Chapter 1
Stochastics

1.1 Ordinary Differential Equation

Let $\mathcal{M}$ a smooth manifold, and $f: \mathbb{R} \times \mathcal{M} \to T\mathcal{M}$, we have ordinary differential equation[1.1]

$$\frac{dx^a}{dt}(t) = f^a(t, x). \tag{1.1}$$

This ordinary differential equation induces a push–forward operator, $\hat{T}_{t \to t'}: \mathbb{R} \times \mathbb{R} \to \mathcal{M} \to \mathcal{M}$ [1.2], which pushes the particle on position on $\mathcal{M}$ at time t to another position on $\mathcal{M}$ at time t'.

Let $\Omega^k(\mathcal{M})$ the space of k–forms on $\mathcal{M}$, where $k \leqslant \dim(\mathcal{M})$. This ordinary differential equation also induces a pull–back operator on k–forms[1.3], $\hat{T}^*_{t \to t'}: \mathbb{R} \times \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$, which describes the transition of k–form from time t to t'.

Precisely, consider $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$, we can write it explicitly by indices, as

$$\psi^{(k)}(x) = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}, \tag{1.2}$$

where the indices in $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ is anti–symmetric. Regardless of the $1/k!$ factor, the $\psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)$ can be viewed as the local density at x and the $dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$ as area or volume unit. Thus, $\psi^{(k)}(x)$, as a whole, is the mass unit. $\hat{T}^*$ is thus is a forward pushing transition of mass unit.

Lemma 1.1. *Explicitly, we have*

$$(\hat{T}^*_{t \to t'}\, \psi^{(k)})(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\, x') \bigwedge_{i=1}^{k} (\mathcal{D}\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\, dx'^{\beta_i}, \tag{1.3}$$

where $\mathcal{D}\hat{T}_{t' \to t}$ *denotes the Jacobian of* $\hat{T}_{t' \to t}$.

Proof. Let $x' = \hat{T}_{t \to t'}\, x$ and $\psi'^{(k)} = \hat{T}^*_{t \to t'}\, \psi^{(k)}$, by conservation of mass during the pushing, we have

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, dx'^{\alpha_1} \wedge \cdots \wedge dx'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(x)\, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}.$$

Replace x by $x = \hat{T}_{t' \to t}\, x'$, we get

$$\psi'^{(k)}_{\alpha_1 \cdots \alpha_k}(x')\, dx'^{\alpha_1} \wedge \cdots \wedge dx'^{\alpha_k} = \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\, x') \bigwedge_{i=1}^{k} (\mathcal{D}\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\, dx'^{\beta_i}.$$

Inserting back the $(1/k!)$ factor, we arrive at

$$(\hat{T}^*_{t \to t'}\, \psi^{(k)})(x') = (1/k!)\, \psi^{(k)}_{\alpha_1 \cdots \alpha_k}(\hat{T}_{t' \to t}\, x') \bigwedge_{i=1}^{k} (\mathcal{D}\hat{T}_{t' \to t})^{\alpha_i}_{\beta_i}(x')\, dx'^{\beta_i}.$$ □

1.1.1 Infinitesimal Pull–back

Now, we try to derive the explicit expression of $\hat{T}^*_{t \to t'}$ depending on f^a in the limit $t' \to t$. This infinitesimal version of pull–back can be described by Lie derivative.

Definition 1.2. [*Lie Derivative*] *Given* $f: \mathbb{R} \times \mathcal{M} \to T\mathcal{M}$, *Lie derivative* $\hat{L}_f: \mathbb{R} \to \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$ *is defined as*

$$\hat{L}_f(t) := \lim_{\Delta t \to 0} \frac{1 - \hat{T}^*_{t \to t + \Delta t}}{\Delta t}, \tag{1.4}$$

where $\hat{T}^*$ *is the pull–back induced by* f.

Some useful definitions in exterior algebra are recalled. Operators $\hat{d} := dx^\alpha \wedge \partial_\alpha$ [1.4] and, for $\forall f$, $\hat{i}_f := f^\alpha\, i_\alpha$, where i_α is the interior product[1.5]. Let A and B compositions of $dx^a \wedge$ and i_a, then $[A, B]$ is commutator if both A and B have balanced $dx^a \wedge$ and i_a, otherwise anti–commutator.

