1 Stochastics

1 Ordinary Differential Equation

Let $\mathcal{M}$ a smooth manifold, and $f\colon \mathbb{R}\times\mathcal{M}\rightarrow T\mathcal{M}$, we have ordinary differential equation[1]

$$\frac{\mathrm{d}x^a}{\mathrm{d}t}(t)=f^a(t,x). \tag{1}$$

This ordinary differential equation induces a push-forward operator, $\hat{T}_{t\rightarrow t'}\colon \mathbb{R}\times\mathbb{R}\rightarrow\mathcal{M}\rightarrow\mathcal{M}$ [2], which pushes the particle on position on $\mathcal{M}$ at time t to another position on $\mathcal{M}$ at time t'.

Let $\Omega^k(\mathcal{M})$ the space of k-forms on $\mathcal{M}$, where $k\leqslant\dim(\mathcal{M})$. This ordinary differential equation also induces a pull-back operator on k-forms[3], $\hat{T}^*_{t\rightarrow t'}\colon \mathbb{R}\times\mathbb{R}\rightarrow\Omega^k(\mathcal{M})\rightarrow\Omega^k(\mathcal{M})$, which describes the transition of k-form from time t to t'.

Precisely, consider $\forall\psi^{(k)}\in\Omega^k(\mathcal{M})$, we can write it explicitly by indices, as

$$\psi^{(k)}(x)=(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}, \tag{2}$$

where the indices in $\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)$ is anti-symmetric. Regardless of the $1/k!$ factor, the $\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)$ can be viewed as the local density at x and the $\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}$ as area or volume unit. Thus, $\psi^{(k)}(x)$, as a whole, is the mass unit. $\hat{T}^*$ is thus is a forward pushing transition of mass unit.

Lemma 1. *Explicitly, we have*

$$(\hat{T}^*_{t\rightarrow t'}\,\psi^{(k)})(x')=(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(\hat{T}_{t'\rightarrow t}\,x')\bigwedge_{i=1}^{k}\,(T\hat{T}_{t'\rightarrow t})^{\alpha_i}_{\beta_i}(x')\,\mathrm{d}x'^{\beta_i}, \tag{3}$$

where $T\hat{T}_{t'\rightarrow t}$ denotes the Jacobian of $\hat{T}_{t'\rightarrow t}$.

Proof. Let $x'=\hat{T}_{t\rightarrow t'}\,x$ and $\psi'^{(k)}=\hat{T}^*_{t\rightarrow t'}\,\psi^{(k)}$, by conservation of mass during the pushing, we have

$$\psi'^{(k)}_{\alpha_1\cdots\alpha_k}(x')\,\mathrm{d}x'^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x'^{\alpha_k}=\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}.$$

Replace x by $x=\hat{T}_{t'\rightarrow t}\,x'$, we get

$$\psi'^{(k)}_{\alpha_1\cdots\alpha_k}(x')\,\mathrm{d}x'^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x'^{\alpha_k}=\psi^{(k)}_{\alpha_1\cdots\alpha_k}(\hat{T}_{t'\rightarrow t}\,x')\bigwedge_{i=1}^{k}\,(T\hat{T}_{t'\rightarrow t})^{\alpha_i}_{\beta_i}(x')\,\mathrm{d}x'^{\beta_i}.$$

Inserting back the $(1/k!)$ factor, we arrive at

$$(\hat{T}^*_{t\rightarrow t'}\,\psi^{(k)})(x')=(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(\hat{T}_{t'\rightarrow t}\,x')\bigwedge_{i=1}^{k}\,(T\hat{T}_{t'\rightarrow t})^{\alpha_i}_{\beta_i}(x')\,\mathrm{d}x'^{\beta_i}.$$ □

1.1 Infinitesimal Pull-back

Now, we try to derive the explicit expression of $\hat{T}^*_{t\rightarrow t'}$ depending on f^a in the limit $t'\rightarrow t$. This infinitesimal version of pull-back can be described by Lie derivative.

Definition 2. *[Lie Derivative] Given $f\colon \mathbb{R}\times\mathcal{M}\rightarrow T\mathcal{M}$, Lie derivative $\hat{L}_f\colon \mathbb{R}\rightarrow\Omega^k(\mathcal{M})\rightarrow\Omega^k(\mathcal{M})$ is defined as*

$$\hat{L}_f(t):=\lim_{\Delta t\rightarrow 0}\frac{1-\hat{T}^*_{t\rightarrow t+\Delta t}}{\Delta t}, \tag{4}$$

where $\hat{T}^*$ is the pull-back induced by f.

