Search Smooths Discontinuities in Discrete/Continuous Problems

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Abstract

When used to solve models with both discrete and continuous choices, the Endogenous Gridpoint Method (EGM), in addition to finding the optimal solution, can also generate points in the solution that are suboptimal. This paper shows that incorporating search into such problems can smooth out the discontinuities in the decision rules, which are responsible for producing the suboptimal points. With ample search frictions, the suboptimal points can be eliminated entirely. In addition to providing greater control over the smoothness of the problem, search offers the added benefit of rationalizing the smoothness into the agent's choice problem.

Keywords Consumption, Computational Economics, Discrete Continuous Choice, Endogenous Gridpoint Method, Search

JEL codes C13, C63

Repo: `https://github.com/llorracc/dcegmSearch`
PDF: `http://econ.jhu.edu/people/ccarroll/papers/dcegmSearch.pdf`
Appendix: `http://econ.jhu.edu/people/ccarroll/papers/dcegmSearch/#Appendix`
Web: `http://econ.jhu.edu/people/ccarroll/papers/dcegmSearch/`
Slides: Versions to View or Print

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The computational results in this paper were constructed using tools in the Econ-ARK/HARK toolkit. The toolkit can be cited by its digital object identifier, 10.5281/zenodo.1001068, as is done in the paper's own references as Carroll, White, and Econ-ARK (2017). Thanks to Matthew White for initial discussions of this subject.

1 Introduction

This paper introduces new methods for efficiently solving dynamic optimization problems with both discrete and continuous choices (DC models). These methods extend the Endogenous Gridpoint Method (EGM) (Carroll (2006)) by including exogenous outcome probabilities, search frictions, and taste shocks to '*concavify*' the value function of the optimization problem. Compared to existing extensions of the EGM for DC models, the methods introduced in this paper have the added advantage of not only providing greater smoothness, but also rationalizing the smoothness into the agent's choice problem.

For dynamic stochastic optimization problems with continuous choice variables, the Endogeneous Gridpoint Method (EGM) is significantly more efficient compared to standard root-finding methods in finding the optimal decision rules. In these problems, the value functions are typically concave and the Euler equation is a necessary and sufficient condition for the optimal decision rule. In contrast, when the choice variables include both continuous and discrete choices (DC models), the Euler equation has multiple solutions for the continuous choice. Therefore, the Euler equation is only a necessary, but not sufficient, condition for the optimal decision rule.

The complication in solving a dynamic optimization problem arises due to the kinks in the value function and discontinuities in the decision rule at the points in the state space where discrete choices are made. These discontinuities make the decision rule non-monotonic, which leads to the Euler equation producing suboptimal points. In addition, each period's kinks and discontinuities are propagated back in time as the solution is solved by backward iteration, thus, exacerbating the problem. Fella (2014), Druedahl and Jørgensen (2017), and Iskhakov, Jørgensen, Rust, and Schjerning (2017) provide methods for finding and discarding these suboptimal points from the final solution.

In order to mitigate the complications arising from the accumulation of the discontinuities, Iskhakov, Jørgensen, Rust, and Schjerning (2017) introduce Extreme Value Type I taste shocks that affect the likelihood of the discrete choice outcomes. These taste shocks have the advantage of smoothing out the expected value function and the expected marginal utility function. With larger taste shocks, the value function can be 'concavified' to a greater degree. Such taste shocks, however, are in fact 'behavioral' shocks that add a degree of randomization to the discrete choice outcomes.

In contrast, this paper introduces search frictions whereby agents exert search effort that determines the likelihood of changing the discrete state. In addition, it includes log-normally distributed taste shocks that affect the utility that agents derive from making discrete choices. Combined, these two features not only lead to greater smoothness of the decision rule, they also rationalize the smoothness arising from the taste shocks, as these shocks determine the search effort that the agents exert.

To illustrate the properties of search as a smoothing mechanism and to compare it to other modeling features that can also provide smoothness, we use a standard consumption-saving model in which the agent can also receive the 'option' to make a binary retirement decision, depending on the search effort that they make. If the agent makes no search effort, they receive the option to retire with exogenous probability p and if they make the maximum effort, they receive the option to retire with certainty. Effort, however, is costly and the cost function is assumed to be a convex function of effort. In addition, households receive taste shocks that affect the relative value they derive from working and are also exposed to income uncertainty.

All four of the aforementioned modeling features, i.e. search, exogenous switching probability, taste shocks, and income shocks, serve to smooth out the decision rule in the DC model with varying degrees of effectiveness. While search provides the highest degree of smoothness, in terms of reducing the size of the discontinuities in the decision rule, it has to be combined with taste shocks to yield uniform smoothness around the discontinuities. Moreover, each of the smoothing mechanisms have different economic interpretations and affect the decision rules differently. While greater smoothness from search and exogenous switching probability shifts the consumption function upwards, greater smoothness from taste shocks and income shocks shifts the consumption function downwards.

2 Model

In this section, we build a consumption-saving model and extend it to include the option to retire and search frictions. Agents in this model derive utility from consuming non-durable goods c_t and get disutility from working. The disutility parameter is denoted by δ. Each period, there is a possibility for the agents to receive an 'option' to retire. The likelihood of receiving the option depends on the amount of search effort ε_{t-1} that the agent exerted in the previous period. Search effort, however, is costly and we assume that the search cost is a convex function of effort, with a functional form given by $\theta(\varepsilon_t) = \sigma^s \left(\varepsilon_t + (1-\varepsilon_t)\log\left(1-\varepsilon_t\right)\right)$. This cost function implies that the initial marginal effort is cheap; however, greater effort that increases the likelihood of receiving the option to switch the discrete state leads to increasingly larger costs. Assuming logarithmic utility from consumption, and denoting the choice to retire and the choice to work by $d_t = 0$ and $d_t = 1$, respectively, the utility function for the agent is given by

$$u(c_t, d_t, \varepsilon_t) = \log(c_t) - \delta d_t - \theta(\varepsilon_t). \tag{1}$$

At the end of each period, households can save their end-of-period liquid assets A_t at a risk free rate R. Working in the current period determines whether the agent receives income y_t in the following period. Therefore, the beginning-of-period assets are given by $M_t = RA_t + y_t d_{t-1}$. Furthermore, we assume that $y_t = y\eta_t$, where η_t follows a mean-one log-normal distribution, $\eta_t \sim \mathcal{N}(-\sigma_\eta^2/2, \sigma_\eta^2)$.

