笔记前言：

本笔记的内容是去掉步骤的概述后，视频的所有内容。

本猴觉得，自己的步骤概述写的太啰嗦，大家自己做笔记时，

应该每个人都有自己的最舒服最简练的写法，所以没给大家写。

再是本猴觉得，不给大家写这个概述的话，大家会记忆的更深，

掌握的更好！

所以老铁！一定要过呀！不要辜负本猴的心意！~~~

【祝逢考必过，心想事成~~~~】

【一定能过！！！！！】

计算重极限

例1. 求 $\lim\limits_{(x,y)\to(0,1)}\frac{x+y}{x^2+y^2}$

原式 $=\frac{0+1}{0^2+1^2}$

$=\frac{1}{1}$

$=1$

例2. 求 $\lim\limits_{\substack{x\to0\\y\to0}}\sqrt{xy+4}+2$

原式 $=\sqrt{0\cdot0+4}+2$

$=\sqrt{4}+2$

$=4$

例3. 求 $\lim\limits_{(x,y)\to(0,0)}\frac{xy}{\sqrt{xy+4}-2}$（直接代入后是 $\frac{0}{0}$ 型）

设 xy=r　$(x,y)\to(0,0)\implies r=xy\to0\cdot0=0$（后边的题同理）

$\lim\limits_{r\to0}\frac{r}{\sqrt{r+4}-2}$　$\frac{0}{0}$ 型，分子分母同乘 $\left(\sqrt{r+4}+2\right)$

$=\lim\limits_{r\to0}\frac{r\cdot\left(\sqrt{r+4}+2\right)}{\left(\sqrt{r+4}-2\right)\cdot\left(\sqrt{r+4}+2\right)}$

$=\lim\limits_{r\to0}\frac{r\cdot\left(\sqrt{r+4}+2\right)}{r+4-4}$

$=\lim\limits_{r\to0}\frac{r\cdot\left(\sqrt{r+4}+2\right)}{r}$

$=\lim\limits_{r\to0}\left(\sqrt{r+4}+2\right)$

$=\sqrt{0+4}+2$

$=4$

三种解题方法:

① 设xy=r
(多为式子中的 x 和 y 均为 xy 时)

② 将 lim 后的式子拆分成 C·0
(多为式子中包含 $\frac{C_1\cdot(x^2)^a+C_2\cdot(y^2)^a}{(x^2+y^2)^a}$ 时)

③ 设 $\begin{cases}x=x_0+r\cos\theta\\y=y_0+r\sin\theta\end{cases}$
(多为式子中的 x 和 y 均为 x^2+y^2 时)

例4. 求 $\lim\limits_{(x,y)\to(0,1)}\frac{\sin(xy)}{xy}$

设 xy=r

$\lim\limits_{r\to0}\frac{\sin r}{r}$　$\frac{0}{0}$ 型，$r\to0$时，$\sin r\to r$

$=\lim\limits_{r\to0}\frac{r}{r}$

$=\lim\limits_{r\to0}1$

$=1$

例5. 求 $\lim\limits_{(x,y)\to(0,1)} \dfrac{\sin(xy)}{x}$

原式 $= \lim\limits_{(x,y)\to(0,1)} \left[\dfrac{\sin(xy)}{xy} \cdot y\right]$

$= \boxed{\lim\limits_{(x,y)\to(0,1)} \dfrac{\sin(xy)}{xy}} \cdot \lim\limits_{(x,y)\to(0,1)} y$ 例4算过了

$= \quad 1 \quad \cdot \quad 1$

$= 1$

例6. 求 $\lim\limits_{\substack{x\to0\\y\to0}} \dfrac{x^2y}{x^2+y^2}$

$\because \substack{x\to0\\y\to0}$ 时，

$x^2>0$，$y^2>0$

$\therefore x^2+y^2>x^2>0$

$\therefore 0<\dfrac{x^2}{x^2+y^2}<1$

原式 $= \lim\limits_{\substack{x\to0\\y\to0}} \left(\dfrac{x^2}{x^2+y^2} \cdot y\right)$

$\quad C \cdot 0 = 0$

$= 0$

例7. 求 $\lim\limits_{\substack{x\to0\\y\to0}} \dfrac{(|x|+|y|)\cdot y^3}{\sqrt{x^2+y^2}}$

$\dfrac{|x|+|y|}{\sqrt{x^2+y^2}} = \dfrac{|x|}{\sqrt{x^2+y^2}} + \dfrac{|y|}{\sqrt{x^2+y^2}}$

$\because \substack{x\to0\\y\to0}$ 时，

$x^2>0$，$y^2>0$

$\therefore x^2+y^2>x^2>0$

$\therefore 0<\dfrac{x^2}{x^2+y^2}<1$ $\qquad \therefore 0<\dfrac{y^2}{x^2+y^2}<1$

$\therefore \sqrt{0}<\sqrt{\dfrac{x^2}{x^2+y^2}}<\sqrt{1}$ $\qquad \therefore \sqrt{0}<\sqrt{\dfrac{y^2}{x^2+y^2}}<\sqrt{1}$

即 $0<\dfrac{|x|}{\sqrt{x^2+y^2}}<1$ $\qquad$ 即 $0<\dfrac{|y|}{\sqrt{x^2+y^2}}<1$

$\therefore 0<\dfrac{|x|}{\sqrt{x^2+y^2}}+\dfrac{|y|}{\sqrt{x^2+y^2}}<2$

即 $0<\dfrac{|x|+|y|}{\sqrt{x^2+y^2}}<2$

原式 $= \lim\limits_{\substack{x\to0\\y\to0}} \left(\dfrac{|x|+|y|}{\sqrt{x^2+y^2}} \cdot y^3\right)$

$\quad C \cdot 0 = 0$

$= 0$

例8. 已知 $\varphi(0,0)=0$，求 $\lim\limits_{\substack{x\to0\\y\to0}}\frac{|x-y|\cdot\varphi(x,y)}{\sqrt{x^2+y^2}}$

$|x\pm y|\le|x|+|y|$

$$\text{原式}=\lim_{\substack{x\to0\\y\to0}}\left[\underbrace{\frac{|x-y|}{\sqrt{x^2+y^2}}}_{C}\cdot\underbrace{\varphi(x,y)}_{0}\right]$$

$C\cdot0=0$

$=0$

$\because \substack{x\to0\\y\to0}$ 时，

$|x-y|\ge0$，$\sqrt{x^2+y^2}>0$

$\therefore \frac{|x-y|}{\sqrt{x^2+y^2}}\ge0$

$\because |x-y|\le|x|+|y|$

$\therefore \frac{|x-y|}{\sqrt{x^2+y^2}}\le\boxed{\frac{|x|+|y|}{\sqrt{x^2+y^2}}}<2 \longrightarrow$ 例7分析过

$\therefore 0\le\frac{|x-y|}{\sqrt{x^2+y^2}}<2$

例9. 求 $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)\cdot\ln(x^2+y^2)$

设 $\begin{cases}x=0+r\cos\theta\\y=0+r\sin\theta\end{cases}\Rightarrow\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}\Rightarrow x^2+y^2=r^2$

$\lim\limits_{r\to0}r^2\cdot\ln r^2 \longrightarrow$ $\boxed{0\cdot\infty\text{ 型，将 }r^2\text{ 变为 }\frac{1}{\frac{1}{r^2}}}$

$=\lim\limits_{r\to0}\frac{\ln r^2}{\frac{1}{r^2}} \longrightarrow$ $\boxed{\frac{\infty}{\infty}\text{ 型，利用洛必达法则来做}}$

$=\lim\limits_{r\to0}\frac{(\ln r^2)'}{\left(\frac{1}{r^2}\right)'}$

$=\lim\limits_{r\to0}\frac{\frac{2r}{r^2}}{\frac{-2}{r^3}}$

$=\lim\limits_{r\to0}(-r^2)$

$=-0^2$

$=0$

证明重极限不存在

例1：请证明 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4}$ 不存在

① 找出一些线 y =?x (或 x=?y)，使其经过 (0 , 0)

如 $y = x$，$y = \sqrt{x}$，~~$y = 0$，$x = 0$~~

② 当 $y = x$ 时，原式= $\lim\limits_{\substack{x\to 0\\ x\to 0}} \frac{x\cdot x^2}{x^2+x^4} = \lim\limits_{x\to 0} \frac{x\cdot x^2}{x^2+x^4}$

$$= \lim_{x\to 0} \frac{x}{1+x^2}$$

$$= \frac{0}{1+0^2}$$

$$= 0$$

当 $y = \sqrt{x}$ 时，原式= $\lim\limits_{\substack{x\to 0\\ \sqrt{x}\to 0}} \frac{x(\sqrt{x})^2}{x^2+(\sqrt{x})^4} = \lim\limits_{\substack{x\to 0\\ x\to 0}} \frac{x(\sqrt{x})^2}{x^2+(\sqrt{x})^4}$

$$= \lim_{x\to 0} \frac{x(\sqrt{x})^2}{x^2+(\sqrt{x})^4}$$

$$= \lim_{x\to 0} \frac{x\cdot x}{x^2+x^2}$$

$$= \lim_{x\to 0} \frac{x^2}{2x^2}$$

$$= \lim_{x\to 0} \frac{1}{2}$$

$$= \frac{1}{2}$$

~~当 $y = 0$ 时，原式= $\lim\limits_{\substack{x\to 0\\ 0\to 0}} \frac{x\cdot 0^2}{x^2+0^4} = \lim\limits_{x\to 0} \frac{x\cdot 0^2}{x^2+0^4}$~~

~~$= \lim\limits_{x\to 0} \frac{0}{x^2}$~~

~~$= \lim\limits_{x\to 0} 0$~~

~~$= 0$~~

划掉的地方其实不用写，只是保留下来给大家做个提示

~~当 $x = 0$ 时，原式= $\lim\limits_{\substack{0\to 0\\ y\to 0}} \frac{0\cdot y^2}{0^2+y^4} = \lim\limits_{y\to 0} \frac{0\cdot y^2}{0^2+y^4}$~~

~~$= \lim\limits_{y\to 0} \frac{0}{y^2}$~~

~~$= \lim\limits_{y\to 0} 0$~~

~~$= 0$~~

③ $\because 0 \neq \frac{1}{2}$

$\therefore$ 重极限 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4}$ 不存在

例2：请证明 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2\cdot \sin\frac{1}{x^2+y^2}+y^2}{x^2+y^2}$ 不存在

① 找出一些线 y =?x (或 x=?y)，使其经过 (0, 0)

如 $y=0$，$y=x$，$x=0$

② 当 $y=0$ 时，原式$=\lim\limits_{\substack{x\to 0\\ 0\to 0}} \dfrac{x^2\cdot \sin\frac{1}{x^2+0^2}+0^2}{x^2+0^2}$

$$=\lim_{x\to 0} \frac{x^2\cdot \sin\frac{1}{x^2+0^2}+0^2}{x^2+0^2}$$

$$=\lim_{x\to 0} \frac{x^2\cdot \sin\frac{1}{x^2}}{x^2}$$

$$=\lim_{x\to 0} \sin\frac{1}{x^2}$$

$$=\sin\frac{1}{0^2}$$

$=\sin\infty$　极限不存在

③ $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2\cdot \sin\frac{1}{x^2+y^2}+y^2}{x^2+y^2}$ 不存在

求偏导 (简单情况)

例1 . 设 $u=\sqrt{x^2+y^2}$ ，$v=\frac{x}{\sqrt{x^2+y^2}}$ ，$w=\frac{y}{\sqrt{x^2+y^2}}$ 求 $\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial y}$，$\frac{\partial v}{\partial x}$，$\frac{\partial v}{\partial y}$，$\frac{\partial w}{\partial y}$

$$\frac{\partial u}{\partial x}=\boxed{\frac{\partial(\sqrt{x^2+y^2})}{\partial x}}=\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y}=\boxed{\frac{\partial(\sqrt{x^2+y^2})}{\partial y}}=\frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial x}=\boxed{\frac{\partial(\frac{x}{\sqrt{x^2+y^2}})}{\partial x}}=\frac{y^2}{\sqrt{(x^2+y^2)^3}}$$

$$\frac{\partial v}{\partial y}=\boxed{\frac{\partial(\frac{x}{\sqrt{x^2+y^2}})}{\partial y}}=-\frac{xy}{\sqrt{(x^2+y^2)^3}}$$

$$\frac{\partial w}{\partial y}=\boxed{\frac{\partial(\frac{y}{\sqrt{x^2+y^2}})}{\partial y}}=\frac{x^2}{\sqrt{(x^2+y^2)^3}}$$

$$\begin{aligned}\left(\sqrt{t^2+2^2}\right)'&=\frac{1}{2}(t^2+2^2)^{(\frac{1}{2}-1)}\cdot(t^2+2^2)'\\&=\frac{1}{2}(t^2+2^2)^{(-\frac{1}{2})}\cdot 2t\\&=(t^2+2^2)^{(-\frac{1}{2})}\cdot t\\&=\frac{t}{\sqrt{t^2+2^2}}\end{aligned}$$

(提示)

$$\begin{aligned}\left(\sqrt{2^2+y^2}\right)'&=\frac{1}{2}(2^2+y^2)^{(\frac{1}{2}-1)}\cdot(2^2+y^2)'\\&=\frac{1}{2}(2^2+y^2)^{(-\frac{1}{2})}\cdot 2y\\&=(2^2+y^2)^{(-\frac{1}{2})}\cdot y\\&=\frac{y}{\sqrt{2^2+y^2}}\end{aligned}$$

(提示)

$$\begin{aligned}\left(\frac{t}{\sqrt{t^2+2^2}}\right)'&=\frac{t'\cdot\sqrt{t^2+2^2}-t\cdot(\sqrt{t^2+2^2})'}{(\sqrt{t^2+2^2})^2}\\&=\frac{1\cdot\sqrt{t^2+2^2}-t\cdot\frac{1}{2}(t^2+2^2)^{(\frac{1}{2}-1)}\cdot(t^2+2^2)'}{t^2+2^2}\\&=\frac{\sqrt{t^2+2^2}-t\cdot\frac{1}{2}(t^2+2^2)^{(\frac{1}{2}-1)}\cdot 2t}{t^2+2^2}\\&=\frac{(t^2+2^2)^{(-\frac{1}{2})}\cdot[(t^2+2^2)^1-t^2]}{t^2+2^2}\\&=\frac{2^2}{(t^2+2^2)^{\frac{1}{2}}\cdot(t^2+2^2)}\\&=\frac{2^2}{\sqrt{(t^2+2^2)^3}}\end{aligned}$$

(提示)

$$\begin{aligned}\left(\frac{2}{\sqrt{2^2+t^2}}\right)'&=-\frac{1}{2}\cdot 2\cdot(2^2+t^2)^{(-\frac{1}{2}-1)}\cdot(2^2+t^2)'\\&=-\frac{1}{2}\cdot 2\cdot(2^2+t^2)^{(-\frac{3}{2})}\cdot 2t\\&=-2t(2^2+t^2)^{(-\frac{3}{2})}\\&=-\frac{2t}{\sqrt{(2^2+t^2)^3}}\end{aligned}$$

(提示)

$$\begin{aligned}\left(\frac{t}{\sqrt{2^2+t^2}}\right)'&=\frac{t'\cdot\sqrt{2^2+t^2}-t\cdot(\sqrt{2^2+t^2})'}{(\sqrt{2^2+t^2})^2}\\&=\frac{1\cdot\sqrt{2^2+t^2}-t\cdot\frac{1}{2}(2^2+t^2)^{(\frac{1}{2}-1)}\cdot(2^2+t^2)'}{2^2+t^2}\\&=\frac{\sqrt{2^2+t^2}-t\cdot\frac{1}{2}(2^2+t^2)^{(\frac{1}{2}-1)}\cdot 2t}{2^2+t^2}\\&=\frac{(2^2+t^2)^{(-\frac{1}{2})}\cdot[(2^2+t^2)^1-t^2]}{2^2+t^2}\\&=\frac{2^2}{(2^2+t^2)^{\frac{1}{2}}\cdot(2^2+t^2)}\\&=\frac{2^2}{\sqrt{(2^2+t^2)^3}}\end{aligned}$$

(提示)

例2 . $z=x^3y^2+3xy$，且z具有二阶连续偏导，求 $\frac{\partial^2 z}{\partial x\partial y}$，$\frac{\partial^2 z}{\partial y\partial x}$

若一个多元函数 $z=f(x,y)$ 具有二阶连续偏导，则 $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial(\frac{\partial z}{\partial x})}{\partial y}=\frac{\partial(3x^2y^2+3y)}{\partial y}=6x^2y+3$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=6x^2y+3$$

例3 . 已知 $f(u,v)$ 满足$f[\,xg(y),y\,]=x+g(y)$，其中 $g(y)$ 可导且 $g(y)\neq 0$ 求 $\frac{\partial^2 f}{\partial u\partial v}$

$$\frac{\partial^2 f}{\partial u\partial v}=\frac{\partial(\frac{\partial f}{\partial u})}{\partial v}$$

$$\text{令}\left.\begin{array}{l} u=xg(y) \\ v=y \end{array}\right\}\begin{array}{l} x=\frac{u}{g(y)}=\frac{u}{g(v)} \\ g(y)=g(v) \end{array}\Rightarrow x+g(y)=\frac{u}{g(v)}+g(v)$$

可知 $f(u,v)=x+g(y)$

$$\frac{\partial f}{\partial u}=\frac{\partial[x+g(y)]}{\partial u}=\frac{\partial[\frac{u}{g(v)}+g(v)]}{\partial u}=\frac{1}{g(v)}$$

$$\frac{\partial^2 f}{\partial u\partial v}=\frac{d[\frac{1}{g(v)}]}{dv}=[\frac{1}{g(v)}]'$$

$$=\frac{(1)'g(v)-1\cdot g'(v)}{g^2(v)}$$

$$=-\frac{g'(v)}{g^2(v)}$$

例4 . $z=f(x)\cdot 3x^2y+g(x)\cdot(5x+y)$，求 $\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x}=\frac{\partial[\,f(x)\cdot 3x^2y+g(x)\cdot(5x+y)\,]}{\partial x}$$

$$=\frac{\partial[f(x)\cdot 3x^2y\,]}{\partial x}+\frac{\partial[g(x)\cdot(5x+y)]}{\partial x}$$

$$=\frac{d[f(x)\,]}{dx}\cdot 3x^2y+f(x)\cdot\frac{\partial(3x^2y)}{\partial x}+\frac{d[g(x)]}{dx}\cdot(5x+y)+g(x)\cdot\frac{\partial[(5x+y)]}{\partial x}$$

$$=f'(x)\cdot 3x^2y+f(x)\cdot 6xy+g'(x)\cdot(5x+y)+g(x)\cdot 5$$

$$=f'(x)\cdot 3x^2y+f(x)\cdot 6xy+g'(x)\cdot(5x+y)+5g(x)$$

$$\begin{cases} \text{公式①}\ (A+B+C)'=A'+B'+C' \\ \text{公式②}\ (A\cdot B)'=A'B+AB' \\ \text{公式③}\ \left(\frac{A}{B}\right)'=\frac{A'B-AB'}{B^2} \end{cases}$$

$$\Longrightarrow\begin{cases} \text{公式①}\ \frac{\partial(A+B+C)}{\partial D}=\frac{\partial A}{\partial D}+\frac{\partial B}{\partial D}+\frac{\partial C}{\partial D} \\ \text{公式②}\ \frac{\partial(A\cdot B)}{\partial D}=\frac{\partial A}{\partial D}\cdot B+A\cdot\frac{\partial B}{\partial D} \\ \text{公式③}\ \frac{\partial(\frac{A}{B})}{\partial D}=\frac{\frac{\partial A}{\partial D}\cdot B-A\cdot\frac{\partial B}{\partial D}}{B^2} \end{cases}$$

例5 . $z=\frac{3x^2y}{5x+y}$，求 $\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x}=\frac{\partial(\frac{3x^2y}{5x+y})}{\partial x}$$

$$=\frac{\frac{\partial(3x^2y)}{\partial x}\cdot(5x+y)-3x^2y\cdot\frac{\partial(5x+y)}{\partial x}}{(5x+y)^2}$$

$$=\frac{6xy\cdot(5x+y)-3x^2y\cdot 5}{(5x+y)^2}$$

$$=\frac{30x^2y+6xy^2-15x^2y}{(5x+y)^2}$$

$$=\frac{15x^2y+6xy^2}{(5x+y)^2}$$

$$\begin{cases} \text{公式①}\ (A+B+C)'=A'+B'+C' \\ \text{公式②}\ (A\cdot B)'=A'B+AB' \\ \text{公式③}\ \left(\frac{A}{B}\right)'=\frac{A'B-AB'}{B^2} \end{cases}$$

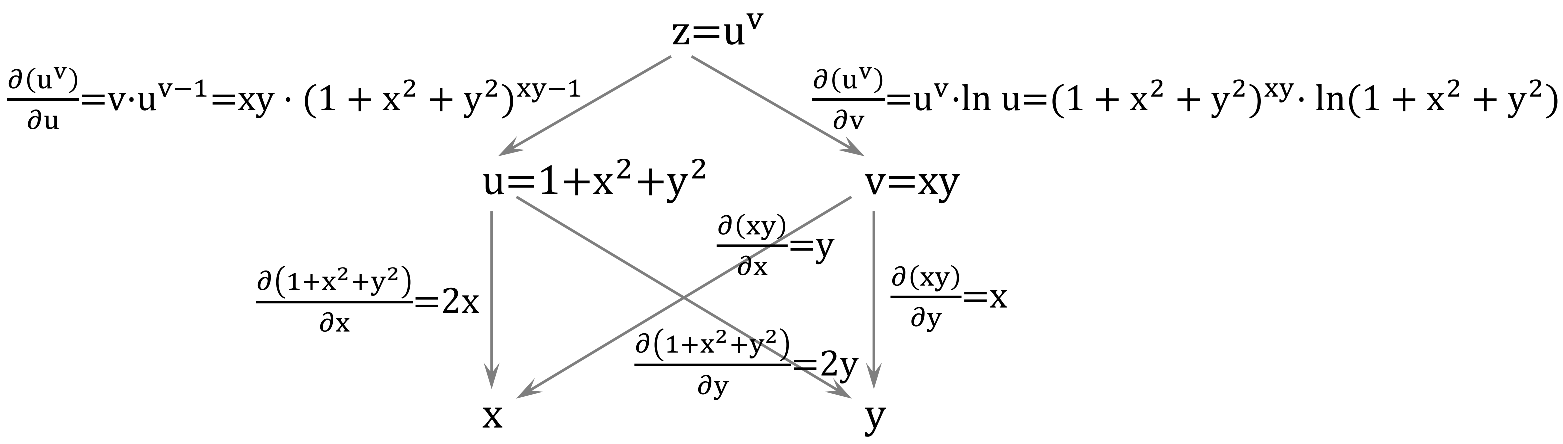
$$\Longrightarrow\begin{cases} \text{公式①}\ \frac{\partial(A+B+C)}{\partial D}=\frac{\partial A}{\partial D}+\frac{\partial B}{\partial D}+\frac{\partial C}{\partial D} \\ \text{公式②}\ \frac{\partial(A\cdot B)}{\partial D}=\frac{\partial A}{\partial D}\cdot B+A\cdot\frac{\partial B}{\partial D} \\ \text{公式③}\ \frac{\partial(\frac{A}{B})}{\partial D}=\frac{\frac{\partial A}{\partial D}\cdot B-A\cdot\frac{\partial B}{\partial D}}{B^2} \end{cases}$$

求偏导（复杂情况）

例1. 已知 $z=f(u,v)=u^v$，$u=1+x^2+y^2$，$v=xy$，求 $\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$

①

②



③ $\frac{\partial z}{\partial x}=xy\cdot(1+x^2+y^2)^{xy-1}\cdot 2x+(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)\cdot y$

$=2x^2y\cdot(1+x^2+y^2)^{xy-1}+y\cdot(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)$

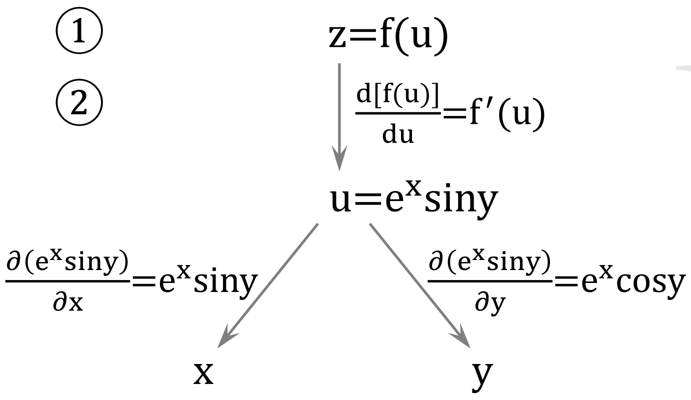
$\frac{\partial z}{\partial y}=xy\cdot(1+x^2+y^2)^{xy-1}\cdot 2y+(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)\cdot x$

$=2xy^2\cdot(1+x^2+y^2)^{xy-1}+x\cdot(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)$

例2. 已知 $z=f(u)$，$f(u)$具有一阶导数，$u=e^x\sin y$，求 $\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$

①

②



③ $\frac{\partial z}{\partial x}=f'(u)\cdot e^x\sin y$

$\frac{\partial z}{\partial y}=f'(u)\cdot e^x\cos y$

例3. 已知 z=f(u)，f(u)具有二阶导数，$u=\sqrt{x^2+y^2}$，求 $\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$、$\frac{\partial^2 z}{\partial x^2}$、$\frac{\partial^2 z}{\partial y^2}$、$\frac{\partial^2 z}{\partial x\partial y}$

①　　z=f(u)

②　　$\downarrow \frac{d[f(u)]}{du}=f'(u)$

$u=\sqrt{x^2+y^2}$

$\frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$　　$\frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$

x　　y

③ $\frac{\partial z}{\partial x}=f'(u)\cdot\frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial z}{\partial y}=f'(u)\cdot\frac{y}{\sqrt{x^2+y^2}}$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x}$$
$$=\frac{\partial\left[f'(u)\cdot\frac{x}{\sqrt{x^2+y^2}}\right]}{\partial x}$$
$$=\frac{\partial[f'(u)]}{\partial x}\cdot\frac{x}{\sqrt{x^2+y^2}}+f'(u)\cdot\frac{\partial\left[\frac{x}{\sqrt{x^2+y^2}}\right]}{\partial x}$$
$$=\boxed{\frac{\partial[f'(u)]}{\partial x}}\cdot\frac{x}{\sqrt{x^2+y^2}}+f'(u)\cdot\frac{y^2}{\sqrt{(x^2+y^2)^3}}$$
$$=f''(u)\cdot\frac{x}{\sqrt{x^2+y^2}}\cdot\frac{x}{\sqrt{x^2+y^2}}+f'(u)\cdot\frac{y^2}{\sqrt{(x^2+y^2)^3}}$$
$$=f''(u)\cdot\frac{x^2}{x^2+y^2}+f'(u)\cdot\frac{y^2}{\sqrt{(x^2+y^2)^3}}$$