1. We employ Einstein's convension of summation thoroughly.

2. The notation $A\rightarrow B\rightarrow C$ in declarations always means $A\rightarrow(B\rightarrow C)$. Further, $A\rightarrow B\rightarrow\cdots$ means $A\rightarrow(B\rightarrow(\cdots))$. This is a useful convension from Haskell.

3. Even though we call it something-back, but it pushes forward the k-forms. The name comes from the fact that forward pushing of k-forms is equivalent to backward pushing the mass unit, as the following discussion shows.

Some useful definitions in exterior algebra are recalled. Operators $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$ [4] and, for $\forall f$, $\hat{i}_f := f^\alpha\, i_\alpha$, where i_α is the interior product[5]. Let A and B compositions of $\mathrm{d}x^a\wedge$ and i_a, then $[A, B]$ is commutator if both A and B have balanced $\mathrm{d}x^a\wedge$ and i_a, otherwise anti-commutator.

With these definitions, we conclude the explicit relatoin between f and $\hat{L}_f$, as follow.

Theorem 3. *[Cartan's Magic Formula] We have*

$$\hat{L}_f = [\hat{\mathrm{d}}, \hat{i}_f]. \tag{6}$$

Proof. As $t' = t + \Delta t$ with Δt tiny, we have $\hat{T}_{t'\to t}\, x' = x' - f(t', x')\,\Delta t$. Then, $T\hat{T}_{t'\to t} = 1 - Tf\,\Delta t$, where Tf denotes the Jacobian of f. Now, insert this two expressions into the definition of $\hat{T}^*_{t\to t'}\,\psi^{(k)}$, we find

$$(\hat{T}^*_{t\to t'}\,\psi^{(k)})(x') = (1/k!)\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x' - f(t', x')\,\Delta t)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\,\Delta t)\mathrm{d}x'^{\beta_i}.$$

First, consider the expansion of $(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\,\Delta t)\mathrm{d}x'^{\beta_i}$, up to $\mathcal{O}(\Delta t)$,

$$\begin{aligned}
&(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\,\Delta t)\mathrm{d}x'^{\beta_i}\\
&= (1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}\\
&- \Delta t\sum_{i=1}^{k}(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\,(\partial_{\beta_i} f^{\alpha_i})(t', x')\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge(\mathrm{d}x^{\alpha_i}\to\mathrm{d}x^{\beta_i})\wedge\cdots\wedge\mathrm{d}x^{\alpha_k},
\end{aligned}$$

where $A\to B$ means that the original A is replaced by B. Now, we show that summation in the last line equals to $\partial_\beta f^\alpha(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha\,\psi^{(k)}(x)$. Recall that

$$i_\alpha\,\psi^{(k)}(x) := (1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_i}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k},$$

where A means the original A is deleted. Indeed,

$$\begin{aligned}
&(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha\,\psi^{(k)}(x)\\
&= (\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge(1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_i}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}\\
&= (\partial_\beta f^\alpha)(t', x')\sum_{i=1}^{k}(1/k!)\,\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge(\mathrm{d}x^{\alpha_i}\to\mathrm{d}x^\beta)\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}\\
&= \sum_{i=1}^{k}(\partial_{\beta_i} f^{\alpha_i})(t', x')\,(1/k!)\,\psi_{\alpha_1\cdots\alpha_i\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge(\mathrm{d}x^{\alpha_i}\to\mathrm{d}x^{\beta_i})\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}\\
&= \sum_{i=1}^{k}(\partial_{\beta_i} f^{\alpha_i})(t', x')\,(1/k!)\,\psi_{\alpha_1\cdots\alpha_k}(x)\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge(\mathrm{d}x^{\alpha_i}\to\mathrm{d}x^{\beta_i})\wedge\cdots\wedge\mathrm{d}x^{\alpha_k},
\end{aligned}$$

where in the last two lines, we replaced the dummy indices $\alpha\to\alpha_i$ and $\beta\to\beta_i$, and then found that $\psi_{\alpha_1\cdots\alpha_i\cdots\alpha_k}$ can be written back to $\psi_{\alpha_1\cdots\alpha_k}$. Thus,

$$(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i} - (\partial_{\beta_i} f^{\alpha_i})(t', x')\,\Delta t)\mathrm{d}x'^{\beta_i} = (1 - \Delta t\,(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}(x).$$

