Transition to Theoretical Mathematics

Proof Portfolio

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Contents

1 Preface

The following is a compilation that summarize my progress in my Transition to Theoretical Mathematics course (Amherst MATH221).

2 Reflection

When I was making the decision to enroll in this class I was thinking about discrete math as a tool for other courses, mainly computer science and statistics. I wanted to understand some of the more complicated theory in these fields, and I had a vague sense of what type of mathematical background I had to aim for. Now, I realize that I stopped looking at math through the lens of a first year student coming to Multivariable Calculus and rushing into Linear Algebra the second semester. Instead, I now regard math as a natural extension to my way of thinking. Without meaning to sound too dramatic, math – and specifically proof writing – has given me an application to some of the formal argumentation that I had learned in my philosophy logic class.

In the beginning of the semester I regarded proofs as building blocks that were created and then reused to form more complex structures and arguments. This was not a bad idea. However, this changed somewhat throughout the semester. Now, I regard proofs as tangible examples of the growth of humanity, an account of our progress. Math is in every other science. To realize that every proof that we studied and reproduced was a careful addition to Man's body of knowledge was humbling. We could go over a proof in class that had taken years to be originally formulated. We studied proofs and learned how to write them, but it somehow did not feel like creating anymore. I have come to the view that the proofs were already out there, they always have, but only after we take the next leaps do we consider ourselves able to absorb the result into our huge body of knowledge and call it ours, and possibly name it after someone of of us.

For all purposes, proofs are contracts with ourselves. A proof leads to trust in the claim. Some of the problems people struggled centuries ago are now solved and in turn, newer and more complicated problems are discovered. Students like me can catch up to centuries of knowledge by looking over these proofs, testaments of human achievement.

Taking a less dramatic tone now, I think that proofs have given me the ability to understand that there are always many logical steps that follows from a situation, but only the careful selection of these steps will logically lead to the claim where I want to go. Not only in mathematics, but in writing, speech, programming, and others. Problems can be solved with a finite amounts of techniques because of the patterns they exhibit, and simple rules can tackle complexity. This is incredibly important for me because I like tackling complexity in the computer science field. My ability to take a problem and abstract it away matured as I was taking this course, and frequently, there would be "Ahhh" moments where I would understand what my professors in Linear Algebra or Multivariable where trying to say but I was not able to understand then.

Math is not my field. That does not mean that I cannot enjoy its benefits, nor that I have to stay away from it. Learning about proof techniques, mathematical results, and proof writing open doors to other mathematics that I will now be able to read and comprehend.

3 Proof By Cases (Triangle Inequality)

Theorem 1 (`The Triangle Inequality`)**.** *For each pair of real numbers x and y, one has that:*

$$|x + y| \leq |x| + |y|.$$

3.1 Why this proof?

The reason this proof is included is that the Triangle Inequality is the clearest example of the proof by cases technique. The way that the proof generally follows from the definition of the absolute value makes cases clear and it is easy to see they are exhaustive. Additionally, I thought that it was important that the fourth case does not need to be proven because it is the same as the third case but with the variables switched.

The first time I did the proof I was not able to write it down, and when I had it in the homework I did not end up feeling comfortable about it. Thus, revisiting this proof and understanding what is going on, as well as revisiting the proof technique used here, was important as the course closed off.

Proof. We proceed by cases.

Case 1: $x \geq 0$ and $y \geq 0$.

Note that by the definition of absolute value and since $x \geq 0$ and $y \geq 0$, we have that $|x| = x$ and $|y| = y$ and $|x + y| = x + y$ (since $x + y \geq 0$).

Thus it follows that $|x + y| \leq |x| + |y|$ (since $|x + y| = |x| + |y|$).

Case 2: $x < 0$ and $y < 0$.

Note that by the definition of absolute value and since $x < 0$ and $y < 0$, we have that $|x| = -x$ and $|y| = -y$ and $|x + y| = -(x + y) = -x - y$ (since $x + y < 0$).

Thus it follows that $|x + y| \leq |x| + |y|$ (since $|x + y| = |x| + |y|$).

Case 3: $x \geq 0$ and $y \geq 0$.

Note that by the definition of absolute value and since $x \geq 0$ and $y < 0$, we have that $|x| = x$ and $|y| = -y$. Now we need to consider two subcases in which $|x + y|$ can satisfy $|x + y| \geq 0$ or $|x + y| < 0$.

Suppose that $|x + y| \geq 0$. By the definition of absolute value it follows that $|x + y| = x + y$. Note that $x - y \geq x + y$ because y is negative. By substituting we get that $|x + y| \leq |x| + |y|$.