2.1 Retiree's sub-problem

Retirement is assumed to be self-absorbing, so the problem reduces to the simple consumption-saving problem, which in Bellman form is given by:

$$V_t^{Ret}(M_t) = \max_{0 \le c_t \le M_t} \{u(c_t, 0, 0) + \beta V_{t+1}^{Ret}(M_{t+1})\}, \tag{2}$$

where $M_{t+1} = A_t R$ and the end-of-period assets $A_t = M_t - c_t$. Using the gothic script V ($\mathfrak{V}$) to represent the value of ending the period with assets A_t, the end-of-period value for a retired agent is, by definition

$$\mathfrak{V}_t^{Ret}(A_t) \equiv V_{t+1}^{Ret}(M_{t+1}(A_t)) = V_{t+1}^{Ret}(A_t R) \tag{3}$$

Thus, the problem in terms of the end-of-period assets can be restated as

$$\mathfrak{V}_{t-1}^{Ret}(A_{t-1}) = \max_{0 \le c_t \le A_{t-1} R} \{\log(c_t) + \beta \mathfrak{V}_t^{Ret}(A_t)\}. \tag{4}$$

2.2 Worker's sub-problem

Each period, a worker either receives the option to retire or has to continue working. There is an exogenous probability p of receiving the option to retire. In addition, a worker can increase the likelihood of receiving the option to retire by exerting search effort. The search effort is made at the end of a period and affects the likelihood of receiving the option to retire in the following period. When period t starts, an income shock is realized. This is followed by the realization of a taste shock, ξ_t, which is a multiplicative shock that affects the utility that a worker derives from working. The taste shocks are assumed to follow a mean-one log-normal distribution, i.e. $\xi_t \sim \mathcal{N}(-\sigma_\xi^2/2, \sigma_\xi^2)$. These taste shocks are different from the Extreme Value Type I shocks in Iskhakov, Jørgensen, Rust, and Schjerning (2017) in two major ways. Firstly, the assumption of log-normal taste shocks means that these shocks can take up multiple distinct values in the discrete approximation of the distribution. Secondly, these shocks are not the purely behavioral shocks that randomize the discrete choice outcome. Instead, these taste shocks affect the likelihood of the discrete outcome by determining the search effort that the agent exerts.

Based on the search effort made in the previous period, if the agent receives the option to switch, the agent decides whether to retire or continue working. If the agent does not receive the option to switch, then they continue to work. Having made the

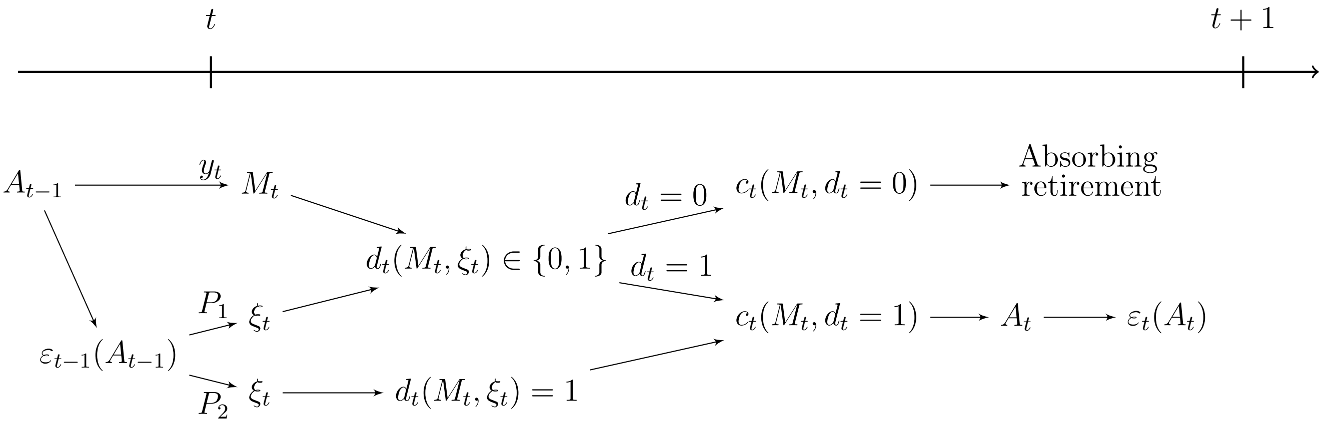


Figure 1 Timing of shock realizations and decisions in the worker's problem.

discrete choice, the agent makes a consumption decision. If the agent chooses to retire, they stay as a retiree, because the assumption that retirement is self-absorbing. If the agent chooses to continue working, they exert search effort, which then determines the likelihood of receiving the option to switch in the next period. Fig.1 outlines the timeline of the realization of various shocks and the decisions in the worker's problem.

Let the set of possible choice sets for the worker's discrete choice be given by $\mathcal{D}_t \in \big\{\{\text{Retire}, \text{Work}\}, \{\text{Work}\}\big\} \equiv \big\{\{d_t = 0, d_t = 1\}, \{d_t = 1\}\big\}$. The first element of this set is the case when the agent has the option to switch and the second element is the case when the agent does not have the option to switch and must continue working. For the case when $\mathcal{D}_t = \{W\}$, i.e. the worker does not have the option to switch, the Bellman equation is given by:

$$V_t(M_t, \{W\}, \xi_t) = \xi_t. \max_{\substack{0 \le c_t \le M_t \\ 0 \le \varepsilon_t \le 1}} \left\{u(c_t, 1, \varepsilon_t) + \beta P_1(\varepsilon_t)\mathfrak{V}_t^1(A_t) + \beta P_2(\varepsilon_t)\mathfrak{V}_t^2(A_t)\right\} \tag{5}$$

where $\mathfrak{V}_t^1(A_t)$ is the expected value of saving A_t when having the option to retire in $t+1$, $P_1(\varepsilon_t) = (1-\varepsilon_t)p+\varepsilon_t$ is the probability receiving the option to retire in $t+1$, given the search effort ε_t in the current period, $\mathfrak{V}_t^2(A_t)$ is the expected value of saving A_t when not having the option to retire in $t+1$, and $P_2(\varepsilon_t) = (1-\varepsilon_t)(1-p)$ is the probability of not receiving the option to retire in $t+1$, given the search effort ε_t in the current period.