So, we find,

$$\begin{aligned}
(\hat{T}^*_{t\to t+\Delta t}\,\psi^{(k)})(x') &= (1 - \Delta t\,(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}(x)\\
\{x = x' - f(t', x')\,\Delta t\} &= (1 - \Delta t\,(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}(x' - f(t', x')\,\Delta t)\\
&= (1 - \Delta t\,(\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,(1 - \Delta t\,f^\alpha(t', x')\,\partial_\alpha)\,\psi^{(k)}(x')\\
&= \psi^{(k)}(x') - (f^a(t', x')\partial_a + (\partial_\beta f^\alpha)(t', x')\,\mathrm{d}x^\beta\wedge i_\alpha)\,\psi^{(k)}(x')\,\Delta t + \mathcal{O}(\Delta t^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{L}_f &:= \lim_{\Delta t\to 0}\frac{1 - \hat{T}^*_{t\to t+\Delta t}}{\Delta t}\\
&= f^a\partial_a + (\partial_\beta f^\alpha)\,\mathrm{d}x^\beta\wedge i_\alpha.
\end{aligned}$$

4. Operator ∂_α is short for $\partial/\partial x^\alpha$.

5. Interior product $i_a\colon \Omega^k(\mathcal{M})\to\Omega^{k-1}(\mathcal{M})$ is defined as, for $\forall\psi^{(k)}\in\Omega^k(\mathcal{M})$,

$$i_\alpha\,\psi^{(k)} := (1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_i}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}, \tag{5}$$

where A means that A is deleted, and $A\to B$ means that the original A is replaced by B. That is, it annihilates a $\mathrm{d}x^a\wedge$. The most useful property of interior product is the anti-commutator $[\mathrm{d}x^\alpha\wedge, i_\beta]_+ = \delta^\alpha_\beta$.

Since $\hat{\mathrm{d}} := \mathrm{d}x^\alpha \wedge \partial_\alpha$ and $\hat{i}_f := f^\alpha\, i_\alpha$, we have

$$
\begin{aligned}
[\hat{\mathrm{d}}, \hat{i}_f] &= \mathrm{d}x^\alpha \wedge \partial_\alpha\, f^\beta\, i_\beta + f^\beta\, i_\beta \mathrm{d}x^\alpha \wedge \partial_\alpha \\
&= (\partial_\alpha\, f^\beta) \mathrm{d}x^\alpha \wedge i_\beta + f^\beta\, \mathrm{d}x^\alpha \wedge i_\beta\, \partial_\alpha \\
\{[\mathrm{d}x^a \wedge, i_\beta] = \delta_\beta^a\} &+ f^\beta\, \delta_\beta^\alpha \partial_\alpha - f^\beta\, \mathrm{d}x^\alpha \wedge i_\beta\, \partial_\alpha \\
&= (\partial_\alpha\, f^\beta) \mathrm{d}x^\alpha \wedge i_\beta + f^a \partial_a,
\end{aligned}
$$

which is $\hat{L}_f$. □

2 Stochastic Differential Equation

A direct generalization of ordinary differential equatoin is adding a Gaussian noise, as stochastic differential equation

$$
\frac{\mathrm{d}x^a}{\mathrm{d}t}(t) = f^a(t, x) + g_\beta^a(t, x)\, \eta^\beta(t), \tag{7}
$$

where, for $\forall t$ and α, $\eta^\alpha(t) \sim \mathcal{N}(0, 1/\mathrm{d}t)$. Thus, $\eta: \mathbb{R} \to V$ with V an Euclidean space, and $g: \mathbb{R} \times \mathcal{M} \to V \to T\mathcal{M}$.