Now suppose that $|x + y| < 0$. Observe that $|x + y| \leq x$. Since y is negative we have that $|x + y| < x - y$. After substituting equivalent values we have that $|x + y| \leq |x| + |y|$.

Since in Case 3 we have that x and y are arbitrary, we have exhausted the cases and we have that $|x + y| \leq |x| + |y|$ for all $\mathbb{R}$.

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3.2 Reflection

Looking back at the original version of this proof I can see that I have clearly improved the proof's readibility. These changes in the narrative of the proof were included because I now know that it is important for a proof to be transparent to a new reader as well as correct.

I changed the cases a bit in order to make the proof technique stand out more and flow better one into the other.

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4 Set Proof

Let $f : A \to B$ be a function. Prove the following statements.

Claim 1. Let $C, D \subseteq B$. Then $f^{-1}(C) \cup f^{-1}(D) = f^{-1}(C \cup D)$.

4.1 Why this proof?

Set proofs are one of the first types of proofs that we encountered in this course. Starting out with simple set proofs, we saw them again in the other topics like functions, countable sets, supremums, and so on. The reason I included the following proof is that it has a lot of moving parts that are straightforward, but easy to lose along the narrative. I think that the point of including this proof is showing that the flow of my arguments has improven over time. At every step, explaining why we are performing it, is very important in terms of readibility.

Proof. We will show that Claim 1 is true by showing $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}(C \cup D)$ and $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

We first show that $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}(C \cup D)$. By definition of a subset, given an arbitrary element x, the statement we are trying to show is equivalent to the statement that if $x \in f^{-1}(C) \cup f^{-1}(D)$ then $x \in f^{-1}(C \cup D)$.

Let us assume that $x \in f^{-1}(C) \cup f^{-1}(D)$. We assume that it is the case that $x \in f^{-1}(C)$ then by definition of the preimage we know that $f(x) \in C$. Logically, it would also be true that $f(x) \in C \cup f(x) \in D$. Then observe that since $f(x) \in (C \cup D)$, again, by the definition of preimage it follows that $x \in f^{-1}(C \cup D)$. The argument for the case in which $x \in f^{-1}(D)$ follows the same argument. Therefore, we have that $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}(C \cup D)$ as desired.

Now, we proceed to show that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$. Given an element x, assume that $x \in f^{-1}(C \cup D)$. We will show that $x \in f^{-1}(C) \cup f^{-1}(D)$.

By the definition of preimage we have that $f(x) \in (C \cup D)$. However, if $f(x)$ is in the union of C and D then logically it follows that $f(x) \in C \cup f(x) \in D$. By the definition of preimage it holds true that $x \in f^{-1}(C) \cup f^{-1}(D)$. Therefore it is true that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

Since we have shown that $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}(C \cup D)$ is true and that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$ is also true, then by definition we have that $f^{-1}(C) \cup f^{-1}(D) = f^{-1}(C \cup D)$ as desired.

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4.2 Reflection

Looking back, most of the changes to this version of the proof revolve around explaining at each step the reason we are taking that step. I think that this makes the proof friendlier and more organized.

5   Direct Proof

Claim 1. $\displaystyle\lim_{n\to\infty} \frac{3n}{n+2} = 3$

5.1   Why this proof?

Epsilon-N proofs are a very clear example of what direct proof tries to achieve. To prove the statement one must work backwards in order to understand what this forward and direct path is. It is very clear what the technique is trying to achieve, and in just a couple of steps we can prove a big statement about how a series behaves when n approaches infinity. The proof technique here adheres neatly to the definition, and it seems to be the easiest and clearest way to prove that a sequence converges to a given value.

Proof. To show Claim 1 we will first fix $\varepsilon \in (0, \infty)$ arbitrary. Then, we define $N := \lceil 6/\varepsilon \rceil + 1 \in \mathbb{N}$. Now we fix $n \in \mathbb{N}$, arbitrary and assume that $n \geq \lceil 6/\varepsilon \rceil + 1$.

Observe that from the previous statement it follows that $n \geq 6/\varepsilon + 1$ and $n > 6/\varepsilon$. We can also write that $\varepsilon > 6/n$. Observe that $\varepsilon > \dfrac{6}{n+2}$. We can rewrite the equivalent statement $\left| \dfrac{3n}{n+2} - \dfrac{3n+6}{n+2} \right| < \varepsilon$ and from this we have that $\left| \dfrac{3n}{n+2} - 3 \right| < \varepsilon$.

