1 Stochastics

1 Ordinary Differential Equation

Let $\mathcal{M}$ a smooth manifold, and $f\colon \mathcal{M} \times \mathbb{R} \to T\mathcal{M}$, we have ordinary differential equation[1]

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(x, t). \tag{1}$$

This ordinary differential equation induces a push-forward operator, $\hat{T}_{t \to t'}\colon \mathbb{R} \times \mathbb{R} \to \mathcal{M} \to \mathcal{M}$[2], which pushes the particle on position on $\mathcal{M}$ at time t to another position on $\mathcal{M}$ at time t'.

Let $\Omega^k(\mathcal{M})$ the space of k-forms on $\mathcal{M}$, where $k \leqslant \dim(\mathcal{M})$. This ordinary differential equation also induces a pull-back operator on k-forms[3], $\hat{T}^*_{t \to t'}\colon \mathbb{R} \times \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$, which describes the transition of k-form from time t to t'.

Precisely, consider $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$, we can write it explicitly by indices, as

$$\psi^{(k)}(x) = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \tag{2}$$

where the indices in $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ is anti-symmetric. Regardless of the $1/k!$ factor, the $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ can be viewed as the local density at x and the $\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}$ as area or volume unit. Thus, $\psi^{(k)}(x)$, as a whole, is the mass unit. $\hat{T}^*$ is thus is a forward pushing transition of mass unit.

Lemma 1. *Explicitly, we have*

$$\hat{T}^*_{t \to t'}\, \psi^{(k)}(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\, x') \bigwedge_{i=1}^{k} (T\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}, \tag{3}$$

where $T\hat{T}_{t' \to t}$ denotes the Jacobian of $\hat{T}_{t' \to t}$.

Proof. Let $x' = \hat{T}_{t \to t'}\, x$ and $\psi'^{(k)} = \hat{T}^*_{t \to t'}\, \psi^{(k)}$, by conservation of mass during the pushing, we have

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, \mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}.$$

Replace x by $x = \hat{T}_{t' \to t}\, x'$, we get

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, \mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\, x') \bigwedge_{i=1}^{k} (T\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}.$$

Inserting back the $(1/k!)$ factor, we arrive at

$$\hat{T}^*_{t \to t'}\, \psi^{(k)}(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\, x') \bigwedge_{i=1}^{k} (T\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}. \qquad \square$$

1.1 Infinitesimal Pull-back

Now, we try to derive the explicit expression of $\hat{T}^*_{t \to t'}$ depending on f^a in the limit $t' \to t$. This infinitesimal version of pull-back can be described by Lie derivative.

Definition 2. *[Lie Derivative] Given $f\colon \mathcal{M} \times \mathbb{R} \to T\mathcal{M}$, Lie derivative $\hat{L}_f\colon \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$ is defined as*

$$\hat{L}_f(t) := \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t \to t + \Delta t}}{\Delta t}, \tag{4}$$

---

1. We employ Einstein's convention of summation thoroughly.

2. The notation $A \to B \to C$ in declarations always means $A \to (B \to C)$. Further, $A \to B \to \cdots$ means $A \to (B \to (\cdots))$. This is a useful convension from Haskell.

3. Even though we call it something-back, but it pushes forward the k-forms. The name comes from the fact that forward pushing of k-forms is equivalent to backward pushing the mass unit, as the following discussion shows.

where $\hat{T}^*$ is the pull-back induced by f.

Some useful definitions in exterior algebra are recalled. Operators $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$[4] and, for $\forall f$, $\hat{i}_f := f^\alpha i_\alpha$, where i_α is the interior product[5]. Let A and B compositions of $\mathrm{d}x^a \wedge$ and i_a, then $[A, B]$ is commutator if both A and B have balanced $\mathrm{d}x^a \wedge$ and i_a, otherwise anti-commutator.

With these definitions, we conclude the explicit relatoin between f and $\hat{L}_f$, as follow.

