Seminar Thesis for Methods of Algorithm Design, Summer Term 2021
Department of Informatics, Universität Hamburg

On Lower Bounds on the Competitiveness for the Online Facility Location Problem and its Multi-commodity Extension

Marcus Rottschäfer

September 2, 2021

Facility Location (FL) is a widely applicable optimization problem in computational geometry dealing with the optimal placement and opening of facilities in order to serve some set of clients under minimal cost. Service costs based on the client-facility distance, and construction costs for opening new facilities influence the overall cost function. In the Multi-commodity extension of the problem called Multi-commodity Facility Location (MCFL), clients can additionally ask for a subset of commodities which needs to be served by a number of possibly multiple facilities jointly. We regard the online versions of both problems in which client request locations are not known in advance and analyse both problems under the notion of Competitive Analysis (CA).

In our findings, we present two lower-bounds on the Competitive Ratios (CRs). Any randomized online algorithm for Online Facility Location can not be better than $\Omega(\frac{log\ n}{log\ log\ n})$-competitive against an oblivious adversary, even if the problem is defined in a line-segment metric space. With S denoting the set of all commodities, no randomized algorithm for the Online Multi-commodity Facility Location Problem can be better than $\Omega(\sqrt{|S|}+\frac{log\ n}{log\ log\ n})$-competitive against an oblivious adversary on the line segment. We prove lower-bounds by providing malicious request sequence distributions and conclude by arguing tightness of the lower-bounds based on existing algorithms for the problems.

Contents

1 Introduction

Where to place hospitals in a city is a crucial choice for optimal patient healthcare. Ideally, ambulance travel time should be kept minimal, while building a hospital in every street block is unrealistic. Naturally, similar modeling arises for other facilities like train stations, supermarkets, or even garbage dumps. Apart from municipal location modeling, one could ask where to best place servers so that the overall latency and provisioning cost for an online service is minimized.

All these examples are instances of the Facility Location Problem (FLP). In the FLP, we are given a set of possible *facility* locations (imagine possible hospital locations) and a set of *client* locations (i.e. patient locations), both in some space M. We are allowed to open facilities at any possible facility location m, incurring a *construction cost* of f_m. Additionally, we need to serve the given clients, specified by their location. A client at location i is served when it is connected to a facility at location j (when a patient is assigned to a hospital), incurring a service cost or *provisioning cost* of c_{ij}.

We consider an instance of the FLP as solved when we serve all the clients while minimizing a total cost defined as the sum of all construction costs and provisioning costs.

An important variant for this problem is the metric FLP. For example to model locations in 2D Euclidean space, it seems natural to let M be a metric space (M, d) with a proper distance metric d. Hence we talk about a metric FLP and the provisioning cost c_{ij} corresponds to the induced distance $d(i, j)$. Further, we do not allow the facilities to have a limit to how many clients can be served. The facilities are therefore *uncapacitated*. For the rest of the paper, when referring to the FLP, we implicitly talk about the metric, uncapacitated version.

For some real-world problems one tries to model with the FLP, it's assumptions turn out to be to strict. Consider for example the task of placing a set of warehouses (facilities) that are meant to supply a set of supermarkets (clients). Supermarkets turn out to have different needs in commodities, largely due to location-dependent features like customer-distribution, store-size, etc. Hence serving a supermarket i might require providing different commodities from a warehouse j than serving i'. Introducing the notion of different commodities into the problem allows to model more fine-grained construction costs (and provisioning costs). For example, it might be inappropriate to construct a warehouse offering the full range of a hypothetical 1 million commodities, when all supermarkets in the area request only a little as 50.000 different products. The so-called Multi-commodity Facility Location Problem (MCFLP) has been proposed by Ravi et al. [RS04] and allows to model settings in which there are a finite number of different commodities in play. A client i now has an additional request $s_i \subseteq S$ for a subset of all *commodities* S, which has to be served by any number of open facilities jointly offering the commodities requested in s_i. When constructing a facility f, any subset of commodities must be specified to be available in f. The construction cost is now also depending on the commodities associated with the facility, typically further assumptions (e.g. linearity or subadditivity) apply. Provisioning costs arise as the sum of distances connecting a client i to all the open facilities jointly serving the client's requested commodities s_i.

Originally, both the FLP and its extension MCFLP have been proposed as offline optimization problems where an algorithm is tasked with building facilities to minimize the overall cost for a given, finite set of clients. In this paper, we analyse both of the problems in their online variant. The Online Facility Location Problem (OFLP) and the OMCFLP respectively model situations in which the set of client requests is not known beforehand. An algorithm is given a single client request at a time and then has to decide, if and where to build a facility (in the extension, with which commodity configuration), and how to serve the given client request. Following the notion of online algorithms, we now talk about a request sequence as input. Decisions have to be made after each client request in the sequence and are not revocable.

In this paper, we further analyse the OFLP and the OMCFLP under the notion of the CR against an oblivious adversary. We present two lower-bounds for the OFLP and OMCFLP respectively, following work by Fotakis [Fot08] and Castenow et al. [Cas+20]. Finally, we discuss results obtained for both the OFLP and the OMCFLP in terms of implications for algorithm design.

1.1 Formal Problem Definition & Notation

For the rest of the paper, we use the following notation. The metric space $\mathcal{M} = (M, d)$ with point set M and metric $d : M \times M \to \mathbb{R}^+$ models our universe, in which the set of open facilities F and the set of client requests R are placed. Note that facilities and requests can be placed anywhere in M and are in this online setting not known beforehand. In case of the MCFLP, we additionally have S as a finite set of all commodities.

Let $r \in R$ be a single client request, anywhere in the request sequence. While in the OFLP, requests are just given as the request location $r \in M$, in the OMCFLP we formally denote a request as a tuple $r = (m_r, s_r)$ with $m_r \in M, s_r \subseteq S$ to accommodate for the (non-empty) subset of commodities asked for. Nonetheless, for convenience, we sometimes refer to r as the client request location itself, with s_r being the subset of commodities asked for in the request. Note that in the OMCFLP, $r \in R$ is strictly speaking not only a location, only we use it to keep the notation more consistent to the OFLP.