For the case when $\mathcal{D}_t = \{R, W\}$, i.e. the case when the worker has the option to switch, the Bellman equation is given by:

$$V_t(M_t, \{R, W\}, \xi_t) = \max_{d_t \in \{0,1\}} \Big[V_t^{Ret}(M_t); V_t(M_t, \{W\}, \xi_t)\Big], \tag{6}$$

or

$$V_t(M_t, \{R, W\}, \xi_t) = \max_{d_t \in \{0,1\}} \Big[\max_{0 \le c_t \le M_t} \big\{ u(c_t, 0, 0) + \beta \mathfrak{V}_t^{Ret}(A_t) \big\}; \\ \xi_t \cdot \max_{\substack{0 \le c_t \le M_t \\ 0 \le \varepsilon_t \le 1}} \big\{ u(c_t, 1, \varepsilon_t) + \beta P_1(\varepsilon_t) \mathfrak{V}_t^1(A_t) + \beta P_2(\varepsilon_t) \mathfrak{V}_t^2(A_t) \big\} \Big]. \tag{7}$$

The expected value of saving A_t when having the option to retire in $t+1$, i.e. $\mathfrak{V}_t^1(A_t)$, is given by

$$\mathfrak{V}_t^1(A_t) = \int_{y_{t+1}} \int_{\xi_{t+1}} V_{t+1}(M_{t+1}(A_t), \{R, W\}, \xi_{t+1}) dF(y_{t+1}) dF(\xi_{t+1}), \tag{8}$$

where $M_{t+1} = A_t R + d_t y$. Similarly, the expected value of saving A_t when not having the option to retire in $t+1$, i.e. $\mathfrak{V}_t^2(A_t)$, is given by

$$\mathfrak{V}_t^2(A_t) = \int_{y_{t+1}} \int_{\xi_{t+1}} V_{t+1}(M_{t+1}(A_t), \{W\}, \xi_{t+1}) dF(y_{t+1}) dF(\xi_{t+1}). \tag{9}$$

2.3 Optimal effort and consumption functions

Since we assume that the consumption decision is made before the effort decision, (5) can be split into two sequential sub-period problems and rewritten as:

$$V_t(M_t, \{W\}, \xi_t) = \xi_t \max_{0 \le c_t \le M_t} \Big[\mathcal{V}_t(A_t) \Big] \tag{10}$$

where

$$\mathcal{V}_t(A_t) = \max_{0 \le \varepsilon_t \le 1} \big\{ u(c_t, 1, \varepsilon_t) + \beta P_1(\varepsilon_t) \mathfrak{V}_t^1(A_t) + \beta P_2(\varepsilon_t) \mathfrak{V}_t^2(A_t) \big\} \tag{11}$$

Thus, the first order condition of (11) gives the optimal effort as a function of the end-of-period assets that are determined after the optimal consumption decision has been made in the previous sub-period. The optimal effort is given by

$$\frac{\partial \theta(\varepsilon_t)}{\partial \varepsilon_t} = \beta \mathfrak{V}_t^1(A_t) \frac{\partial P_1(\varepsilon_t)}{\partial \varepsilon_t} + \beta \mathfrak{V}_t^2(A_t) \frac{\partial P_2(\varepsilon_t)}{\partial \varepsilon_t} \tag{12}$$

$$-\sigma^s \log(1 - \varepsilon_t^*) = \beta(1-p) \mathfrak{V}_t^1(A_t) - \beta(1-p) \mathfrak{V}_t^2(A_t) \tag{13}$$

$$\varepsilon_t^* = 1 - \exp\left(-\frac{\beta(1-p)}{\sigma^s} \left(\mathfrak{V}_t^1(A_t) - \mathfrak{V}_t^2(A_t) \right) \right). \tag{14}$$

This expression states that optimal effort is an increasing function of the excess value of receiving the option to switch in the next period over not receiving the option to switch.

Next, we derives the expressions for the optimal consumption functions for the retiree and the worker. Since retirement is self-absorbing, the retiree's consumption function is relatively straightforward to derive and is given by:

$$u'(c_t, 0, 0) = R\beta u'(c_{t+1}, 0, 0) \tag{15}$$

or

$$c_t^{Ret} = \left(R\beta\left(c_{t+1}^{Ret}(A_t R)\right)^{-1}\right)^{-1}. \tag{16}$$

To derive the consumption function for the worker, we start with the solution to the problem of a worker who does not have the option to retire, i.e. $\mathcal{D}_t = \{W\}$. The first-order-condition for problem (10), with respect to c_t, is given by

$$u'(c_t, 1, \varepsilon_t^*) = \beta\Big[P_1(\varepsilon_t^*)\frac{\partial\mathfrak{V}_t^1(A_t)}{\partial A_t} + P_2(\varepsilon_t^*)\frac{\partial\mathfrak{V}_t^2(A_t)}{\partial A_t}\Big], \tag{17}$$

where ε_t^* is the optimal effort function, which is determined in (14), and the derivates of the end-of-period values are given by

$$\frac{\partial\mathfrak{V}_t^1(A_t)}{\partial A_t} = \mathbb{E}_t\Big\{\frac{\partial V_{t+1}^{R,W}(M_{t+1}, \xi_{t+1})}{\partial M_{t+1}}\frac{\partial M_{t+1}}{\partial A_t}\Big\} = R\mathbb{E}_t\Big\{\frac{\partial V_{t+1}^{R,W}(M_{t+1}, \xi_{t+1})}{\partial M_{t+1}}\Big\} \tag{18}$$

and

$$\frac{\partial\mathfrak{V}_t^2(A_t)}{\partial A_t} = R\mathbb{E}_t\Big\{\frac{\partial V_{t+1}^{W}(M_{t+1}, \xi_{t+1})}{\partial M_{t+1}}\Big\}, \tag{19}$$