1.1. We employ Einstein℘s convension of summation thoroughly.

1.2. The notation $A \to B \to C$ in declarations always means $A \to (B \to C)$. Further, $A \to B \to \cdots$ means $A \to (B \to (\cdots))$. This is a useful convension from Haskell.

1.3. Even though we call it something–back, but it pushes forward the k–forms. The name comes from the fact that forward pushing of k–forms is equivalent to backward pushing the mass unit, as the following discussion shows.

1.4. Operator ∂_α is short for $\partial/\partial x^\alpha$.

1.5. Interior product $i_a: \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ is defined as, for $\forall \psi^{(k)} \in \Omega^k(\mathcal{M})$,

$$i_\alpha\, \psi^{(k)} := (1/k!) \sum_{i=1}^{k} (-1)^{i-1} \psi_{\alpha_1 \cdots (\alpha_i \to \alpha) \cdots \alpha_k}\, dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_i} \wedge \cdots \wedge dx^{\alpha_k}, \tag{1.5}$$

where A means that A is deleted, and $A \to B$ means that the original A is replaced by B. That is, it annihilates a $dx^a \wedge$. The most useful property of interior product is the anti–commutator $[dx^\alpha \wedge, i_\beta]_+ = \delta^\alpha_\beta$.

With these definitions, we conclude the explicit relatoin between f and $\hat{L}_f$, as follow.

Theorem 1.3. [*Cartanφs Magic Formula*] *We have*

$$\hat{L}_f=[\hat{d},\hat{\imath}_f]. \tag{1.6}$$

Proof. As $t'=t+\Delta t$ with Δt tiny, we have $\hat{T}_{t'\to t}\,x'=x'-f(t',x')\,\Delta t$. Then, $\mathcal{D}\hat{T}_{t'\to t}=1-\mathcal{D}f\,\Delta t$, where $\mathcal{D}f$ denotes the Jacobian of f. Now, insert this two expressions into the definition of $\hat{T}^*_{t\to t'}\psi^{(k)}$, we find

$$(\hat{T}^*_{t\to t'}\psi^{(k)})(x')=(1/k!)\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x'-f(t',x')\,\Delta t)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i}-(\partial_{\beta_i}f^{\alpha_i})(t',x')\,\Delta t)dx'^{\beta_i}.$$

First, consider the expansion of $(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i}-(\partial_{\beta_i}f^{\alpha_i})(t',x')\,\Delta t)dx'^{\beta_i}$, up to $\mathcal{O}(\Delta t)$,

$$\begin{aligned}
&(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i}-(\partial_{\beta_i}f^{\alpha_i})(t',x')\,\Delta t)dx'^{\beta_i}\\
&=(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_k}\\
&-\Delta t\sum_{i=1}^{k}(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\,(\partial_{\beta_i}f^{\alpha_i})(t',x')\,dx^{\alpha_1}\wedge\cdots\wedge(dx^{\alpha_i}\to dx^{\beta_i})\wedge\cdots\wedge dx^{\alpha_k},
\end{aligned}$$

where $A\to B$ means that the original A is replaced by B. Now, we show that summation in the last line equals to $\partial_\beta f^\alpha(t',x')\,dx^\beta\wedge i_\alpha\,\psi^{(k)}(x)$. Recall that

$$i_\alpha\,\psi^{(k)}(x):=(1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}(x)\,dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_i}\wedge\cdots\wedge dx^{\alpha_k},$$

where A means the original A is deleted. Indeed,

$$\begin{aligned}
&(\partial_\beta f^\alpha)(t',x')\,dx^\beta\wedge i_\alpha\,\psi^{(k)}(x)\\
&=(\partial_\beta f^\alpha)(t',x')\,dx^\beta\wedge(1/k!)\sum_{i=1}^{k}(-1)^{i-1}\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}(x)\,dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_i}\wedge\cdots\wedge dx^{\alpha_k}\\
&=(\partial_\beta f^\alpha)(t',x')\sum_{i=1}^{k}(1/k!)\,\psi_{\alpha_1\cdots(\alpha_i\to\alpha)\cdots\alpha_k}(x)\,dx^{\alpha_1}\wedge\cdots\wedge(dx^{\alpha_i}\to dx^{\beta})\wedge\cdots\wedge dx^{\alpha_k}\\
&=\sum_{i=1}^{k}(\partial_{\beta_i}f^{\alpha_i})(t',x')\,(1/k!)\,\psi_{\alpha_1\cdots\alpha_i\cdots\alpha_k}(x)\,dx^{\alpha_1}\wedge\cdots\wedge(dx^{\alpha_i}\to dx^{\beta_i})\wedge\cdots\wedge dx^{\alpha_k}\\
&=\sum_{i=1}^{k}(\partial_{\beta_i}f^{\alpha_i})(t',x')\,(1/k!)\,\psi_{\alpha_1\cdots\alpha_k}(x)\,dx^{\alpha_1}\wedge\cdots\wedge(dx^{\alpha_i}\to dx^{\beta_i})\wedge\cdots\wedge dx^{\alpha_k},
\end{aligned}$$

