

Period

$$T = \frac{2\pi}{\omega_0}$$
$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

Periodicity of Multiple Signals

$$\frac{T_1}{T_2} = \frac{k}{l} \text{ where } k, l \in \mathbb{Z}$$

$$T_0 = LCM(T_1, T_2, \dots)$$
$$LCM\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}\right) = \frac{LCM(a_1, a_2, \dots, a_n)}{GCD(b_1, b_2, \dots, b_n)}$$

Support

- Smallest closed set where signal is non-zero,
- Include boundaries and ignore simple intersections

$$\text{supp}(x) = [0, 1] \cup [2, \infty)$$

Even and Odd

$$x_{\text{even}}(t) = \frac{1}{2}(x(t) + x(-t))$$
$$x_{\text{odd}}(t) = \frac{1}{2}(x(t) - x(-t))$$

Special Signals

Unit Step: CT \approx DT

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Impulse: CT is derived from integral

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Sifting:

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$
$$\therefore \int_{-\infty}^{\infty} x(t)\delta(t - t_0) dt = x(t_0)$$

Sinusoids

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{N_0}$$

DT: $e^{j\omega n}$ is periodic N_0 if ω is an integer multiple of $\frac{2\pi}{N}$

Frequency Periodicity:

$$e^{j\omega n} = e^{j\omega n + 2\pi}$$

$$e^{j\pi n} = (-1)^n$$

Theorem 2.3:

Let $N_0 \in \mathbb{Z}_{\geq 1}$ be a desired period. There are exactly N_0 distinct DT complex exponential signals of period N_0 given by

$$\phi_k[n] = e^{jk\omega_0 n}, \quad k \in \{0, 1, \dots, N_0 - 1\}$$

CT Fourier Series

Existence:

If x has finite action, $x \in L_1^{per}$
then α_k are well defined and

$$\lim_{k \rightarrow \pm\infty} |\alpha_k| = 0$$

Convergence:

Let $e_k = \hat{x}_k - x$ be the error of approximation of x
 \hat{x} converges to x

- i. pointwise if $\lim_{k \rightarrow \infty} e_k(t_0) = 0$, t_0 is the point of interest
- ii. uniformly if $\lim_{k \rightarrow \infty} \|e_k\|_{\infty} = 0$, amplitude of error goes to 0
- iii. in energy if $\lim_{k \rightarrow \infty} \|e_k\|_2 = 0$, energy of error goes to 0

i. if x has finite action and $\frac{d}{dt}x$ is cont. at t_0 , then \hat{x} converges to x

OR if there is a finite discontinuity (left and right limits exist) \hat{x} converges to the midpoint of the discontinuities, *Gibbs Phenomenon*

ii. if x has finite action and $\frac{d}{dt}x$ is cont. everywhere, then \hat{x} converges to x uniformly

iii. if x has finite energy, $x \in L_2^{per}$

1. \hat{x} converges to x in energy

2. $\{\alpha_k\}$ have finite energy, $\alpha \in l_2^{per}$

3. the signal and coefficients satisfy *Parseval's Relation*:

$$\frac{1}{T_0} \|x\|_2^2 = \|\alpha\|_2^2$$

$$\frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |\alpha_k|^2$$

energy of x = energy of α_k
scaled by period

Finding α_k from sinusoidal signal:

Given $x(t) = \sin(16\pi t) + \cos(12\pi t)$

$$\text{Recall } \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}, \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore x(t) = \frac{1}{2}je^{-16j\pi t} - \frac{1}{2}je^{16j\pi t} + \frac{1}{2}e^{-12j\pi t} + \frac{1}{2}e^{12j\pi t}$$

$$x(t) = \frac{1}{2}je^{-4j4\pi t} - \frac{1}{2}je^{4j4\pi t} + \frac{1}{2}e^{-3j4\pi t} + \frac{1}{2}e^{3j4\pi t}$$

$$\therefore \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \alpha_{-4} = \frac{1}{2}j, & \alpha_4 = \frac{-1}{2}j, & \alpha_{-3} = \frac{1}{2}, & \alpha_3 = \frac{1}{2} \end{matrix}$$

$\alpha_0 = 0$, otherwise