Transition to Theoretical Mathematics

Proof Portfolio

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18 December, 2018

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1 Preface

The following are a set of proofs that summarize my progress in my Transition to Theoretical Mathematics (Amherst MATH221).

2 Proof By Cases (Triangle Inequality)

Theorem 1 (`The Triangle Inequality`). *For each pair of real numbers x and y, one has that:*

$$|x + y| \leq |x| + |y|.$$

(Hint: Consider cases like $x, y \geq 0$ or $x, y \leq 0$ or $x \geq 0$ and $y < 0$, etc.. These cases will help you evaluate the absolute value using its definition.)

Proof. We proceed by cases.

Case 1: $x \geq 0$ and $y \geq 0$.

Note that by the definition of absolute value and since $x \geq 0$ and $y \geq 0$, we have that $|x| = x$ and $|y| = y$ and $|x + y| = x + y$ (since $x + y \geq 0$).

Thus it follows that $|x + y| \leq |x| + |y|$ (since $|x + y| = |x| + |y|$).

Case 2: $x < 0$ and $y < 0$.

Note that by the definition of absolute value and since $x < 0$ and $y < 0$, we have that $|x| = -x$ and $|y| = -y$ and $|x + y| = -(x + y) = -x - y$ (since $x + y < 0$).

Thus it follows that $|x + y| \leq |x| + |y|$ (since $|x + y| = |x| + |y|$).

Case 3: $x \geq 0$ and $y \geq 0$.

Note that by the definition of absolute value and since $x \geq 0$ and $y < 0$, we have that $|x| = x$ and $|y| = -y$. Now we need to consider two subcases in which $|x + y|$ can satisfy $|x + y| \geq 0$ or $|x + y| < 0$.

Suppose that $|x + y| \geq 0$. By the definition of absolute value it follows that $|x + y| = x + y$. Note that $x - y \geq x + y$ because y is negative. By substituting we get that $|x + y| \leq |x| + |y|$.

Now suppose that $|x + y| < 0$. Observe that $|x + y| \leq x$. Since y is negative we have that $|x + y| < x - y$. After substituting equivalent values we have that $|x + y| \leq |x| + |y|$.

Since in Case 3 we have that x and y are arbitrary, we have exhausted the cases and we have that $|x + y| \leq |x| + |y|$ for all $\mathbb{R}$.

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3 Set Proof

Let $f : A \to B$ be a function. Prove the following statements.

Claim 1. Let $C, D \subseteq B$. Then $f^{-1}(C) \cup f^{-1}(D) = f^{-1}\big(C \cup D\big)$.

Proof. We will show that Claim 1 is true by showing $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}\big(C \cup D\big)$ and $f^{-1}\big(C \cup D\big) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

We first show that $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}\big(C \cup D\big)$. By definition of a subset, given an arbitrary element x, the statement we are trying to show is equivalent to the statement that if $x \in f^{-1}(C) \cup f^{-1}(D)$ then $x \in f^{-1}\big(C \cup D\big)$.

Let us assume that $x \in f^{-1}(C) \cup f^{-1}(D)$. We assume that it is the case that $x \in f^{-1}(C)$ then by definition of the preimage we know that $f(x) \in C$. Logically, it would also be true that $f(x) \in C \cup f(x) \in D$. Then observe that since $f(x) \in \big(C \cup D\big)$, again, by the definition of preimage it follows that $x \in f^{-1}\big(C \cup D\big)$. The argument for the case in which $x \in f^{-1}(D)$ follows the same argument. Therefore, we have that $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}\big(C \cup D\big)$ as desired.

Now, we proceed to show that $f^{-1}\big(C \cup D\big) \subseteq f^{-1}(C) \cup f^{-1}(D)$. Given an element x, assume that $x \in f^{-1}\big(C \cup D\big)$. We will show that $x \in f^{-1}(C) \cup f^{-1}(D)$.

