

(2)

Let $\prod_{n=1}^{\infty} \mathbb{N} := \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots \times \mathbb{N} := \{(a_1, a_2, \dots, a_n, \dots) : a_n \in \mathbb{N}, \forall n \in \mathbb{N}\}$.

We use Cantor's diagonalization argument to show that the set $\prod_{n=1}^{\infty} \mathbb{N}$ is uncountable.

proof: Assume to the contrary that $f: \mathbb{N} \rightarrow \prod_{n=1}^{\infty} \mathbb{N}$ is a bijection (thus $\prod_{n=1}^{\infty} \mathbb{N}$ is countable). We will show that f is not surjective, ~~that is~~.

We know that $f(n) \in \prod_{n=1}^{\infty} \mathbb{N}$, $\forall n \in \mathbb{N}$, and we can write $f(n) = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, a_4^{(n)}, \dots, a_m^{(n)}, \dots)$ where $n \in \mathbb{N}$ and $\forall m \in \mathbb{N}$.

Now, define $x \in \prod_{n=1}^{\infty} \mathbb{N}$ by setting $a_m := \begin{cases} 3 & \text{if } a_m^{(n)} \neq 3 \\ 4 & \text{if } a_m^{(n)} = 3 \end{cases}$.

So we have that $x \neq f(n)$, $\forall n \in \mathbb{N}$ since $a_m \neq a_m^{(n)}$ by construction.

Thus $x \in \prod_{n=1}^{\infty} \mathbb{N}$ but $\forall n \in \mathbb{N}$, $x \notin f(n)$. Thus, $f: \mathbb{N} \rightarrow \prod_{n=1}^{\infty} \mathbb{N}$ is not surjective, which contradicts our assumptions.

So we have shown that $\prod_{n=1}^{\infty} \mathbb{N}$ is uncountable.