where the expectation is over next period income and taste shocks, and $V_{t+1}^W(M_{t+1}, \xi_{t+1})$ and $V_{t+1}^{R,W}(M_{t+1}, \xi_{t+1})$, are shorthands for $V_{t+1}(M_{t+1}, \{W\}, \xi_{t+1})$ and $V_{t+1}(M_{t+1}, \{R, W\}, \xi_{t+1})$, respectively. Substituting these expression into (17) yields

$$u'(c_t, 1, \varepsilon_t^*) = R\beta\Big[P_1(\varepsilon_t^*)\mathbb{E}_t\Big\{\frac{\partial V_{t+1}^{R,W}(M_{t+1}, \xi_{t+1})}{\partial M_{t+1}}\Big\} + P_2(\varepsilon_t^*)\mathbb{E}_t\Big\{\frac{\partial V_{t+1}^{W}(M_{t+1}, \xi_{t+1})}{\partial M_{t+1}}\Big\}\Big]. \tag{20}$$

The Envelope Condition for (10) is given by

$$\frac{\partial V_t^W(M_t,\xi_t)}{\partial M_t} = \xi_t\beta\Big[P_1(\varepsilon_t^*)\frac{\partial \mathfrak{V}_t^1(A_t)}{\partial A_t} + P_2(\varepsilon_t^*)\frac{\partial \mathfrak{V}_t^2(A_t)}{\partial A_t}\Big] \tag{21}$$

or

$$\frac{\partial V_t^W(M_t,\xi_t)}{\partial M_t} = \xi_t u'(c_t,1,\varepsilon_t^*) \tag{22}$$

which can be iterated forward to yield

$$\frac{\partial V_{t+1}^W(M_{t+1},\xi_{t+1})}{\partial M_{t+1}} = \xi_{t+1} u'(c_{t+1},1,\varepsilon_{t+1}^*). \tag{23}$$

To derive a similar expression for the derivative of $V_{t+1}^{R,W}$, we note that for the worker who has the option to retire, i.e. $\mathcal{D}_t = \{R,W\}$, the value function is simply the maximum of the value function of the retiree and the value function of the worker who continues to work

$$V_t^{R,W}(M_t,\xi_t) = \max\Big\{V_t^{Ret}(M_t);V_t^W(M_t,\xi_t)\Big\}. \tag{24}$$

Therefore, it can be shown that

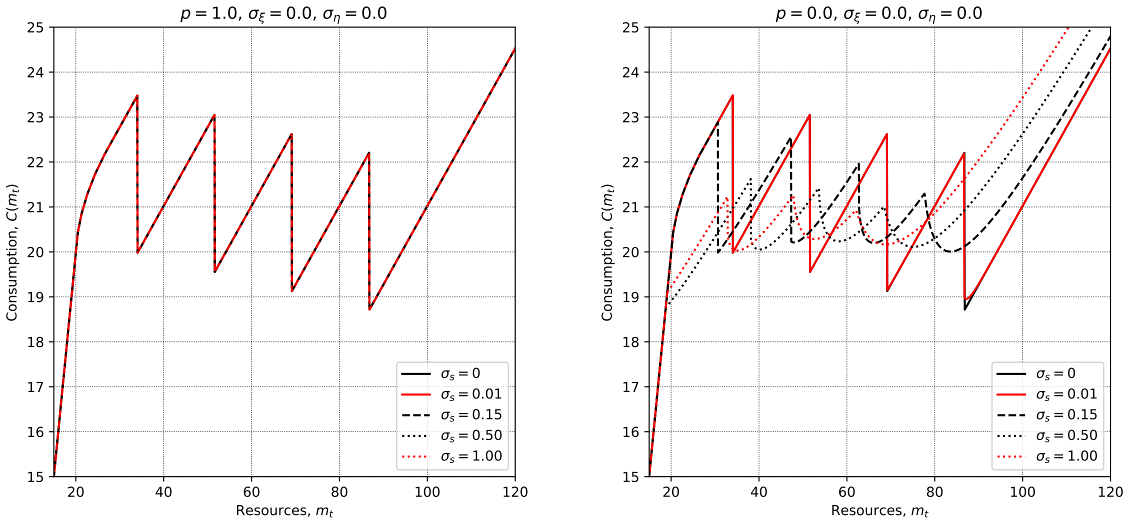
$$\frac{\partial V_{t+1}^{R,W}(M_{t+1},\xi_{t+1})}{\partial M_{t+1}} = \mathbb{1}_{t+1}^0.u'(c_{t+1},0,0) + \mathbb{1}_{t+1}^1.\xi_{t+1}u'(c_{t+1},1,\varepsilon_{t+1}^*), \tag{25}$$

where $\mathbb{1}_{t+1}^0 \equiv \mathbb{1}_{t+1}^0(M_{t+1},\xi_{t+1}) = 1$ if $V_{t+1}^{Ret}(M_{t+1}) \geq V_{t+1}^W(M_{t+1},\xi_{t+1})$ and 0 otherwise, and $\mathbb{1}_{t+1}^1 \equiv \mathbb{1}_{t+1}^1(M_{t+1},\xi_{t+1}) = 1$ if $V_{t+1}^{Ret}(M_{t+1}) < V_{t+1}^W(M_{t+1},\xi_{t+1})$ and 0 otherwise.

