1 Stochastics

1 Ordinary Differential Equation

Let $\mathcal{M}$ a smooth manifold, and $f\colon \mathcal{M} \times \mathbb{R} \to T\mathcal{M}$, we have ordinary differential equation[1]

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(x, t). \tag{1}$$

This ordinary differential equation induces a push-forward operator, $\hat{T}_{t\to t'}\colon \mathbb{R} \times \mathbb{R} \to \mathcal{M} \to \mathcal{M}$[2], which pushes the particle on position on $\mathcal{M}$ at time t to another position on $\mathcal{M}$ at time t'.

Let $\Omega^k(\mathcal{M})$ the space of k-forms on $\mathcal{M}$, where $k \leqslant \dim(\mathcal{M})$. This ordinary differential equation also induces a pull-back operator on k-forms[3], $\hat{T}^*_{t\to t'}\colon \mathbb{R} \times \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$, which describes the transition of k-form from time t to t'.

Precisely, consider $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$, we can write it explicitly by indices, as

$$\psi^{(k)}(x) = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \tag{2}$$

where the indices in $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ is anti-symmetric. Regardless of the $1/k!$ factor, the $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ can be viewed as the local density at x and the $\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}$ as area or volume unit. Thus, $\psi^{(k)}(x)$, as a whole, is the mass unit. $\hat{T}^*$ is thus is a forward pushing transition of mass unit.

Lemma 1. *Explicitly, we have*

$$\hat{T}^*_{t\to t'}\, \psi^{(k)}(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t'\to t}\, x') \bigwedge_{i=1}^{k} (T\hat{T}_{t'\to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}, \tag{3}$$

where $T\hat{T}_{t'\to t}$ denotes the Jacobian of $\hat{T}_{t'\to t}$.

Proof. Let $x' = \hat{T}_{t\to t'}\, x$ and $\psi'^{(k)} = \hat{T}^*_{t\to t'}\, \psi^{(k)}$, by conservation of mass during the pushing, we have

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, \mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}.$$

Replace x by $x = \hat{T}_{t'\to t}\, x'$, we get

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, \mathrm{d}x'^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t'\to t}\, x') \bigwedge_{i=1}^{k} (T\hat{T}_{t'\to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}.$$

Inserting back the $(1/k!)$ factor, we arrive at

$$\hat{T}^*_{t\to t'}\, \psi^{(k)}(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t'\to t}\, x') \bigwedge_{i=1}^{k} (T\hat{T}_{t'\to t})^{\alpha_i}_{\beta_i}(x')\, \mathrm{d}x'^{\beta_i}.$$ □

1.1 Infinitesimal Pull-back

Now, we try to derive the explicit expression of $\hat{T}^*_{t\to t'}$ depending on f^a in the limit $t' \to t$. This infinitesimal version of pull-back can be described by Lie derivative.

Definition 2. *[Lie Derivative] Given $f\colon \mathcal{M} \times \mathbb{R} \to T\mathcal{M}$, Lie derivative $\hat{L}_f\colon \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$ is defined as*

$$\hat{L}_f(t) := \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t\to t+\Delta t}}{\Delta t}, \tag{4}$$

1. We employ Einstein's convension of summation thoroughly.

2. The notation $A \to B \to C$ in declarations always means $A \to (B \to C)$. Further, $A \to B \to \cdots$ means $A \to (B \to (\cdots))$. This is a useful convension from Haskell.

3. Even though we call it something-back, but it pushes forward the k-forms. The name comes from the fact that forward pushing of k-forms is equivalent to backward pushing the mass unit, as the following discussion shows.

where $\hat{T}^*$ is the pull-back induced by f.

Some useful definitions in exterior algebra are recalled. Operators $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$[4] and, for $\forall f$, $\hat{i}_f := f^\alpha i_\alpha$, where i_α is the interior product[5]. Let A and B compositions of $\mathrm{d}x^a \wedge$ and i_a, then $[A, B]$ is commutator if both A and B have balanced $\mathrm{d}x^a \wedge$ and i_a, otherwise anti-commutator.

With these definitions, we conclude the explicit relatoin between f and $\hat{L}_f$, as follow.

