On Lower Bounds for the Online Facility Location Problem and its Multi-commodity extension

Seminar Talk for Methods of Algorithm Design

Marcus Rottschäfer

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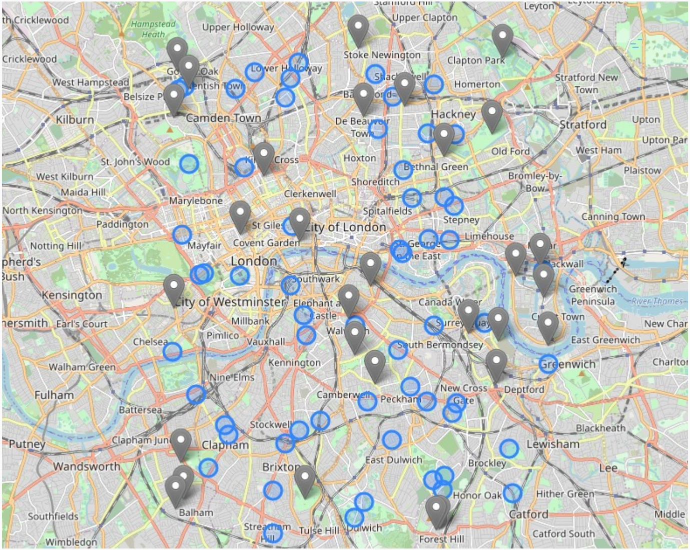
Department of Informatics, University of Hamburg

What is Facility Location good for?



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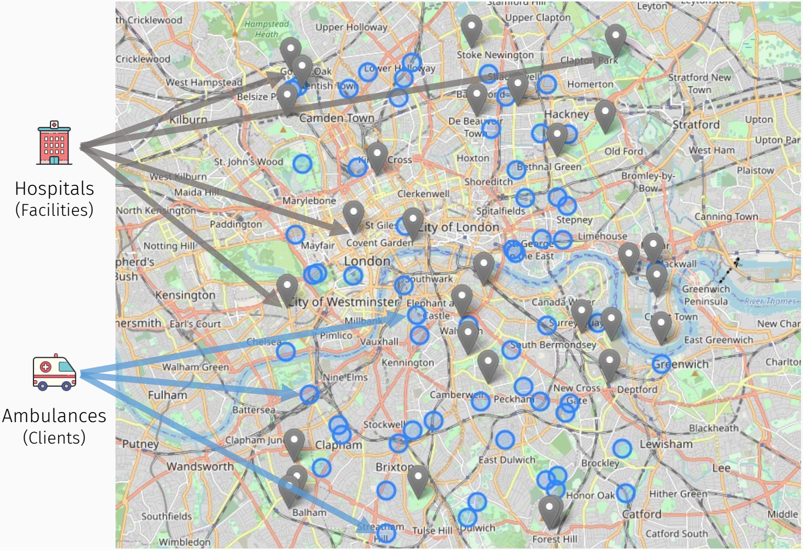
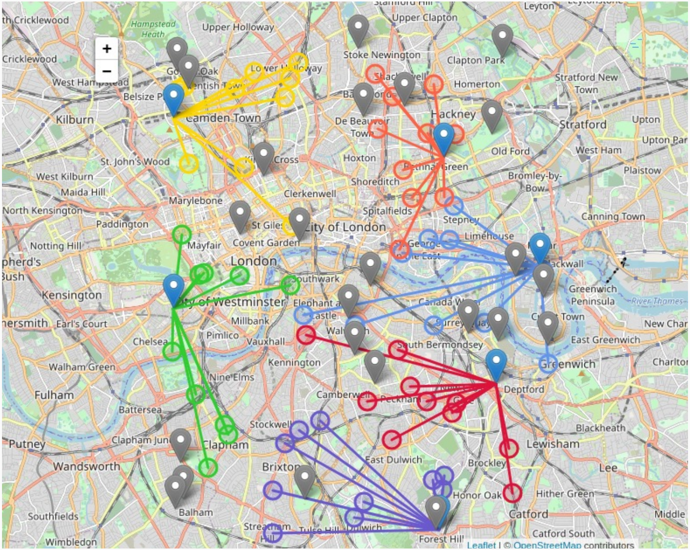


What is Facility Location good for?