Substituting (23) and (25) into (20) yields the consumption function for the worker who continues to work

$$\begin{aligned} u'(c_t,1,\varepsilon_t^*) = R\beta\Big[P_1(\varepsilon_t^*)\int_{y_{t+1}}\int_{\xi_{t+1}}\big[\mathbb{1}_{t+1}^0.u'(c_{t+1},0,0) + \mathbb{1}_{t+1}^1.\xi_{t+1}u'(c_{t+1},1,\varepsilon_{t+1}^*)\big]dF(y_{t+1})dF(\xi_{t+1}) \\ + P_2(\varepsilon_t^*)\int_{y_{t+1}}\int_{\xi_{t+1}}\big[\xi_{t+1}u'(c_{t+1},1,\varepsilon_{t+1}^*)\big]dF(y_{t+1})dF(\xi_{t+1})\Big] \end{aligned} \tag{26}$$

or



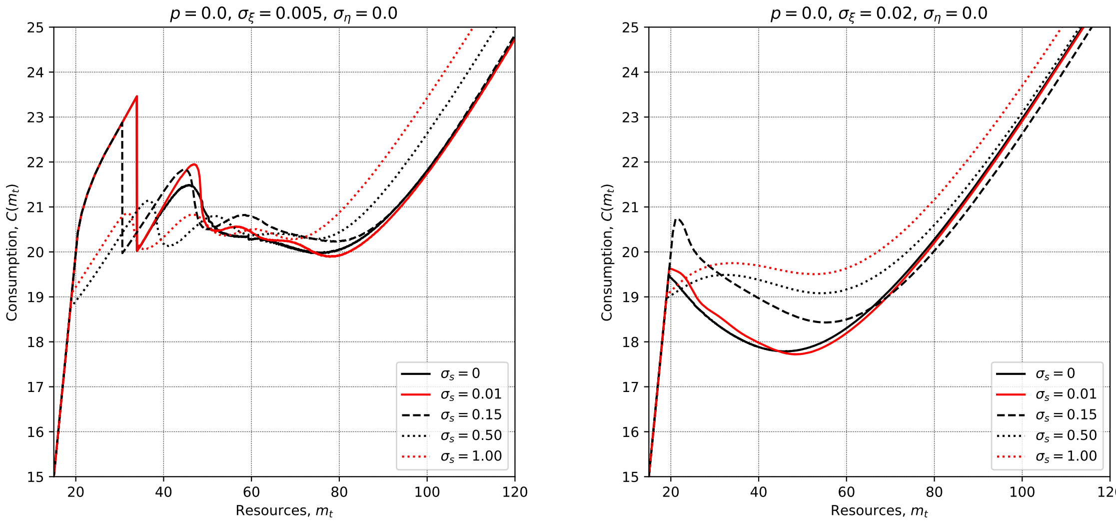
ve switching option with certainty, $p = 1$. **(b)** Stay as worker with certainty without search effort, $p = 0$.

Figure 2 The plots show optimal consumption rules of the workers who decides to continue working in period $t = T - 5$ for a set of search cost scales σ_s in the absence of income uncertainty or taste shocks, for the case with $p = 1.0$ (left panel) with $p = 0.0$ (right panel). The rest of the model parameters are $R = 1$, $\beta = 0.98$, $y = 20$.

$$c_t^W = \Bigg(R\beta \Big[P_1(\varepsilon_t^*) \int_{y_{t+1}} \int_{\xi_{t+1}} \big[\mathbb{1}_{t+1}^0 \cdot \left(c_{t+1}^{Ret}(A_t R) \right)^{-1} + \mathbb{1}_{t+1}^1 \cdot \xi_{t+1} \left(c_{t+1}^W(A_t R + y_{t+1}) \right)^{-1} \big] dF(y_{t+1}) dF(\xi_{t+1}) \\ + P_2(\varepsilon_t^*) \int_{y_{t+1}} \int_{\xi_{t+1}} \big[\xi_{t+1} \left(c_{t+1}^W(A_t R + y_{t+1}) \right)^{-1} \big] dF(y_{t+1}) dF(\xi_{t+1}) \Big] \Bigg)^{-1}. \tag{27}$$

2.4 Results

Having established the solution of the retirement problem, we demonstrate how varying the different smoothness parameters affects the solution of the problem. Fig. 2a shows the optimal consumption rule of the worker who continues to work in period $T - 5$, for the case in which $p = 1.0$. The plot shows the consumption rule for a set of search cost scales, in the absence of income uncertainty and taste shocks. In this scenario, there is no smoothing and the discontinuities that arise from discrete choices being made in periods $T - 4$ through $T - 1$ are distinctly visible. This is because when $p = 1.0$, the worker receives the option to switch with certainty. In such a case the worker does not exert any search effort, regardless of the search frictions. Consequently, in the case with $p = 1.0$, for all levels of search frictions, we get the same optimal consumption rule,



taste shocks, $\sigma_\xi = 0.005$.

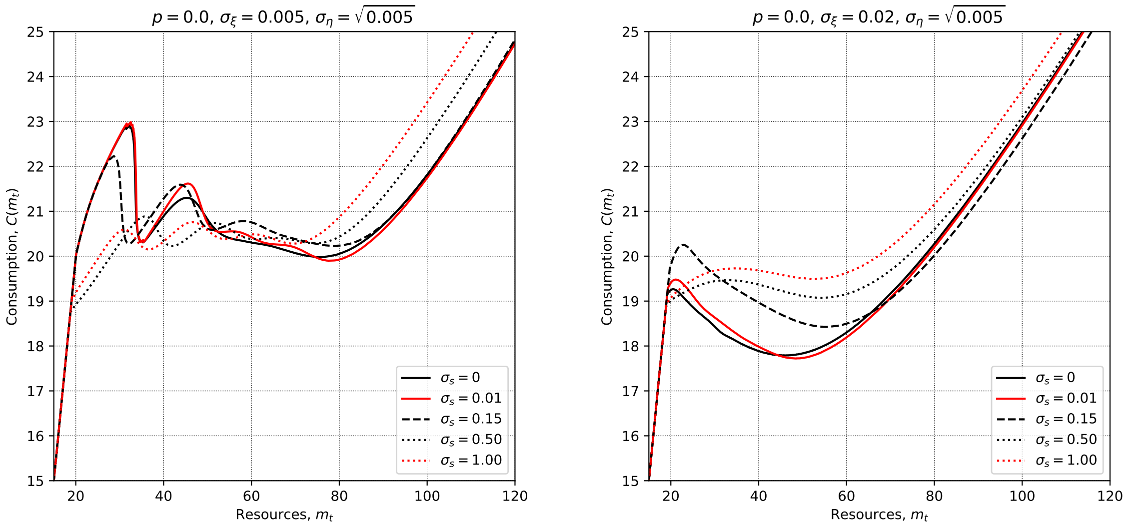
(b) Large taste shocks, $\sigma_\xi = 0.02$.