①　　f′(u)=?u

②　　$\downarrow \frac{d[f'(u)]}{du}=f''(u)$

$u=\sqrt{x^2+y^2}$

$\frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$　　$\frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$

x　　y

③ $\frac{\partial[f'(u)]}{\partial x}=f''(u)\cdot\frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial[f'(u)]}{\partial y}=f''(u)\cdot\frac{y}{\sqrt{x^2+y^2}}$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial y}$$
$$=\frac{\partial\left[f'(u)\cdot\frac{y}{\sqrt{x^2+y^2}}\right]}{\partial y}$$
$$=\frac{\partial[f'(u)]}{\partial y}\cdot\frac{y}{\sqrt{x^2+y^2}}+f'(u)\cdot\frac{\partial\left[\frac{y}{\sqrt{x^2+y^2}}\right]}{\partial y}$$
$$=\boxed{\frac{\partial[f'(u)]}{\partial y}}\cdot\frac{y}{\sqrt{x^2+y^2}}+f'(u)\cdot\frac{x^2}{\sqrt{(x^2+y^2)^3}}$$
$$=f''(u)\cdot\frac{y}{\sqrt{x^2+y^2}}\cdot\frac{y}{\sqrt{x^2+y^2}}+f'(u)\cdot\frac{x^2}{\sqrt{(x^2+y^2)^3}}$$
$$=f''(u)\cdot\frac{y^2}{x^2+y^2}+f'(u)\cdot\frac{x^2}{\sqrt{(x^2+y^2)^3}}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y}$$
$$=\frac{\partial\left[f'(u)\cdot\frac{x}{\sqrt{x^2+y^2}}\right]}{\partial y}$$
$$=\boxed{\frac{\partial[f'(u)]}{\partial y}}\cdot\frac{x}{\sqrt{x^2+y^2}}+f'(u)\cdot\frac{\partial\left[\frac{x}{\sqrt{x^2+y^2}}\right]}{\partial y}$$
$$=f''(u)\cdot\frac{y}{\sqrt{x^2+y^2}}\cdot\frac{x}{\sqrt{x^2+y^2}}+f'(u)\cdot\frac{\partial\left[\frac{x}{\sqrt{x^2+y^2}}\right]}{\partial y}$$
$$=f''(u)\cdot\frac{y}{\sqrt{x^2+y^2}}\cdot\frac{x}{\sqrt{x^2+y^2}}+f'(u)\cdot\left(-\frac{xy}{\sqrt{(x^2+y^2)^3}}\right)$$
$$=f''(u)\cdot\frac{xy}{x^2+y^2}-f'(u)\cdot\frac{xy}{\sqrt{(x^2+y^2)^3}}$$

例4. 设 $z=\frac{1}{x}\cdot f(xy)+y\cdot\varphi(x+y)$，f、φ 具有二阶连续导数，试求 $\frac{\partial z}{\partial x}$

设 $u=xy$，$v=x+y$　　则 $z=\frac{1}{x}\cdot f(u)+y\cdot\varphi(v)$

$$
\begin{aligned}
\frac{\partial z}{\partial x}&=\frac{\partial\left[\frac{1}{x}\cdot f(u)+y\cdot\varphi(v)\right]}{\partial x}\\
&=\frac{\partial\left[\frac{1}{x}\cdot f(u)\right]}{\partial x}+\frac{\partial[y\cdot\varphi(v)]}{\partial x}\\
&=\frac{d\left(\frac{1}{x}\right)}{dx}\cdot f(u)+\frac{1}{x}\cdot\frac{\partial[f(u)]}{\partial x}+\frac{\partial y}{\partial x}\cdot\varphi(v)+y\cdot\frac{\partial[\varphi(v)]}{\partial x}\\
&=-\frac{1}{x^2}\cdot f(u)+\frac{1}{x}\cdot\boxed{\frac{\partial[f(u)]}{\partial x}}+0\cdot\varphi(v)+y\cdot\frac{\partial[\varphi(v)]}{\partial x}\\
&=-\frac{1}{x^2}\cdot f(u)+\frac{1}{x}\cdot f'(u)\cdot y+0\cdot\varphi(v)+y\cdot\boxed{\frac{\partial[\varphi(v)]}{\partial x}}\\
&=-\frac{1}{x^2}\cdot f(u)+\frac{1}{x}\cdot f'(u)\cdot y+0\cdot\varphi(v)+y\cdot\varphi'(v)\\
&=-\frac{f(u)}{x^2}+\frac{y}{x}\cdot f'(u)+y\cdot\varphi'(v)\\
&=-\frac{f(xy)}{x^2}+\frac{y}{x}\cdot f'(xy)+y\cdot\varphi'(x+y)
\end{aligned}
$$

① $f(u)=?u$

② $\frac{d[f(u)]}{du}=f'(u)$

$u=xy$

$\frac{\partial(xy)}{\partial x}=y$　x　　$\frac{\partial(xy)}{\partial y}=x$　y

③ $\frac{\partial[f(u)]}{\partial x}=f'(u)\cdot y$

① $\varphi(v)=?v$

② $\frac{d[\varphi(v)]}{dv}=\varphi'(v)$

$v=x+y$

$\frac{\partial(x+y)}{\partial x}=1$　x　　$\frac{\partial(x+y)}{\partial y}=1$　y

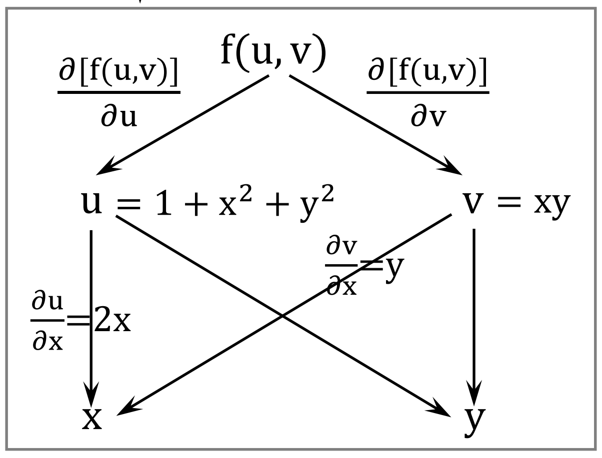
③ $\frac{\partial[\varphi(v)]}{\partial x}=\varphi'(v)\cdot 1=\varphi'(v)$

用 f' 表示部分偏导

例1．$f(u,v)$有一阶偏导数，$z=f(1+x^2+y^2,xy)$，求$\frac{\partial z}{\partial x}$、$\frac{\partial^2 z}{\partial x\partial y}$

设 $u=1+x^2+y^2$，$v=xy$ $\Longrightarrow z=f(u,v)$

$$\frac{\partial z}{\partial x}=\boxed{\frac{\partial[f(u,v)]}{\partial x}}=\frac{\partial[f(u,v)]}{\partial u}\cdot 2x+\frac{\partial[f(u,v)]}{\partial v}\cdot y$$



$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial(\frac{\partial z}{\partial x})}{\partial y}$$

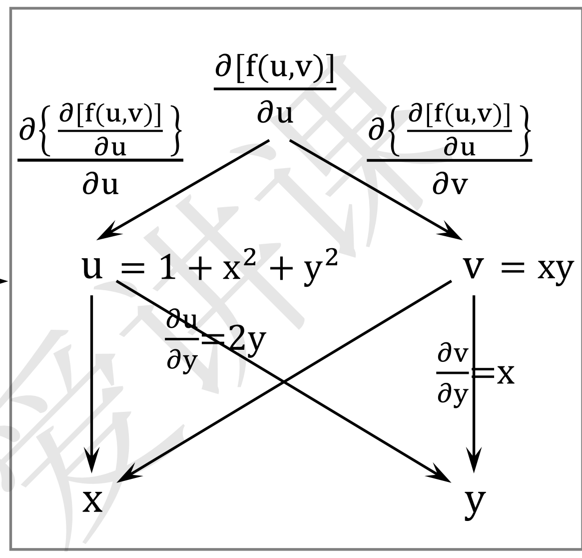
$$=\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial u}\cdot 2x+\frac{\partial[f(u,v)]}{\partial v}\cdot y\right\}}{\partial y}$$

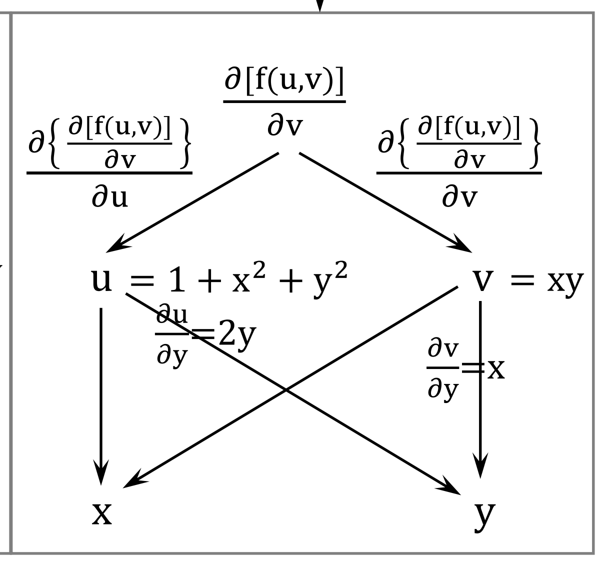
$$=\boxed{\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial u}\cdot 2x\right\}}{\partial y}}+\boxed{\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial v}\cdot y\right\}}{\partial y}}$$

$$=\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial u}\right\}}{\partial y}\cdot 2x+\frac{\partial[f(u,v)]}{\partial u}\cdot\frac{\partial(2x)}{\partial y}+\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial v}\right\}}{\partial y}\cdot y+\frac{\partial[f(u,v)]}{\partial v}\cdot\frac{dy}{dy}$$

$$=\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial u}\right\}}{\partial y}\cdot 2x+\frac{\partial[f(u,v)]}{\partial u}\cdot 0+\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial v}\right\}}{\partial y}\cdot y+\frac{\partial[f(u,v)]}{\partial v}\cdot 1$$

$$=\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial u}\right\}}{\partial y}\cdot 2x+\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial v}\right\}}{\partial y}\cdot y+\frac{\partial[f(u,v)]}{\partial v}$$





$$=\left[\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial u}\right\}}{\partial u}\cdot 2y+\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial u}\right\}}{\partial v}\cdot x\right]\cdot 2x+\left[\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial v}\right\}}{\partial u}\cdot 2y+\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial v}\right\}}{\partial v}\cdot x\right]\cdot y+\frac{\partial[f(u,v)]}{\partial v}$$

$$=\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial u}\right\}}{\partial u}\cdot 4xy+\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial u}\right\}}{\partial v}\cdot 2x^2+\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial v}\right\}}{\partial u}\cdot 2y^2+\frac{\partial\left\{\frac{\partial[f(u,v)]}{\partial v}\right\}}{\partial v}\cdot xy+\frac{\partial[f(u,v)]}{\partial v}$$

$$=f''_{11}(u,v)\cdot 4xy+f''_{12}(u,v)\cdot 2x^2+f''_{21}(u,v)\cdot 2y^2+f''_{22}(u,v)\cdot xy+f'_2(u,v)$$

$$=f''_{11}\cdot 4xy+f''_{12}\cdot 2x^2+f''_{21}\cdot 2y^2+f''_{22}\cdot xy+f'_2$$

例2．设 $z=f\left(xy,\frac{x}{y}\right)+g\left(\frac{y}{x}\right)$，其中f具有二阶连续偏导数，g具有二阶连续导数，求 $\frac{\partial^2 z}{\partial x\partial y}$

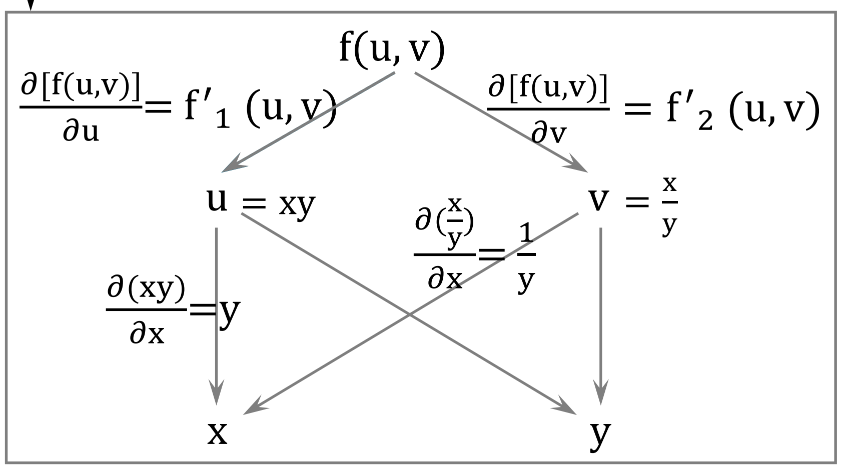
设 $u=xy$，$v=\frac{x}{y}$，$w=\frac{y}{x}\Longrightarrow z=f(u,v)+g(w)$ $\quad \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial(\frac{\partial z}{\partial x})}{\partial y}$

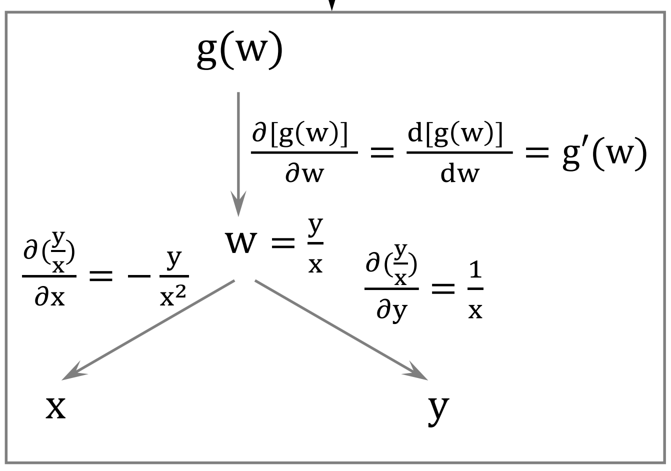
$$\frac{\partial z}{\partial x}=\frac{\partial[f(u,v)+g(w)]}{\partial x}$$

$$=\boxed{\frac{\partial[f(u,v)]}{\partial x}}+\boxed{\frac{\partial[g(w)]}{\partial x}}$$

$$=f'_1(u,v)\cdot y+f'_2(u,v)\cdot\frac{1}{y}+g'(w)\cdot(-\frac{y}{x^2})$$

$$=f'_1(u,v)\cdot y+f'_2(u,v)\cdot\frac{1}{y}-g'(w)\cdot\frac{y}{x^2}$$





$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial[\,f'_1(u,v)\cdot y+f'_2(u,v)\cdot\frac{1}{y}-g'(w)\cdot\frac{y}{x^2}]}{\partial y}$$

$$=\frac{\partial[f'_1(u,v)\cdot y]}{\partial y}+\frac{\partial[f'_2(u,v)\cdot\frac{1}{y}]}{\partial y}-\frac{\partial[\,g'(w)\cdot\frac{y}{x^2}]}{\partial y}$$

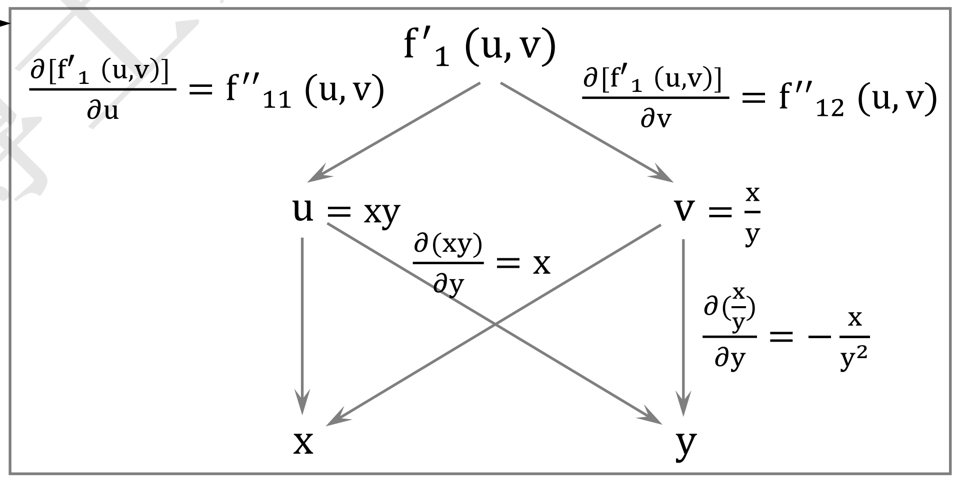
$$=\frac{\partial[f'_1(u,v)]}{\partial y}\cdot y+f'_1(u,v)\cdot\frac{dy}{dy}+\frac{\partial[f'_2(u,v)]}{\partial y}\cdot\frac{1}{y}+f'_2(u,v)\cdot\frac{d(\frac{1}{y})}{dy}-[\frac{\partial[g'(w)]}{\partial y}\cdot\frac{y}{x^2}+g'(w)\cdot\frac{\partial(\frac{y}{x^2})}{\partial y}]$$

$$=\frac{\partial[f'_1(u,v)]}{\partial y}\cdot y+f'_1(u,v)\cdot 1+\frac{\partial[f'_2(u,v)]}{\partial y}\cdot\frac{1}{y}+f'_2(u,v)\cdot(-\frac{1}{y^2})-[\frac{\partial[g'(w)]}{\partial y}\cdot\frac{y}{x^2}+g'(w)\cdot\frac{1}{x^2}]$$

$$=\frac{\partial[f'_1(u,v)]}{\partial y}\cdot y+f'_1(u,v)+\frac{\partial[f'_2(u,v)]}{\partial y}\cdot\frac{1}{y}-f'_2(u,v)\cdot\frac{1}{y^2}-\frac{\partial[g'(w)]}{\partial y}\cdot\frac{y}{x^2}-g'(w)\cdot\frac{1}{x^2}$$

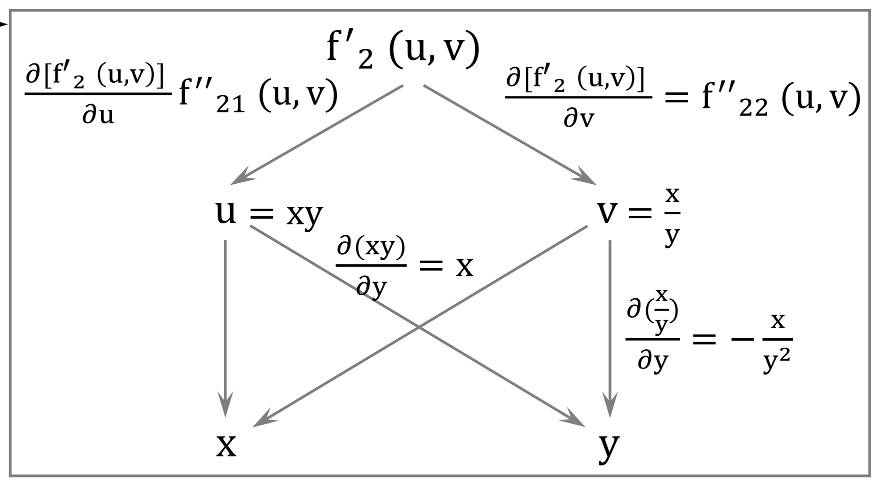
$$\boxed{\frac{\partial[f'_1(u,v)]}{\partial y}}=f''_{11}(u,v)\cdot x+f''_{12}(u,v)\cdot(-\frac{x}{y^2})$$

$$=f''_{11}(u,v)\cdot x-f''_{12}(u,v)\cdot\frac{x}{y^2}$$

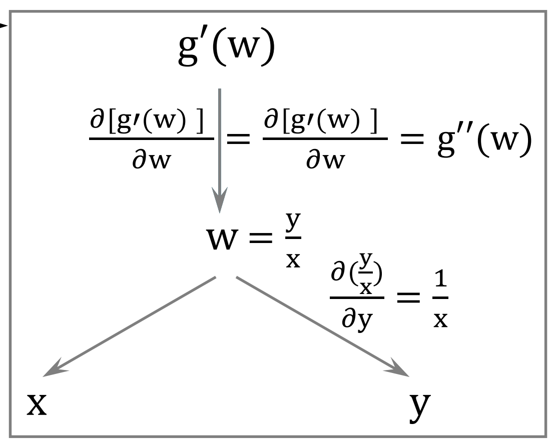


$$\boxed{\frac{\partial[f'_2(u,v)]}{\partial y}}=f''_{21}(u,v)\cdot x+f''_{22}(u,v)\cdot(-\frac{x}{y^2})$$

$$=f''_{21}(u,v)\cdot x-f''_{22}(u,v)\cdot\frac{x}{y^2}$$



$$\boxed{\frac{\partial[g'(w)]}{\partial y}}=g''(w)\cdot\frac{1}{x}$$



$$\frac{\partial^2 z}{\partial x\partial y}=\left[f''_{11}(u,v)\cdot x-f''_{12}(u,v)\cdot\frac{x}{y^2}\right]\cdot y+f'_1(u,v)+\left[f''_{21}(u,v)\cdot x-f''_{22}(u,v)\cdot\frac{x}{y^2}\right]\cdot\frac{1}{y}-f'_2(u,v)\cdot\frac{1}{y^2}-g''(w)\cdot\frac{1}{x}\cdot\frac{y}{x^2}-g'(w)\cdot\frac{1}{x^2}$$

$$=f''_{11}(u,v)\cdot xy-f''_{12}(u,v)\cdot\frac{x}{y}+f'_1(u,v)+f''_{21}(u,v)\cdot\frac{x}{y}-f''_{22}(u,v)\cdot\frac{x}{y^3}-f'_2(u,v)\cdot\frac{1}{y^2}-g''(w)\cdot\frac{y}{x^3}-g'(w)\cdot\frac{1}{x^2}$$

$$=f''_{11}\cdot xy-f''_{12}\cdot\frac{x}{y}+f'_1+\boxed{f''_{21}}\cdot\frac{x}{y}-f''_{22}\cdot\frac{x}{y^3}-f'_2\cdot\frac{1}{y^2}-g''\cdot\frac{y}{x^3}-g'\cdot\frac{1}{x^2}$$

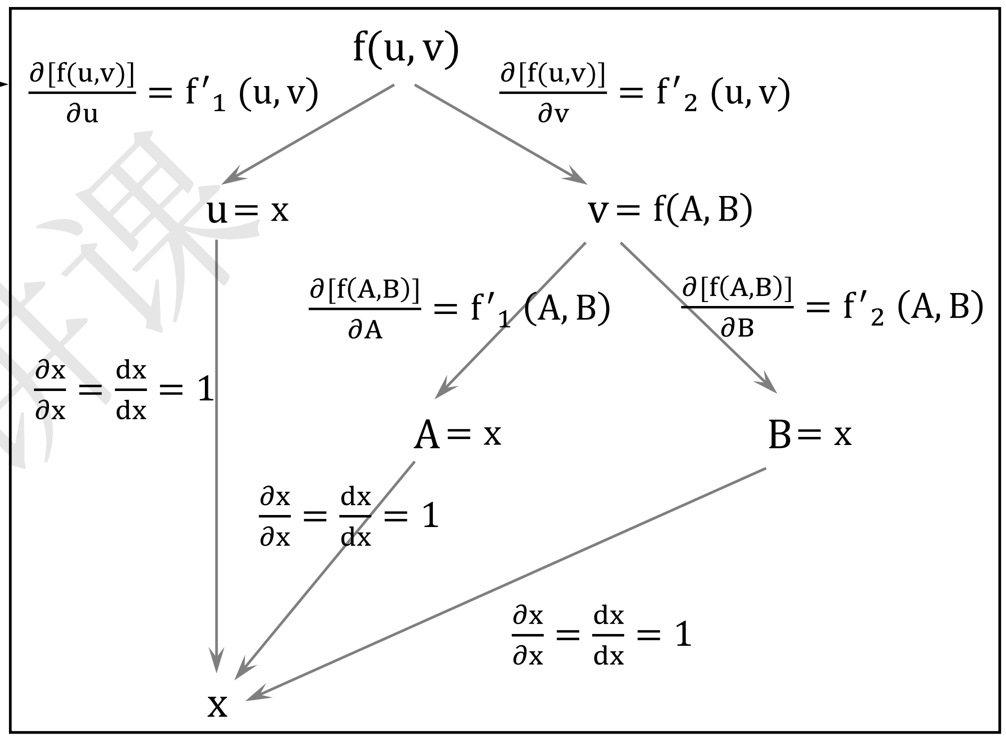
∵ f具有二阶连续偏导数
∴ $f''_{12}=f''_{21}$

$$=f''_{11}\cdot xy-f''_{12}\cdot\frac{x}{y}+f'_1+f''_{12}\cdot\frac{x}{y}-f''_{22}\cdot\frac{x}{y^3}-f'_2\cdot\frac{1}{y^2}-g''\cdot\frac{y}{x^3}-g'\cdot\frac{1}{x^2}$$

$$=f''_{11}\cdot xy+f'_1-f''_{22}\cdot\frac{x}{y^3}-f'_2\cdot\frac{1}{y^2}-g''\cdot\frac{y}{x^3}-g'\cdot\frac{1}{x^2}$$

例3．$f(x,y)$有一阶偏导数，$\varphi(x)=f[x,f(x,x)]$，求$\frac{d\varphi(x)}{dx}$

设$u=x$，$v=f(A,B)$，$A=x$，$B=x\Rightarrow\varphi=f(u,v)$　　$\frac{d\varphi(x)}{dx}=\boxed{\frac{d[f(u,v)]}{dx}}$



$$\frac{d\varphi(x)}{dx}=f'_1(u,v)\cdot 1+f'_2(u,v)\cdot f'_1(A,B)\cdot 1+f'_2(u,v)\cdot f'_2(A,B)\cdot 1$$

$$=f'_1(u,v)+f'_2(u,v)\cdot f'_1(A,B)+f'_2(u,v)\cdot f'_2(A,B)$$

$$=f'_1[x,f(A,B)]+f'_2[x,f(A,B)]\cdot f'_1(x,x)+f'_2[x,f(A,B)]\cdot f'_2(x,x)$$

$$=f'_1[x,f(x,x)]+f'_2[x,f(x,x)]\cdot f'_1(x,x)+f'_2[x,f(x,x)]\cdot f'_2(x,x)$$

例4 . 设 $z=f(x,y)$ 在点(1,1)处可微，且 $f(1,1)=1$，$\left.\frac{\partial f}{\partial x}\right|_{(1,1)}=2$，$\left.\frac{\partial f}{\partial y}\right|_{(1,1)}=3$，$\varphi(x)=f[x,f(x,x)]$，求 $\left.\frac{d\varphi^3(x)}{dx}\right|_{x=1}$

$$
\begin{aligned}
\frac{d\varphi^3(x)}{dx} &= [\varphi^3(x)]' \\
&= 3\varphi^2(x)\cdot\varphi'(x) \\
&= 3\varphi^2(x)\cdot\boxed{\frac{d\varphi(x)}{dx}} \longrightarrow (\text{例3算过}) \\
&= 3\varphi^2(x)\cdot\{f'_1[x,f(x,x)]+f'_2[x,f(x,x)]\cdot f'_1(x,x)+f'_2[x,f(x,x)]\cdot f'_2(x,x)\}
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{d\varphi^3(x)}{dx}\right|_{x=1} &= [\,3\varphi^2(x)\cdot\{f'_1[x,f(x,x)]+f'_2[x,f(x,x)]\cdot f'_1(x,x)+f'_2[x,f(x,x)]\cdot f'_2(x,x)\}\,]|_{x=1} \\
&= 3\varphi^2(1)\cdot\{f'_1[1,f(1,1)]+f'_2[1,f(1,1)]\cdot f'_1(1,1)+f'_2[1,f(1,1)]\cdot f'_2(1,1)\} \\
&= 3[\varphi(1)]^2\cdot\{f'_1[1,f(1,1)]+f'_2[1,f(1,1)]\cdot f'_1(1,1)+f'_2[1,f(1,1)]\cdot f'_2(1,1)\} \\
&= 3\{f[1,f(1,1)]\}^2\cdot\{f'_1[1,f(1,1)]+f'_2[1,f(1,1)]\cdot f'_1(1,1)+f'_2[1,f(1,1)]\cdot f'_2(1,1)\} \\
&= 3[f(1,1)]^2\cdot[f'_1(1,1)+f'_2(1,1)\cdot f'_1(1,1)+f'_2(1,1)\cdot f'_2(1,1)] \\
&= 3\cdot[f'_1(1,1)+f'_2(1,1)\cdot f'_1(1,1)+f'_2(1,1)\cdot f'_2(1,1)]
\end{aligned}
$$

$$\frac{\partial f}{\partial x}=\frac{\partial[f(x,y)]}{\partial x}=f'_1(x,y)$$

$$f'_1(x,y)|_{(1,1)}=2$$

$$\Longrightarrow\ f'_1(1,1)\quad=2$$

$$\frac{\partial f}{\partial y}=\frac{\partial[f(x,y)]}{\partial y}=f'_2(x,y)$$

$$f'_2(x,y)|_{(1,1)}=3$$

$$\Longrightarrow\ f'_2(1,1)\quad=3$$

$$
\begin{aligned}
\left.\frac{d\varphi^3(x)}{dx}\right|_{x=1} &= 3\cdot[2+3\cdot2+3\cdot3] \\
&= 3\times17 \\
&= 51
\end{aligned}
$$