By the definition of preimage we have that $f(x) \in \big(C \cup D\big)$. However, if $f(x)$ is in the union of C and D then logically it follows that $f(x) \in C \cup f(x) \in D$. By the definition of preimage it holds true that $x \in f^{-1}(C) \cup f^{-1}(D)$. Therefore it is true that $f^{-1}\big(C \cup D\big) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

Since we have shown that $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}\big(C \cup D\big)$ is true and that $f^{-1}\big(C \cup D\big) \subseteq f^{-1}(C) \cup f^{-1}(D)$ is also true, then by definition we have that $f^{-1}(C) \cup f^{-1}(D) = f^{-1}\big(C \cup D\big)$ as desired.

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Claim 2. Let $C, D \subseteq A$. If f is injective then $f(C) \cap f(D) \subseteq f\big(C \cap D\big)$.

Proof.

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4 Direct Proof

Claim 1. $\lim_{n\to\infty} \frac{3n}{n+2} = 3$

Proof. To show Claim1 we will first fix $\varepsilon \in (0, \infty)$ arbitrary. Then, we define $N := \lceil 6/\varepsilon \rceil + 1 \in \mathbb{N}$. Now we fix $n \in \mathbb{N}$, arbitrary and assume that $n \geq \lceil 6/\varepsilon \rceil + 1$.

Observe that from the previous statement it follows that $n \geq 6/\varepsilon + 1$ and $n > 6/\varepsilon$. We can also write that $\varepsilon > 6/n$. Observe that $\varepsilon > \dfrac{6}{n+2}$. We can rewrite the equivalent statement $\left|\dfrac{3n}{n+2} - \dfrac{3n+6}{n+2}\right|$ and from this we have that $\left|\dfrac{3n}{n+2} - 3\right| < \varepsilon$.

Since we had fixed n and ε arbitrary, then by the definition of a convergent sequence it follows that $\lim_{n\to\infty} \dfrac{3n}{n+2} = 3$ as desired.

$\blacksquare$

5 Contradiction Proof

Let $\prod_{n=1}^{\infty} \mathbb{N} := \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \times \cdots := \big\{(a_1, a_2, \ldots, a_n, \ldots) : a_n \in \mathbb{N},\ \forall\, n \in \mathbb{N}\big\}$.

Claim 1. $\prod_{n=1}^{\infty} \mathbb{N}$ is uncountable.

Proof. We proceed to show the original statement is true by contradiction.

Assume to the contrary that $\prod_{n=1}^{\infty} \mathbb{N}$ is countable. By definition it follows that there exists a bijection $\varphi : \mathbb{N} \to \prod_{n=1}^{\infty} \mathbb{N}$. Seeking a contradiction, we will show how φ cannot be surjective.

Since φ is bijective, we know that $f(n) \in \prod_{n=1}^{\infty} \mathbb{N}$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we can write $f(n) = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)} \ldots, a_m^{(n)}, \ldots)$, where $m \in \mathbb{N}$.

Let's define $x \in \prod_{n=1}^{\infty} \mathbb{N}$ by setting $a_m^{(n)} := \begin{cases} 3 \text{ if } a_n^{(n)} \neq 3 \\ 4 \text{ if } a_n^{(n)} = 3 \end{cases} \quad$ for every $k \in \mathbb{N}$.

Thus, by construction we have that $x \neq f(n)$ for every $n \in \mathbb{N}$ since $a_m^{(n)} \neq a_n^{(n)}$.

At this point we have that $x \in \prod_{n=1}^{\infty} \mathbb{N}$ but also for every $n \in \mathbb{N}$ we have that $x \notin f(n)$. This means that by definition, *varphi* cannot be surjective, and since φ had been said to be bijective, is a contradiction.

Hence, we have that shown that the original statement that $\prod_{n=1}^{\infty} \mathbb{N}$ is uncountable is true.

$\blacksquare$

6 Contraposition Proof

Claim 1.

Proof. ∎

7 Induction Proof

Proof the following claims.

Claim 1. Show that $2^n \leq 3^n$ for each $n \in \mathbb{N}$.