Figure 3 The plots show optimal consumption rules of the workers who decides to continue working in period $t = T - 5$ for the case with $p = 0$, in the presence of small taste shocks (left panel) and large taste shocks (right panel), without income uncertainty.

without any smoothing.

In contrast, when $p = 0.0$ (Fig. 2b), the worker who does not exert search effort, continues to stay as a worker with certainty. This gives the worker a motive to exert search effort, which determines the likelihood of receiving the option to switch. For a given level of search frictions, decreasing p increases the smoothness of the solution. Moreover, for a given level of p, increasing the search frictions (higher value of σ_s) also increases smoothness. Fig. 2b also shows the asymmetric smoothness offered by search, when there are no taste or income shocks. The smoothness happens to the right of the discontinuities. This is because for low liquid resources, workers optimally choose to continue working and hence do not exert any search effort. Workers only exert positive effort to receive the option to retire when they have ample liquid resources. Therefore, the smoothness from search frictions only happens to the right of the discontinuities.

To address this asymmetric smoothness, we need to introduce taste shocks. Since the taste shocks can take a range of different values, there are certain realizations of the taste shock that make the worker want to have the option to switch, even at levels of liquid assets at which, in the absence of a taste shock, the worker would not want to switch. In other words, with the introduction of taste shocks, the range of liquid assets around the discontinuities over which the worker exerts search effort expands. Fig.3a shows that with the incorporation of taste shocks, the consumption function now smoothes out more uniformly around the discontinuities. Moreover, introducing taste



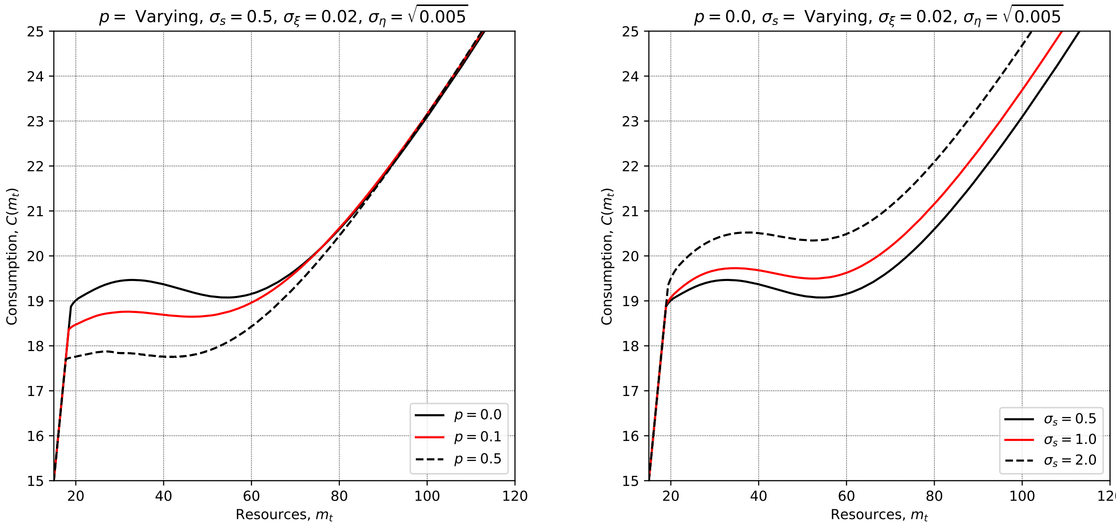
taste shocks, $\sigma_\xi = 0.005$. **(b)** Large taste shocks, $\sigma_\xi = 0.02$.

Figure 4 The plots show optimal consumption rules of the workers who decides to continue working in period $t = T - 5$ for the case with $p = 0$, in the presence of small taste shocks (left panel) and large taste shocks (right panel), with income uncertainty.

shocks allows the search frictions to significantly smooths out most of the discontinuities in the optimal consumption function. This holds even for a very small smoothness parameter for the taste shocks, $\sigma_\xi = 0.005$. This is because now there is a positive likelihood of a worker receiving a taste shock that makes them exert search effort even for values of M_t for which the worker would not exert any search effort in the absence of taste shocks. Increasing the magnitude of the taste shocks to $\sigma_\xi = 0.02$ (Fig.3b) significantly reduces the size of all the discontinuities in the consumption function. Moreover, only moderately sized search frictions are need to virtually completely smooth out the consumption function.

The results discussed so far do not include any income uncertainty. Adding income uncertainty even into standard consumption-saving models adds curvature and smoothness to the consumption rules. We study the impact of incorporating income uncertainty to the retirement problem on the smoothness of the decision rules in Figs.4a and 4b. These figures show that while income uncertainty adds smoothness, it does not play a significant role in reducing the size of the discontinuities. Therefore, search frictions and taste shocks play the dominant role in smoothing out the problem. An appropriate combination of search frictions, taste shocks, and income uncertainty can yield a monotonically increasing outcome-specific consumption function by smoothing out the discontinuities that propagate through each period while iterating backwards.

It is important to note that although each of these smoothness mechanisms serve

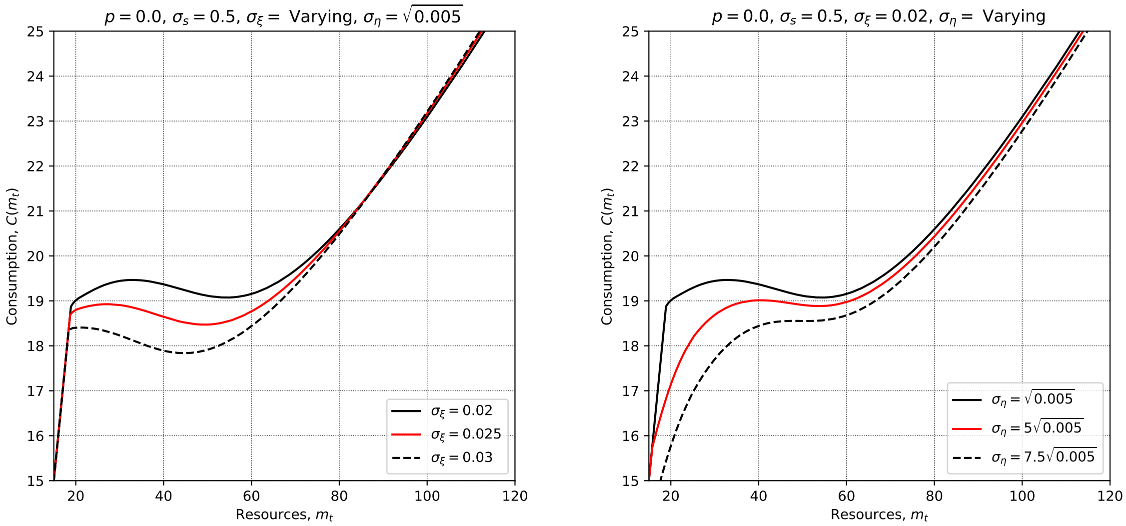


ng the exogenous probability of receiving the optio **(b)** Varying search friction intensity.