For any functional $F[\eta]$, split the time interval $[t, t']$ by $t = t_1 < t_2 < \cdots < t_N = t'$, with $t_{i+1} - t_i \equiv \Delta t$, then define the expectation as

$$
\begin{aligned}
\langle F \rangle &:= \int D[\eta] \exp\!\left(-\frac{1}{2} \int \mathrm{d}t \delta_{\alpha\beta}\, \eta^a(t)\, \eta^\beta(t) \right) F[\eta] \\
&:= \lim_{\Delta t \to 0} Z^{-1} \int \mathrm{d}\eta(t_1) \cdots \mathrm{d}\eta(t_N) \exp\!\left(-\frac{1}{2} \sum_i\; \Delta t \delta_{\alpha\beta}\, \eta^a(t_i)\, \eta^\beta(t_i) \right) F[\eta],
\end{aligned}
$$

where Z the normalization factor so that $\langle 1 \rangle = 1$. Notice that the dimension of η and that of f may not equal.

2.1 Infinitesimal Pull-back Expectation

For any configuration η given, the stochastic differential equation is reduced to an ordinary differential equation. In this case, the pull-back depends on η, that is, $\hat{T}^*_{t \to t'}[\eta]$. We care about the expectation $\langle \hat{T}^*_{t \to t'}[\eta] \rangle$ over all possible configuration of η, especially it's infinitesimal version.

As before, define the stochastic version of Lie derivative, as

$$
\hat{H}_{(f,g)}(t) := \lim_{\Delta t \to 0} \frac{1 - \langle \hat{T}^*_{t \to t + \Delta t}[\cdot] \rangle}{\Delta t}, \tag{8}
$$

where $\hat{T}^*_{t \to t + \Delta t}$ depends on the configuration η.

Theorem 4. *We have*

$$
\hat{H}_{(f,g)} = \hat{L}_f - \frac{1}{2} \hat{L}_g^2, \tag{9}
$$

where $\hat{L}_g^2 := \delta^{\alpha\beta}\, \hat{L}_{g_\alpha} \hat{L}_{g_\beta}$.

Proof. Given configuration of η, let $F_\eta^\alpha(t, x) := f^\alpha(t, x) + g_\beta^\alpha(t, x) \eta^\beta(t)$. Directly, we have

$$
\hat{L}_{F_\eta} = \hat{L}_f + \hat{L}_{g_\beta}\, \eta^\beta.
$$

Then, we have $\hat{T}^*_{t \to t + \Delta t}[\eta] = \exp(-\hat{L}_{F_\eta} \Delta t)$.[6] Then,

$$
\begin{aligned}
\langle \hat{T}^*_{t \to t + \Delta t}[\cdot] \rangle &= \int \mathrm{d}\eta(t) \exp\!\left(-\frac{1}{2} \Delta t\, \delta_{\alpha\beta}\, \eta^\alpha(t)\, \eta^\beta(t) \right) \exp(-\hat{L}_{F_\eta} \Delta t) \\
&= \int \mathrm{d}\eta(t) \exp\!\left(-\frac{1}{2} \Delta t\, \delta_{\alpha\beta}\, \eta^\alpha(t)\, \eta^\beta(t) \right) \left(1 - \hat{L}_{F_\eta} \Delta t + \frac{1}{2} (\hat{L}_{F_\eta} \Delta t)^2 + \cdots \right) \\
&= 1 - \langle \hat{L}_{F_\eta}\, \Delta t \rangle + \frac{1}{2} \langle (\hat{L}_{F_\eta} \Delta t)^2 \rangle + \cdots
\end{aligned}
$$

Since $\langle \eta^\beta \rangle = 0$, $\langle \hat{L}_{F_\eta}\, \Delta t \rangle = \langle \hat{L}_f + \hat{L}_{g_\beta}\, \eta^\beta \rangle \Delta t = \hat{L}_f\, \Delta t$. And since $\langle \eta^\alpha\, \eta^\beta \rangle = \delta^{\alpha\beta} / \Delta t$,

$$
\begin{aligned}
&\langle (\hat{L}_{F_\eta}\, \Delta t)^2 \rangle \\
&= \langle (\hat{L}_f + \hat{L}_{g_\alpha}\, \eta^\alpha)\, (\hat{L}_f + \hat{L}_{g_\beta}\, \eta^\beta) \rangle\, \Delta t^2 \\
\{\langle \eta^\beta \rangle = 0\} &= \hat{L}_f^2\, \Delta t^2 + \hat{L}_{g_\alpha}\, \hat{L}_{g_\beta}\, \langle \eta^\alpha\, \eta^\beta \rangle\, \Delta t^2 \\
\{\langle \eta^\alpha\, \eta^\beta \rangle = \delta^{\alpha\beta} / \Delta t\} &= \hat{L}_f^2\, \Delta t^2 + \hat{L}_{g_\alpha}\, \hat{L}_{g_\beta}\, \delta^{\alpha\beta}\, \Delta t \\
\{\hat{L}_g^2 := \cdots\} &= \hat{L}_f^2\, \Delta t^2 + \hat{L}_g^2\, \Delta t.
\end{aligned}
$$