Theorem 3. *[Cartan's Magic Formula] We have*

$$\hat{L}_f = [\hat{\mathrm{d}}, \hat{i}_f]. \tag{6}$$

Proof. As $t' = t + \Delta t$ with Δt tiny, we have $\hat{T}_{t' \to t}\, x' = x' - f\, \Delta t$. Then, $T\hat{T}_{t' \to t} = 1 - Tf\, \Delta t$. Now we insert this two expressions into the definition of $\hat{T}^*_{t \to t'}\, \psi^{(k)}$, that is,

$$(1/k!) \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x' - f(x')\, \Delta t) \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(x')\, \Delta t) \mathrm{d}x'^{\beta_i}.$$

First, consider the expansion of $(1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k} \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(x')\, \Delta t) \mathrm{d}x'^{\beta_i}$, up to $\mathcal{O}(\Delta t)$,

$$\begin{aligned} & (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k} \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(x')\, \Delta t) \mathrm{d}x'^{\beta_i} \\ = & (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k} \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\ - & \Delta t \sum_{i=1}^{k} (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k} (\partial_{\beta_i} f^{\alpha_i})(x')\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \end{aligned}$$

where $A \to B$ means that the original A is replaced by B. Now, we show that summation in the last line equals to $\partial_\beta f^\alpha(x')\, \mathrm{d}x^\beta \wedge i_\alpha\, \psi^{(k)}$. Recall that

$$i_\alpha\, \psi^{(k)} := (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_i} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k},$$

where A means the original A is deleted. Indeed,

$$\begin{aligned} & (\partial_\beta f^\alpha)(x')\, \mathrm{d}x^\beta \wedge i_\alpha\, \psi^{(k)} \\ = & (\partial_\beta f^\alpha)(x')\, \mathrm{d}x^\beta \wedge (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_i} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\ = & (\partial_\beta f^\alpha)(x') \sum_{i=1}^{k} (1/k!)\, \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^\beta) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\ = & \sum_{i=1}^{k} (\partial_{\beta_i} f^{\alpha_i})(x')\, (1/k!)\, \psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k} \\ = & \sum_{i=1}^{k} (\partial_{\beta_i} f^{\alpha_i})(x')\, (1/k!)\, \psi_{\alpha_1 \cdots \alpha_k}\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha_i} \to \mathrm{d}x^{\beta_i}) \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \end{aligned}$$

where in the last two lines, we replaced the dummy indices $\alpha \to \alpha_i$ and $\beta \to \beta_i$, and then found that $\psi_{\alpha_1 \cdots \alpha_i \cdots \alpha_k}$ can be written back to $\psi_{\alpha_1 \cdots \alpha_k}$. Thus,

$$(1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k} \bigwedge_{i=1}^{k} (\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(x')\, \Delta t) \mathrm{d}x'^{\beta_i} = (1 - \Delta t\, (\partial_\beta f^\alpha)(x')\, \mathrm{d}x^\beta \wedge i_\alpha)\, \psi^{(k)}.$$

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4. Operator ∂_α is short for $\partial / \partial x^\alpha$.

5. Interior product $i_a: \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ is defined as, for $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$,

$$i_\alpha\, \psi^{(k)} := (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_i} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}, \tag{5}$$

where A means that A is deleted, and $A \to B$ means that the original A is replaced by B. That is, it annihilates a $\mathrm{d}x^a \wedge$. The most useful property of interior product is the anti-commutator $[\mathrm{d}x^\alpha \wedge, i_\beta]_+ = \delta^\alpha_\beta$.