Hospitals
(Facilities)

Ambulances
(Clients)



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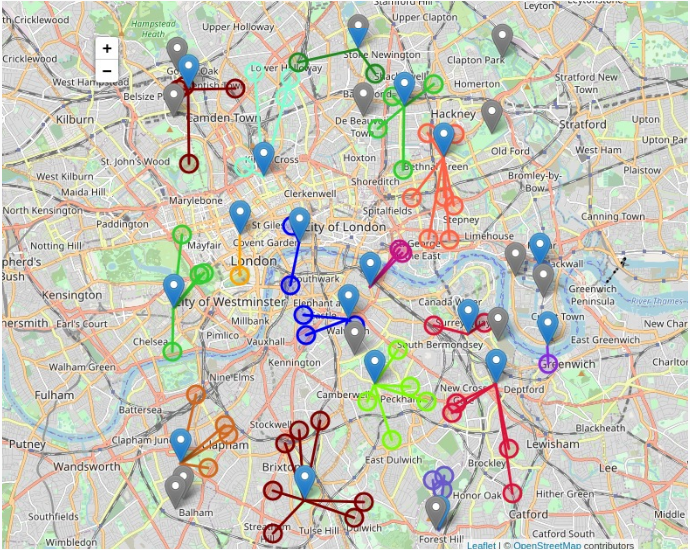
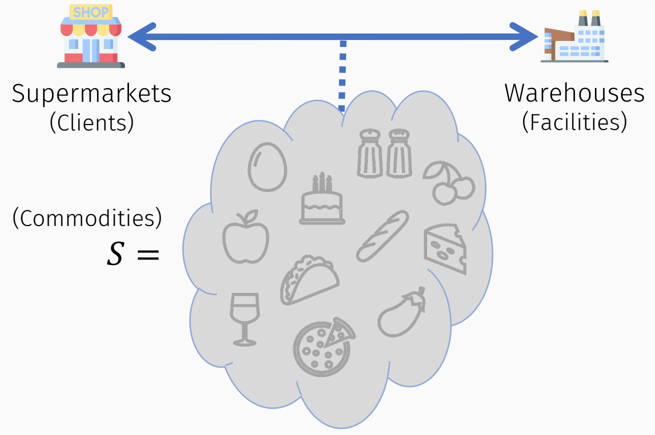


Why should we extend it to Multi-commodities?



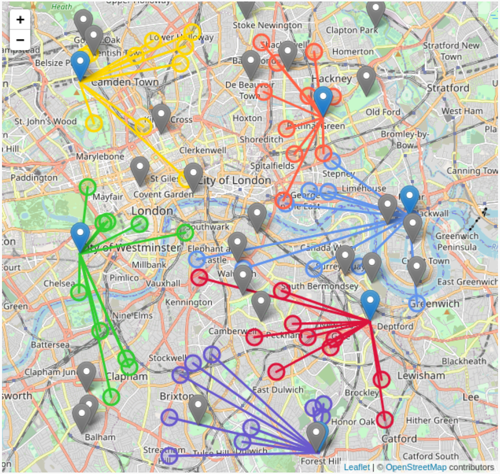
Multi-commodity Facility Location vs. Facility Location

- Clients can have **multiple** needs $s \subseteq S$
- Facilities provide **commodities** $\sigma \subseteq S$

Table of contents

Formalities

Online Facility Location Problem (OFLP)



- Metric space
  $\mathcal{M} = (M, d)$
  $d : M \times M \to \mathbb{R}^+$
- Requests & Facilities
  $r \in R, f \in F$, anywhere in M
  Request sequence σ
- Construction costs
  f_m: Open f at $m \in M$
- Provisioning costs
  $d(r, f)$: Serve r with f

Serve $\sigma = (r_1, r_2, \ldots)$ and construct facilities F such that
$\mathcal{A}(\sigma) := \sum_{f_m \in F} f_m + \sum_{r \in \sigma} d(r, f)$ is minimized.

