Overview. My research interests lie in the area of numerical analysis and computational sciences. My graduate work focuses on developing *novel analytical tools* and *efficient implementations* for hybridizable discontinuous Galerkin (HDG) methods. The tools I developed can be used to *unify the analysis of a large class of HDG methods and to systematically devise and analyze new methods*; see (1) and (2) for more details. I also have rich experience in implementing HDG methods for various types of problems; see (3) for more details. In the future, I am open to work on different research projects to further extend my vision and abilities.

(1) Projection-based analysis of HDG+ methods

HDG+ methods [18, 20, 21] are an important class of HDG methods possessing several distinguishing properties such as the superconvergence on general polyhedral meshes and the easiness of implementation. Despite these advantages, most of the current error analyses of HDG+ methods do not use specifically devised projections to make the analysis simple, concise and unified. In [12, 13, 14], we devised *HDG+ projections enabling a unified error analysis of the standard HDG and HDG+ methods*. As a consequence, many current analyses of HDG+ methods are rendered more concise. More importantly, the projections enable us to systematically reuse existing analysis techniques of standard HDG methods to analyze HDG+ methods. For instance, by combining an HDG+ projection with certain analysis techniques used for an HDG method for acoustic waves, we devised a new HDG+ method for transient elastic waves and proved its uniform-in-time superconvergence. See Section 1 for more details.

(2) Unified error analysis of HDG methods for Maxwell equations

HDG methods for Maxwell equations are much less understood than HDG methods for elliptic/elastic problems. One reason is the lack of a theory of M-decomposition [7] for curl-curl operators in $\mathbb{R}^3$. Another reason is that HDG methods for Maxwell equations usually need to discretize both the curl-curl and the grad-div operators. Recently, many HDG variants for Maxwell equations have been proposed. However, the connections between these variants were unclear and the comparison was hard to conduct. In [15], we proposed *a framework that allows a unified error analysis of most HDG methods for Maxwell equations*. By using the framework, we analyzed four variants of HDG methods in one simple analysis, where different projections were devised to capture the unique structures of each variant. We recovered the error estimates of two known HDG variants under more *general* conditions on the meshes and the stabilization functions. We also discovered *two new variants* where both were proved to be optimal and one achieves superconvergence. The connections between the four variants were made clear by investigating the projections used to analyze them, and the comparison was also rendered obvious. See Section 2 for more details.

(3) High-order HDG implementations for elastic/viscoelastic and electromagnetic waves

My coding projects have been developed based on HDG3D, a fully functional *high-order HDG Matlab library with user-friendly documentations*. I have a rich experience with the library and am proficient with most modules, from the most fundamental level, such as mesh generation and subdivision, basis functions, quadrature rules, tensor acceleration, to middle level including bookkeeping, assembly, local solver, global solver, and finally to front end modules such as post-processing, error analysis, and visualization. As an active member of the team, I extended the library to include modules simulating elastic/viscoelastic waves by using HDG+ for spacial and convolution quadrature (CQ) for temporal discretization. Besides, I implemented various variants of HDG methods for Maxwell equations, with a detailed yet easy-to-access documentation.

More discussion about future work can be found in Section 3.

1 Projection-based analysis of HDG+ methods

1.1 Background

HDG projection. Classical HDG projection [9] plays a key role in the design and analysis of a wide class of HDG methods for various types of PDEs, such as steady-state diffusion, Helmholtz equations, fractional/ellptic diffusion and acoustic waves. The projection is deeply connected to the more well-known Raviart-Thomas (RT) and the Brezzi-Douglas-Marini projections [22, 1]. For either the HDG or the mixed methods, their projections make the error analysis simple, concise, and unified. This analysis method is often referred to as the "projection-based error analysis".

HDG+ method. The standard HDG methods can lose optimal convergence when the meshes are non-simplicial [10], or in the case of elastic problems where the symmetry of the stress is enforced strongly in the approximations [16]. HDG+ methods, however, were shown to be optimal in all these cases at least for steady-state systems [20, 21]. The implementation of HDG+ is also straightforward: from standard HDG to HDG+, it only requires to (1) increase the polynomial degree for the primal variable by one and (2) insert an L^2 projection in the numerical flux. Despite the many advantages of HDG+ methods, most of their current analysis were conducted by L^2 projections, rendering the analysis very different and less concise from the projection-based error analysis of the standard HDG methods. More importantly, extending the analysis of the HDG+ methods to evolutionary and Helmholtz equations becomes highly non-trivial without first finding the "HDG+ projection".

