# COMP90038 Algorithms and Complexity

## More Divide-and-Conquer Algorithms

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# Divide and Conquer

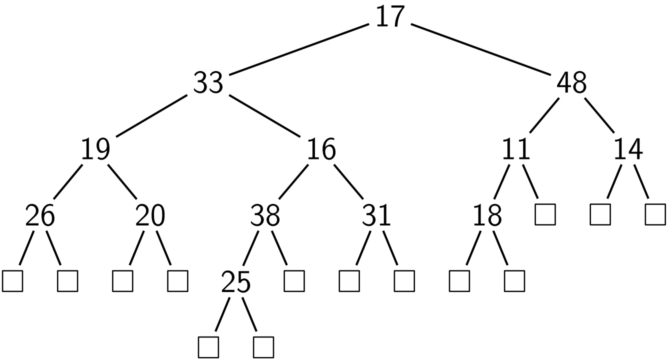
In the last lecture we studied the archetypal divide-and-conquer sorting algorithms: mergesort and quicksort.

We also introduced the powerful master theorem, providing solutions to a large class of recurrence relations, for free.

Now we shall look at tree traversal, and then a final example of divide-and-conquer, giving a better solution to the closest-pair problem.

# Binary Trees Again

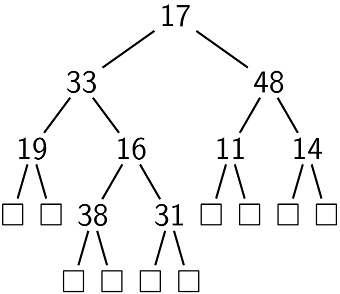
An example of a binary tree, with empty subtrees marked with □:



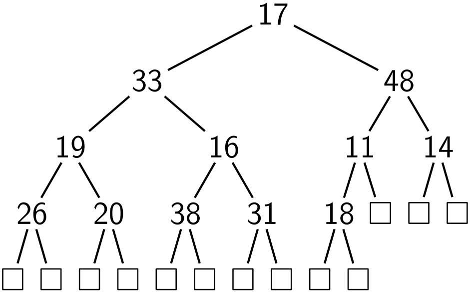

This tree has height 4, the empty tree having height -1.

# Binary Tree Concepts

Special trees have their external nodes □ only at level $h$ and $h+1$ for some $h$:



A full binary tree: Each node has 0 or 2 children.



A complete tree: Each level filled left to right.

# Binary Tree Concepts

A non-empty tree $T$ has a root $T_{root}$, a left subtree $T_{left}$, and a right subtree $T_{right}$.

Recursion is the natural way of calculating the height:

**function** $\text{HEIGHT}(T)$
    **if** $T$ is empty **then**
        **return** $-1$
    **else**
        **return** $max(\text{HEIGHT}(T_{left}), \text{HEIGHT}(T_{right})) + 1$

# Binary Tree Concepts

It is not hard to prove that the number $x$ of external nodes $\Box$ is always one greater than the number $n$ of internal nodes.

The function HEIGHT makes a tree comparison (empty or non-empty?) per node (internal and external), so altogether $2n+1$ comparisons.

# Binary Tree Traversal

Preorder traversal visits the root, then the left subtree, and finally the right subtree.

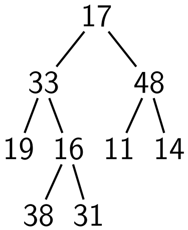
Inorder traversal visits the left subtree, then the root, and finally the right subtree.

Postorder traversal visits the left subtree, the right subtree, and finally the root.

Level-order traversal visits the nodes, level by level, starting from the root.

# Binary Tree Traversal: Preorder

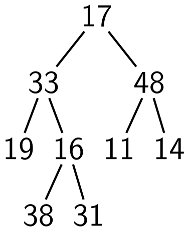
**function** PREORDERTRAVERSE($T$)
    **if** $T$ is non-empty **then**
        visit $T_{root}$
        PREORDERTRAVERSE($T_{left}$)
        PREORDERTRAVERSE($T_{right}$)




Visit order for the example: 17, 33, 19, 16, 38, 31, 48, 11, 14.

# Binary Tree Traversal: Inorder

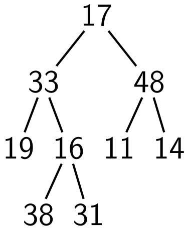
**function** InorderTraverse($T$)
  **if** $T$ is non-empty **then**
    InorderTraverse($T_{left}$)
    visit $T_{root}$
    InorderTraverse($T_{right}$)




Visit order for the example: 19, 33, 38, 16, 31, 17, 11, 48, 14.

# Binary Tree Traversal: Postorder

**function** POSTORDERTRAVERSE($T$)
  **if** $T$ is non-empty **then**
    POSTORDERTRAVERSE($T_{left}$)
    POSTORDERTRAVERSE($T_{right}$)
    visit $T_{root}$




Visit order for the example: 19, 38, 31, 16, 33, 11, 14, 48, 17.

# Preorder Traversal Using a Stack

We could also implement preorder traversal of $T$ by maintaining a stack explicitly.