Online Multi-commodity Facility Location Problem (OMCFLP)

What changes with the Multi-commodity extension?

- Commodities
  predefined set S
- Requests $r \in R$
  additional $s_r \subseteq S$
- Facilities
  f^{ψ} with $\psi \subseteq S$
  (configuration of f)

Construction costs:

- depend on configuration ψ : f_m^{ψ}:

Provisioning costs:

- Sum of sums (all facilities jointly serving s_r)

Formalities

Further Remarks

- Start without facilities
- Can’t close facilities
- Requests only connected to facilities capable of serving them
- Always serve r with the closest facility
- Consider the metric, uncapacitated case only

Related Work

Offline Cases

- NP-hard on general graphs :(
- 1.463-approximation lower bound [3]
- Li [4] gave 1.488-approximation algorithm
- Multi-commodity: $\Theta(log\ |S|)$-approximation [6], but constant approximation [7] (linear facility cost function)

Lower Bounds

Overview of the Results

What Competitive Ratio can we expect?

- Bad news: we can't hope for a *constant* CR :(

Theorem (Online Facility Location Problem)

No randomized algorithm for the OFLP can achieve a CR better than $\Omega(\frac{\log n}{\log \log n})$ against an oblivious adversary, even if the metric space is a line segment.

Theorem (Online Multi-commodity Facility Location Problem)

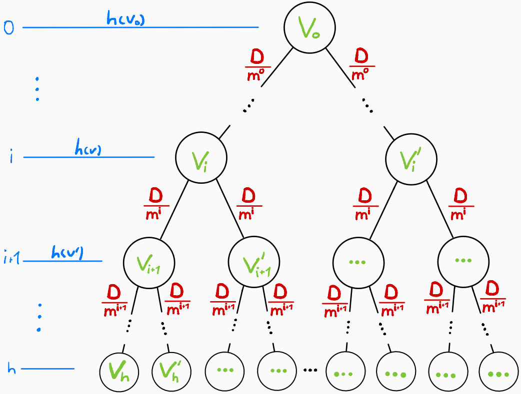
No randomized algorithm for the OMCFLP can achieve a CR better than $\Omega(\sqrt{|S|} + \frac{\log n}{\log \log n})$ against an oblivious adversary, even if the metric space is a line segment.

Lower Bound for OFL: Proof Overview

Structure of Proof:

- Limit to uniform facility costs & single-facility solutions only
- Construct special tree-structure metric space T
- Provide input σ s.t. lower bound holds on T
- Project T on any line segment

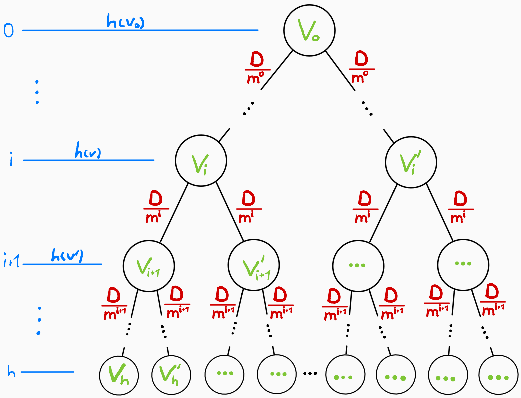
(Tree-structure) Metric space T



Let our metric space T be a full, binary m−HST.

Lower Bound for OFL: Proof (1/5))

(Tree-structure) Metric space T



Let our metric space T be a full, binary m—HST.

Properties

- Both children have same distance
- Distance drops by factor m
- All leaves at height h
- Represents finite metric space

Lower Bound for OFL: Proof (2/5)

Claim (Distances in T)

Let T_v be the subtree rooted at node $v \in T$. Then the distance from v to any node $u \notin T_v$ is at least $\frac{D}{m^{h(v)-1}}$. The distance from v to any node $w \in T_v$ is not larger than $\frac{m}{m-1}\frac{D}{m^{h(v)}}$.

