笔记前言：

本笔记的内容是去掉步骤的概述后，视频的所有内容。

本猴觉得，自己的步骤概述写的太啰嗦，大家自己做笔记时，

应该每个人都有自己的最舒服最简练的写法，所以没给大家写。

再是本猴觉得，不给大家写这个概述的话，大家会记忆的更深，

掌握的更好！

所以老铁！一定要过呀！不要辜负本猴的心意！~~~

【祝逢考必过，心想事成~~~~】

【一定能过！！！！！】

计算二次积分

例1. 计算 $\int_1^2 dx \int_1^x xy\,dy$

$$\int_{\cdots}^{\cdots} A\,du \int_{\cdots}^{\cdots} B\,dv = \int_{\cdots}^{\cdots}\left(\int_{\cdots}^{\cdots} AB\,dv\right)du$$

$$\begin{aligned}\int_1^2 1\,dx \int_1^x xy\,dy &= \int_1^2\left(\int_1^x 1\cdot xy\,dy\right)dx\\ &= \int_1^2\left(\boxed{\int_1^x xy\,dy}\right)dx\\ &= \int_1^2\left(\frac{x^3}{2}-\frac{x}{2}\right)dx\\ &= \left(\frac{x^4}{8}-\frac{x^2}{4}\right)\Bigg|_{x=1}^{x=2}\\ &= \left(\frac{2^4}{8}-\frac{2^2}{4}\right)-\left(\frac{1^4}{8}-\frac{1^2}{4}\right)\\ &= \frac{9}{8}\end{aligned}$$

$$\begin{aligned}\int_1^x xy\,dy &= \left(\frac{x}{2}\cdot y^2\right)\Bigg|_{y=1}^{y=x}\\ &= \frac{x}{2}\cdot x^2-\frac{x}{2}\cdot 1^2\\ &= \frac{x^3}{2}-\frac{x}{2}\end{aligned}$$

例2. 设 $f(x)$ 为连续函数，$F(t)=\int_1^t dy\int_y^t f(x)\,dx$，则 $F'(2)=$ <u>B</u>

$$\int_{\cdots}^{\cdots} A\,du \int_{\cdots}^{\cdots} B\,dv = \int_{\cdots}^{\cdots}\left(\int_{\cdots}^{\cdots} AB\,dv\right)du$$

A. $2f(2)$　　B. $f(2)$　　C. $-f(2)$　　D. 0

$F(t)=\int_1^t 1\,dy\int_y^t f(x)\,dx=\int_1^t\left(\int_y^t 1\cdot f(x)\,dx\right)dy=\int_1^t\left(\int_y^t f(x)\,dx\right)dy$

设 $a'(x)=f(x)$，则：

$$\begin{aligned}\int_y^t f(x)\,dx &= a(x)|_{x=y}^{x=t}\\ &= a(t)-a(y)\end{aligned}$$

$$\begin{aligned}F(t) &= \int_1^t[a(t)-a(y)]\,dy\\ &= \int_1^t a(t)\,dy-\int_1^t a(y)\,dy\\ &= a(t)\int_1^t dy-\int_1^t a(y)\,dy\\ &= a(t)\cdot y|_{y=1}^{y=t}-\int_1^t a(y)\,dy\\ &= a(t)\cdot(t-1)-\int_1^t a(y)\,dy\\ &= a(t)\cdot(t-1)-b(y)|_{y=1}^{y=t}\\ &= a(t)\cdot(t-1)-[b(t)-b(1)]\\ &= a(t)\cdot t-b(t)-a(t)+b(1)\end{aligned}$$

设 $b'(y)=a(y)$

$$\begin{aligned}F'(t) &= [a(t)\cdot t-b(t)-a(t)+b(1)]'\\ &= [a(t)\cdot t]'-[b(t)]'-[a(t)]'+0\\ &= a'(t)\cdot t+a(t)-b'(t)-a'(t)\\ &= f(t)\cdot t+a(t)-b'(t)-f(t)\\ &= f(t)\cdot t+a(t)-a(t)-f(t)\\ &= f(t)\cdot t-f(t)\end{aligned}$$

将 $t=2$ 代入 $F'(t)=f(t)\cdot t-f(t)$，则：

$$\begin{aligned}F'(2) &= f(2)\cdot 2-f(2)\\ &= f(2)\end{aligned}$$

$$\int_{\cdots}^{\cdots} A\,du \int_{\cdots}^{\cdots} B\,dv = \int_{\cdots}^{\cdots}\left(\int_{\cdots}^{\cdots} AB\,dv\right)du$$

例3. 设函数 f(x) 在区间 [0,1] 上连续，且 $\int_0^1 f(x)\,dx=A$，求 $\int_0^1 dx \int_x^1 f(x)f(y)\,dy$

$$\int_0^1 1\,dx \int_x^1 f(x)f(y)\,dy = \int_0^1 \left[\int_x^1 1\cdot f(x)f(y)\,dy\right]dx = \int_0^1 \left[\int_x^1 f(x)f(y)\,dy\right]dx$$

$$\begin{aligned}\int_x^1 f(x)f(y)\,dy &= f(x)\int_x^1 f(y)\,dy \qquad \boxed{\text{设 } F'(y)=f(y)}\\ &= f(x)\left[F(y)\big|_{y=x}^{y=1}\right]\\ &= f(x)[F(1)-F(x)]\\ &= f(x)F(1)-f(x)F(x)\end{aligned}$$

$$\begin{aligned}\therefore \text{原式} &= \int_0^1 [f(x)F(1)-f(x)F(x)]\,dx\\ &= \int_0^1 f(x)F(1)\,dx - \int_0^1 f(x)F(x)\,dx\\ &= F(1)\cdot\int_0^1 f(x)\,dx - \int_0^1 f(x)F(x)\,dx\\ &= F(1)\cdot A - \int_0^1 f(x)F(x)\,dx\\ &= F(1)\cdot A - \boxed{\int_0^1 F'(x)F(x)\,dx}\\ &= F(1)\cdot A - \frac{1}{2}\cdot[F(x)\cdot F(x)]\Big|_0^1\\ &= F(1)\cdot A - \left[\frac{F^2(1)-F^2(0)}{2}\right]\\ &= F(1)\cdot A - \left[\frac{[F(1)-F(0)]\cdot[F(1)+F(0)]}{2}\right]\\ &= F(1)\cdot A - \left[\frac{A\cdot[F(1)+F(0)]}{2}\right]\\ &= A\cdot\left[F(1)-\frac{F(1)+F(0)}{2}\right]\\ &= A\cdot\frac{\boxed{F(1)-F(0)}}{2}\\ &= A\cdot\frac{A}{2}\\ &= \frac{A^2}{2}\end{aligned}$$

$$\int_b^a U'\cdot U\,dx = (U\cdot U)\,\big|_b^a - \int_b^a U\cdot U'\,dx$$

$$2\cdot\int_b^a U'\cdot U\,dx = (U\cdot U)\,\big|_b^a$$

$$\int_b^a U'\cdot U\,dx = \frac{1}{2}\cdot(U\cdot U)\,\big|_b^a$$

当 $F'(x)=f(x)$ 时

$$\begin{aligned}\int_0^1 f(x)\,dx &= F(x)\big|_0^1\\ &= F(1)-F(0)\end{aligned}$$

$\because \int_0^1 f(x)\,dx = A$

$\therefore F(1)-F(0)=A$

常用的公式：

$$\int_a^{a+b} du \int_e^f f(u,v)\,dv + \int_{a+b}^{a+b+c} du \int_e^f f(u,v)\,dv + \int_{a+b+c}^{a+b+c+d} du \int_e^f f(u,v)\,dv = \int_a^{a+b+c+d} du \int_e^f f(u,v)\,dv$$

$$\int_e^f du \int_a^{a+b} f(u,v)\,dv + \int_e^f du \int_{a+b}^{a+b+c} f(u,v)\,dv + \int_e^f du \int_{a+b+c}^{a+b+c+d} f(u,v)\,dv = \int_e^f du \int_a^{a+b+c+d} f(u,v)\,dv$$

求二重积分

例1．计算$\iint\limits_{D} xy\,d\sigma$，其中 D 是由直线 y=1，x=2，y=x 所围成的区域

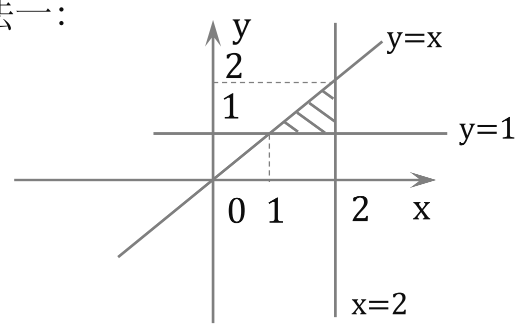
方法一公式：

$$\iint\limits_{D} f(x,y)\,d\sigma = \int_a^b dx \int_{y_{下}(x)}^{y_{上}(x)} f(x,y)\,dy$$

方法二公式：

$$\iint\limits_{D} f(x,y)\,d\sigma = \int_c^d dy \int_{x_{左}(y)}^{x_{右}(y)} f(x,y)\,dx$$

方法一:

① 

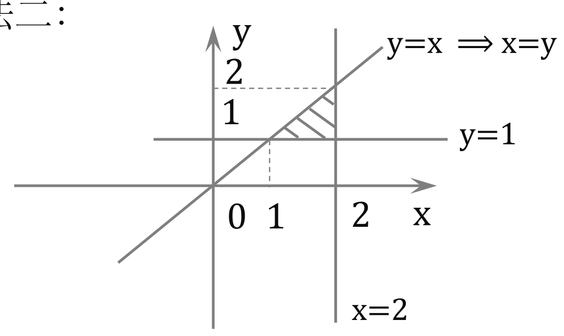
② 确定区域内 x 的取值范围 [1,2]

③ 确定区域的下边界 $y = 1$与上边界 $y = x$

④ $\iint\limits_{D} f(x,y)\,d\sigma = \int_a^b dx \int_{y_{下}(x)}^{y_{上}(x)} f(x,y)\,dy$

$$\begin{aligned}\iint\limits_{D} xy\,d\sigma &= \int_1^2 dx \int_1^x xy\,dy \\ &= \int_1^2 \left(\int_1^x xy\,dy\right) dx \\ &= \int_1^2 \left(\frac{x^3}{2} - \frac{x}{2}\right) dx \\ &= \left(\frac{x^4}{8} - \frac{x^2}{4}\right)\Big|_{x=1}^{x=2} \\ &= \left(\frac{2^4}{8} - \frac{2^2}{4}\right) - \left(\frac{1^4}{8} - \frac{1^2}{4}\right) \\ &= \frac{9}{8}\end{aligned}$$

方法二:

① 

② 确定区域内 y 的取值范围 [1,2]

③ 确定区域的左边界 $x = y$ 与右边界 $x = 2$

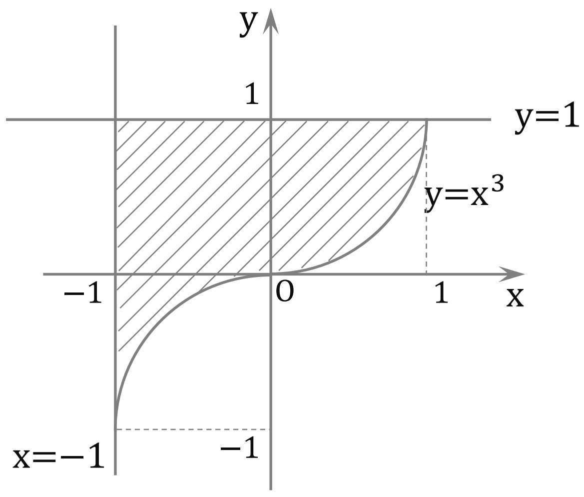
④ $\iint\limits_{D} f(x,y)\,d\sigma = \int_c^d dy \int_{x_{左}(y)}^{x_{右}(y)} f(x,y)\,dx$

$$\begin{aligned}\iint\limits_{D} xy\,d\sigma = \int_1^2 dy \int_y^2 xy\,dx &= \int_1^2 \left(\int_y^2 xy\,dx\right) dy \\ &= \int_1^2 \left(\frac{1}{2}yx^2\Big|_y^2\right) dy \\ &= \int_1^2 \left[\frac{1}{2}y\cdot 2^2 - \frac{1}{2}y\cdot y^2\right] dy \\ &= \int_1^2 (2y - \frac{1}{2}y^3)dy \\ &= \frac{1}{2}\int_1^2 (4y - y^3)dy \\ &= \frac{1}{2}\left[\left(2y^2 - \frac{1}{4}y^4\right)\Big|_1^2\right] \\ &= \frac{1}{2}(2\cdot 2^2 - \frac{1}{4}\cdot 2^4 - 2\cdot 1^2 + \frac{1}{4}\cdot 1^4) \\ &= \frac{1}{2}(8 - 4 - 2 + \frac{1}{4}) \\ &= \frac{9}{8}\end{aligned}$$

例2．计算$\iint\limits_D x\,dxdy$，其中 D 是由 $y=x^3$、$y=1$、$x=-1$ 所围成的区域

方法一:

①



② 确定区域内 x 的取值范围 $[-1,1]$

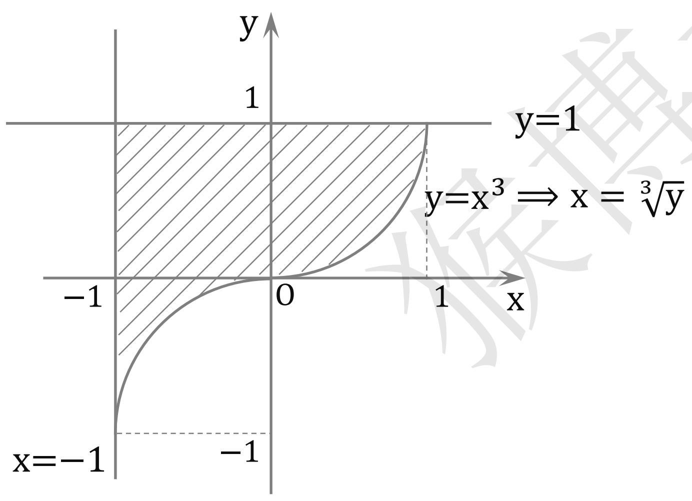
③ 确定区域的下边界 $y=x^3$与上边界 $y=1$

④ $\iint\limits_D f(x,y)\,d\sigma=\int_a^b dx\int_{y_{下}(x)}^{y_{上}(x)} f(x,y)\,dy$

$$\begin{aligned}\iint\limits_D x\,d\sigma=\int_{-1}^1 dx\int_{x^3}^1 x\,dy&=\int_{-1}^1\left(\int_{x^3}^1 x\,dy\right)dx\\&=\int_{-1}^1\left(xy\Big|_{y=x^3}^{y=1}\right)dx\\&=\int_{-1}^1(x\cdot 1-x\cdot x^3)\,dx\\&=\int_{-1}^1(x-x^4)\,dx\\&=\left(\frac{x^2}{2}-\frac{x^5}{5}\right)\Big|_{-1}^1\\&=\frac{1^2}{2}-\frac{1^5}{5}-\left[\frac{(-1)^2}{2}-\frac{(-1)^5}{5}\right]\\&=\frac{1}{2}-\frac{1}{5}-\frac{1}{2}-\frac{1}{5}\\&=-\frac{2}{5}\end{aligned}$$

方法二:

①



② 确定区域内 y 的取值范围 $[-1,1]$

③ 确定区域的左边界 $x=-1$与右边界 $x=\sqrt[3]{y}$

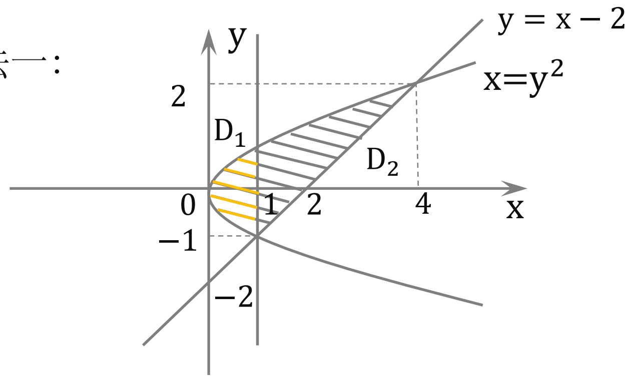
④ $\iint\limits_D f(x,y)\,d\sigma=\int_c^d dy\int_{x_{左}(y)}^{x_{右}(y)} f(x,y)\,dx$

$$\begin{aligned}\iint\limits_D x\,d\sigma=\int_{-1}^1 dy\int_{-1}^{\sqrt[3]{y}} x\,dx&=\int_{-1}^1\left(\int_{-1}^{\sqrt[3]{y}} x\,dx\right)dy\\&=\int_{-1}^1\left(\frac{x^2}{2}\Big|_{-1}^{\sqrt[3]{y}}\right)dy\\&=\int_{-1}^1\left[\frac{(\sqrt[3]{y})^2}{2}-\frac{(-1)^2}{2}\right]dy\\&=\int_{-1}^1\left(\frac{1}{2}y^{\frac{2}{3}}-\frac{1}{2}\right)dy\\&=\left(\frac{3}{10}y^{\frac{5}{3}}-\frac{1}{2}y\right)\Big|_{-1}^1\\&=\frac{3}{10}\cdot 1^{\frac{5}{3}}-\frac{1}{2}\cdot 1-\left[\frac{3}{10}\cdot(-1)^{\frac{5}{3}}-\frac{1}{2}\cdot(-1)\right]\\&=\frac{3}{10}-\frac{1}{2}+\frac{3}{10}-\frac{1}{2}\\&=-\frac{2}{5}\end{aligned}$$

例3．计算$\iint\limits_D xy\,d\sigma$，其中 D 是由 $x=y^2$ 与 $y=x-2$ 所围成的区域

方法一：

①



D_1：② 确定区域内 x 的取值范围 [0,1]

③ 确定区域的下边界 $y=-\sqrt{x}$与上边界 $y=\sqrt{x}$

④ $\iint\limits_D f(x,y)\,d\sigma=\int_a^b dx\int_{y_{下}(x)}^{y_{上}(x)} f(x,y)\,dy$

$$\iint\limits_{D_1} xy\,d\sigma=\int_0^1 dx\int_{-\sqrt{x}}^{\sqrt{x}} xy\,dy=\int_0^1\left(\int_{-\sqrt{x}}^{\sqrt{x}} xy\,dy\right)dx$$

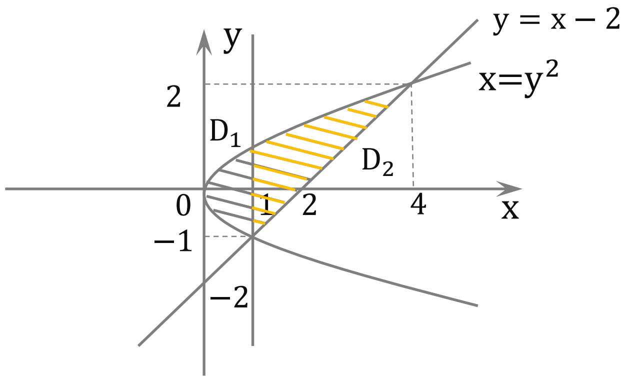
$$=\int_0^1\left(\frac{1}{2}xy^2\Big|_{-\sqrt{x}}^{\sqrt{x}}\right)dx$$

$$=\int_0^1\left[\frac{1}{2}x(\sqrt{x})^2-\frac{1}{2}x(-\sqrt{x})^2\right]dx$$

$$=\int_0^1 0\,dx$$

$$=0$$

①



D_2：② 确定区域内 x 的取值范围 [1,4]

③ 确定区域的下边界 $y=x-2$ 与上边界 $y=\sqrt{x}$

④ $\iint\limits_D f(x,y)\,d\sigma=\int_a^b dx\int_{y_{下}(x)}^{y_{上}(x)} f(x,y)\,dy$

$$\iint\limits_{D_2} xy\,d\sigma=\int_1^4 dx\int_{x-2}^{\sqrt{x}} xy\,dy$$

$$=\int_1^4\left(\int_{x-2}^{\sqrt{x}} xy\,dy\right)dx$$

$$=\int_1^4\left(\frac{1}{2}xy^2\Big|_{x-2}^{\sqrt{x}}\right)dx$$

$$=\int_1^4\left[\frac{1}{2}x(\sqrt{x})^2-\frac{1}{2}x(x-2)^2\right]dx$$

$$=\int_1^4\left(\frac{1}{2}x^2-\frac{1}{2}x^3+2x^2-2x\right)dx$$

$$=\int_1^4\left(-\frac{1}{2}x^3+\frac{5}{2}x^2-2x\right)dx$$

$$=\left(-\frac{1}{8}x^4+\frac{5}{6}x^3-x^2\right)\Big|_1^4$$

$$=-\frac{1}{8}\cdot 4^4+\frac{5}{6}\cdot 4^3-4^2-\left(-\frac{1}{8}\cdot 1^4+\frac{5}{6}\cdot 1^3-1^2\right)$$

$$=-\frac{1}{8}\cdot 256+\frac{5}{6}\cdot 64-16-\left(-\frac{1}{8}+\frac{5}{6}-1\right)$$

$$=-32+\frac{160}{3}-16+\frac{1}{8}-\frac{5}{6}+1\,)$$

$$=\frac{19}{3}+\frac{1}{8}-\frac{5}{6}$$

$$=\frac{155}{24}-\frac{20}{24}$$

$$=\frac{45}{8}$$

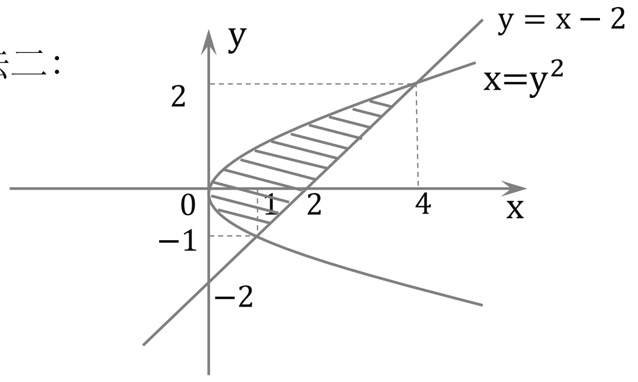
$$\iint\limits_D xy\,d\sigma=\iint\limits_{D_1} xy\,d\sigma+\iint\limits_{D_2} xy\,d\sigma$$

$$=0+\frac{45}{8}$$

$$=\frac{45}{8}$$

例3．计算$\iint\limits_D xy\,d\sigma$，其中D是由$x=y^2$与$y=x-2$所围成的区域

方法二：

① 

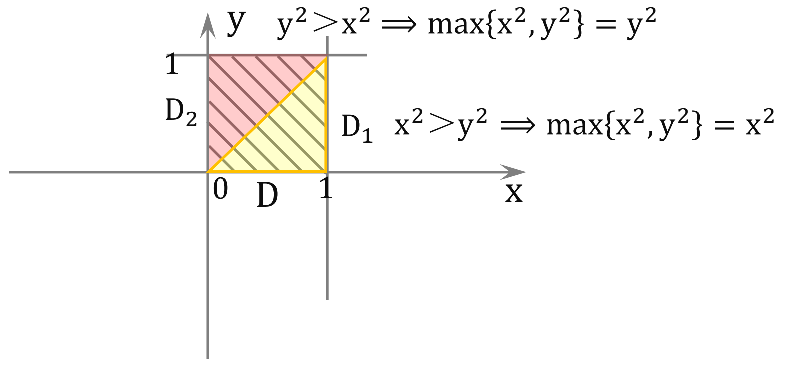
② 确定区域内y的取值范围$[-1,2]$

③ 确定区域的左边界$x=y^2$与右边界$x=y+2$

④ $\iint\limits_D f(x,y)\,d\sigma=\int_c^d dy\int_{x_{左}(y)}^{x_{右}(y)} f(x,y)\,dx$

$$
\begin{aligned}
\iint\limits_D xy\,d\sigma=\int_{-1}^{2}dy\int_{y^2}^{y+2}xy\,dx &=\int_{-1}^{2}\left(\int_{y^2}^{y+2}xy\,dx\right)dy\\
&=\int_{-1}^{2}\left(\frac{1}{2}yx^2\Big|_{x=y^2}^{x=y+2}\right)dy\\
&=\int_{-1}^{2}\left[\frac{1}{2}y(y+2)^2-\frac{1}{2}y(y^2)^2\right]dy\\
&=\int_{-1}^{2}\left(\frac{1}{2}y^3+2y^2+2y-\frac{1}{2}y^5\right)dy\\
&=\left(\frac{1}{8}y^4+\frac{2}{3}y^3+y^2-\frac{1}{12}y^6\right)\Big|_{-1}^{2}\\
&=\frac{1}{8}\cdot 2^4+\frac{2}{3}\cdot 2^3+2^2-\frac{1}{12}\cdot 2^6-\left[\frac{1}{8}\cdot(-1)^4+\frac{2}{3}\cdot(-1)^3+(-1)^2-\frac{1}{12}\cdot(-1)^6\right]\\
&=2+\frac{16}{3}+4-\frac{16}{3}-\frac{1}{8}+\frac{2}{3}-1+\frac{1}{12}\\
&=5-\frac{1}{8}+\frac{2}{3}+\frac{1}{12}\\
&=\frac{45}{8}
\end{aligned}
$$

例4．计算二重积分$\iint\limits_{D} e^{\max\{x^2,y^2\}}\,d\sigma$，其中$D=\{(x,y)|0\le x\le 1,0\le y\le 1\}$

① 

D_1：② 确定区域内 x 的取值范围 [0,1]

③ 确定区域的下边界 $y=0$与上边界 $y=x$

④ $\iint\limits_{D} f(x,y)\,d\sigma=\int_a^b dx\int_{y_{下}(x)}^{y_{上}(x)} f(x,y)\,dy$

$$\iint\limits_{D_1} e^{\max\{x^2,y^2\}}\,d\sigma=\int_0^1 dx\int_0^x e^{\max\{x^2,y^2\}}\,dy=\int_0^1 dx\int_0^x e^{x^2}\,dy$$
$$=\int_0^1\left(\int_0^x e^{x^2}\,dy\right)dx$$
$$=\int_0^1\left(e^{x^2}y\Big|_0^x\right)dx$$
$$=\int_0^1 e^{x^2}x\,dx$$
$$=\frac{e^{x^2}}{2}\Big|_0^1$$
$$=\frac{e^{1^2}}{2}-\frac{e^{0^2}}{2}$$
$$=\frac{e-1}{2}$$

D_2：② 确定区域内 y 的取值范围 [0,1]

③ 确定区域的左边界 $x=0$与右边界 $x=y$

④ $\iint\limits_{D} f(x,y)\,d\sigma=\int_c^d dy\int_{x_{左}(y)}^{x_{右}(y)} f(x,y)\,dx$

$$\iint\limits_{D_2} e^{\max\{x^2,y^2\}}\,d\sigma=\int_0^1 dy\int_0^y e^{\max\{x^2,y^2\}}\,dx=\int_0^1 dy\int_0^y e^{y^2}\,dx$$
$$=\int_0^1\left(\int_0^y e^{y^2}\,dx\right)dy$$
$$=\int_0^1\left(e^{y^2}x\Big|_0^y\right)dy$$
$$=\int_0^1 e^{y^2}y\,dy$$
$$=\frac{e^{y^2}}{2}\Big|_0^1$$
$$=\frac{e^{1^2}}{2}-\frac{e^{0^2}}{2}$$
$$=\frac{e-1}{2}$$

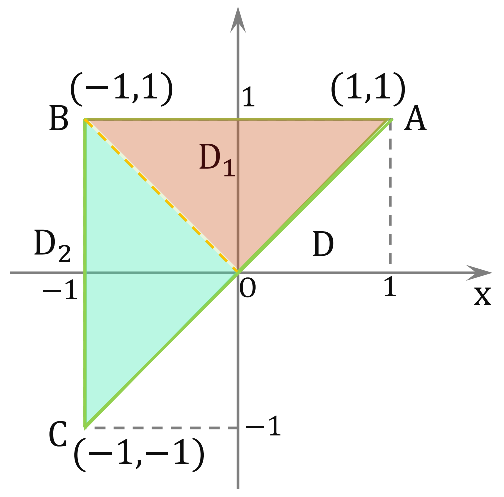
$$\iint\limits_{D} e^{\max\{x^2,y^2\}}\,d\sigma=\iint\limits_{D_1} e^{\max\{x^2,y^2\}}\,d\sigma+\iint\limits_{D_2} e^{\max\{x^2,y^2\}}\,d\sigma$$
$$=\frac{e-1}{2}+\frac{e-1}{2}$$
$$=e-1$$

补充：

① 当 $D=D_1+D_2+D_3+\cdots+D_n$ 时

$$\iint_D f(x,y)\,d\sigma=\iint_{D_1} f(x,y)\,d\sigma+\iint_{D_2} f(x,y)\,d\sigma+\iint_{D_3} f(x,y)\,d\sigma+\cdots+\iint_{D_n} f(x,y)\,d\sigma$$

② $\iint_D [\,f_1(x,y)+f_2(x,y)+f_3(x,y)+\cdots+f_n(x,y)\,]d\sigma=\iint_D f_1(x,y)\,d\sigma+\iint_D f_2(x,y)\,d\sigma+\iint_D f_3(x,y)\,d\sigma+\cdots+\iint_D f_n(x,y)\,d\sigma$



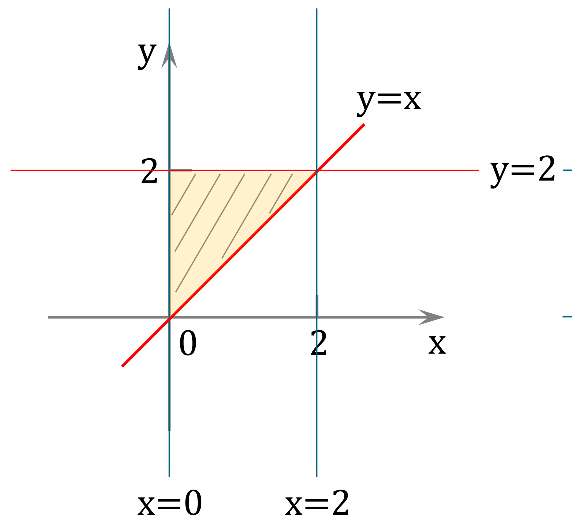
$$\iint_D (xy+\cos x\sin y)\,dxdy=\iint_{D_1}(xy+\cos x\sin y)\,dxdy+\iint_{D_2}(xy+\cos x\sin y)\,dxdy$$

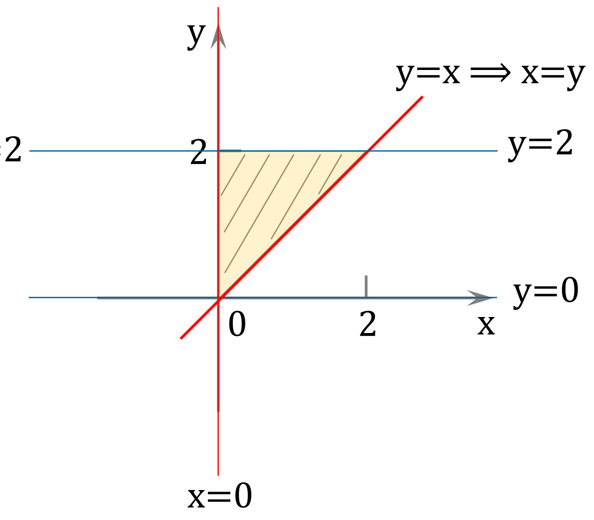
$D=D_1+D_2$

$$=\iint_{D_1} xy\,dxdy+\iint_{D_1}\cos x\sin y\,dxdy+\iint_{D_2} xy\,dxdy+\iint_{D_2}\cos x\sin y\,dxdy$$