The construction cost for a facility f is only depending on the building site location $m \in M$ and is denoted as f_m. When referring to the OMCFLP, construction costs are additionally depending on the kind of commodities offered by that factory. Let $\sigma \subseteq S$ denote the (non-empty) subset of commodities offered at facility f, then f_m^σ denotes the construction cost in the OMCFLP. Construction costs are known for all $m \in M$ and $\sigma \subseteq S$ at all times.

Provisioning costs are incurred by connecting a client request $r \in R$ to all open facilities $F' \subseteq F$ such that the union over all commodities of facilities in F' is at least s_r. Costs then correspond to the sum of all distances connecting a request location to all the serving facilities F' locations. Note that we can safely assume a request r not to be connected to a facility when that facility offers none of the commodities asked for in s_r, because we always look at the minimal cost. In the OFLP, $|s_r| = |S| = 1$, hence costs are written as $d(r, m)$ when facility f serves r and is built at $m \in M$.

We task an algorithm for the OFLP (and the OMCFLP) with building facilities and assigning requests to these facilities so that each request is served and such that overall construction costs plus provisioning costs are minimized.

1.2 Further Remarks to the Problem Definition

Initially, the set of open facilities is empty. Despite implausible in reality, we allow the algorithm to build multiple facilities at the same location [Cas+20]. Note that the demanded irreversibly of the online algorithm's actions together with the problem definition also imply that facilities can not be closed once they have been opened [Fot08]. Further note that we can always assume an algorithm to connect a request r to the closest facility that can serve r [Fot08]. This is because we are looking at the minimal cost and facilities can serve requests indefinitely and instantly.

Lastly, we assume the construction cost function to be subadditive w.r.t the configuration. This assumtion only applies to the OMCFLP. For any point $m \in M$ and any configuration σ with $a, b \subseteq \sigma, a \cup b = \sigma$, we assume

$$f_m^\sigma \leq f_m^a + f_m^b \tag{1.1}$$

So constructing a single facility with joint configuration $a \cup b$ is at most as expensive as splitting the same configuration into two distinct facilities on the same location. As we are looking at the minimal cost, we can hence always assume to build a single larger facility, instead of multiple smaller facilities when considering a single point m.

1.3 Our Results

In this paper, we present proofs for lower bounds on the CR for randomized algorithms for both the OFLP and the OMCFLP. Fotakis showed that no randomized algorithm for the OFLP can achieve a CR better than $\Omega(\frac{log\ n}{log\ log\ n})$ [Fot08]. Castenow et al. built up on that work and proved a lower bound of $\Omega(\sqrt{|S|}+\frac{log\ n}{log\ log\ n})$ for the CR of any randomized algorithm for the OMCFLP [Cas+20], given a commodity set S. We enhance the proofs presented by the corresponding authors by more detailed explanations and additional visualizations. A discussion of implications of the proofs for algorithm design and a conclusion on the tightness of the lower-bounds follow.

2 Related Work

We regard the offline case for the FLP first. Unsurprisingly, the problem is difficult; Guha et al. [GK99] showed that the uncapacitated FLP is NP-hard, by a reduction from the B-vertex cover problem. Further they showed that there can't exist a polynomial-time approximation algorithm with an approximation ratio better than 1.463 unless $NP \subseteq DTIME[n^{\mathcal{O}(log\ log\ n)}]$. More strongly, that lower-bound has later been shown to hold unless $P = NP$ [Svi02]. The lower-bound approximation ratio is currently not matched; Li [Li13] gave the up-to-date best polynomial approximation with a ratio of 1.488.

The multi-commodity extension MCFLP with commodity set S has a polynomial-time approximation ratio of $\Theta(log|S|)$, as has been shown by Ravi et al. [RS10] by both an $\mathcal{O}(log|S|)$-approximation algorithm and an $\Omega(log|S|)$-approximation lower-bound. This holds under the subadditivity assumption from Equation (1.1). Constant-factor polynomial-time approximation is only possible under assuming linear construction cost functions and a cost-based ordering on the facilities, as shown by Shmoys [SSL04] with a primal-dual scheme-based approximation algorithm.

Results regarding OFL have first been shown by Meyerson [Mey01] who gave an $\mathcal{O}(\frac{log\ n}{log\ log\ n})$-competitive randomized algorithm for the OFLP. A constant CR is possible when the oblivious adversary's power is weakened to produce requests in random order [Mey01]. Meyerson also gave a first non-constant lower-bound on the CR of $\Omega(log^*n)$. log^*n denotes the inverse Ackermann function. Fotakis [Fot08] was able to raise that lower-bound to $\Omega(\frac{log\ n}{log\ log\ n})$ and in fact we will present his proof for this bound in Section 3.1. So Meyerson's randomized algorithm [Mey01] matches the lower-bound up to a constant. Fotakis [Fot08] additionally gave a deterministic algorithm matching his lower-bound, determining the CR for the OFLP to be in $\Theta(\frac{log\ n}{log\ log\ n})$.

As mentioned by Fotakis himself [Fot07], this deterministic algorithm is complicated in analysis and implementation and emits high constants, making it unsuited for use in practice. He gave a second deterministic algorithm with very good practical applicability and a slightly worse CR not larger than $4log(n+1)+2$ [Fot07]. The algorithm is based on the primal-dual scheme of Linear Programming and serves as the foundation for a deterministic online algorithm for the OMCFLP by Castenow et al. [Cas+20].

Previous work on deterministic algorithms for the OFLP resulted in a fast and simple $\Theta(2^d log\ n)$-competitive algorithm by Anagnostopoulos et al. [Ana+04], but it works only in d-dimensional Euclidean spaces and is restricted to uniform cost functions only.

The online variant of the MCFLP has been introduced by Castenow et al. [Cas+20] who gave a lower-bound on the CR and both a randomized- and a deterministic algorithm. Their proof of the $\Omega(\sqrt{|S|}+\frac{log\ n}{log\ log\ n})$-competitiveness lower-bound is presented in Section 3.2. Contrary to the OFLP, it is unclear whether that lower-bound on the CR is tight. Castenow et al. [Cas+20] gave both an $\mathcal{O}(\sqrt{|S|}\frac{log\ n}{log\ log\ n})$-competitive randomized algorithm and an $\mathcal{O}(\sqrt{|S|}log\ n)$-competitive deterministic algorithm based on the primal-dual online algorithm by Fotakis [Fot07]. Note the multiplication instead of a summation in the lower-bound. To the best of our knowledge, these are currently the only online algorithms for the multi-commodity extension of the problem.