Figure 5 The plots show the impact of increasing the smoothness parameters for the exogenous option probability (left panel) and increasing the magnitude of the search frictions (right panel).

to smooth out the solution for the worker's problem, they have different economic interpretations and affect the optimal consumption rule differently. While lowering the exogenous probability of receive the option to switch, p, and increasing the search frictions, σ_s, shift the consumption function upwards, large taste shocks, σ_ξ, and income shocks, σ_η, shift the consumption function downwards. Ceteris paribus, decreasing p or increasing σ_s, reduce the likelihood of the worker switching their discrete state successfully. This means that the likelihood of the worker retiring in a given period decreases. As a result, the expected number of years that an agent works increases and they save less in a given period. In contrast, when the size of the taste shocks is increased, the uncertainty surrounding the likelihood of a worker's retirement decision increases. Consequently, the agents save more in a given period, because unexpectedly retiring early would limit the resources that the agent can consume for the remainder of their life. Similarly, when the size of the income shocks increases, households increase their buffer stock savings, and the consumption function shifts down. Therefore, although each of the smoothness mechanisms achieve the common goal of 'concavifying' the solution, they have different interpretations and different impacts on the optimal consumption rule.

To illustrate these points graphically, we start from the baseline consumption function that has the discontinuities completely smoothed out using smoothness parameters $p = 0$, $\sigma_\xi = 0.02$, $\sigma_s = 0.5$, and $\sigma_\eta = \sqrt{0.005}$. Figs.5a shows the impact of lowering p from the baseline level to $p = 0.1$ and $p = 0.5$, which decreases smoothness, and Fig.5b shows the impact of increasing the search frictions from the baseline level to $\sigma_s = 1.0$ and $\sigma_s = 2.0$,



ng the magnitude of the taste shocks. **(b)** Varying the magnitude of income shocks.

Figure 6 The plots show the impact of increasing the smoothness parameters for the taste shocks (left panel) and increasing the magnitude of the income shocks (right panel).

which increases smoothness. As mentioned above, increasing smoothness by adjusting either of the two mechanisms shifts the optimal consumption rule upwards. On the other hand, Fig.6a shows the impact of increasing the size of the taste shocks from the baseline level to $\sigma_\xi = 0.025$ and $\sigma_\xi = 0.03$, and Fig.6b shows the impact of increasing the size of the income shocks from the baseline level to $\sigma_\eta = 5\sqrt{0.005}$ and $\sigma_\eta = 7.5\sqrt{0.005}$. These figures demonstrate that, as expected, increasing the magnitude of the taste or income shocks shifts the optimal consumption function downwards. Therefore, while a combination of these smoothing mechanisms will smooth out the discontinuities, it is also important to note how each of them shift the decision rule differently.

3 Algorithm

This section outlines the algorithm for solving the problem of the worker who continues to work in the current period. The problem for the retiree can be pre-solved relatively easily. With the solution of the worker who continues to work and the retiree at hand, it is straightforward to compute the solution of the worker with the option to switch. With ample smoothness, the EGM-step should not produce any suboptimal points. However, if there is not enough smoothness, then an additional 'upper envelope' step is required to remove the suboptimal points. The upper envelope refinement step in Iskhakov, Jørgensen, Rust, and Schjerning (2017) is an example of an algorithm that can be employed at the end of the EGM-step to achieve this.

