(b)

i) Show: $\forall n \in \mathbb{N}, 2^n \leq 3^n$.

We proceed by induction. Note that our base case of $n=1$ holds because $2 \leq 3$. We then proceed to show that for every $n \in \mathbb{N}$, if $2^n \leq 3^n$ then $2^{n+1} \leq 3^{n+1}$.

We now fix n, arbitrary and assume $2^n \leq 3^n$. However, note that since $2^n \leq 3^n$, we have:

$$2 \cdot 2^n \leq 2 \cdot 3^n$$

and since $2 < 3$,

$$2 \cdot 2^n \leq 2 \cdot 3^n \leq 3 \cdot 3^n,$$

which means that:

$$2^{n+1} \leq 3^{n+1}.$$

By PMI, we conclude that the statement is true.

$\boxtimes$

ii) Use $\forall n \in \mathbb{N}$, $2^n \leq 3^n$ to show: $\forall n \in \mathbb{N}$, $2^n + 1 \leq 3^n$.

We proceed by induction. Let our base case be $2^1 + 1 \leq 3^1$ ✓.

Now our induction step is $\forall n \in \mathbb{N}$, $2^n + 1 \leq 3^n \Rightarrow 2^{n+1} + 1 \leq 3^{n+1}$.

Fix n.

observe that $2^n + 1 \leq 3^n$ and

$$2(2^n + 1) \leq 3(3^n)$$

$$2^{n+1} + 2 \leq 3^{n+1}$$

$$2 \leq 3^{n+1} - 2^{n+1}$$

However, since $1 \leq 2$, $1 \leq 3^{n+1} - 2^{n+1}$,

which means $2^{n+1} + 1 \leq 3^{n+1}$.

By PMI the statement is true.

iii) Is it true that $2^n + n \leq 3^n$ for every $n \in \mathbb{N}$.

We proceed by induction.

proof: base case $2^1 + 1 \leq 3^1$ $3 \leq 3$ ✓.

our induction step $\forall n \in \mathbb{N}$, $2^n + n \leq 3^n \Rightarrow 2^{n+1} + (n+1) \leq 3^{n+1}$.

Fix n.

observe: $2(2^n + n) \leq 3^n \cdot 3$ since $2 \leq 3$.