So, we find

$$\begin{aligned}\hat{T}^*_{t\to t+\Delta t}\,\psi^{(k)} &= (1-\Delta t\,(\partial_\beta f^\alpha)(x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}(x'-f\,\Delta t)\\ &= (1-\Delta t\,(\partial_\beta f^\alpha)(x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,(1-\Delta t\,f^\alpha(x')\,\partial_\alpha)\psi^{(k)}(x')\\ &= \psi^{(k)} - (f^a\partial_a + (\partial_\beta f^\alpha)\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}\Delta t + \mathcal{O}(\Delta t^2).\end{aligned}$$

Thus,

$$\begin{aligned}\hat{L}_f &:= \lim_{\Delta t\to 0}\frac{1-\hat{T}^*_{t\to t+\Delta t}}{\Delta t}\\ &= f^a\partial_a + (\partial_\beta f^\alpha)\,\mathrm{d}x^\beta\wedge i_\alpha.\end{aligned}$$

Since $\hat{\mathrm{d}} := \mathrm{d}x^\alpha\wedge\partial_\alpha$ and $\hat{i}_f := f^\alpha\, i_\alpha$, we have

$$\begin{aligned}[\hat{\mathrm{d}},\hat{i}_f] &= \mathrm{d}x^\alpha\wedge\partial_\alpha\, f^\beta\, i_\beta + f^\beta\, i_\beta \mathrm{d}x^\alpha\wedge\partial_\alpha\\ &= (\partial_\alpha f^\beta)\mathrm{d}x^\alpha\wedge i_\beta + f^\beta\,\mathrm{d}x^\alpha\wedge i_\beta\,\partial_\alpha\\ \{[\mathrm{d}x^a\wedge, i_\beta] = \delta^\alpha_\beta\} &+ f^\beta\,\delta^\alpha_\beta\partial_\alpha - f^\beta\,\mathrm{d}x^\alpha\wedge i_\beta\,\partial_\alpha\\ &= (\partial_\alpha f^\beta)\mathrm{d}x^\alpha\wedge i_\beta + f^a\partial_a,\end{aligned}$$

which is $\hat{L}_f$. □

2 Stochastic Differential Equation

A direct generalization of ordinary differential equatoin is adding a Gaussian noise, as stochastic differential equation

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(x,t) + g^a_\beta(x,t)\,\eta^\beta(t), \tag{7}$$

where, for $\forall t$ and α, $\eta^\alpha(t)\sim\mathcal{N}(0,1/\mathrm{d}t)$. For any functional $F[\eta]$, split the time interval $[t,t']$ by $t=t_1 < t_2 < \cdots < t_N = t'$, with $t_{i+1}-t_i\equiv\Delta t$, then define the expectation as

$$\begin{aligned}\langle F\rangle &:= \int D[\eta]\exp\!\left(-\frac{1}{2}\int \mathrm{d}t\,\delta_{\alpha\beta}\,\eta^a(t)\,\eta^\beta(t)\right)F[\eta]\\ &:= \lim_{\Delta t\to 0} Z^{-1}\int \mathrm{d}\eta(t_1)\cdots\mathrm{d}\eta(t_N)\exp\!\left(-\frac{1}{2}\sum_i\,\Delta t\,\delta_{\alpha\beta}\,\eta^a(t_i)\,\eta^\beta(t_i)\right)F[\eta],\end{aligned}$$

where Z the normalization factor so that $\langle 1\rangle = 1$. Notice that the dimension of η and that of f may not equal.

2.1 Infinitesimal Pull-back Expectation

For any configuration η given, the stochastic differential equation is reduced to an ordinary differential equation. In this case, the pull-back depends on η, that is, $\hat{T}^*_{t\to t'}[\eta]$. We care about the expectation $\langle\hat{T}^*_{t\to t'}[\eta]\rangle$ over all possible configuration of η, especially it's infinitesimal version.

As before, define the stochastic version of Lie derivative, as

$$\hat{H}_{(f,g)}(t) := \lim_{\Delta t\to 0}\frac{1-\langle\hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle}{\Delta t}, \tag{8}$$

where $\hat{T}^*_{t\to t+\Delta t}$ depends on the configuration η.

Theorem 4. *We have*

$$\hat{H}_{(f,g)} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2, \tag{9}$$

where $\hat{L}_g^2 := \delta^{\alpha\beta}\,\hat{L}_{g_\alpha}\hat{L}_{g_\beta}$.