用公式法求隐函数的偏导　　类型一

例1. 若 $z=z(x,y)$ 由 $e^z+xyz+x+\cos x=2$ 确定，求 $\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$

① $e^z+xyz+x+\cos x-2=0$

$F=e^z+xyz+x+\cos x-2$

② $$\frac{\partial F}{\partial x}=\frac{\partial(e^z+xyz+x+\cos x-2)}{\partial x}$$
$$=\frac{\partial(e^z)}{\partial x}+\frac{\partial(xyz)}{\partial x}+\frac{dx}{dx}+\frac{d(\cos x)}{dx}-\frac{\partial 2}{\partial x}$$
$$=0+yz+1+(-\sin x)-0$$
$$=yz+1-\sin x$$

$$\frac{\partial F}{\partial y}=\frac{\partial(e^z+xyz+x+\cos x-2)}{\partial y}$$
$$=\frac{\partial(e^z)}{\partial y}+\frac{\partial(xyz)}{\partial y}+\frac{\partial x}{\partial y}+\frac{\partial(\cos x)}{\partial y}-\frac{\partial 2}{\partial y}$$
$$=0+xz+0+0-0$$
$$=xz$$

$$\frac{\partial F}{\partial z}=\frac{\partial(e^z+xyz+x+\cos x-2)}{\partial z}$$
$$=\frac{d(e^z)}{dz}+\frac{\partial(xyz)}{\partial z}+\frac{\partial x}{\partial z}+\frac{\partial(\cos x)}{\partial z}-\frac{\partial 2}{\partial z}$$
$$=e^z+xy+0+0-0$$
$$=e^z+xy$$

③ $$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$
$$=-\frac{yz+1-\sin x}{e^z+xy}$$
$$=\frac{\sin x-yz-1}{e^z+xy}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$
$$=-\frac{xz}{e^z+xy}$$

例2. 设 f(u,v) 可微，z=z(x,y) 由 $(x+1)\cdot z-y^2=x^2\cdot f(x-z,y)$

确定，求 $\frac{\partial z}{\partial x}$、$\frac{\partial z}{\partial y}$

① $(x+1)\cdot z-y^2-x^2\cdot f(x-z,y)=0$

$F=(x+1)\cdot z-y^2-x^2\cdot f(x-z,y)$

② $$\frac{\partial F}{\partial x}=\frac{\partial[(x+1)\cdot z-y^2-x^2\cdot f(x-z,y)]}{\partial x}$$
$$=\frac{\partial[x\cdot z+z-y^2-x^2\cdot f(x-z,y)]}{\partial x}$$
$$=\frac{\partial(x\cdot z)}{\partial x}+\frac{\partial z}{\partial x}-\frac{\partial(y^2)}{\partial x}-\frac{\partial[x^2\cdot f(x-z,y)]}{\partial x}$$
$$=z-\frac{\partial[x^2\cdot f(x-z,y)]}{\partial x}$$
$$=z-\left[\frac{\partial(x^2)}{\partial x}\cdot f(x-z,y)+x^2\cdot\frac{\partial[f(x-z,y)]}{\partial x}\right]$$
$$=z-\left[2x\cdot f(x-z,y)+x^2\cdot\boxed{\frac{\partial[f(x-z,y)]}{\partial x}}\right]$$
$$=z-[2x\cdot f(x-z,y)+x^2\cdot f'_1]$$
$$=z-2x\cdot f(x-z,y)-x^2\cdot f'_1$$

$$\frac{\partial F}{\partial y}=\frac{\partial[(x+1)\cdot z-y^2-x^2\cdot f(x-z,y)]}{\partial y}$$
$$=\frac{\partial[x\cdot z+z-y^2-x^2\cdot f(x-z,y)]}{\partial y}$$
$$=\frac{\partial(x\cdot z)}{\partial y}+\frac{\partial z}{\partial y}-\frac{d(y^2)}{dy}-\frac{\partial[x^2\cdot f(x-z,y)]}{\partial y}$$
$$=-2y-\frac{\partial[x^2\cdot f(x-z,y)]}{\partial y}$$
$$=-2y-\left[\frac{\partial(x^2)}{\partial y}\cdot f(x-z,y)+x^2\cdot\frac{\partial[f(x-z,y)]}{\partial y}\right]$$
$$=-2y-x^2\cdot\boxed{\frac{\partial[f(x-z,y)]}{\partial y}}$$
$$=-2y-x^2\cdot f'_2$$

$$\frac{\partial F}{\partial z}=\frac{\partial[(x+1)\cdot z-y^2-x^2\cdot f(x-z,y)]}{\partial z}$$
$$=\frac{\partial[x\cdot z+z-y^2-x^2\cdot f(x-z,y)]}{\partial z}$$
$$=\frac{\partial(x\cdot z)}{\partial z}+\frac{dz}{dz}-\frac{\partial(y^2)}{\partial z}-\frac{\partial[x^2\cdot f(x-z,y)]}{\partial z}$$
$$=x+1-\frac{\partial[x^2\cdot f(x-z,y)]}{\partial z}$$
$$=x+1-\left[\frac{\partial(x^2)}{\partial z}\cdot f(x-z,y)+x^2\cdot\frac{\partial[f(x-z,y)]}{\partial z}\right]$$
$$=x+1-x^2\cdot\boxed{\frac{\partial[f(x-z,y)]}{\partial z}}$$
$$=x+1-x^2\cdot(-f'_1)$$
$$=x+1+x^2\cdot f'_1$$

设 u=x−z，v=y

则 $\frac{\partial[f(x-z,y)]}{\partial x}=\frac{\partial[f(u,v)]}{\partial x}$

f(u,v)

$\frac{\partial[f(u,v)]}{\partial u}=f'_1$　　$\frac{\partial[f(u,v)]}{\partial v}=f'_2$

u=x−z　　v=y

$\frac{\partial(x-z)}{\partial x}=1$　　$\frac{\partial(x-z)}{\partial z}=-1$　　$\frac{\partial y}{\partial y}=\frac{dy}{dy}=1$

x　　z　　y　　z同x,y

无关

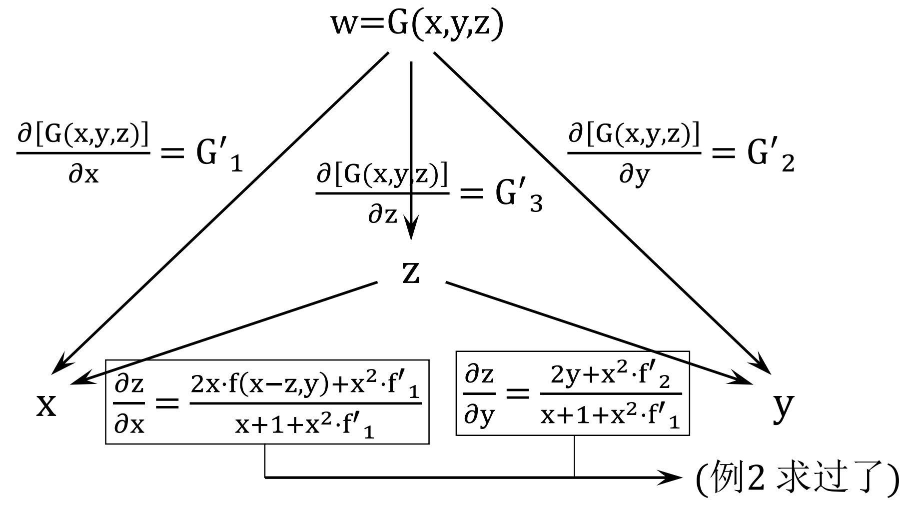
$$\frac{\partial[f(x-z,y)]}{\partial x}=\frac{\partial[f(u,v)]}{\partial x}=f'_1\cdot 1=f'_1$$
$$\frac{\partial[f(x-z,y)]}{\partial y}=\frac{\partial[f(u,v)]}{\partial y}=f'_2\cdot 1=f'_2$$
$$\frac{\partial[f(x-z,y)]}{\partial z}=\frac{\partial[f(u,v)]}{\partial z}=f'_1\cdot(-1)=-f'_1$$

③ $$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$
$$=-\frac{z-2x\cdot f(x-z,y)-x^2\cdot f'_1}{x+1+x^2\cdot f'_1}$$
$$=\frac{2x\cdot f(x-z,y)+x^2\cdot f'_1-z}{x+1+x^2\cdot f'_1}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$
$$=-\frac{-2y-x^2\cdot f'_2}{x+1+x^2\cdot f'_1}$$
$$=\frac{2y+x^2\cdot f'_2}{x+1+x^2\cdot f'_1}$$

例3. 设 $w=G(x,y,z)$ 有一阶连续偏导数，$f(u,v)$ 可微，

$z=z(x,y)$ 由 $(x+1)\cdot z-y^2=x^2\cdot f(x-z,y)$ 确定，求 $\frac{\partial w}{\partial x}$、$\frac{\partial w}{\partial y}$



$$\frac{\partial w}{\partial x}=G'_1+G'_3\cdot\frac{2x\cdot f(x-z,y)+x^2\cdot f'_1-z}{x+1+x^2\cdot f'_1}$$

$$\frac{\partial w}{\partial y}=G'_3\cdot\frac{2y+x^2\cdot f'_2}{x+1+x^2\cdot f'_1}+G'_2$$

例4. 设 $xy-z\cdot\ln y+e^{xz}=1$，存在点 $(0,1,1)$ 的一个邻域，

在此邻域内，该方程 ___D___

(A) 只能确定一个具有连续偏导数的隐函数 $z=z(x,y)$

(B) 可确定两个具有连续偏导数的隐函数 $y=y(x,z)$ 和 $z=z(x,y)$

(C) 可确定两个具有连续偏导数的隐函数 $x=x(y,z)$ 和 $z=z(x,y)$

(D) 可确定两个具有连续偏导数的隐函数 $x=x(y,z)$ 和 $y=y(x,z)$

$\left(\frac{\partial F}{\partial z}\neq 0\right)$ ×	$\left(\frac{\partial F}{\partial x}\neq 0\right)$ √	$\left(\frac{\partial F}{\partial y}\neq 0\right)$ √
$z=z(x,y)$	$x=x(y,z)$	$y=y(x,z)$

① $xy-z\cdot\ln y+e^{xz}-1=0$

$F=xy-z\cdot\ln y+e^{xz}-1$

② $\frac{\partial F}{\partial x}=\frac{\partial(xy-z\cdot\ln y+e^{xz}-1)}{\partial x}$

$=\frac{\partial(xy)}{\partial x}-\frac{\partial(z\cdot\ln y)}{\partial x}+\frac{\partial(e^{xz})}{\partial x}-\frac{\partial 1}{\partial x}$

$=y-0+z\cdot e^{xz}-0$

$=y+z\cdot e^{xz}$

$\frac{\partial F}{\partial y}=\frac{\partial(xy-z\cdot\ln y+e^{xz}-1)}{\partial y}$

$=\frac{\partial(xy)}{\partial y}-\frac{\partial(z\cdot\ln y)}{\partial y}+\frac{\partial(e^{xz})}{\partial y}-\frac{\partial 1}{\partial y}$

$=x-z\cdot\frac{1}{y}+0-0$

$=x-\frac{z}{y}$

$\frac{\partial F}{\partial z}=\frac{\partial(xy-z\cdot\ln y+e^{xz}-1)}{\partial z}$

$=\frac{\partial(xy)}{\partial z}-\frac{\partial(z\cdot\ln y)}{\partial z}+\frac{\partial(e^{xz})}{\partial z}-\frac{\partial 1}{\partial z}$

$=0-\ln y+x\cdot e^{xz}-0$

$=x\cdot e^{xz}-\ln y$

$\frac{\partial F}{\partial x}=y+z\cdot e^{xz}$

$\frac{\partial F}{\partial y}=x-\frac{z}{y}$

$\frac{\partial F}{\partial z}=x\cdot e^{xz}-\ln y$

$\left.\frac{\partial F}{\partial x}\right|_{\substack{x=0\\y=1\\z=1}}=1+1\cdot e^{0\cdot 1}=2\neq 0$

$\left.\frac{\partial F}{\partial y}\right|_{\substack{x=0\\y=1\\z=1}}=0-\frac{1}{1}=-1\neq 0$

$\left.\frac{\partial F}{\partial z}\right|_{\substack{x=0\\y=1\\z=1}}=0\cdot e^{0\cdot 1}-\ln 1=0$

类型二

例5. 若 $z=z(x,y)$ 由 $\varphi(x^2,e^y,z)=0$ 确定，$y=\sin x$，其中 φ 具有一阶连续偏导数，求 $\frac{dz}{dx}$

① $F=\varphi(x^2,e^y,z)$

② $\frac{\partial F}{\partial x}=\boxed{\frac{\partial[\varphi(x^2,e^y,z)]}{\partial x}}=\varphi'_1\cdot 2x+\varphi'_2\cdot e^{\sin x}\cdot\cos x$

$\frac{\partial F}{\partial z}=\boxed{\frac{\partial[\varphi(x^2,e^y,z)]}{\partial z}}=\varphi'_3$

③ $\frac{dz}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$

$=-\frac{\varphi'_1\cdot 2x+\varphi'_2\cdot e^{\sin x}\cdot\cos x}{\varphi'_3}$

设 $u=x^2$，$v=e^y=e^{\sin x}$，$w=z$

则 $\frac{\partial[\varphi(x^2,e^y,z)]}{\partial x}=\frac{\partial[\varphi(u,v,w)]}{\partial x}$

$\varphi(u,v,w)$

$\frac{\partial[\varphi(u,v,w)]}{\partial u}=\varphi'_1$　$\frac{\partial[\varphi(u,v,w)]}{\partial v}=\varphi'_2$　$\frac{\partial[\varphi(u,v,w)]}{\partial w}=\varphi'_3$

$u=x^2$　$v=e^{\sin x}$　$w=z$

$\frac{\partial(x^2)}{\partial x}=\frac{d(x^2)}{dx}=2x$　$\frac{\partial(e^{\sin x})}{\partial x}=\frac{d(e^{\sin x})}{dx}=e^{\sin x}\cdot\cos x$　$\frac{\partial z}{\partial z}=\frac{dz}{dz}=1$

x　z

z同x无关

$\frac{\partial[\varphi(x^2,e^y,z)]}{\partial x}=\frac{\partial[\varphi(u,v,w)]}{\partial x}=\varphi'_1\cdot 2x+\varphi'_2\cdot e^{\sin x}\cdot\cos x$

$\frac{\partial[\varphi(x^2,e^y,z)]}{\partial z}=\frac{\partial[\varphi(u,v,w)]}{\partial z}=\varphi'_3\cdot 1=\varphi'_3$

例6. 设 $\varphi(x^2,e^y,z)=0$，$y=\sin x$，其中 φ 具有一阶连续偏导数，且 $\frac{\partial\varphi}{\partial z}\neq 0$，求 $\frac{dz}{dx}$

(解题过程和例5一样)

例7. 设 $u=f(x,y,z)$，$\varphi(x^2,e^y,z)=0$，$y=\sin x$，其中 f、φ 都具有一阶连续偏导数，且 $\frac{\partial\varphi}{\partial z}\neq 0$，求 $\frac{du}{dx}$

$u=f(x,y,z)$

$\frac{\partial[f(x,y,z)]}{\partial y}=f'_2$　$\frac{\partial[f(x,y,z)]}{\partial x}=f'_1$　$\frac{\partial[f(x,y,z)]}{\partial z}=f'_3$

$y=\sin x$　z

$\frac{\partial(\sin x)}{\partial x}=\frac{d(\sin x)}{dx}=\cos x$　$\frac{\partial z}{\partial x}=\boxed{\frac{dz}{dx}=-\frac{\varphi'_1\cdot 2x+\varphi'_2\cdot e^{\sin x}\cdot\cos x}{\varphi'_3}}$ (例6算过了)

x

$\frac{du}{dx}=f'_2\cdot\cos x+f'_1+f'_3\cdot\left(-\frac{\varphi'_1\cdot 2x+\varphi'_2\cdot e^{\sin x}\cdot\cos x}{\varphi'_3}\right)$

$=f'_1+f'_2\cdot\cos x-f'_3\cdot\frac{\varphi'_1\cdot 2x+\varphi'_2\cdot e^{\sin x}\cdot\cos x}{\varphi'_3}$

用两边同求偏导法求隐函数的偏导

例1. $z=z(x,y)$由$e^z+xyz+x+\cos x=2$确定，求$\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$，$\frac{\partial^2 z}{\partial y\partial x}$

$$\frac{\partial(e^z+xyz+x+\cos x)}{\partial x}=\frac{\partial 2}{\partial x}$$

$$\underline{\frac{\partial(e^z)}{\partial x}}+\underline{\frac{\partial(xyz)}{\partial x}}+\frac{dx}{dx}+\frac{d(\cos x)}{dx}=0$$

$$\begin{tikzcd} & e^z \arrow[d, "\frac{\partial(e^z)}{\partial z}=\frac{d(e^z)}{dz}=e^z"'] & \\ & z=z(x,y) \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\ x & & y \end{tikzcd}$$

$$\frac{\partial(e^z)}{\partial x}=e^z\cdot\frac{\partial z}{\partial x}$$

$$e^z\cdot\frac{\partial z}{\partial x}+y\cdot z+xy\cdot\frac{\partial z}{\partial x}+1+(-\sin x)=0$$

$$(e^z+xy)\cdot\frac{\partial z}{\partial x}+yz+1-\sin x=0$$

$$(e^z+xy)\cdot\frac{\partial z}{\partial x}=\sin x-yz-1$$

$$\frac{\partial z}{\partial x}=\frac{\sin x-yz-1}{e^z+xy}$$

$$\begin{aligned}\frac{\partial(xyz)}{\partial x}&=\frac{\partial[xy\cdot z(x,y)]}{\partial x}=\frac{\partial(xy)}{\partial x}\cdot z(x,y)+xy\cdot\frac{\partial[z(x,y)]}{\partial x}\\&=y\cdot z(x,y)+xy\cdot\frac{\partial[z(x,y)]}{\partial x}\\&=y\cdot z(x,y)+xy\cdot\frac{\partial z}{\partial x}\\&=yz+xy\cdot\frac{\partial z}{\partial x}\end{aligned}$$

$$\frac{\partial(e^z+xyz+x+\cos x)}{\partial y}=\frac{\partial 2}{\partial y}$$

$$\underline{\frac{\partial(e^z)}{\partial y}}+\underline{\frac{\partial(xyz)}{\partial y}}+\frac{\partial x}{\partial y}+\frac{\partial(\cos x)}{\partial y}=0$$

$$\begin{tikzcd} & e^z \arrow[d, "\frac{\partial(e^z)}{\partial z}=\frac{d(e^z)}{dz}=e^z"'] & \\ & z=z(x,y) \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\ x & & y \end{tikzcd}$$

$$\frac{\partial(e^z)}{\partial y}=e^z\cdot\frac{\partial z}{\partial y}$$

$$e^z\cdot\frac{\partial z}{\partial y}+x\cdot z+xy\cdot\frac{\partial z}{\partial y}+0+0=0$$

$$(e^z+xy)\cdot\frac{\partial z}{\partial y}+xz=0$$

$$(e^z+xy)\cdot\frac{\partial z}{\partial y}=-xz$$

$$\frac{\partial z}{\partial y}=-\frac{xz}{e^z+xy}$$

$$\begin{aligned}\frac{\partial(xyz)}{\partial y}&=\frac{\partial[xy\cdot z(x,y)]}{\partial y}=\frac{\partial(xy)}{\partial y}\cdot z(x,y)+xy\cdot\frac{\partial[z(x,y)]}{\partial y}\\&=x\cdot z(x,y)+xy\cdot\frac{\partial[z(x,y)]}{\partial y}\\&=x\cdot z(x,y)+xy\cdot\frac{\partial z}{\partial y}\\&=xz+xy\cdot\frac{\partial z}{\partial y}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y\partial x}&=\frac{\partial(\frac{\partial z}{\partial y})}{\partial x}\\&=\frac{\partial(-\frac{xz}{e^z+xy})}{\partial x}\\&=-\frac{\partial(\frac{xz}{e^z+xy})}{\partial x}\\&=-\frac{\boxed{\frac{\partial(xz)}{\partial x}}\cdot(e^z+xy)-xz\cdot\boxed{\frac{\partial(e^z+xy)}{\partial x}}}{(e^z+xy)^2}\\&=-\frac{\left(z+x\cdot\frac{\sin x-yz-1}{e^z+xy}\right)\cdot(e^z+xy)-xz\cdot\left(e^z\cdot\frac{\sin x-yz-1}{e^z+xy}+y\right)}{(e^z+xy)^2}\\&=-\frac{\left(z+x\cdot\frac{\sin x-yz-1}{e^z+xy}\right)\cdot(e^z+xy)}{(e^z+xy)^2}+\frac{xz\cdot\left(e^z\cdot\frac{\sin x-yz-1}{e^z+xy}+y\right)}{(e^z+xy)^2}\\&=-\frac{z+x\cdot\frac{\sin x-yz-1}{e^z+xy}}{e^z+xy}+\frac{e^z xz\cdot\frac{\sin x-yz-1}{e^z+xy}+xyz}{(e^z+xy)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial(e^z+xy)}{\partial x}&=\frac{\partial(e^z)}{\partial x}+\frac{\partial(xy)}{\partial x}\\&=e^z\cdot\frac{\partial z}{\partial x}+y\\&=e^z\cdot\frac{\sin x-yz-1}{e^z+xy}+y\end{aligned}$$

$$\begin{aligned}\frac{\partial(xz)}{\partial x}&=\frac{\partial[x\cdot z(x,y)]}{\partial x}=\frac{\partial x}{\partial x}\cdot z(x,y)+x\cdot\frac{\partial[z(x,y)]}{\partial x}\\&=1\cdot z(x,y)+x\cdot\frac{\partial[z(x,y)]}{\partial x}\\&=z(x,y)+x\cdot\frac{\partial z}{\partial x}\\&=z(x,y)+x\cdot\frac{\sin x-yz-1}{e^z+xy}\\&=z+x\cdot\frac{\sin x-yz-1}{e^z+xy}\end{aligned}$$

用两边同求偏导法求隐函数的偏导

例2．设 $f(u,v)$ 可微，$z=z(x,y)$ 由 $(x+1)z-y^2=x^2\cdot f(x-z,y)$ 确定，求 $\frac{\partial z}{\partial x}$

$$\frac{\partial[(x+1)z-y^2]}{\partial x}=\frac{\partial[x^2\cdot f(x-z,y)]}{\partial x}$$

$$\boxed{\frac{\partial[(x+1)z]}{\partial x}}-\frac{\partial(y^2)}{\partial x}=\frac{d(x^2)}{dx}\cdot f(x-z,y)+x^2\cdot\frac{\partial[f(x-z,y)]}{\partial x}$$

$$z+(x+1)\cdot\frac{\partial z}{\partial x}-0=2x\cdot f(x-z,y)+x^2\cdot(f'_1-f'_1\cdot\frac{\partial z}{\partial x})$$

$$z+(x+1)\cdot\frac{\partial z}{\partial x}=2x\cdot f(x-z,y)+x^2\cdot f'_1-x^2\cdot f'_1\cdot\frac{\partial z}{\partial x}$$

$$(x+1+x^2\cdot f'_1)\cdot\frac{\partial z}{\partial x}=2x\cdot f(x-z,y)+x^2\cdot f'_1-z$$

$$\frac{\partial z}{\partial x}=\frac{2x\cdot f(x-z,y)+x^2\cdot f'_1-z}{x+1+x^2\cdot f'_1}$$

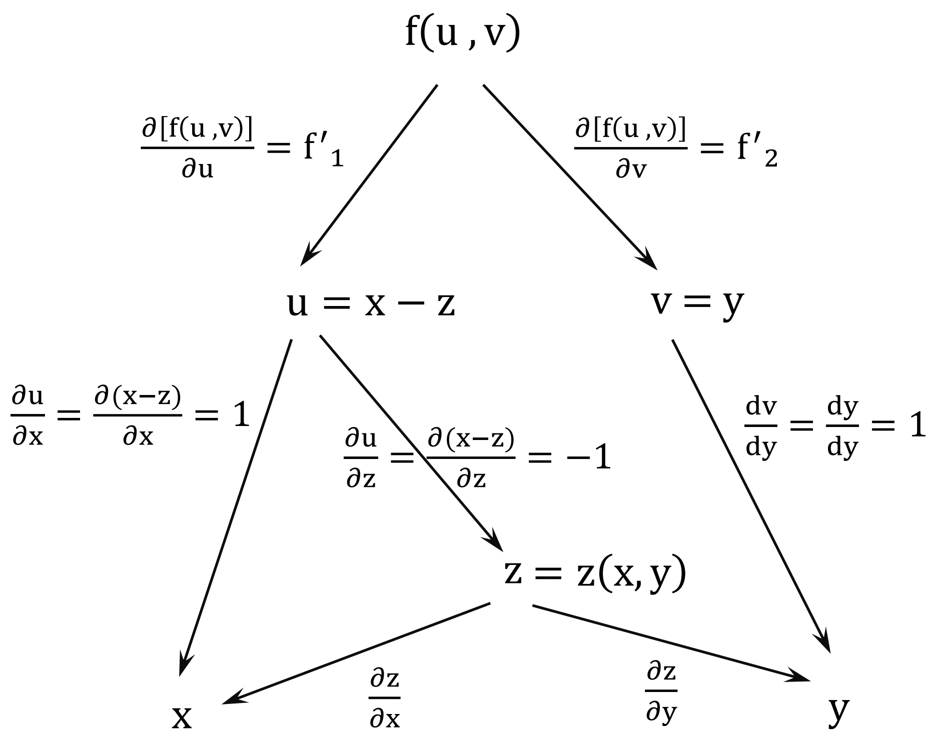
$$\frac{\partial[(x+1)z]}{\partial x}=\frac{\partial[(x+1)\cdot z(x,y)]}{\partial x}=\frac{\partial(x+1)}{\partial x}\cdot z(x,y)+(x+1)\cdot\frac{\partial[z(x,y)]}{\partial x}$$

$$=1\cdot z(x,y)+(x+1)\cdot\frac{\partial[z(x,y)]}{\partial x}$$

$$=z(x,y)+(x+1)\cdot\frac{\partial z}{\partial x}$$

$$=z+(x+1)\cdot\frac{\partial z}{\partial x}$$

设 u=x−z，v=y　则 $\frac{\partial[f(x-z,y)]}{\partial x}=\frac{\partial[f(u,v)]}{\partial x}$



$$\frac{\partial[f(x-z,y)]}{\partial x}=\frac{\partial[f(u,v)]}{\partial x}=f'_1\cdot 1+f'_1\cdot(-1)\cdot\frac{\partial z}{\partial x}=f'_1-f'_1\cdot\frac{\partial z}{\partial x}$$

求某点的偏导值

方法一：

使劲算！！

方法二：

$$f'_x(x_0,y_0)=\frac{d[f(x,y_0)]}{dx}\bigg|_{x=x_0}$$

$$f'_y(x_0,y_0)=\frac{d[f(x_0,y)]}{dy}\bigg|_{y=y_0}$$

当$\begin{cases}\text{方法一、方法二 求不出来时}\\ \text{求的是分段函数分段点处的偏导时}\end{cases}$

方法三：

$$f'_x(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$f'_y(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

例1. 已知 $f(x,y)=x^2+2y^2+(y-2)\cdot\arcsin\frac{x}{1+xy}$，求 $f'_x(0,2)$、$f'_y(0,2)$