where in the last two lines, we replaced the dummy indices $\alpha\to\alpha_i$ and $\beta\to\beta_i$, and then found that $\psi_{\alpha_1\cdots\alpha_i\cdots\alpha_k}$ can be written back to $\psi_{\alpha_1\cdots\alpha_k}$. Thus,

$$(1/k!)\,\psi^{(k)}_{\alpha_1\cdots\alpha_k}(x)\bigwedge_{i=1}^{k}(\delta^{\alpha_i}_{\beta_i}-(\partial_{\beta_i}f^{\alpha_i})(t',x')\,\Delta t)dx'^{\beta_i}=(1-\Delta t\,(\partial_\beta f^\alpha)(t',x')\,dx^\beta\wedge i_\alpha)\,\psi^{(k)}(x).$$

So, we find,

$$\begin{aligned}
(\hat{T}^*_{t\to t+\Delta t}\,\psi^{(k)})(x')&=(1-\Delta t\,(\partial_\beta f^\alpha)(t',x')\,dx^\beta\wedge i_\alpha)\,\psi^{(k)}(x)\\
\{x=x'-f(t',x')\,\Delta t\}&=(1-\Delta t\,(\partial_\beta f^\alpha)(t',x')\,dx^\beta\wedge i_\alpha)\,\psi^{(k)}(x'-f(t',x')\,\Delta t)\\
&=(1-\Delta t\,(\partial_\beta f^\alpha)(t',x')\,dx^\beta\wedge i_\alpha)\,(1-\Delta t\,f^\alpha(t',x')\,\partial_\alpha)\psi^{(k)}(x')\\
&=\psi^{(k)}(x')-(f^a(t',x')\partial_a+(\partial_\beta f^\alpha)(t',x')\,dx^\beta\wedge i_\alpha)\,\psi^{(k)}(x')\,\Delta t+\mathcal{O}(\Delta t^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{L}_f&:=\lim_{\Delta t\to 0}\frac{1-\hat{T}^*_{t\to t+\Delta t}}{\Delta t}\\
&=f^a\partial_a+(\partial_\beta f^\alpha)\,dx^\beta\wedge i_\alpha.
\end{aligned}$$

Since $\hat{d}:=dx^\alpha\wedge\partial_\alpha$ and $\hat{\imath}_f:=f^\alpha\,i_\alpha$, we have

$$\begin{aligned}
[\hat{d},\hat{\imath}_f]&=dx^\alpha\wedge\partial_\alpha\,f^\beta\,i_\beta+f^\beta\,i_\beta dx^\alpha\wedge\partial_\alpha\\
&=(\partial_\alpha\,f^\beta)dx^\alpha\wedge i_\beta+f^\beta\,dx^\alpha\wedge i_\beta\,\partial_\alpha\\
\{[dx^a\wedge,i_\beta]=\delta^\alpha_\beta\}&+f^\beta\,\delta^\alpha_\beta\partial_\alpha-f^\beta\,dx^\alpha\wedge i_\beta\,\partial_\alpha\\
&=(\partial_\alpha\,f^\beta)dx^\alpha\wedge i_\beta+f^a\partial_a,
\end{aligned}$$

which is $\hat{L}_f$. □

1.2 Stochastic Differential Equation

A direct generalization of ordinary differential equatoin is adding a Gaussian noise, as stochastic differential equation

$$\frac{dx^a}{dt}(t)=f^a(t,x)+g^a_\beta(t,x)\,\eta^\beta(t), \tag{1.7}$$

where, for $\forall t$ and α, $\eta^\alpha(t)\sim\mathcal{N}(0,1/dt)$. Thus, $\eta\colon\mathbb{R}\to V$ with V an Euclidean space, and $g\colon\mathbb{R}\times\mathcal{M}\to V\to T\mathcal{M}$.