3 Lower Bounds on the Competitive Ratio

Here we present lower bounds for both the OFLP (Section 3.1) and the OMCFLP (Section 3.2) regarding the CR. The following proofs are taken from the original papers respectively and are extended for better understandability and readability.

3.1 Online Facility Location Problem

The following theorem and its proof are taken from Fotakis [Fot08] and adapted to fit the notation and to be more comprehensible. We assume the facility construction cost function to be uniform, that means that the construction cost for a facility f_m is the same on all locations m. Further we consider only problem instances in which the optimal algorithm would construct a single facility. As we want to prove a lower bound, it is fine to consider just this subset of problem instances.

Theorem 1. *No randomized algorithm for the Online Facility Location Problem can achieve a Competitive Ratio better than $\Omega(\frac{log\ n}{log\ log\ n})$ against an oblivious adversary, even if the metric space is a line segment.*

Proof. To proof the theorem holds, we first consider a different metric space called a Hierarchically Well-Separated Tree (HST). A $m-$HST is a special weighted tree in which (i) The distance from a node to each of its children is the same and (ii) Going from the root to any node on the shortest path, the edge weight drops by a factor m on every edge [Bar96]. Later, we give an embedding of the $m-$HST into a line segment to show the result.

We let $T=(V,E,w)$ be a binary $m-$HST with an initial edge weight of D from the root to every child. As T is a weighted connected graph representation of our finite metric space, we interpret the edge weight between two nodes as the induced distance between two points. The height for node v, denoted $h(v)$, is the minimum number of edges to get back to the root node. It follows that the distance from node v to both children is $\frac{D}{m^{h(v)}}$. Further, let T be a full tree. That is, every non-leaf node has exactly two children and every leaf is of the same height $h=\max_{v\in V} h(v)$.

Figure 1 depicts T. Every node corresponds to a possible location in M. We observe two important inequalities.

Claim 2. *Let T_v be the subtree rooted at node $v \in T$. Then the distance from v to any node $w \in T_v$ is not larger than $\frac{m}{m-1}\frac{D}{m^{h(v)}}$. The distance from v to any node $u \notin T_v$ is at least $\frac{D}{m^{h(v)-1}}$.*

Proof. The second inequality can easily be verified by looking at Figure 1. The closest node $u \notin T_v$ is the parent of v. The distance is then at least $\frac{D}{m^{h(v)-1}}$.

For the first inequality, we calculate the path to any leaf node. Due to non-negativity of the distances, this must be the longest path to any $w \in T_v$. We have

$$\begin{aligned}\max_{w \in T_v} d(v,w) &= \sum_{i=h(v)}^{h-1} \frac{D}{m^i} = \frac{D}{m^{h(v)}} \sum_{i=h(v)}^{h-1} \frac{1}{m^{i-h(v)}} = \frac{D}{m^{h(v)}} \sum_{i=0}^{h-1-h(v)} \frac{1}{m^i} \\ &= \frac{D}{m^{h(v)}} \frac{(\frac{1}{m})^{h-h(v)} - 1}{\frac{1}{m} - 1} = \frac{D}{m^{h(v)}} \frac{m - \frac{m}{m^{h-h(v)}}}{m-1} \leq \frac{D}{m^{h(v)}} \frac{m}{m-1}\end{aligned}$$

which proofs the claim.

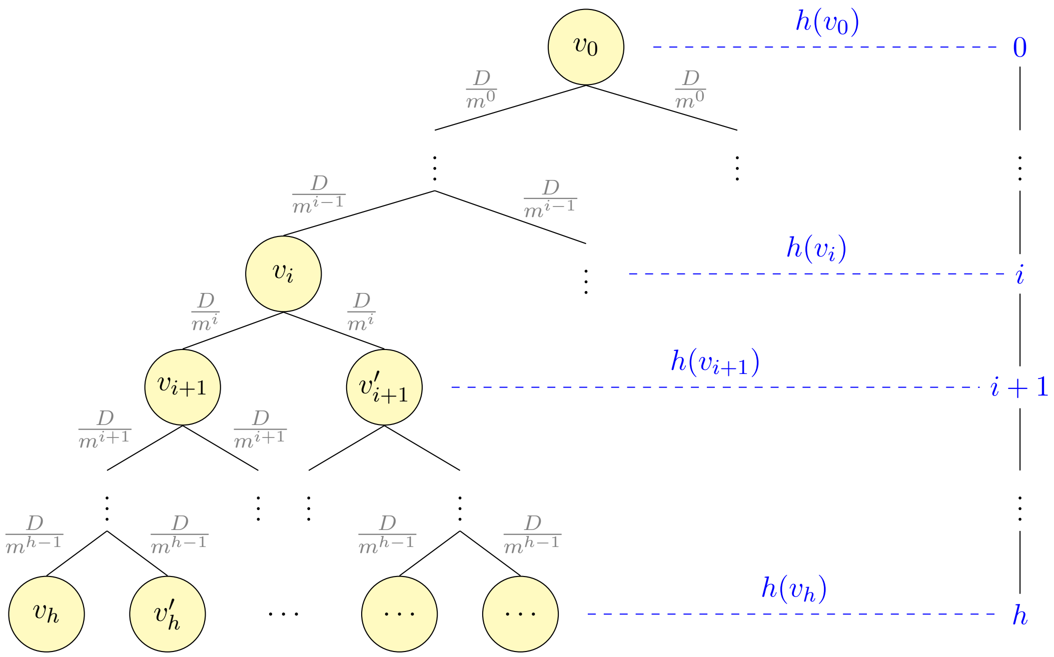


Figure 1: Sketch of the m−HST used in the proof. Blue numbers indicate the height of the nodes. All leaves are on the same height h.