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Proof.

1. See picture.
2. The longest path must end at leaf node. Hence

$$\max_{w \in T_v} d(v,w) = \sum_{i=h(v)}^{h-1} \frac{D}{m^i} = \frac{D}{m^{h(v)}} \sum_{i=0}^{h-1-h(v)} \frac{1}{m^i} \leq \frac{D}{m^{h(v)}} \frac{m}{m-1}$$

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- Now we'd construct bad (for any randomized algorithm) input sequence σ.
- Instead we apply Yao's Principle.

Lower Bound for OFL: Proof (3/5)

Bad Request Sequence

$$\sigma = (\underbrace{v_0}_{1}\ \underbrace{v_1 \ldots v_1}_{\text{m}} \quad \ldots \quad \underbrace{v_i \ldots v_i}_{m^i}\ \underbrace{v_{i+1} \ldots v_{i+1}}_{m^{i+1}} \quad \ldots \quad \underbrace{v_h \ldots v_h}_{m^h})$$

- At phase transition $v_i \rightarrow v_{i+1}$: Choose v_{i+1} u.i.r from v_i's children
- Yields request sequence distribution $\mathbb{Q}$

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Optimal Algorithm's Costs (Upper Bound):

- $\mathcal{A}$: (1) Open facility at leaf v_h (2) Serve σ.

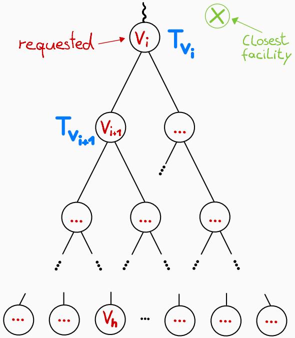
$$\mathcal{OPT}(\sigma) \leq \mathcal{A}(\sigma) = f_{v_h} + \sum_{i=0}^{h} \sum_{j=1}^{m^i} d(v_h, \sigma_{ij}) \overset{3}{\leq} f_{v_h} + \sum_{i=0}^{h-1} m^i \frac{D}{m^i} \frac{m}{m-1}$$

$$= f + hD\frac{m}{m-1}$$

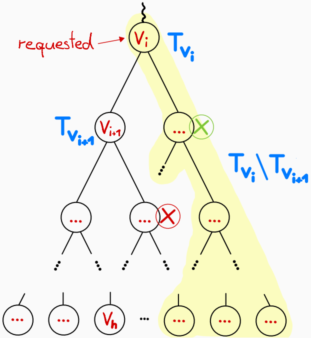
Lower Bound for OFL: Proof (4/5)

Deterministic Algorithm's Cost (Lower Bound):

- Let $\mathcal{ALG}$ be ANY deterministic algorithm run on $\mathbb{Q}$
- Right before processing phase $i+1$ (fixes T_{v_i}):
- Consider expected costs for T_{v_i}, but not for $T_{v_{i+1}}$: $\mathbb{E}[\mathcal{ALG}(\sigma_i)|T_{v_i}]$



(a) No facility in T_{v_i}



(b) At least one facility in T_{v_i}

(a) exp. costs $\geq \frac{D}{m^{i-1}} m^i + 0 = Dm$

(b) exp. costs $\geq 0 + \frac{1}{2} f + \frac{1}{2} 0 = \frac{f}{2}$

Lower Bound for OFL: Proof (5/5)

- Hence for the expected costs of $\mathcal{ALG}$:

$$\mathbb{E}[\mathcal{ALG}(\sigma_i)|T_{v_i}] \geq min(Dm, \frac{f}{2}) + \mathbb{E}[\mathcal{ALG}(\sigma_{i-1})|T_{v_{i-1}}]$$

