Algorithm Design and Analysis

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算法设计与分析

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This week's content

这周的内容

- Today Wednesday:
  - Chapter 29: Linear Programming
    - 29.1–29.3
  - Exercises
- Tomorrow Thursday:
  - Exercise solutions
  - Chapter 29: Linear Programming
    - 29.4–

- 今天周三：
  - 第29章：线性规划
  - 练习
- 明天周四：
  - 练习题解答
  - 第29章：线性规划

Algorithm Design and Analysis

Linear Programming

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线性规划

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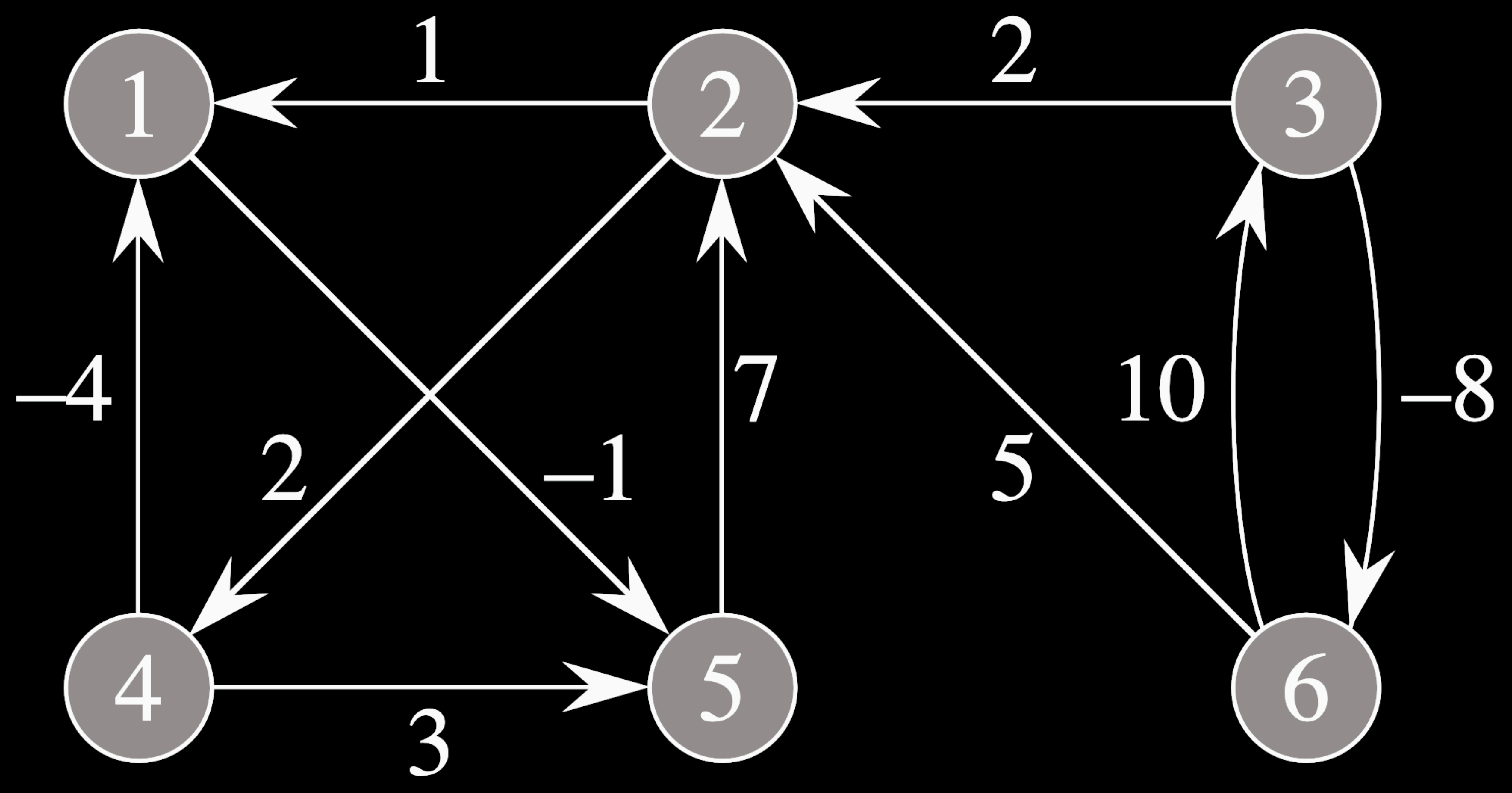
Ch. 29

29章

25.2-1

Run the Floyd–Warshall algorithm on the weighted, directed graph of Figure 25.2. Show the matrix $D^{(k)}$ that results for each iteration of the outer loop.

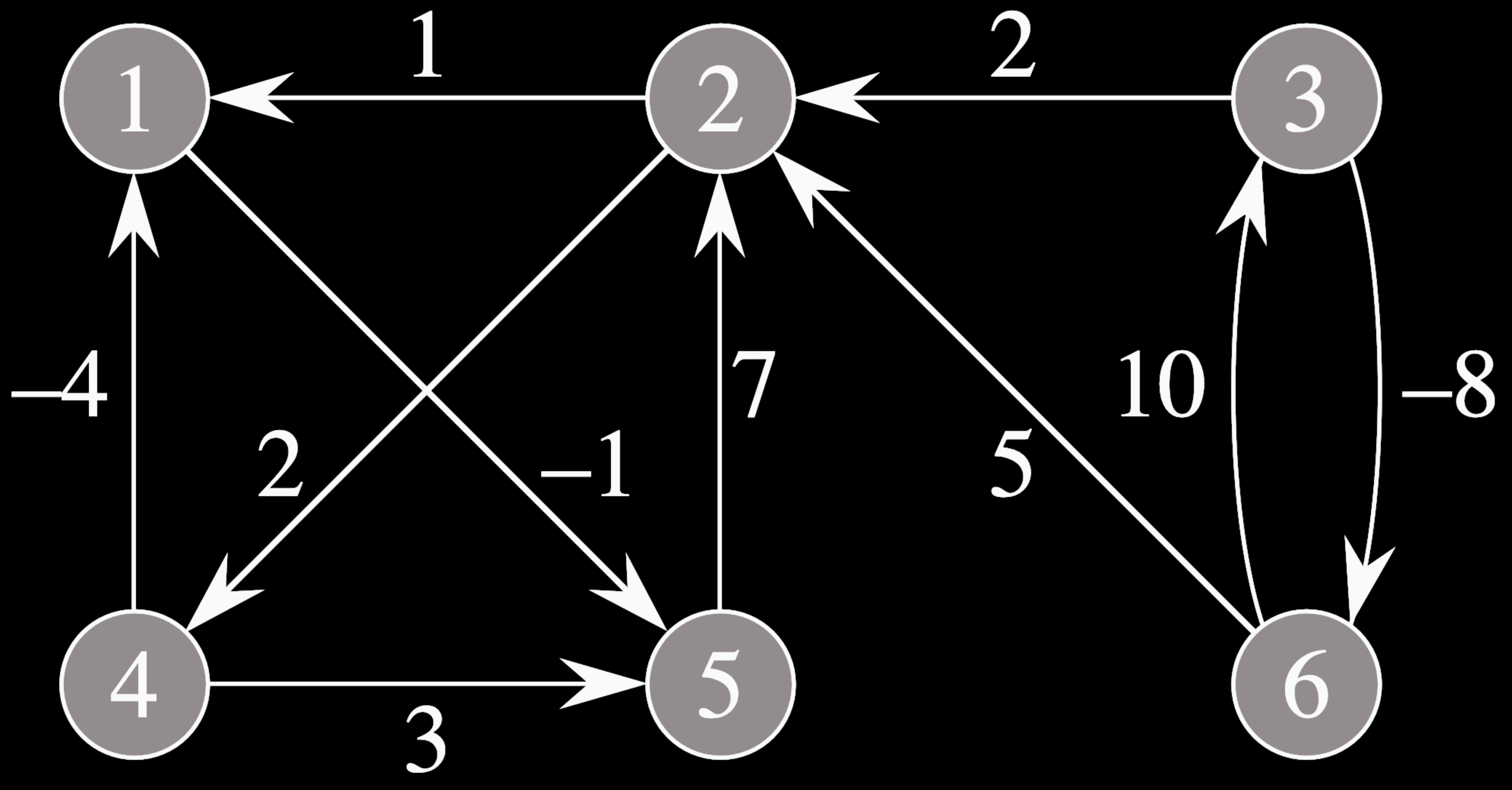
在图25-2所示的带权重的有向图上运行Floyd–Warshall算法，给出外层循环的每一次迭代所生成的矩阵 $D^{(k)}$。



Solution 25.2-1

$$D^{(0)} = W = \begin{pmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ 1 & 0 & \infty & 2 & \infty & \infty \\ \infty & 2 & 0 & \infty & \infty & -8 \\ -4 & \infty & \infty & 0 & 3 & \infty \\ \infty & 7 & \infty & \infty & 0 & \infty \\ \infty & 5 & 10 & \infty & \infty & 0 \end{pmatrix}$$

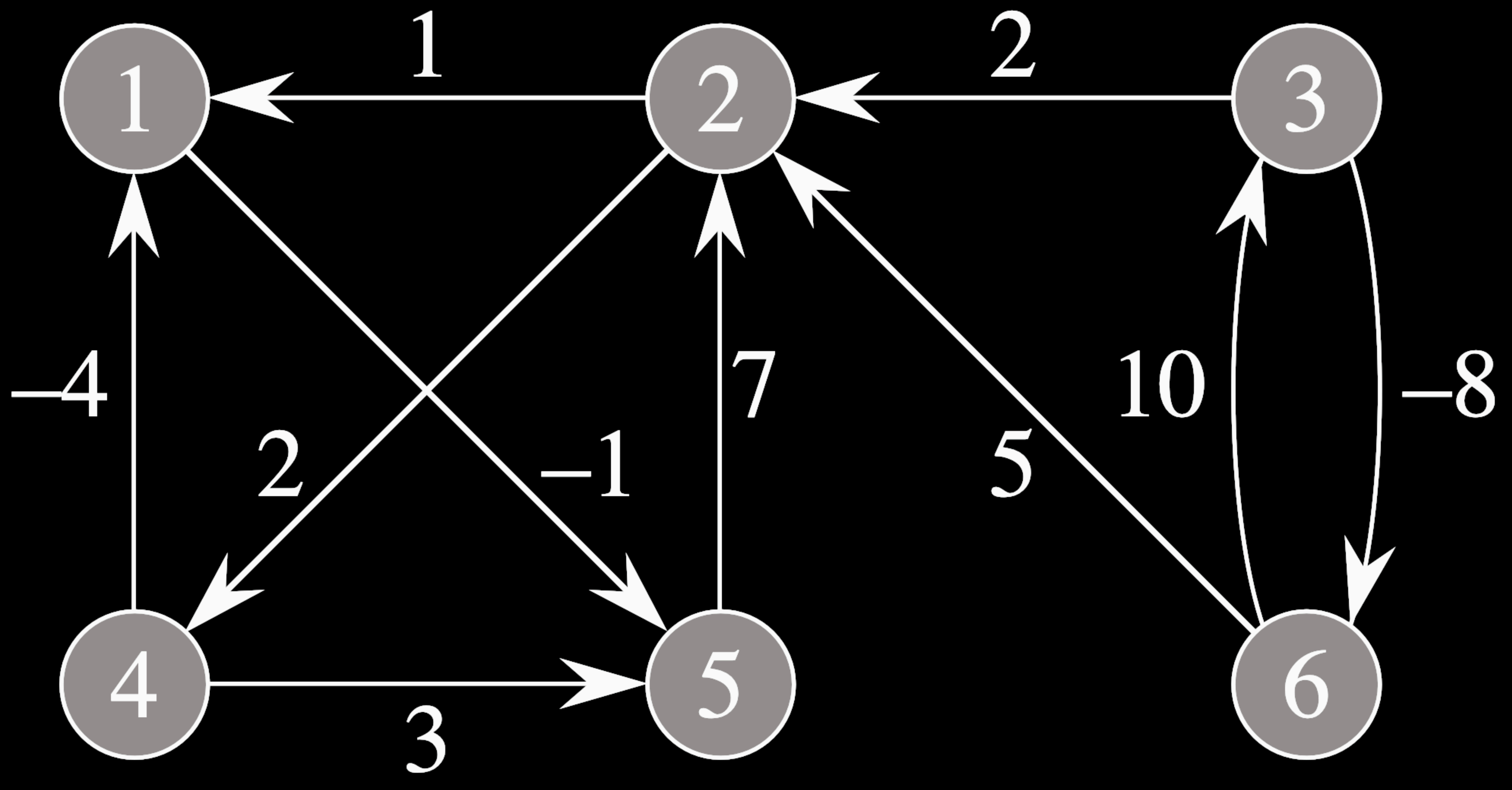
$D^{(1)}_{25} = \min\{D^{(0)}_{25}, D^{(0)}_{21} + D^{(0)}_{15}\}$



$$D^{(1)} = \begin{pmatrix} 0 & \infty & \infty & \infty & \boxed{-1} & \infty \\ \boxed{1} & 0 & \infty & 2 & 0 & \infty \\ \infty & 2 & 0 & \infty & \infty & -8 \\ \boxed{-4} & \infty & \infty & 0 & -5 & \infty \\ \infty & 7 & \infty & \infty & 0 & \infty \\ \infty & 5 & 10 & \infty & \infty & 0 \end{pmatrix}$$