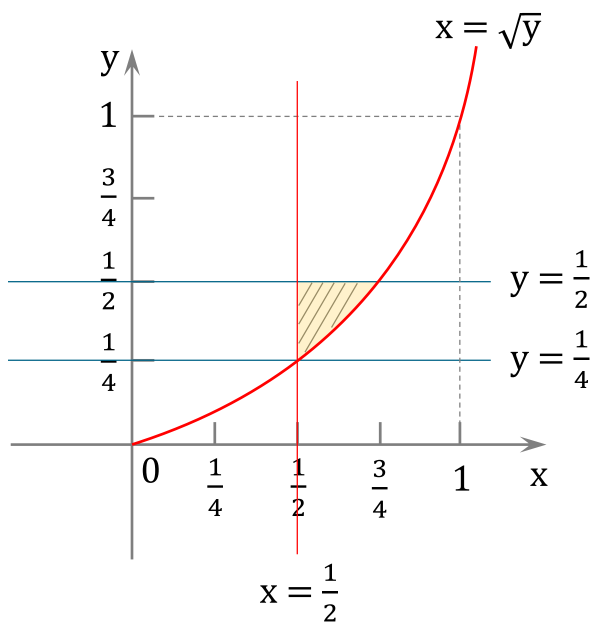
交换二次积分的积分次序

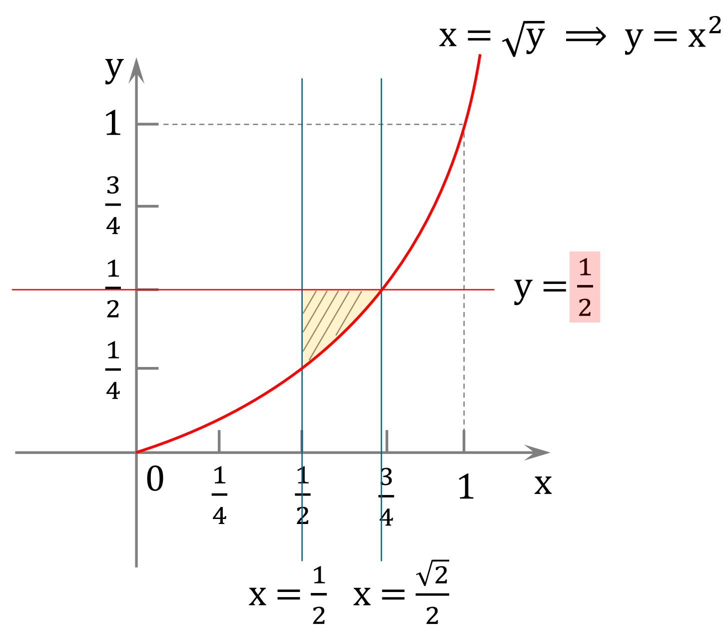
例1. 交换二次积分的积分次序：$\int_0^2 dx \int_x^2 e^{-y^2} dy = \underline{\int_0^2 dy\ \int_0^y e^{-y^2} dx}$。





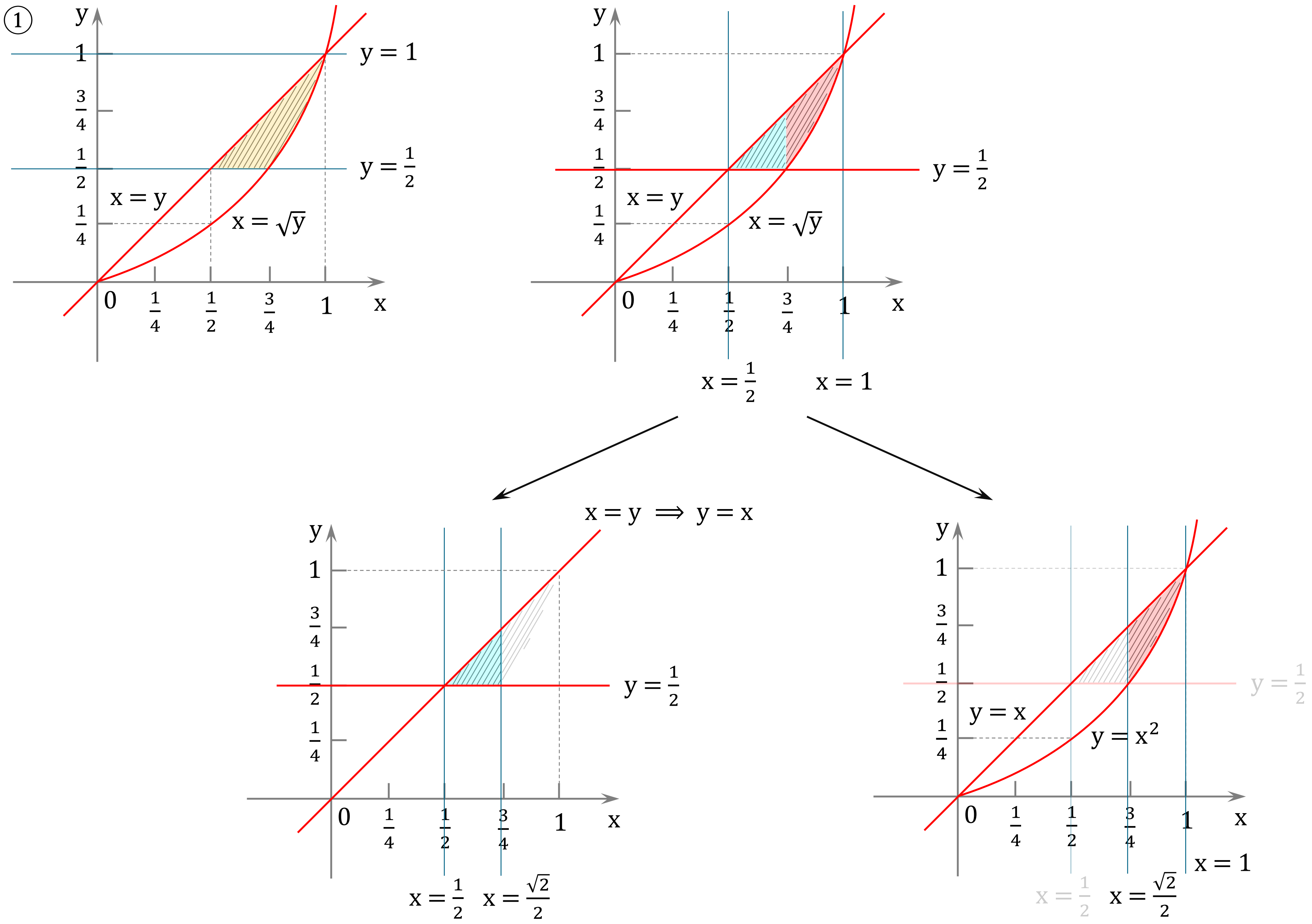
例2. 交换二次积分的积分次序：$\int_{\frac{1}{4}}^{\frac{1}{2}} dy \int_{\frac{1}{2}}^{\sqrt{y}} e^{\frac{y}{x}} dx = \underline{\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} dx\ \int_{x^2}^{\frac{1}{2}} e^{\frac{y}{x}} dy}$。





例3. 交换二次积分的积分次序：$\int_{\frac{1}{2}}^{1} dy \int_{y}^{\sqrt{y}} e^{\frac{y}{x}} dx$

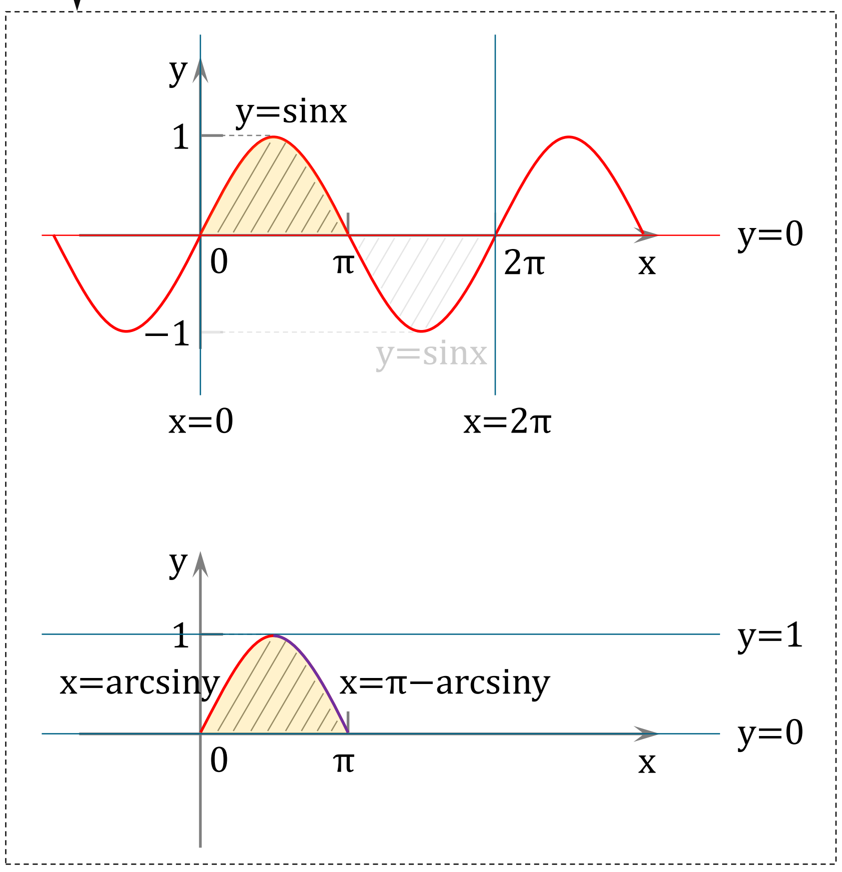
$$= \underline{\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} dx \int_{\frac{1}{2}}^{x} e^{\frac{y}{x}} dy + \int_{\frac{\sqrt{2}}{2}}^{1} dx \int_{x^2}^{x} e^{\frac{y}{x}} dy}\text{。}$$

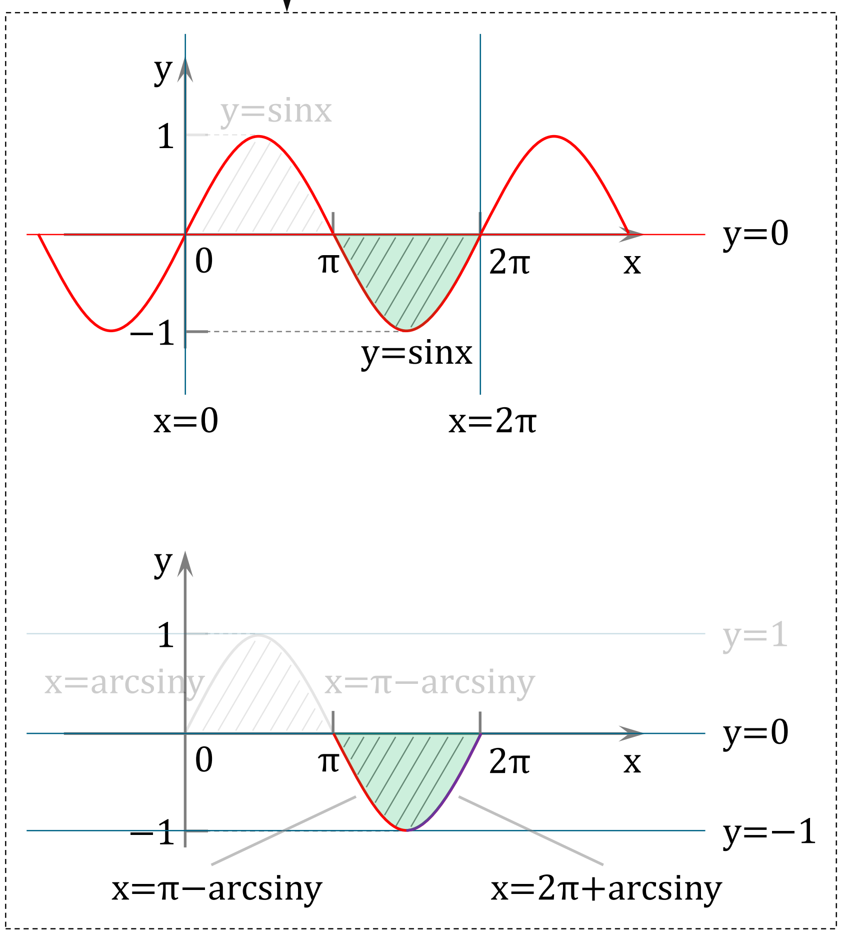


例4. 交换$\int_0^{2\pi} dx \int_0^{\sin x} f(x,y)\, dy$的积分次序，其中$f(x,y)$为连续函数。

$$原式 = \boxed{\int_0^1 dy \int_{\arcsin y}^{\pi-\arcsin y} f(x,y)\, dx} + (-1)\cdot \boxed{\int_{-1}^0 dy \int_{\pi-\arcsin y}^{2\pi+\arcsin y} f(x,y)\, dx}$$

$$= \int_0^1 dy \int_{\arcsin y}^{\pi-\arcsin y} f(x,y)\, dx - \int_{-1}^0 dy \int_{\pi-\arcsin y}^{2\pi+\arcsin y} f(x,y)\, dx$$





通过交换二次积分的积分次序来计算积分

例1. 计算积分$\int_0^2 dx \int_x^2 e^{-y^2}\, dy$

$$\int_0^2 dx \int_x^2 e^{-y^2}\, dy = \int_0^2 dy \int_0^y e^{-y^2}\, dx \longleftarrow \text{(上一课例1讲过)}$$

$$= \int_0^2 \left(\int_0^y e^{-y^2}\, dx\right) dy$$

$$= \int_0^2 \left(e^{-y^2} \cdot \int_0^y 1\, dx\right) dy$$

$$= \int_0^2 \left(e^{-y^2} \cdot x\Big|_{x=0}^{x=y}\right) dy$$

$$= \int_0^2 \left[e^{-y^2} \cdot (y-0)\right] dy$$

$$= \int_0^2 (e^{-y^2} \cdot y)\, dy$$

$$= \left(-\frac{1}{2} e^{-y^2}\right)\Big|_0^2$$

$$= \left(-\frac{1}{2} \cdot e^{-2^2}\right) - \left(-\frac{1}{2} \cdot e^{-0^2}\right)$$

$$= \left(-\frac{1}{2} \cdot e^{-4}\right) - \left(-\frac{1}{2} \cdot 1\right)$$

$$= \frac{1-e^{-4}}{2}$$

例2. 计算积分 $\int_0^1 dx \int_x^1 \sin(\pi u^2)\, du$

$$\text{原式} = \int_0^1 du \int_0^u \sin(\pi u^2)\, dx \longrightarrow$$

$$= \int_0^1 \left[\int_0^u \sin(\pi u^2)\, dx\right] du$$

$$= \int_0^1 \left[\sin(\pi u^2) \cdot \int_0^u 1\, dx\right] du$$

$$= \int_0^1 \left[\sin(\pi u^2) \cdot x\Big|_{x=0}^{x=u}\right] du$$

$$= \int_0^1 [\sin(\pi u^2) \cdot (u-0)]\, du$$

$$= \int_0^1 [\sin(\pi u^2) \cdot u]\, du$$

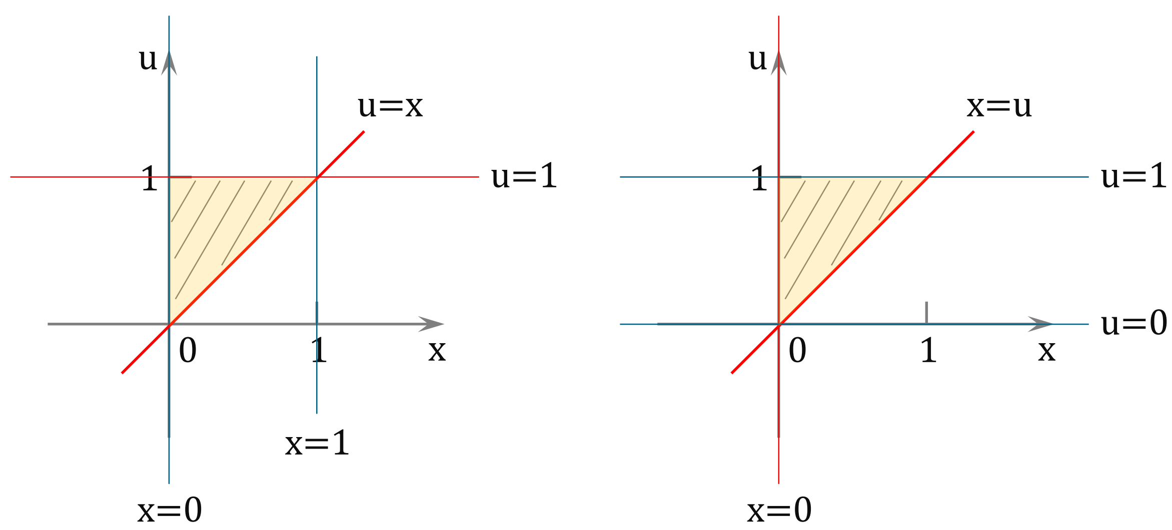
$$= \left[-\frac{1}{2\pi} \cos(\pi u^2)\right]\Big|_0^1$$

$$= \left[-\frac{1}{2\pi} \cos(\pi \cdot 1^2)\right] - \left[-\frac{1}{2\pi} \cos(\pi \cdot 0^2)\right]$$

$$= \left(-\frac{1}{2\pi} \cos\pi\right) - \left(-\frac{1}{2\pi} \cos 0\right)$$

$$= \left[-\frac{1}{2\pi} \cdot (-1)\right] - \left(-\frac{1}{2\pi} \cdot 1\right)$$

$$= \frac{1}{\pi}$$



$$\text{原式} = \int_0^1 du \int_0^u \sin(\pi u^2)\, dx$$

通过极坐标变换来计算积分

例1. 设 D 是第一象限中，由 曲线 $2xy=1$、$4xy=1$，与直线 $y=x$、$y=\sqrt{3}x$ 围成的平面区域，函数 $f(x,y)$ 在 D 上连续，则 $\iint\limits_D xy\,dxdy=$ (B)