As our goal is to prove a lower bound, we have to come up with a request sequence σ for which the expected cost of any randomized algorithm in ratio to the optimal cost is no less than $\Omega(\frac{log\ n}{log\ log\ n})$. Instead of providing such a σ, we apply Yao's principle [BE98, Chapter 8.3]. The principle allows us to lower-bound the worst-case expected cost of any randomized algorithm by the expected cost of the best deterministic algorithm against any request sequence distribution. Hence we only have to provide a distribution $\mathbb{Q}$ over request sequences and prove that no deterministic algorithm can be better than $\Omega(\frac{log\ n}{log\ log\ n})$ - competitive against $\mathbb{Q}$.

For the construction of $\mathbb{Q}$, consider a separation of a request sequence σ into phases. We divide σ into $h+1$ phases, corresponding to the number of nodes from the root v_0 to any leaf v_h in T. Let σ be

$$\sigma = (\underbrace{v_0}_{1}\underbrace{v_1 ... v_1}_{\text{m}} \quad ... \quad \underbrace{v_i ... v_i}_{m^i}\underbrace{v_{i+1} ... v_{i+1}}_{m^{i+1}} \quad ... \quad \underbrace{v_h ... v_h}_{m^h}) \tag{3.1}$$

so that, for a node v, we request that node $m^{h(v)}$ times, from the root down to a leaf. At each phase transition $v_i \to v_{i+1}$, we choose v_{i+1} uniformly at random from the two children of v_i, yielding our probability distribution $\mathbb{Q}$.

It is easy to verify that $|\sigma| \leq m^h \frac{m}{m-1} :\leq n$. We upper-bound the cost of the optimal deterministic algorithm $\mathcal{OPT}$ with the following algorithm $\mathcal{A}$: (1) Open a single facility at v_h and (2) Serve σ. With σ_{ij} denoting the j-th request in phase i, we have

$$\mathcal{OPT}(\sigma) \leq \mathcal{A}(\sigma) = f_{v_h} + \sum_{i=0}^{h}\sum_{j=1}^{m^i} d(v_h, \sigma_{ij}) \overset{2}{\leq} f_{v_h} + \sum_{i=0}^{h-1} m^i \frac{D}{m^i}\frac{m}{m-1} = f + hD\frac{m}{m-1} \tag{3.2}$$

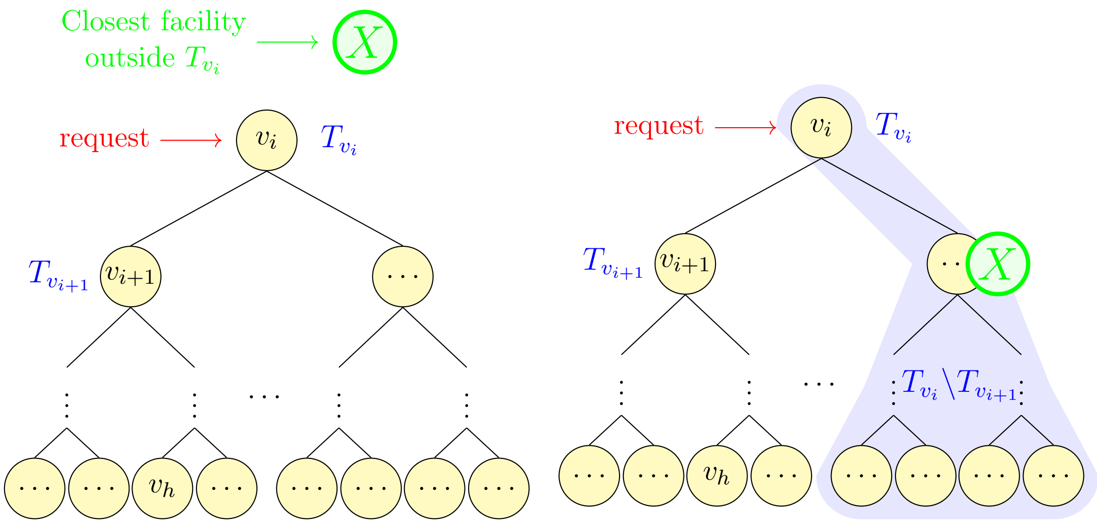
where f is the (uniform) construction cost and $hD\frac{m}{m-1}$ is a bound on the total provisioning cost for σ.

Now we are left to show a lower bound on the expected cost on $\mathbb{Q}$ for any deterministic algorithm. Let $\mathcal{ALG}$ be any such deterministic online algorithm. Let $\sigma_i = (v_0, ..., v_i)$ denote the sequence of nodes the adversary chose up to phase i according to $\mathbb{Q}$. Note that $0 \leq i \leq h-1$. This fixes the subtree T_{v_i} because we fixed an arbitrary path σ_i. Given subtree T_{v_i}, we are now interested in the expected cost of $\mathcal{ALG}$ for requests and facility construction up to (including) the i-th phase, which we denote as $\mathbb{E}[\mathcal{ALG}(\sigma_i)|T_{v_i}]$. Note that in this expected cost, we explicitly omit the provisioning and construction costs caused by $\mathcal{ALG}$ for all future requests in $T_{v_{i+1}}$.

Let v_{i+1} denote the first request in phase $i+1$. Right before v_{i+1}, we distinct between two cases. Figure 2 visualizes these two cases.

Case 1. No facilities in T_{v_i}.
Because facilities cannot be deconstructed, we must have that, for all m^i requests v_i in phase i, any facility $f \notin T_{v_i}$ must have served v_i. Using Theorem 2, $\mathcal{ALG}$ provisioning costs in that phase were at least $\frac{D}{m^{i-1}}m^i = Dm$.

Case 2. At least one facility in T_{v_i}.
At least a single facility was constructed somewhere in T_{v_i}. Provisioning costs for all v_i where hence at least 0. Now, as we choose v_{i+1} uniformly and independently at random from the two children of v_i, we find a facility in $T_{v_i} \setminus T_{v_{i+1}}$ with probability at least $\frac{1}{2}$. If at least one facility is in the branch $T_{v_i} \setminus T_{v_{i+1}}$, we count its construction cost f to this phase i, otherwise it is located in $T_{v_{i+1}}$ and we do not regard it in $\mathbb{E}[\mathcal{ALG}(\sigma_i)|T_{v_i}]$.