For any functional $F[\eta]$, split the time interval $[t, t']$ by $t = t_1 < t_2 < \cdots < t_N = t'$, with $t_{i+1} - t_i \equiv \Delta t$, then define the expectation as

$$\begin{aligned}\langle F \rangle &:= \int D[\eta] \exp\left(-\frac{1}{2}\int dt\, \delta_{\alpha\beta}\, \eta^a(t)\, \eta^\beta(t)\right) F[\eta] \\ &:= \lim_{\Delta t \to 0} Z^{-1} \int d\eta(t_1) \cdots d\eta(t_N) \exp\left(-\frac{1}{2}\sum_i \Delta t\, \delta_{\alpha\beta}\, \eta^a(t_i)\, \eta^\beta(t_i)\right) F[\eta],\end{aligned}$$

where Z the normalization factor so that $\langle 1 \rangle = 1$. Notice that the dimension of η and that of f may not equal.

1.2.1 Infinitesimal Pull–back Expectation

For any configuration η given, the stochastic differential equation is reduced to an ordinary differential equation. In this case, the pull–back depends on η, that is, $\hat{T}^*_{t \to t'}[\eta]$. We care about the expectation $\langle \hat{T}^*_{t \to t'}[\eta] \rangle$ over all possible configuration of η, especially itφs infinitesimal version.

As before, define the stochastic version of Lie derivative, as

$$\hat{H}_{(f,g)}(t) := \lim_{\Delta t \to 0} \frac{1 - \langle \hat{T}^*_{t \to t+\Delta t}[\cdot] \rangle}{\Delta t}, \tag{1.8}$$

where $\hat{T}^*_{t \to t+\Delta t}$ depends on the configuration η.

Theorem 1.4. *We have*

$$\hat{H}_{(f,g)} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2, \tag{1.9}$$

where $\hat{L}_g^2 := \delta^{\alpha\beta}\, \hat{L}_{g_\alpha} \hat{L}_{g_\beta}$.

Proof. Given configuration of η, let $F^\alpha_\eta(t, x) := f^\alpha(t, x) + g^\alpha_\beta(t, x)\eta^\beta(t)$. Directly, we have

$$\hat{L}_{F_\eta} = \hat{L}_f + \hat{L}_{g_\beta}\, \eta^\beta.$$

Then, we have $\hat{T}^*_{t \to t+\Delta t}[\eta] = \exp(-\hat{L}_{F_\eta}\Delta t)$.[1.6] Then,

$$\begin{aligned}\langle \hat{T}^*_{t \to t+\Delta t}[\cdot] \rangle &= \int d\eta(t) \exp\left(-\frac{1}{2}\Delta t\, \delta_{\alpha\beta}\, \eta^\alpha(t)\, \eta^\beta(t)\right) \exp(-\hat{L}_{F_\eta}\Delta t) \\ &= \int d\eta(t) \exp\left(-\frac{1}{2}\Delta t\, \delta_{\alpha\beta}\, \eta^\alpha(t)\, \eta^\beta(t)\right)\left(1 - \hat{L}_{F_\eta}\Delta t + \frac{1}{2}(\hat{L}_{F_\eta}\Delta t)^2 + \cdots\right) \\ &= 1 - \langle \hat{L}_{F_\eta}\, \Delta t \rangle + \frac{1}{2}\langle (\hat{L}_{F_\eta}\Delta t)^2 \rangle + \cdots\end{aligned}$$

Since $\langle \eta^\beta \rangle = 0$, $\langle \hat{L}_{F_\eta}\, \Delta t \rangle = \langle \hat{L}_f + \hat{L}_{g_\beta}\, \eta^\beta \rangle \Delta t = \hat{L}_f\, \Delta t$. And since $\langle \eta^\alpha\, \eta^\beta \rangle = \delta^{\alpha\beta} / \Delta t$,