Theorem 3. *[Cartan's Magic Formula] We have*

$$\hat{L}_f = [\hat{\mathrm{d}}, \hat{i}_f]. \tag{6}$$

Proof. As $t' = t + \Delta t$ with Δt tiny, we have $\hat{T}_{t' \to t}\, x' = x' - f\,\Delta t$. Then, $T\hat{T}_{t' \to t} = 1 - Tf\,\Delta t$. Now we insert this two expressions into the definition of $\hat{T}^*_{t \to t'}\, \psi^{(k)}$, that is,

$$(1/k!)\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x' - f(x')\,\Delta t)\bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(x')\,\Delta t)\mathrm{d}x'^{\beta_i}.$$

First, consider the expansion of $(1/k!)\,\psi^{(k)}_{\alpha_1 \cdots \alpha_k}\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i}f^{\alpha_i})(x')\,\Delta t)\mathrm{d}x'^{\beta_i}$, up to $\mathcal{O}(\Delta t)$,

$$\begin{aligned}
&(1/k!)\,\psi^{(k)}_{\alpha_1 \cdots \alpha_k}\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i}f^{\alpha_i})(x')\,\Delta t)\mathrm{d}x'^{\beta_i} \\
&= (1/k!)\,\psi^{(k)}_{\alpha_1 \cdots \alpha_k}\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&\quad - \Delta t \sum_{i=1}^{k}(1/k!)\,\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\partial_{\beta_i}f^{\alpha_i})(x')\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},
\end{aligned}$$

where $A \to B$ means that the original A is replaced by B. Now, we show that summation in the last line equals to $\partial_\beta f^\alpha(x')\,\mathrm{d}x^\beta \wedge i_\alpha\,\psi^{(k)}$. Recall that

$$i_\alpha\,\psi^{(k)} := (1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \cancel{\mathrm{d}x^{\alpha_i}} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},$$

where $\cancel{A}$ means the original A is deleted. Indeed,

$$\begin{aligned}
&(\partial_\beta f^\alpha)(x')\,\mathrm{d}x^\beta \wedge i_\alpha\,\psi^{(k)} \\
&= (\partial_\beta f^\alpha)(x')\,\mathrm{d}x^\beta \wedge (1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \cancel{\mathrm{d}x^{\alpha_i}} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&= (\partial_\beta f^\alpha)(x')\sum_{i=1}^{k}(1/k!)\,\psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^\beta) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&= \sum_{i=1}^{k}(\partial_{\beta_i}f^{\alpha_i})(x')\,(1/k!)\,\psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\
&= \sum_{i=1}^{k}(\partial_{\beta_i}f^{\alpha_i})(x')\,(1/k!)\,\psi_{\alpha_1 \cdots \alpha_k}\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},
\end{aligned}$$

where in the last two lines, we replaced the dummy indices $\alpha \to \alpha_i$ and $\beta \to \beta_i$, and then found that $\psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}$ can be written back to $\psi_{\alpha_1 \cdots \alpha_k}$. Thus,

$$(1/k!)\,\psi^{(k)}_{\alpha_1 \cdots \alpha_k}\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i}f^{\alpha_i})(x')\,\Delta t)\mathrm{d}x'^{\beta_i} = (1 - \Delta t\,(\partial_\beta f^\alpha)(x')\,\mathrm{d}x^\beta \wedge i_\alpha)\,\psi^{(k)}.$$

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4. Operator ∂_α is short for $\partial/\partial x^\alpha$.

5. Interior product $i_a: \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ is defined as, for $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$,

$$i_\alpha\,\psi^{(k)} := (1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \cancel{\mathrm{d}x^{\alpha_i}} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \tag{5}$$

where $\cancel{A}$ means that A is deleted, and $A \to B$ means that the original A is replaced by B. That is, it annihilates a $\mathrm{d}x^a \wedge$. The most useful property of interior product is the anti-commutator $[\mathrm{d}x^\alpha \wedge, i_\beta]_+ = \delta^\alpha_\beta$.

So, we find

$$
\begin{aligned}
\hat{T}^*_{t \to t+\Delta t}\, \psi^{(k)} &= (1 - \Delta t\, (\partial_\beta f^\alpha)(x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}(x' - f\Delta t) \\
&= (1 - \Delta t\, (\partial_\beta f^\alpha)(x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, (1 - \Delta t f^\alpha(x')\, \partial_\alpha) \psi^{(k)}(x') \\
&= \psi^{(k)} - (f^a \partial_a + (\partial_\beta f^\alpha)\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)} \Delta t + \mathcal{O}(\Delta t^2).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\hat{L}_f &:= \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t \to t+\Delta t}}{\Delta t} \\
&= f^a \partial_a + (\partial_\beta f^\alpha)\, \mathrm{d}x^\beta \wedge i_\alpha.
\end{aligned}
$$