1.2 What we did

HDG+ projection and boundary remainder. To infuse HDG+ methods with the analytical advantages of the projection-based error analysis, we devised the HDG+ projections. The novelty of any HDG+ projection is that we pair the projection with an associated boundary remainder term. This new idea of using the "projection-remainder pair" instead of using only the projection greatly generalizes the classical projection-based analysis and makes vastly more HDG methods, including the HDG+ methods analyzable using HDG projections. The boundary remainder has the following key properties:

1. The classical HDG projection has a vanishing boundary remainder, and becomes a special case in the new projection-remainder setting.

2. A small enough boundary remainder, instead of a vanishing one, is *sufficient* to guarantee the optimal convergence.

3. Introducing the boundary remainder does NOT complicate the analysis.

To see these properties more clearly, we write the energy identity (steady-state diffusion) obtained by using the classical HDG projection:

$$\boxed{\mathrm{HDG} \qquad a^2 + b^2 = ae \Longrightarrow a + b \lesssim e} \tag{1}$$

where a are b are the error terms, and e is the HDG projection interpolation error. To compare, we write the energy identity obtained by using the new HDG+ projection:

$$\boxed{\mathrm{HDG+} \qquad a^2 + b^2 = ae + b\delta \Longrightarrow a + b \lesssim e + \delta} \tag{2}$$

where e is the HDG+ projection interpolation error, and δ is the boundary remainder. By comparing (1) and (2), the three properties (a), (b) and (c) are easily justified. For all HDG+ methods, the boundary remainders are small enough to guarantee that the error terms a and b converge in optimal order.

We devised the HDG+ projections for elastic problems in [13] and for elliptic problems in [12]. The projections were initially devised on simplicial elements and then generalized to general polyhedral elements in [14].

Applications of HDG+ projections. In [13], we devised the HDG+ projection for elasticity. We demonstrated the power of the projection by using it to analyze the HDG methods for steady-state elasticity, time-harmonic elastodynamics, and transient elastic waves. We proved that the three methods are optimal and the analysis for elastic waves is the first in the literature. Our analyses were rendered simple and concise by using the HDG+ projection. Moreover, they were systematically obtained by recycling certain existing analysis techniques of standard HDG methods:

systematically recycle existing analysis	$\text{analysis of standard HDG for} \begin{cases} \text{steady-state diffusion[9]} \\ \text{Helmholtz equations[17]} \\ \text{acoustic waves[8]} \end{cases}$ $\Big\downarrow$ HDG+ projection[13] $\text{analysis of HDG+ for} \begin{cases} \text{steady-state elasticity[13]} \\ \text{time-harmonic elastodynamics[13]} \\ \text{transient elastic waves[13]} \end{cases}$

In [12], we devised the HDG+ projection for elliptic problems and applied it to design and analyze the HDG+ methods for elliptic diffusion and acoustic waves. Similar to the elastic case, the HDG+ projection makes the analysis concise and facilitates the reuse of existing analysis of standard HDG methods in a systematic way.

2 Unified error analysis of HDG methods for Maxwell equations

2.1 Background

HDG methods for Maxwell equations. HDG methods for Maxwell equations seek the approximations

$$\mathbf{w}_h \to \mathbf{w}, \quad \mathbf{u}_h \to \mathbf{u}, \quad p_h \to p, \quad \widehat{\mathbf{u}}_h \to \mathbf{n} \times \mathbf{u} \times \mathbf{n}, \quad \widehat{p}_h \to \gamma p,$$

where $\mathbf{w}$ and $\mathbf{u}$ are the magnetic and the electric fields, p is the Lagrange multiplier introduced to better control $\nabla \cdot \mathbf{u}$. To specify an HDG method, we need to specify the approximation spaces:

$$\mathbf{w}_h\big|_K \in W(K), \quad \mathbf{u}_h\big|_K \in V(K), \quad p_h\big|_K \in Q(K), \quad \widehat{\mathbf{u}}_h\big|_F \in N(F), \quad \widehat{p}_h\big|_F \in M(F).$$

In addition, we need to specify two stablization functions: τ_t and τ_n. Compared to elliptic/elastic problems, where only three approximation spaces and one stabilization function show up, more possible variants exist for HDG methods for Maxwell equations, and their analyses are somewhat more involved. To date, many HDG variants have been proposed and analyzed (see, for instance, [19, 5, 4, 3]). However, these results were obtained independently and no unified understanding has been provided to compare and understand the connections between these variants.

