Department of Computing and Information Systems
COMP90038 Algorithms and Complexity Tutorial Week 2

Possible Responses and Discussion

The exercises

1. Consider the usual (unsigned) binary representation of integers. For example, `10110010` represents 178, and `000011` represents 3.

   (a) If we call the bits in an n-bit word $x_{n-1}, x_{n-2}, \ldots, x_2, x_1, x_0$ (so x_0 is the *least significant* bit), which natural number is denoted by $x_{n-1}x_{n-2}\cdots x_2x_1x_0$?

   (b) Describe, in English, an algorithm for converting from binary to decimal notation.

   (c) Write the algorithm in (pseudo-) code.

   (d) Describe, in English, how to convert the decimal representation to binary.

   Response.

   (a) Assuming unsigned representation, n bits allows us to represent the integers from 0 to $2^n - 1$, inclusive. The bit-string $x_{n-1}x_{n-2}\cdots x_2x_1x_0$ denotes $\sum_{i=0}^{n-1} 2^i \cdot x_i$.

   (b) Here is one method, expressed in English. We build the decimal-notation number by visiting the binary digits from left to right, constructing the result in an "accumulator". Start with the accumulator being 0. As long as there is a next bit to process, double the value of the accumulator, and add the value of that next bit.

   (c) Notice how all sorts of ambiguities creep in when we use natural languages. You might easily get the impression that what was meant with the previous answer was "as long as there is a next bit, double the accumulator, and then, after all that doubling, add something." It isn't clear from the structure of the English sentence that "and add the value" is part of what should be done for each bit (and the use of a comma before "and" didn't help). In pseudo-code this should be made unambiguous:

   function BINTODEC($x_{n-1}x_{n-2}\cdots x_2x_1x_0$)
   $a \leftarrow 0$
   for $i \leftarrow 0$ to $n-1$ **do**
   $a \leftarrow 2a + x_{n-i-1}$
   return a

   (d) To convert decimal representation d to binary, the natural way is to generate the bits from right to left. To get the rightmost bit, calculate the parity of d, that is, find d mod 2. Then halve d (rounding down). Now repeat this process, to get the remaining bits. More precisely:

   function DECTOBIN(d)
   $n \leftarrow 0$
   while $d \neq 0$ **do**
   $x_n \leftarrow d \mod 2$
   $d \leftarrow \lfloor d/2 \rfloor$
   $n \leftarrow n + 1$
   return $x_{n-1}x_{n-2}\cdots x_2x_1x_0$

   This works for non-negative n.

2. Which of the following can be considered an algorithm for computing the area of a triangle whose side lengths are given positive numbers a, b, and c?

   (a) $S = \sqrt{p(p-a)(p-b)(p-c)}$, where $p = (a+b+c)/2$

   (b) $S = \frac{1}{2}bc\sin A$, where A is the angle between sides b and c

   (c) $S = \frac{1}{2}ah_a$, where h_a is the height to base a

   Response.

   (a) This is a fine algorithm, as long as the square root operation is a primitive operation available on our computing device (or we know how to calculate square roots).

   (b) This is a problematic formulation, because, even if the sine function is considered a primitive, there is no indication of how to compute the angle A.

   (c) Again, the formula says "the height to base a", without any indication of how to find that. So we can't really call that an algorithm.

3. (a) Show the stack after each operation of the following sequence that starts with the empty stack:

   `push(a), push(b), pop, push(c), push(d), pop`

   Response

   $< a >$
   $< b, a >$
   $< a >$
   $< c, a >$
   $< d, c, a >$
   $< c, a >$

   (b) Show the queue after each operation of the following sequence that starts with the empty queue:

   `enqueue(a), enqueue(b), dequeue, enqueue(c), enqueue(d), dequeue`

   Response

   $< a >$
   $< b, a >$
   $< b >$
   $< c, b >$
   $< d, c, b, >$
   $< d, c >$

4. Consider the following problem: You are to design an algorithm to determine the best route for a subway passenger to take from one station to another in a city such as Kolkata or Tokyo.

   (a) Discuss ways of making the problem statement less vague. In particular, what is "best" supposed to mean?