方法一：

$$\begin{aligned}
f'_x&=\frac{\partial f}{\partial x}\\
&=\frac{\partial\left[x^2+2y^2+(y-2)\cdot\arcsin\frac{x}{1+xy}\right]}{\partial x}\\
&=\frac{d(x^2)}{dx}+\frac{\partial(2y^2)}{\partial x}+\frac{\partial\left[(y-2)\cdot\arcsin\frac{x}{1+xy}\right]}{\partial x}\\
&=2x+\frac{\partial\left[(y-2)\cdot\arcsin\frac{x}{1+xy}\right]}{\partial x}\\
&=2x+\frac{\partial(y-2)}{\partial x}\cdot\arcsin\frac{x}{1+xy}+(y-2)\cdot\frac{\partial\left(\arcsin\frac{x}{1+xy}\right)}{\partial x}\\
&=2x+(y-2)\cdot\boxed{\frac{\partial\left(\arcsin\frac{x}{1+xy}\right)}{\partial x}}\\
&=2x+(y-2)\cdot\frac{1}{\sqrt{1-\left(\frac{x}{1+xy}\right)^2}}\cdot\frac{1}{(1+xy)^2}
\end{aligned}$$

$$\begin{aligned}
\left(\arcsin\frac{x}{1+x\cdot3}\right)'&=\frac{1}{\sqrt{1-\left(\frac{x}{1+x\cdot3}\right)^2}}\cdot\left(\frac{x}{1+x\cdot3}\right)'\\
&=\frac{1}{\sqrt{1-\left(\frac{x}{1+x\cdot3}\right)^2}}\cdot\frac{x'\cdot(1+x\cdot3)-x\cdot(1+x\cdot3)'}{(1+x\cdot3)^2}\\
&=\frac{1}{\sqrt{1-\left(\frac{x}{1+x\cdot3}\right)^2}}\cdot\frac{1\cdot(1+x\cdot3)-x\cdot3}{(1+x\cdot3)^2}\\
&=\frac{1}{\sqrt{1-\left(\frac{x}{1+x\cdot3}\right)^2}}\cdot\frac{1}{(1+x\cdot3)^2}
\end{aligned}$$

（提示）

$$\begin{aligned}
f'_x(0,2)&=2\cdot0+(2-2)\cdot\frac{1}{\sqrt{1-\left(\frac{0}{1+0\cdot2}\right)^2}}\cdot\frac{1}{(1+0\cdot2)^2}\\
&=0+0\\
&=0
\end{aligned}$$

$$\begin{aligned}
f'_y&=\frac{\partial f}{\partial y}\\
&=\frac{\partial\left[x^2+2y^2+(y-2)\cdot\arcsin\frac{x}{1+xy}\right]}{\partial y}\\
&=\frac{\partial(x^2)}{\partial y}+\frac{d(2y^2)}{dy}+\frac{\partial\left[(y-2)\cdot\arcsin\frac{x}{1+xy}\right]}{\partial y}\\
&=4y+\frac{\partial\left[(y-2)\cdot\arcsin\frac{x}{1+xy}\right]}{\partial y}\\
&=4y+\frac{d(y-2)}{dy}\cdot\arcsin\frac{x}{1+xy}+(y-2)\cdot\frac{\partial\left(\arcsin\frac{x}{1+xy}\right)}{\partial y}\\
&=4y+\arcsin\frac{x}{1+xy}+(y-2)\cdot\boxed{\frac{\partial\left(\arcsin\frac{x}{1+xy}\right)}{\partial y}}\\
&=4y+\arcsin\frac{x}{1+xy}+(y-2)\cdot\left[-\frac{1}{\sqrt{1-\left(\frac{x}{1+xy}\right)^2}}\cdot\frac{x^2}{(1+xy)^2}\right]\\
&=4y+\arcsin\frac{x}{1+xy}-(y-2)\cdot\frac{1}{\sqrt{1-\left(\frac{x}{1+xy}\right)^2}}\cdot\frac{x^2}{(1+xy)^2}
\end{aligned}$$

$$\begin{aligned}
\left(\arcsin\frac{3}{1+3\cdot y}\right)'&=\frac{1}{\sqrt{1-\left(\frac{3}{1+3\cdot y}\right)^2}}\cdot\left(\frac{3}{1+3\cdot y}\right)'\\
&=\frac{1}{\sqrt{1-\left(\frac{3}{1+3\cdot y}\right)^2}}\cdot[3\cdot(1+3\cdot y)^{-1}]'\\
&=\frac{1}{\sqrt{1-\left(\frac{3}{1+3\cdot y}\right)^2}}\cdot3\cdot(1+3\cdot y)^{-2}\cdot(3\cdot y)'\\
&=-\frac{1}{\sqrt{1-\left(\frac{3}{1+3\cdot y}\right)^2}}\cdot\frac{3^2}{(1+3\cdot y)^2}
\end{aligned}$$

（提示）

$$\begin{aligned}
f'_y(0,2)&=4\cdot2+\arcsin\frac{0}{1+0\cdot2}-(2-2)\cdot\frac{1}{\sqrt{1-\left(\frac{0}{1+0\cdot2}\right)^2}}\cdot\frac{0^2}{(1+0\cdot2)^2}\\
&=8+\arcsin0-0\\
&=8
\end{aligned}$$

（过程比较繁琐）

方法二：

$$\begin{aligned}
f'_x(0,2)&=\frac{d[f(x,2)]}{dx}\bigg|_{x=0}\\
&=\frac{d(x^2+8)}{dx}\bigg|_{x=0}\\
&=2x\,|_{x=0}\\
&=2\cdot0\\
&=0
\end{aligned}$$

$$\begin{aligned}
f(x,2)&=x^2+2\cdot2^2+(2-2)\cdot\arcsin\frac{x}{1+x\cdot2}\\
&=x^2+8+0\\
&=x^2+8
\end{aligned}$$

$$\begin{aligned}
f'_y(0,2)&=\frac{d[f(0,y)]}{dy}\bigg|_{y=2}\\
&=\frac{d(2y^2)}{dy}\bigg|_{y=2}\\
&=4y\,|_{y=2}\\
&=4\cdot2\\
&=8
\end{aligned}$$

$$\begin{aligned}
f(0,y)&=0^2+2y^2+(y-2)\cdot\arcsin\frac{0}{1+0\cdot y}\\
&=0+2y^2+(y-2)\cdot\arcsin0\\
&=0+2y^2+0\\
&=2y^2
\end{aligned}$$

例2. 已知 $f(x,y)=\frac{y^2}{1+xy^2}$，求 $f'_x(0,1)$、$f'_y(0,1)$

方法二:

$$f'_x(0,1)=\left.\frac{d[f(x,1)]}{dx}\right|_{x=0} \longrightarrow f(x,1)=\frac{1^2}{1+x\cdot 1^2}=\frac{1}{x+1}$$

$$=\left.\boxed{\frac{d\left(\frac{1}{x+1}\right)}{dx}}\right|_{x=0} \longrightarrow \frac{d\left(\frac{1}{x+1}\right)}{dx}=\frac{d\left[(x+1)^{-1}\right]}{dx}=-1\cdot(x+1)^{-2}=-\frac{1}{(x+1)^2}$$

$$=-\left.\frac{1}{(x+1)^2}\right|_{x=0}$$

$$=-\frac{1}{(0+1)^2}$$

$$=-1$$

$$f'_y(0,1)=\left.\frac{d[f(0,y)]}{dy}\right|_{y=1} \longrightarrow f(0,y)=\frac{y^2}{1+0\cdot y^2}=y^2$$

$$=\left.\frac{d(y^2)}{dy}\right|_{y=1}$$

$$=2y\,|_{y=1}$$

$$=2\cdot 1$$

$$=2$$

方法一:

使劲算！！

方法二:

$$f'_x(x_0,y_0)=\left.\frac{d[f(x,y_0)]}{dx}\right|_{x=x_0}$$

$$f'_y(x_0,y_0)=\left.\frac{d[f(x_0,y)]}{dy}\right|_{y=y_0}$$

当 $\begin{cases}\text{方法一、方法二 求不出来时}\\ \text{求的是分段函数分段点处的偏导时}\end{cases}$

方法三:

$$f'_x(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$f'_y(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

例3. 已知 $z=\ln(1+xy^2)$，试求 $\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(0,1)}=$ ______

方法二:

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y} \longrightarrow \frac{\partial z}{\partial x}=\frac{\partial[\ln(1+xy^2)]}{\partial x}=\frac{y^2}{1+xy^2}$$

$$[\ln(1+x\cdot 3^2)]'=\frac{1}{1+x\cdot 3^2}\cdot(1+x\cdot 3^2)'=\frac{1}{1+x\cdot 3^2}\cdot 3^2=\frac{3^2}{1+x\cdot 3^2}\quad\text{（提示）}$$

$$=\frac{\partial\left(\frac{y^2}{1+xy^2}\right)}{\partial y}\qquad \boxed{\text{设 } f(x,y)=\frac{y^2}{1+xy^2}}$$

$$=\frac{\partial[f(x,y)]}{\partial y}$$

$$=f'_y(x,y)$$

$$\therefore \left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(0,1)}=f'_y(x,y)\big|_{(0,1)}$$

$$=f'_y(0,1)$$

$$=\left.\frac{d[f(0,y)]}{dy}\right|_{y=1} \longrightarrow f(0,y)=\frac{y^2}{1+0\cdot y^2}=y^2$$

$$=\left.\frac{d(y^2)}{dy}\right|_{y=1}$$

$$=2y\,|_{y=1}$$

$$=2\cdot 1$$

$$=2$$

方法一:

使劲算！！

方法二:

$$f'_x(x_0,y_0)=\left.\frac{d[f(x,y_0)]}{dx}\right|_{x=x_0}$$

$$f'_y(x_0,y_0)=\left.\frac{d[f(x_0,y)]}{dy}\right|_{y=y_0}$$

当 $\begin{cases}\text{方法一、方法二 求不出来时}\\ \text{求的是分段函数分段点处的偏导时}\end{cases}$

方法三:

$$f'_x(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$f'_y(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

例4. 设 $z=f[xy,y\cdot g(x)]$，其中 f 具有二阶连续偏导数，$g(x)$ 可导，

且在 $x=1$ 处取得极值 $g(1)=1$，试求 $\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{\substack{x=1\\y=1}}$

方法一：

使劲算！！

方法二：

$$f'_x(x_0,y_0)=\left.\frac{d[f(x,y_0)]}{dx}\right|_{x=x_0}$$

$$f'_y(x_0,y_0)=\left.\frac{d[f(x_0,y)]}{dy}\right|_{y=y_0}$$

当 $\begin{cases}\text{方法一、方法二 求不出来时}\\ \text{求的是分段函数分段点处的偏导时}\end{cases}$

方法三：

$$f'_x(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

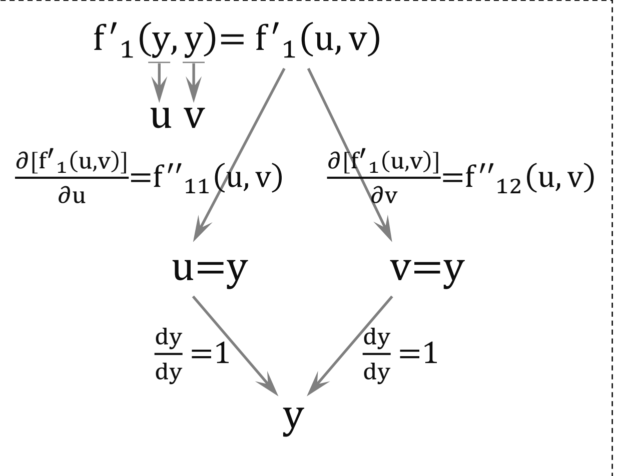
$$f'_y(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

方法二：

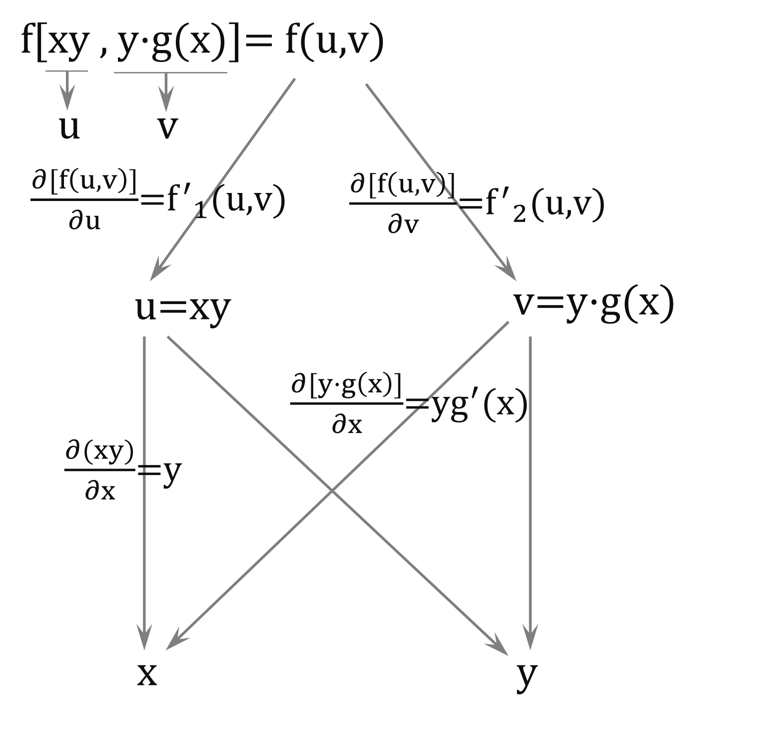
$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y}&=\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y}\\&=\frac{\partial\{f'_1[xy,y\cdot g(x)]\cdot y+f'_2[xy,y\cdot g(x)]\cdot y\cdot g'(x)\}}{\partial y}\\&=\frac{\partial[G(x,y)]}{\partial y}\qquad \boxed{\text{设 } G(x,y)=f'_1[xy,y\cdot g(x)]\cdot y+f'_2[xy,y\cdot g(x)]\cdot y\cdot g'(x)}\\&=G'_y(x,y)\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{\substack{x=1\\y=1}}&=G'_y(x,y)\Big|_{\substack{x=1\\y=1}}\\&=G'_y(1,1)\\&=\left.\frac{d[G(1,y)]}{dy}\right|_{y=1}\\&=\left.\frac{d[f'_1(y,y)\cdot y]}{dy}\right|_{y=1}\\&=\left[\frac{d[f'_1(y,y)]}{dy}\cdot y+f'_1(y,y)\cdot\frac{dy}{dy}\right]\Big|_{y=1}\\&=\left[\boxed{\frac{d[f'_1(y,y)]}{dy}}\cdot y+f'_1(y,y)\right]\Big|_{y=1}\\&=\left[[f''_{11}(y,y)+f''_{12}(y,y)]\cdot y+f'_1(y,y)\right]\Big|_{y=1}\\&=[f''_{11}(1,1)+f''_{12}(1,1)]\cdot 1+f'_1(1,1)\\&=f''_{11}(1,1)+f''_{12}(1,1)+f'_1(1,1)\end{aligned}$$

$$\begin{aligned}G(1,y)&=f'_1[1\cdot y,y\cdot g(1)]\cdot y+f'_2[1\cdot y,y\cdot g(1)]\cdot y\cdot g'(1)\\&=f'_1[y,y\cdot g(1)]\cdot y+f'_2[y,y\cdot g(1)]\cdot y\cdot g'(1)\\&=f'_1(y,y\cdot 1)\cdot y+f'_2(y,y\cdot 1)\cdot y\cdot 0\\&=f'_1(y,y)\cdot y\end{aligned}$$



$$\begin{aligned}\frac{d[f'_1(y,y)]}{dy}&=f''_{11}(u,v)\cdot 1+f''_{12}(u,v)\cdot 1\\&=f''_{11}(y,y)+f''_{12}(y,y)\end{aligned}$$



$$\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial\{f[xy,y\cdot g(x)]\}}{\partial x}\\&=f'_1(u,v)\cdot y+f'_2(u,v)\cdot yg'(x)\\&=f'_1[xy,y\cdot g(x)]\cdot y+f'_2[xy,y\cdot g(x)]\cdot yg'(x)\end{aligned}$$

例5. 已知 $f(x,y)=\sqrt[3]{x^2-y^3}$，试求 $f_x'(0,0)$

方法三：

$$f_x'(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0} \qquad \boxed{f(x,0)=\sqrt[3]{x^2-0^3}=\sqrt[3]{x^2}=x^{\frac{2}{3}}}$$

$$=\lim_{x\to0}\frac{x^{\frac{2}{3}}-f(0,0)}{x-0} \qquad \boxed{f(0,0)=\sqrt[3]{0^2-0^3}=\sqrt[3]{0}=0}$$

$$=\lim_{x\to0}\frac{x^{\frac{2}{3}}-0}{x-0}$$

$$=\lim_{x\to0}\frac{x^{\frac{2}{3}}}{x}$$

$$=\lim_{x\to0}\frac{1}{x^{\frac{1}{3}}}$$

$$=\lim_{x\to0}\frac{1}{\sqrt[3]{x}}$$

$=\infty$ (即 不存在)

$\therefore f_x'(0,0)$ 不存在

方法一：

使劲算！！

方法二：

$$f_x'(x_0,y_0)=\frac{d[f(x,y_0)]}{dx}\bigg|_{x=x_0}$$

$$f_y'(x_0,y_0)=\frac{d[f(x_0,y)]}{dy}\bigg|_{y=y_0}$$

当 $\begin{cases}\text{方法一、方法二 求不出来时}\\ \text{求的是分段函数分段点处的偏导时}\end{cases}$

方法三：

$$f_x'(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$f_y'(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

例6. 已知 $f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$，请判断

$f_x'(0,0)$、$f_y'(0,0)$ 是否存在

方法三：

$$f_x'(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}$$

$$=\lim_{x\to0}\frac{f(x,0)-0}{x-0} \qquad \boxed{f(x,0)=\frac{x^2\cdot0}{x^2+0^2}=0}$$

$$=\lim_{x\to0}\frac{0-0}{x-0}$$

$$=\lim_{x\to0}\frac{0}{x}$$

$$=\lim_{x\to0}0$$

$=0$

$$f_y'(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}$$

$$=\lim_{y\to0}\frac{f(0,y)-0}{y-0} \qquad \boxed{f(0,y)=\frac{0^2\cdot y}{0^2+y^2}=0}$$

$$=\lim_{y\to0}\frac{0-0}{y-0}$$

$$=\lim_{y\to0}\frac{0}{y}$$

$$=\lim_{y\to0}0$$

$=0$

$\therefore (0,0)$ 处两个偏导 $f_x'(0,0)$、$f_y'(0,0)$ 都存在，均为 0

方法一：

使劲算！！

方法二：

$$f_x'(x_0,y_0)=\frac{d[f(x,y_0)]}{dx}\bigg|_{x=x_0}$$

$$f_y'(x_0,y_0)=\frac{d[f(x_0,y)]}{dy}\bigg|_{y=y_0}$$

当 $\begin{cases}\text{方法一、方法二 求不出来时}\\ \text{求的是分段函数分段点处的偏导时}\end{cases}$

方法三：

$$f_x'(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$f_y'(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

例7. 已知 $f(x,y)=\begin{cases}\frac{\sqrt{|x|}}{x^2+y^2}\cdot\sin(x^2+y^2),\ (x,y)\neq(0,0)\\ 0\quad,\ (x,y)=(0,0)\end{cases}$,

试求 $f_x'(0,0)$、$f_y'(0,0)$

方法三:

$$f_x'(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}$$

$$\boxed{f(x,0)=\frac{\sqrt{|x|}}{x^2+0^2}\cdot\sin(x^2+0^2)=\frac{\sqrt{|x|}}{x^2}\cdot\sin x^2}$$

$$=\lim_{x\to0}\frac{f(x,0)-0}{x-0}$$

$$=\lim_{x\to0}\frac{\frac{\sqrt{|x|}}{x^2}\cdot\sin x^2-0}{x-0}$$

$$=\lim_{x\to0}\frac{\sqrt{|x|}\cdot\sin x^2}{x^3}$$

$$=\lim_{x\to0}\frac{\sqrt{|x|}\cdot x^2}{x^3}$$

$$=\lim_{x\to0}\frac{\sqrt{|x|}}{x}$$

$$\lim_{x\to0^+}\frac{\sqrt{|x|}}{x}=\lim_{x\to0^+}\frac{\sqrt{x}}{x}$$

$$=\lim_{x\to0^+}\frac{1}{\sqrt{x}}$$

$$=\frac{1}{\sqrt{0^+}}$$

$$=\frac{1}{0}$$

$$=\infty$$

$\therefore\ \lim\limits_{x\to0}\frac{\sqrt{|x|}}{x}$ 不存在

即 $f_x'(0,0)$ 不存在

$$f_y'(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}$$

$$\boxed{f(0,y)=\frac{\sqrt{|0|}}{0^2+y^2}\cdot\sin(0^2+y^2)=0}$$

$$=\lim_{y\to0}\frac{f(0,y)-0}{y-0}$$

$$=\lim_{y\to0}\frac{0-0}{y-0}$$

$$=\lim_{y\to0}0$$

$$=0$$

综上: $f_x'(0,0)$ 不存在, $f_y'(0,0)=0$

方法一:

使劲算!!

方法二:

$$f_x'(x_0,y_0)=\frac{d[f(x,y_0)]}{dx}\Big|_{x=x_0}$$

$$f_y'(x_0,y_0)=\frac{d[f(x_0,y)]}{dy}\Big|_{y=y_0}$$

当 $\begin{cases}\text{方法一、方法二 求不出来时}\\ \text{求的是分段函数分段点处的偏导时}\end{cases}$

方法三:

$$f_x'(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$f_y'(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

例8. 已知 $f(x,y)$ 满足 $\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0$，试求 $f_x'(0,0)$、$f_y'(0,0)$

方法三:

$$f_x'(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}$$

$$f_y'(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y}$$

$$\lim_{\substack{x\to0\\y\to0}}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0$$

$$\Rightarrow\lim_{x\to0}\frac{f(x,0)-f(0,0)}{\sqrt{x^2+0^2}}=0 \qquad \Rightarrow\lim_{y\to0}\frac{f(0,y)-f(0,0)}{\sqrt{0^2+y^2}}=0$$

$$\Rightarrow\lim_{x\to0}\frac{f(x,0)-f(0,0)}{\sqrt{x^2}}=0 \qquad \Rightarrow\lim_{y\to0}\frac{f(0,y)-f(0,0)}{\sqrt{y^2}}=0$$

$$\Rightarrow\lim_{x\to0}\frac{f(x,0)-f(0,0)}{|x|}=0 \qquad \Rightarrow\lim_{y\to0}\frac{f(0,y)-f(0,0)}{|y|}=0$$

$$\Rightarrow\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}=0 \qquad \Rightarrow\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y}=0$$

$\therefore f_x'(0,0)=0$ $\qquad$ $\therefore f_y'(0,0)=0$

方法一:

使劲算！！

方法二:

$$f_x'(x_0,y_0)=\frac{d[f(x,y_0)]}{dx}\bigg|_{x=x_0}$$

$$f_y'(x_0,y_0)=\frac{d[f(x_0,y)]}{dy}\bigg|_{y=y_0}$$

当 $\begin{cases}\text{方法一、方法二 求不出来时}\\\text{求的是分段函数分段点处的偏导时}\end{cases}$

方法三:

$$f_x'(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$f_y'(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

公式拓展:

题干中为 $f(x,y,z)\to$

$$f_x'(x_0,y_0,z_0)=\frac{d[f(x,y_0,z_0)]}{dx}\bigg|_{x=x_0}$$

$$f_y'(x_0,y_0,z_0)=\frac{d[f(x_0,y,z_0)]}{dy}\bigg|_{y=y_0}$$

$$f_z'(x_0,y_0,z_0)=\frac{d[f(x_0,y_0,z)]}{dz}\bigg|_{z=z_0}$$

方法二:

题干中为 $f(x,y)\to$

$$f_x'(x_0,y_0)=\frac{d[f(x,y_0)]}{dx}\bigg|_{x=x_0}$$

$$f_y'(x_0,y_0)=\frac{d[f(x_0,y)]}{dy}\bigg|_{y=y_0}$$

题干中为 $f(x,y,z)\to$

$$f_x'(x_0,y_0,z_0)=\lim_{x\to x_0}\frac{f(x,y_0,z_0)-f(x_0,y_0,z_0)}{x-x_0}$$

$$f_y'(x_0,y_0,z_0)=\lim_{y\to y_0}\frac{f(x_0,y,z_0)-f(x_0,y_0,z_0)}{y-y_0}$$

$$f_z'(x_0,y_0,z_0)=\lim_{z\to z_0}\frac{f(x_0,y_0,z)-f(x_0,y_0,z_0)}{z-z_0}$$

方法三:

题干中为 $f(x,y)\to$

$$f_x'(x_0,y_0)=\lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$$

$$f_y'(x_0,y_0)=\lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$$

题干中为 $f(x,y,z,w)\to$

$$f_x'(x_0,y_0,z_0,w_0)=\frac{d[f(x,y_0,z_0,w_0)]}{dx}\bigg|_{x=x_0}$$

$$f_y'(x_0,y_0,z_0,w_0)=\frac{d[f(x_0,y,z_0,w_0)]}{dy}\bigg|_{y=y_0}$$

$$f_z'(x_0,y_0,z_0,w_0)=\frac{d[f(x_0,y_0,z,w_0)]}{dz}\bigg|_{z=z_0}$$

$$f_w'(x_0,y_0,z_0,w_0)=\frac{d[f(x_0,y_0,z_0,w)]}{dw}\bigg|_{w=w_0}$$

题干中为 $f(x,y,z,w)\to$

$$f_x'(x_0,y_0,z_0,w_0)=\lim_{x\to x_0}\frac{f(x,y_0,z_0,w_0)-f(x_0,y_0,z_0,w_0)}{x-x_0}$$

$$f_y'(x_0,y_0,z_0,w_0)=\lim_{y\to y_0}\frac{f(x_0,y,z_0,w_0)-f(x_0,y_0,z_0,w_0)}{y-y_0}$$

$$f_z'(x_0,y_0,z_0,w_0)=\lim_{z\to z_0}\frac{f(x_0,y_0,z,w_0)-f(x_0,y_0,z_0,w_0)}{z-z_0}$$

$$f_w'(x_0,y_0,z_0,w_0)=\lim_{w\to w_0}\frac{f(x_0,y_0,z_0,w)-f(x_0,y_0,z_0,w_0)}{w-w_0}$$

已知偏导数，通过积分求表达式

已知	则
$\frac{\partial^2 z}{\partial x^2}=\frac{\partial(\frac{\partial z}{\partial x})}{\partial x}$	$\int \frac{\partial^2 z}{\partial x^2}dx=\frac{\partial z}{\partial x}+\varphi(y)$
$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial(\frac{\partial z}{\partial y})}{\partial x}$	$\int \frac{\partial^2 z}{\partial y\partial x}dx=\frac{\partial z}{\partial y}+\varphi(y)$
$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial(\frac{\partial z}{\partial x})}{\partial y}$	$\int \frac{\partial^2 z}{\partial x\partial y}dy=\frac{\partial z}{\partial x}+\varphi(x)$
$\frac{\partial^2 z}{\partial y^2}=\frac{\partial(\frac{\partial z}{\partial y})}{\partial y}$	$\int \frac{\partial^2 z}{\partial y^2}dy=\frac{\partial z}{\partial y}+\varphi(x)$
$\frac{\partial z}{\partial x}$	$\int \frac{\partial z}{\partial x}dx=z+\varphi(y)$
$\frac{\partial z}{\partial y}$	$\int \frac{\partial z}{\partial y}dy=z+\varphi(x)$