(A) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta\int_{\frac{1}{2\sin2\theta}}^{\frac{1}{\sin2\theta}}r^3\cos\theta\sin\theta\,dr$ (B) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta\int_{\frac{1}{\sqrt{2\sin2\theta}}}^{\frac{1}{\sqrt{\sin2\theta}}}r^3\cos\theta\sin\theta\,dr$

(C) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta\int_{\frac{1}{2\sin2\theta}}^{\frac{1}{\sin2\theta}}r^2\cos\theta\sin\theta\,dr$ (D) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta\int_{\frac{1}{\sqrt{2\sin2\theta}}}^{\frac{1}{\sqrt{\sin2\theta}}}r^2\cos\theta\sin\theta\,dr$

$$\iint\limits_D xy\,dxdy=\boxed{\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta\int_{\frac{1}{\sqrt{2\sin2\theta}}}^{\frac{1}{\sqrt{\sin2\theta}}}r\cos\theta\, r\sin\theta\, r\,dr}=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta\int_{\frac{1}{\sqrt{2\sin2\theta}}}^{\frac{1}{\sqrt{\sin2\theta}}}r^3\cos\theta\sin\theta\,dr$$

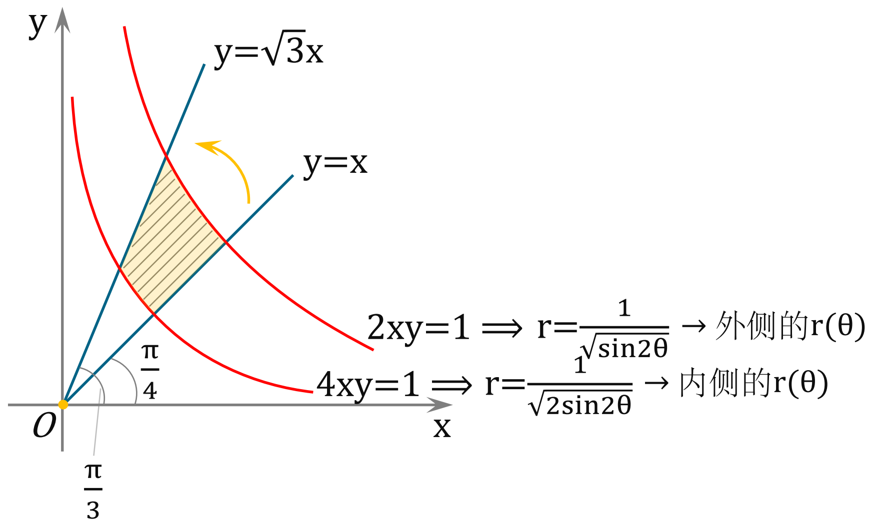
通过极坐标变换来计算积分的情况：

① 积分区域是被两条从原点 O 发出的射线切割出来的

② 积分区域为圆润图形

当满足以上两种情况时，使用公式：

$$\iint\limits_D f(x,y)\,dxdy=\int_?^?d\theta\int_{\text{内侧的}r(\theta)}^{\text{外侧的}r(\theta)}f(r\cos\theta,r\sin\theta)\,r\,dr$$



外侧的 $r(\theta)$	内侧的 $r(\theta)$
$2r\cos\theta\cdot r\sin\theta=1$	$4r\cos\theta\cdot r\sin\theta=1$
$\Longrightarrow 2r^2\sin\theta\cos\theta=1$	$\Longrightarrow 4r^2\sin\theta\cos\theta=1$
$\Longrightarrow r^2=\frac{1}{2\sin\theta\cos\theta}$	$\Longrightarrow r^2=\frac{1}{4\sin\theta\cos\theta}$
$\Longrightarrow r=\frac{1}{\sqrt{2\sin\theta\cos\theta}}$	$\Longrightarrow r=\frac{1}{\sqrt{4\sin\theta\cos\theta}}$
$\Longrightarrow r=\frac{1}{\sqrt{\sin2\theta}}$	$\Longrightarrow r=\frac{1}{\sqrt{2\sin2\theta}}$

例2. 计算 $\iint\limits_D \frac{1}{1+x^2+y^2}\,dxdy$，其中

$D=\{(x,y)|x^2+y^2\le 1, x\ge 0, y\ge 0\}$

$$\iint\limits_D \frac{1}{1+x^2+y^2}\,dxdy=\boxed{\int_0^{\frac{\pi}{2}} d\theta\int_0^1 \frac{1}{1+(r\cos\theta)^2+(r\sin\theta)^2}\,r\,dr}$$

$$=\int_0^{\frac{\pi}{2}}\left[\int_0^1 \frac{1}{1+(r\cos\theta)^2+(r\sin\theta)^2}\,r\,dr\right]d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left[\int_0^1 \frac{1}{1+r^2\cos^2\theta+r^2\sin^2\theta}\,r\,dr\right]d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left[\int_0^1 \frac{1}{1+r^2(\cos^2\theta+\sin^2\theta)}\,r\,dr\right]d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left(\int_0^1 \frac{1}{1+r^2\cdot 1}\,r\,dr\right)d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left(\int_0^1 \frac{r}{1+r^2}\,dr\right)d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left[\frac{\ln(1+r^2)}{2}\Big|_0^1\right]d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left[\frac{\ln(1+1^2)}{2}-\frac{\ln(1+0^2)}{2}\right]d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left[\frac{\ln 2}{2}-\frac{\ln 1}{2}\right]d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\ln 2}{2}\,d\theta$$

$$=\frac{\ln 2}{2}\cdot\int_0^{\frac{\pi}{2}}1\,d\theta$$

$$=\frac{\ln 2}{2}\cdot\theta\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{\ln 2}{2}\cdot\left(\frac{\pi}{2}-0\right)$$

$$=\frac{\pi\ln 2}{4}$$

通过极坐标变换来计算积分的情况：

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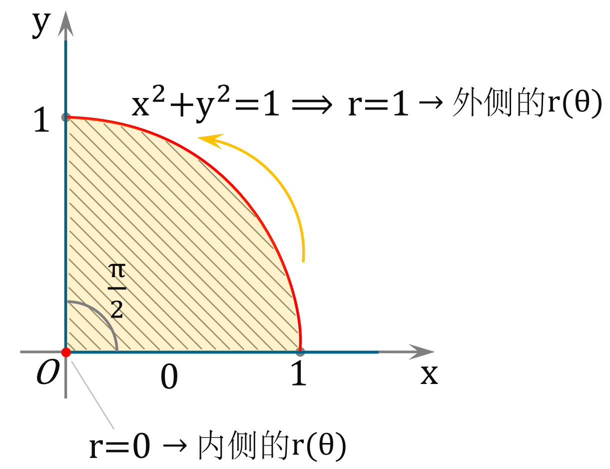
当满足以上两种情况时，使用公式：

$$\iint\limits_D f(x,y)\,dxdy=\int_?^? d\theta\int_{\text{内侧的}r(\theta)}^{\text{外侧的}r(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr$$



外侧的$r(\theta)$

$$(r\cos\theta)^2+(r\sin\theta)^2=1$$

$$\Longrightarrow\ r^2\cos^2\theta+r^2\sin^2\theta=1$$

$$\Longrightarrow\ r^2(\cos^2\theta+\sin^2\theta)=1$$

$$\Longrightarrow\ r^2\cdot 1=1$$

$$\Longrightarrow\ r=1$$

例3. 设区域 D 为 $x^2+y^2\le R^2$，则 $\iint\limits_D\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dxdy=$____。

$$\iint\limits_D\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dxdy=\boxed{\int_0^{2\pi}d\theta\int_0^R\left[\frac{(r\cos\theta)^2}{a^2}+\frac{(r\sin\theta)^2}{b^2}\right]r\,dr}$$

$$=\int_0^{2\pi}\left[\int_0^R\left[\frac{(r\cos\theta)^2}{a^2}+\frac{(r\sin\theta)^2}{b^2}\right]r\,dr\right]d\theta$$

$$=\int_0^{2\pi}\left[\int_0^R\left(\frac{r^2\cos^2\theta}{a^2}+\frac{r^2\sin^2\theta}{b^2}\right)r\,dr\right]d\theta$$

$$=\int_0^{2\pi}\left[\int_0^R\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)r^3\,dr\right]d\theta$$

$$=\int_0^{2\pi}\left[\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\int_0^R r^3\,dr\right]d\theta$$

$$=\int_0^{2\pi}\left[\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\frac{r^4}{4}\bigg|_0^R\right]d\theta$$

$$=\int_0^{2\pi}\left[\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\left(\frac{R^4}{4}-\frac{0^4}{4}\right)\right]d\theta$$

$$=\int_0^{2\pi}\left[\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\frac{R^4}{4}\right]d\theta$$

$$\cos^2\theta=\frac{1}{2}\cdot\cos2\theta+\frac{1}{2}$$
$$\sin^2\theta=-\frac{1}{2}\cdot\cos2\theta+\frac{1}{2}$$

$$=\frac{R^4}{4}\cdot\int_0^{2\pi}\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)d\theta$$

$$=\frac{R^4}{4}\cdot\int_0^{2\pi}\left(\frac{\frac{1}{2}\cdot\cos2\theta+\frac{1}{2}}{a^2}+\frac{-\frac{1}{2}\cdot\cos2\theta+\frac{1}{2}}{b^2}\right)d\theta$$

$$=\frac{R^4}{4}\cdot\int_0^{2\pi}\left(\frac{1}{2}\cdot\frac{\cos2\theta+1}{a^2}+\frac{1}{2}\cdot\frac{-\cos2\theta+1}{b^2}\right)d\theta$$

$$=\frac{R^4}{8}\cdot\int_0^{2\pi}\left(\frac{\cos2\theta+1}{a^2}+\frac{-\cos2\theta+1}{b^2}\right)d\theta$$

$$=\frac{R^4}{8}\cdot\left(\frac{\sin2\theta+2\theta}{2a^2}+\frac{-\sin2\theta+2\theta}{2b^2}\right)\bigg|_0^{2\pi}$$

$$=\frac{R^4}{8}\cdot\left[\left(\frac{\sin(2\cdot2\pi)+2\cdot2\pi}{2a^2}+\frac{-\sin(2\cdot2\pi)+2\cdot2\pi}{2b^2}\right)-\left(\frac{\sin(2\cdot0)+2\cdot0}{2a^2}+\frac{-\sin(2\cdot0)+2\cdot0}{2b^2}\right)\right]$$

$$=\frac{R^4}{8}\cdot\left[\left(\frac{0+4\pi}{2a^2}+\frac{0+4\pi}{2b^2}\right)-0\right]$$

$$=\frac{R^4}{8}\cdot\left(\frac{2\pi}{a^2}+\frac{2\pi}{b^2}\right)$$

$$=\frac{\pi R^4}{4}\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

通过极坐标变换来计算积分的情况：

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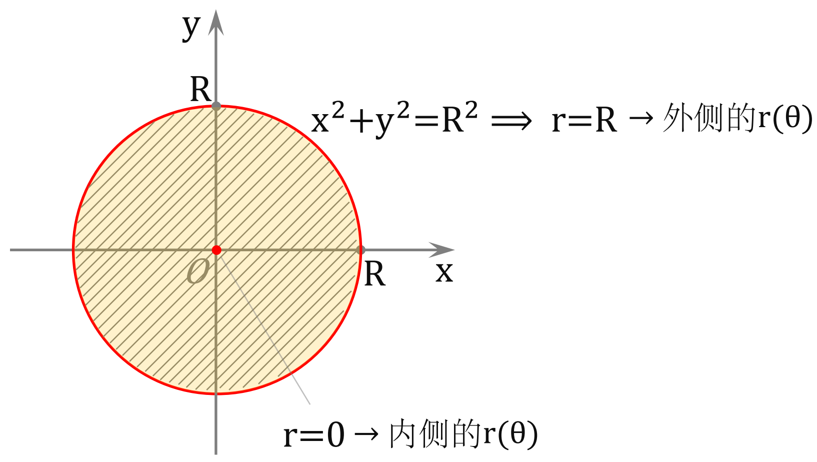
② 积分区域为圆润图形

当满足以上两种情况时，使用公式：

$$\iint\limits_D f(x,y)\,dxdy=\int_?^?d\theta\int_{内侧的r(\theta)}^{外侧的r(\theta)}f(r\cos\theta,r\sin\theta)\,r\,dr$$



外侧的r(θ)

$$(r\cos\theta)^2+(r\sin\theta)^2=R^2$$
$$\Longrightarrow\ r^2\cos^2\theta+r^2\sin^2\theta=R^2$$
$$\Longrightarrow r^2(\cos^2\theta+\sin^2\theta)=R^2$$
$$\Longrightarrow\ r^2\cdot1=R^2$$
$$\Longrightarrow\ r=R$$

例4. 计算 $\iint\limits_D \sqrt{x^2+y^2}\,dxdy$，其中 D 由 曲线 $x^2+y^2=6y$ 所围成

$$\iint\limits_D \sqrt{x^2+y^2}\,dxdy=\boxed{\int_0^{\pi} d\theta \int_0^{6\sin\theta} \sqrt{(r\cos\theta)^2+(r\sin\theta)^2}\, r\,dr}$$

$$=\int_0^{\pi}\left[\int_0^{6\sin\theta}\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}\,r\,dr\right]d\theta$$

$$=\int_0^{\pi}\left[\int_0^{6\sin\theta}\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}\,r\,dr\right]d\theta$$

$$=\int_0^{\pi}\left[\int_0^{6\sin\theta}\sqrt{r^2(\cos^2\theta+\sin^2\theta)}\,r\,dr\right]d\theta$$

$$=\int_0^{\pi}\left(\int_0^{6\sin\theta}\sqrt{r^2\cdot 1}\,r\,dr\right)d\theta$$

$$=\int_0^{\pi}\left(\int_0^{6\sin\theta}r^2\,dr\right)d\theta$$

$$=\int_0^{\pi}\left(\frac{r^3}{3}\Big|_0^{6\sin\theta}\right)d\theta$$

$$=\int_0^{\pi}\left[\frac{(6\sin\theta)^3}{3}-\frac{0^3}{3}\right]d\theta$$

$$=\int_0^{\pi}72\sin^3\theta\,d\theta$$

$$=72\cdot\boxed{\int_0^{\pi}\sin^3\theta\,d\theta}$$

$$=72\cdot\left(\frac{\cos3\theta}{12}-\frac{3\cos\theta}{4}\right)\Big|_0^{\pi}$$

$$=72\cdot\left[\left(\frac{\cos3\pi}{12}-\frac{3\cos\pi}{4}\right)-\left(\frac{\cos(3\cdot0)}{12}-\frac{3\cos0}{4}\right)\right]$$

$$=72\cdot\left[\left(-\frac{1}{12}+\frac{3}{4}\right)-\left(\frac{1}{12}-\frac{3}{4}\right)\right]$$

$$=72\cdot\frac{4}{3}$$

$$=96$$

通过极坐标变换来计算积分的情况：

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$$\iint\limits_D f(x,y)\,dxdy=\int_?^? d\theta\int_{\text{内侧的}r(\theta)}^{\text{外侧的}r(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr$$