Solution 25.2-1

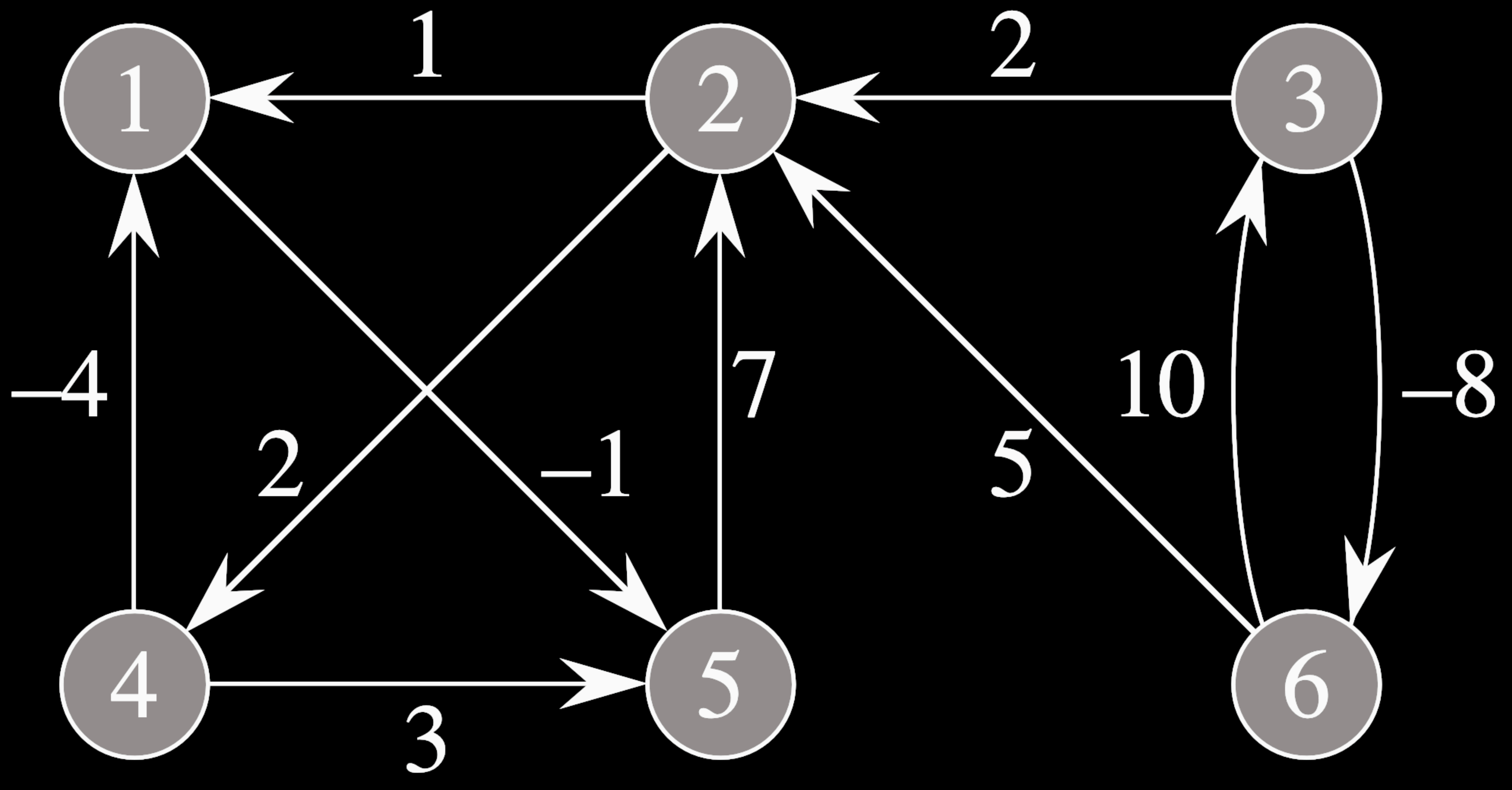
$$D^{(1)} = \begin{pmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ 1 & 0 & \infty & 2 & 0 & \infty \\ \infty & 2 & 0 & \infty & \infty & -8 \\ -4 & \infty & \infty & 0 & -5 & \infty \\ \infty & 7 & \infty & \infty & 0 & \infty \\ \infty & 5 & 10 & \infty & \infty & 0 \end{pmatrix}$$



$$D^{(2)} = \begin{pmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ \boxed{1} & 0 & \infty & \boxed{2} & \boxed{0} & \infty \\ 3 & \boxed{2} & 0 & 4 & 2 & -8 \\ -4 & \infty & \infty & 0 & -5 & \infty \\ 8 & \boxed{7} & \infty & 9 & 0 & \infty \\ 6 & \boxed{5} & 10 & 7 & 5 & 0 \end{pmatrix}$$

Solution 25.2-1

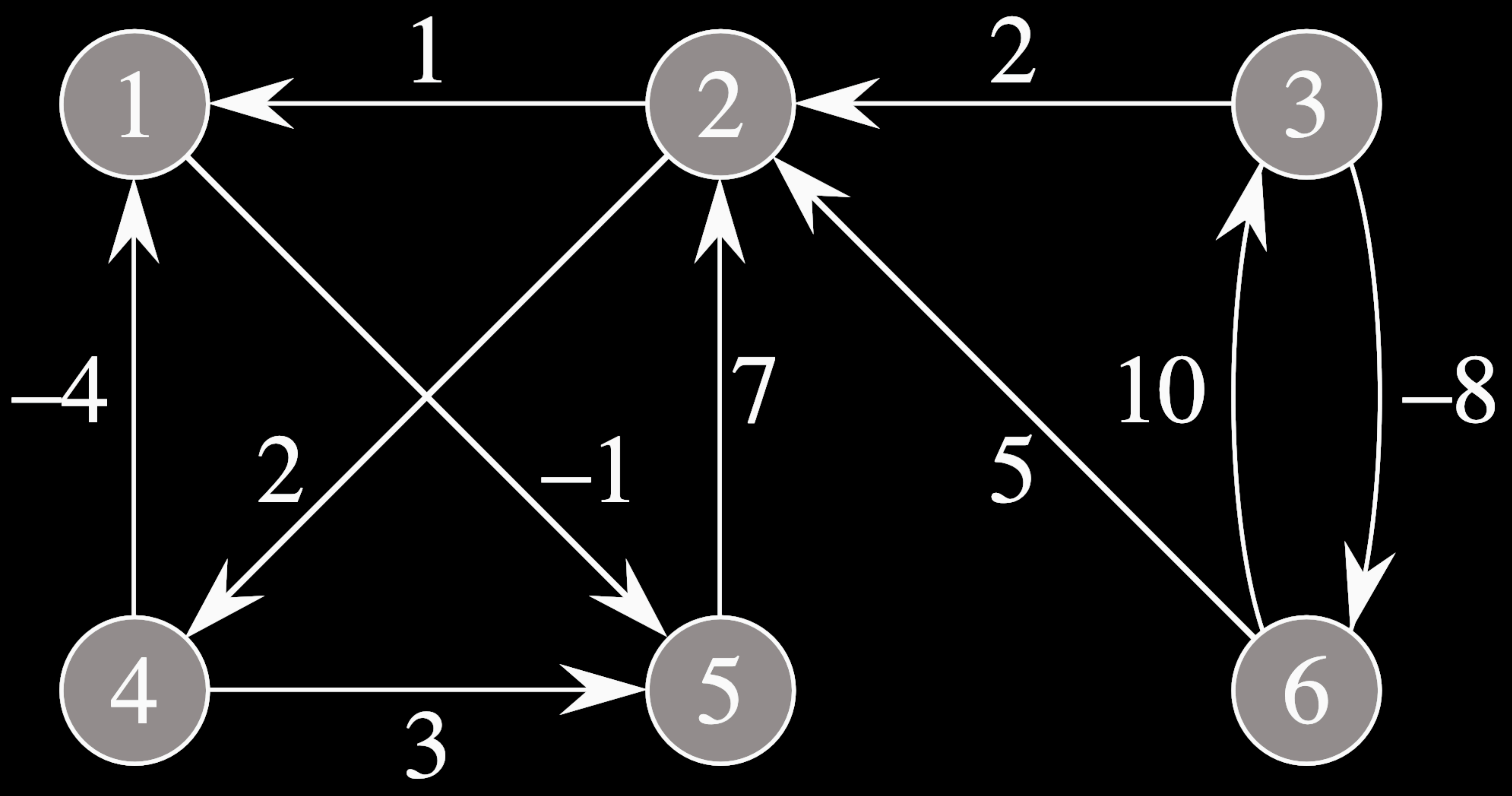
$$D^{(2)} = \begin{pmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ 1 & 0 & \infty & 2 & 0 & \infty \\ 3 & 2 & 0 & 4 & 2 & -8 \\ -4 & \infty & \infty & 0 & -5 & \infty \\ 8 & 7 & \infty & 9 & 0 & \infty \\ 6 & 5 & 10 & 7 & 5 & 0 \end{pmatrix}$$



$$D^{(3)} = \begin{pmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ 1 & 0 & \infty & 2 & 0 & \infty \\ \boxed{3} & \boxed{2} & 0 & \boxed{4} & \boxed{2} & \boxed{-8} \\ -4 & \infty & \infty & 0 & -5 & \infty \\ 8 & 7 & \infty & 9 & 0 & \infty \\ 6 & 5 & \boxed{10} & 7 & 5 & 0 \end{pmatrix}$$

Solution 25.2-1

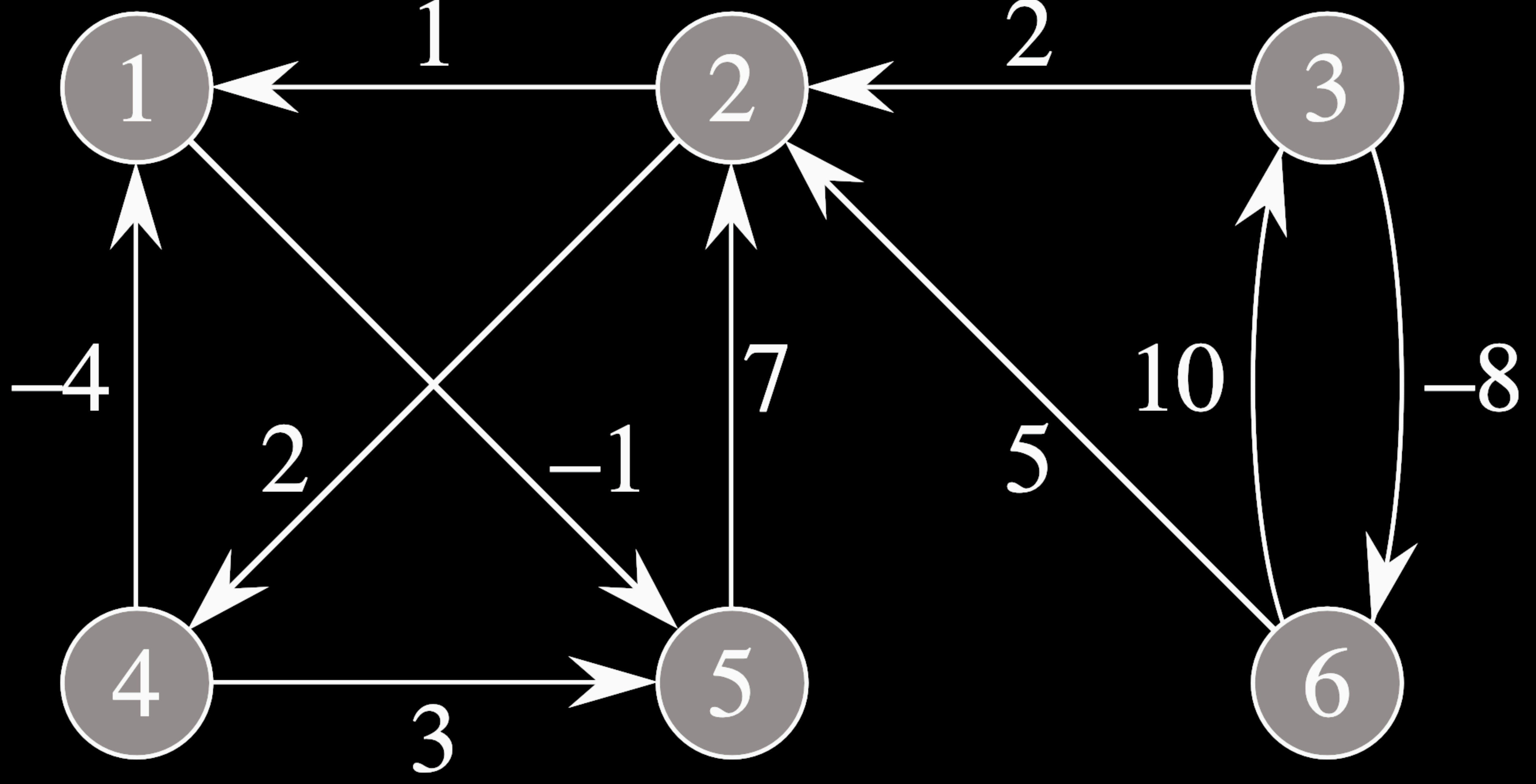
$$D^{(3)} = \begin{pmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ 1 & 0 & \infty & 2 & 0 & \infty \\ 3 & 2 & 0 & 4 & 2 & -8 \\ -4 & \infty & \infty & 0 & -5 & \infty \\ 8 & 7 & \infty & 9 & 0 & \infty \\ 6 & 5 & 10 & 7 & 5 & 0 \end{pmatrix}$$