例1．若 $z=f(x,y)$ 满足 $\frac{\partial z}{\partial y}=2y+x-1$，且 $f(x,1)=x+2$，求 $f(x,y)$

$$\int \frac{\partial z}{\partial y}dy=z+\varphi(x)$$

$$\int(2y+x-1)\,dy=z+\varphi(x)$$

$$y^2+xy-y=z+\varphi(x)$$

$$y^2+xy-y=f(x,y)+\varphi(x)$$

$$f(x,y)=y^2+xy-y-\varphi(x)$$

$$f(x,1)=1^2+x\cdot1-1-\varphi(x)$$

$$x+2=1^2+x\cdot1-1-\varphi(x)$$

$$\varphi(x)=-2$$

$$f(x,y)=y^2+xy-y-(-2)$$

$$=y^2+xy-y+2$$

例2．若 $z=f(x,y)$ 满足 $\frac{\partial^2 z}{\partial y^2}=2$，且 $f(x,1)=x+2$，$f'_y(x,1)=x+1$，求 $f(x,y)$

$$\int \frac{\partial^2 z}{\partial y^2}dy=\frac{\partial z}{\partial y}+\varphi(x)$$

$$\Rightarrow\quad \int 2\,dy=\frac{\partial z}{\partial y}+\varphi(x)$$

$$2y=\frac{\partial z}{\partial y}+\varphi(x)$$

$$2y=\frac{\partial f(x,y)}{\partial y}+\varphi(x)$$

$$2y=f'_y(x,y)+\varphi(x)$$

令 $y=1$

$$2\cdot1=f'_y(x,1)+\varphi(x)$$

$$2\cdot1=x+1+\varphi(x)$$

$$\varphi(x)=1-x$$

$$2y=\frac{\partial z}{\partial y}+1-x$$

$$\frac{\partial z}{\partial y}=2y+x-1$$

$$\int \frac{\partial z}{\partial y}dy=z+\varphi(x)$$

$$\int(2y+x-1)\,dy=z+\varphi(x)$$

$$y^2+xy-y=z+\varphi(x)$$

$$y^2+xy-y=f(x,y)+\varphi(x)$$

$$f(x,y)=y^2+xy-y-\varphi(x)$$

$$f(x,1)=1^2+x\cdot1-1-\varphi(x)$$

$$x+2=1^2+x\cdot1-1-\varphi(x)$$

$$\varphi(x)=-2$$

$$f(x,y)=y^2+xy-y-(-2)$$

$$=y^2+xy-y+2$$

已知	则
$\frac{\partial^2 z}{\partial x^2}=\frac{\partial(\frac{\partial z}{\partial x})}{\partial x}$	$\int \frac{\partial^2 z}{\partial x^2}dx=\frac{\partial z}{\partial x}+\varphi(y)$
$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial(\frac{\partial z}{\partial y})}{\partial x}$	$\int \frac{\partial^2 z}{\partial y\partial x}dx=\frac{\partial z}{\partial y}+\varphi(y)$
$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial(\frac{\partial z}{\partial x})}{\partial y}$	$\int \frac{\partial^2 z}{\partial x\partial y}dy=\frac{\partial z}{\partial x}+\varphi(x)$
$\frac{\partial^2 z}{\partial y^2}=\frac{\partial(\frac{\partial z}{\partial y})}{\partial y}$	$\int \frac{\partial^2 z}{\partial y^2}dy=\frac{\partial z}{\partial y}+\varphi(x)$
$\frac{\partial z}{\partial x}$	$\int \frac{\partial z}{\partial x}dx=z+\varphi(y)$
$\frac{\partial z}{\partial y}$	$\int \frac{\partial z}{\partial y}dy=z+\varphi(x)$

例3．已知 $\frac{\partial^2 z}{\partial x\partial y}=1$，且 $z(0,y)=\sin y$，$z'_x(x,0)=\cos x$，求 $z(x,y)$

$\int\frac{\partial^2 z}{\partial x\partial y}dy=\frac{\partial z}{\partial x}+\varphi(x)$

$\Rightarrow\quad \int 1\,dy=\frac{\partial z}{\partial x}+\varphi(x)$

$y=\frac{\partial z}{\partial x}+\varphi(x)$

$y=\frac{\partial z(x,y)}{\partial x}+\varphi(x)$

$y=z'_x(x,y)+\varphi(x)$

$z'_x(x,y)=y-\varphi(x)$

$z'_x(x,0)=0-\varphi(x)$

$\cos x=0-\varphi(x)$

$\varphi(x)=-\cos x$

$z'_x(x,y)=y-(-\cos x)$

$\frac{\partial z}{\partial x}=y+\cos x$

$\int\frac{\partial z}{\partial x}dx=z+\varphi(y)$

$\int(y+\cos x)\,dx=z+\varphi(y)$

$xy+\sin x=z+\varphi(y)$

$xy+\sin x=z(x,y)+\varphi(y)$

$z(x,y)=xy+\sin x-\varphi(y)$

$z(0,y)=0\cdot y+\sin\cdot 0-\varphi(y)$

$\sin y=0\cdot y+\sin\cdot 0-\varphi(y)$

$\varphi(y)=-\sin y$

$z(x,y)=xy+\sin x-(-\sin y)$

$=xy+\sin x+\sin y$

已知	则
$\frac{\partial^2 z}{\partial x^2}=\frac{\partial(\frac{\partial z}{\partial x})}{\partial x}$	$\int\frac{\partial^2 z}{\partial x^2}dx=\frac{\partial z}{\partial x}+\varphi(y)$
$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial(\frac{\partial z}{\partial y})}{\partial x}$	$\int\frac{\partial^2 z}{\partial y\partial x}dx=\frac{\partial z}{\partial y}+\varphi(y)$
$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial(\frac{\partial z}{\partial x})}{\partial y}$	$\int\frac{\partial^2 z}{\partial x\partial y}dy=\frac{\partial z}{\partial x}+\varphi(x)$
$\frac{\partial^2 z}{\partial y^2}=\frac{\partial(\frac{\partial z}{\partial y})}{\partial y}$	$\int\frac{\partial^2 z}{\partial y^2}dy=\frac{\partial z}{\partial y}+\varphi(x)$
$\frac{\partial z}{\partial x}$	$\int\frac{\partial z}{\partial x}dx=z+\varphi(y)$
$\frac{\partial z}{\partial y}$	$\int\frac{\partial z}{\partial y}dy=z+\varphi(x)$

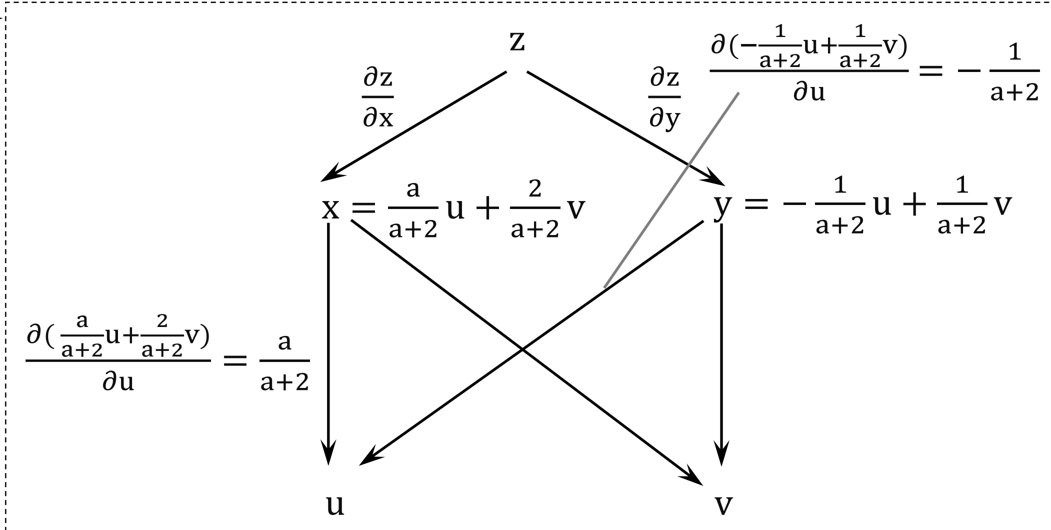
变量代换下化简偏导数满足的关系式

（注．默认$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$）

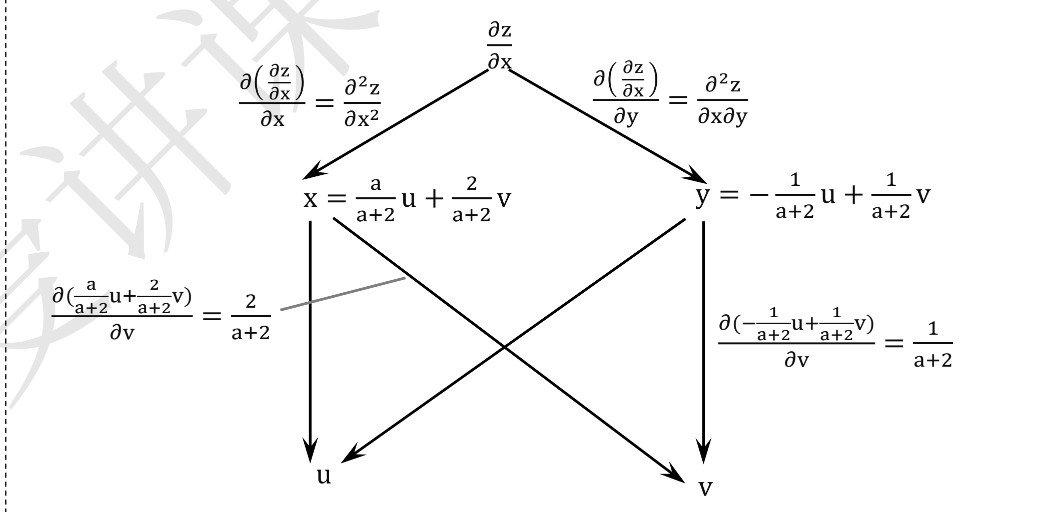
例1．设变换$\begin{cases}u=x-2y\\v=x+ay\end{cases}$可把方程$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$简化为$\frac{\partial^2 z}{\partial u\partial v}=0$，求常数a

$$\begin{cases}u=x-2y\\v=x+ay\end{cases}\Rightarrow\begin{cases}x=\frac{a}{a+2}u+\frac{2}{a+2}v\\y=-\frac{1}{a+2}u+\frac{1}{a+2}v\end{cases}$$

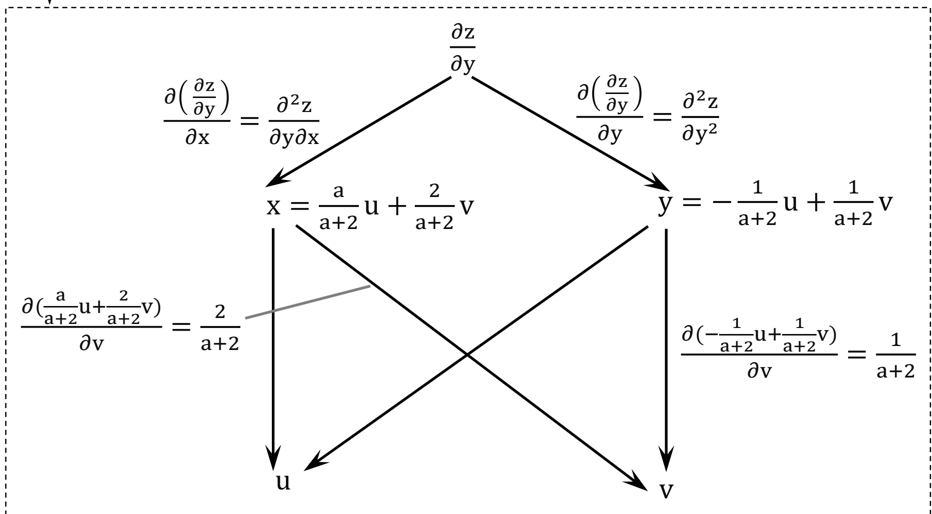
$$\begin{aligned}\frac{\partial^2 z}{\partial u\partial v}&=\frac{\partial\left(\frac{\partial z}{\partial u}\right)}{\partial v}\\&=\frac{\partial\left(\frac{\partial z}{\partial x}\cdot\frac{a}{a+2}-\frac{\partial z}{\partial y}\cdot\frac{1}{a+2}\right)}{\partial v}\\&=\frac{\partial\left(\frac{\partial z}{\partial x}\cdot\frac{a}{a+2}\right)}{\partial v}-\frac{\partial\left(\frac{\partial z}{\partial y}\cdot\frac{1}{a+2}\right)}{\partial v}\\&=\frac{a}{a+2}\cdot\boxed{\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial v}}-\frac{1}{a+2}\cdot\boxed{\frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial v}}\\&=\frac{a}{a+2}\cdot\left(\frac{\partial^2 z}{\partial x^2}\cdot\frac{2}{a+2}+\frac{\partial^2 z}{\partial x\partial y}\cdot\frac{1}{a+2}\right)-\frac{1}{a+2}\cdot\left(\frac{\partial^2 z}{\partial y\partial x}\cdot\frac{2}{a+2}+\frac{\partial^2 z}{\partial y^2}\cdot\frac{1}{a+2}\right)\\&=\frac{2a}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial x^2}+\frac{a}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}-\frac{2}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial y\partial x}-\frac{1}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial y^2}\\&=\frac{2a}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial x^2}+\frac{a}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}-\frac{2}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}-\frac{1}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial y^2}\\&=\frac{2a}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial x^2}+\frac{a-2}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}-\frac{1}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial y^2}\end{aligned}$$



$$\begin{aligned}\frac{\partial z}{\partial u}&=\frac{\partial z}{\partial x}\cdot\frac{a}{a+2}+\frac{\partial z}{\partial y}\cdot\left(-\frac{1}{a+2}\right)\\&=\frac{\partial z}{\partial x}\cdot\frac{a}{a+2}-\frac{\partial z}{\partial y}\cdot\frac{1}{a+2}\end{aligned}$$



$$\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial v}=\frac{\partial^2 z}{\partial x^2}\cdot\frac{2}{a+2}+\frac{\partial^2 z}{\partial x\partial y}\cdot\frac{1}{a+2}$$



$$\frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial v}=\frac{\partial^2 z}{\partial y\partial x}\cdot\frac{2}{a+2}+\frac{\partial^2 z}{\partial y^2}\cdot\frac{1}{a+2}$$

$\therefore\ \frac{2a}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial x^2}+\frac{a-2}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}-\frac{1}{(a+2)^2}\cdot\frac{\partial^2 z}{\partial y^2}=0$

$\because\ 6\cdot\frac{\partial^2 z}{\partial x^2}+1\cdot\frac{\partial^2 z}{\partial x\partial y}-1\cdot\frac{\partial^2 z}{\partial y^2}=0$

$\therefore\ \frac{\frac{2a}{(a+2)^2}}{6}=\frac{\frac{a-2}{(a+2)^2}}{1}=\frac{-\frac{1}{(a+2)^2}}{-1}$

$\Longrightarrow\ \frac{2a}{6}=a-2=1$

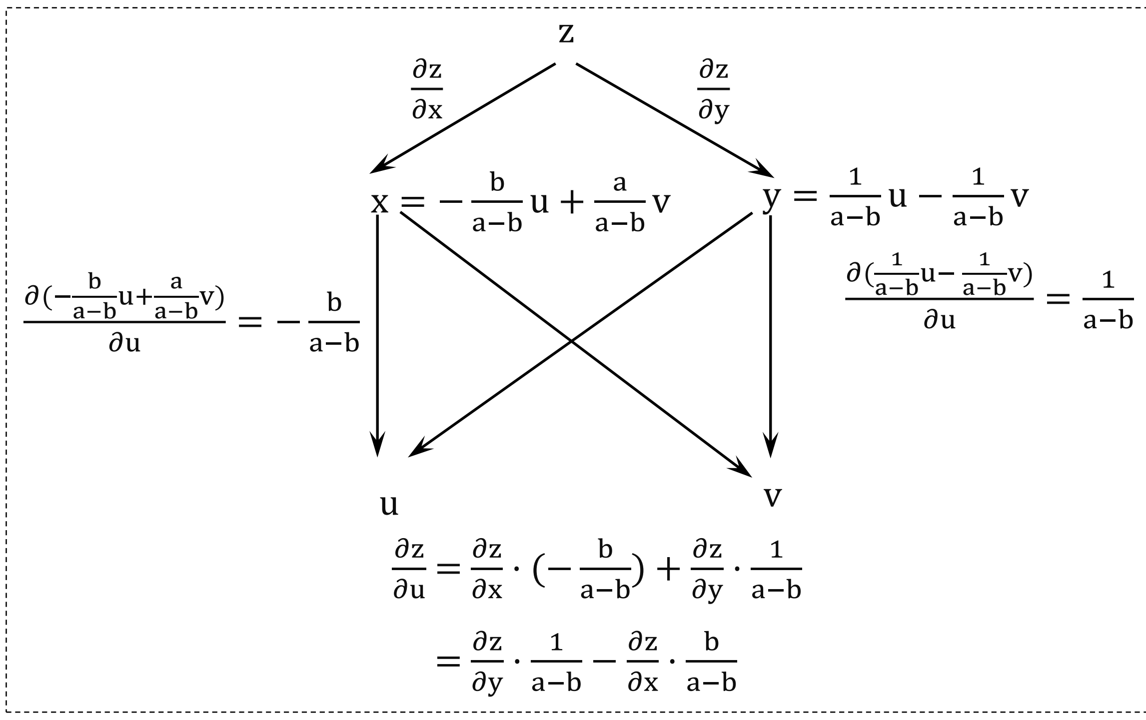
$\Longrightarrow\ a=3$

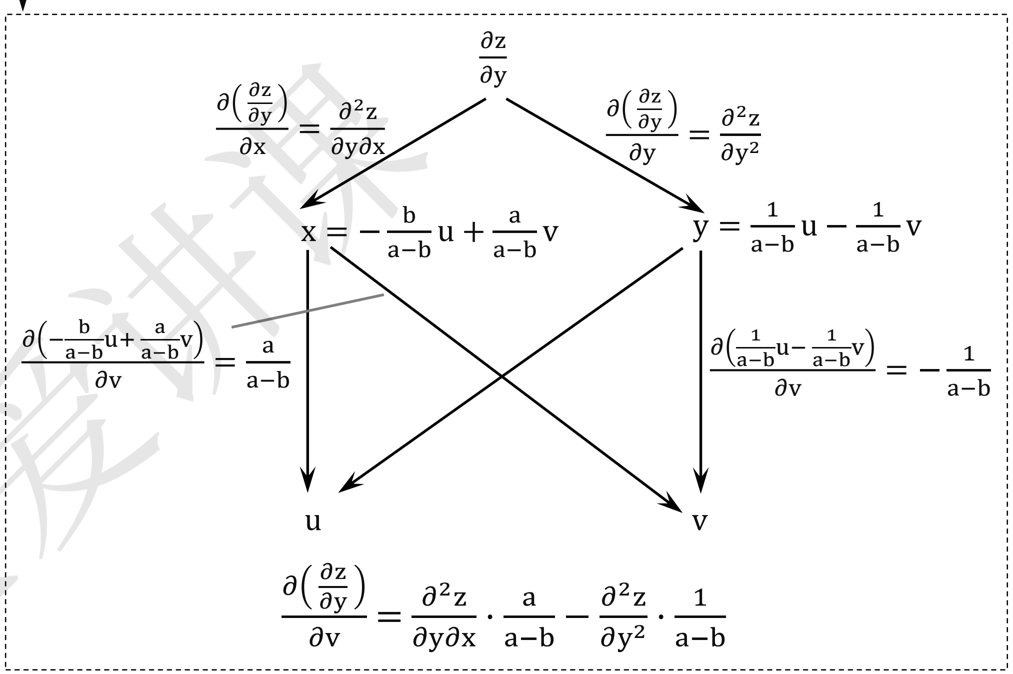
例2．设 $z=f(x,y)$ 具有二阶连续偏导数，且满足 $4\frac{\partial^2 z}{\partial x^2}+12\frac{\partial^2 z}{\partial x\partial y}+5\frac{\partial^2 z}{\partial y^2}=0$， （注．默认 $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$）

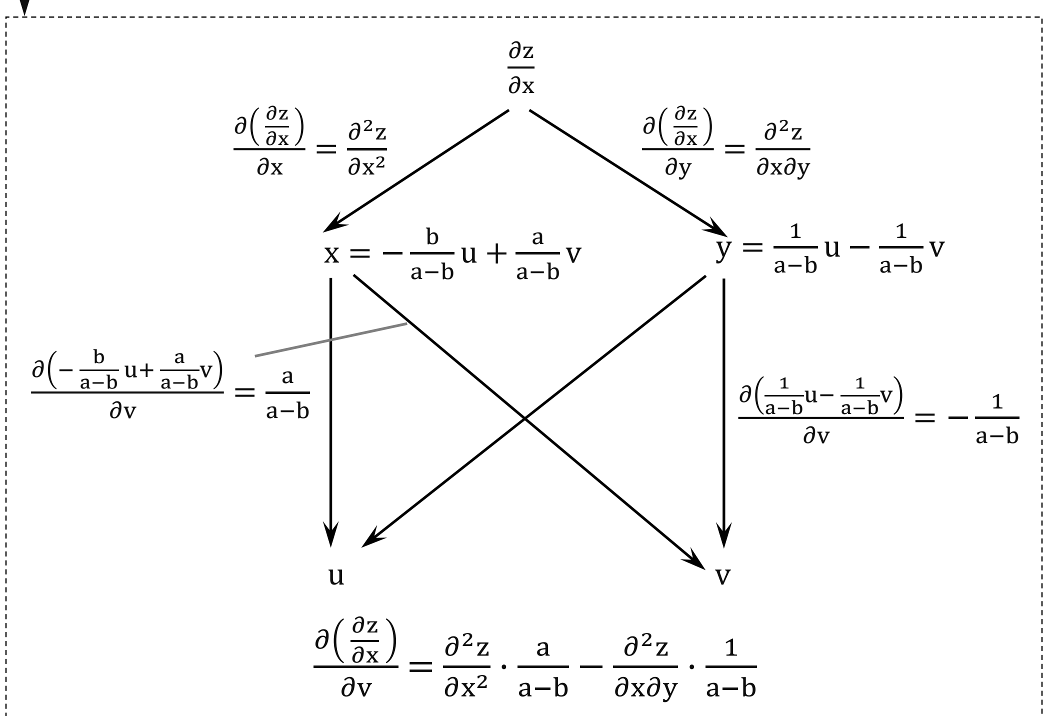
确定 a,b 的值使上式在 $\begin{cases}u=x+ay\\v=x+by\end{cases}$ 下简化为 $\frac{\partial^2 z}{\partial u\partial v}=0$

$$\begin{cases}u=x+ay\\v=x+by\end{cases}\Rightarrow\begin{cases}x=-\frac{b}{a-b}u+\frac{a}{a-b}v\\y=\frac{1}{a-b}u-\frac{1}{a-b}v\end{cases}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial u\partial v}&=\frac{\partial\left(\frac{\partial z}{\partial u}\right)}{\partial v}\\&=\frac{\partial\left(\frac{\partial z}{\partial y}\cdot\frac{1}{a-b}-\frac{\partial z}{\partial x}\cdot\frac{b}{a-b}\right)}{\partial v}\\&=\frac{\partial\left(\frac{\partial z}{\partial y}\cdot\frac{1}{a-b}\right)}{\partial v}-\frac{\partial\left(\frac{\partial z}{\partial x}\cdot\frac{b}{a-b}\right)}{\partial v}\\&=\frac{1}{a-b}\cdot\boxed{\frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial v}}-\frac{b}{a-b}\cdot\boxed{\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial v}}\\&=\frac{1}{a-b}\cdot\left(\frac{\partial^2 z}{\partial y\partial x}\cdot\frac{a}{a-b}-\frac{\partial^2 z}{\partial y^2}\cdot\frac{1}{a-b}\right)-\frac{b}{a-b}\cdot\left(\frac{\partial^2 z}{\partial x^2}\cdot\frac{a}{a-b}-\frac{\partial^2 z}{\partial x\partial y}\cdot\frac{1}{a-b}\right)\\&=\frac{a}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial y\partial x}-\frac{1}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial y^2}-\frac{ab}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial x^2}+\frac{b}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}\\&=\frac{a}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}-\frac{1}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial y^2}-\frac{ab}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial x^2}+\frac{b}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}\\&=-\frac{ab}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial x^2}+\frac{a+b}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}-\frac{1}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial y^2}\end{aligned}$$







$\therefore\ -\frac{ab}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial x^2}+\frac{a+b}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial x\partial y}-\frac{1}{(a-b)^2}\cdot\frac{\partial^2 z}{\partial y^2}=0$

$\because\quad 4\cdot\frac{\partial^2 z}{\partial x^2}+12\cdot\frac{\partial^2 z}{\partial x\partial y}+5\cdot\frac{\partial^2 z}{\partial y^2}=0$

$\therefore\ \frac{-\frac{ab}{(a-b)^2}}{4}=\frac{\frac{a+b}{(a-b)^2}}{12}=\frac{-\frac{1}{(a-b)^2}}{5}$

$\Rightarrow\ \frac{-ab}{4}=\frac{a+b}{12}=\frac{-1}{5}$

$\Rightarrow\begin{cases}a=-\frac{2}{5}\\b=-2\end{cases}$ 或 $\begin{cases}a=-2\\b=-\frac{2}{5}\end{cases}$

求全微分

若 z 可微 且 z=z(x,y)，
则 $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

例1. 已知 $z=(1+x^2+y^2)^{xy}$，试求 dz

$$dz=\boxed{\frac{\partial z}{\partial x}}dx+\boxed{\frac{\partial z}{\partial y}}dy$$

$$=[2x^2y\cdot(1+x^2+y^2)^{xy-1}+y\cdot(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)]\,dx+[2xy^2\cdot(1+x^2+y^2)^{xy-1}+x\cdot(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)]\,dy$$

$$\begin{tikzcd}
& z=(1+x^2+y^2)^{xy}=u^v \arrow[dl, "xy\cdot(1+x^2+y^2)^{xy-1}"'] \arrow[dr, "(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)"] & \\
u=1+x^2+y^2 \arrow[d, "2x"'] \arrow[drr, "2y" near start] & & v=xy \arrow[d, "x"] \arrow[dll, "y"' near start] \\
x & & y
\end{tikzcd}$$

$$\frac{\partial z}{\partial x}=xy\cdot(1+x^2+y^2)^{xy-1}\cdot 2x+(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)\cdot y$$
$$=2x^2y\cdot(1+x^2+y^2)^{xy-1}+y\cdot(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)$$

$$\frac{\partial z}{\partial y}=xy\cdot(1+x^2+y^2)^{xy-1}\cdot 2y+(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)\cdot x$$
$$=2xy^2\cdot(1+x^2+y^2)^{xy-1}+x\cdot(1+x^2+y^2)^{xy}\cdot\ln(1+x^2+y^2)$$

例2. 若 z=z(x,y) 由 $e^z+xyz+x+\cos x=2$ 确定，试求 dz

若 z 可微 且 z=z(x,y)，
则 $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$$dz=\boxed{\frac{\partial z}{\partial x}}dx+\boxed{\frac{\partial z}{\partial y}}dy$$

$$=\frac{\sin x-yz-1}{e^z+xy}dx+\left(-\frac{xz}{e^z+xy}\right)dy$$

$$=\frac{\sin x-yz-1}{e^z+xy}dx-\frac{xz}{e^z+xy}dy$$

$e^z+xyz+x+\cos x-2=0$

$F=e^z+xyz+x+\cos x-2$

$$\frac{\partial F}{\partial x}=\frac{\partial(e^z+xyz+x+\cos x-2)}{\partial x}$$
$$=\frac{\partial(e^z)}{\partial x}+\frac{\partial(xyz)}{\partial x}+\frac{dx}{dx}+\frac{d(\cos x)}{dx}-\frac{\partial 2}{\partial x}$$
$$=\ 0\ +\ yz\ +1\ +(-\sin x)-\ 0$$
$$=yz+1-\sin x$$

$$\frac{\partial F}{\partial y}=\frac{\partial(e^z+xyz+x+\cos x-2)}{\partial y}$$
$$=\frac{\partial(e^z)}{\partial y}+\frac{\partial(xyz)}{\partial y}+\frac{\partial x}{\partial y}+\frac{\partial(\cos x)}{\partial y}-\frac{\partial 2}{\partial y}$$
$$=\ 0\ +\ xz\ +0+\ 0\ -0$$
$$=xz$$

$$\frac{\partial F}{\partial z}=\frac{\partial(e^z+xyz+x+\cos x-2)}{\partial z}$$
$$=\frac{d(e^z)}{dz}+\frac{\partial(xyz)}{\partial z}+\frac{\partial x}{\partial z}+\frac{\partial(\cos x)}{\partial z}-\frac{\partial 2}{\partial z}$$
$$=\ e^z\ +\ xy\ +0+\ 0\ -0$$
$$=e^z+xy$$