$\sin^3\theta=\sin\theta\cdot\sin\theta\cdot\sin\theta$

$\because \sin\alpha\cdot\sin\beta=-\frac{1}{2}\cdot[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$

$\therefore \quad =-\frac{1}{2}\cdot[\cos(\theta+\theta)-\cos(\theta-\theta)]\cdot\sin\theta$

$\quad =-\frac{1}{2}\cdot(\cos2\theta-1)\cdot\sin\theta$

$\quad =-\frac{1}{2}\cdot\sin\theta\cdot\cos2\theta+\frac{1}{2}\cdot\sin\theta$

$\because \sin\alpha\cdot\cos\beta=\frac{1}{2}\cdot[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

$\therefore \quad =-\frac{1}{2}\cdot\frac{1}{2}\cdot[\sin(\theta+2\theta)+\sin(\theta-2\theta)]+\frac{1}{2}\cdot\sin\theta$

$\quad =-\frac{1}{4}\cdot\sin3\theta+\frac{3}{4}\cdot\sin\theta$

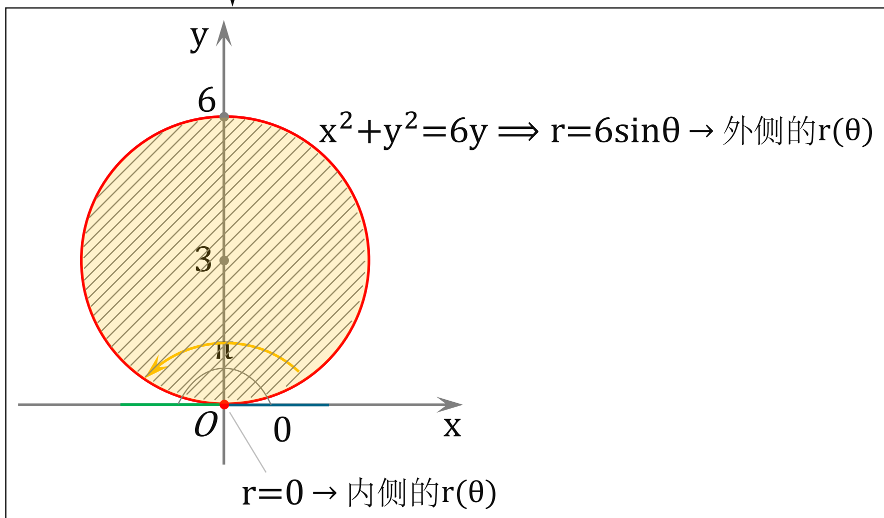
$\therefore \int\sin^3\theta\,d\theta=\int\left(-\frac{1}{4}\cdot\sin3\theta+\frac{3}{4}\cdot\sin\theta\right)d\theta$

$\quad =\int-\frac{1}{4}\cdot\sin3\theta\,d\theta+\int\frac{3}{4}\cdot\sin\theta\,d\theta$

$\quad =-\frac{1}{4}\cdot\int\sin3\theta\,d\theta+\frac{3}{4}\cdot\int\sin\theta\,d\theta$

$\quad =-\frac{1}{4}\cdot\frac{-\cos3\theta}{3}+\frac{3}{4}\cdot(-\cos\theta)$

$\quad =\frac{\cos3\theta}{12}-\frac{3\cos\theta}{4}$



外侧的$r(\theta)$

$(r\cos\theta)^2+(r\sin\theta)^2=6r\sin\theta$

$\Longrightarrow\ r^2\cos^2\theta+r^2\sin^2\theta=6r\sin\theta$

$\Longrightarrow\ r^2(\cos^2\theta+\sin^2\theta)=6r\sin\theta$

$\Longrightarrow\ r^2\cdot1=6r\sin\theta$

$\Longrightarrow\ r=6\sin\theta$

例5. 计算 $\iint\limits_D (x+y)\,dxdy$，$D=\left\{(x,y)\,\middle|\,\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\le 1\right\}$

设 $X=x-\frac{1}{2}$，$Y=y-\frac{1}{2}$

则 $x=X+\frac{1}{2}$，$y=Y+\frac{1}{2}$

$$dx=\frac{1}{X'}dX=\frac{1}{\left(x-\frac{1}{2}\right)'}dX=dX \qquad dy=\frac{1}{Y'}dY=\frac{1}{\left(y-\frac{1}{2}\right)'}dY=dY$$

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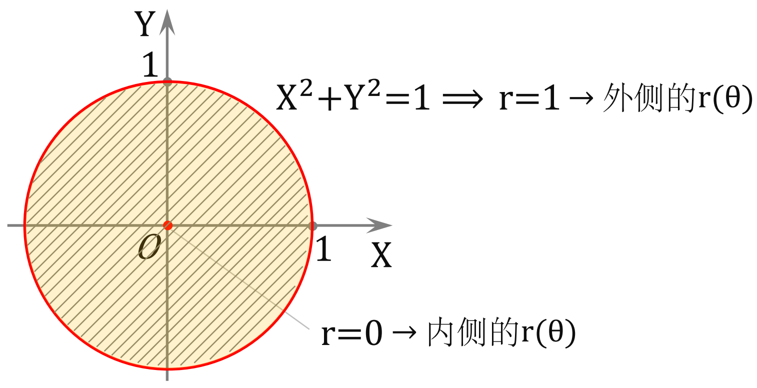
原题干 即 计算 $\iint\limits_D (X+Y+1)\,dXdY$，$D=\{(X,Y)|X^2+Y^2\le 1\}$

$$\begin{aligned}
\iint\limits_D (X+Y+1)\,dxdy&=\boxed{\int_0^{2\pi}d\theta\int_0^1(r\cos\theta+r\sin\theta+1)\,r\,dr}\\
&=\int_0^{2\pi}\left[\int_0^1(r\cos\theta+r\sin\theta+1)\,r\,dr\right]d\theta\\
&=\int_0^{2\pi}\left[\int_0^1(r^2\cos\theta+r^2\sin\theta+r)\,dr\right]d\theta\\
&=\int_0^{2\pi}\left[\int_0^1[r^2(\cos\theta+\sin\theta)+r]\,dr\right]d\theta\\
&=\int_0^{2\pi}\left[\left[\frac{r^3}{3}(\cos\theta+\sin\theta)+\frac{r^2}{2}\right]\Bigg|_{r=0}^{r=1}\right]d\theta\\
&=\int_0^{2\pi}\left[\left[\frac{1^3}{3}(\cos\theta+\sin\theta)+\frac{1^2}{2}\right]-\left[\frac{0^3}{3}(\cos\theta+\sin\theta)+\frac{0^2}{2}\right]\right]d\theta\\
&=\int_0^{2\pi}\left[\frac{1}{3}(\cos\theta+\sin\theta)+\frac{1}{2}\right]d\theta\\
&=\left[\frac{1}{3}(\sin\theta-\cos\theta)+\frac{1}{2}\theta\right]\Bigg|_0^{2\pi}\\
&=\left[\frac{1}{3}[\sin(2\pi)-\cos(2\pi)]+\frac{1}{2}\cdot(2\pi)\right]-\left[\frac{1}{3}[\sin 0-\cos 0]+\frac{1}{2}\cdot 0\right]\\
&=\pi
\end{aligned}$$



外侧的$r(\theta)$

$$\begin{aligned}
&(r\cos\theta)^2+(r\sin\theta)^2=1\\
\Longrightarrow\ &r^2\cos^2\theta+r^2\sin^2\theta=1\\
\Longrightarrow\ &r^2(\cos^2\theta+\sin^2\theta)=1\\
\Longrightarrow\ &r^2\cdot 1=1\\
\Longrightarrow\ &r=1
\end{aligned}$$

$\therefore \iint\limits_D (x+y)\,dxdy=\iint\limits_D (X+Y+1)\,dxdy=\pi$

通过直角坐标变换来计算积分

$$\int_?^? d\theta \int_{内侧的r(\theta)}^{外侧的r(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr = \iint\limits_D f(x,y)\, dxdy$$

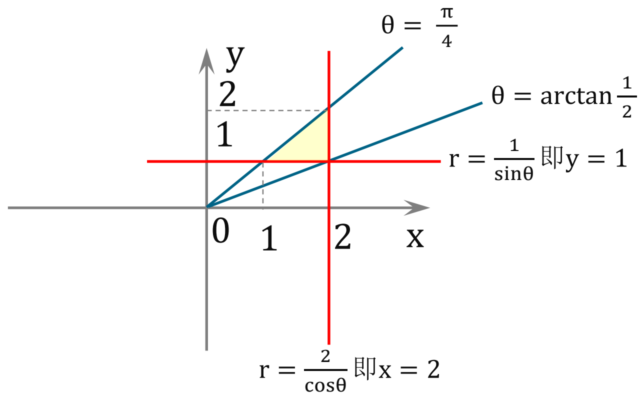
例1. 计算二次积分 $\int_{\arctan\frac{1}{2}}^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\sin\theta}}^{\frac{2}{\cos\theta}} \frac{r^2\sin2\theta}{2} rdr$

① a. $1 \cdot \frac{r^2\sin2\theta}{2} r = \frac{r^2\sin2\theta}{2} r$

b. $\frac{\frac{r^2\sin2\theta}{2} r}{r} = \frac{r^2\sin2\theta}{2}$

c. $\frac{r^2\sin2\theta}{2} = \frac{r^2 2\sin\theta\cos\theta}{2} = r^2\sin\theta\cos\theta = r\cos\theta \cdot r\sin\theta = xy \quad f(x,y) = xy$

②



$r = \frac{2}{\cos\theta} \Longrightarrow r\cos\theta = 2 \Longrightarrow x = 2$

$r = \frac{1}{\sin\theta} \Longrightarrow r\sin\theta = 1 \Longrightarrow y = 1$

③ $\int_{\arctan\frac{1}{2}}^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\sin\theta}}^{\frac{2}{\cos\theta}} \frac{r^2\sin2\theta}{2} rdr = \iint\limits_D xydxdy = \frac{9}{8}$

【注意】

若待求的是二次积分 $\int ?\, d\theta \int ?\, dr$，

则结果前面可能有−1，

判断方法是：

观察积分区域边界

若 θ的上限 对应的是 积分下限，

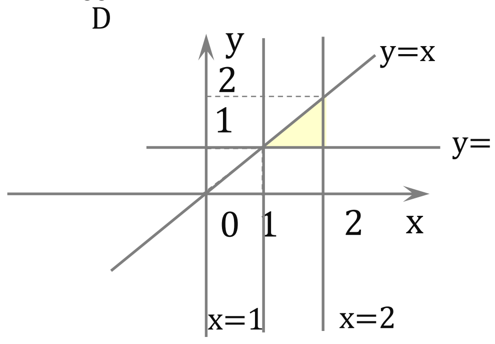
则 在结果前 乘(−1)；

若 r的上限 对应的是 积分下限，

则 在结果前 乘(−1)。

例1. 计算 $\iint\limits_D xy\, dxdy$，其中 D 为如下区域

①



② 确定范围内 x 的取值范围 [1,2]

③ 确定范围的下边界 y = 1与上边界 y =x

④ $\iint\limits_D f(x,y)\, d\sigma = \int_a^b dx \int_{y_{下}(x)}^{y_{上}(x)} f(x,y)\, dy$

$$\iint\limits_D xy\, d\sigma = \int_1^2 dx \int_1^x xy\, dy = \int_1^2 \left(\int_1^x 1 \cdot xy\, dy\right) dx = \int_1^2 \left(\int_1^x xy\, dy\right) dx$$

$$= \int_1^2 \left(\frac{x^3}{2} - \frac{x}{2}\right) dx$$

$$= \left(\frac{x^4}{8} - \frac{x^2}{4}\right)\Big|_{x=1}^{x=2}$$

$$= \left(\frac{2^4}{8} - \frac{2^2}{4}\right) - \left(\frac{1^4}{8} - \frac{1^2}{4}\right)$$

$$= \frac{9}{8}$$

例2. 计算二重积分$\iint\limits_D r^2\sin\theta\sqrt{1-r^2\cos 2\theta}\,drd\theta$，其中

$D=\left\{(r,\theta)\middle|0\le r\le\sec\theta,0\le\theta\le\frac{\pi}{4}\right\}$

① a. $r^2\sin\theta\sqrt{1-r^2\cos2\theta}$

b. $\frac{r^2\sin\theta\sqrt{1-r^2\cos2\theta}}{r}=r\sin\theta\sqrt{1-r^2\cos2\theta}$

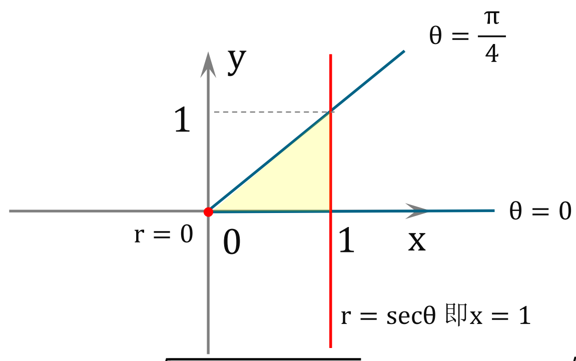
c. $r\sin\theta\sqrt{1-r^2\cos2\theta}=r\sin\theta\sqrt{1-r^2(\cos^2\theta-\sin^2\theta)}=r\sin\theta\sqrt{1-r^2\cos^2\theta+r^2\sin^2\theta}$

$$=r\sin\theta\sqrt{1-(r\cos\theta)^2+(r\sin\theta)^2}$$

$$=y\sqrt{1-x^2+y^2}$$

$$f(x,y)=y\sqrt{1-x^2+y^2}$$

②

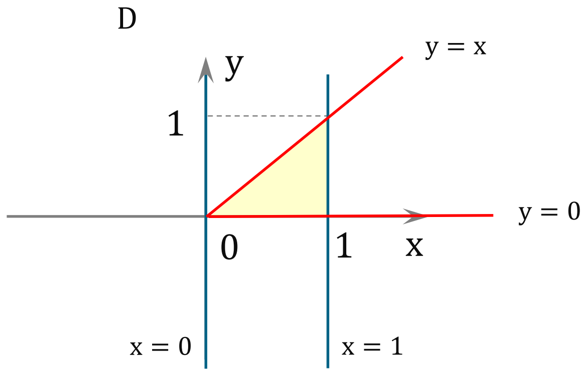


$r=\sec\theta=\frac{1}{\cos\theta}\Longrightarrow r\cos\theta=1\Longrightarrow x=1$

③ $\iint\limits_D r^2\sin\theta\sqrt{1-r^2\cos2\theta}\,drd\theta=\underline{\iint\limits_D y\sqrt{1-x^2+y^2}\,dxdy}=\frac{1}{3}-\frac{\pi}{16}$

例2. 计算$\iint\limits_D y\sqrt{1-x^2+y^2}\,dxdy$，其中 D 为如下区域

①



② 确定范围内 x 的取值范围 [0,1]