$$D^{(4)} = \begin{pmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ -2 & 0 & \infty & \boxed{2} & -3 & \infty \\ 0 & 2 & 0 & \boxed{4} & -1 & -8 \\ \boxed{-4} & \infty & \infty & 0 & \boxed{-5} & \infty \\ 5 & 7 & \infty & \boxed{9} & 0 & \infty \\ 3 & 5 & 10 & \boxed{7} & 2 & 0 \end{pmatrix}$$

Solution 25.2-1

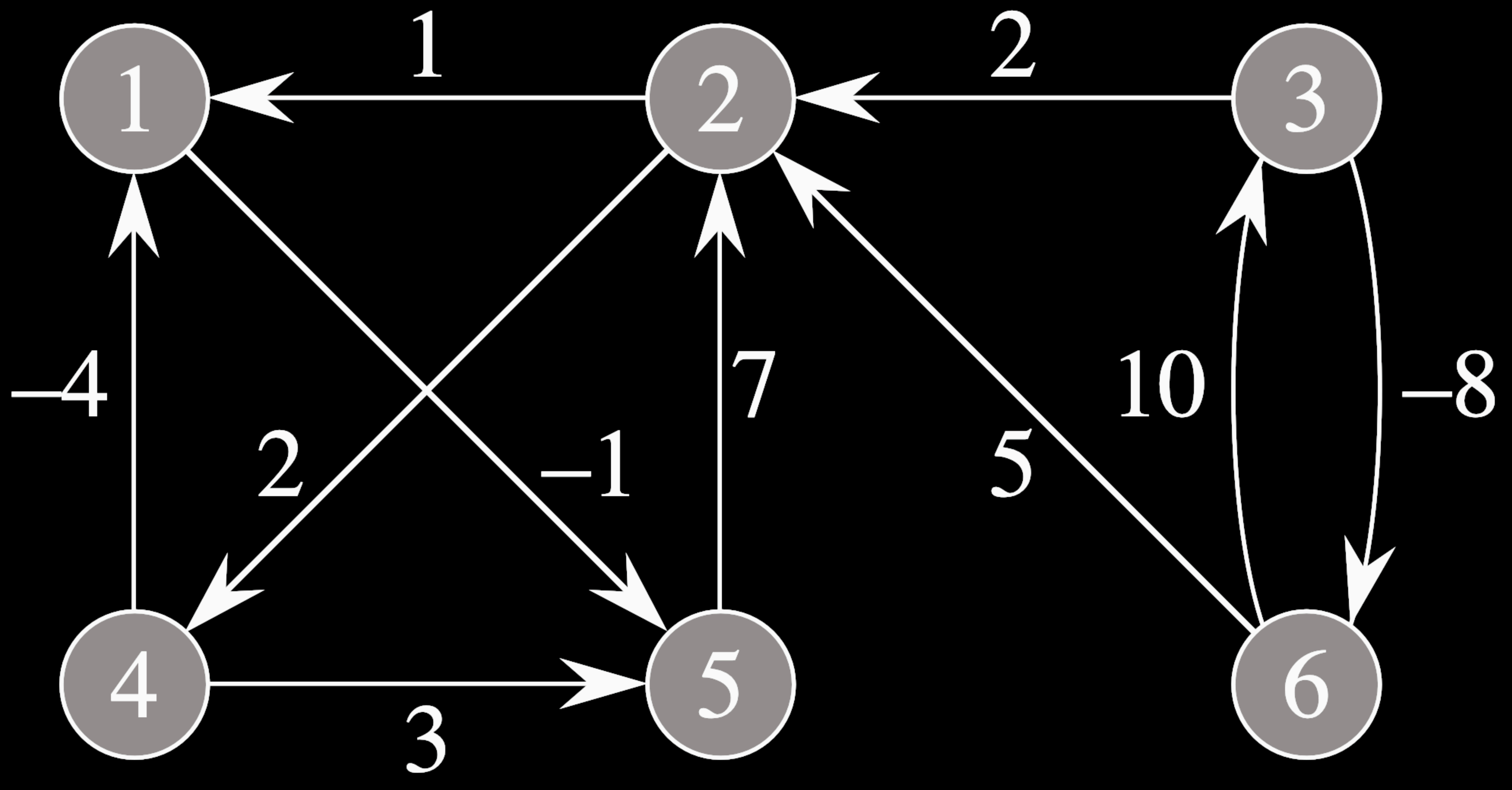
$$D^{(4)} = \begin{pmatrix} 0 & \infty & \infty & \infty & -1 & \infty \\ -2 & 0 & \infty & 2 & -3 & \infty \\ 0 & 2 & 0 & 4 & -1 & -8 \\ -4 & \infty & \infty & 0 & -5 & \infty \\ 5 & 7 & \infty & 9 & 0 & \infty \\ 3 & 5 & 10 & 7 & 2 & 0 \end{pmatrix}$$



$$D^{(5)} = \begin{pmatrix} 0 & 6 & \infty & 8 & \boxed{-1} & \infty \\ -2 & 0 & \infty & 2 & \boxed{-3} & \infty \\ 0 & 2 & 0 & 4 & \boxed{-1} & -8 \\ -4 & 2 & \infty & 0 & \boxed{-5} & \infty \\ \boxed{5} & \boxed{7} & \infty & \boxed{9} & 0 & \infty \\ 3 & 5 & 10 & 7 & \boxed{2} & 0 \end{pmatrix}$$

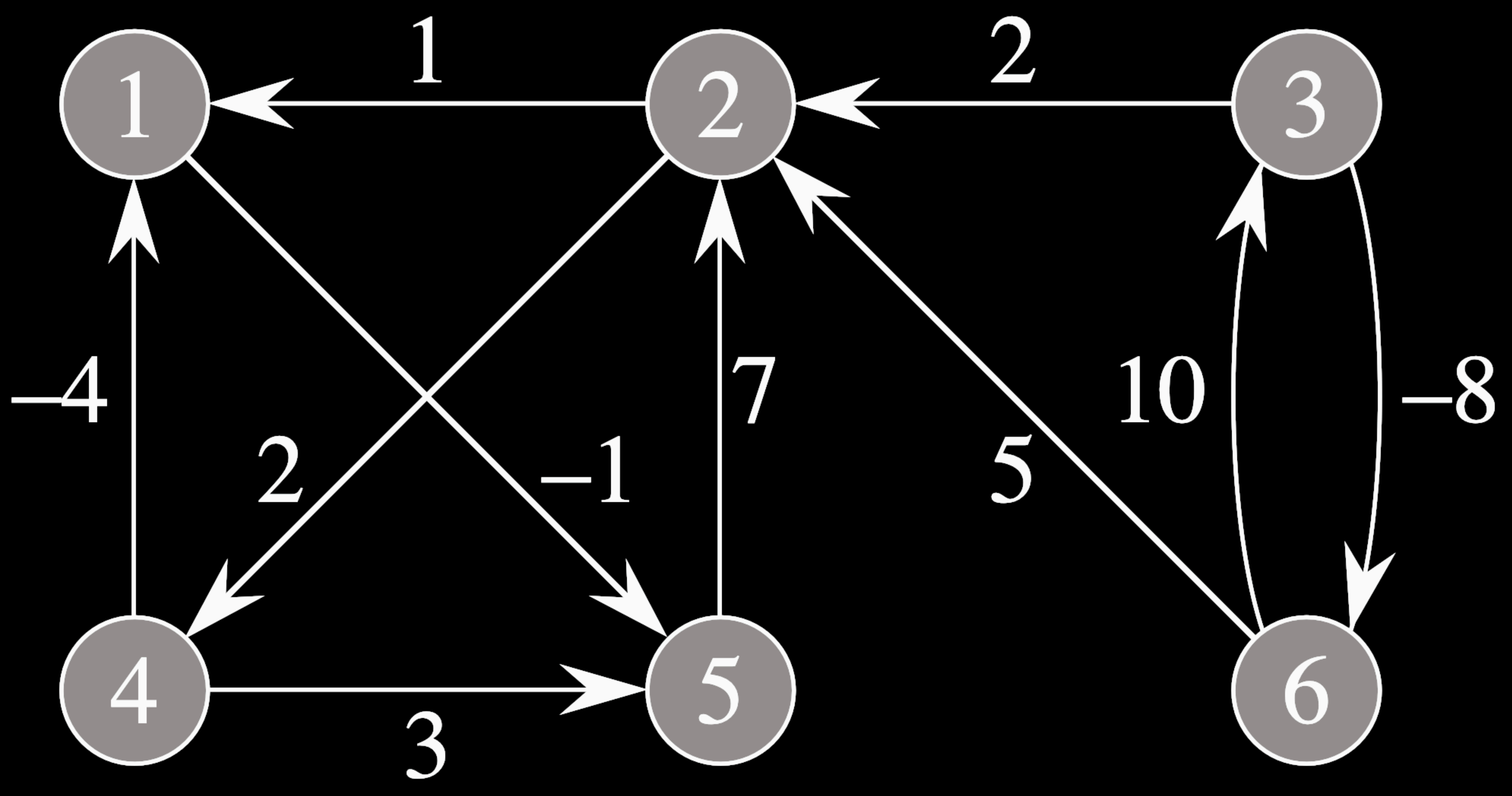
Solution 25.2-1

$$D^{(5)} = \begin{pmatrix} 0 & 6 & \infty & 8 & -1 & \infty \\ -2 & 0 & \infty & 2 & -3 & \infty \\ 0 & 2 & 0 & 4 & -1 & -8 \\ -4 & 2 & \infty & 0 & -5 & \infty \\ 5 & 7 & \infty & 9 & 0 & \infty \\ 3 & 5 & 10 & 7 & 2 & 0 \end{pmatrix}$$



$$D^{(6)} = \begin{pmatrix} 0 & 6 & \infty & 8 & -1 & \infty \\ -2 & 0 & \infty & 2 & -3 & \infty \\ -5 & -3 & 0 & -1 & -6 & \boxed{-8} \\ -4 & 2 & \infty & 0 & -5 & \infty \\ 5 & 7 & \infty & 9 & 0 & \infty \\ \boxed{3} & \boxed{5} & \boxed{10} & \boxed{7} & \boxed{2} & 0 \end{pmatrix}$$