Since we had fixed n and ε arbitrary, then by the definition of a convergent sequence it follows that $\displaystyle\lim_{n\to\infty} \frac{3n}{n+2} = 3$ as desired.

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5.2   Reflection

The most important change in the proof is that the original one had a big mistake when choosing N. In the original version N could become a negative number due to an extra term that was being subtracted from it. After doing the same mistake in the second midterm, I understood the problem and learned how to avoid it. Other changes included a little bit more transition words that makes the argument flow more like a paragraph instead of a series of disjoint mathematical facts.

6   Contradiction Proof

Let $\prod_{n=1}^{\infty} \mathbb{N} := \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \times \cdots := \big\{ (a_1, a_2, \ldots, a_n, \ldots) : a_n \in \mathbb{N}, \ \forall\, n \in \mathbb{N} \big\}.$

Claim 1. $\prod_{k=1}^{\infty} \mathbb{N}$ is uncountable.

6.1   Why this proof?

I chose this proof because I felt very proud of the work I put into proving the statement. I remember that in class I asked whether this statement was true, and the next thing I knew, I was doing it in the homework assignment! I feel that this proof has a lot of moving parts but it is very easy to see that the general intent is to find a contradiction. The proof by contradiction technique gives us material to work with, and even the shortness of the claim seems to hint at the importance of the usage of this technique.

Proof. We proceed to show the original statement is true by contradiction.

Assume to the contrary that $\prod_{k=1}^{\infty} \mathbb{N}$ is countable. By definition it follows that there exists a bijection $\varphi : \mathbb{N} \to \prod_{k=1}^{\infty} \mathbb{N}$. Seeking a contradiction, we will show how φ cannot be surjective.

Since φ is bijective, we know that $\varphi(m) \in \prod_{k=1}^{\infty} \mathbb{N}$ for every $m \in \mathbb{N}$. For every $m \in \mathbb{N}$ we can write $\varphi(m) = (a_1^{(m)}, a_2^{(m)}, a_3^{(m)} \ldots, a_n^{(m)}, \ldots)$, where $n \in \mathbb{N}$.

Let's define $x \in \prod_{k=1}^{\infty} \mathbb{N}$ by setting $a_n^l := \begin{cases} 3 \text{ if } a_n^{(n)} \neq 3 \\ 4 \text{ if } a_n^{(n)} = 3 \end{cases}$ for every $n \in \mathbb{N}$.

Thus, by construction we have that $x \neq f(m)$ for every $m \in \mathbb{N}$ since $a_n^{(m)} \neq a_n^{(l)}$.

At this point we have that $x \in \prod_{n=1}^{\infty} \mathbb{N}$ but also for every $m \in \mathbb{N}$ we have that $x \notin f(m)$. This means that by definition, φ cannot be surjective, and since φ had been said to be bijective, it is a contradiction.

Hence, we have that shown that the original statement that $\prod_{k=1}^{\infty} \mathbb{N}$ is uncountable is true.

$\blacksquare$

6.2   Reflection

Again, the main changes that I included in this proof are geared towards improving its readibility. I think that the notation of subscripts was confusing the first time I saw it and I tried to clarify which parts were which in the final version by using different letter names. Overall, the proof is clearer and more friendly.

7 Contraposition Proof

Claim 1. Let $n, m \in \mathbb{Z}$. If $n^2 * (m + 3)$ is even then n is even or m is odd.

7.1 Why this proof?

I had trouble picking out what proof by contraposition I wanted to include in this portfolio. The trouble was mainly because this was the least common method that I used. The only place I used contraposition by the end of the semester was within supremum proofs, but the proof technique did not shine through because it was in just a subproof after all (proving that the supremum is the least upper bound). In summary, this proof, while simple, is a great example of the power of contraposition because of it's consequent. I think that it is clear that by taking the path of proof by contrapositive we are working with more manageable pieces of the puzzle and the solution lends itself to the way the contrapositive is laid out.

Proof. We will assume that it is not the case that n is even or m is odd and show that $n^2 * (m + 3)$ must be odd. Our assumption that it is not the case that n is even or m is odd is equivalent to the statement that n is odd and m is even. By definition, this means that we can write $n = 2k + 1$ for some $k \in \mathbb{Z}$ and $m = 2l$ for some $l \in \mathbb{Z}$.

Now, observe that we have the following:
$$n^2 * (m + 3)$$
$$(2k + 1)^2 * (2l + 3) =$$
$$(4k^2 + 4k + 1)(2l + 3) =$$
$$8k^2l + 12k^2 + 8kl + 12k + 2l + 3 =$$
$$2(4kl + 6k^2 + 4kl + 6k + l + 1) + 1.$$

Since $4kl + 6k^2 + 4kl + 6k + l + 1 \in \mathbb{Z}$ then by definition we have that $n^2 * (m + 3)$ is odd as desired.