Proof. We proceed to prove the statement by induction.

Let us consider the base case when $n = 1$. Observe that we have that $2^1 \leq 3^1$ which is true.

Now we have to show that for every $n \in \mathbb{N}$ it follows that if $2^n \leq 3^n$ then $2^{n+1} \leq 3^{n+1}$.

We now fix n arbitrary and assume that $2^n \leq 3^n$.

It follows from the inductive hypothesis that $2 * 2^n \leq 2 * 3^n$. Observe that since $2 < 3$ we have that $2 * 2^n \leq 2 * 3^n \leq 3 * 3^n$.

Thus it follows that $2^{n+1} \leq 3^{n+1}$. Since n was fixed arbitrary, the same holds for every $n \in \mathbb{N}$.

By the Principle of Mathematical Induction we conclude Claim 1 as desired.

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Claim 2.

Use Claim 1 to prove that $2^n + 1 \leq 3^n$ for every $n \in \mathbb{N}$.

Proof. We will show that Claim 2 is true by induction.

Consider our base case when $n = 1$. Note that $2^1 + 1 \leq 3^1$ is true.

In our inductive step we now show that for every $n \in \mathbb{N}$ we have that if $2^n + 1 \leq 3^n$ then $2^{n+1} + 1 \leq 3^{n+1}$.

Fix n arbitrary. We assume $2^n + 1 \leq 3^n$. From the inductive hypothesis it follows that $2 * (2^n + 1) \leq 3 * (3^n)$ because $2 < 3$. Then, it is true that $2^{n+1} + 2 \leq 3^{n+1}$. By rearranging the terms we get that $2 \leq 3^{n+1} - 2^{n+1}$. At this point observe that $1 < 2$, which means that $1 \leq 3^{n+1} - 2^{n+1}$. After rearranging the terms again we now get that $2^{n+1} + 1 \leq 3^{n+1}$. Recall that n was fixed arbitrary, which means that we have shown the inductive step is true as desired.

Thus, we have shown Claim 2 is true by the Principle of Mathematical Induction.

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Claim 3.

Is it true that $2^n + n \leq 3^n$ for every $n \in \mathbb{N}$?

Proof. Proceeding by induction, we consider the base case when $n = 1$. We know that $2^1 + 1 \leq 3^1 = 3 \leq 3$

Focusing now on the inductive step, we have to show that for every $n \in \mathbb{N}$ it follows that if $2^n + n \leq 3^n$ then it is also true that $2^{n+1} + (n+1) \leq 3^{n+1}$.

After fixing n arbitrary, assume that $2^n + n \leq 3^n$. In a similar vein as in the previous proof, we use the fact that $2 < 3$ and our inductive hypothesis to get that $2 * (2^n + n) \leq 3 * (3^n)$. After simplifying and rearranging the terms we have that $2n \leq 3^{n+1} - 2^{n+1}$. Seeking to show the inductive step we use the fact that $1 \leq n$ for every $n \in \mathbb{N}$ to deduce that for every $n \in \mathbb{N}$ we have that $n + 1 \leq 2n$. Using this fact, we can now rewrite our argument as $n + 1 \leq 3^{n+1} - 2^{n+1}$. Finally, after rearranging the terms we can see that $2^{n+1} + (n+1) \leq 3^{n+1})$. Since n was fixed arbitrary, we know that the same statement holds true for every $n \in \mathbb{N}$ as desired.

By the Principle of Mathematical Induction it follows that $2^n + n \leq 3^n$ for every $n \in \mathbb{N}$.

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8 Proof Evaluation 30

Claim 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and suppose that $A \subseteq \mathbb{R}$ is a nonempty set which is bounded from above. Then $f(A)$ is bounded from above.

Proof. Let $u \in \mathbb{R}$ be an upper bound for A. Then $a \le u$ for all $a \in A$. This implies $f(a) \le f(u)$ for all $a \in A$. Hence $f(u) \in \mathbb{R}$ is an upper bound for $f(A)$.