Solution 25.2-1

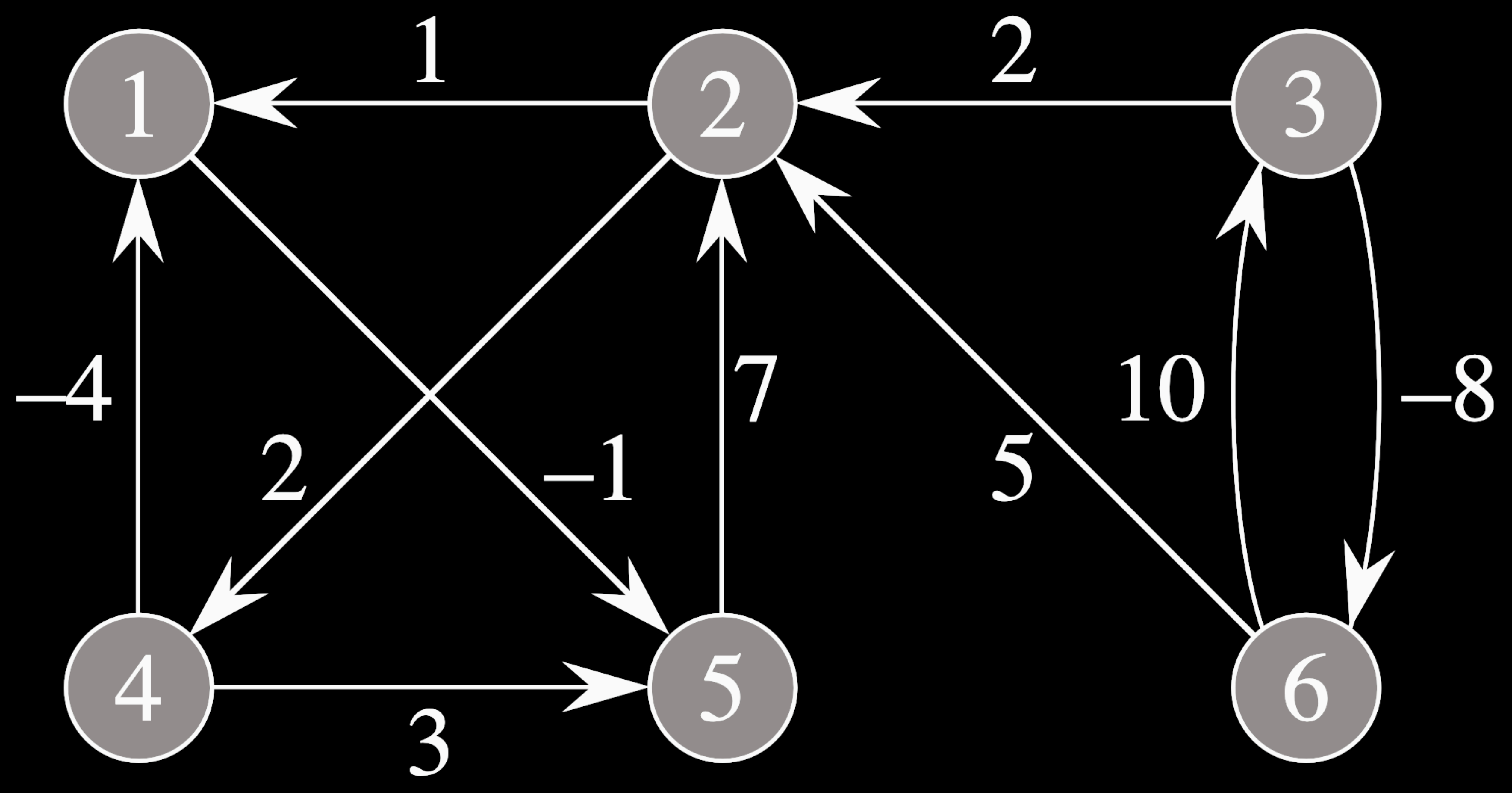


$$D^{(6)} = \begin{pmatrix} 0 & 6 & \infty & 8 & -1 & \infty \\ -2 & 0 & \infty & 2 & -3 & \infty \\ -5 & -3 & 0 & -1 & -6 & -8 \\ -4 & 2 & \infty & 0 & -5 & \infty \\ 5 & 7 & \infty & 9 & 0 & \infty \\ 3 & 5 & 10 & 7 & 2 & 0 \end{pmatrix}$$

25.3-1

Use Johnson's algorithm to find the shortest paths between all pairs of vertices in the graph of Figure 25.2. Show the values of h and $\hat{w}$ computed by the algorithm.

请在图25-2上使用Johnson算法来找到所有结点对之间的最短路径。给出算法计算出的 h 和 $\hat{w}$ 值。

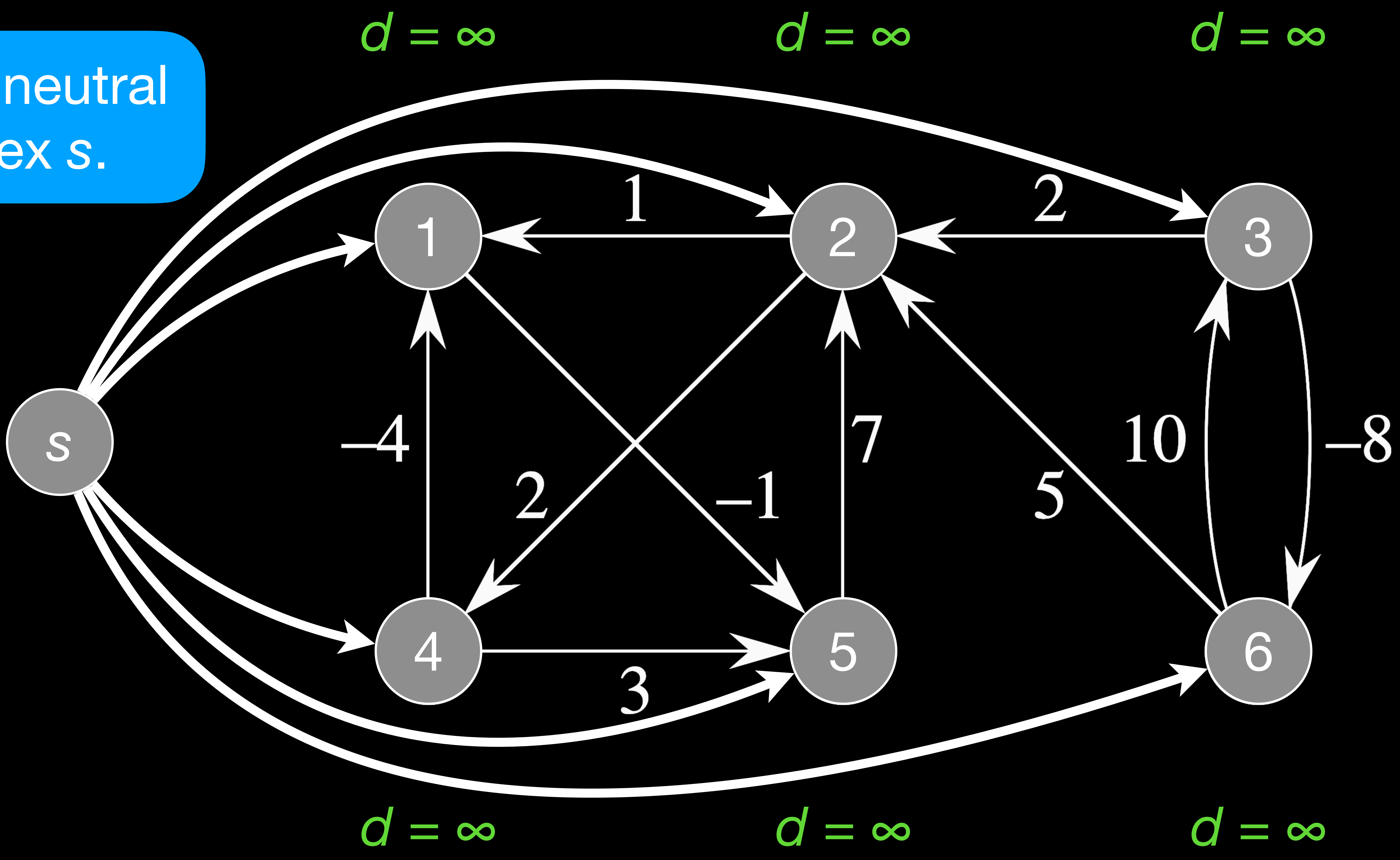


Johnson's Algorithm

JOHNSON(G,w)
Let $G' = (G.V \cup \{s\}, G.E \cup \{(s,v) \mid v \in G.V\})$ and $w(s,v) = 0$
BELLMAN-FORD(G',w,s)
if there is a negative-weight cycle
    return "There is a negative-weight cycle."
for each vertex v $\in G.V$
    $h(v) = v.d$
for each edge $(u,v) \in G.E$
    $\hat{w}(u,v) = w(u,v) + h(u) - h(v)$
Let $D = (d_{uv})$ be a new $|G.V| \times |G.V|$-matrix
for each vertex $u \in G.V$
    DIJKSTRA(G,$\hat{w}$,u)
    for each vertex $v \in G.V$
        $d_{uv} = v.d + h(v) - h(u)$
return D

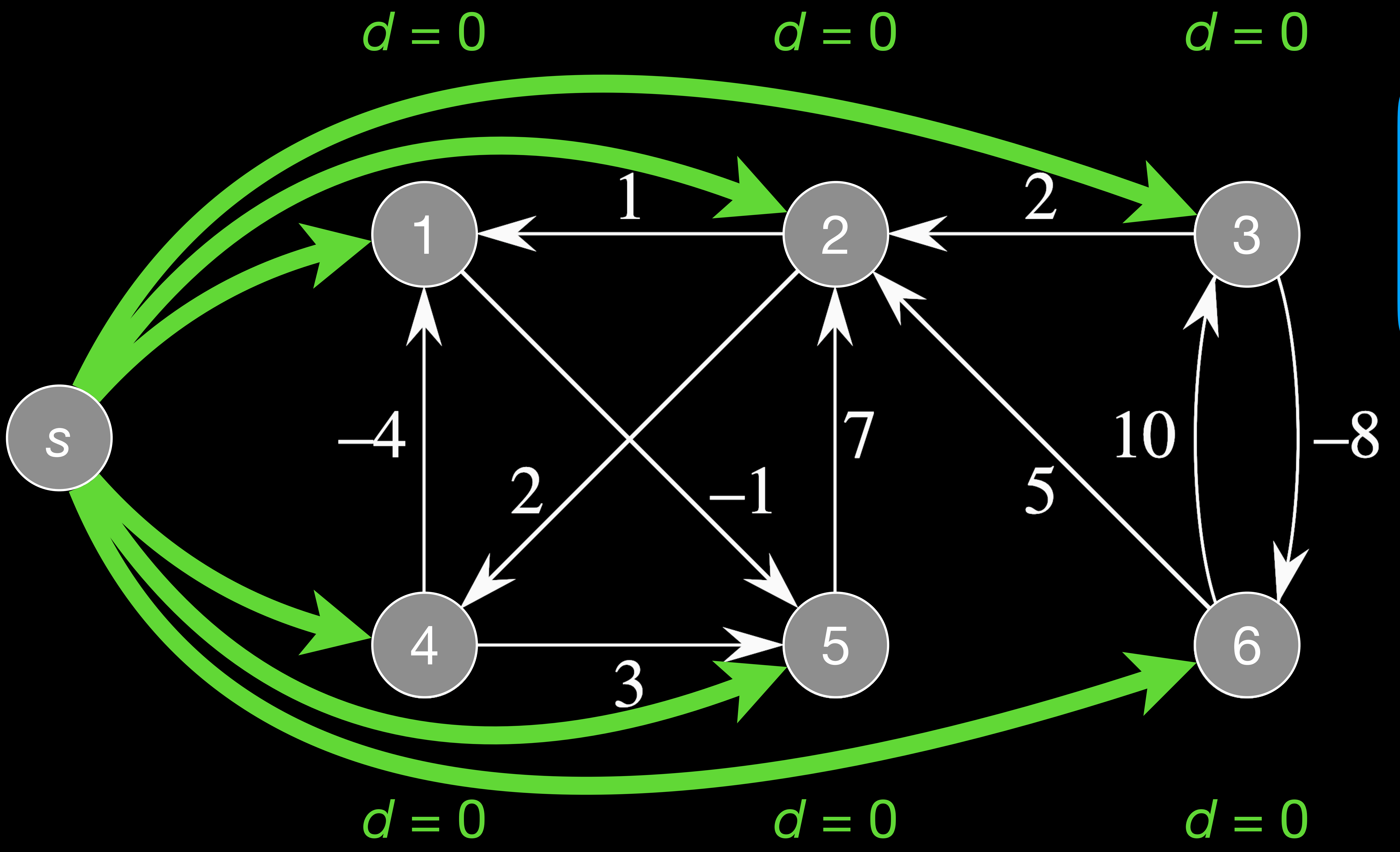
Solution 25.3-1

Add a neutral vertex *s*.