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7.2 Reflection

I did not change this proof too much from the original. The changes that I included were in the beginning, were I was making it even clearer that this is a proof by contraposition (without stating it explicitly because that somehow felt less elegant). I also didn't skip the step where I applied DeMorgan's Laws to translate the negation of the OR statement to an AND statement.

8 Induction Proof

8.1 Why this proof?

These are actually several subproblems that were included in one problem of the homework assignment. I remember that I had trouble understanding how to manipulate the terms in order to show the inductive step was true while using the inductive hypothesis. Induction seems to be the most straightforward method to solve these problems, and these proofs are evidence (haha) that I have learned how to reason through induction.

Proof the following claims.

Claim 1. Show that $2^n \leq 3^n$ for each $n \in \mathbb{N}$.

Proof. We proceed to prove the statement by induction.

Let us consider the base case when $n = 1$. Observe that we have that $2^1 \leq 3^1$ which is true.

Now we have to show that for every $n \in \mathbb{N}$ it follows that if $2^n \leq 3^n$ then $2^{n+1} \leq 3^{n+1}$.

We now fix n arbitrary and assume that $2^n \leq 3^n$.

It follows from the inductive hypothesis that $2 * 2^n \leq 2 * 3^n$. Observe that since $2 < 3$ we have that $2 * 2^n \leq 2 * 3^n \leq 3 * 3^n$.

Thus it follows that $2^{n+1} \leq 3^{n+1}$. Since n was fixed arbitrary, the same holds for every $n \in \mathbb{N}$.

By the Principle of Mathematical Induction we conclude Claim 1 as desired.

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Claim 2.

Use Claim 1 to prove that $2^n + 1 \leq 3^n$ for every $n \in \mathbb{N}$.

Proof. We will show that Claim 2 is true by induction.

Consider our base case when $n = 1$. Note that $2^1 + 1 \leq 3^1$ is true.

In our inductive step we now show that for every $n \in \mathbb{N}$ we have that if $2^n + 1 \leq 3^n$ then $2^{n+1} + 1 \leq 3^{n+1}$.

Fix n arbitrary. We assume $2^n + 1 \leq 3^n$. From the inductive hypothesis it follows that $2 * (2^n + 1) \leq 3 * (3^n)$ because $2 < 3$. Then, it is true that $2^{n+1} + 2 \leq 3^{n+1}$. By rearranging the terms we get that $2 \leq 3^{n+1} - 2^{n+1}$. At this point observe that $1 < 2$, which means that $1 \leq 3^{n+1} - 2^{n+1}$. After rearranging the terms again we now get that $2^{n+1} + 1 \leq 3^{n+1}$. Recall that n was fixed arbitrary, which means that we have shown the inductive step is true as desired.

Thus, we have shown Claim 2 is true by the Principle of Mathematical Induction.

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Claim 3.

Is it true that $2^n + n \leq 3^n$ for every $n \in \mathbb{N}$?

Proof. Proceeding by induction, we consider the base case when $n = 1$. We know that $2^1 + 1 \leq 3^1 = 3 \leq 3$

Focusing now on the inductive step, we have to show that for every $n \in \mathbb{N}$ it follows that if $2^n + n \leq 3^n$ then it is also true that $2^{n+1} + (n + 1) \leq 3^{n+1}$.

After fixing n arbitrary, assume that $2^n + n \leq 3^n$. In a similar vein as in the previous proof, we use the fact that $2 < 3$ and our inductive hypothesis to get that $2 * (2^n + n) \leq 3 * (3^n)$. After simplifying and rearranging the terms we have that $2n \leq 3^{n+1} - 2^{n+1}$. Seeking to show the inductive step we use the fact that $1 \leq n$ for every $n \in \mathbb{N}$ to deduce that for every $n \in \mathbb{N}$ we have that $n + 1 \leq 2n$. Using this fact, we can now rewrite our argument as $n + 1 \leq 3^{n+1} - 2^{n+1}$. Finally, after rearranging the terms we can see that $2^{n+1} + (n+1) \leq 3^{n+1})$. Since n was fixed arbitrary, we know that the same statement holds true for every $n \in \mathbb{N}$ as desired.

By the Principle of Mathematical Induction it follows that $2^n + n \leq 3^n$ for every $n \in \mathbb{N}$.