Proof. Given configuration of η, let $F^\alpha_\eta(x,t) := f^\alpha(x,t) + g^\alpha_\beta(x,t)\eta^\beta(t)$. We have

$$\hat{L}_F = \hat{L}_f + \hat{L}_{g_\beta}\eta^\beta.$$

Then, we have $\hat{T}^*_{t\to t+\Delta t}[\eta] = \exp(-\hat{L}_{F_\eta}\Delta t)$. Then,

$$\begin{aligned}
\langle \hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle &= \int \mathrm{d}\eta(t)\exp\!\left(-\frac{1}{2}\Delta t\,\delta_{\alpha\beta}\,\eta^\alpha(t)\,\eta^\beta(t)\right)\exp(-\hat{L}_{F_\eta}\Delta t)\\
&= \int \mathrm{d}\eta(t)\exp\!\left(-\frac{1}{2}\Delta t\,\delta_{\alpha\beta}\,\eta^\alpha(t)\,\eta^\beta(t)\right)\left(1-\hat{L}_{F_\eta}\Delta t+\frac{1}{2}(\hat{L}_{F_\eta}\Delta t)^2+\cdots\right)\\
&= 1-\langle \hat{L}_{F_\eta}\rangle\Delta t+\frac{1}{2}\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle+\cdots
\end{aligned}$$

Since $\langle\eta^\beta\rangle = 0$, $\langle \hat{L}_F\rangle = \hat{L}_f$. And since $\langle \eta^\alpha\,\eta^\beta\rangle = \delta^{\alpha\beta}/\Delta t$, $\langle \hat{L}_{g_\alpha}\eta^\alpha\,\hat{L}_{g_\beta}\eta^\beta\rangle = \hat{L}_{g_\alpha}\,\hat{L}_{g_\beta}\,\langle\eta^\alpha\,\eta^\beta\rangle = \hat{L}_g^2/\Delta t$. Thus,

$$\begin{aligned}
\langle \hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle &= 1-\langle \hat{L}_{F_\eta}\rangle\Delta t+\frac{1}{2}\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle+\cdots\\
&= 1-\hat{L}_f\,\Delta t+\frac{1}{2}\,\langle\hat{L}_g^2\rangle\Delta t+o(\Delta t).
\end{aligned}$$

Thus

$$\hat{H}_{(f,g)} := \lim_{\Delta t\to 0}\frac{\hat{1}-\langle \hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle}{\Delta t} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2.$$

□

2.2 Symmetry (TODO)

Definition 5. *Given A, B is called A-exact if there exists X such that*

$$B = [A, X]. \tag{10}$$

Lemma 6. *If B is A-exact, then $[A,B]=0$.*

Proof.

$$\begin{aligned}
[A,B] &= [A,[A,X]]\\
&= [X,[A,A]]+[A,[X,A]]\\
&= ?
\end{aligned}$$

□

Lemma 7. *We have decomposition*

$$\hat{H} = [\hat{\mathrm{d}}, \hat{j}], \tag{11}$$

where $\hat{j} := \hat{i}_f - \frac{1}{2}\hat{L}_{g_\beta}\eta^\beta$.

That is, $\hat{H}$ is $\hat{\mathrm{d}}$-exact, thus,

$$[\hat{\mathrm{d}}, \hat{H}] = 0. \tag{12}$$

$\hat{\mathrm{d}}\psi^{(k)} = 0$, but there isn't $\varphi^{(k-1)}$ s.t. $\psi^{(k)} = \hat{\mathrm{d}}\varphi^{(k-1)}$. Symmetric state.

$\hat{\mathrm{d}}\psi^{(k)} \neq 0$. $\hat{\mathrm{d}}^2\psi^{(k)} = 0$. Symmetry breaking state. $\mathrm{tr}(\langle\hat{T}^*_{t\to t'}[\cdot]\rangle) = \sum_n \exp(-E_n(t'-t)) = \exp(\lambda(t'-t))$.

$\hat{j}^2 = 0$. $[\hat{j}, \hat{H}] = 0$.

j is the flux. That is, for $\psi^{(D)}$, where $D = \dim(\mathcal{M})$, $\hat{H}\,\psi^{(D)} = \hat{j}\,\psi^{(D)}$, so that $\partial_t\psi^{(D)} + \hat{j}\,\psi^{(D)} = 0$, indicating that $\hat{j}$ is the flux.