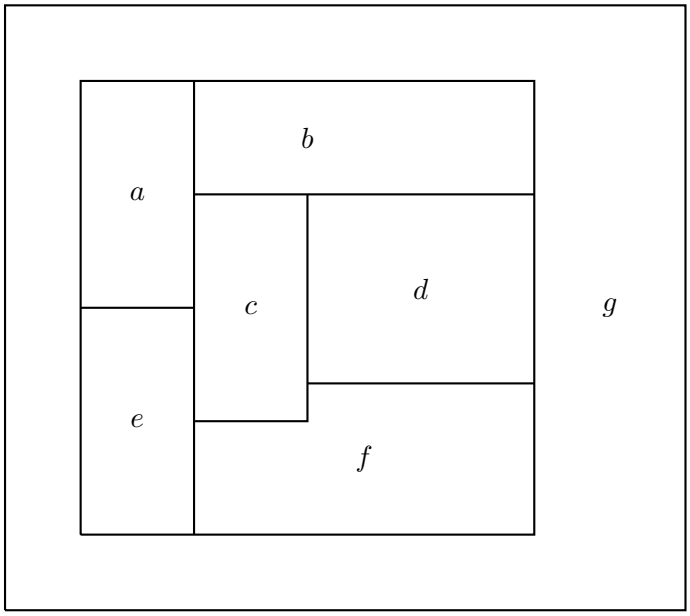
   (b) How would you model this problem by a graph?

   Response.

   (a) In this context, "best" can mean many things. We may want to minimize the travel time, the number of train stops, the number of train changes, or some combination of these.

(b) The natural choice is to let nodes correspond to stations. Then there is an edge between two nodes iff the stations that correspond to the incident nodes are directly connected by a train line. If travel time is important (part of the definition of "best" route) then we need a weighted graph. In this case, we may also need to indicate how long it takes to change train, noting that a station may be on several lines. That information could be kept separately, or as annotations to stations, or we could do what some subway maps do: have several nodes for the same station.

5. Consider the following map:



(a) A cartographer wants to colour the map so that no two neighbouring countries have the same colour. How few colours can she get away with?

(b) Show how to reduce the problem to a graph-colouring problem.

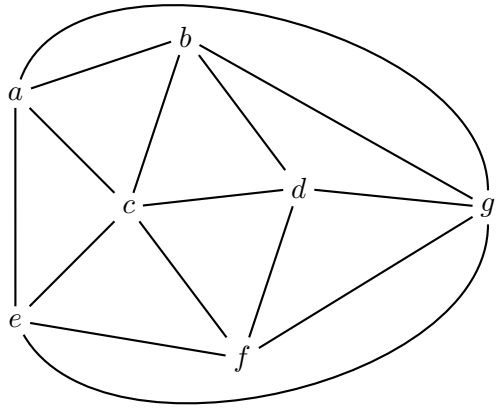
Response.

(a) It turns out that four colours suffice for *any* planar map, no matter how complicated the map. Of course, for some maps fewer colours may be enough. The one given here does require four.

The result about the four colours has an interesting history. It has been conjectured to hold since 1852, and many incorrect proofs of the theorem have been given. Natural approaches to a proof involve extensive case analysis, too many cases to go through by hand. In 1976, Appel and Haken from the University of Illinois at Urbana-Champaign completed a proof by programming a computer to handle most of the tedious case analysis. At the time, there was much debate about the validity of such a proof: If it is too long for anybody to follow by reading, is it really a proof? Who says the program they used worked correctly—surely we also need a proof of *its* correctness. Since then, many independent computer-assisted proofs have been produced, and there is now a general consensus that the so-called four-colour theorem holds.

It is easy to determine whether a map can be coloured with one, two, or four colours (in the last case, the decision procedure can say 'yes' without even looking at its input). However, the case of three colours appears to be hard. Technically it is "NP-complete", a concept we will discuss towards the end of this course.

(b) We can generate an undirected planar graph (*planar* meaning one that has no edges crossing), by placing a node in each of the "countries" a–g and connecting two nodes iff the corresponding countries have a common border. This is a general construction; it works for all maps. Now the question "how many colours are needed for the map?" becomes "how many colours are needed to colour the nodes of the graph, so that no two neighboring nodes have the same colour?" For our example, the graph looks like this:



6. You have to search for a given number n in a *sorted* list of numbers.

   (a) How can you take advantage of knowing that the list is represented as a linked (and sorted) list?

   (b) How can you take advantage of knowing the list is represented as an array?

   Response.

   (a) We can stop searching as soon as we find (n or) a number greater than n.

   (b) With an array we can use binary search.

7. Let A be the adjacency matrix of an undirected graph. How can you tell from the matrix whether the graph

   (a) is complete?

   (b) has a loop, that is, an edge connecting a node to itself?

   (c) has an isolated node?

   Response.

   (a) All elements (except those on the main diagonal) are 1.

   (b) Some element on the main diagonal is 1.

   (c) Some row (and hence some column) has all zeros.

8. Design an algorithm to check whether two given words are anagrams, that is, whether one can be obtained from the other by permuting its letters. For example, *garner* and *ranger* are anagrams.

   Response. Note that we cannot just say "for each letter in the first word, check that it occurs in the second, and vice versa." (Why not?)

   A nice solution sorts the letters in each word and simply compares the two results. This solution scales well, that is, has good performance even as the words involved grow longer.