Since $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$ and $\hat{i}_f := f^\alpha i_\alpha$, we have

$$
\begin{aligned}
[\hat{\mathrm{d}}, \hat{i}_f] &= \mathrm{d}x^\alpha \wedge \partial_\alpha\, f^\beta i_\beta + f^\beta i_\beta \mathrm{d}x^\alpha \wedge \partial_\alpha \\
&= (\partial_\alpha f^\beta) \mathrm{d}x^\alpha \wedge i_\beta + f^\beta\, \mathrm{d}x^\alpha \wedge i_\beta \partial_\alpha \\
\{[\mathrm{d}x^a \wedge, i_\beta] = \delta^\alpha_\beta\} &+ f^\beta \delta^\alpha_\beta \partial_\alpha - f^\beta\, \mathrm{d}x^\alpha \wedge i_\beta \partial_\alpha \\
&= (\partial_\alpha f^\beta) \mathrm{d}x^\alpha \wedge i_\beta + f^a \partial_a,
\end{aligned}
$$

which is $\hat{L}_f$. $\qquad\square$

2 Stochastic Differential Equation

A direct generalization of ordinary differential equatoin is adding a Gaussian noise, as stochastic differential equation

$$
\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(x, t) + g^a_\beta(x, t)\, \eta^\beta(t), \tag{7}
$$

where, for $\forall t$ and α, $\eta^\alpha(t) \sim \mathcal{N}(0, 1/\mathrm{d}t)$. For any functional $F[\eta]$, split the time interval $[t, t']$ by $t = t_1 < t_2 < \cdots < t_N = t'$, with $t_{i+1} - t_i \equiv \Delta t$, then define the expectation as

$$
\begin{aligned}
\langle F \rangle &:= \int D[\eta] \exp\!\left(-\frac{1}{2} \int \mathrm{d}t\, \delta_{\alpha\beta}\, \eta^a(t)\, \eta^\beta(t) \right) F[\eta] \\
&:= \lim_{\Delta t \to 0} Z^{-1} \int \mathrm{d}\eta(t_1) \cdots \mathrm{d}\eta(t_N) \exp\!\left(-\frac{1}{2} \sum_i \Delta t\, \delta_{\alpha\beta}\, \eta^a(t_i)\, \eta^\beta(t_i) \right) F[\eta],
\end{aligned}
$$

where Z the normalization factor so that $\langle 1 \rangle = 1$. Notice that the dimension of η and that of f may not equal.

2.1 Infinitesimal Pull-back Expectation

For any configuration η given, the stochastic differential equation is reduced to an ordinary differential equation. In this case, the pull-back depends on η, that is, $\hat{T}^*_{t \to t'}[\eta]$. We care about the expectation $\langle \hat{T}^*_{t \to t'}[\eta] \rangle$ over all possible configuration of η, especially it's infinitesimal version.

As before, define the stochastic version of Lie derivative, as

$$
\hat{H}_{(f, g)}(t) := \lim_{\Delta t \to 0} \frac{1 - \langle \hat{T}^*_{t \to t+\Delta t}[\cdot] \rangle}{\Delta t}, \tag{8}
$$

where $\hat{T}^*_{t \to t+\Delta t}$ depends on the configuration η.

Theorem 4. *We have*

$$
\hat{H}_{(f, g)} = \hat{L}_f - \frac{1}{2} \hat{L}^2_g, \tag{9}
$$

where $\hat{L}^2_g := \delta^{\alpha\beta}\, \hat{L}_{g_\alpha} \hat{L}_{g_\beta}$.

Proof. Given configuration of η, let $F^\alpha_\eta(x, t) := f^\alpha(x, t) + g^\alpha_\beta(x, t) \eta^\beta(t)$. Directly, we have

$$
\hat{L}_{F_\eta} = \hat{L}_f + \hat{L}_{g_\beta} \eta^\beta.
$$

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Then, we have $\hat{T}^*_{t\to t+\Delta t}[\eta] = \exp(-\hat{L}_{F_\eta}\Delta t)$.[6] Then,

$$\langle\hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle = \int d\eta(t)\exp\left(-\frac{1}{2}\Delta t\,\delta_{\alpha\beta}\,\eta^\alpha(t)\,\eta^\beta(t)\right)\exp(-\hat{L}_{F_\eta}\Delta t)$$

$$= \int d\eta(t)\exp\left(-\frac{1}{2}\Delta t\,\delta_{\alpha\beta}\,\eta^\alpha(t)\,\eta^\beta(t)\right)\left(1 - \hat{L}_{F_\eta}\Delta t + \frac{1}{2}(\hat{L}_{F_\eta}\Delta t)^2 + \cdots\right)$$

$$= 1 - \langle\hat{L}_{F_\eta}\Delta t\rangle + \frac{1}{2}\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle + \cdots$$