③ 确定范围的下边界 y = 0与上边界 y =x

④ $\iint\limits_D f(x,y)\,d\sigma=\int_a^b dx\int_{y_下(x)}^{y_上(x)}f(x,y)\,dy$

$$\iint\limits_D y\sqrt{1-x^2+y^2}\,d\sigma=\int_0^1 dx\int_0^x y\sqrt{1-x^2+y^2}\,dy$$

$$=\int_0^1\left(\int_0^x y\sqrt{1-x^2+y^2}\,dy\right)dx$$

$$=\int_0^1\left[\frac{(1-x^2+y^2)^{\frac{3}{2}}}{3}\Bigg|_{y=0}^{y=x}\right]dx$$

$$=\int_0^1\left[\frac{(1-x^2+x^2)^{\frac{3}{2}}}{3}-\frac{(1-x^2+0^2)^{\frac{3}{2}}}{3}\right]dx$$

$$=\int_0^1\left[\frac{1}{3}-\frac{(1-x^2)^{\frac{3}{2}}}{3}\right]dx$$

$$=\boxed{\int_0^1\left[\frac{1-(1-x^2)^{\frac{3}{2}}}{3}\right]dx}$$

$$=\int_0^{\frac{\pi}{2}}\frac{1-\cos^3t}{3}\cdot\cos t\,dt$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{\cos t}{3}-\frac{\cos^4t}{3}\right)dt$$

$$=\boxed{\frac{1}{3}\int_0^{\frac{\pi}{2}}\cos t\,dt-\frac{1}{3}\int_0^{\frac{\pi}{2}}\cos^4t\,dt}$$

$$=\frac{1}{3}\cdot1-\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$=\frac{1}{3}-\frac{\pi}{16}$$

$\boxed{\text{令}x=\sin t\quad\text{则}(1-x^2)^{\frac{3}{2}}=\cos^3t\text{、}dx=\cos t\,dt\quad x:0\to1\ \Rightarrow t:0\to\frac{\pi}{2}}$

如果遗忘请复习高数上第三章第3课：用第二类换元法计算不定积分

$$\boxed{\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}，\ n\text{为偶数}\\\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}，\ n\text{为大于1的奇数}\\1，\ n=1\end{cases}}$$

如果遗忘请复习高数上第三章第5课：计算定积分、广义积分

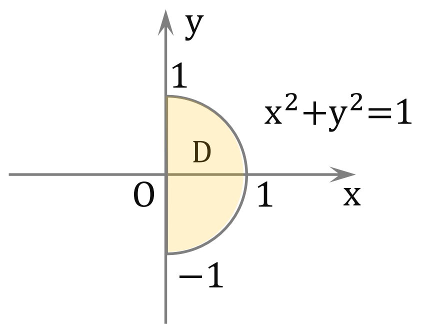
通过对称性来计算积分

解题方法:

可通过对称性来计算的情形		计算方法
积分区域关于y轴对称	$f(-x,y)=-f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=0$
	$f(-x,y)=f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=2\iint\limits_{\frac{D}{2}} f(x,y)\,dxdy$
积分区域关于x轴对称	$f(x,-y)=-f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=0$
	$f(x,-y)=f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=2\iint\limits_{\frac{D}{2}} f(x,y)\,dxdy$
积分区域关于原点对称	$f(-x,-y)=-f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=0$
	$f(-x,-y)=f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=2\iint\limits_{\frac{D}{2}} f(x,y)\,dxdy$

例1 .设区域D=$\{(x,y)|x^2+y^2\le 1,x\ge 0\}$，计算二重积分

$$I=\iint\limits_D \frac{xy}{1+x^2+y^2}\,dxdy$$



$$f(x,y)=\frac{xy}{1+x^2+y^2}$$

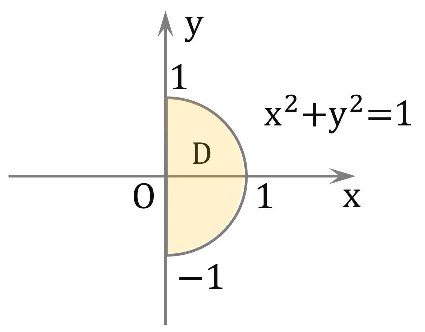
$$f(x,-y)=\frac{x\cdot(-y)}{1+x^2+(-y)^2}=\frac{-x\cdot y}{1+x^2+y^2}$$

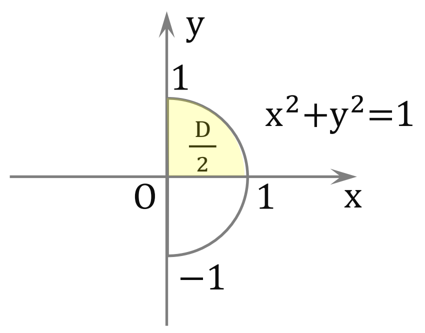
$$f(x,-y)=-f(x,y)$$

$$\iint\limits_D f(x,y)\,dxdy=0$$

例2 .设区域D=$\{(x,y)|x^2+y^2\le 1,x\ge 0\}$，计算二重积分

$$I=\iint\limits_D \frac{1}{1+x^2+y^2}\,dxdy = 2\iint\limits_{\frac{D}{2}} \frac{1}{1+x^2+y^2}\,dxdy$$

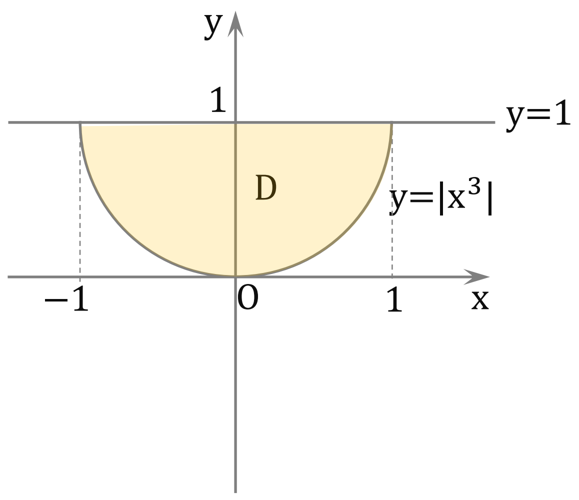




$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^1\frac{1}{1+(r\cos\theta)^2+(r\sin\theta)^2}rdr$$
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^1\frac{1}{1+r^2}rdr$$
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^1\frac{1}{1+r^2}rdr\right)d\theta$$
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{\ln(1+r^2)}{2}\Big|_0^1\right]d\theta$$
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{\ln(1+1^2)}{2}-\frac{\ln(1+0^2)}{2}\right]d\theta$$
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\ln2}{2}d\theta$$
$$=\frac{\ln2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1\,d\theta$$
$$=\frac{\ln2}{2}\left(\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$
$$=\frac{\ln2}{2}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]$$
$$=\frac{\pi\ln2}{2}$$

$$=2\int_0^{\frac{\pi}{2}}d\theta\int_0^1\frac{1}{1+(r\cos\theta)^2+(r\sin\theta)^2}rdr$$
$$=2\int_0^{\frac{\pi}{2}}d\theta\int_0^1\frac{1}{1+r^2}rdr$$
$$=2\int_0^{\frac{\pi}{2}}\left(\int_0^1\frac{1}{1+r^2}rdr\right)d\theta$$
$$=2\int_0^{\frac{\pi}{2}}\left[\frac{\ln(1+r^2)}{2}\Big|_0^1\right]d\theta$$
$$=2\int_0^{\frac{\pi}{2}}\left[\frac{\ln(1+1^2)}{2}-\frac{\ln(1+0^2)}{2}\right]d\theta$$
$$=2\int_0^{\frac{\pi}{2}}\frac{\ln2}{2}d\theta$$
$$=2\cdot\frac{\ln2}{2}\int_0^{\frac{\pi}{2}}1\,d\theta$$
$$=\ln2\left(\theta\Big|_0^{\frac{\pi}{2}}\right)$$
$$=\ln2\left(\frac{\pi}{2}-0\right)$$
$$=\frac{\pi\ln2}{2}$$

例3 .计算$\iint\limits_{D} xy \cdot \rho(x^2+y^2)\,dxdy$，其中 D 是由 $y=|x^3|$、$y=1$所围成的区域，ρ 为连续函数



$f(x,y)=xy\cdot\rho(x^2+y^2)$

$f(-x,y)=-xy\cdot\rho[(-x)^2+y^2]=-xy\cdot\rho(x^2+y^2)$

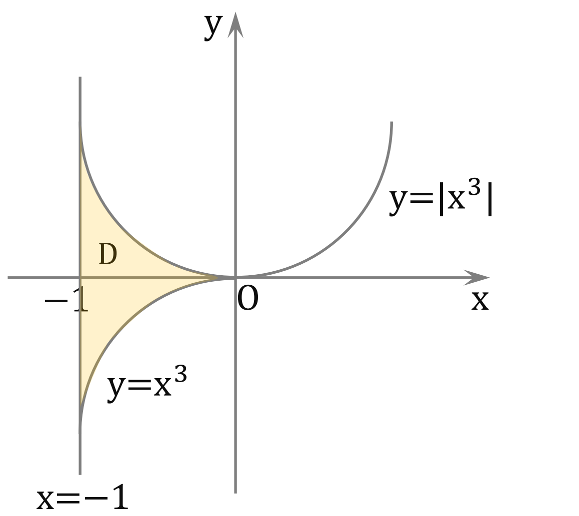
$f(-x,y)=-f(x,y)$

$\iint\limits_{D} f(x,y)\,dxdy=0$

解题方法:

可通过对称性来计算的情形		计算方法
积分区域关于y轴对称	$f(-x,y)=-f(x,y)$	$\iint\limits_{D} f(x,y)\,dxdy=0$
	$f(-x,y)=f(x,y)$	$\iint\limits_{D} f(x,y)\,dxdy=2\iint\limits_{\frac{D}{2}} f(x,y)\,dxdy$
积分区域关于x轴对称	$f(x,-y)=-f(x,y)$	$\iint\limits_{D} f(x,y)\,dxdy=0$
	$f(x,-y)=f(x,y)$	$\iint\limits_{D} f(x,y)\,dxdy=2\iint\limits_{\frac{D}{2}} f(x,y)\,dxdy$
积分区域关于原点对称	$f(-x,-y)=-f(x,y)$	$\iint\limits_{D} f(x,y)\,dxdy=0$
	$f(-x,-y)=f(x,y)$	$\iint\limits_{D} f(x,y)\,dxdy=2\iint\limits_{\frac{D}{2}} f(x,y)\,dxdy$

例4 .计算$\iint\limits_{D} xy \cdot \rho(x^2+y^2)\,dxdy$，其中 D 是由 $y=|x^3|$、$y=x^3$ 、$x=-1$所围成的区域，ρ 为连续函数



$f(x,y)=xy\cdot\rho(x^2+y^2)$

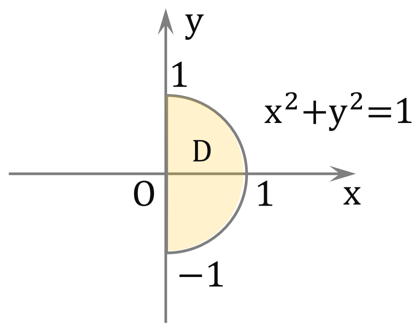
$f(x,-y)=x(-y)\cdot\rho[x^2+(-y)^2]=-xy\cdot\rho(x^2+y^2)$

$f(x,-y)=-f(x,y)$

$\iint\limits_{D} f(x,y)\,dxdy=0$

例5 .设区域D=$\{(x,y)|x^2+y^2\le 1, x\ge 0\}$，计算二重积分

$$I=\iint\limits_D \frac{1+xy}{1+x^2+y^2}dxdy$$

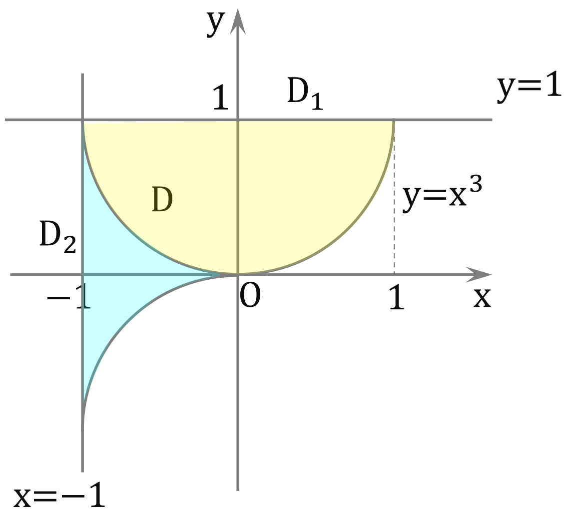


$$\begin{aligned}\iint\limits_D \frac{1+xy}{1+x^2+y^2}dxdy &= \iint\limits_D \left(\frac{1}{1+x^2+y^2}+\frac{xy}{1+x^2+y^2}\right)dxdy\\ &= \underbrace{\iint\limits_D \frac{1}{1+x^2+y^2}dxdy}_{\text{例2求过}} + \underbrace{\iint\limits_D \frac{xy}{1+x^2+y^2}dxdy}_{\text{例1求过}}\\ &= \frac{\pi\ln 2}{2}+0\\ &= \frac{\pi\ln 2}{2}\end{aligned}$$

解题方法:

可通过对称性来计算的情形		计算方法
积分区域关于y轴对称	$f(-x,y)=-f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=0$
	$f(-x,y)=f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=2\iint\limits_{\frac{D}{2}} f(x,y)\,dxdy$
积分区域关于x轴对称	$f(x,-y)=-f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=0$
	$f(x,-y)=f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=2\iint\limits_{\frac{D}{2}} f(x,y)\,dxdy$
积分区域关于原点对称	$f(-x,-y)=-f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=0$
	$f(-x,-y)=f(x,y)$	$\iint\limits_D f(x,y)\,dxdy=2\iint\limits_{\frac{D}{2}} f(x,y)\,dxdy$

例6 .计算$\iint\limits_D xy\cdot\rho(x^2+y^2)\,dxdy$，其中D是由$y=x^3$、$y=1$、$x=-1$所围成的区域，ρ为连续函数



$$\begin{aligned}\iint\limits_D xy\cdot\rho(x^2+y^2)\,dxdy &= \underbrace{\iint\limits_{D_1} xy\cdot\rho(x^2+y^2)\,dxdy}_{\text{例3求过}} + \underbrace{\iint\limits_{D_2} xy\cdot\rho(x^2+y^2)\,dxdy}_{\text{例4求过}}\\ &= 0 + 0\\ &= 0\end{aligned}$$

通过轮换对称性来计算积分

情况一

特征：

$\begin{cases}\text{① 积分区域关于直线 } y=x \text{ 对称} \\ \text{② } [f(x,y)+f(y,x)] \text{ 的积分更好求}\end{cases}$

公式：

$$\iint\limits_D f(x,y)\,dxdy=\frac{1}{2}\iint\limits_D [f(x,y)+f(y,x)]\,dxdy$$

例1. 设区域 D={(x,y)|$x^2+y^2\le 1, x\ge 0, y\ge 0$}，

f(x) 为 D 上正值连续函数，a、b 为常数，

求 $\iint\limits_D \frac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}\,dxdy$

