## 1 Logic

A particular way of thinking about something **OR** Logic is the systematic study of the form of valid inference
**Types of Logic:**: 1. **1. Deductive Logic**: Properties of a Set are applied to the individual members of the Set. This is also known as **Mathematical Logic.**
**2. Inductive Logic**: Properties of Individual Members are applied to the whole Set.

**Propositions**: English phrases which are Primary Bearers of Truth Value(true/False)
**Atomic Propositions**: "Ram is married"(p), "Ram has a wife"(q)
**Logical Connectives**: Symbol/Word connecting 2 related ideas. ∧(AND), ∨(OR), →(IMPLIES)
**Compound Propositions**: Atomic Propositions + Logical Connectives.
**Example**: **p→q**(Ram is married IMPLIES Ram has a wife) **[Logical Representation]**
If Ram is married Then Ram has a wife**[Non Logical Representation]**

**Variable**: x="Ram" or more Generally x="Man".
**Predicate**: $P_1(x)$=Man is married, $P_2(x)$=Man has a wife. Statement that may be true or false depending on the values of its variables.
**Quantifiers**: Specifies quantity of specimens in domain of discourse that satisfy an open formula. ∀(for each/all)**[Universal Quantification]**, ∃(there exists some)**[Existential Quantification]**
**Example**: $\forall x(P_1(x)\rightarrow P_2(x))$. For all men being married IMPLIES he has a wife**[Logical Representation]**. For all men If Man is married Then Man has a Wife**[Non Logical Representation]**

Statement is a ***Declarative Sentence*** that's **either true** or **false** but **NOT both.**
**Example**:
>> 3+7=10.
>> $\forall(x,y)\in\mathbb{R}((x+y)^2=x^2+2xy+y^2)$)
[For all x and y belonging to Real Numbers $(x+y)^2=(x^2+2xy+y^2)$]

**Definition**: RoI provide the Templates or Guidelines for constructing Valid Arguments from the Statements that we already have.
**Valid Argument**: Argument is Valid if Conclusion follows from the truth values of the Premises.
**Argument**:Argument is a sequence of Statements.
**Example**: **Modus Ponens**[((p→q)∧p)→q), **Modus Tollens**[((p→q)∧¬q)→¬p)

p → q
**1. Propositional Logic (Zeroth Order Logic)**

Propositional Logic
+ Quantifiers

∀x(P₁(x)) → P₂(x))
**2. Predicate Logic (First Order Logic)**

**3. Mathematical Statement**

Used to construct RoI. A.K.A Logical Form: A precisely defined Semantic version of Syntactic Expression

**4. Rules of Inference**

## 3 Proof

Logic

p → q
1. Propositional Logic (Zeroth Order Logic)

2. Predicate Logic (First Order Logic)

Rules of Inferences**(Logic)** supply **Templates** to construct syntactically correct **Arguments** while Logic(**Propositional/Predicate**) also supply **Templates** to construct syntactically correct **Statements** **(SYNTAX=Proper Structure)**

Inferential argument for a mathematical statement, showing that the stated assumptions logically guarantee the conclusion.

Logic proves the validity of various Proof Techniques

Premises
p

Contains

Conjecture
**p→q**

Contains

Conclusion
q

**Proof Technique**

**Direct**

**Indirect**

**TYPES**
**> Simple Direct: Assume p Prove q**
**> Proof by Exhaustion. Prove all cases arising from statement**
**> Proof by Mathematical Induction**

**TYPES**
**> Proof By Contradiction. Assume ¬q and reach a Contradiction.**
**> Proof by Contraindication.¬q→¬p**
**> Proof by Infinite Descent.**

These Statements use Logical Constructs to form Valid Arguments

Start with

Premises
p

(Which depend on Proof Technique used)

**Argument**
(Set of Statements)

Statement 1
Statement 2
Statement 3
.
.
.
.
Statement m

This whole activity is known as **Reasoning.** In Mathematics we use **Deductive Reasoning.**

Arrive at

(which depends On chosen Proof Technique)

Conclusion
q

Proven

**Theorem/ Lemma/ Corollary**

These Statements use Mathematical Objects

Definition
Axiom(s)
Lemma
Theorem
Corollary

**Meaning** to/the **Arguments/Statements** comes from the terms defined in the **Formal System/Language (SEMANTIC=Proper Meaning)**

## 2 Structure Of Math

Various simple objects/structures at atomic levels combine to evolve into complex structures/objects.

**No Definition. No Proof**
**Example**: Successor Function

Primitive Function

Primitive Notion

**Used by**

**No Definition, No Proofs**
**Example**: Zero, Set

Definition

**No Proofs**
**Example: Circle** is set of points in a plane that are at a given distance from a given point, the centre.

**Used By**

Conjecture

Axiom(s)

An axiom or postulate is a statement that is taken to be true, to serve as a premise or starting point for further reasoning and arguments.
**No Proofs**
**Example**: Peano Axioms(Natural Numbers):0 is a natural number.

**Proven Conjecture:** A PROVEN Conjecture(Mathematical Statement).

**Proven Into**

**Formal System**
**Example:Axiomatic System**
**Example: Natural Number System(Peano Axioms)**

Lemma

**Proof Required**
A minor theorem; acts as helping fact in proving other theorems

Theorem

**May Result in**

Corollary

**Proof Required**
A Statement that follows trivially from any Theorem, Lemma, or other Corollary. A result in which the proof relies heavily on a given theorem (we often say that "this is a corollary of Theorem A").