Since $\langle\eta^\beta\rangle = 0$, $\langle\hat{L}_{F_\eta}\Delta t\rangle = \langle\hat{L}_f + \hat{L}_{g_\beta}\eta^\beta\rangle\Delta t = \hat{L}_f\,\Delta t$. And since $\langle\eta^\alpha\eta^\beta\rangle = \delta^{\alpha\beta}/\Delta t$,

$$\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle$$
$$= \langle(\hat{L}_f + \hat{L}_{g_\alpha}\eta^\alpha)\,(\hat{L}_f + \hat{L}_{g_\beta}\eta^\beta)\rangle\,\Delta t^2$$
$$\{\langle\eta^\beta\rangle = 0\} = \hat{L}_f^2\Delta t^2 + \hat{L}_{g_\alpha}\hat{L}_{g_\beta}\langle\eta^\alpha\eta^\beta\rangle\,\Delta t^2$$
$$\{\langle\eta^\alpha\eta^\beta\rangle = \delta^{\alpha\beta}/\Delta t\} = \hat{L}_f^2\Delta t^2 + \hat{L}_{g_\alpha}\hat{L}_{g_\beta}\delta^{\alpha\beta}\,\Delta t$$
$$\{\hat{L}_g^2 := \cdots\} = \hat{L}_f^2\Delta t^2 + \hat{L}_g^2\,\Delta t.$$

Thus,

$$\langle\hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle = 1 - \langle\hat{L}_{F_\eta}\rangle\Delta t + \frac{1}{2}\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle + \cdots$$

$$= 1 - \hat{L}_f\,\Delta t + \frac{1}{2}\langle\hat{L}_g^2\rangle\Delta t + o(\Delta t).$$

So, finally,

$$\hat{H}_{(f,g)} := \lim_{\Delta t\to 0}\frac{\hat{1} - \langle\hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle}{\Delta t} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2. \qquad\qquad \square$$

2.2 Symmetry (TODO)

Definition 5. *Given A, B is called A-exact if there exists X such that*

$$B = [A, X]. \qquad\qquad (10)$$

Lemma 6. *If B is A-exact, then $[A, B] = 0$.*

Proof.

$$[A, B] = [A, [A, X]]$$
$$= [X, [A, A]] + [A, [X, A]]$$
$$= ?$$

$$\square$$

Lemma 7. *We have decomposition*

$$\hat{H} = [\hat{d}, \hat{j}], \qquad\qquad (11)$$

where $\hat{j} := \hat{i}_f - \frac{1}{2}\hat{L}_{g_\beta}\eta^\beta$.

That is, $\hat{H}$ is $\hat{d}$-exact, thus,

$$[\hat{d}, \hat{H}] = 0. \qquad\qquad (12)$$

$\hat{d}\psi^{(k)} = 0$, but there isn't $\varphi^{(k-1)}$ s.t. $\psi^{(k)} = \hat{d}\varphi^{(k-1)}$. Symmetric state.

$\hat{d}\psi^{(k)} \neq 0$. $\hat{d}^2\psi^{(k)} = 0$. Symmetry breaking state. $\text{tr}(\langle\hat{T}^*_{t\to t'}[\cdot]\rangle) = \sum_n \exp(-E_n(t'-t)) = \exp(\lambda(t'-t))$.

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6. TODO: needs some proof.

$\hat{j}^2 = 0.$ $[\hat{j}, \hat{H}] = 0.$

j is the flux. That is, for $\psi^{(D)}$, where $D = \dim(\mathcal{M})$, $\hat{H}\psi^{(D)} = \hat{j}\psi^{(D)}$, so that $\partial_t \psi^{(D)} + \hat{j}\psi^{(D)} = 0$, indicating that $\hat{j}$ is the flux.