(a) No facility in T_{v_i} (b) At least one facility in T_{v_i}

Figure 2: Two cases regarding facility locations right before processing v_{i+1}. Green markers denote facility locations. Subtrees are marked in blue.

Hence, on expectation, we have for this case that $\mathcal{ALG}$ pays at least 0 (provisioning) plus $\frac{1}{2}f + \frac{1}{2}0 = \frac{f}{2}$ (construction).

Note that when we are in the last phase h and there is at least one facility in T_{v_h}, we can't further postpone the construction cost into $T_{v_{h+1}}$. Hence the expected cost is then at least f.

Combining the cases, the costs for phase $i, 0 \leq i \leq h-1$ are at least $min(Dm, \frac{f}{2})$. The cost for phase h is at least $min(Dm, f)$. Therefore,

$$\mathbb{E}[\mathcal{ALG}(\sigma_i)|T_{v_i}] \geq min(Dm, \frac{f}{2}) + \mathbb{E}[\mathcal{ALG}(\sigma_{i-1})|T_{v_{i-1}}]$$

Because T_{v_i} was arbitrary but fixed, we can remove the condition from the expected costs. Hence, $\mathbb{E}[\mathcal{ALG}(\sigma)] \geq h\ min(Dm, \frac{f}{2}) + min(Dm, f)$.

Since m and D are just parameters, we can choose $m = h$ and $D = \frac{f}{h}$, yielding

$$\mathbb{E}[\mathcal{ALG}(\sigma)] \geq h\ min(\frac{f}{h}h, \frac{f}{2}) + min(\frac{f}{h}h, f) = h\frac{f}{2} + f = hD\frac{h+2}{2} \tag{3.3}$$

while

$$\mathcal{OPT}(\sigma) \overset{3.2}{\leq} f + hD\frac{m}{m-1} = hD + hD\frac{h}{h-1} = hD\frac{2h-1}{h-1} \tag{3.4}$$

From our upper bound $|\sigma| \leq m^h \frac{m}{m-1} :\leq n$, we obtain a further condition $\frac{h^{h+1}}{h-1} \leq n$. Now in order to maximize the expected competitive ratio under the given constraint, we set $h = \left\lfloor \frac{log\ n}{log\ log\ n} \right\rfloor$, which results in the claimed lower bound on the CR.

Lastly, it remains to show that the result holds for line segment metric spaces. One can embed the tree-metric space T into a line segment by collapsing all non-leaf nodes to height h. More formally, one maps the root node v_0 to 0. Recursively, with a node v of height i mapped to $\tilde{v}$, mapping the left child of v to $\tilde{v} - \frac{D}{m^i}$ and mapping the right child of v to $\tilde{v} + \frac{D}{m^i}$, we obtain an embedding into the line that preserves the properties of T (Theorem 2) for any $m > 2$ [Fot08].

□

3.2 Online Multi-commodity Facility Location Problem

Results from the previous section tell us that even single-commodity OFL has a non-constant lower bound on the CR. Hence we can't expect a CR better than $\Omega(\frac{log\ n}{log\ log\ n})$ for the Multi-commodity extension; and indeed, extending the problem to include multiple commodities makes the lower bound even worse. The following theorem by Castenow et al. [Cas+20] formalizes the statement.

Theorem 3. *No randomized algorithm for the Online Multi-commodity Facility Location Problem can achieve a Competitive Ratio better than $\Omega(\sqrt{|S|} + \frac{log\ n}{log\ log\ n})$ against an oblivious adversary, even if the metric space is a line segment.*

Proof. For the proof of this theorem, we follow the structure presented by Castenow et al. [Cas+20]. First, we show a general lower bound on the CR of $\Omega(\sqrt{|S|})$, even when the metric space is only a single point. Then, using Theorem 1, we derive the claimed lower bound of Theorem 3.

For the first part of the proof, assume our whole problem setting takes place on a single point $m \in M$. We define the following (uniform) construction cost function

$$f_m^\sigma := g(|\sigma|) = \left\lceil \frac{|\sigma|}{\sqrt{|S|}} \right\rceil \tag{3.5}$$

only depending on the number of commodities offered at facility f. Further assume that, for readability, $\sqrt{|S|} \in \mathbb{N}$. As for the proof of Theorem 1, we use Yao's principle [BE98, Chapter 8.3] to lower-bound the expected cost of any randomized algorithm by the expected cost of the best-performing deterministic online algorithm against any input request sequence distribution $\mathbb{Q}$.

We define the distribution $\mathbb{Q}$ as follows. Sample from S uniformly and independently at random until $\sqrt{|S|}$ different commodities are sampled. We denote that set as S', $|S'| = \sqrt{|S|} := l$. Let $(s_1, s_2, ..., s_l)$ denote any permutation of the set S', then a single request sequence sampled from $\mathbb{Q}$ looks like

$$\sigma = (m^{s_1}, m^{s_2}, ..., m^{s_l}) \tag{3.6}$$

where m^{s_i} denotes a request at m demanding a single commodity $s_i \in S'$.

The optimal algorithm $\mathcal{OPT}$ would construct a single facility with a configuration of S' for cost

$$\mathcal{OPT}(\sigma) \leq f_m^{S'} = g(|S'|) = \left\lceil \frac{\sqrt{|S|}}{\sqrt{|S|}} \right\rceil = 1 \tag{3.7}$$

Let $\mathcal{ALG}$ be any deterministic online algorithm for OMCFLP. Now we need to show that the expected cost of $\mathcal{ALG}$ on distribution $\mathbb{Q}$ is at least $\Omega(\sqrt{|S|})$.

Interestingly, we need $\mathcal{ALG}$ to predict commodities that were not yet requested when building a facility. Processing any request sampled from $\mathbb{Q}$ without prediction would lead to $\mathcal{ALG}$ constructing $\sqrt{|S|}$ different facilities, each serving the uncovered commodity at least, incurring a cost of at least $\sqrt{|S|}g(1) = \sqrt{|S|}$. Hence we look at the case where $\mathcal{ALG}$ is predicting.