2.2 What we did

A framework for unified error analysis. In [15], we proposed a framework to analyze HDG methods for Maxwell equations that enables us to clearly decouple the error analysis techniques into two groups – those related to the PDEs and those related to the HDG variants. The framework is built on two assumptions. The first assumption is about the approximation spaces:

$$\boxed{\text{As}} \begin{cases} \nabla \times V \subset W, \quad \nabla \cdot V \subset Q, \quad \nabla \times W + \nabla Q \subset V, \\ \mathbf{n} \times W \subset N, \quad V \cdot \mathbf{n} + \gamma Q \subset M. \end{cases}$$

The second assumption is about the existence of a projection Π:

$$\boxed{\text{Ap}} \begin{cases} \exists(\Pi\mathbf{w}, \Pi\mathbf{u}, \Pi p) \in W \times V \times Q \quad \text{s.t.} \\ (\Pi\mathbf{w} - \mathbf{w}, \nabla \times \mathbf{v})_K = \langle \mathbf{n} \times \mathbf{w} - \mathrm{P}_N(\mathbf{n} \times \mathbf{w}), \mathbf{v} \rangle_{\partial K} \quad \forall \mathbf{v} \in \mathbf{V}, \\ (\Pi\mathbf{u} - \mathbf{u}, \nabla Q + \nabla \times W)_K = 0, \quad (\Pi p - p, \nabla \cdot V)_K = 0. \end{cases}$$

These two assumptions are very general and satisfied by most existing HDG methods for Maxwell equations. We proved that for any HDG variant fitting into this framework, there holds the following energy identity (static case):

$$\boxed{a^2 + b^2 + c^2 = ae + b\delta_t + c\delta_n \Longrightarrow a + b + c \lesssim e + \delta_t + \delta_n}$$

where a, b, c are the error terms, while e, δ_t, δ_n are the interpolation error and the two boundary remainders (see Section 1.2 for details about boundary remainder). Since e, δ_t, δ_n are completely determined by the projection Π, the error analysis of the HDG method is now reduced to the following much easier problem:

$$\boxed{\text{reduced problem} \qquad \text{Find } \Pi \text{ satisfying } \boxed{\text{Ap}} \quad \text{s.t.} \quad e,\ \delta_t,\ \delta_n \text{ are small}}$$

The reduced problem is easy since we only need to find a projection Π so that e, δ_t, δ_n are small enough (for instance $\mathcal{O}(h^{k+1})$). This will immediately give the optimal estimates for the error terms a, b, c.

Recover existing results and discover new methods. By using the framework, we analyzed four HDG variants using one simple analysis but with different projections Π to capture the unique structures of the approximation spaces for each variant. See Tabel 1 for an overview. Among the four variants, variants $\mathfrak{B}+$ and $\mathfrak{H}$ were proposed in [5, 3] and variants $\mathfrak{H}+$ and $\mathfrak{B}$ are new variants we discovered. We proved that all four variants are optimal and variants $\mathfrak{B}+$ and $\mathfrak{H}+$ achieve superconvergence. We also improved the existing estimates of the variants $\mathfrak{B}+$ and $\mathfrak{H}$ so that the estimates hold under more general conditions on the meshes $\mathcal{T}_h$ and the stabilization function τ_n.

