① (a) Prove $|x+y| \leq |x|+|y|$ for $\forall x,y \in \mathbb{R}$.

we proceed by cases:

First consider when $x, y \geq 0$.

we have that $|x+y| = |x|+|y|$.

Then consider when $x, y \leq 0$.

we have: $|x+y| = -(x+y) = -x-y = |x|+|y|$.

Now we examine the cases where x is ≥ 0 and $y < 0$, or the other way around.

without a loss of generality we assume $|x| \leq |y|$.

If $y > 0$ then $|x+y| = -|x|+|y| = |y|-|x|$. This is less than $|y|$ and also less than $|y|+|x|$ so $|x+y| \leq |x|+|y|$.

If $y < 0$ then again $= |y|-|x|$. By the same argument $|x+y| \leq |x|+|y|$.

Thus $\forall x,y \in \mathbb{R}$, $|x+y| \leq |x|+|y|$.

QED