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$
$$=-\frac{yz+1-\sin x}{e^z+xy}$$
$$=\frac{\sin x-yz-1}{e^z+xy}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$
$$=-\frac{xz}{e^z+xy}$$

例3. 已知 $f(x,y)=x^2+2y^2+(y-2)\cdot\arcsin\frac{x}{1+xy}$，试求 $df(0,2)$

若 f 可微 且 f=f(x,y)，
则 $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$

$$df(x_0,y_0)=\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}dx+\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}dy$$

$$\begin{aligned}df(0,2)&=\left.\frac{\partial f}{\partial x}\right|_{(0,2)}dx+\left.\frac{\partial f}{\partial y}\right|_{(0,2)}dy\\&=\boxed{f_x'(0,2)}dx+\boxed{f_y'(0,2)}dy\\&=\ \ 0\ \ dx+\ \ 8\ \ dy\\&=8\,dy\end{aligned}$$

$$\begin{aligned}f_x'(0,2)&=\left.\frac{d[f(x,2)]}{dx}\right|_{x=0}\\&=\left.\frac{d(x^2+8)}{dx}\right|_{x=0}\\&=2x\,|_{x=0}\\&=2\cdot 0\\&=0\end{aligned}$$

$$\begin{aligned}f_y'(0,2)&=\left.\frac{d[f(0,y)]}{dy}\right|_{y=2}\\&=\left.\frac{d(2y^2)}{dy}\right|_{y=2}\\&=4y\,|_{y=2}\\&=4\cdot 2\\&=8\end{aligned}$$

判断题：

例4. 已知 $f'_x(0,0)=3$、$f'_y(0,0)=1$，

则必有 $df(0,0)=3\,dx+dy$　（ ✗ ）

若 f 可微 且 f=f(x,y)，
则 $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$

$$df(x_0,y_0)=\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}dx+\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}dy$$

公式拓展：

f=f(x,y) ⟶

若 f 可微 且 f=f(x,y)，
则 $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$

$$df(x_0,y_0)=\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}dx+\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}dy$$

u=u(x,y,z) ⟶

若 u 可微 且 u=u(x,y,z)，
则 $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

$$du(x_0,y_0,z_0)=\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0,z_0)}dx+\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0,z_0)}dy+\left.\frac{\partial u}{\partial z}\right|_{(x_0,y_0,z_0)}dz$$

v=v(x,y,z,w) ⟶

若 v 可微 且 v=v(x,y,z,w)，
则 $dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz+\frac{\partial v}{\partial w}dw$

$$dv(x_0,y_0,z_0,w_0)=\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0,z_0,w_0)}dx+\left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0,z_0,w_0)}dy+\left.\frac{\partial v}{\partial z}\right|_{(x_0,y_0,z_0,w_0)}dz+\left.\frac{\partial v}{\partial w}\right|_{(x_0,y_0,z_0,w_0)}dw$$

已知全微分，求全微分里的未知数

公式：

$dz = P\,dx + Q\,dy$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

例1. 已知 $(x^2 + axy)\,dx + (x^2 + 3y^2)\,dy$ 为某函数全微分，

试确定 a 的值

① $P = x^2 + axy$

$Q = x^2 + 3y^2$

② 令 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

则 $\frac{\partial(x^2+axy)}{\partial y} = \frac{\partial(x^2+3y^2)}{\partial x}$

$\Rightarrow \quad ax = 2x$

$\Rightarrow \quad a = 2$

例2. 已知 $(a \cdot x \cdot y^3 - y^2 \cdot \cos x)\,dx + (1 + b \cdot y \cdot \sin x + 3x^2 \cdot y^2)\,dy$

是某函数的全微分，求 a、b

① $P = a \cdot x \cdot y^3 - y^2 \cdot \cos x$

$Q = 1 + b \cdot y \cdot \sin x + 3x^2 \cdot y^2$

② 令 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

则 $\frac{\partial(a\cdot x\cdot y^3 - y^2\cdot\cos x)}{\partial y} = \frac{\partial(1+b\cdot y\cdot\sin x+3x^2\cdot y^2)}{\partial x}$

$\Rightarrow 3a\cdot x\cdot y^2 - 2y\cdot\cos x = b\cdot y\cdot\cos x + 6x\cdot y^2$

$\Rightarrow 3a\cdot x\cdot y^2 - 2y\cdot\cos x - b\cdot y\cdot\cos x - 6x\cdot y^2 = 0$

$\Rightarrow (3a-6)\cdot x\cdot y^2 - (2+b)\cdot y\cdot\cos x = 0$

$\Rightarrow \begin{cases} 3a - 6 = 0 \\ 2 + b = 0 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = -2 \end{cases}$

判断函数在点 (x_0,y_0) 处是否可微

公式：

$$\lim_{\substack{x\to x_0\\y\to y_0}}\frac{[f(x,y)-f(x_0,y_0)]-[f'_x(x_0,y_0)\cdot(x-x_0)+f'_y(x_0,y_0)\cdot(y-y_0)]}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

例1. 已知 $f(x,y)$ 满足 $\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0$，请判断 $f(x,y)$ 在 $(0,0)$ 处是否可微

① $f'_x(0,0)$ 与 $f'_y(0,0)$ 都存在 且 $f'_x(0,0)=f'_y(0,0)=0$

$$f'_x(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}$$

$$f'_y(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y}$$

$$\lim_{\substack{x\to0\\y\to0}}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0$$

$$\Rightarrow\lim_{x\to0}\frac{f(x,0)-f(0,0)}{\sqrt{x^2+0^2}}=0 \qquad \Rightarrow\lim_{y\to0}\frac{f(0,y)-f(0,0)}{\sqrt{0^2+y^2}}=0$$

$$\Rightarrow\lim_{x\to0}\frac{f(x,0)-f(0,0)}{\sqrt{x^2}}=0 \qquad \Rightarrow\lim_{y\to0}\frac{f(0,y)-f(0,0)}{\sqrt{y^2}}=0$$

$$\Rightarrow\lim_{x\to0}\frac{f(x,0)-f(0,0)}{|x|}=0 \qquad \Rightarrow\lim_{y\to0}\frac{f(0,y)-f(0,0)}{|y|}=0$$

$$\Rightarrow\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}=0 \qquad \Rightarrow\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y}=0$$

$\therefore f'_x(0,0)=0$ $\qquad$ $\therefore f'_y(0,0)=0$

② $\lim\limits_{\substack{x\to0\\y\to0}}\frac{[f(x,y)-f(0,0)]-[f'_x(0,0)\cdot(x-0)+f'_y(0,0)\cdot(y-0)]}{\sqrt{(x-0)^2+(y-0)^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{[f(x,y)-f(0,0)]-[f'_x(0,0)\cdot x+f'_y(0,0)\cdot y]}{\sqrt{x^2+y^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{[f(x,y)-f(0,0)]-[0\cdot x+0\cdot y]}{\sqrt{x^2+y^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}$

$=0$

$\therefore f(x,y)$ 在 $(0,0)$ 处可微

例2. 已知 $f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$，请判断

$f(x,y)$ 在 $(0,0)$ 处是否可微

① $f'_x(0,0)$ 与 $f'_y(0,0)$ 都存在 且 $f'_x(0,0)=f'_y(0,0)=0$

$$f'_x(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to0}\frac{f(x,0)-0}{x-0}=\lim_{x\to0}\frac{0-0}{x-0}=\lim_{x\to0}\frac{0}{x}=\lim_{x\to0}0=0$$

$$f'_y(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to0}\frac{f(0,y)-0}{y-0}=\lim_{y\to0}\frac{0-0}{y-0}=\lim_{y\to0}\frac{0}{y}=\lim_{y\to0}0=0$$

∴ $(0,0)$ 处两个偏导 $f'_x(0,0)$、$f'_y(0,0)$ 都存在，均为 0

② $\lim\limits_{\substack{x\to0\\y\to0}}\frac{[f(x,y)-f(0,0)]-[f'_x(0,0)\cdot(x-0)+f'_y(0,0)\cdot(y-0)]}{\sqrt{(x-0)^2+(y-0)^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{[f(x,y)-f(0,0)]-[f'_x(0,0)\cdot x+f'_y(0,0)\cdot y]}{\sqrt{x^2+y^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{[f(x,y)-f(0,0)]-[0\cdot x+0\cdot y]}{\sqrt{x^2+y^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)-0}{\sqrt{x^2+y^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{\frac{x^2y}{x^2+y^2}-0}{\sqrt{x^2+y^2}}$

$=\lim\limits_{\substack{x\to0\\y\to0}}\frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}}$ 不存在 ⟶

① 找出一些线 ，使其经过 $(0,0)$　　∴ 找到的线为：$y=x$，$y=\sqrt{x}$

② 当 $y=x$ 时，

用x替换原式中的y

原式 $=\lim\limits_{\substack{x\to0\\x\to0}}\frac{x^2x}{(x^2+x^2)^{\frac{3}{2}}}$

即 $=\lim\limits_{x\to0}\frac{x^3}{(2x^2)^{\frac{3}{2}}}=\lim\limits_{x\to0}\frac{x^3}{2^{\frac{3}{2}}\cdot x^3}=\lim\limits_{x\to0}\frac{1}{2^{\frac{3}{2}}}=\frac{\sqrt{2}}{4}$

当 $y=\sqrt{x}$ 时，

用$\sqrt{x}$替换原式中的y

原式 $=\lim\limits_{\substack{x\to0\\\sqrt{x}\to0}}\frac{x^2\sqrt{x}}{\left[x^2+(\sqrt{x})^2\right]^{\frac{3}{2}}}$

即 $=\lim\limits_{x\to0}\frac{x^{\frac{5}{2}}}{[x(x+1)]^{\frac{3}{2}}}=\lim\limits_{x\to0}\frac{x}{(x+1)^{\frac{3}{2}}}=\frac{0}{(0+1)^{\frac{3}{2}}}=0$

③ ∵ $\frac{\sqrt{2}}{4}\neq0$

∴ $\lim\limits_{\substack{x\to0\\y\to0}}\frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}}$ 不存在

做题方法

① 找出一些线 $y=?x$ (或 $x=?y$)，
　使其经过 (x_0, y_0)
② 用 $?x$ (或 $?y$)替换式子中的 y (或 x)，
　算出对应的极限
③ 若 ② 中算出的极限值 $\begin{cases}\text{不全相等}\\\text{有不存在的}\end{cases}$，
　则重极限不存在

∴ $f(x,y)$ 在 $(0,0)$ 处不可微

例3. 连续函数 $z=f(x,y)$ 满足 $\lim\limits_{\substack{x\to 0\\ y\to 1}}\frac{f(x,y)-2x+y-2}{\sqrt{x^2+(y-1)^2}}=0$，

试求 $dz(0,1)$

$$\boxed{dz(0,1)=\left.\frac{\partial z}{\partial x}\right|_{(0,1)}dx+\left.\frac{\partial z}{\partial y}\right|_{(0,1)}dy}$$

$$=f'_x(0,1)\,dx+f'_y(0,1)\,dy$$

$$\lim_{\substack{x\to 0\\ y\to 1}}\frac{[f(x,y)-f(0,1)]-[f'_x(0,1)\cdot(x-0)+f'_y(0,1)\cdot(y-1)]}{\sqrt{(x-0)^2+(y-1)^2}}=0$$

$$\lim_{\substack{x\to 0\\ y\to 1}}\frac{[f(x,y)-\boxed{f(0,1)}]-f'_x(0,1)\cdot x-f'_y(0,1)\cdot(y-1)}{\sqrt{x^2+(y-1)^2}}=0$$

$$\lim_{\substack{x\to 0\\ y\to 1}}\frac{[f(x,y)-1]-f'_x(0,1)\cdot x-f'_y(0,1)\cdot(y-1)}{\sqrt{x^2+(y-1)^2}}=0$$

若 z 可微 且 $z=z(x,y)$，
则 $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$$dz(x_0,y_0)=\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}dx+\left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}dy$$

$$\lim_{\substack{x\to 0\\ y\to 1}}\frac{f(x,y)-2x+y-2}{\sqrt{x^2+(y-1)^2}}=0$$

$$\Longrightarrow\quad \frac{f(0,1)-2\cdot 0+1-2}{\sqrt{0^2+(1-1)^2}}=0$$

$$\Longrightarrow\quad \frac{f(0,1)-1}{0}=0$$

$\frac{\text{不为0的数}}{0}=\infty$，$\frac{0}{0}$才可能 $=0$

$$\Longrightarrow\quad f(0,1)-1=0$$

$$\Longrightarrow\quad f(0,1)=1$$

令 $[f(x,y)-1]-f'_x(0,1)\cdot x-f'_y(0,1)\cdot(y-1)=f(x,y)-2x+y-2$

$$\Longrightarrow\quad f(x,y)-1-f'_x(0,1)\cdot x-f'_y(0,1)\cdot y+f'_y(0,1)=f(x,y)-2x+y-2$$

$$\Longrightarrow[-f'_x(0,1)+2]\cdot x-[f'_y(0,1)+1]\cdot y+[f'_y(0,1)+1]=0$$

$$\therefore\begin{cases}-f'_x(0,1)+2=0\\ f'_y(0,1)+1=0\\ f'_y(0,1)+1=0\end{cases}\Longrightarrow\begin{cases}f'_x(0,1)=2\\ f'_y(0,1)=-1\end{cases}$$

$dz(0,1)=2\,dx+(-1)\,dy$

$\quad=2\,dx-dy$

判断函数在点 (x_0,y_0) 处是否连续

公式：

若 $\lim\limits_{\substack{x\to x_0\\y\to y_0}} f(x,y)=f(x_0,y_0)$，

则 函数在点 (x_0,y_0) 处连续；

否则，不连续

例1. 已知 $f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$，请判断

f(x,y) 在点 (0,0) 处是否连续

① $\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)=\boxed{\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{x^2y}{x^2+y^2}}=0$

$f(0,0)=0$

② $\because \lim\limits_{\substack{x\to0\\y\to0}} f(x,y)=f(0,0)$

$\therefore$ f(x,y) 在点 (0,0) 处连续

原式 $=\lim\limits_{\substack{x\to0\\y\to0}}\left(\dfrac{x^2}{x^2+y^2}\cdot y\right)$

$C\cdot 0=0$

$=0$

例2. 已知 $f(x,y)=\begin{cases}(x^2+y^2)\ln(x^2+y^2), & x^2+y^2\neq0\\ 0, & x^2+y^2=0\end{cases}$，

请判断 f(x,y) 在点 (0,0) 处是否连续

① $\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)=\boxed{\lim\limits_{\substack{x\to0\\y\to0}}(x^2+y^2)\ln(x^2+y^2)}=0$

$f(0,0)=0$

② $\because \lim\limits_{\substack{x\to0\\y\to0}} f(x,y)=f(0,0)$

$\therefore$ f(x,y) 在点 (0,0) 处连续

设 $\begin{cases}x=0+r\cos\theta\\y=0+r\sin\theta\end{cases}\Rightarrow\begin{cases}x=r\cos\theta\\y=r\sin\theta\end{cases}\Rightarrow x^2+y^2=r^2$

$\lim\limits_{r\to0} r^2\cdot\ln r^2$ → $\boxed{0\cdot\infty \text{ 型，将 } r^2 \text{ 变为 } \frac{1}{\frac{1}{r^2}}}$

$=\lim\limits_{r\to0}\dfrac{\ln r^2}{\frac{1}{r^2}}$ → $\boxed{\frac{\infty}{\infty}\text{ 型，利用洛必达法则来做}}$

$=\lim\limits_{r\to0}\dfrac{(\ln r^2)'}{\left(\frac{1}{r^2}\right)'}$

$=\lim\limits_{r\to0}\dfrac{\frac{2r}{r^2}}{\frac{-2}{r^3}}$

$=\lim\limits_{r\to0}(-r^2)$

$=-0^2$

$=0$

例3. 已知 $f(x,y)=\begin{cases} \dfrac{x^2\cdot\sin\frac{1}{x^2+y^2}+y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$，

请判断 f(x,y) 在点 (0,0) 处是否连续

① $\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)=\boxed{\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{x^2\cdot\sin\frac{1}{x^2+y^2}+y^2}{x^2+y^2}}$ (不存在)

$f(0,0)=0$

② $\because \lim\limits_{\substack{x\to0\\y\to0}} f(x,y)$ 不存在

$\therefore \lim\limits_{\substack{x\to0\\y\to0}} f(x,y)=f(0,0)$ 不可能成立

$\therefore$ f(x,y) 在点 (0,0) 处不连续

① 找出一些线 y =?x (或 x=?y)，使其经过 (0, 0)

如 y = 0，y = x，x = 0

② 当 y = 0 时，原式$=\lim\limits_{\substack{x\to0\\0\to0}}\dfrac{x^2\cdot\sin\frac{1}{x^2+0^2}+0^2}{x^2+0^2}$

$=\lim\limits_{x\to0}\dfrac{x^2\cdot\sin\frac{1}{x^2+0^2}+0^2}{x^2+0^2}$

$=\lim\limits_{x\to0}\dfrac{x^2\cdot\sin\frac{1}{x^2}}{x^2}$

$=\lim\limits_{x\to0}\sin\dfrac{1}{x^2}$

$=\sin\dfrac{1}{0^2}$

$=\sin\infty$ 极限不存在

③ $\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{x^2\cdot\sin\frac{1}{x^2+y^2}+y^2}{x^2+y^2}$ 不存在

例4. 已知 $f(x,y)=\begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y)\neq(0,0) \\ a, & (x,y)=(0,0) \end{cases}$，且 f(x,y) 在

点 (0,0) 处连续，试求 a

$\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)=f(0,0)$

$\Rightarrow \boxed{\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{x^2y}{x^2+y^2}}=a$

$\Rightarrow 0=a$

$\therefore a=0$

原式$=\lim\limits_{\substack{x\to0\\y\to0}}\left(\dfrac{x^2}{x^2+y^2}\cdot y\right)$

$C \cdot 0=0$

$=0$

连续、可导、可微的关系

公式:

$f(x,y)$ 在 (x_0,y_0) 处:

$$
\begin{tikzcd}
 & \text{一阶偏导数连续} \arrow[d, shift right] & \\
 & \text{可微} \arrow[u, shift right, "\times"'] \arrow[dl, shift right] \arrow[dr, shift left] & \\
\text{连续} \arrow[ur, shift right, "\times"'] & & \text{一阶偏导数存在} \arrow[ul, shift left, "\times"]
\end{tikzcd}
$$

例1. 二元函数 f(x,y) 在 (x_0,y_0) 处的两个偏导数 $f'_x(x_0,y_0)$、$f'_y(x_0,y_0)$ 存在，是 f(x,y) 在该点连续的 (D)

(A) 充分条件而非必要条件　　(B) 必要条件而非充分条件

(C) 充分必要条件　　(D) 既非充分条件又非必要条件

例2. 考虑二元函数 f(x,y) 的下面 4 条性质:

① f(x,y) 在点 (x_0,y_0) 处连续　　② f(x,y) 在点 (x_0,y_0) 处的两个偏导数连续

③ f(x,y) 在点 (x_0,y_0) 处可微　　④ f(x,y) 在点 (x_0,y_0) 处的两个偏导数存在

若用 "$P \Rightarrow Q$" 表示可由性质 P 推出性质 Q，则有 (D)

(A) ③⟹①⟹④　　(B) ③⟹②⟹①　　(C) ③⟹④⟹①　　(D) ②⟹③⟹①

例3. 函数 f(x,y) 在点 (x_0,y_0) 处的偏导数存在，是 f(x,y) 在该点处 (C)

(A) 连续的充分条件　　(B) 连续的必要条件

(C) 可微的必要条件　　(D) 可微的充分条件

若 $A \Rightarrow B$，
则 A 是 B 的充分条件，
B 是 A 的必要条件

公式拓展:

$f(x,y)$ 在 (x_0,y_0) 处:

$$
\begin{tikzcd}
 & \text{一阶偏导数连续} \arrow[d, shift right] & \\
 & \text{可微} \arrow[u, shift right, "\times"'] \arrow[dl, shift right] \arrow[dr, shift left] & \\
\text{连续} \arrow[ur, shift right, "\times"'] & & \text{一阶偏导数存在} \arrow[ul, shift left, "\times"]
\end{tikzcd}
$$

一阶偏导数连续 $\left(即 \begin{cases} \lim\limits_{\substack{x\to x_0\\ y\to y_0}} f'_x(x,y) = f'_x(x_0,y_0) \\ \lim\limits_{\substack{x\to x_0\\ y\to y_0}} f'_y(x,y) = f'_y(x_0,y_0) \end{cases} \right)$

可微 $\left(即 \begin{cases} f'_x(x_0,y_0) \text{ 与 } f'_y(x_0,y_0) \text{ 存在} \\ \lim\limits_{\substack{x\to x_0\\ y\to y_0}} \dfrac{[f(x,y)-f(x_0,y_0)]-[f'_x(x_0,y_0)\cdot(x-x_0)+f'_y(x_0,y_0)\cdot(y-y_0)]}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0 \end{cases} \right)$

$\left(即 \lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0,y_0) \right)$ 连续

一阶偏导数存在 (即 $f'_x(x_0,y_0)$ 与 $f'_y(x_0,y_0)$ 存在)

一般函数求无条件极值

例1. 求 $z=x\cdot e^{-\frac{x^2+y^2}{2}}$ 的极值

① $\frac{\partial z}{\partial x}=\frac{\partial\left(x\cdot e^{-\frac{x^2+y^2}{2}}\right)}{\partial x}$

$=(1-x^2)\cdot e^{-\frac{x^2+y^2}{2}}$

$\frac{\partial z}{\partial y}=\frac{\partial\left(x\cdot e^{-\frac{x^2+y^2}{2}}\right)}{\partial y}$

$=-xy\cdot e^{-\frac{x^2+y^2}{2}}$

② $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}\Rightarrow\begin{cases}(1-x^2)\cdot e^{-\frac{x^2+y^2}{2}}=0\\ -xy\cdot e^{-\frac{x^2+y^2}{2}}=0\end{cases}\Rightarrow\begin{cases}1-x^2=0\\ -xy=0\end{cases}\Rightarrow\begin{cases}x=\pm1\\ y=0\end{cases}\Rightarrow\begin{cases}x_1=-1\\ y_1=0\end{cases}$ 或 $\begin{cases}x_2=1\\ y_2=0\end{cases}$

$\therefore$ 满足 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}$ 的两组解：$(-1,0)$、$(1,0)$

③ $A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{(-1,0)}$

$=\left.\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x}\right|_{(-1,0)}$

$=\left.\frac{\partial\left[(1-x^2)\cdot e^{-\frac{x^2+y^2}{2}}\right]}{\partial x}\right|_{(-1,0)}$

$=\left[(x^3-3x)\cdot e^{-\frac{x^2+y^2}{2}}\right]\Big|_{(-1,0)}$

$=[(-1)^3-3\cdot(-1)]\cdot e^{-\frac{(-1)^2+0^2}{2}}$

$=2\cdot e^{-\frac{1}{2}}$

$B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(-1,0)}$

$=\left.\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y}\right|_{(-1,0)}$

$=\left.\frac{\partial\left[(1-x^2)\cdot e^{-\frac{x^2+y^2}{2}}\right]}{\partial y}\right|_{(-1,0)}$

$=\left[-(1-x^2)\cdot y\cdot e^{-\frac{x^2+y^2}{2}}\right]\Big|_{(-1,0)}$

$=-[1-(-1)^2]\cdot 0\cdot e^{-\frac{(-1)^2+0^2}{2}}$

$=0$

$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{(-1,0)}$

$=\left.\frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial y}\right|_{(-1,0)}$

$=\left.\frac{\partial\left[-xy\cdot e^{-\frac{x^2+y^2}{2}}\right]}{\partial y}\right|_{(-1,0)}$

$=\left[-x\cdot(1-y^2)\cdot e^{-\frac{x^2+y^2}{2}}\right]\Big|_{(-1,0)}$

$=-(-1)\cdot(1-0^2)\cdot e^{-\frac{(-1)^2+0^2}{2}}$

$=e^{-\frac{1}{2}}$

$A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{(1,0)}$

$=\left.\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x}\right|_{(1,0)}$

$=\left.\frac{\partial\left[(1-x^2)\cdot e^{-\frac{x^2+y^2}{2}}\right]}{\partial x}\right|_{(1,0)}$

$=\left[(x^3-3x)\cdot e^{-\frac{x^2+y^2}{2}}\right]\Big|_{(1,0)}$

$=(1^3-3\cdot 1)\cdot e^{-\frac{1^2+0^2}{2}}$

$=-2\cdot e^{-\frac{1}{2}}$

$B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(1,0)}$

$=\left.\frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y}\right|_{(1,0)}$

$=\left.\frac{\partial\left[(1-x^2)\cdot e^{-\frac{x^2+y^2}{2}}\right]}{\partial y}\right|_{(1,0)}$

$=\left[-(1-x^2)\cdot y\cdot e^{-\frac{x^2+y^2}{2}}\right]\Big|_{(1,0)}$

$=-(1-1^2)\cdot 0\cdot e^{-\frac{1^2+0^2}{2}}$

$=0$

$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{(1,0)}$

$=\left.\frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial y}\right|_{(1,0)}$

$=\left.\frac{\partial\left[-xy\cdot e^{-\frac{x^2+y^2}{2}}\right]}{\partial y}\right|_{(1,0)}$

$=\left[-x\cdot(1-y^2)\cdot e^{-\frac{x^2+y^2}{2}}\right]\Big|_{(1,0)}$

$=-1\cdot(1-0^2)\cdot e^{-\frac{1^2+0^2}{2}}$

$=-e^{-\frac{1}{2}}$

④ 解为 $(-1,0)$ 时：

$B^2-AC=0^2-\left(2\cdot e^{-\frac{1}{2}}\right)\cdot e^{-\frac{1}{2}}$

$=-2\cdot e^{-1}$

$=-\frac{2}{e}$

$\therefore(-1,0)$ 为极小值点

$z_{极小值}=(-1)\cdot e^{-\frac{(-1)^2+0^2}{2}}=-e^{-\frac{1}{2}}$

解为 $(1,0)$ 时：

$B^2-AC=0^2-\left(-2\cdot e^{-\frac{1}{2}}\right).\left(-e^{-\frac{1}{2}}\right)$

$=-2\cdot e^{-1}$

$=-\frac{2}{e}$

$\therefore(1,0)$ 为极大值点

$z_{极大值}=1\cdot e^{-\frac{1^2+0^2}{2}}=e^{-\frac{1}{2}}$

方法：

求满足 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}$ 的所有解 (x_i,y_i)

求出各组解的

$A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{(x_i,y_i)}$

$B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(x_i,y_i)}$

$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{(x_i,y_i)}$

求 B^2-AC

$B^2-AC\begin{cases}<0\begin{cases}A<0 & 极大值点\\ A>0 & 极小值点\end{cases}\\ =0 \quad 不确定\\ >0 \quad 不是极值点\end{cases}$

例2. 求 $z=\left(y+\frac{x^3}{3}\right)\cdot e^{x+y}$ 的极值

① $\frac{\partial z}{\partial x}=\frac{\partial\left[\left(y+\frac{x^3}{3}\right)\cdot e^{x+y}\right]}{\partial x}$

$=\left(x^2+y+\frac{x^3}{3}\right)\cdot e^{x+y}$

$\frac{\partial z}{\partial y}=\frac{\partial\left[\left(y+\frac{x^3}{3}\right)\cdot e^{x+y}\right]}{\partial y}$