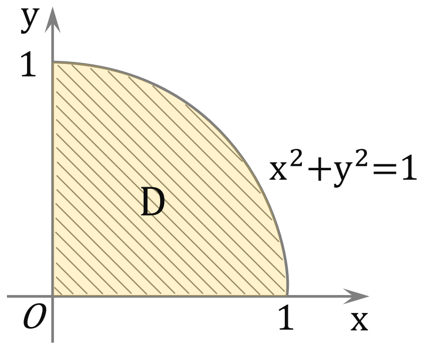
$$\iint\limits_D \frac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}\,dxdy=\frac{1}{2}\iint\limits_D\left(\frac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}+\frac{a\sqrt{f(y)}+b\sqrt{f(x)}}{\sqrt{f(y)}+\sqrt{f(x)}}\right)dxdy$$

$$=\frac{1}{2}\iint\limits_D (a+b)\,dxdy$$

$$=\frac{1}{2}(a+b)\iint\limits_D 1\,dxdy$$

$$=\frac{1}{2}(a+b)\cdot\frac{\pi}{4}$$

$$=\frac{\pi(a+b)}{8}$$



例2. 设区域 D 为 $x^2+y^2\le R^2$，

则 $\iint\limits_D\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dxdy=$____。

$$\iint\limits_D\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dxdy$$

$$=\frac{1}{2}\iint\left[\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\left(\frac{y^2}{a^2}+\frac{x^2}{b^2}\right)\right]dxdy$$

$$=\frac{1}{2}\iint\limits_D\left(\frac{1}{a^2}+\frac{1}{b^2}\right)(x^2+y^2)\,dxdy$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\iint\limits_D (x^2+y^2)\,dxdy \quad \text{(通过极坐标变换)}$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_0^{2\pi}d\theta\int_0^R[(r\cos\theta)^2+(r\sin\theta)^2]\,r\,dr$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_0^{2\pi}d\theta\int_0^R r^3\,dr$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_0^{2\pi}\left(\int_0^R r^3\,dr\right)d\theta$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_0^{2\pi}\left(\left.\frac{r^4}{4}\right|_0^R\right)d\theta$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_0^{2\pi}\left(\frac{R^4}{4}-\frac{0^4}{4}\right)d\theta$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_0^{2\pi}\frac{R^4}{4}\,d\theta$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\cdot\frac{R^4}{4}\cdot\int_0^{2\pi}1\,d\theta$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\cdot\frac{R^4}{4}\cdot\theta\Big|_0^{2\pi}$$

$$=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\cdot\frac{R^4}{4}\cdot(2\pi-0)$$

$$=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\cdot\frac{\pi R^4}{4}$$

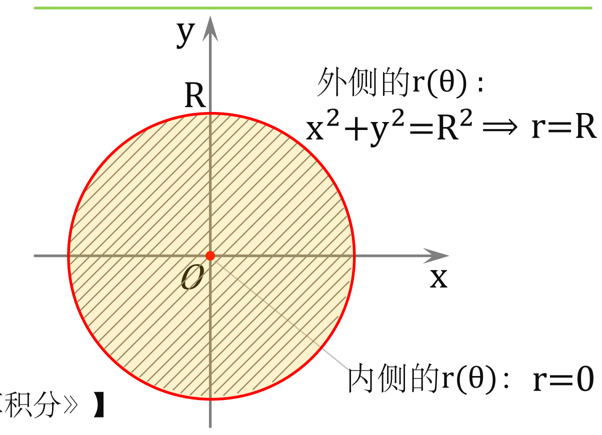
∵ 积分区域D 为 $x^2+y^2\le R^2$，为圆润图形

∴ 可用极坐标方法来求，即

$$\iint\limits_D f(x,y)\,dxdy=\int_?^? d\theta\int_{\text{内侧的}r(\theta)}^{\text{外侧的}r(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr$$

∵ 本题中，积分区域包裹了圆心，内外侧r(θ)如下图，

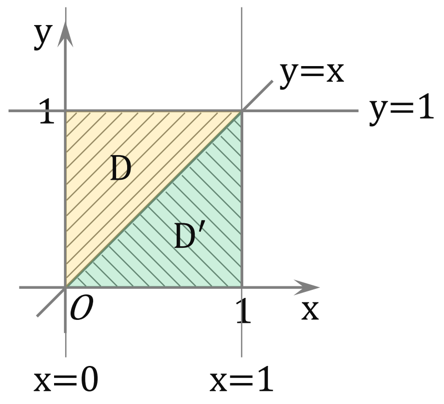
$$\therefore \iint\limits_D (x^2+y^2)\,dxdy=\int_0^{2\pi}d\theta\int_0^R[(r\cos\theta)^2+(r\sin\theta)^2]\,r\,dr$$

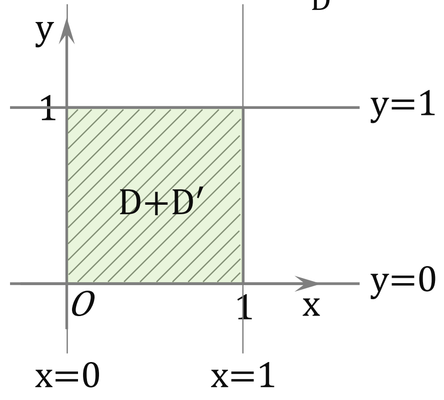


【如不熟悉，请复习《通过极坐标变换来计算积分》】

情况二

例1. 设区域 D=$\{(x,y)|0 \le x \le 1, x \le y \le 1\}$，函数 $f(x)$ 在区间 $[0,1]$ 上连续，且 $\int_0^1 f(x)\,dx=A$，求 $\iint\limits_D f(x)f(y)\,dxdy$





$$
\begin{aligned}
\iint\limits_D f(x)f(y)\,dxdy &= \frac{1}{2}\iint\limits_{D+D'} f(x)f(y)\,dxdy \\
&= \frac{1}{2}\int_0^1 dy\int_0^1 f(x)f(y)\,dx \\
&= \frac{1}{2}\int_0^1 \left[\int_0^1 f(x)f(y)\,dx\right] dy \\
&= \frac{1}{2}\int_0^1 f(y)\left[\int_0^1 f(x)\,dx\right] dy \\
&= \frac{1}{2}\int_0^1 f(y)\cdot A\,dy \\
&= \frac{A}{2}\int_0^1 f(y)\,dy \\
&= \frac{A}{2}\cdot A \\
&= \frac{A^2}{2}
\end{aligned}
$$

题干已知：$\int_0^1 f(x)\,dx = A$

$\therefore \int_0^1 f(y)\,dy = A$

特征：

$\begin{cases} \text{① } f(x,y) = f(y,x) \\ \text{② 积分区域加上其关于直线 } y=x \text{ 对称后的区域后，积分更好求} \end{cases}$

公式：

$$\iint\limits_D f(x,y)\,dxdy = \frac{1}{2}\iint\limits_{D+D'} f(x,y)\,dxdy$$

通过积分区域的形心来计算积分

可通过积分区域的形心来计算积分的情形：

① $\begin{cases}\text{被积函数}=?x+?y\\ \text{积分区域面积好求}\\ \text{积分区域形心好找}\end{cases}$

② $\begin{cases}\text{被积函数}=?y\\ \text{积分区域面积好求}\\ \text{积分区域关于 } y=\text{某数 对称}\end{cases}$

③ $\begin{cases}\text{被积函数}=?x\\ \text{积分区域面积好求}\\ \text{积分区域关于 } x=\text{某数 对称}\end{cases}$

解题方法：

$\iint\limits_D(ax+by)\,dxdy=(a\bar{x}+b\bar{y})\cdot D$的面积

其中 $(\bar{x},\bar{y})$ 是 D 的形心

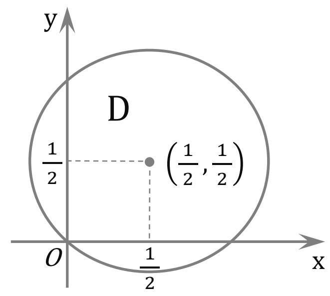
例1．计算$\iint\limits_D(x+y)\,dxdy$，其中D由 $x^2+y^2\le x+y$ 所确定

$x^2+y^2\le x+y$

$\Rightarrow x^2-x+y^2-y\le 0$

$\Rightarrow\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+\left(y-\frac{1}{2}\right)^2-\frac{1}{4}\le 0$

$\Rightarrow\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{2}$



$S_D=\pi r^2=\pi\cdot\frac{1}{2}=\frac{\pi}{2}$

积分区域形心为$\left(\frac{1}{2},\frac{1}{2}\right)$

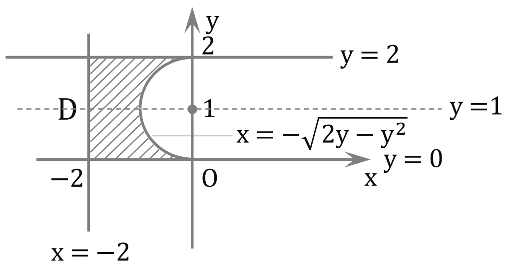
$\iint\limits_D(ax+by)\,dxdy=(a\bar{x}+b\bar{y})\cdot D$的面积（$a=1$，$b=1$，$S_D=\frac{\pi}{2}$）

$\iint\limits_D(x+y)\,dxdy=(\bar{x}+\bar{y})\cdot\frac{\pi}{2}$

$=\left(\frac{1}{2}+\frac{1}{2}\right)\cdot\frac{\pi}{2}$

$=\frac{\pi}{2}$

例2．计算$\iint\limits_D y\,dxdy$，其中D是由直线 $x=-2$、$y=0$、$y=2$ 以及曲线 $x=-\sqrt{2y-y^2}$ 所围成的区域



$S_D=S_{正方形}-S_{半圆}=l^2-\frac{1}{2}\pi r^2=2^2-\frac{1}{2}\pi\cdot 1^2=4-\frac{\pi}{2}$

积分区域关于 $y=1$ 对称

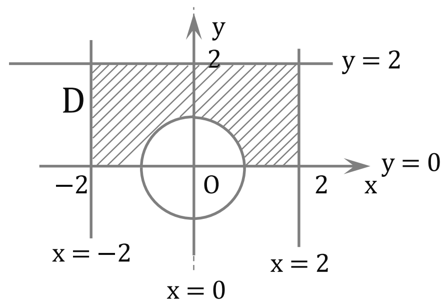
$\iint\limits_D(ax+by)\,dxdy=(a\bar{x}+b\bar{y})\cdot D$的面积（$a=0$，$b=1$，$S_D=4-\frac{\pi}{2}$）

$\iint\limits_D y\,dxdy=\bar{y}\cdot(4-\frac{\pi}{2})$

$=1\cdot(4-\frac{\pi}{2})$

$=4-\frac{\pi}{2}$

例3．计算$\iint\limits_D 3x\,dxdy$，其中D是由直线 $x=-2$、$x=2$、$y=2$ 、$y=0$ 以及曲线 $x^2+y^2=1$ 所围成的区域



$S_D=S_{长方形}-S_{半圆}=长\times宽-\frac{1}{2}\pi r^2=4\times 2-\frac{1}{2}\pi\cdot 1^2=8-\frac{\pi}{2}$

积分区域关于 $x=0$ 对称

$\iint\limits_D(ax+by)\,dxdy=(a\bar{x}+b\bar{y})\cdot D$的面积（$a=3$，$b=0$，$S_D=8-\frac{\pi}{2}$）

$\iint\limits_D 3x\,dxdy=3\bar{x}\cdot(8-\frac{\pi}{2})$

$=3\cdot 0\cdot(8-\frac{\pi}{2})$

$=0$

比较二重积分的大小

公式：

$\iint\limits_{D} 3\,dxdy > \iint\limits_{D} 2\,dxdy > \iint\limits_{D} 1\,dxdy$

$\iint\limits_{D_1+D_2} f(x,y)\,dxdy > \iint\limits_{D_1} f(x,y)\,dxdy$

[注：$f(x,y) \ge 0$]

例1. 设区域 D 为 $x^2+y^2\le1$，请比较

$\iint\limits_{D} 5\,dxdy$ 、$\iint\limits_{D} 3\,dxdy$ 、$\iint\limits_{D} 2\,dxdy$ 的大小

$\because 5 > 3 > 2$

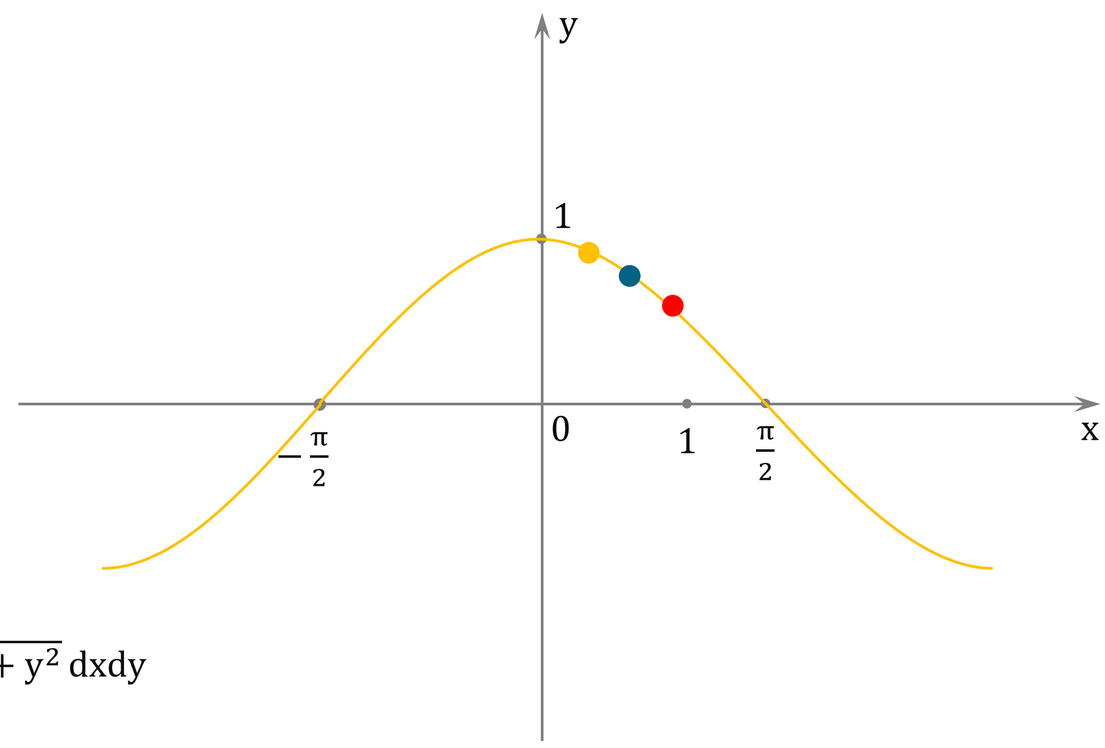
$\therefore \iint\limits_{D} 5\,dxdy > \iint\limits_{D} 3\,dxdy > \iint\limits_{D} 2\,dxdy$

例2. 设区域 D 为 $x^2+y^2\le1$，请比较

$\iint\limits_{D} \cos(x^2+y^2)^2\,dxdy$ 、

$\iint\limits_{D} \cos(x^2+y^2)\,dxdy$ 、

$\iint\limits_{D} \cos\sqrt{x^2+y^2}\,dxdy$ 的大小



比较 $\cos(x^2+y^2)^2$ 、$\cos(x^2+y^2)$、$\cos\sqrt{x^2+y^2}$ 的大小

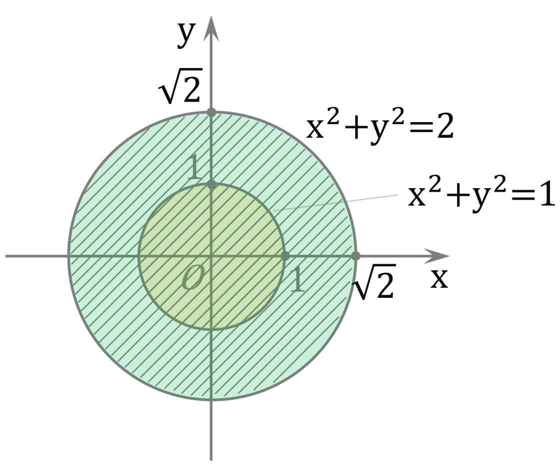
$\because\ 0 \le (x^2+y^2)^2 \le (x^2+y^2) \le \sqrt{x^2+y^2} \le 1$

$\therefore \cos(x^2+y^2)^2 \ge \cos(x^2+y^2) \ge \cos\sqrt{x^2+y^2}$

$\therefore \iint\limits_{D} \cos(x^2+y^2)^2\,dxdy > \iint\limits_{D} \cos(x^2+y^2)\,dxdy > \iint\limits_{D} \cos\sqrt{x^2+y^2}\,dxdy$

例3. 请比较 $I_1=\iint\limits_{x^2+y^2\le1}(x^2+y^2)\,dxdy$ 、

$I_2=\iint\limits_{x^2+y^2\le2}(x^2+y^2)\,dxdy$ 的大小



$\because$ 积分区域 $x^2+y^2\le2$ 包含 积分区域 $x^2+y^2\le1$

$\therefore \iint\limits_{x^2+y^2\le2}(x^2+y^2)\,dxdy > \iint\limits_{x^2+y^2\le1}(x^2+y^2)\,dxdy$