- We go through $h+1$ phases: $0 \leq i \leq h-1$ + last phase h

$$\mathbb{E}[\mathcal{ALG}(\sigma)] \geq h\ min(Dm, \frac{f}{2}) + min(Dm, f)$$

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Competitive Ratio

- We choose $m = h$ and $D = \frac{f}{h}$:

$$\mathbb{E}[\mathcal{ALG}(\sigma)] \geq h\ min(\frac{f}{h}h, \frac{f}{2}) + min(\frac{f}{h}h, f) = h\frac{f}{2} + f = hD\frac{h+2}{2}$$

$$\mathcal{OPT}(\sigma) \overset{12}{\leq} f + hD\frac{m}{m-1} = hD + hD\frac{h}{h-1} = hD\frac{2h-1}{h-1}$$

- Since $|\sigma| \leq m^h \frac{m}{m-1} :\leq n \overset{m=h}{\Longrightarrow} \frac{h^{h+1}}{h-1} \leq n$: Choose $h = \left\lfloor \frac{log\ n}{log\ log\ n} \right\rfloor$ to obtain competitive ratio of $\Omega(\frac{log\ n}{log\ log\ n})$.

Lower Bound for OMCFL: Overview

Theorem (Online Multi-commodity Facility Location Problem)

No randomized algorithm for the Online Multi-commodity Facility Location Problem can achieve a competitive ratio better than $\Omega(\sqrt{|S|})$, even on a single point.

Corollary (Online Multi-commodity Facility Location Problem)

No randomized algorithm for the Online Multi-commodity Facility Location Problem can achieve a competitive ratio better than $\Omega(\sqrt{|S|} + \frac{\log n}{\log \log n})$, even on a line metric.

- No proof here
- Corollary follows from combining theorem with OFLP lower bound

What more is there?

Remarks on the Lower Bounds

Lower bounds matched?

- Fotakis [2] gave $\mathcal{O}(\frac{log\ n}{log\ log\ n})$-competitive deterministic algorithm (OFLP)
- Meyerson [5] gave $\mathcal{O}(\frac{log\ n}{log\ log\ n})$-competitive randomized algorithm (OFLP)
- $\mathcal{O}(\sqrt{|S|} * log\ n)$-competitive deterministic algorithm [1] (OMCFLP)
- $\mathcal{O}(\sqrt{|S|} * \frac{log\ n}{log\ log\ n})$-competitive randomized algorithm [1] (OMCFLP)

Concluding

Conclusion

(Metric Spaces, Uncapacitated Problems)

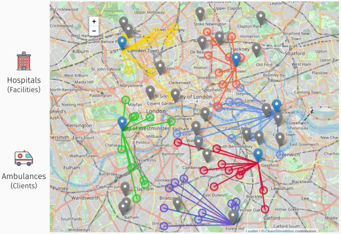
OFLP

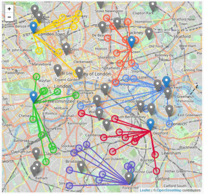
- $\Theta(\frac{log\ n}{log\ log\ n})$-competitive lower bound (Theorem 1)
- Matched by both deterministic and randomized algorithm

OMCFLP

- $\Omega(\sqrt{|S|} + \frac{log\ n}{log\ log\ n})$ lower bound on CR (Theorem 2)
- $\mathcal{O}(\sqrt{|S|} * log\ n)$-competitive deterministic algorithm
- $\mathcal{O}(\sqrt{|S|} * \frac{log\ n}{log\ log\ n})$-competitive randomized algorithm

Your Questions





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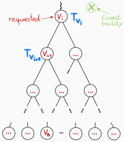

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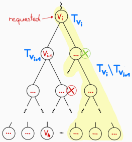

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OMCFLP

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- $\mathcal{O}(\sqrt{|S|} * log\ n)$-competitive deterministic algorithm
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Back-Up Slides

Back up - Yao's Principle

- Σ: set of all input sequences
- $\mathbb{Q}$: distribution over input sequences
- $\mathcal{R}$: any randomized algorithm
- $\mathcal{D}$: set of all deterministic algorithms

Yao's Principle [8]

Lower bound on the (worst-case) expected cost of any randomized algorithm $\mathcal{R}$.

$$\max_{\sigma \in \Sigma} \mathbb{E}[\mathcal{R}(\sigma)] \geq \min_{d \in \mathcal{D}} \mathbb{E}[d(\mathbb{Q})]$$