Variant	k	Q	N	τ_n
$\mathfrak{B}$(new)	$k \geqslant 0$	$\mathcal{P}_k$	$\mathcal{P}_{k+1}^t$	$\lesssim h_K \quad K \in \mathcal{T}_h^s$ $\approx h_K \quad K \in \mathcal{T}_h^p$
$\mathfrak{H}$([3])	$k \geqslant 0$	$\mathcal{P}_{k+1}$	$\mathcal{P}_{k+1}^t$	$0 \neq \tau_n,\ \tau_n^{\text{sec}} \lesssim h_K \quad K \in \mathcal{T}_h^s$ $\approx h_K \quad K \in \mathcal{T}_h^p$
$\mathfrak{B}+$([5])	$k \geqslant 1$	$\mathcal{P}_k$	$\mathcal{P}_k^t \oplus \nabla_F \widetilde{\mathcal{P}}_{k+2}$	$\lesssim h_K \quad K \in \mathcal{T}_h^s$ $\approx h_K \quad K \in \mathcal{T}_h^p$
$\mathfrak{H}+$(new)	$k \geqslant 1$	$\mathcal{P}_{k+1}$	$\mathcal{P}_k^t \oplus \nabla_F \widetilde{\mathcal{P}}_{k+2}$	$0 \neq \tau_n,\ \tau_n^{\text{sec}} \lesssim h_K \quad K \in \mathcal{T}_h^s$ $\approx h_K \quad K \in \mathcal{T}_h^p$

Table 1: Approximation spaces and stabilization functions of variants $\mathfrak{B}$, $\mathfrak{H}$, $\mathfrak{B}+$, $\mathfrak{H}+$. For all the four variants, $W = \mathcal{P}_k^3$, $V = \mathcal{P}_{k+1}^3$, $M = \mathcal{P}_{k+1}$, and $\tau_t \approx h_K^{-1}$.

3 Future work

In the future, I am open to work on projects that are different from my current research areas to further extend my vision and abilities. I have broad interests in computational sciences while my current major strength lies in numerical analysis. My graduate work has provided a solid theoretical foundation to develop and analyze robust and efficient numerical methods for simulating a wide range of physical phenomenons including viscoelastic behavior, elastic-electromagnetic interactions and waves propagation. I will next discuss three possible future directions that are based on my current work.

3.1 Exploring the stabilization functions of HDG methods

HDG with hybrid high-order stabilization (HHO-HDG). As shown in [6], HDG+ methods can be regarded as special cases of HHO-HDG methods. A natural question arises: can we devise projections for general HHO-HDG methods so that their analyses are also incorporated into the projection-based error analysis setting?

3.2 Efficient time integrator

HDG with semi-Lagrangian for convection-diffusion. For the problems that over-resolution of temporal scale is enforced by the CFL condition, semi-Lagrangian methods become very useful since they relax the time-step restriction. Some recent work [11] about semi-Lagrangian methods for convection-diffusion problems use explicit DG for convection and implicit LDG for diffusion. An interesting future work is to use HDG instead for diffusion to increase the efficiency; according to my best knowledge, there is no work that has been done in this direction.

HDG with exponential-integrator for stiff problems. Modern developments in numerical linear algebra make possible efficient evaluation of exponential integrator using techniques such as Krylov space method. Exponential integrator is accurate since it integrates the linear part of the source term exactly. In addition, it possesses good stability properties similar to implicit methods, allowing integration with large time steps. I am interested in combing exponential-integrator with HDG methods to design and analyze full space-time discretization achieving good robustness and efficiency.

3.3 Elastic/viscoelastic and electromagnetic waves

HDG+ for viscoelastic waves. In [2], we considered the mathematical foundation for wave prorogation in solids modeled with a wide collection of viscoelastic laws where various types of stability bounds were established in the Laplace and time domain. I am interested in combining these results with the novel analytical tools we developed for HDG+ methods to design robust and efficient numerical methods for simulating viscoelastic wave propagation.

Electromagnetic waves. Our unified framework [15] was proposed for static Maxwell equations. However, many tools we developed in the paper, such as the projections we devised, can be reused to extend the unified analysis framework to electromagnetic waves.

References

[1] Franco Brezzi, Jim Douglas, Jr., and L. D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numer. Math.*, 47(2):217–235, 1985.

[2] Thomas S. Brown, Shukai Du, Hasan Eruslu, and Francisco-Javier Sayas. Analysis of models for viscoelastic wave propagation. *Appl. Math. Nonlinear Sci.*, 3(1):55–96, 2018.