Castenow et al. regard so-called rounds [Cas+20]. A round starts when a commodity s' is requested that is not yet covered by any open facility in m. So at the beginning of a new round i, $\mathcal{ALG}$ has to construct a facility that serves at least s' and optionally $t_i \in \mathbb{N}$ additional commodities out of the set S as a prediction. In total, there are X rounds until all commodities needed for any $\sigma \sim \mathbb{Q}$ are covered. $\mathcal{ALG}$ may also have predicted commodities that were not requested at all, because $\mathcal{ALG}$ has no prior knowledge of S'. Figure 3 depicts the round-based view on $\mathcal{ALG}$'s behavior. Note that this is not a partitioning of the request into phases. The round view only depicts the process of covering commodities, requiring an (unknown) quantity of X rounds to serve the full request sequence. $T := \sum_i t_i$, denoting the total number of commodities predicted by $\mathcal{ALG}$, could well be larger than $|S'| - X$, if prediction is bad.

By construction of σ, each commodity is requested exactly once. Firstly, note that because we are looking at the best-performing deterministic $\mathcal{ALG}$, and due to Equation (1.1), we can assume $\mathcal{ALG}$ to construct a single facility at the beginning of round i, instead of multiple facilities with disjoint configurations. Secondly, note that we can assume $\mathcal{ALG}$ to only construct new facilities when a rounds begins, which is equivalent to only when an uncovered commodity is requested. To see this, assume that $\mathcal{ALG}$ constructs facilities when any covered commodity is requested. Then whatever $\mathcal{ALG}$ constructed can be constructed at the beginning of the next round instead, without changing the overall cost.

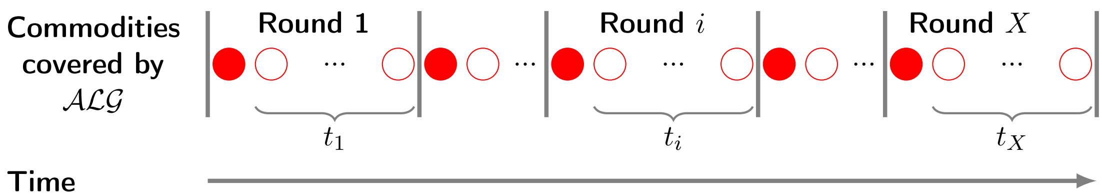


Figure 3: Round-based view on the coverage of commodities by $\mathcal{ALG}$, inspired by [Cas+20]. A new facility must be constructed at the beginning of each round (filled circle marker). t_i denotes the number of additionally covered commodities in that construction.

Provisioning costs are 0 for all requests. Since there are X rounds, $\mathcal{ALG}$ pays at least $Xg(1) = X$ for the construction costs. Since the construction costs only depend on the number of offered commodities, $\mathcal{ALG}$ pays at least $g(T)$. So we have a first lower bound on $\mathcal{ALG}$'s cost of $\max(X, \frac{T}{\sqrt{|S|}})$. We further distinct two cases.

Case 1. $X \geq \frac{\sqrt{|S|}}{2}$.

At least $\frac{\sqrt{|S|}}{2}$ facilities where built for a cost of at least 1 each, hence the overall expected costs of $\mathcal{ALG}$ are at least $\Omega(\sqrt{|S|})$.

Case 2. $X < \frac{\sqrt{|S|}}{2}$.

When less than $\frac{\sqrt{|S|}}{2}$ facilities where built, we show that T is large on expectation, leading to our claim. This trade-off appears natural: the less facilities are being build, the more prediction is needed and hence more difficult it gets for $\mathcal{ALG}$.

Let $S_a^{'} \subset S'$ denote the set of all commodities requested at the beginning of a round. This is the set of all requested, thought unpredicted commodities and it is of cardinality X. Then $S_b^{'} = S' \setminus S_a^{'}$ denotes the set of all requested and successfully predicted commodities. By premise of this case $|S_a^{'}| = X < \frac{\sqrt{|S|}}{2}$, therefor $\sqrt{|S|} > |S_b^{'}| \geq \frac{\sqrt{|S|}}{2}$. As S' is sampled uniformly and independently at random, $\mathcal{ALG}$ has no prior knowledge of what is good to predict and hence we can abstract from the actual commodities to the cardinalities of commodities instead. We model $\mathcal{ALG}$'s prediction of the T commodities as (uniformly) drawing T times without replacement from the set $S \setminus S_a^{'} := S_b$. Note that $|S_b| = |S| - |S_a^{'}| \geq |S| - \frac{\sqrt{|S|}}{2} \geq \frac{|S|}{2}$.

We now ask for the expected number of draws from S_b until all commodities of $S_b^{'}$ are covered. This is

$$\begin{aligned}\mathbb{E}[T] &= \sum_{i=|S_b^{'}|}^{|S_b|} i\Pr[T=i] \geq \sum_{i=|S_b^{'}|}^{\frac{|S|}{2}} i\Pr[T=i] \geq \sum_{i=\frac{|S|}{c}}^{\frac{|S|}{2}} \frac{|S|}{c}\Pr[T=i] \\ &= \frac{|S|}{c}\Pr[T \geq \frac{|S|}{c}] \geq \frac{|S|}{c}\Pr[T > \frac{|S|}{c}]\end{aligned} \tag{3.8}$$

In the best case, we predict exactly the commodities requested in $S_b^{'}$, in which case $T = |S_b^{'}|$, which happens with probability $\Pr[T = |S_b^{'}|]$. In the worst case, we predict all of S_b until $S_b^{'}$ is covered. $\frac{|S|}{c}$ upper-bounds $|S_b^{'}|$ for a constant $c \geq 4$.

The expectation on T can then be lower-bounded by $\frac{|S|}{c}\Pr[T > \frac{|S|}{c}]$. Now we prove this case by showing that $\Pr[T > \frac{|S|}{c}]$ is a non-zero probability, leading to $\mathbb{E}[T] = \Omega(|S|)$.

Consider the hypergeometrically-distributed random variable $Y \sim \mathit{Hypergeo}(\frac{|S|}{2}, \frac{\sqrt{|S|}}{2}, \frac{|S|}{c})$. The distribution models drawing, without replacement, $\frac{|S|}{c}$ times out of $\frac{|S|}{2} \leq |S_b|$ distinct commodities from which $\frac{\sqrt{|S|}}{2} \leq |S_b^{'}|$ commodities are marked (requested); Y then models the number of requested commodities that were drawn.