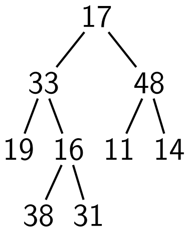
$push(T)$
**while** the stack is non-empty **do**
  $T \leftarrow pop$
  visit $T_{root}$
  **if** $T_{right}$ is non-empty **then**
    $push(T_{right})$
  **if** $T_{left}$ is non-empty **then**
    $push(T_{left})$

In an implementation, the elements placed on the stack would not be whole trees, but pointers to the corresponding internal nodes.

# Tree Traversal Using a Queue: Level-Order

Level-order traversal results if we replace the stack with a queue.

*inject*$(T)$
**while** the queue is non-empty **do**
  $T \leftarrow$ *eject*
  visit $T_{root}$
  **if** $T_{left}$ is non-empty **then**
    *inject*$(T_{left})$
  **if** $T_{right}$ is non-empty **then**
    *inject*$(T_{right})$




Visit order for the example: 17, 33, 48, 19, 16, 11, 14, 38, 31.

# The Closest Pair Problem Revisited

In Lecture 5 we gave a brute-force algorithm for the closest pair problem: Given $n$ points in the Cartesian plane, find a pair with minimal distance.

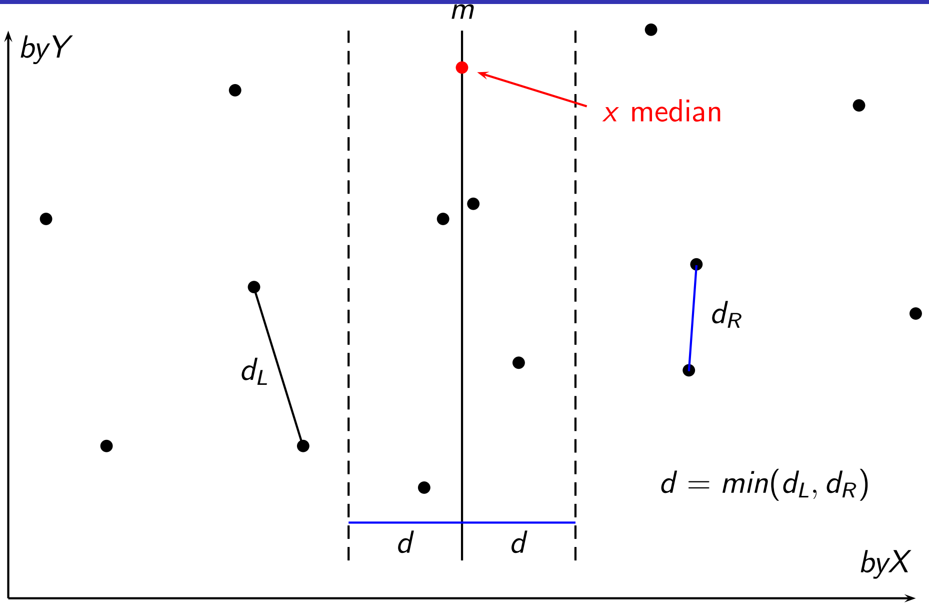
The brute-force method had complexity $\Theta(n^2)$. We can use divide-and-conquer to do better, namely $\Theta(n \log n)$.

First, sort the points by $x$ value and store the result in array $byX$.

Also sort the points by $y$ value and store the result in array $byY$.

Now we can identify the $x$ median, and recursively process the set $P_L$ of points with lower $x$ values, as well as the set $P_R$ with higher $x$ values.

# The Closest Pair Problem Revisited



# The Closest Pair Problem Revisited

The recursive calls will identify $d_L$, the shortest distance for pairs in $P_L$, and $d_R$, the shortest distance for pairs in $P_R$.

Let $m$ be the $x$ median and let $d = min(d_L, d_R)$. This $d$ is a candidate for the smallest distance.

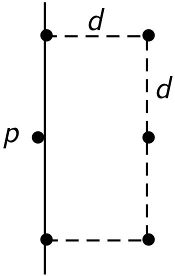
But $d$ may not be the global minimum—there could be some close pair whose points are on opposite sides of the median line $x = m$.

For candidates that may improve on $d$ we only need to look at those in the band $m - d \leq x \leq m + d$.

So pick out, from array $byY$, each point $p$ with $x$-coordinate between $m - d$ and $m + d$, and keep these in array $S$.

For each point in $S$, consider just its "close" neighbours.

# The Closest Pair Problem Revisited




The following calculates the smallest distance and leaves the (square of the) result in *minsq*.

It can be shown that the while loop can execute at most 5 times for each $i$ value—see diagram.

$minsq \leftarrow d^2$
copy all points of $byY$ with $|x - m| < d$ to array $S$
$k \leftarrow |S|$
**for** $i \leftarrow 0$ to $k - 2$ **do**
  $j \leftarrow i + 1$
  **while** $j \leq k - 1$ and $(S[j].y - S[i].y)^2 < minsq$ **do**
    $minsq \leftarrow min(minsq, (S[j].x - S[i].x)^2 + (S[j].y - S[i].y)^2)$
    $j \leftarrow j + 1$