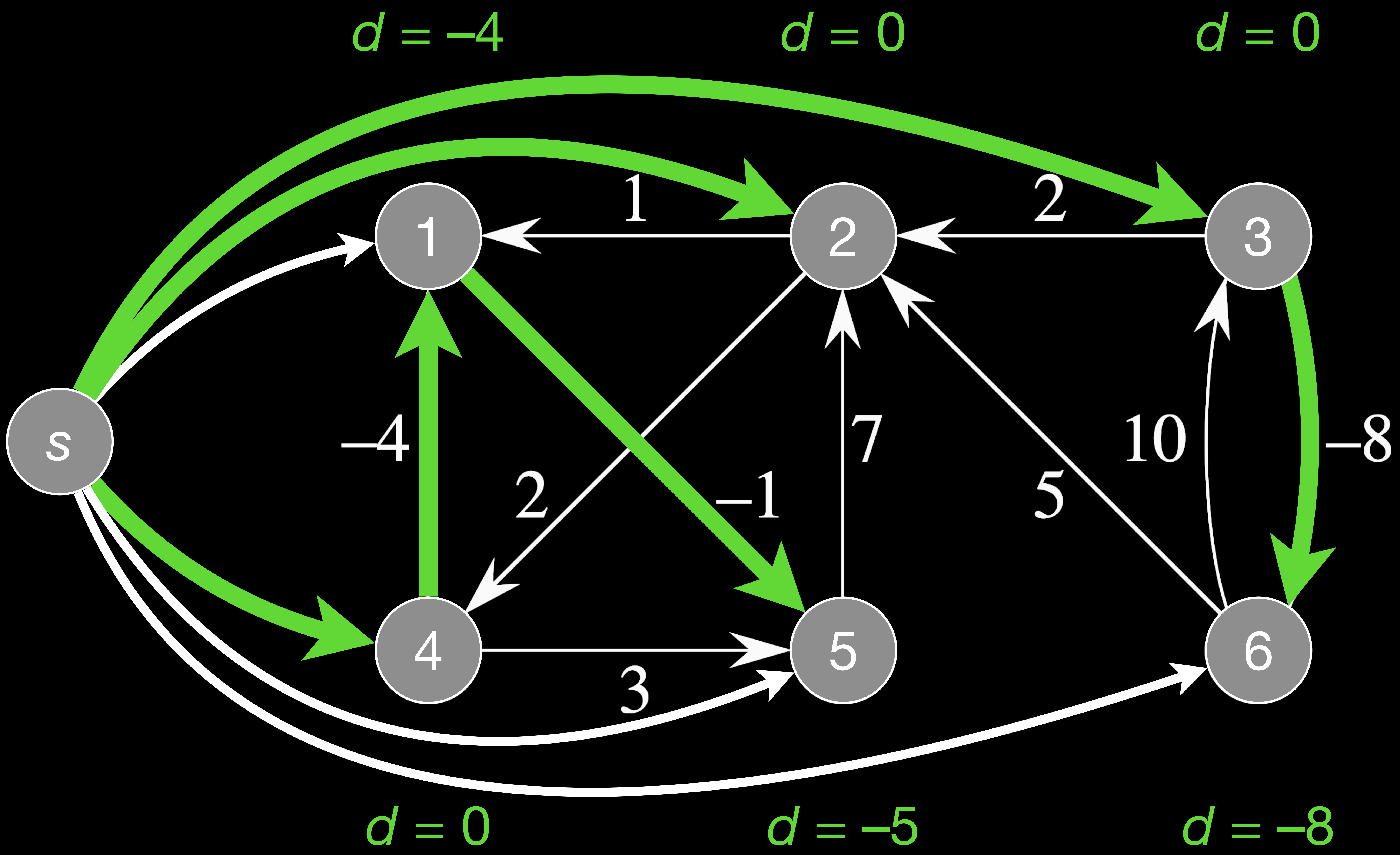
Use Bellman–Ford to find the distance from *s* to every other vertex.

Solution 25.3-1



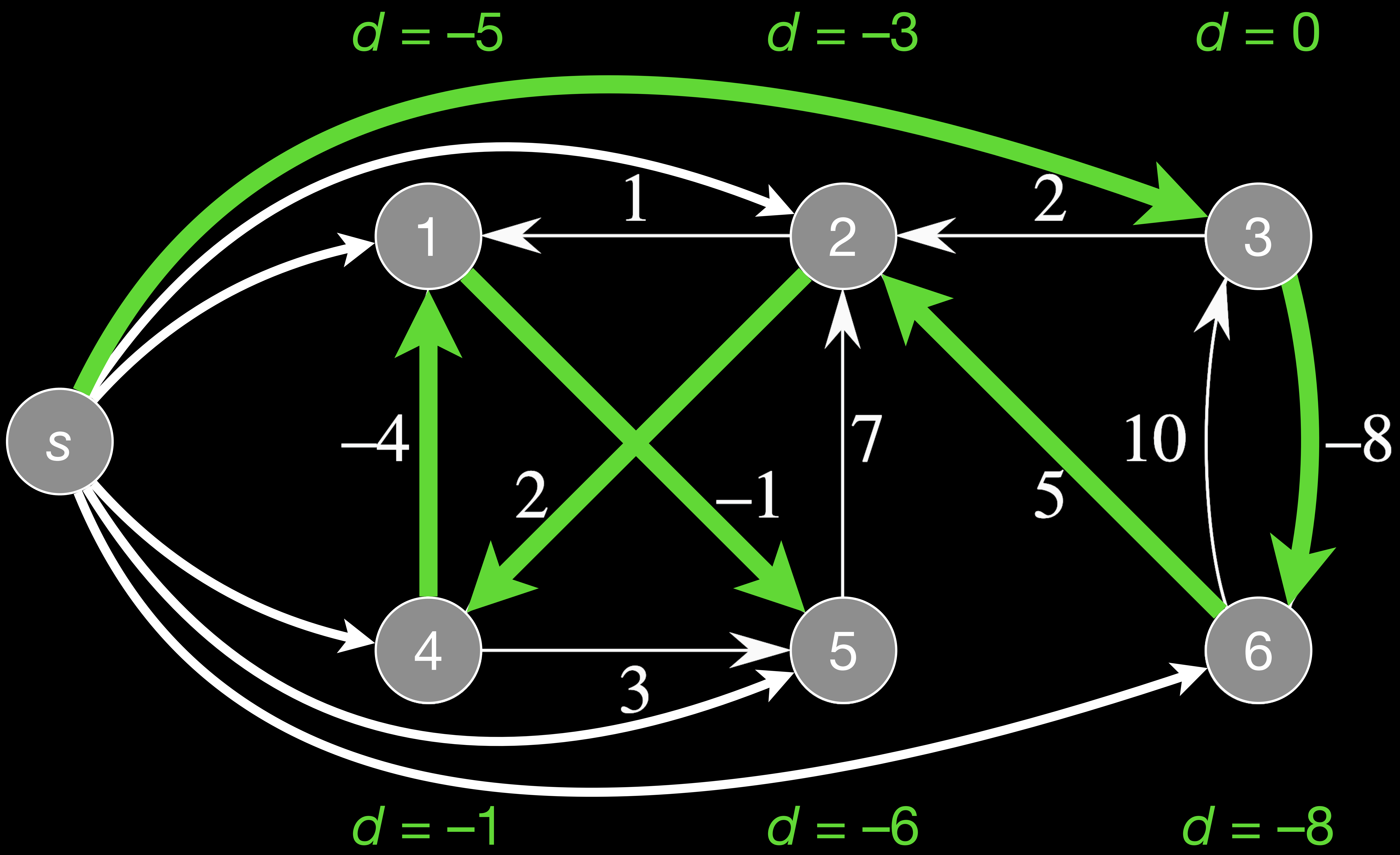
Use Bellman–Ford to find the distance from *s* to every other vertex.

Solution 25.3-1



Use Bellman–Ford to find the distance from *s* to every other vertex.

Solution 25.3-1

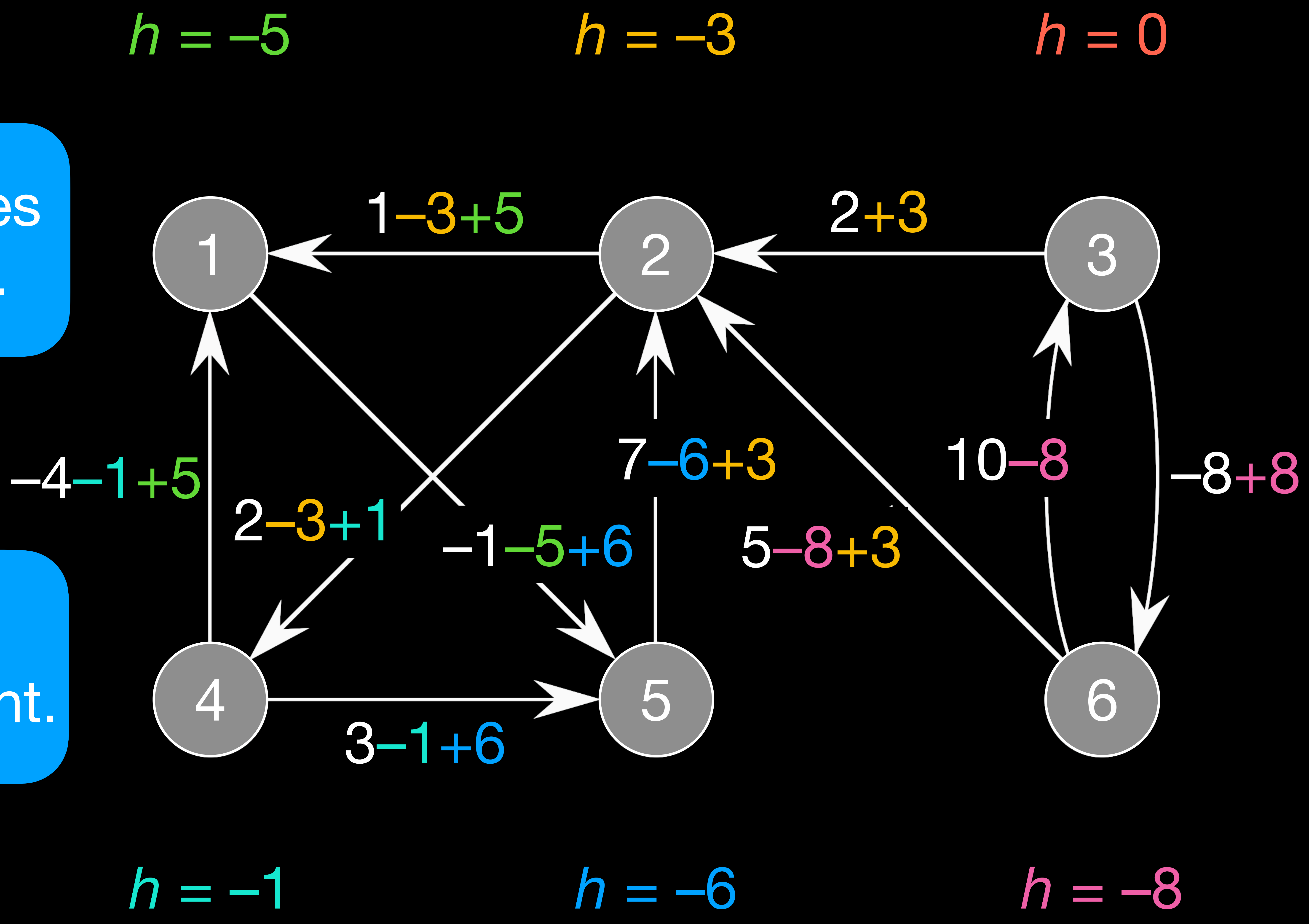


Use Bellman–Ford to find the distance from *s* to every other vertex.