The event $T > \frac{|S|}{c}$ denotes that $\frac{|S|}{c}$ draws from the hypergeometric distribution were not enough to cover $S_b^{'}$, because $\mathcal{ALG}$'s sample size T (to cover $S_b^{'}$) was really larger. Note that the best-performing deterministic $\mathcal{ALG}$ would not sample from S_b when it doesn't need to predict anymore. Hence the actual number of sampled, requested commodities Y (obtained from $\frac{|S|}{c}$ draws) must have been smaller than $\frac{\sqrt{|S|}}{2} \leq |S_b^{'}|$ This equivalence helps to further lower-bound the expectation of T from Equation (3.8):

$$\begin{aligned}
\frac{|S|}{c}\Pr[T > \frac{|S|}{c}] &= \frac{|S|}{c}\Pr[Y < \frac{\sqrt{|S|}}{2}] = \frac{|S|}{c}(1 - \Pr[Y \geq \frac{\sqrt{|S|}}{2}]) \\
&= \frac{|S|}{c}(1 - \Pr[Y \geq \frac{\sqrt{|S|}}{c} - \frac{\sqrt{|S|}}{c} + \frac{\sqrt{|S|}}{2}])
\end{aligned}$$

Since $\mathbb{E}[Y] = \frac{\sqrt{|S|}}{c}$:

$$\begin{aligned}
&= \frac{|S|}{c}(1 - \Pr[Y \geq \mathbb{E}[Y] - \frac{2\sqrt{|S|} + c\sqrt{|S|}}{2c}]) \\
&= \frac{|S|}{c}(1 - \Pr[Y \geq \mathbb{E}[Y] + \frac{(c-2)|S|}{2\sqrt{|S|}c}])
\end{aligned}$$

and with bounds from Hoeffding and Chvátal [Hoe94; Chv79] and $c \geq 4$:

$$\begin{aligned}
&\geq \frac{|S|}{c}(1 - e^{-2\frac{(c-2)^2}{4|S|}\frac{|S|}{c}}) = \frac{|S|}{c}(1 - e^{-\frac{(c-2)^2}{2c}}) \geq \frac{|S|}{c}(1 - e^{-\frac{1}{2}}) \\
&\geq \frac{|S|}{c}(\frac{1}{4}) = \frac{|S|}{16}
\end{aligned} \tag{3.9}$$

So when less than $\frac{\sqrt{|S|}}{2}$ facilities are build by $\mathcal{ALG}$, the number of expected predictions until all commodities in S_b' are covered is at least $\frac{|S|}{16}$. This leads to a lower bound on $\mathcal{ALG}$'s cost of

$$\mathcal{ALG}(\sigma) \geq \max(X, g(\mathbb{E}[T])) \geq \max(X, \frac{\frac{|S|}{16}}{\sqrt{|S|}}) \geq \frac{\sqrt{|S|}}{16} = \Omega(\sqrt{|S|}) \tag{3.10}$$

combined with Equation (3.7) giving us the CR lower-bound of $\Omega(\sqrt{|S|})$. Now at least this CR can always be obtained by a malicious adversary against any (randomized) online algorithm, even on a single point. As the adversary chooses the requests, this lower-bound also holds on a line-segment metric space. Further, an instance of the FLP can be seen as an instance of the MCFLP with just a single commodity. Now we can combine two malicious request sequences from Theorem 1 and this proof to obtain a request sequence that causes any deterministic online algorithm to pay at least both lower bounds, yielding a competitive ratio of $\Omega(\sqrt{|S|} + \frac{log\ n}{log\ log\ n})$, and proving our claim.

□

4 Discussion

We discuss implications of the proofs and theorems presented in the last section. We start by deriving a prediction rule for an asymptotically tight CR on single-point metric spaces. Afterwards, we explain incorporation of that rule into algorithms for the OMCFLP and show current results regarding a tight lower-bound.

4.1 Prediction Rule

An interesting result from the proof of Theorem 3 is that in order to come anywhere near the $\Omega(\sqrt{|S|})$-competitiveness lower-bound, an online algorithm $\mathcal{ALG}$ for the OMCFLP has to predict yet unrequested commodities. Otherwise, the adversary can always ask for an uncovered commodity until all S is requested, in which case $\mathcal{ALG}$ constructs $|S|$ facilities. Under suitable construction cost functions, this can lead to $\Theta(|S|)$-competitiveness (e.g. $\mathcal{OPT}$ constructs a single facility). Predicting on unseen commodities is the only way to escape this malicious strategy.

Still, it remains the question on how to incorporate prediction into algorithms for the OMCFLP. Based on the lower-bound in Theorem 3, Castenow et al. [Cas+20] give a simple heuristic that enables $\mathcal{ALG}$, at least on single-point metric spaces, to be $\Theta(\sqrt{|S|})$-competitive. The idea is to categorize facilities into *small* and *large* facilities. A small facility is constructed with a single-commodity configuration only. Large facilities are configured to offer all commodities S. Based on Castenow's rule, no other facility configurations are needed.

The following prediction strategy is proposed [Cas+20]. While $\sqrt{|S|}$ or less distinct commodities are requested, construct small facilities serving the requests. In the worst case, this causes $\mathcal{ALG}$ to construct $\sqrt{|S|}$ as many facilities as $\mathcal{OPT}$. If the $\sqrt{|S|}+1$th distinct commodity is requested, construct a single, large facility. This might be expensive and the adversary might stop right after that request, but $\mathcal{OPT}$ has to serve at least $\sqrt{|S|}+1$ commodities as well. With a construction cost function depending on the number of configured commodities and the assumption from Equation (1.1), this leads to $\mathcal{O}(\sqrt{|S|})$-competitiveness [Cas+20].

