

(2)

Let  $\prod_{n=1}^{\infty} \mathbb{N} := \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots \times \mathbb{N} := \{(a_1, a_2, \dots, a_n, \dots) : a_n \in \mathbb{N}, \forall n \in \mathbb{N}\}$ .

Use Cantor's diagonalization argument to show that the set  $\prod_{n=1}^{\infty} \mathbb{N}$  is uncountable.

~~proof:~~ Assume to the contrary that  $f: \mathbb{N} \rightarrow \prod_{n=1}^{\infty} \mathbb{N}$  is a bijection (thus  $\prod_{n=1}^{\infty} \mathbb{N}$  is countable). We will show that  $f$  is not surjective. ~~so~~

We know that  $f(n) \in \prod_{n=1}^{\infty} \mathbb{N}$ ,  $\forall n \in \mathbb{N}$ , and we can write  $f(n) = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, a_4^{(n)}, \dots, a_m^{(n)}, \dots)$  ~~both~~ where  $n \in \mathbb{N}$  and  $\forall m \in \mathbb{N}$ .

Now, define  $x \in \prod_{n=1}^{\infty} \mathbb{N}$  by setting  $a_m := \begin{cases} 3 & \text{if } a_m^{(n)} \neq 3 \\ 4 & \text{if } a_m^{(n)} = 3 \end{cases}$ .

So we have that  $x \neq f(n)$ ,  $\forall n \in \mathbb{N}$  since  $a_m \neq a_m^{(n)}$  by construction.

Thus  $x \in \prod_{n=1}^{\infty} \mathbb{N}$  but  $\forall n \in \mathbb{N}$ ,  $x \notin f(n)$ . Thus,

$f: \mathbb{N} \rightarrow \prod_{n=1}^{\infty} \mathbb{N}$  is not surjective, which contradicts our assumptions.

So we have shown that  $\prod_{n=1}^{\infty} \mathbb{N}$  is uncountable.