6. TODO: needs some proof.

Thus,

$$\begin{aligned}\langle \hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle &= 1 - \langle \hat{L}_{F_\eta}\rangle \Delta t + \frac{1}{2}\langle(\hat{L}_{F_\eta}\Delta t)^2\rangle + \cdots \\ &= 1 - \hat{L}_f\,\Delta t + \frac{1}{2}\,\langle \hat{L}_g^2\rangle \Delta t + o(\Delta t).\end{aligned}$$

So, finally,

$$\hat{H}_{(f,g)} := \lim_{\Delta t\to 0}\frac{\hat{1} - \langle \hat{T}^*_{t\to t+\Delta t}[\cdot]\rangle}{\Delta t} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2.$$

□

Example 5. [Fokker-Planck Equation] In the case $g^\alpha_\beta \equiv \sqrt{2T}\,\delta^\alpha_\beta$,

$$\hat{H} = (\partial_\alpha f^\beta)\,\mathrm{d}x^\alpha \wedge i_\beta + f^\alpha \partial_\alpha - T\partial^2, \tag{10}$$

where $\partial^2 := \delta^{\alpha\beta}\partial_\alpha\partial_\beta$. Applying on $\psi^{(D)}$ with $D = \dim(\mathcal{M})$, since $\mathrm{d}x^\alpha \wedge \psi^{(D)} = 0$, and thus $\mathrm{d}x^\alpha \wedge i_\beta\,\psi^{(D)} = \delta^\alpha_\beta\,\psi^{(D)}$, we find

$$\hat{H}\psi^{(D)} = (\partial_\alpha f^\alpha - T\partial^2)\psi^{(D)}, \tag{11}$$

which is the Fokker-Planck equation.

2.2 Symmetry (TODO)

Definition 6. *Given A, B is called A-exact if there exists X such that*

$$B = [A, X]. \tag{12}$$

Lemma 7. *If B is A-exact, then $[A, B] = 0$.*

Proof.

$$\begin{aligned}[A,B] &= [A,[A,X]] \\ &= [X,[A,A]] + [A,[X,A]] \\ &= ?\end{aligned}$$

□

Lemma 8. *We have decomposition*

$$\hat{H} = [\hat{\mathrm{d}}, \hat{j}], \tag{13}$$

where $\hat{j} := \hat{i}_f - \frac{1}{2}\hat{L}_{g_\beta}\eta^\beta$.

That is, $\hat{H}$ is $\hat{\mathrm{d}}$-exact, thus,

$$[\hat{\mathrm{d}}, \hat{H}] = 0. \tag{14}$$

$\hat{\mathrm{d}}\psi^{(k)} = 0$, but there isn't $\varphi^{(k-1)}$ s.t. $\psi^{(k)} = \hat{\mathrm{d}}\varphi^{(k-1)}$. Symmetric state.

$\hat{\mathrm{d}}\psi^{(k)} \neq 0$. $\hat{\mathrm{d}}^2\psi^{(k)} = 0$. Symmetry breaking state. $\mathrm{tr}(\langle \hat{T}^*_{t\to t'}[\cdot]\rangle) = \sum_n \exp(-E_n(t'-t)) = \exp(\lambda(t'-t))$.

$\hat{j}^2 = 0$. $[\hat{j}, \hat{H}] = 0$.

j is the flux. That is, for $\psi^{(D)}$, where $D = \dim(\mathcal{M})$, $\hat{H}\,\psi^{(D)} = \hat{j}\,\psi^{(D)}$, so that $\partial_t\psi^{(D)} + \hat{j}\,\psi^{(D)} = 0$, indicating that $\hat{j}$ is the flux.