$=\left(1+y+\frac{x^3}{3}\right)\cdot e^{x+y}$

② $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}\Longrightarrow\begin{cases}\left(x^2+y+\frac{x^3}{3}\right)\cdot e^{x+y}=0\\ \left(1+y+\frac{x^3}{3}\right)\cdot e^{x+y}=0\end{cases}\Longrightarrow\begin{cases}x^2+y+\frac{x^3}{3}=0 & (1)\\ 1+y+\frac{x^3}{3}=0 & (2)\end{cases}$

用 (1)式 − (2)式：$x^2-1=0\Longrightarrow x=1$ 或 -1

当 $x=1$ 时，
将 $x=1$ 代入 (2)式：
$1+y+\frac{1^3}{3}=0\Longrightarrow y=-\frac{4}{3}$

当 $x=-1$ 时，
将 $x=-1$ 代入 (2)式：
$1+y+\frac{(-1)^3}{3}=0\Longrightarrow y=-\frac{2}{3}$

∴ 满足 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}$ 的两组解：$\left(1,-\frac{4}{3}\right)$、$\left(-1,-\frac{2}{3}\right)$

③ $A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{\left(1,-\frac{4}{3}\right)}=3\cdot e^{-\frac{1}{3}}$　　$B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{\left(1,-\frac{4}{3}\right)}=e^{-\frac{1}{3}}$　　$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{\left(1,-\frac{4}{3}\right)}=e^{-\frac{1}{3}}$

$A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{\left(-1,-\frac{2}{3}\right)}=-e^{-\frac{5}{3}}$　　$B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{\left(-1,-\frac{2}{3}\right)}=e^{-\frac{5}{3}}$　　$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{\left(-1,-\frac{2}{3}\right)}=e^{-\frac{5}{3}}$

④ 解为 $\left(1,-\frac{4}{3}\right)$ 时：

$B^2-AC=\left(e^{-\frac{1}{3}}\right)^2-\left(3\cdot e^{-\frac{1}{3}}\right)\cdot e^{-\frac{1}{3}}$

$=-2\cdot e^{-\frac{2}{3}}$

∴ $\left(1,-\frac{4}{3}\right)$ 为极小值点

$z_{极小值}=\left(-\frac{4}{3}+\frac{1^3}{3}\right)\cdot e^{1+\left(-\frac{4}{3}\right)}=-e^{-\frac{1}{3}}$

解为 $\left(-1,-\frac{2}{3}\right)$ 时：

$B^2-AC=\left(e^{-\frac{5}{3}}\right)^2-\left(-e^{-\frac{5}{3}}\right)\cdot e^{-\frac{5}{3}}$

$=2\cdot e^{-\frac{10}{3}}$

∴ $\left(-1,-\frac{2}{3}\right)$ 不是极值点

例3 . 求函数 $z=x^2y+xy^2$ 的极值

① $\frac{\partial z}{\partial x}=\frac{\partial(x^2y+xy^2)}{\partial x}=2xy+y^2$

$\frac{\partial z}{\partial y}=\frac{\partial(x^2y+xy^2)}{\partial y}=x^2+2xy$

② $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}\Longrightarrow\begin{cases}2xy+y^2=0 & (1)\\ x^2+2xy=0 & (2)\end{cases}$

$\xlongequal{(1)-(2)\text{得}}$ $y^2-x^2=0 \Longrightarrow x=\pm y$ (3)

$\xlongequal{(3)\text{带入}(2)\text{得}}$ $y^2\pm 2y^2=0 \Longrightarrow y=0$ (4)

$\xlongequal{(4)\text{带入}(3)\text{得}}$ $\begin{cases}y=0\\ x=0\end{cases}$ ∴满足 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}$ 的一组解：(0,0)

③ $A=\frac{\partial^2 z}{\partial x^2}\Big|_{(0,0)}=\frac{\partial(2xy+y^2)}{\partial x}\Big|_{(0,0)}=(2y)|_{(0,0)}=0$

$B=\frac{\partial^2 z}{\partial x\partial y}\Big|_{(0,0)}=\frac{\partial(2xy+y^2)}{\partial y}\Big|_{(0,0)}=(2x+2y)|_{(0,0)}=0$

$C=\frac{\partial^2 z}{\partial y^2}\Big|_{(0,0)}=\frac{\partial(x^2+2xy)}{\partial y}\Big|_{(0,0)}=(2x)|_{(0,0)}=0$

④ 解为 (0,0) 时：

$B^2-AC=0^2-0\cdot 0$
$=0$

∴不确定 (0,0) 是不是极值点

例4. 设函数 f(x) 具有二阶连续导数，且 f(x)>0，f′(0)=0，则函数

$z=f(x)\cdot\ln f(y)$ 在点(0,0) 处取得极小值的一个充分条件是 (A)

(A) f(0)>1，f′′(0)>0　　(B) f(0)>1，f′′(0)<0

(C) f(0)<1，f′′(0)>0　　(D) f(0)<1，f′′(0)<0

充分条件：在点(0,0) 处，$B^2-AC<0$ 且 A>0

$$A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{(0,0)}=\left.\frac{\partial\boxed{\frac{\partial z}{\partial x}}}{\partial x}\right|_{(0,0)}=\left.\frac{\partial[\ln f(y)\cdot f'(x)]}{\partial x}\right|_{(0,0)}=\left[\ln f(y)\cdot\frac{df'(x)}{dx}\right]\Big|_{(0,0)}=[\ln f(y)\cdot f''(x)]|_{(0,0)}=\ln f(0)\cdot f''(0)$$

$$B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(0,0)}=\left.\frac{\partial\boxed{\frac{\partial z}{\partial x}}}{\partial y}\right|_{(0,0)}=\left.\frac{\partial[\ln f(y)\cdot f'(x)]}{\partial y}\right|_{(0,0)}=\left[f'(x)\cdot\frac{d\ln f(y)}{dy}\right]\Big|_{(0,0)}=\left\{\frac{1}{f(y)}\cdot[f(y)]'\cdot f'(x)\right\}\Big|_{(0,0)}=\frac{[f'(0)]^2}{f(0)}=\frac{[0]^2}{f(0)}=0$$

$$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{(0,0)}=\left.\frac{\partial\boxed{\frac{\partial z}{\partial y}}}{\partial y}\right|_{(0,0)}=\left.\frac{\partial\left[\frac{f(x)}{f(y)}\cdot f'(y)\right]}{\partial y}\right|_{(0,0)}=\left[f(x)\cdot\frac{d\frac{f'(y)}{f(y)}}{dy}\right]\Big|_{(0,0)}=\left[f(x)\cdot\frac{f''(y)\cdot f(y)-f'(y)\cdot f'(y)}{f^2(y)}\right]\Big|_{(0,0)}$$

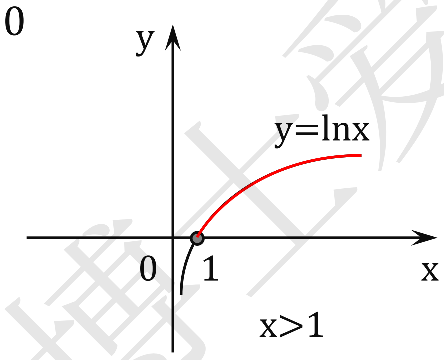
$$=f''(0)-\frac{1}{f(0)}\cdot[f'(0)]^2=f''(0)-\frac{1}{f(0)}\cdot[0]^2=f''(0)$$

$$B^2-AC<0 \text{ 且 } A>0 \Rightarrow \begin{cases}0^2-\ln f(0)\cdot f''(0)\cdot f''(0)<0\\ \ln f(0)\cdot f''(0)>0\end{cases}$$

$$\Rightarrow \begin{cases}\ln f(0)\cdot[f''(0)]^2>0\\ \ln f(0)\cdot f''(0)>0\end{cases}$$

$$\Rightarrow \begin{cases}\ln f(0)>0\\ f''(0)>0\end{cases}$$

$$\Rightarrow \begin{cases}f(0)>1\\ f''(0)>0\end{cases}$$



z=f(x)·ln f(y)=u·ln v

$\frac{\partial(u\cdot\ln v)}{\partial u}=\ln v=\ln f(y)$　　$\frac{\partial(u\cdot\ln v)}{\partial v}=\frac{u}{v}=\frac{f(x)}{f(y)}$

u=f(x)　　v=f(y)

$\frac{d[f(x)]}{dx}=f'(x)$　　$\frac{d[f(y)]}{dy}=f'(y)$

x　　y

$\frac{\partial z}{\partial x}=\ln f(y)\cdot f'(x)$　　$\frac{\partial z}{\partial y}=\frac{f(x)}{f(y)}\cdot f'(y)$

例5. 设 $z=z(x,y)$ 是由 $x^2-6xy+10y^2-2yz-z^2+18=0$ 确定的函数，

求 $z=z(x,y)$ 的极值点和极值

(1) 将等式变成 ?=0 的形式，令 F=?

(2) 求 $\frac{\partial F}{\partial x}$、$\frac{\partial F}{\partial y}$、$\frac{\partial F}{\partial z}$ (当作 z 同 x、y 无关)

(3) $\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$，$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$ $\left(\frac{\partial F}{\partial z}\neq 0\right)$

① (1) $x^2-6xy+10y^2-2yz-z^2+18=0$

$F=x^2-6xy+10y^2-2yz-z^2+18$

(2)
$$\begin{aligned}\frac{\partial F}{\partial x}&=\frac{\partial\left(x^2-6xy+10y^2-2yz-z^2+18\right)}{\partial x}\\&=\frac{d\left(x^2\right)}{dx}-\frac{\partial(6xy)}{\partial x}+\frac{\partial\left(10y^2\right)}{\partial x}-\frac{\partial(2yz)}{\partial x}-\frac{\partial\left(z^2\right)}{\partial x}+\frac{\partial(18)}{\partial x}\\&=2x-6y+0-0-0+0\\&=2x-6y\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial y}&=\frac{\partial\left(x^2-6xy+10y^2-2yz-z^2+18\right)}{\partial y}\\&=\frac{\partial\left(x^2\right)}{\partial y}-\frac{\partial(6xy)}{\partial y}+\frac{d\left(10y^2\right)}{dy}-\frac{\partial(2yz)}{\partial y}-\frac{\partial\left(z^2\right)}{\partial y}+\frac{\partial(18)}{\partial y}\\&=0-6x+20y-2z-0+0\\&=-6x+20y-2z\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial z}&=\frac{\partial\left(x^2-6xy+10y^2-2yz-z^2+18\right)}{\partial z}\\&=\frac{\partial\left(x^2\right)}{\partial z}-\frac{\partial(6xy)}{\partial z}+\frac{\partial\left(10y^2\right)}{\partial z}-\frac{\partial(2yz)}{\partial z}-\frac{d\left(z^2\right)}{dz}+\frac{\partial(18)}{\partial z}\\&=0-0+0-2y-2z+0\\&=-2y-2z\end{aligned}$$

(3)
$$\begin{aligned}\frac{\partial z}{\partial x}&=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\\&=-\frac{2x-6y}{-2y-2z}\\&=\frac{x-3y}{y+z}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y}&=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\\&=-\frac{-6x+20y-2z}{-2y-2z}\\&=\frac{-3x+10y-z}{y+z}\end{aligned}$$

② $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\\ \text{原方程}\end{cases}\Rightarrow\begin{cases}\frac{x-3y}{y+z}=0\\ \frac{-3x+10y-z}{y+z}=0\\ x^2-6xy+10y^2-2yz-z^2+18=0\end{cases}\Rightarrow\begin{cases}x-3y=0\\ -3x+10y-z=0\\ x^2-6xy+10y^2-2yz-z^2+18=0\end{cases}\Rightarrow\begin{cases}x_1=9\\ y_1=3\\ z_1=3\end{cases}$ 或 $\begin{cases}x_2=-9\\ y_2=-3\\ z_2=-3\end{cases}$

∴ 满足 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\\ \text{原方程}\end{cases}$ 的两组解：(9,3,3)、(−9,−3,−3)

③ $A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{(9,3,3)}=\frac{1}{6}$　　$B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(9,3,3)}=-\frac{1}{2}$　　$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{(9,3,3)}=\frac{5}{3}$

$A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{(-9,-3,-3)}=-\frac{1}{6}$　　$B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(-9,-3,-3)}=\frac{1}{2}$　　$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{(-9,-3,-3)}=-\frac{5}{3}$

④ 解为 (9,3,3) 时:

$$\begin{aligned}B^2-AC&=\left(-\frac{1}{2}\right)^2-\frac{1}{6}\cdot\frac{5}{3}\\&=-\frac{1}{36}\end{aligned}$$

∴ (9,3) 为极小值点

$z_{\text{极小值}}=3$

解为 (−9,−3,−3) 时:

$$\begin{aligned}B^2-AC&=\left(\frac{1}{2}\right)^2-\left(-\frac{1}{6}\right)\cdot\left(-\frac{5}{3}\right)\\&=-\frac{1}{36}\end{aligned}$$

∴ (−9, − 3) 为极大值点

$z_{\text{极大值}}=-3$

利用定义判断极值点

例1．求函数 $z=x^2y+xy^2$ 的极值

① $\frac{\partial z}{\partial x}=\frac{\partial(x^2y+xy^2)}{\partial x}=2xy+y^2$

$\frac{\partial z}{\partial y}=\frac{\partial(x^2y+xy^2)}{\partial y}=x^2+2xy$

② $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases} \Rightarrow \begin{cases}2xy+y^2=0 & (1)\\ x^2+2xy=0 & (2)\end{cases}$

$\xlongequal{(1)-(2)\text{得}}$ $y^2-x^2=0 \ \Longrightarrow\ x=\pm y$ (3)

$\xlongequal{(3)\text{带入}(2)\text{得}}$ $y^2\pm 2y^2=0 \Rightarrow y=0$ (4)

$\xlongequal{(4)\text{带入}(3)\text{得}}$ $\begin{cases}y=0\\ x=0\end{cases}$ ∴满足 $\begin{cases}\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial y}=0\end{cases}$ 的一组解：(0,0)

③ $A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{(0,0)}=0$　　$B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(0,0)}=0$　　$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{(0,0)}=0$

④ 解为 (0,0) 时：

$$\begin{aligned}B^2-AC&=0^2-0\cdot 0\\&=0\end{aligned}$$

∴不确定 (0,0) 是不是极值点

$f(0,0)=0^2\cdot 0+0\cdot 0^2=0$

$\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim\limits_{\substack{x\to 0\\ y\to 0}}(x^2y+xy^2)=0^2\cdot 0+0\cdot 0^2=0$

$\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=f(0,0)$

$\lim\limits_{\substack{x\to 0^+\\ y\to 0^+}} f(x,y)=\lim\limits_{\substack{x\to 0^+\\ y\to 0^+}}(x^2y+xy^2)>f(0,0)$

$\lim\limits_{\substack{x\to 0^-\\ y\to 0^-}} f(x,y)=\lim\limits_{\substack{x\to 0^-\\ y\to 0^-}}(x^2y+xy^2)<f(0,0)$

∴ f(0,0)不是极值

∴函数 $z=x^2y+xy^2$ 没有极值

$$\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$$

$$\Downarrow$$

$$\begin{cases}\lim\limits_{\substack{x\to 0^+\\ y\to 0^+}} f(x,y)\\ \lim\limits_{\substack{x\to 0^+\\ y\to 0^-}} f(x,y)\\ \lim\limits_{\substack{x\to 0^-\\ y\to 0^+}} f(x,y)\\ \lim\limits_{\substack{x\to 0^-\\ y\to 0^-}} f(x,y)\end{cases}$$

分析(0,0)周围的值是不是

都比f(0,0)大　　极小值

或者

都比f(0,0)小　　极大值

或者

有的比f(0,0)大，有的比f(0,0)小　　不是极值

例2．设 z=f(x,y) 在点 (0,0) 处连续，且 $\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)}{\sin(x^2+y^2)}=-1$，则判断 (0,0) 点是否是极值点

$f(0,0)$

$\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)}{\sin(x^2+y^2)}=-1$

$\Longrightarrow \frac{f(0,0)}{\sin(0^2+0^2)}=-1$

$\Longrightarrow \frac{f(0,0)}{0}=-1$

$\Longrightarrow f(0,0)=0$

$\lim\limits_{\substack{x\to0\\y\to0}}f(x,y)$

$\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)}{\sin(x^2+y^2)}<0$

$\therefore \lim\limits_{\substack{x\to0\\y\to0}}f(x,y)<0$

$\therefore \lim\limits_{\substack{x\to0\\y\to0}}f(x,y)<f(0,0)$

∴ f(0,0)是极大值

∴ (0,0)点是极大值点

$$\lim_{\substack{x\to0\\y\to0}}f(x,y)$$

$$\Downarrow$$

$$\begin{cases}\lim\limits_{\substack{x\to0^+\\y\to0^+}}f(x,y)\\ \lim\limits_{\substack{x\to0^+\\y\to0^-}}f(x,y)\\ \lim\limits_{\substack{x\to0^-\\y\to0^+}}f(x,y)\\ \lim\limits_{\substack{x\to0^-\\y\to0^-}}f(x,y)\end{cases}$$

分析(0,0)周围的值是不是

都比f(0,0)大　　极小值

或者

都比f(0,0)小　　极大值

或者

有的比f(0,0)大，有的比 f(0,0)小　　不是极值

例3．已知函数 f(x,y) 在点 (0,0) 的某个邻域内连续，且$\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)-xy}{(x^2+y^2)^2}=1$，判断(0,0)点是不是极值点

$$\lim_{\substack{x\to0\\y\to0}} f(x,y)$$

$$\Downarrow$$

$$\begin{cases}\lim\limits_{\substack{x\to0^+\\y\to0^+}} f(x,y)\\ \lim\limits_{\substack{x\to0^+\\y\to0^-}} f(x,y)\\ \lim\limits_{\substack{x\to0^-\\y\to0^+}} f(x,y)\\ \lim\limits_{\substack{x\to0^-\\y\to0^-}} f(x,y)\end{cases}$$

分析(0,0)周围的值是不是

都比f(0,0)大　　极小值

或者

都比f(0,0)小　　极大值

或者

有的比f(0,0)大，有的比f(0,0)小　　不是极值

f(0,0)

$$\lim_{\substack{x\to0\\y\to0}}\frac{f(x,y)-xy}{(x^2+y^2)^2}=1 \Rightarrow \frac{f(0,0)-0\cdot0}{(0^2+0^2)^2}=1$$

$$\Rightarrow \frac{f(0,0)}{0}=1 \Rightarrow f(0,0)=0$$

$\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)$　　$\lim\limits_{\substack{x\to0^+\\y\to0^+}} f(x,y)$、$\lim\limits_{\substack{x\to0^+\\y\to0^-}} f(x,y)$、$\lim\limits_{\substack{x\to0^-\\y\to0^+}} f(x,y)$、$\lim\limits_{\substack{x\to0^-\\y\to0^-}} f(x,y)$

$$\boxed{\lim_{\substack{x\to0\\y\to0}}\frac{f(x,y)-xy}{(x^2+y^2)^2}=1 \qquad \frac{f(x,y)-xy}{(x^2+y^2)^2}=1+\alpha,\ \lim_{\substack{x\to0\\y\to0}}\alpha=0} \longrightarrow$$

lim 带f(x,y)的式子 = A

则　带f(x,y)的式子 = A + α，limα = 0

$$\frac{f(x,y)-xy}{(x^2+y^2)^2}=1+\alpha \Rightarrow f(x,y)-xy=(1+\alpha)(x^2+y^2)^2$$

$$\Rightarrow f(x,y)=xy+(1+\alpha)(x^2+y^2)^2$$

$$\begin{aligned}\lim_{\substack{x\to0\\y\to0}} f(x,y) &= \lim_{\substack{x\to0\\y\to0}}[\,xy+(1+\alpha)(x^2+y^2)^2\,]\\ &= \lim_{\substack{x\to0\\y\to0}}[\,xy+(1+0)(x^2+y^2)^2\,]\\ &= 0\cdot0+(1+0)(0^2+0^2)^2\\ &= 0\end{aligned}$$

$$\lim_{\substack{x\to0\\y\to0}} f(x,y)=f(0,0)$$

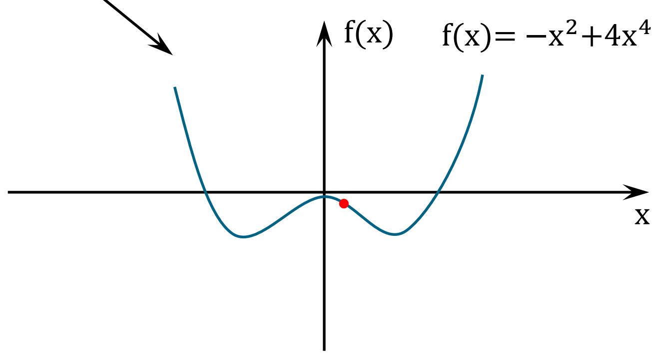
$$\lim_{\substack{x\to0^+\\y\to0^+}} f(x,y)=\lim_{\substack{x\to0^+\\y\to0^+}}[xy+(1+0)(x^2+y^2)^2]>f(0,0)$$

$\boxed{设\ y=x\ 或\ y=-x}$

$$\lim_{\substack{x\to0^+\\y\to0^-}} f(x,y)=\lim_{\substack{x\to0^+\\y\to0^-}}[xy+(1+0)(x^2+y^2)^2]=\lim_{\substack{x\to0^+\\-x\to0^-}}\{x(-x)+[x^2+(-x)^2]^2\}=\lim_{x\to0^+}(-x^2+4x^4)<f(0,0)$$

$$\lim_{\substack{x\to0^-\\y\to0^+}} f(x,y)=\lim_{\substack{x\to0^-\\y\to0^+}}[xy+(1+0)(x^2+y^2)^2]=\lim_{\substack{x\to0^-\\-x\to0^+}}\{x(-x)+[x^2+(-x)^2]^2\}$$

$$\lim_{\substack{x\to0^-\\y\to0^-}} f(x,y)=\lim_{\substack{x\to0^-\\y\to0^-}}[xy+(1+0)(x^2+y^2)^2]>f(0,0)$$



∴ f(0,0)不是极值

∴ (0,0)点不是极值点

在约束条件下找出可能的极值点

例1．找出$f=\frac{1}{13}(2x+3y-6)^2$在约束条件 $x^2+4y^2=4$ 下可能的极值点

① $x^2+4y^2=4 \implies x^2+4y^2-4=0$ 令 $\varphi=x^2+4y^2-4$

② 令 $F(x,y,\lambda)=\frac{1}{13}(2x+3y-6)^2+\lambda(x^2+4y^2-4)$

$$=\frac{1}{13}(4x^2+9y^2+36+12xy-24x-36y)+\lambda x^2+4\lambda y^2-4\lambda$$

$$=\frac{4}{13}x^2+\frac{9}{13}y^2+\frac{36}{13}+\frac{12}{13}xy-\frac{24}{13}x-\frac{36}{13}y+\lambda x^2+4\lambda y^2-4\lambda$$

③ $$\frac{\partial F}{\partial x}=\frac{\partial\left(\frac{4}{13}x^2+\frac{9}{13}y^2+\frac{36}{13}+\frac{12}{13}xy-\frac{24}{13}x-\frac{36}{13}y+\lambda x^2+4\lambda y^2-4\lambda\right)}{\partial x}=\frac{8}{13}x+\frac{12}{13}y-\frac{24}{13}+2\lambda x=\frac{4}{13}(2x+3y-6)+2\lambda x$$

$$\frac{\partial F}{\partial y}=\frac{\partial\left(\frac{4}{13}x^2+\frac{9}{13}y^2+\frac{36}{13}+\frac{12}{13}xy-\frac{24}{13}x-\frac{36}{13}y+\lambda x^2+4\lambda y^2-4\lambda\right)}{\partial y}=\frac{18}{13}y+\frac{12}{13}x-\frac{36}{13}+8\lambda y=\frac{6}{13}(2x+3y-6)+8\lambda y$$

$$\frac{\partial F}{\partial \lambda}=\frac{\partial\left(\frac{4}{13}x^2+\frac{9}{13}y^2+\frac{36}{13}+\frac{12}{13}xy-\frac{24}{13}x-\frac{36}{13}y+\lambda x^2+4\lambda y^2-4\lambda\right)}{\partial \lambda}=x^2+4y^2-4$$

$$\begin{cases}\frac{\partial F}{\partial x}=0\\ \frac{\partial F}{\partial y}=0\\ \frac{\partial F}{\partial \lambda}=0\end{cases} \implies \boxed{\begin{cases}\frac{4}{13}(2x+3y-6)+2\lambda x=0\\ \frac{6}{13}(2x+3y-6)+8\lambda y=0\\ x^2+4y^2-4=0\end{cases}} \implies \begin{cases}x_1=\frac{8}{5},\ y_1=\frac{3}{5}\\ x_2=-\frac{8}{5},\ y_2=-\frac{3}{5}\end{cases}$$

$\therefore \left(\frac{8}{5},\frac{3}{5}\right)$ 与 $\left(-\frac{8}{5},-\frac{3}{5}\right)$ 可能是极值点

$$\begin{cases}\frac{4}{13}(2x+3y-6)+2\lambda x=0\\ \frac{6}{13}(2x+3y-6)+8\lambda y=0\\ x^2+4y^2-4=0\end{cases} \implies \begin{cases}\frac{4}{13}(2x+3y-6)=-2\lambda x & ①\\ \frac{6}{13}(2x+3y-6)=-8\lambda y & ②\\ x^2+4y^2-4=0 & ③\end{cases}$$

$\frac{①}{②}$ 得 $\frac{\frac{4}{13}(2x+3y-6)}{\frac{6}{13}(2x+3y-6)}=\frac{-2\lambda x}{-8\lambda y} \implies \frac{2}{3}=\frac{x}{4y} \implies y=\frac{3}{8}x$

将$y=\frac{3}{8}x$代入③即可求得答案

例2．找出$f=5x^2+5y^2-8xy$在约束条件$x^2+y^2-xy=75$下可能的极值点

① $x^2+y^2-xy=75 \Rightarrow x^2+y^2-xy-75=0$ 令 $\varphi=x^2+y^2-xy-75$

② 令 $F(x,y,\lambda)=5x^2+5y^2-8xy+\lambda(x^2+y^2-xy-75)$

$$=5x^2+5y^2-8xy+\lambda x^2+\lambda y^2-\lambda xy-75\lambda$$

③ $\frac{\partial F}{\partial x}=\frac{\partial(5x^2+5y^2-8xy+\lambda x^2+\lambda y^2-\lambda xy-75\lambda)}{\partial x}=10x-8y+2\lambda x-\lambda y$

$\frac{\partial F}{\partial y}=\frac{\partial(5x^2+5y^2-8xy+\lambda x^2+\lambda y^2-\lambda xy-75\lambda)}{\partial y}=10y-8x+2\lambda y-\lambda x$

$\frac{\partial F}{\partial \lambda}=\frac{\partial(5x^2+5y^2-8xy+\lambda x^2+\lambda y^2-\lambda xy-75\lambda)}{\partial \lambda}=x^2+y^2-xy-75$

$$\begin{cases}\frac{\partial F}{\partial x}=0\\ \frac{\partial F}{\partial y}=0\\ \frac{\partial F}{\partial \lambda}=0\end{cases} \Rightarrow \boxed{\begin{cases}10x-8y+2\lambda x-\lambda y=0\\ 10y-8x+2\lambda y-\lambda x=0\\ x^2+y^2-xy-75=0\end{cases}} \Rightarrow \begin{cases}x_1=5\sqrt{3}，y_1=5\sqrt{3}\\ x_2=-5\sqrt{3}，y_2=-5\sqrt{3}\\ x_3=-5，y_3=5\\ x_4=5，y_4=-5\end{cases}$$

$\therefore (5\sqrt{3},5\sqrt{3})$、$(-5\sqrt{3},-5\sqrt{3})$、$(-5,5)$、$(5,-5)$可能是极值点

$$\begin{cases}10x-8y+2\lambda x-\lambda y=0\\ 10y-8x+2\lambda y-\lambda x=0\\ x^2+y^2-xy-75=0\end{cases} \Rightarrow \begin{cases}10x-8y=\lambda(y-2x) & ①\\ 10y-8x=\lambda(x-2y) & ②\\ x^2+y^2-xy-75=0 & ③\end{cases}$$