# 1 Logic

1. **Propositional Logic (Zeroth Order Logic)** → Use to Construct → 2. **Predicate Logic (First Order Logic)** → Used to Construct → 3. **Mathematical Statement** → Used to construct → 4. **Rules of Inference**

| Logical/Mathematical Symbol | Logical Connectives | Propositional Logic: Proposition | Propositional Logic: Expression | Predicate Logic: Quantifier | Predicate Logic: Predicate | Predicate Logic: Expression | Mathematical Statement |
|---|---|---|---|---|---|---|---|
| | | | | | $P_1$ | | is even |
| | | p | | | $P_1(n)$ | | Integer n is even |
| | | | | | $P_2$ | | '=2k, k is some integer' |
| | | q | | | $P_2(n)$, $k \in Z$ | | Integer n=2k, k is some integer |
| | $\leftrightarrow$ | | | | | | If and Only If |
| | | | $p \leftrightarrow q$ | | | $P_1(n) \leftrightarrow P_2(n)$ | Integer n is even If and Only If n=2k, k is some integer |
| | | | | $\exists$ | | | There exists |
| | | | | | | $\exists k$ | There exists an integer k |
| $\in$ | | | | | | | Belongs to |
| $\ni$, $\mid$ | | | | | | | Such That |
| | | | | | | $P_1(n) \leftrightarrow \exists k \mid P_2(n), k \in Z$ | An integer number n is **even** if and only if there exists a number k such that Integer **n = 2k**, k is some Integer |

Equivalent as Truth Values are same.

| Boolean Logic/Algebra | | | | Direct | Contrapositive |
|---|---|---|---|---|---|
| p | q | ¬p | ¬q | p→q(¬p∨q) | ¬q→¬p(q∨¬p) |
| T | T | F | F | T | T |
| T | F | F | T | F | F |
| F | T | T | F | T | T |
| F | F | T | T | T | T |

Validity of these two proofs is provided by Logic using Truth Table

| Rules of Inference | |
|---|---|
| p: it's raining, q: it's wet | |
| **Modus Ponens** | **Modus Tollens** |
| p→q<br>p<br>----------<br>∴ q | p→q<br>¬q<br>----------<br>∴ ¬p |
| ((p→q)∧p)→q | ((p→q)∧¬q)→¬p |
| If it's raining Then it's wet AND it's raining Therefore it's wet. | If it's raining Then it's wet AND it's NOT wet Therefore it's NOT raining. |

These templates are used in Proof argument

# 3 Proof

**Conjecture**

If a and b are consecutive integers, then the sum a + b is odd.

Contains

**Premises**
p
a and b are consecutive integers

Contains

**Conclusion**
q
the sum a + b is odd

Proof Technique

## Direct

**Direct Proof**
**p→q (Constructive Proof)**
**If a and b are consecutive integers, then the sum a + b is odd.**

1. Assume(p) that a and b are consecutive integers.
2. Because a and b are consecutive we know that b = a + 1, **(From Definition 3 below)**
3. Thus, the sum a + b may be re-written as a + (a+1) =2a+1.
4. Thus, there exists a number k such that a+b = 2k +1.
5. Hence the sum a+b is odd. **(Using point 4 above and Definition 3 below).**
6. Since Premise(p) logically leads to Conclusion(q) hence proven that **If a and b are consecutive integers, then the sum a + b is odd.**

## Indirect

**Indirect Proof**
**¬q→¬p (Proof by Contraposition)**
**If the sum a + b is not odd, then a and b are not consecutive integers.**

1. Assume(¬q) that the sum of the integers a and b is **NOT** odd.
2. Then, there exists no integer k such that a + b = 2k + 1. **(From Contraposition of Definition 2 below)**
3. Thus, a + b ≠ k + (k + 1) [*Rewriting 2k+1*] for all integers k.
4. If k +1 is the successor of k, **(From Definition 3 below)**
5. This implies that a + b is not equal to sum of any two consecutive integers.
6. This implies(¬p) that a and b cannot be consecutive integers.
7. Since negation of Consequence leads to negation of Premise, hence the original assumption p→q is correct, i.e. **If a and b are consecutive integers, then the sum a + b is odd.**

# 2 Structure Of Math

DFn

1. An integer number n is even if and only if there exists a number k such that n = 2k, k is some integer.
2. An integer number n is odd if and only if there exists a number k such that n = 2k + 1.
3. Two integers a and b are consecutive if and only if b = a + 1.
4. Let $n^+$ be the successor of n, that is the number following n in the natural numbers, so $0^+=1$, $1^+=2$.
Define a + 0 = a. Define the general sum recursively by $a + (b^+) = (a + b)^+$. Hence $1 + 1 = 1 + 0^+ = (1 + 0)^+ = 1^+ = 2$.

Uses Primitive Recursive Function

PFn Used by

PNtn

Used By

We don't need to prove that **a+b=integer** because of this axiom.

Axm

**1. Closure of ∗ & +: a∗b and a+b are integers. 2.** Commutativity of +: a+b=b+a
**3.** Associativity of +: (a+b)+c=a+(b+c) **4.** Commutativity of ∗ : a∗b=b∗a
**5.** Associativity of ∗ : (a∗ b)∗ c=a∗ (b∗ c). **6.** Distributivity: a∗ (b+c)=a∗ b+a∗ c
**7.** Trichotomy: Either a<0, a=0, or a>0. **8.** Well-Ordered Principle: Every non-empty set of positive integers has a least element. (This is equivalent to induction.)

Conjecture

**If a and b are consecutive integers, then the *sum a + b is* odd *integer*.**
**p(Premise)=a and b are consecutive integers**
**q(Conclusion)=the sum a + b is odd integer.**

Proof

**Direct Proof**
**(Constructive Proof)**
**p→q**

Proof

**Indirect Proof**
**(Proof by Contraposition)**
**¬q→¬p**
**If the sum a + b is not odd, then a and b are not consecutive integers.**

Proven

**Theorem**