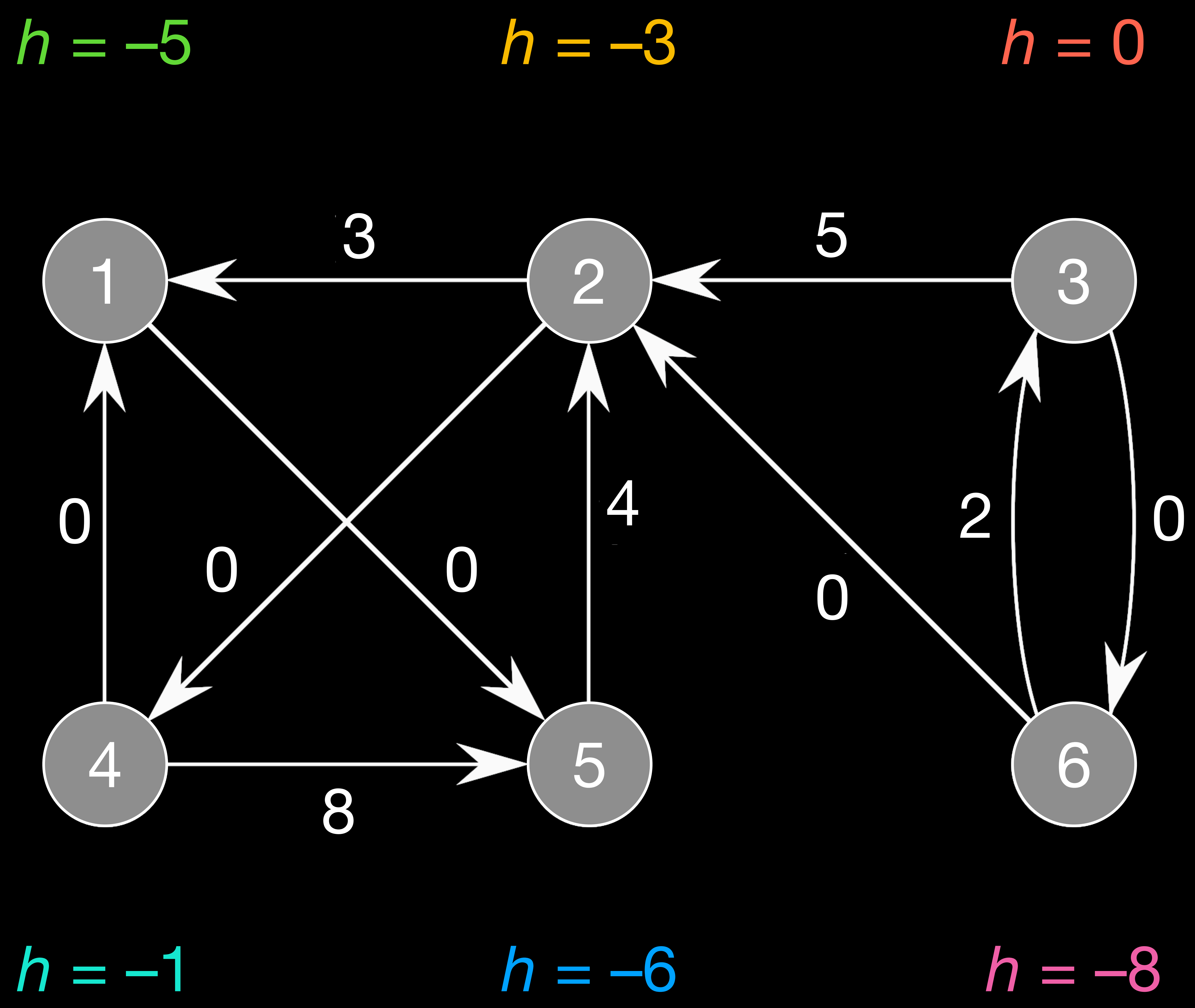
Solution 25.3-1

Distance becomes height of vertex.

Adapt weights according to height.



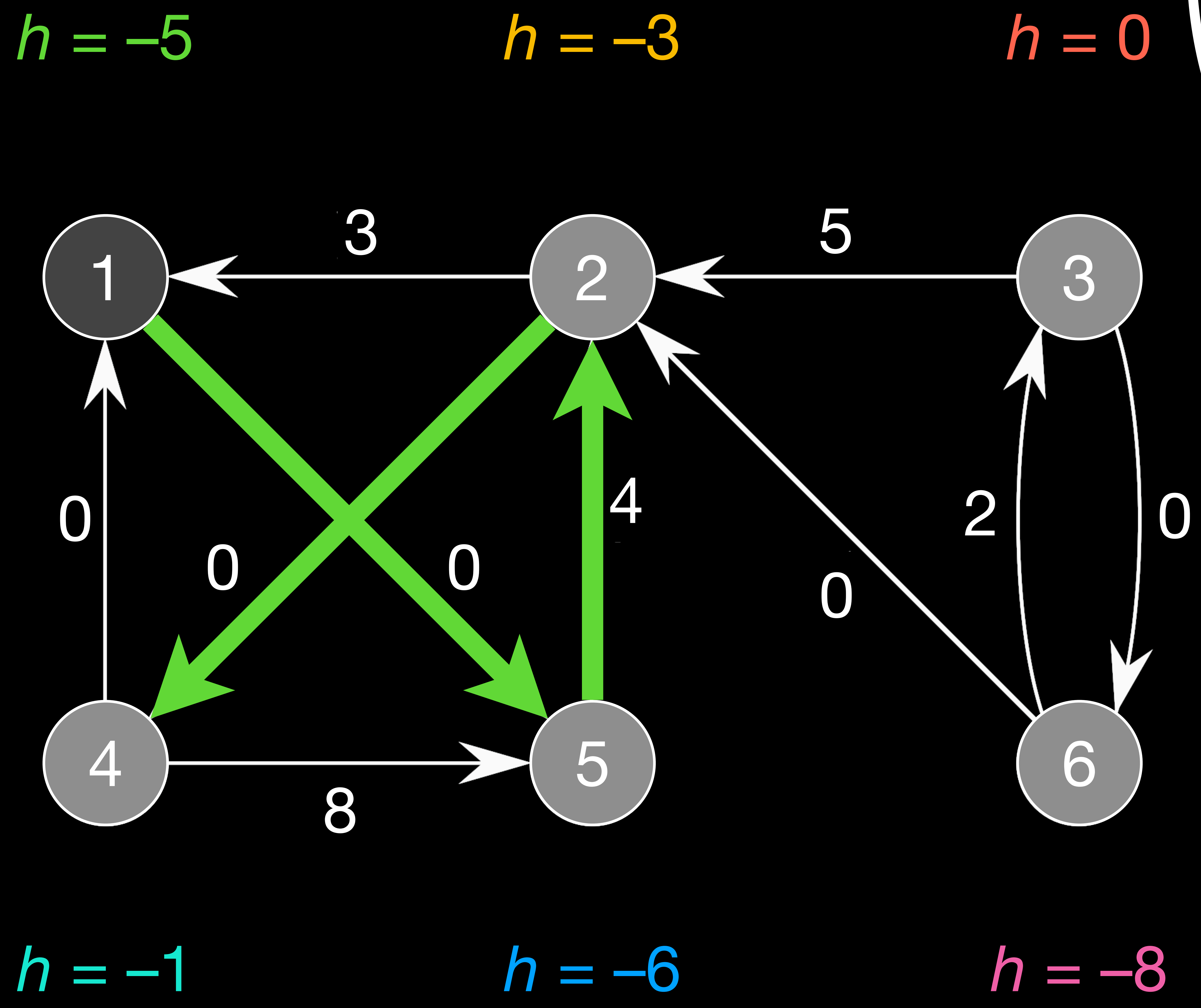
Solution 25.3-1



Run Dijkstra for every source vertex.

Solution 25.3-1

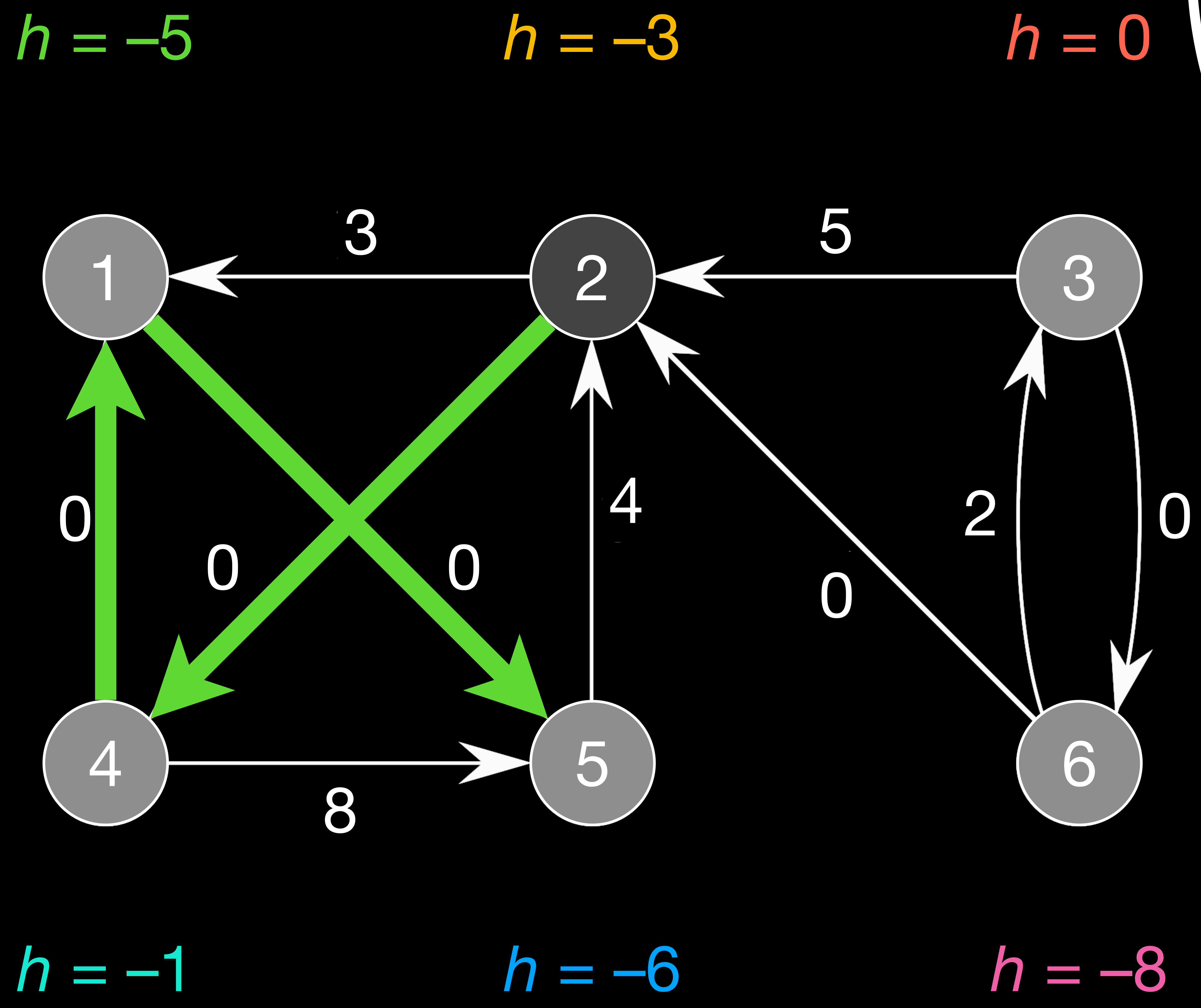
$$\hat{D} = \begin{pmatrix} 0 & 4 & \infty & 4 & 0 & \infty \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{pmatrix}$$



Run Dijkstra for every source vertex.