$\blacksquare$

8.2   Reflection

I don't think the original version of these proofs had mistakes, but I think that they were just drafts. In choosing these proofs I wanted to polish those proof drafts (with $\implies$ and $\forall$) and make them neat and readable.

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9 Proof Evaluation 30

Claim 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and suppose that $A \subseteq \mathbb{R}$ is a nonempty set which is bounded from above. Then $f(A)$ is bounded from above.

Proof. Let $u \in \mathbb{R}$ be an upper bound for A. Then $a \leq u$ for all $a \in A$. This implies $f(a) \leq f(u)$ for all $a \in A$. Hence $f(u) \in \mathbb{R}$ is an upper bound for $f(A)$.

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Reflection What's wrong with the proof?. The proof above is not correct. To show this consider the function $f := \{(x, -(x^3)) \in R \times R : x \in A\}$. Observe how $-2 < 2$ but $f(-2) > f(2)$. Thus in the proof above the statement that $a \leq u$ for all $a \in A$ does not imply that $f(a) \leq f(u)$ for all $a \in A$.

Here is an ammended claim and its proof.

Claim 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function and suppose that $A \subseteq \mathbb{R}$ is a nonempty set which is bounded from above. Then $f(A)$ is bounded from above.

Proof. Let $u \in \mathbb{R}$ be an upper bound for A. Then $a \leq u$ for all $a \in A$. This implies $f(a) \leq f(u)$ for all $a \in A$ by the definition of a nondecreasing function. Hence $f(u) \in \mathbb{R}$ is an upper bound for $f(A)$.

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10 Proof Evaluation 18

Claim 1. Given $a \in \mathbb{R} \setminus \{0\}$, one has that $a^n = 1$ for all $n \in \mathbb{N}_0$

Proof. If $n = 0$ then $a^0 = 1$. Thus the claim is true for the base case. Next, fix $n \in \mathbb{N} \cup \{0\}$ and assume that the claim is true for all $k \in \mathbb{N}_0$ with $k \leq n$, i.e., assume that $a^k = 1$ for all $k \in \mathbb{N}_0$ with $k \leq n$. We want to show that the claim is true for $n + 1$. Observe that

$$a^{n+1} = \frac{a^n * a^n}{a^{n-1}} = \frac{1 * 1}{1} = 1$$

,

where we have used the induction hypothesis in the second equality. Thus the claim is true for $n + 1$ and by PMI we can now conclude that the claim is true for all $n \in \mathbb{N}_0$.

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Reflection What's wrong with the proof?. The error in this proof lies in the inductive step where we are indirectly assuming that $k \geq 1$ in order for us to have a^{n-1} in the denominator. When $k < 1$ we cannot use the inductive hypothesis because the exponent would not be a nonnegative number. We would have to check the base case at $n = 1$ in order to be able to assume that $k \geq 1$, but at $n = 1$ the statement is not true anymore. Thus the statement is a false one.

11 Proof Evaluation 29

Claim 1. Let $A \subseteq \mathbb{R}$ be a nonempty set and suppose that $\alpha := sup(A) \in \mathbb{R}$ exists. If $\varepsilon \in (0, \infty)$ then there exists $a \in A$ such that $\alpha - \varepsilon < a$.

Proof. Let $a := \alpha - \dfrac{\varepsilon}{2}$. Then $a < \alpha$ which implies $a \in A$. By construction $\alpha - \varepsilon < a$.

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Reflection What's wrong with the proof?. We think that the claim above is correct, but the proof is inadequate. The statement that if $a < \alpha$ then $a \in A$ needs to be proven before being used. It is not even clear to the reader what type of proof method they are engaging in.

Below is a correct proof for this statement.

Proof. We will show that the equivalent statement that if $\alpha - \varepsilon < a$ for every $a \in A$ then it follows that $\varepsilon \notin (0, \infty)$, is true.

We assume that $\alpha - \varepsilon < a$ for every $a \in A$ and try to deduce $\varepsilon \notin (0, \infty)$. By the definition of the supremum of a set we know that for every $a \in A$ it is true that $\alpha \geq a$.

Seeking a contradiction, we will assume that $\varepsilon \in (0, \infty)$. Observe that since $\varepsilon > 0$ then $\alpha \geq \alpha - \varepsilon \geq a$ holds for every $a \in A$.

By the definition of an upper bound, we have that $\alpha - \varepsilon$ is an upperbound for A. However, this contradicts the fact that $\alpha = sup(A)$ because $\alpha \geq \alpha - \varepsilon$. It follows that our assumption was false and we have that $\varepsilon \notin (0, \infty)$ as desired.

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