$\frac{①}{②}$ 得 $\frac{10x-8y}{10y-8x}=\frac{\lambda(y-2x)}{\lambda(x-2y)} \Rightarrow \frac{10x-8y}{10y-8x}=\frac{y-2x}{x-2y}$

$$(10x-8y)(x-2y)=(10y-8x)(y-2x)$$

$$10x^2-20xy-8xy+16y^2=10y^2-20xy-8xy+16x^2$$

$$6y^2=6x^2$$

$$x^2=y^2$$

$$x=y \text{ 或 } x=-y$$

将$x=y$或$x=-y$代入③即可求得答案

例3．找出$f=\frac{x^2}{4\pi}+\frac{y^2}{16}+\frac{\sqrt{3}}{36}z^2$在约束条件 $x+y+z=2$ 下可能的极值点

① $x+y+z=2 \Longrightarrow x+y+z-2=0$ 令 $\varphi=x+y+z-2$

② 令 $F(x,y,z,\lambda)=\frac{x^2}{4\pi}+\frac{y^2}{16}+\frac{\sqrt{3}}{36}z^2+\lambda(x+y+z-2)$

$$=\frac{x^2}{4\pi}+\frac{y^2}{16}+\frac{\sqrt{3}}{36}z^2+\lambda x+\lambda y+\lambda z-2\lambda$$

③ $\frac{\partial F}{\partial x}=\frac{\partial(\frac{x^2}{4\pi}+\frac{y^2}{16}+\frac{\sqrt{3}}{36}z^2+\lambda x+\lambda y+\lambda z-2\lambda)}{\partial x}=\frac{x}{2\pi}+\lambda$

$$\frac{\partial F}{\partial y}=\frac{\partial(\frac{x^2}{4\pi}+\frac{y^2}{16}+\frac{\sqrt{3}}{36}z^2+\lambda x+\lambda y+\lambda z-2\lambda)}{\partial y}=\frac{y}{8}+\lambda$$

$$\frac{\partial F}{\partial z}=\frac{\partial(\frac{x^2}{4\pi}+\frac{y^2}{16}+\frac{\sqrt{3}}{36}z^2+\lambda x+\lambda y+\lambda z-2\lambda)}{\partial z}=\frac{\sqrt{3}}{18}z+\lambda$$

$$\frac{\partial F}{\partial \lambda}=\frac{\partial(\frac{x^2}{4\pi}+\frac{y^2}{16}+\frac{\sqrt{3}}{36}z^2+\lambda x+\lambda y+\lambda z-2\lambda)}{\partial \lambda}=x+y+z-2$$

$$\begin{cases}\frac{\partial F}{\partial x}=0\\ \frac{\partial F}{\partial y}=0\\ \frac{\partial F}{\partial z}=0\\ \frac{\partial F}{\partial \lambda}=0\end{cases} \Longrightarrow \boxed{\begin{cases}\frac{x}{2\pi}+\lambda=0\\ \frac{y}{8}+\lambda=0\\ \frac{\sqrt{3}}{18}z+\lambda=0\\ x+y+z-2=0\end{cases}} \Longrightarrow \begin{cases}x_1=\frac{2\pi}{\pi+4+3\sqrt{3}}\\ y_1=\frac{8}{\pi+4+3\sqrt{3}}\\ z_1=\frac{6\sqrt{3}}{\pi+4+3\sqrt{3}}\end{cases}$$

$\therefore \left(\frac{2\pi}{\pi+4+3\sqrt{3}},\frac{8}{\pi+4+3\sqrt{3}},\frac{6\sqrt{3}}{\pi+4+3\sqrt{3}}\right)$是可能的极值点

$$\begin{cases}\frac{x}{2\pi}+\lambda=0\\ \frac{y}{8}+\lambda=0\\ \frac{\sqrt{3}}{18}z+\lambda=0\\ x+y+z-2=0\end{cases} \Longrightarrow \begin{cases}x+2\pi\lambda=0\\ y+8\lambda=0\\ z+6\sqrt{3}\lambda=0\\ x+y+z-2=0\end{cases} \Longrightarrow \begin{cases}x=-2\pi\lambda & ①\\ y=-8\lambda & ②\\ z=-6\sqrt{3}\lambda & ③\\ x+y+z-2=0 & ④\end{cases}$$

$\frac{①}{②}$ 得 $\frac{x}{y}=\frac{-2\pi\lambda}{-8\lambda} \Longrightarrow y=\frac{4}{\pi}x$

$\frac{①}{③}$ 得 $\frac{x}{z}=\frac{-2\pi\lambda}{-6\sqrt{3}\lambda} \Longrightarrow z=\frac{3\sqrt{3}}{\pi}x$

将$y=\frac{4}{\pi}x$与$z=\frac{3\sqrt{3}}{\pi}x$代入④中即可求得答案

例4．找出 $f=z^2$ 在约束条件 $\begin{cases}x^2+y^2-2z^2=0\\ x+y+3z=5\end{cases}$ 下可能的极值点

① $\begin{cases}x^2+y^2-2z^2=0\\ x+y+3z=5\end{cases} \Longrightarrow \begin{cases}x^2+y^2-2z^2=0\\ x+y+3z-5=0\end{cases}$ 令 $\begin{cases}\varphi_1=x^2+y^2-2z^2\\ \varphi_2=x+y+3z-5\end{cases}$

② 令 $F(x,y,z,\lambda_1,\lambda_2)=z^2+\lambda_1(x^2+y^2-2z^2)+\lambda_2(x+y+3z-5)$

$$=z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2$$

③ $\frac{\partial F}{\partial x}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial x}=2\lambda_1x+\lambda_2$

$\frac{\partial F}{\partial y}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial y}=2\lambda_1y+\lambda_2$

$\frac{\partial F}{\partial z}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial z}=2z-4\lambda_1z+3\lambda_2$

$\frac{\partial F}{\partial \lambda_1}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial \lambda_1}=x^2+y^2-2z^2$

$\frac{\partial F}{\partial \lambda_2}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial \lambda_2}=x+y+3z-5$

$$\begin{cases}\frac{\partial F}{\partial x}=0\\ \frac{\partial F}{\partial y}=0\\ \frac{\partial F}{\partial z}=0\\ \frac{\partial F}{\partial \lambda_1}=0\\ \frac{\partial F}{\partial \lambda_2}=0\end{cases} \Longrightarrow \boxed{\begin{cases}2\lambda_1x+\lambda_2=0\\ 2\lambda_1y+\lambda_2=0\\ 2z-4\lambda_1z+3\lambda_2=0\\ x^2+y^2-2z^2=0\\ x+y+3z-5=0\end{cases}} \Longrightarrow \begin{cases}x_1=-5,\ y_1=-5,\ z_1=5\\ x_2=1,\ y_2=1,\ z_2=1\end{cases}$$

$\therefore(-5,-5,5)$、$(1,1,1)$ 两个点可能是极值点

$$\begin{cases}2\lambda_1x+\lambda_2=0\\ 2\lambda_1y+\lambda_2=0\\ 2z-4\lambda_1z+3\lambda_2=0\\ x^2+y^2-2z^2=0\\ x+y+3z-5=0\end{cases} \Longrightarrow \begin{cases}2\lambda_1x=-\lambda_2 & ①\\ 2\lambda_1y=-\lambda_2 & ②\\ 2z-4\lambda_1z+3\lambda_2=0 & ③\\ x^2+y^2-2z^2=0 & ④\\ x+y+3z-5=0 & ⑤\end{cases}$$

$\frac{①}{②}$ 得 $\frac{2\lambda_1x}{2\lambda_1y}=\frac{-\lambda_2}{-\lambda_2} \Longrightarrow \frac{x}{y}=1 \Longrightarrow x=y$

将 $x=y$ 代入④⑤中即可求得答案

在约束条件下求最值、最值点

例1. 求在椭圆 $x^2+4y^2=4$ 上求一$\boxed{点}$，使其到直线 $2x+3y-6=0$ 的距离最短

(x, y)

点（x,y）到直线 $2x+3y-6=0$ 的距离为 $d=\frac{|2x+3y-6|}{\sqrt{2^2+3^2}}=\frac{|2x+3y-6|}{\sqrt{13}}$

距离 $=\frac{|2x+3y-6|}{\sqrt{13}}$

$$距离^2=\left(\frac{|2x+3y-6|}{\sqrt{13}}\right)^2$$

$$=\frac{1}{13}(2x+3y-6)^2$$

找出 $f=\frac{1}{13}(2x+3y-6)^2$ 在约束条件 $x^2+4y^2=4$ 下可能的极值点

$x^2+4y^2=4 \implies x^2+4y^2-4=0$　令 $\varphi=x^2+4y^2-4$

$$令\ F(x,y,\lambda)=\frac{1}{13}(2x+3y-6)^2+\lambda(x^2+4y^2-4)$$

$$=\frac{1}{13}(4x^2+9y^2+36+12xy-24x-36y)+\lambda x^2+4\lambda y^2-4\lambda$$

$$=\frac{4}{13}x^2+\frac{9}{13}y^2+\frac{36}{13}+\frac{12}{13}xy-\frac{24}{13}x-\frac{36}{13}y+\lambda x^2+4\lambda y^2-4\lambda$$

$$\frac{\partial F}{\partial x}=\frac{\partial\left(\frac{4}{13}x^2+\frac{9}{13}y^2+\frac{36}{13}+\frac{12}{13}xy-\frac{24}{13}x-\frac{36}{13}y+\lambda x^2+4\lambda y^2-4\lambda\right)}{\partial x}=\frac{8}{13}x+\frac{12}{13}y-\frac{24}{13}+2\lambda x=\frac{4}{13}(2x+3y-6)+2\lambda x$$

$$\frac{\partial F}{\partial y}=\frac{\partial\left(\frac{4}{13}x^2+\frac{9}{13}y^2+\frac{36}{13}+\frac{12}{13}xy-\frac{24}{13}x-\frac{36}{13}y+\lambda x^2+4\lambda y^2-4\lambda\right)}{\partial y}=\frac{18}{13}y+\frac{12}{13}x-\frac{36}{13}+8\lambda y=\frac{6}{13}(2x+3y-6)+8\lambda y$$

$$\frac{\partial F}{\partial \lambda}=\frac{\partial\left(\frac{4}{13}x^2+\frac{9}{13}y^2+\frac{36}{13}+\frac{12}{13}xy-\frac{24}{13}x-\frac{36}{13}y+\lambda x^2+4\lambda y^2-4\lambda\right)}{\partial \lambda}=x^2+4y^2-4$$

$$\begin{cases}\frac{\partial F}{\partial x}=0\\ \frac{\partial F}{\partial y}=0\\ \frac{\partial F}{\partial \lambda}=0\end{cases}\implies\begin{cases}\frac{4}{13}(2x+3y-6)+2\lambda x=0\\ \frac{6}{13}(2x+3y-6)+8\lambda y=0\\ x^2+4y^2-4=0\end{cases}\implies\begin{cases}x_1=\frac{8}{5},\ y_1=\frac{3}{5}\\ x_2=-\frac{8}{5},\ y_2=-\frac{3}{5}\end{cases}$$

$\therefore \left(\frac{8}{5},\frac{3}{5}\right)$ 与 $\left(-\frac{8}{5},-\frac{3}{5}\right)$ 可能是极值点

$$距离|_{\left(\frac{8}{5},\frac{3}{5}\right)}=\frac{\left|2\cdot\frac{8}{5}+3\cdot\frac{3}{5}-6\right|}{\sqrt{13}}=\frac{1}{\sqrt{13}}$$

$$距离|_{\left(-\frac{8}{5},-\frac{3}{5}\right)}=\frac{\left|2\cdot\left(-\frac{8}{5}\right)+3\cdot\left(-\frac{3}{5}\right)-6\right|}{\sqrt{13}}=\frac{11}{\sqrt{13}}$$

$\therefore$ 点 $\left(\frac{8}{5},\frac{3}{5}\right)$ 到直线 $2x+3y-6=0$ 的距离最短

例2. 已知曲线 C: $\begin{cases} x^2+y^2-2z^2=0 \\ x+y+3z=5 \end{cases}$，求曲线 C 上距离

xOy面 最远的点和最近的点 (x,y,z)

点 (x,y,z) 到 xOy面 的距离为 d=|z|

距离 $=|z|$

距离$^2=|z|^2$

$=z^2$

找出 $f=z^2$ 在约束条件 $\begin{cases} x^2+y^2-2z^2=0 \\ x+y+3z=5 \end{cases}$ 下可能的极值点

$\begin{cases} x^2+y^2-2z^2=0 \\ x+y+3z=5 \end{cases} \Rightarrow \begin{cases} x^2+y^2-2z^2=0 \\ x+y+3z-5=0 \end{cases}$ 令 $\begin{cases} \varphi_1=x^2+y^2-2z^2 \\ \varphi_2=x+y+3z-5 \end{cases}$

令 $F(x,y,z,\lambda_1,\lambda_2)=z^2+\lambda_1(x^2+y^2-2z^2)+\lambda_2(x+y+3z-5)$

$=z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2$

$$\frac{\partial F}{\partial x}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial x}=2\lambda_1x+\lambda_2$$

$$\frac{\partial F}{\partial y}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial y}=2\lambda_1y+\lambda_2$$

$$\frac{\partial F}{\partial z}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial z}=2z-4\lambda_1z+3\lambda_2$$

$$\frac{\partial F}{\partial \lambda_1}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial \lambda_1}=x^2+y^2-2z^2$$

$$\frac{\partial F}{\partial \lambda_2}=\frac{\partial(z^2+\lambda_1x^2+\lambda_1y^2-2\lambda_1z^2+\lambda_2x+\lambda_2y+3\lambda_2z-5\lambda_2)}{\partial \lambda_2}=x+y+3z-5$$

$$\begin{cases} \frac{\partial F}{\partial x}=0 \\ \frac{\partial F}{\partial y}=0 \\ \frac{\partial F}{\partial z}=0 \\ \frac{\partial F}{\partial \lambda_1}=0 \\ \frac{\partial F}{\partial \lambda_2}=0 \end{cases} \Rightarrow \begin{cases} 2\lambda_1x+\lambda_2=0 \\ 2\lambda_1y+\lambda_2=0 \\ 2z-4\lambda_1z+3\lambda_2=0 \\ x^2+y^2-2z^2=0 \\ x+y+3z-5=0 \end{cases} \Rightarrow \begin{cases} x_1=-5,\ y_1=-5,\ z_1=5 \\ x_2=1,\ y_2=1,\ z_2=1 \end{cases}$$

∴$(-5,-5,5)$、$(1,1,1)$ 两个点可能是极值点

距离$|_{(-5,-5,5)}=|5|=5$

距离$|_{(1,1,1)}=|1|=1$

∴ 距离 xOy面 最远的点为：$(-5,-5,5)$，

距离 xOy面 最近的点为：$(1,1,1)$

例3. 将长为 2m 的铁丝分成三段，依次围成圆、正方形与正三角形，求三个图形的面积之和的最小值

设三段铁丝长度分别为 x、y、z

即 圆周长为 x，正方形周长为 y，正三角形周长为 z

$$
\begin{aligned}
和 &= S_{圆} + S_{正方形} + S_{正三角形} \\
&= \pi\cdot(圆半径)^2 + (正方形边长)^2 + \frac{\sqrt{3}}{4}\cdot(正三角形边长)^2 \\
&= \frac{x^2}{4\pi} + \frac{y^2}{16} + \frac{\sqrt{3}}{4}\cdot\left(\frac{z}{3}\right)^2 \\
&= \frac{x^2}{4\pi} + \frac{y^2}{16} + \frac{\sqrt{3}\cdot z^2}{36}
\end{aligned}
$$

找出 $f = \frac{x^2}{4\pi} + \frac{y^2}{16} + \frac{\sqrt{3}}{36}z^2$ 在约束条件 x+y+z=2 下可能的极值点

$x + y + z=2 \Longrightarrow x + y + z - 2 =0$ 令 $\varphi = x + y + z - 2$

$$
\begin{aligned}
令\ F(x,y,z,\lambda) &= \frac{x^2}{4\pi} + \frac{y^2}{16} + \frac{\sqrt{3}}{36}z^2 + \lambda(x + y + z - 2) \\
&= \frac{x^2}{4\pi} + \frac{y^2}{16} + \frac{\sqrt{3}}{36}z^2 + \lambda x+\lambda y+\lambda z - 2\lambda
\end{aligned}
$$

$$\frac{\partial F}{\partial x} = \frac{\partial(\frac{x^2}{4\pi} + \frac{y^2}{16} + \frac{\sqrt{3}}{36}z^2+\lambda x+\lambda y+\lambda z-2\lambda)}{\partial x} = \frac{x}{2\pi} + \lambda$$

$$\frac{\partial F}{\partial y} = \frac{\partial(\frac{x^2}{4\pi} + \frac{y^2}{16} + \frac{\sqrt{3}}{36}z^2+\lambda x+\lambda y+\lambda z-2\lambda)}{\partial y} = \frac{y}{8} + \lambda$$

$$\frac{\partial F}{\partial z} = \frac{\partial(\frac{x^2}{4\pi} + \frac{y^2}{16} + \frac{\sqrt{3}}{36}z^2+\lambda x+\lambda y+\lambda z-2\lambda)}{\partial z} = \frac{\sqrt{3}}{18}z + \lambda$$

$$\frac{\partial F}{\partial \lambda} = \frac{\partial(\frac{x^2}{4\pi} + \frac{y^2}{16} + \frac{\sqrt{3}}{36}z^2+\lambda x+\lambda y+\lambda z-2\lambda)}{\partial \lambda} = x + y + z - 2$$

$$
\begin{cases}
\frac{\partial F}{\partial x} = 0 \\
\frac{\partial F}{\partial y} = 0 \\
\frac{\partial F}{\partial z} = 0 \\
\frac{\partial F}{\partial \lambda} = 0
\end{cases}
\Longrightarrow
\begin{cases}
\frac{x}{2\pi} + \lambda = 0 \\
\frac{y}{8} + \lambda = 0 \\
\frac{\sqrt{3}}{18}z + \lambda = 0 \\
x + y + z - 2 = 0
\end{cases}
\Longrightarrow
\begin{cases}
x_1 = \frac{2\pi}{\pi+4+3\sqrt{3}} \\
y_1 = \frac{8}{\pi+4+3\sqrt{3}} \\
z_1 = \frac{6\sqrt{3}}{\pi+4+3\sqrt{3}}
\end{cases}
$$

$\therefore \left(\frac{2\pi}{\pi+4+3\sqrt{3}}, \frac{8}{\pi+4+3\sqrt{3}}, \frac{6\sqrt{3}}{\pi+4+3\sqrt{3}}\right)$ 是可能的极值点

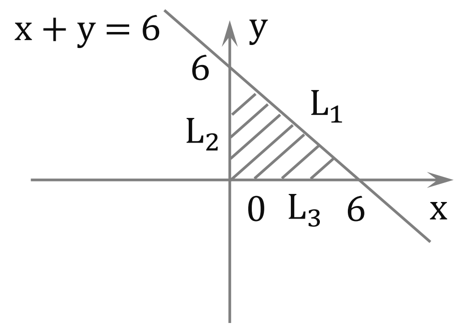
$$
\begin{aligned}
和 &= \frac{\left(\frac{2\pi}{\pi+4+3\sqrt{3}}\right)^2}{4\pi} + \frac{\left(\frac{8}{\pi+4+3\sqrt{3}}\right)^2}{16} + \frac{\sqrt{3}\cdot\left(\frac{6\sqrt{3}}{\pi+4+3\sqrt{3}}\right)^2}{36} \\
&= \frac{1}{\pi+4+3\sqrt{3}}
\end{aligned}
$$

$\therefore$ 三个图形的面积之和的最小值为 $\frac{1}{\pi+4+3\sqrt{3}}\,m^2$

在区域上求最值、最值点

求出满足 $\begin{cases} \frac{\partial f}{\partial x}=0 \\ \frac{\partial f}{\partial y}=0 \end{cases}$ 的 (x_i, y_i)

例1. 求函数 $f=x^2y\,(4-x-y)$ 在由直线 $x+y=6$、x 轴、y 轴所围成的闭区域上的最大值和最小值

① 

② $\begin{cases} \frac{\partial[x^2y(4-x-y)]}{\partial x}=0 \\ \frac{\partial[x^2y(4-x-y)]}{\partial y}=0 \end{cases} \Rightarrow \begin{cases} 2xy(4-x-y)-x^2y=0 \\ x^2(4-x-y)-x^2y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=y \end{cases}$ 或 $\begin{cases} x=4 \\ y=0 \end{cases}$

或 $\begin{cases} x=2 \\ y=1 \end{cases}$

∴满足要求的点为 ~~(0,y)~~，~~(4,0)~~，(2,1)

④ 将(2,1) 代入原函数，求出函数值

$f=2^2\cdot 1\cdot(4-2-1)$

$=4$

⑤ 边界L_1：$y=6-x \quad (0\le x\le 6)$

边界L_2：$x=0 \quad (0\le y\le 6)$

边界L_3：$y=0 \quad (0\le x\le 6)$

⑥ $f_1=x^2(6-x)\,[4-x-(6-x)]$

$=2x^3-12x^2$

求出$f_1=2x^3-12x^2$在$0\le x\le 6$时的最值

最大值为 0，最小值是为-64

$f_2=0^2\cdot y\cdot(4-0-y)$

$=0$

求出$f_2=0$在$0\le y\le 6$时的最值

最大值为0，最小值为0

$f_3=x^2\cdot 0\,(4-x-0)$

$=0$

求出$f_3=0$在$0\le x\le 6$时的最值

最大值为0，最小值为0

① a、 $f_1{}'=(2x^3-12x^2)'$

$=6x^2-24x$

b、 令 $f_1{}'=0$

$\Rightarrow 6x^2-24x=0$

$\Rightarrow x_1=0$、$x_2=4$

c、 $\in[0,6]$ (保留) $\in[0,6]$ (保留)

d、将 $x_1=0$、$x_2=4$ 代入 f

$f_1(0)=2\cdot 0^3-12\cdot 0^2=0$

$f_1(4)=2\cdot 4^3-12\cdot 4^2=-64$

②∵取值范围为$0\le x\le 6$

∴要求 x= 0与 x=6 处 f 的值

$f_1(0)=2\cdot 0^3-12\cdot 0^2=0$

$f_1(6)=2\cdot 6^3-12\cdot 6^2=0$

③ 最大值为 0 最小值为 -64

做题步骤：

① 求出 范围内 f 的极值

a、求出f'

b、令$f'=0$，求出所有的解

c、去掉b中解里，在 范围 之外的解

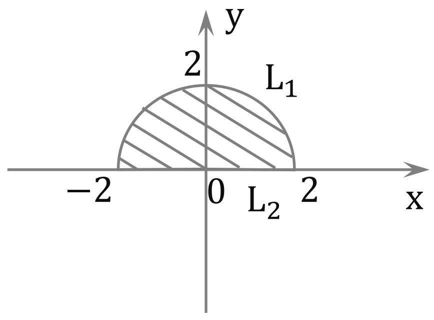
d、将 b中剩余的解代入 f 中，求出对应的 f 值

② 求出 范围边界处 的 f 值

③ 比较 ① 和 ② 中得到的 f 值，这些值里最大的为最大值，最小的为最小值

⑦ 最大值是 4，最小值是 -64

例2. 求函数 $f=x^2+2y^2-x^2y^2$ 在区域 $D=\{(x,y)|x^2+y^2\le 4,\ y\ge 0\}$ 上的最大值和最小值

① 

② $\begin{cases}\dfrac{\partial(x^2+2y^2-x^2y^2)}{\partial x}=0\\ \dfrac{\partial(x^2+2y^2-x^2y^2)}{\partial y}=0\end{cases}\Rightarrow\begin{cases}2x-2xy^2=0\\4y-2x^2y=0\end{cases}\Rightarrow\begin{cases}x_1=\sqrt{2}\\y_1=1\end{cases}$ 或 $\begin{cases}x_2=-\sqrt{2}\\y_2=1\end{cases}$

或 $\begin{cases}x_2=0\\y_2=0\end{cases}$ 或 $\begin{cases}x_2=-\sqrt{2}\\y_2=-1\end{cases}$ 或 $\begin{cases}x_2=\sqrt{2}\\y_2=-1\end{cases}$

∴满足要求的点为 $(\sqrt{2},1)$，$(-\sqrt{2},1)$，$\cancel{(0,0)}$，$\cancel{(-\sqrt{2},-1)}$，$\cancel{(\sqrt{2},-1)}$

④ 将 $(\sqrt{2},1)$，$(-\sqrt{2},1)$ 代入原函数，求出函数值

$f=(\sqrt{2})^2+2\cdot 1^2-(\sqrt{2})^2\cdot 1^2=2$

$f=(-\sqrt{2})^2+2\cdot 1^2-(-\sqrt{2})^2\cdot 1^2=2$

⑤ 边界 L_1：$y=\sqrt{4-x^2}\quad(-2\le x\le 2)$

边界 L_2：$y=0\quad(-2\le x\le 2)$

⑥ $f_1=x^2+2(\sqrt{4-x^2})^2-x^2(\sqrt{4-x^2})^2$
$=x^2+2(4-x^2)-x^2(4-x^2)$
$=x^4-5x^2+8$

求出 $f_1=x^4-5x^2+8$ 在 $-2\le x\le 2$ 时的最值

最大值为 8 最小值为 $\frac{7}{4}$

$f_2=x^2+2\cdot 0^2-x^2\cdot 0^2$
$=x^2$

求出 $f_2=x^2$ 在 $-2\le x\le 2$ 时的最值

最大值为 4 最小值为 0

① a、 $f_1'=(x^4-5x^2+8)'$
$=4x^3-10x$

b、 令 $f_1'=0$

$\Rightarrow 4x^3-10x=0$

$\Rightarrow x_1=0$、$x_2=\sqrt{\frac{5}{2}}$、$x_3=-\sqrt{\frac{5}{2}}$

c、$\in[-2,2]$ (保留)　$\in[-2,2]$ (保留)　$\in[-2,2]$ (保留)

d、将 $x_1=0$、$x_2=\sqrt{\frac{5}{2}}$、$x_3=-\sqrt{\frac{5}{2}}$ 代入 f

$f_1(0)=0^4-5\cdot 0^2+8=8$

$f_1(\sqrt{\frac{5}{2}})=(\sqrt{\frac{5}{2}})^4-5\cdot(\sqrt{\frac{5}{2}})^2+8=\frac{7}{4}$

$f_1(-\sqrt{\frac{5}{2}})=(-\sqrt{\frac{5}{2}})^4-5\cdot(-\sqrt{\frac{5}{2}})^2+8=\frac{7}{4}$

② ∵ 取值范围为 $-2\le x\le 2$
∴ 要求 $x=-2$ 与 $x=2$ 处 f 的值

$f_1(-2)=(-2)^4-5\cdot(-2)^2+8=4$

$f_1(2)=2^4-5\cdot 2^2+8=4$

③ 最大值为 8 最小值为 $\frac{7}{4}$

① a、 $f_2'=(x^2)'$
$=2x$

b、 令 $f_2'=0$

$\Rightarrow 2x=0$

$\Rightarrow x=0$

c、 $\in[-2,2]$ (保留)

d、将 $x=0$ 代入 f

$f_2(0)=0^2=0$

② ∵ 取值范围为 $-2\le x\le 2$
∴ 要求 $x=-2$ 与 $x=2$ 处 f 的值

$f_2(-2)=(-2)^2=4$

$f_2(2)=2^2=4$

③ 最大值为 4 最小值为 0

做题步骤:

① 求出 范围内 f 的极值
  a、求出 f'
  b、令 $f'=0$，求出所有的解
  c、去掉b中解里，在 范围 之外的解
  d、将 b中剩余的解代入 f 中，求出对应的 f 值

② 求出 范围边界处 的 f 值

③ 比较 ① 和 ② 中得到的 f 值，这些值里最大的为最大值，最小的为最小值

⑦ 最大值是 8，最小值是 0