4.2 Incorporating Prediction into Algorithms

Now this rule only really helps on single-point metric spaces. The challenge remains to extend that rule to general metric spaces and to integrate this prediction into online algorithms for the OMCFLP. The distinction between small and large facilities simplifies algorithm design and analysis: algorithms only have to decide between two types of facilities to construct. Both the deterministic, and the randomized algorithm for the OMCFLP proposed by Castenow et al. [Cas+20] make use of this strategy. The deterministic online algorithm called $\mathcal{PD}-\mathcal{OMFLP}$ is inspired by the primal-dual scheme of Fotakis' algorithm in [Fot07]; the randomized online algorithm $\mathcal{RAND}-\mathcal{OMFLP}$ is based on Meyerson's randomized algorithm for the OFLP [Mey01]. $\mathcal{PD}-\mathcal{OMFLP}$'s and $\mathcal{RAND}-\mathcal{OMFLP}$'s CRs are presented in Section 2 and for further details on the algorithms and their analysis, the reader is referred to the original paper [Cas+20]. With these two algorithms currently being the only known non-trivial online algorithms for the OMCFLP, it remains an open question whether the lower-bound presented in Theorem 3 is tight.

4.3 Tight Lower-bound for OMCFLP

We mention here that for a very similar problem, the non-metric Online Facility Location with Service Installation Costs (OFL-SIC), a tight lower-bound is known. Recently, Markarian [Mar21] gave a corresponding result. In the (non-metric) OFL-SIC, requests are configured to demand a subset of a so-called service set, functionally equivalent to the commodity set S in the OMCFLP itself. The only difference is that the construction cost for a facility is now the sum of a so-called fixed *opening cost*, plus, for each service configured for that facility, a *service installation cost*. In the OMCFLP, facility construction costs already combine the opening- and service installation costs, making OFL-SIC be a special case of OMCFLP. OFL-SIC is also known as OMCFLP with linear costs [Mar21].

With m denoting the number of facilities and S denoting the service set, Markarian [Mar21] gave an $\Omega(log(n|S|)\ log\ m)$ lower-bound on the CR and an $\mathcal{O}(log(n|S|)\ log\ m)$-competitive randomized algorithm, determining an asymptotically tight lower-bound. However, this only holds for a special class of OMCFLPs with linear costs, in the non-metric case.

5 Conclusion

We presented two lower-bounds on the Competitive Ratio (CR) for metric Online Facility Location (OFL). In Section 3.1, we saw that no randomized online algorithm for the OFLP can achieve a CR better than $\Omega(\frac{log\ n}{log\ log\ n})$ against an oblivious adversary by a proof based on a malicious request sequence distribution for a special tree-structure metric space, subsequently embedding that space into the line segment. Presentation of the proof was inspired by Fotakis [Fot08]. The lower-bound is known to be tight.

In Section 3.2, we stated that no randomized online algorithm for the Online Multi-commodity Facility Location Problem (OMCFLP) can be better than $\Omega(\sqrt{|S|}+\frac{log\ n}{log\ log\ n})$-competitive by lower-bounding the worst-case expected cost for any online algorithm on a random $\sqrt{|S|}$-sized subset of the commodity set S. The results hold even on a single-point metric space. This lower-bound, as well as both a randomized- and a deterministic online algorithm for the OMCFLP, has been given by Castenow et al. [Cas+20]. A tight lower-bound is only known for a special class of OMCFLPs called (non-metric) Online Facility Location with Service Installation Costs (OFL-SIC). Whether the lower-bound for the general OMCFLP proven in this paper is tight, remains an open question.

References

[Ana+04] Aris Anagnostopoulos et al. “A simple and deterministic competitive algorithm for online facility location.” In: *Information and Computation* 194.2 (2004), pp. 175–202.

[Bar96] Yair Bartal. "Probabilistic approximation of metric spaces and its algorithmic applications." In: *Proceedings of 37th Conference on Foundations of Computer Science*. IEEE. 1996, pp. 184–193.

[BE98] Allan Borodin and Ran El-Yaniv. *Online Computation and Competitive Analysis*. USA: Cambridge University Press, 1998. ISBN: 0521563925.

[Cas+20] Jannik Castenow et al. "The online multi-commodity facility location problem." In: *Proceedings of the 32nd ACM Symposium on Parallelism in Algorithms and Architectures*. 2020, pp. 129–139.

[Chv79] Vasek Chvátal. "The tail of the hypergeometric distribution." In: *Discrete Mathematics* 25.3 (1979), pp. 285–287.

[Fot07] Dimitris Fotakis. "A primal-dual algorithm for online non-uniform facility location." In: *Journal of Discrete Algorithms* 5.1 (2007), pp. 141–148.

[Fot08] Dimitris Fotakis. "On the competitive ratio for online facility location." In: *Algorithmica* 50.1 (2008), pp. 1–57.

[GK99] Sudipto Guha and Samir Khuller. "Greedy strikes back: Improved facility location algorithms." In: *Journal of algorithms* 31.1 (1999), pp. 228–248.

[Hoe94] Wassily Hoeffding. "Probability inequalities for sums of bounded random variables." In: *The collected works of Wassily Hoeffding*. Springer, 1994, pp. 409–426.

[Li13] Shi Li. "A 1.488 approximation algorithm for the uncapacitated facility location problem." In: *Information and Computation* 222 (2013), pp. 45–58.

[Mar21] Christine Markarian. "Online Non-metric Facility Location with Service Installation Costs." In: *Proceedings of the 23rd International Conference on Enterprise Information Systems*. Vol. 1. ICEIS. 2021, pp. 737–743.

[Mey01] Adam Meyerson. "Online facility location." In: *Proceedings 42nd IEEE Symposium on Foundations of Computer Science*. IEEE. 2001, pp. 426–431.

[RS04] Ramamoorthi Ravi and Amitabh Sinha. "Multicommodity facility location." In: *Proceedings of the fifteenth annual ACM-SIAM symposium on Discrete algorithms*. 2004, pp. 342–349.

[RS10] Ramamoorthi Ravi and Amitabh Sinha. "Approximation algorithms for multicommodity facility location problems." In: *SIAM Journal on Discrete Mathematics* 24.2 (2010), pp. 538–551.

[SSL04] David B Shmoys, Chaitanya Swamy, and Retsef Levi. "Facility location with service installation costs." In: *Proceedings of the fifteenth annual ACM-SIAM symposium on Discrete algorithms*. 2004, pp. 1088–1097.

[Svi02] Maxim Sviridenko. "An improved approximation algorithm for the metric uncapacitated facility location problem." In: *International Conference on Integer Programming and Combinatorial Optimization*. Springer. 2002, pp. 240–257.