\partial^1\right)\psi_{12}(x)\,\mathrm{d}x^2 - \left(f^2 - \sqrt{T/2}\,\partial^2\right)\psi_{12}(x)\,\mathrm{d}x^1. \\
\{\text{Compact format}\} &= \left(f^\alpha - \sqrt{T/2}\,\partial^\alpha\right)\psi_{\alpha\beta}(x)\,\mathrm{d}x^\beta.
\end{aligned}$$

If the f of $\hat{H}$ is well-defined, and $\psi_{12}(x)$ decrease to zero as $\|x\| \to +\infty$, then $\int_{\partial\mathbb{R}^2} \hat{j}\psi^{(2)} = 0$. Thus $\lambda = 0$. So, we find that any "good enough" p.d.f. $\psi^{(2)}$ as an eigen-state of $\hat{H}$, then the eigen-value must be zero.

Bibliography

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