$$\begin{aligned}&\langle (\hat{L}_{F_\eta}\, \Delta t)^2 \rangle \\ &= \langle (\hat{L}_f + \hat{L}_{g_\alpha}\, \eta^\alpha)\, (\hat{L}_f + \hat{L}_{g_\beta}\, \eta^\beta) \rangle\, \Delta t^2 \\ \{\langle \eta^\beta \rangle = 0\} &= \hat{L}_f^2\, \Delta t^2 + \hat{L}_{g_\alpha}\, \hat{L}_{g_\beta}\, \langle \eta^\alpha\, \eta^\beta \rangle\, \Delta t^2 \\ \{\langle \eta^\alpha\, \eta^\beta \rangle = \delta^{\alpha\beta} / \Delta t\} &= \hat{L}_f^2\, \Delta t^2 + \hat{L}_{g_\alpha}\, \hat{L}_{g_\beta}\, \delta^{\alpha\beta}\, \Delta t \\ \{\hat{L}_g^2 := \cdots\} &= \hat{L}_f^2\, \Delta t^2 + \hat{L}_g^2\, \Delta t.\end{aligned}$$

Thus,

$$\begin{aligned}\langle \hat{T}^*_{t \to t+\Delta t}[\cdot] \rangle &= 1 - \langle \hat{L}_{F_\eta} \rangle \Delta t + \frac{1}{2}\langle (\hat{L}_{F_\eta}\Delta t)^2 \rangle + \cdots \\ &= 1 - \hat{L}_f\, \Delta t + \frac{1}{2}\, \langle \hat{L}_g^2 \rangle \Delta t + o(\Delta t).\end{aligned}$$

So, finally,

$$\hat{H}_{(f,g)} := \lim_{\Delta t \to 0} \frac{\hat{1} - \langle \hat{T}^*_{t \to t+\Delta t}[\cdot] \rangle}{\Delta t} = \hat{L}_f - \frac{1}{2}\hat{L}_g^2. \qquad \square$$

Example 1.5. [Fokker–Planck Equation] In the case $g^\alpha_\beta \equiv \sqrt{2T}\, \delta^\alpha_\beta$,

$$\hat{H} = (\partial_\alpha f^\beta)\, dx^\alpha \wedge i_\beta + f^\alpha \partial_\alpha - T\partial^2, \tag{1.10}$$

where $\partial^2 := \delta^{\alpha\beta} \partial_\alpha \partial_\beta$. Applying on $\psi^{(D)}$ with $D = \dim(\mathcal{M})$, since $dx^\alpha \wedge \psi^{(D)} = 0$, and thus $dx^\alpha \wedge i_\beta\, \psi^{(D)} = \delta^\alpha_\beta\, \psi^{(D)}$, we find

$$\hat{H}\psi^{(D)} = (\partial_\alpha f^\alpha - T\partial^2)\psi^{(D)}, \tag{1.11}$$

which is the Fokker–Planck equation.

1.2.2 Symmetry (TODO)

Definition 1.6. *Given* A, B *is called* A*–exact if there exists* X *such that*

$$B = [A, X]. \tag{1.12}$$

1.6. TODO. needs some proof.

Lemma 1.7. *If* B *is* $A-$*exact, then* $[A,B]=0$.

Proof.

$$\begin{aligned}[A,B] &= [A,[A,X]] \\ &= [X,[A,A]]+[A,[X,A]] \\ &= ?\end{aligned}$$

□

Lemma 1.8. *We have decomposition*

$$\hat{H}=[\hat{d},\hat{j}], \tag{1.13}$$

where $\hat{j}:=\hat{\imath}_f-\frac{1}{2}\hat{L}_{g_\beta}\eta^\beta$.

That is, $\hat{H}$ is $\hat{d}-$exact, thus,

$$[\hat{d},\hat{H}]=0. \tag{1.14}$$

$\hat{d}\psi^{(k)}=0$, but there isnφt $\varphi^{(k-1)}$ s.t. $\psi^{(k)}=\hat{d}\varphi^{(k-1)}$. Symmetric state.

$\hat{d}\psi^{(k)}\neq 0$. $\hat{d}^2\psi^{(k)}=0$. Symmetry breaking state. $\mathrm{tr}(\langle\hat{T}^*_{t\to t'}[\cdot]\rangle)=\sum_n\exp(-E_n(t'-t))=\exp(\lambda(t'-t))$.

$\hat{j}^2=0$. $[\hat{j},\hat{H}]=0$.

j is the flux. That is, for $\psi^{(D)}$, *where* $D=\dim(\mathcal{M})$, $\hat{H}\psi^{(D)}=\hat{j}\,\psi^{(D)}$, so that $\partial_t\psi^{(D)}+\hat{j}\,\psi^{(D)}=0$, indic– ating that $\hat{j}$ is the flux.