Solution 25.3-1

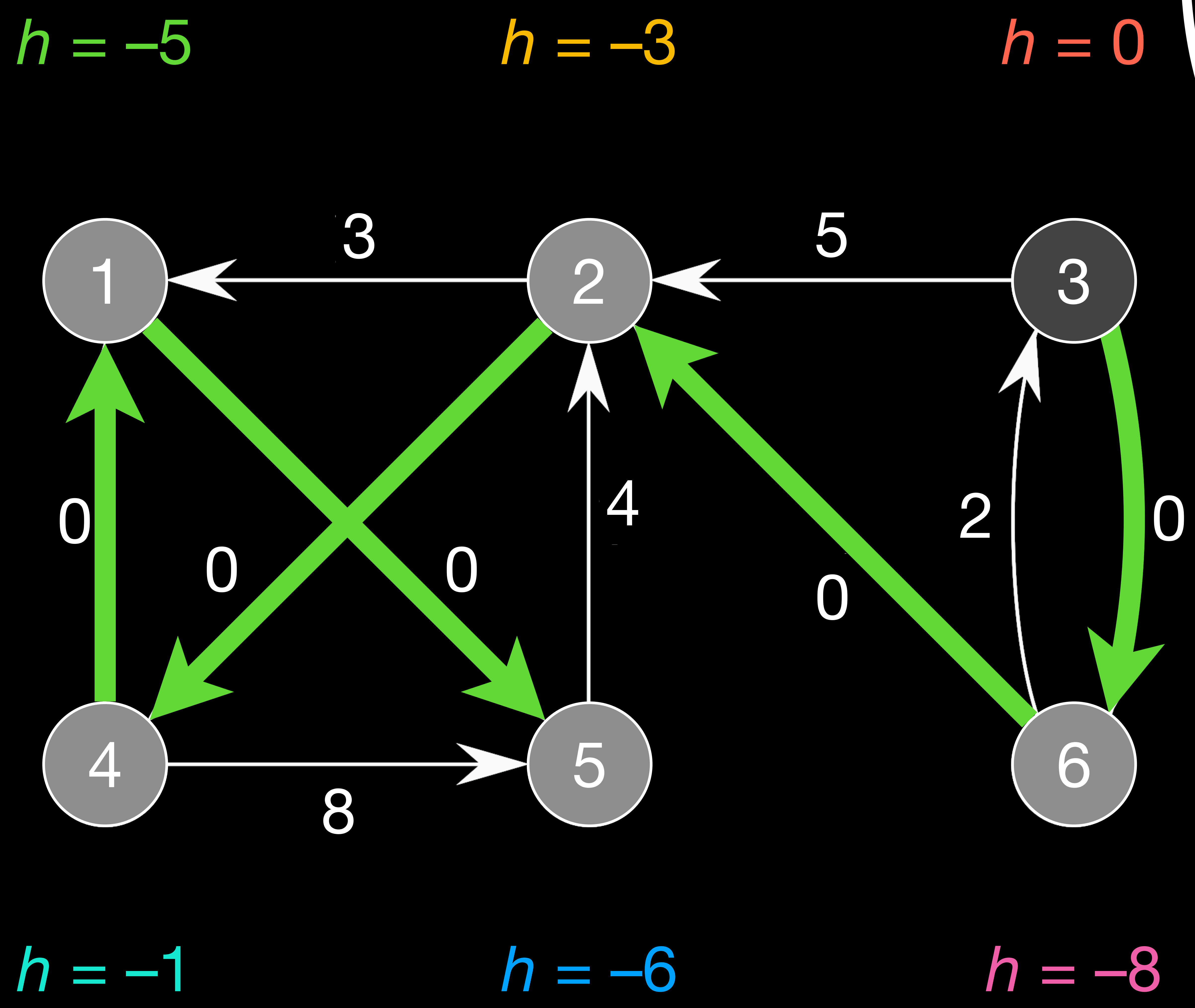
$$\hat{D} = \begin{pmatrix} 0 & 4 & \infty & 4 & 0 & \infty \\ 0 & 0 & \infty & 0 & 0 & \infty \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{pmatrix}$$



Run Dijkstra for every source vertex.

Solution 25.3-1

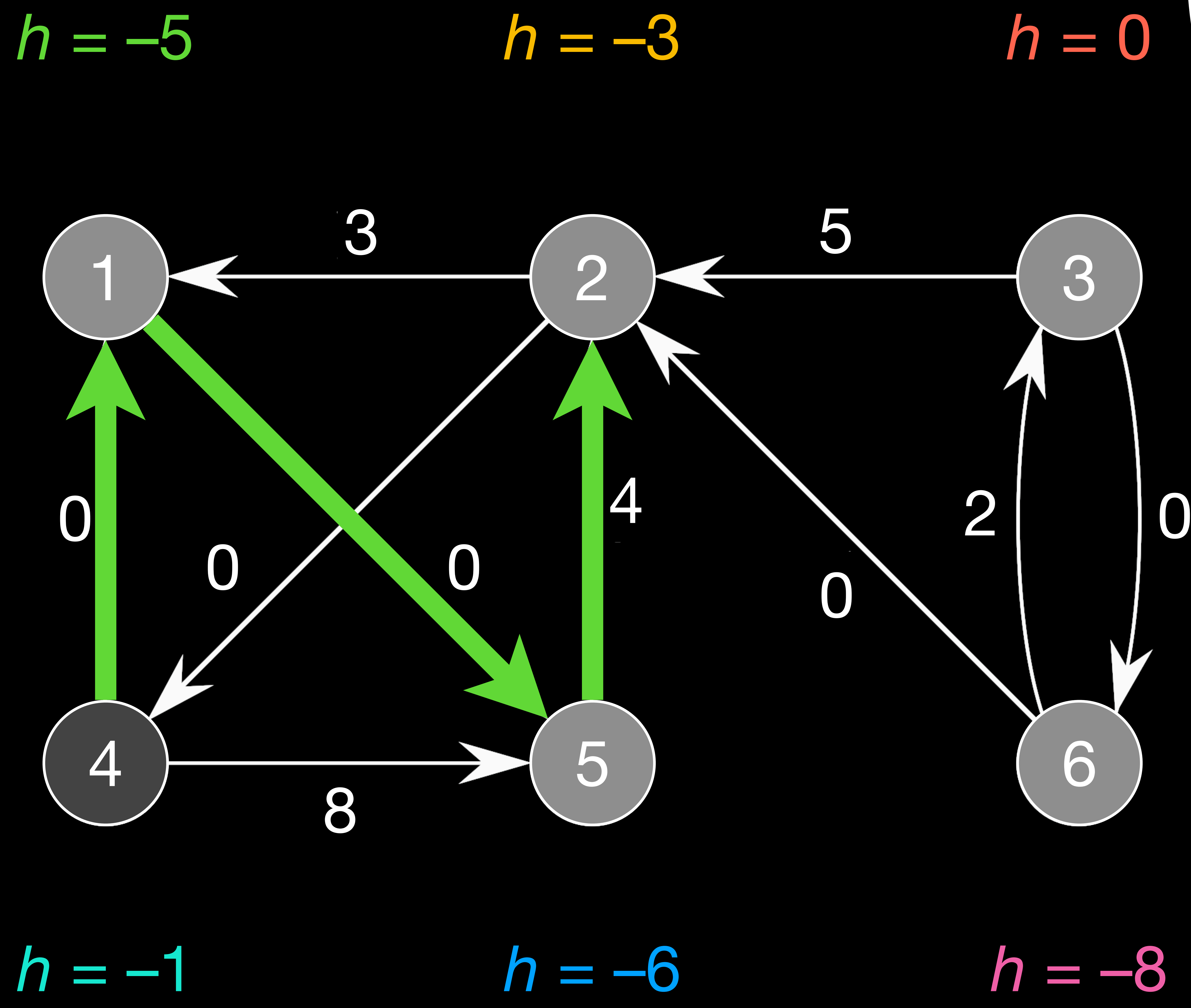
$$\hat{D} = \begin{pmatrix} 0 & 4 & \infty & 4 & 0 & \infty \\ 0 & 0 & \infty & 0 & 0 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{pmatrix}$$



Run Dijkstra for every source vertex.

Solution 25.3-1

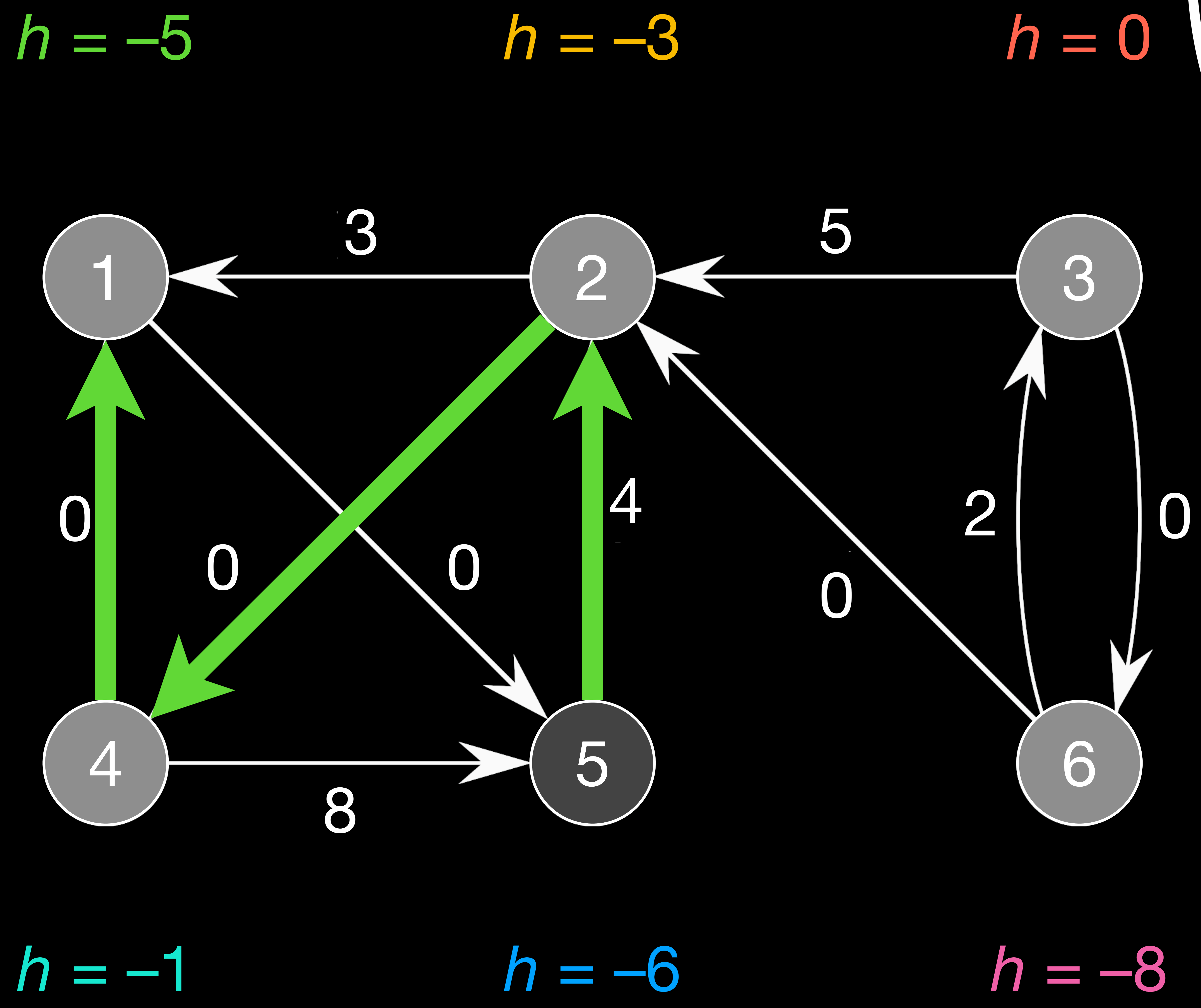
$$\hat{D} = \begin{pmatrix} 0 & 4 & \infty & 4 & 0 & \infty \\ 0 & 0 & \infty & 0 & 0 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & \infty & 0 & 0 & \infty \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{pmatrix}$$



Run Dijkstra for every source vertex.