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Reflection What's wrong with the proof?. The proof above is not correct. To show this consider the function $f := \{(x, -(x^3)) \in R \times R : x \in A\}$. Observe how $-2 < 2$ but $f(-2) > f(2)$. Thus in the proof above the statement that $a \le u$ for all $a \in A$ does not imply that $f(a) \le f(u)$ for all $a \in A$.

Here is an ammended claim and its proof.

Claim 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function and suppose that $A \subseteq \mathbb{R}$ is a nonempty set which is bounded from above. Then $f(A)$ is bounded from above.

Proof. Let $u \in \mathbb{R}$ be an upper bound for A. Then $a \le u$ for all $a \in A$. This implies $f(a) \le f(u)$ for all $a \in A$ by the definition of a nondecreasing function. Hence $f(u) \in \mathbb{R}$ is an upper bound for $f(A)$.

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9 Proof Evaluation 18

Claim 1. Given $a \in \mathbb{R} \setminus \{0\}$, one has that $a^n = 1$ for all $n \in \mathbb{N}_0$

Proof. If $n = 0$ then $a^0 = 1$. Thus the claim is true for the base case. Next, fix $n \in \mathbb{N} \cup \{0\}$ and assume that the claim is true for all $k \in \mathbb{N}_0$ with $k \le n$, i.e., assume that $a^k = 1$ for all $k \in \mathbb{N}_0$ with $k \le n$. We want to show that the claim is true for $n + 1$. Observe that

$$a^{n+1} = \frac{a^n * a^n}{a^{n-1}} = \frac{1 * 1}{1} = 1$$

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where we have used the induction hypothesis in the second equality. Thus the claim is true for $n + 1$ and by PMI we can now conclude that the claim is true for all $n \in \mathbb{N}_0$.

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Reflection What's wrong with the proof?. The error in this proof lies in the inductive step where we are indirectly assuming that $k \ge 1$ in order for us to have a^{n-1} in the denominator. When $k < 1$ we cannot use the inductive hypothesis because the exponent would not be a nonnegative number. We would have to check the base case at $n = 1$ in order to be able to assume that $k \ge 1$, but at $n = 1$ the statement is not true anymore. Thus the statement is a false one.

10 Proof Evaluation 29

Claim 1. Let $A \subseteq \mathbb{R}$ be a nonempty set and suppose that $\alpha := sup(A) \in \mathbb{R}$ exists. If $\varepsilon \in (0, \infty)$ then there exists $a \in A$ such that $\alpha - \varepsilon < a$.

Proof. Let $a := \alpha - \dfrac{\varepsilon}{2}$. Then $a < \alpha$ which implies $a \in A$. By construction $\alpha - \varepsilon < a$.

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Reflection What's wrong with the proof?. We think that the claim above is correct, but the proof is inadequate. The statement that if $a < \alpha$ then $a \in A$ needs to be proven before being used. It is not even clear to the reader what type of proof method they are engaging in.

Below is a correct proof for this statement.

Proof. We will show that the equivalent statement that $\alpha - \varepsilon < a$ for every $a \in A$ then it follows that $\varepsilon \notin (0, \infty)$, is true.

We assume that $\alpha - \varepsilon < a$ for every $a \in A$ and try to deduce $\varepsilon \notin (0, \infty)$. By the definition of the supremum of a set we know that for every $a \in A$ it is true that $\alpha \geq a$.

Seeking a contradiction, we will assume that $\varepsilon \in (0, \infty)$. Observe that since $\varepsilon > 0$ then $\alpha \geq \alpha - \varepsilon \geq a$ holds for every $a \in A$.

By the definition of an upper bound, we have that $\alpha - \varepsilon$ is an upperbound for A. However, this contradicts the fact that $\alpha = sup(A)$ because $\alpha \geq \alpha - \varepsilon$. It follows that our assumption was false and we have that $\varepsilon \notin (0, \infty)$ as desired.

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