即 $I_2 > I_1$

例4. 请比较 $I_1=\iint\limits_{x^2+y^2\le 1}(x^2+y^2)\,dxdy$ 、

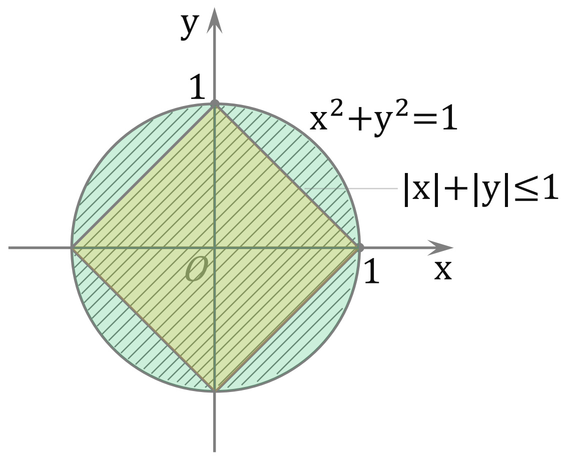
$I_2=\iint\limits_{|x|+|y|\le 1}(x^2+y^2)\,dxdy$ 的大小

第一象限：$x+y\le 1 \Rightarrow y\le 1-x$

第二象限：$-x+y\le 1 \Rightarrow y\le x+1$

第三象限：$(-x)+(-y)\le 1 \Rightarrow y\ge -x-1$

第四象限：$x+(-y)\le 1 \Rightarrow y\ge x-1$



$\because$ 积分区域 $x^2+y^2\le 1$ 包含 积分区域 $|x|+|y|\le 1$

$\therefore \iint\limits_{x^2+y^2\le 1}(x^2+y^2)\,dxdy > \iint\limits_{|x|+|y|\le 1}(x^2+y^2)\,dxdy$

即 $I_1 > I_2$

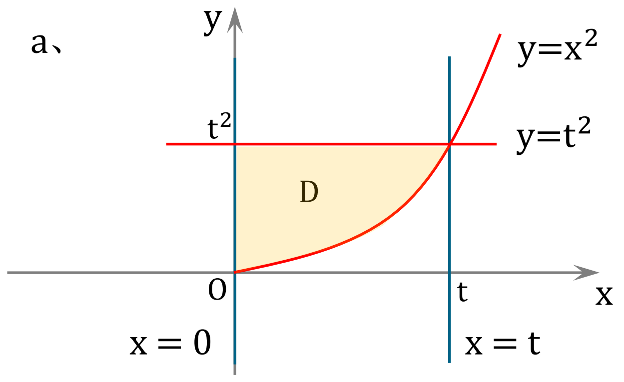
二重积分中值定理

$$\iint\limits_D f(x,y)\,dxdy=f(\xi,\eta)\cdot D\text{的面积}$$

$$\text{即}\iint\limits_D f(x,y)\,dxdy=f(\xi,\eta)\cdot\iint\limits_D dxdy$$

例1 . 求极限 $\lim\limits_{t\to0^+}\frac{1}{t^3}\int_0^t dx\int_{x^2}^{t^2}\arctan\left[\cos\left(3x+5\sqrt{y}\right)\right]dy$

① a、



b、 $\boxed{\iint\limits_D d\sigma=\frac{2t^3}{3}}$

② 设区域D上存在一点(ξ,η)，

可使$\int_0^t dx\int_{x^2}^{t^2}\arctan\left[\cos\left(3x+5\sqrt{y}\right)\right]dy=f(\xi,\eta)\cdot\frac{2t^3}{3}$

$$=f(0,0)\cdot\frac{2t^3}{3}$$

$$=1\cdot\arctan\left[\cos\left(3\cdot0+5\sqrt{0}\right)\right]\cdot\frac{2t^3}{3}$$

$$=\arctan(\cos0)\cdot\frac{2t^3}{3}$$

$$=\arctan1\cdot\frac{2t^3}{3}$$

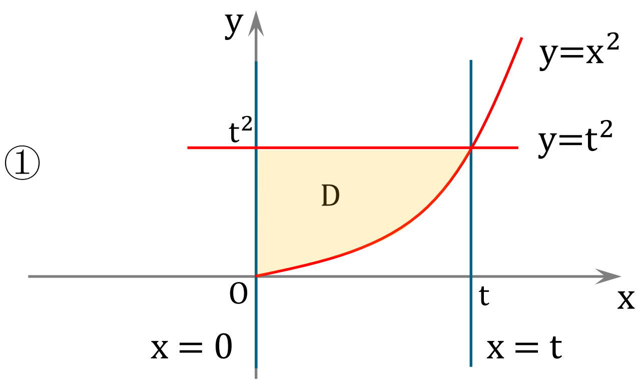
$$=\frac{\pi}{4}\cdot\frac{2t^3}{3}$$

$$=\frac{\pi}{6}t^3$$

⑤ $\lim\limits_{t\to0^+}\frac{1}{t^3}\cdot\frac{\pi}{6}t^3=\lim\limits_{t\to0^+}\frac{\pi}{6}=\frac{\pi}{6}$

计算$\iint\limits_D d\sigma$ ，其中 D 为如下区域

①



② 确定区域内 x 的取值范围 [0,t]

③ 确定区域的下边界 $y=x^2$与上边界 $y=t^2$

④ $\iint\limits_D f(x,y)\,d\sigma=\int_a^b dx\int_{y_{下}(x)}^{y_{上}(x)}f(x,y)\,dy$

$$\iint\limits_D d\sigma=\int_0^t dx\int_{x^2}^{t^2}dy$$

$$=\int_0^t\left(\int_{x^2}^{t^2}dy\right)dx$$

$$=\int_0^t\left(y\Big|_{y=x^2}^{y=t^2}\right)dx$$

$$=\int_0^t(t^2-x^2)\,dx$$

$$=\frac{3t^2x-x^3}{3}\Big|_0^t$$

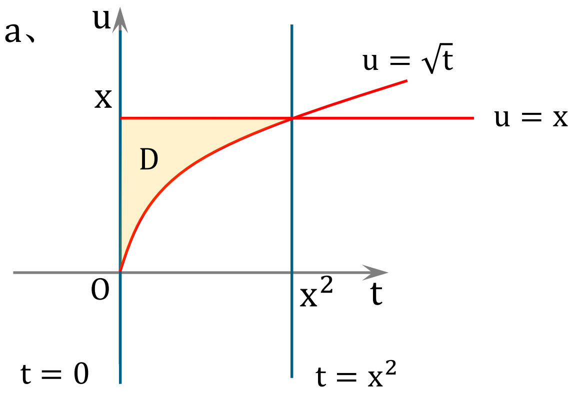
$$=\frac{3t^2\cdot t-t^3}{3}-\frac{3t^2\cdot0-0^3}{3}$$

$$=\frac{2t^3}{3}$$

例2．设f(x, y)是定义在 $0 \le x \le 1$、$0 \le y \le 1$ 上的连续函数，

$f(0,0)=-1$，求 $\lim\limits_{x\to 0^+}\dfrac{\int_0^{x^2}dt\int_x^{\sqrt{t}}f(t,u)du}{1-e^{-x^3}}$

① a、



b、 $\boxed{\iint\limits_D d\sigma = \frac{x^3}{3}}$

② 设区域D上存在一点(ξ,η)，可使$\int_0^{x^2}dt\int_x^{\sqrt{t}}f(t,u)\,du = -f(\xi,\eta)\cdot\frac{x^3}{3}$

$$= -f(0,0)\cdot\frac{x^3}{3}$$

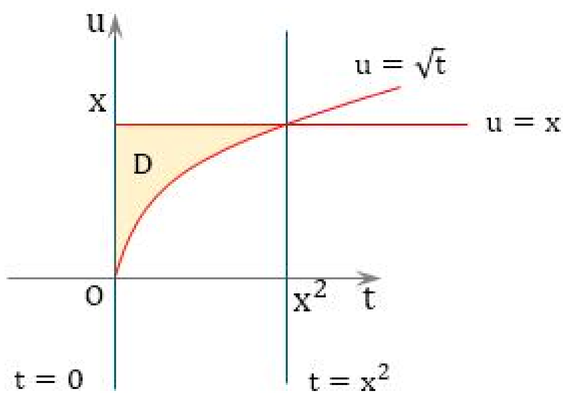
$$= -(-1)\cdot\frac{x^3}{3}$$

$$= \frac{x^3}{3}$$

⑤ $\lim\limits_{x\to 0^+}\dfrac{\frac{x^3}{3}}{1-e^{-x^3}} = \lim\limits_{x\to 0^+}\dfrac{\frac{x^3}{3}}{-\left(e^{-x^3}-1\right)} = \lim\limits_{x\to 0^+}\dfrac{-\frac{x^3}{3}}{e^{-x^3}-1} = \lim\limits_{x\to 0^+}\dfrac{-\frac{x^3}{3}}{-x^3} = \lim\limits_{x\to 0^+}\frac{1}{3} = \frac{1}{3}$

计算$\iint\limits_D d\sigma$ =，其中 D 为如下区域

①



② 确定区域内 t 的取值范围 $[0, x^2]$

③ 确定区域的下边界 $u=\sqrt{t}$ 与上边界 $u = x$

④ $\iint\limits_D f(t,u)\,d\sigma = \int_a^b dt\int_{u_{下}(t)}^{u_{上}(t)} f(t,u)\,du$

$$\iint\limits_D d\sigma = \int_0^{x^2}dt\int_{\sqrt{t}}^{x}du$$

$$= \int_0^{x^2}\left(\int_{\sqrt{t}}^{x}du\right)dt$$

$$= \int_0^{x^2}\left(u\Big|_{u=\sqrt{t}}^{u=x}\right)dt$$

$$= \int_0^{x^2}\left(x-\sqrt{t}\right)dt$$

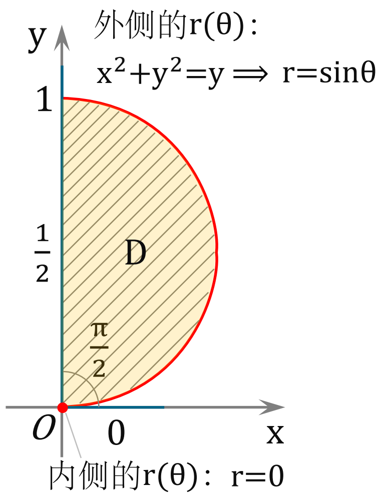
$$= \frac{3xt-2t^{\frac{3}{2}}}{3}\Bigg|_0^{x^2}$$

$$= \frac{3x\cdot x^2-2(x^2)^{\frac{3}{2}}}{3} - \frac{3x\cdot 0-2\cdot 0^{\frac{3}{2}}}{3}$$

$$= \frac{x^3}{3}$$

函数表达式含二重积分

例1. 设区域 D 由 $x^2+y^2\le y$ 和 $x\ge 0$ 所确定，f(x,y) 为 D 上的连续函数，且 $f(x,y)=\sqrt{1-x^2-y^2}-\frac{8}{\pi}\iint\limits_D f(u,v)\,dudv$，求 f(x,y)



① 设 $\iint\limits_D f(x,y)\,dxdy=\iint\limits_D f(u,v)\,dudv=S$

则 $\quad f(x,y)=\sqrt{1-x^2-y^2}-\frac{8}{\pi}S$

② $\iint\limits_D f(x,y)\,dxdy=\iint\limits_D\left(\sqrt{1-x^2-y^2}-\frac{8}{\pi}S\right)dxdy$

$$\Longrightarrow S=\iint\limits_D\sqrt{1-x^2-y^2}\,dxdy-\iint\limits_D\frac{8}{\pi}S\,dxdy$$

$$\Longrightarrow S=\iint\limits_D\sqrt{1-x^2-y^2}\,dxdy-\frac{8}{\pi}S\cdot\iint\limits_D 1\,dxdy$$

$$\Longrightarrow S=\iint\limits_D\sqrt{1-x^2-y^2}\,dxdy-\frac{8}{\pi}S\cdot\frac{\pi}{8}$$

$$\Longrightarrow S=\iint\limits_D\sqrt{1-x^2-y^2}\,dxdy-S$$

$$\Longrightarrow 2S=\iint\limits_D\sqrt{1-x^2-y^2}\,dxdy$$

$$\Longrightarrow S=\frac{1}{2}\cdot\iint\limits_D\sqrt{1-x^2-y^2}\,dxdy$$

$$S=\frac{1}{2}\cdot\iint\limits_D\sqrt{1-x^2-y^2}\,dxdy$$

$$=\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}d\theta\int_0^{\sin\theta}\sqrt{1-(r\cos\theta)^2-(r\sin\theta)^2}\cdot r\,dr$$

【详见强化高数下·二重积分·第五课↓】

∵ 积分区域为圆润图形

∴ 可用极坐标方法来求，即

$$\iint\limits_D f(x,y)\,dxdy=\int_?^? d\theta\int_{\text{内侧的}r(\theta)}^{\text{外侧的}r(\theta)}f(r\cos\theta,r\sin\theta)\,r\,dr$$

∵ 本题中，内外侧r(θ)如右上图

$$\therefore \iint\limits_D\sqrt{1-x^2-y^2}\,dxdy$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{\sin\theta}\sqrt{1-(r\cos\theta)^2-(r\sin\theta)^2}\cdot r\,dr$$

$$=\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}d\theta\int_0^{\sin\theta}\sqrt{1-r^2}\cdot r\,dr$$

$$=\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}\left(\int_0^{\sin\theta}\sqrt{1-r^2}\cdot r\,dr\right)d\theta$$

$$=\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}\left[\left.\frac{(1-r^2)^{\frac{3}{2}}}{-3}\right|_0^{\sin\theta}\right]d\theta$$

$$=\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}\left(\frac{1}{3}-\frac{\cos^3\theta}{3}\right)d\theta$$

$$=\frac{1}{2}\cdot\left[\int_0^{\frac{\pi}{2}}\frac{1}{3}\,d\theta-\int_0^{\frac{\pi}{2}}\frac{\cos^3\theta}{3}\,d\theta\right]$$

$$=\frac{1}{2}\cdot\left[\frac{\pi}{6}-\frac{1}{3}\cdot\int_0^{\frac{\pi}{2}}\cos^3\theta\,d\theta\right]$$

$$=\frac{1}{2}\cdot\left[\frac{\pi}{6}-\frac{1}{3}\int_0^{\frac{\pi}{2}}\left(\frac{1}{4}\cdot\cos3\theta+\frac{3}{4}\cdot\cos\theta\right)d\theta\right]$$

【详见高数上·一元函数积分学·第一课·例8↓】

$\because \cos\alpha\cdot\cos\beta=\frac{1}{2}\cdot[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

$\therefore \cos\theta\cdot\cos\theta=\frac{1}{2}\cdot[\cos(\theta+\theta)+\cos(\theta-\theta)]$

$\qquad=\frac{1}{2}\cdot[\cos2\theta+\cos0]=\frac{1}{2}\cos2\theta+\frac{1}{2}$

$\cos2\theta\cdot\cos\theta=\frac{1}{2}\cdot[\cos(2\theta+\theta)+\cos(2\theta-\theta)]$

$\qquad=\frac{1}{2}\cos3\theta+\frac{1}{2}\cos\theta$

$\therefore \cos^3\theta=(\cos\theta\cdot\cos\theta)\cdot\cos\theta=(\frac{1}{2}\cos2\theta+\frac{1}{2})\cdot\cos\theta$

$\qquad=\frac{1}{2}\cdot\cos2\theta\cdot\cos\theta+\frac{1}{2}\cdot\cos\theta$

$\qquad=\frac{1}{2}\cdot(\frac{1}{2}\cos3\theta+\frac{1}{2}\cos\theta)+\frac{1}{2}\cdot\cos\theta$

$\qquad=\frac{1}{4}\cdot\cos3\theta+\frac{3}{4}\cdot\cos\theta$

$$=\frac{1}{2}\cdot\left[\frac{\pi}{6}-\frac{1}{3}\cdot\left.\left(\frac{1}{12}\cdot\sin3\theta+\frac{3}{4}\cdot\sin\theta\right)\right|_0^{\frac{\pi}{2}}\right]$$

$$=\frac{1}{2}\cdot\left(\frac{\pi}{6}-\frac{2}{9}\right)$$

$$=\frac{3\pi-4}{36}$$

③ $f(x,y)=\sqrt{1-x^2-y^2}-\frac{8}{\pi}\cdot\frac{3\pi-4}{36}$

$\qquad=\sqrt{1-x^2-y^2}-\frac{6\pi-8}{9\pi}$