Algorithm 1: EGM-step with search for the worker who continues to work

Let $\overrightarrow{\eta} = \{\eta^1, ..., \eta^J\}$ and $\overrightarrow{\xi} = \{\xi^1, ..., \xi^K\}$ be vectors of quadrature points with associated weights $\overrightarrow{\omega} = \{\omega^1, ..., \omega^J\}$ and $\overrightarrow{\mu} = \{\mu^1, ..., \mu^K\}$, respectively.
Form an ascending grid over end-of-period wealth, $\overrightarrow{A}_t = \{A_t^1, ..., A_t^I\}$
for $i = 1, ..., I$ **do**
  for $j = 1, ..., J$ **do**
    Compute $M_{t+1}^j(A^i) = RA^i + y\eta_{t+1}^j$
    for $d_{t+1} = 0, 1$ **do**
      Compute $c_{t+1}^{Ret}(M_{t+1}^j(A^i))$ by interpolating $c_{t+1}^{Ret}(\overrightarrow{M}_{t+1})$ at the point $M_{t+1}^j(A^i)$
      Compute $c_{t+1}^{W}(M_{t+1}^j(A^i))$ by interpolating $c_{t+1}^{W}(\overrightarrow{M}_{t+1})$ at the point $M_{t+1}^j(A^i)$
      Compute $V_{t+1}^{Ret}(M_{t+1}^j(A^i))$ by interpolating $V_{t+1}^{Ret}(\overrightarrow{M}_{t+1})$ at the point $M_{t+1}^j(A^i)$
      for $k = 1, ..., K$ **do**
        Compute $V_{t+1}^{W}(M_{t+1}^j(A^i), \xi^k)$ by interpolating $V_{t+1}^{W}(\overrightarrow{M}_{t+1})$ at the point $M_{t+1}^j(A^i)$ and multiplying it by ξ^k
        Compute $V_{t+1}^{RW}(M_{t+1}^j(A^i), \xi^k) = \max\{V_{t+1}^{Ret}(M_{t+1}^j(A^i)), V_{t+1}^{W}(M_{t+1}^j(A^i), \xi^k)\}$
        Compute $\mathbb{1}_{t+1}^0(M_{t+1}^j(A^i), \xi^k) = 1$ if $V_{t+1}^{Ret}(M_{t+1}^j(A^i)) \geq V_{t+1}^{W}(M_{t+1}^j(A^i), \xi^k)$ and 0 otherwise.
        Compute $\mathbb{1}_{t+1}^1(M_{t+1}^j(A^i), \xi^k) = 1 - \mathbb{1}_{t+1}^0(M_{t+1}^j(A^i), \xi^k)$
        Compute $RHS_1^{pre}(M_{t+1}^j(A^i), \xi^k) =$ $\mathbb{1}_{t+1}^0(M_{t+1}^j(A^i), \xi^k).u'(c_{t+1}^{Ret}(M_{t+1}^j(A^i))) + \xi^k.\mathbb{1}_{t+1}^1(M_{t+1}^j(A^i), \xi^k).u'(c_{t+1}^{W}(M_{t+1}^j(A^i)))$
        Compute $RHS_2^{pre}(M_{t+1}^j(A^i), \xi^k) = \xi^k.u'(c_{t+1}^{W}(M_{t+1}^j(A^i)))$
      end
    end
  end
  Compute $\mathfrak{V}_t^1(A^i) = \sum_{j=1}^J \sum_{k=1}^K \omega^j \mu^k.V_{t+1}^{RW}(M_{t+1}^j(A^i), \xi^k)$
  Compute $\mathfrak{V}_t^2(A^i) = \sum_{j=1}^J \sum_{k=1}^K \omega^j \mu^k.V_{t+1}^{W}(M_{t+1}^j(A^i), \xi^k)$
  Compute $\varepsilon_t(A^i) = 1 - \exp\left(-\beta(1-p)(\mathfrak{V}_t^1(A^i) - \mathfrak{V}_t^2(A^i))/\sigma^s\right)$
  Compute $\theta(\varepsilon_t(A^i)) = \sigma^s \left(\varepsilon_t(A^i) + (1 - \varepsilon_t(A^i))\log(1 - \varepsilon_t(A^i))\right)$
  Compute $P_1(\varepsilon_t(A^i)) = (1 - \varepsilon_t(A^i)).p + \varepsilon_t(A^i)$
  Compute $P_2(\varepsilon_t(A^i)) = (1 - \varepsilon_t(A^i)).(1-p)$
  Compute $RHS_1(A^i) = \sum_{j=1}^J \sum_{k=1}^K \omega^j \mu^k.RHS_1^{pre}(M_{t+1}^j(A^i), \xi^k)$
  Compute $RHS_2(A^i) = \sum_{j=1}^J \sum_{k=1}^K \omega^j \mu^k.RHS_2^{pre}(M_{t+1}^j(A^i), \xi^k)$
  Compute $RHS(A^i) = R\beta\left(P_1(\varepsilon_t(A^i))RHS_1(A^i) + P_2(\varepsilon_t(A^i))RHS_2(A^i)\right)$
  Compute expected value function $EV_{t+1}(M_{t+1}(A^i)) = P_1(\varepsilon_t(A^i)).\mathfrak{V}_t^1(A^i) + P_2(\varepsilon_t(A^i)).\mathfrak{V}_t^2(A^i)$
  Compute current consumption $c_t^W(A^i) = u'^{-1}\left(RHS(A^i)\right)$
  Compute value function $V_t^W(M_t(A^i)) = u(c_t^W(A^i), 1, \varepsilon_t(A^i)) + \beta EV_{t+1}(M_{t+1}(A^i))$
  Compute endogenous grid point $M_t(A^i) = c_t^W(A^i) + A^i$
end
Collect the points $M_t(A^i)$, $c_t^W(A^i)$, and $V_t^W(M_t(A^i))$ to form the consumption function, $c_t^W(\overrightarrow{M}_t)$, and value function, $V_t^W(\overrightarrow{M}_t)$.

4 Conclusion

This paper introduces search as a new method for smoothing out discrete-continuous choice (DC) problems. While search can significantly reduce the size of the discontinuities in the decision rule of a DC problem, to get uniform smoothness around the discontinuities, we need to incorporate taste shocks as well. Together, search and taste shocks offer modelers greater control over the degree of smoothness in DC models, while rationalizing the smoothness into the agent's choice problem. With ample smoothness, discontinuities in the decision rule can be smoothed out entirely; thus, eliminating the need for incorporating an additional, computationally costly, step of identifying and removing suboptimal points generated in the EGM step for DC models. This paper studies search in the context of a consumption-saving-retirement problem; however, the framework can be easily applied to models of housing purchase, lumpy firm investments, and a wide array of other DC models.

References

Carroll, Christopher D. (2006): "The method of endogenous gridpoints for solving dynamic stochastic optimization problems," *Economics Letters*, 91(3), 312–320.

Carroll, Christopher D, Matthew N White, and Team Econ-ARK (2017): "econ-ark/HARK: 0.8.0," Available at via doi:10.5281/zenodo.1001068 or at `https://doi.org/10.5281/zenodo.1001068`.

Druedahl, Jeppe, and Thomas Høgholm Jørgensen (2017): "A general endogenous grid method for multi-dimensional models with non-convexities and constraints," *Journal of Economic Dynamics and Control*, 74, 87–107.

Fella, Giulio (2014): "A generalized endogenous grid method for non-smooth and non-concave problems," *Review of Economic Dynamics*, 17(2), 329–344.

Iskhakov, Fedor, Thomas H. Jørgensen, John Rust, and Bertel Schjerning (2017): "The endogenous grid method for discrete-continuous dynamic choice models with (or without) taste shocks," *Quantitative Economics*, 8(2), 317–365.

McFadden, Daniel (1973): "Conditional logit analysis of qualitative choice behavior," in *Frontiers in econometrics*. ed. by P. Zarembka, Academic Press, New York.

White, Matthew N. (2015): "The method of endogenous gridpoints in theory and practice," *Journal of Economic Dynamics and Control*, 60, 26–41.