4)

h) The parameter estimation technique for ordinal likelihood in the paper involves formulating a likelihood function based on the cumulative logit model, taking the logarithm to create a log-likelihood function, and finding the parameter values that maximize this log-likelihood using numerical optimization methods. These estimated parameters help model the relationship between covariates & ordinal responses.

[$\boxed{\text{ordinal likelihood model}}$ → Defining likelihood Funⁿ → $\boxed{\text{Taking Natural log}}$ → $\boxed{\text{Calculate First Derivative}}$ → $\boxed{\text{Set Derivative to zero.}}$

θ: vector parameter trying to estimate

x: explanatory variables

y: ordinal response data, probability of observing these responses based on the model

k: number of ordinal categories or levels in response variables.

n_j: counts representing the number of observations falling into each ordinal category, n_2 in the second category

$\therefore$ $L(\theta / y, x)$: represents the likelihood function [given ordinal data (y) given a set of parameters (θ) and covariates (x). Product of probabilities for each category.

$\therefore L(\theta / y, x) = \prod_{j=1}^{k} \left[y_j^{n_j} \times (1-y_j)^{(n_{j+1}-n_j)} \right]$

log-likelihood

$\therefore \log L(\theta / y, x) = \sum_{j=1}^{k} [n_j \log(y_j) + (n_{j+1} - n_j)\log(1-y_j)]$

taking derivatives with respect to each parameter θ_j:

$$\frac{\partial}{\partial \theta_j} \log L(\theta / y, x) = \frac{\partial}{\partial \theta_j}\left[\sum_{j=1}^{k} (n_j \log(y_j) + (n_{j+1} - n_j)\log(1-y_j))\right]$$

setting log-likelihood function, derivatives equal to zero.

$$\frac{\partial}{\partial \theta_j} \log L(\theta / y, x) = 0$$

$\beta^T x_i \rightarrow$ location for i^{th} row,

$\tau_i \rightarrow$ scale.

$t \rightarrow$ no. of rows.

$\sum \log \tau_i = 0.$

$\therefore \text{link}(\gamma_{ij}) = (\theta_j - \beta^T x_i)/\tau_i$

$\therefore \log \tau_i = \tau^T (x_i - \bar{x})$

$\hookrightarrow \tau$ parameter to be estimated

$[\pi_1^{\,n1}, \pi_{k_1}^{\,n}]$ with the probabilities π_j satisfying,

dealing with cumulative probabilities we define,

$$R_1 = n_1, \quad Z_1 = R_1/n$$

$$R_2 = n_1 + n_2 \quad Z_2 = R_2/n$$

$$\vdots$$

$$R_k = \sum n_j = n; \quad Z_k = R_k/n = 1.$$

→ parameters of the cumulative distribution the likelihood can be written as the product of k-1 quantities.

$$\left\{ \left(\frac{\gamma_1}{\gamma_2}\right)^{R_1} \left(\frac{\gamma_2 - \gamma_1}{\gamma_2}\right)^{R_2 - R_1} \right\} \times$$

$$\left\{ \left(\frac{\gamma_2}{\gamma_3}\right)^{R_2} \left(\frac{\gamma_3 - \gamma_2}{\gamma_3}\right)^{R_3 - R_2} \right\} \cdots \times \left\{ \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^{R_{k-1}} \left(\frac{\gamma_k - \gamma_{k-1}}{\gamma_k}\right)^{R_k - R_{k-1}} \right\}.$$

$$\therefore \phi_j = \log\{\gamma_j / (\gamma_{j+1} - \gamma_j)\} = \text{logit}(\gamma_j / \gamma_{j+1})$$

$$l = n \left[\{Z_1 \phi_1 - Z_2 g(\phi_1)\} + \{Z_2 \phi_2 - Z_3 g(\phi_2)\} + \cdots + \{Z_{k-1}\phi_{k-1} - g(\phi_{k-1})\} \right].$$