[3] Gang Chen, Jintao Cui, and Liwei Xu. Analysis of a hybridizable discontinuous Galerkin method for the Maxwell operator. *ESAIM Math. Model. Numer. Anal.*, 53(1):301–324, 2019.

[4] Huangxin Chen, Weifeng Qiu, and Ke Shi. A priori and computable a posteriori error estimates for an HDG method for the coercive Maxwell equations. *Comput. Methods Appl. Mech. Engrg.*, 333:287–310, 2018.

[5] Huangxin Chen, Weifeng Qiu, Ke Shi, and Manuel Solano. A superconvergent HDG method for the Maxwell equations. *J. Sci. Comput.*, 70(3):1010–1029, 2017.

[6] Bernardo Cockburn, Daniele A. Di Pietro, and Alexandre Ern. Bridging the hybrid high-order and hybridizable discontinuous Galerkin methods. *ESAIM Math. Model. Numer. Anal.*, 50(3):635–650, 2016.

[7] Bernardo Cockburn, Guosheng Fu, and Francisco Javier Sayas. Superconvergence by M-decompositions. Part I: General theory for HDG methods for diffusion. *Math. Comp.*, 86(306):1609–1641, 2017.

[8] Bernardo Cockburn, Zhixing Fu, Allan Hungria, Liangyue Ji, Manuel A. Sánchez, and Francisco-Javier Sayas. Stormer-Numerov HDG methods for acoustic waves. *J. Sci. Comput.*, 75(2):597–624, 2018.

[9] Bernardo Cockburn, Jayadeep Gopalakrishnan, and Francisco-Javier Sayas. A projection-based error analysis of HDG methods. *Math. Comp.*, 79(271):1351–1367, 2010.

[10] Bernardo Cockburn, Weifeng Qiu, and Ke Shi. Conditions for superconvergence of HDG methods for second-order elliptic problems. *Math. Comp.*, 81(279):1327–1353, 2012.

[11] Mingchang Ding, Xiaofeng Cai, Wei Guo, and Jing-Mei Qiu. A semi-lagrangian discontinuous galerkin (dg)–local dg method for solving convection-diffusion-reaction equations. *arXiv preprint arXiv:1907.06117*, 2019.

[12] Shukai Du and Francisco-Javier Sayas. *An invitation to the theory of the Hybridizable Discontinuous Galerkin Method*. SpringerBriefs in Mathematics. 2019.

[13] Shukai Du and Francisco-Javier Sayas. New analytical tools for HDG in elasticity, with applications to elastodynamics. *arXiv preprint arXiv:1903.11766*, 2019. Submitted.

[14] Shukai Du and Francisco-Javier Sayas. A note on devising HDG+ projections on polyhedral elements. 2019. Submitted.

[15] Shukai Du and Francisco-Javier Sayas. A unified error analysis of HDG methods for the static Maxwell equations. *arXiv preprint arXiv:1910.01000*, 2019. Submitted.

[16] G. Fu, B. Cockburn, and H. Stolarski. Analysis of an HDG method for linear elasticity. *Internat. J. Numer. Methods Engrg.*, 102(3-4):551–575, 2015.

[17] Roland Griesmaier and Peter Monk. Error analysis for a hybridizable discontinuous Galerkin method for the Helmholtz equation. *J. Sci. Comput.*, 49(3):291–310, 2011.

[18] Christoph Lehrenfeld. Hybrid discontinuous galerkin methods for solving incompressible flow problems. *Rheinisch-Westfalischen Technischen Hochschule Aachen*, 2010.

[19] N. C. Nguyen, J. Peraire, and B. Cockburn. Hybridizable discontinuous Galerkin methods for the time-harmonic Maxwell's equations. *J. Comput. Phys.*, 230(19):7151–7175, 2011.

[20] Issei Oikawa. A hybridized discontinuous Galerkin method with reduced stabilization. *J. Sci. Comput.*, 65(1):327–340, 2015.

[21] Weifeng Qiu, Jiguang Shen, and Ke Shi. An HDG method for linear elasticity with strong symmetric stresses. *Math. Comp.*, 87(309):69–93, 2018.

[22] P.-A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, pages 292–315. Lecture Notes in Math., Vol. 606. Springer, Berlin, 1977.