Solution 25.3-1

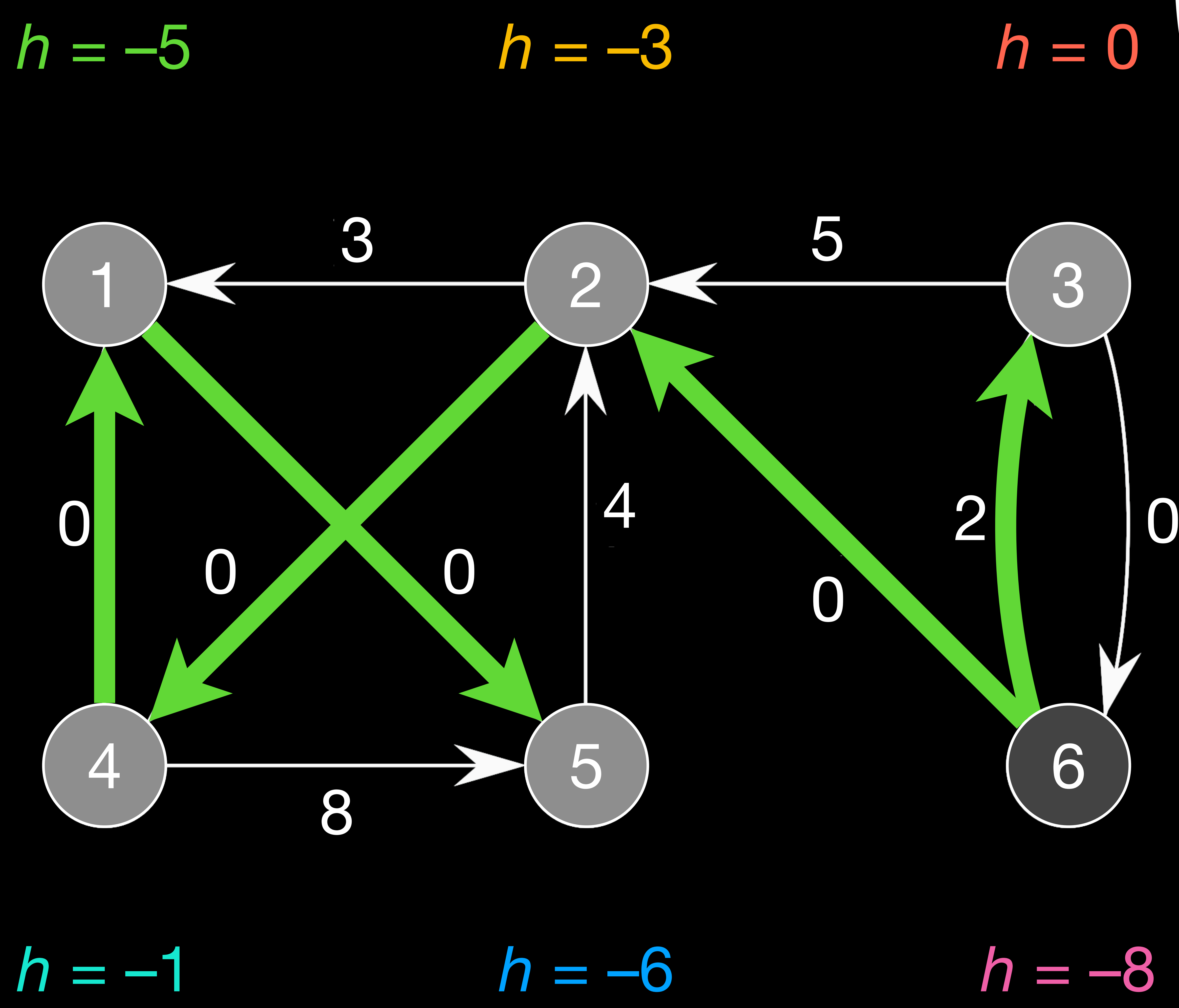
$$\hat{D} = \begin{pmatrix} 0 & 4 & \infty & 4 & 0 & \infty \\ 0 & 0 & \infty & 0 & 0 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & \infty & 0 & 0 & \infty \\ 4 & 4 & \infty & 4 & 0 & \infty \\ ? & ? & ? & ? & ? & ? \end{pmatrix}$$



Run Dijkstra for every source vertex.

Solution 25.3-1

$$\hat{D} = \begin{pmatrix} 0 & 4 & \infty & 4 & 0 & \infty \\ 0 & 0 & \infty & 0 & 0 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & \infty & 0 & 0 & \infty \\ 4 & 4 & \infty & 4 & 0 & \infty \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$



Run Dijkstra for every source vertex.

Solution 25.3-1

$$\hat{D} = \begin{pmatrix} 0 & 4 & \infty & 4 & 0 & \infty \\ 0 & 0 & \infty & 0 & 0 & \infty \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & \infty & 0 & 0 & \infty \\ 4 & 4 & \infty & 4 & 0 & \infty \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

$h = -5$ $h = -3$ $h = 0$

$$D = \begin{pmatrix} 0+5-5 & 4+5-3 & \infty+5-0 & 4+5-1 & 0+5-6 & \infty+5-8 \\ 0+3-5 & 0+3-3 & \infty+3-0 & 0+3-1 & 0+3-6 & \infty+3-8 \\ 0+0-5 & 0+0-3 & 0+0-0 & 0+0-1 & 0+0-6 & 0+0-8 \\ 0+1-5 & 4+1-3 & \infty+1-0 & 0+1-1 & 0+1-6 & \infty+1-8 \\ 4+6-5 & 4+6-3 & \infty+6-0 & 4+6-1 & 0+6-6 & \infty+6-8 \\ 0+8-5 & 0+8-3 & 2+8-0 & 0+8-1 & 0+8-6 & 0+8-8 \end{pmatrix}$$

Correct $\hat{D}$ using heights.

$h = -1$ $h = -6$ $h = -8$

Solution 25.3-1

$$D = \begin{pmatrix} 0 & 6 & \infty & 8 & -1 & \infty \\ -2 & 0 & \infty & 2 & -3 & \infty \\ -5 & -3 & 0 & -1 & -6 & -8 \\ -4 & 2 & \infty & 0 & -5 & \infty \\ 5 & 7 & \infty & 9 & 0 & \infty \\ 3 & 5 & 10 & 7 & 2 & 0 \end{pmatrix}$$

29.1-6

Show that the following linear program is infeasible:

说明下面线性规划是不可解的：

$$
\begin{array}{ll}
\text{maximize 最大化} & 3x_1 - 2x_2 \\
\text{subject to 满足约束} & \\
& \begin{aligned} x_1 + x_2 &\le 2 \\ -2x_1 - 2x_2 &\le -10 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}
\end{array}
$$

Solution 29.1-6

Show that the following linear program is infeasible:

说明下面线性规划是不可解的：

$$
\begin{array}{lrcl}
\text{maximize 最大化} & 3x_1 - 2x_2 & & \\
\text{subject to 满足约束} & x_1 + x_2 & \leq & 2 \\
& -2x_1 - 2x_2 & \leq & -10
\end{array}
$$

$x_1 + x_2 \geq 5$

If any solution (x_1, x_2) exists, then $5 \leq 2$.

29.3-6

Solve the following linear program using SIMPLEX:

采用SIMPLEX求解下面的线性规划：

$$
\begin{array}{ll}
\text{maximize 最大化} & 5x_1 - 3x_2 \\
\text{subject to 满足约束} & \begin{aligned} x_1 - x_2 &\le 1 \\ 2x_1 + x_2 &\le 2 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}
\end{array}
$$

Solution 29.3-6

Slack form: 松弛型：

$c_1 > 0$ ➡ x_1 should become a basic variable

$$
\begin{aligned}
&\text{maximize 最大化} && z = 5x_1 - 3x_2 \\
&\text{subject to 满足约束} && x_3 = 1 - x_1 + x_2 \\
& && x_4 = 2 - 2x_1 - x_2
\end{aligned}
$$

If x_3 becomes nonbasic, then $x_1 = 1 + x_2 - x_3$

If x_4 becomes nonbasic, then $x_1 = 1 - \frac{1}{2}x_2 - \frac{1}{2}x_4$

$$
\begin{aligned}
&\text{maximize 最大化} && z = 5 + 2x_2 - 5x_3 \\
&\text{subject to 满足约束} && x_1 = 1 + x_2 - x_3 \\
& && x_4 = -3x_2 + 2x_3
\end{aligned}
$$

Either variable can become nonbasic.

Solution 29.3-6

$c_2 > 0$ ➡ x_2 should become a basic variable

maximize 最大化 $z = 5 + 2x_2 - 5x_3$

subject to 满足约束 $x_1 = 1 + x_2 - x_3$

$x_4 = -3x_2 + 2x_3$

If x_1 becomes nonbasic, then $x_2 = -1 + x_1 + x_3$

If x_4 becomes nonbasic, then $x_2 = \frac{2}{3}x_3 - \frac{1}{3}x_4$

maximize 最大化 $z = 5 - 3\frac{2}{3}x_3 - \frac{2}{3}x_4$

subject to 满足约束 $x_1 = 1 - \frac{1}{3}x_3 - \frac{1}{3}x_4$

$x_2 = \frac{2}{3}x_3 - \frac{1}{3}x_4$

Only x_4 can become nonbasic.

Solution 29.3-6

The solution is optimal if all constants are ≤ 0.
如果所有的常数≤ 0，则解决最优。

maximize 最大化 $z = 5 - 3\tfrac{2}{3}x_3 - \tfrac{2}{3}x_4$

subject to 满足约束

$$x_1 = 1 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4$$
$$x_2 = \tfrac{2}{3}x_3 - \tfrac{1}{3}x_4$$

Found optimal solution:
nonbasic variables $x_3 = x_4 = 0$
basic variables $x_1 = 1$, $x_2 = 0$

Correctness

正确性

Proof in multiple steps:

1. If the algorithm terminates,
   then the solution is feasible. ✔
2. ~~If the algorithm does not loop, then~~
   it terminates within … PIVOT steps. ✔
3. If the algorithm returns a solution,
   then it is optimal (uses duality).
4. INITIALIZE-SIMPLEX finds an initial feasible solution if one exists (uses optimality).

多步骤证明：

1. 如果算法终止，
   那么回复的解决是可行的。✔
2. ~~如果算法没有循环，~~
   那么它在 … 转换步骤内终止。✔
3. 如果算法回复一个解，
   则它是最优的（使用对偶性）。
4. INITIALIZE-SIMPLEX找到一个
   初始可行的解（如果存在）
   （使用最优性）。

Duality 对偶性

- method to prove optimality
- idea: for a max-LP, find another min-LP with the same optimal value

primal LP 原始线性规划

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \text{subject to} \quad & \mathbf{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \mathbf{0} \end{aligned}$$

n variables
m constraints

dual LP 对偶线性规划

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} \\ \text{subject to} \quad & \mathbf{A}^{\mathsf{T}}\boldsymbol{y} \ge \boldsymbol{c} \\ & \boldsymbol{y} \ge \mathbf{0} \end{aligned}$$

m variables
n constraints

Duality 对偶性: Example

$$\begin{aligned} \text{maximize} \quad & 18x_1 + 12.5x_2 \\ \text{subject to} \quad & 1x_1 + 1x_2 \le 20 \\ & 1x_1 + 0x_2 \le 12 \\ & 0x_1 + 1x_2 \le 16 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 20y_3 + 12y_4 + 16y_5 \\ \text{subject to} \quad & 1y_3 + 1y_4 + 0y_5 \ge 18 \\ & 1y_3 + 0y_4 + 1y_5 \ge 12.5 \\ & y_3 \ge 0 \\ & y_4 \ge 0 \\ & y_5 \ge 0 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 20 \\ 12 \\ 16 \end{pmatrix} \qquad c = \begin{pmatrix} 18 \\ 12.5 \end{pmatrix}$$

Duality 对偶性

- Lemma 29.8: For every feasible solution of the primal LP, its objective function value is ≤ the objective function value of every feasible solution of the dual LP.

- Corollary 29.9: If a solution of the primal LP has the same objective function value as a solution of the dual LP, they are both optimal.

- Theorem 29.10: If SIMPLEX returns a solution, then it is optimal (and a solution for the corresponding dual LP can be derived).

Duality 对偶性

Lemma 29.8: For every feasible solution of the primal LP, its objective function value is ≤ the objective function value of every feasible solution of the dual LP.

Proof:

$b_i \geq \sum_{j=1}^{n} a_{ij}\bar{x}_j$

- Let $\bar{x}_1, \ldots, \bar{x}_n$ be a feasible solution of the primal LP, and $\bar{y}_1, \ldots, \bar{y}_n$ be a feasible solution of the dual LP.

$c_j \leq \sum_{i=1}^{m} a_{ij}\bar{y}_i$

- Then $\underbrace{\sum_{j=1}^{n} c_j\bar{x}_j}_{\text{objective function of primal LP}} \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\bar{y}_i\right)\bar{x}_j = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\bar{x}_j\right)\bar{y}_i \leq \underbrace{\sum_{i=1}^{m} b_i\bar{y}_i}_{\text{objective function of dual LP}}.$

objective function of primal LP

objective function of dual LP

Duality 对偶性

Corollary 29.9: If a solution of the primal LP has the same objective function value as a solution of the dual LP, they are both optimal.

Proof:

- Let v be the value of the objective functions.
  Because of Lemma 29.8, there is no feasible solution of the primal LP with objective function value $> v$.
  Because of Lemma 29.8, there is no feasible solution of the dual LP with objective function value $< v$.

Duality 对偶性

Theorem 29.10: If SIMPLEX returns a solution, then it is optimal (and a solution for the corresponding dual LP can be derived).

Proof idea:

- The transformations of the primal LP produced by SIMPLEX are all equivalent.
- Therefore, all their duals are equivalent.
- Upon termination, all $\hat{b}_i \geq 0$ and all $\hat{c}_j \leq 0$.
  The primal objective function is $z = \hat{v} + \hat{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}$, the dual is $z' = \hat{v}' + \hat{\boldsymbol{b}}^{\mathrm{T}}\boldsymbol{y}$.
- Some bookkeeping (29.99) shows that $\hat{v} = \hat{v}'$ and that the basic solution of the dual problem (generated from the final slack form) is feasible.
- We have found solutions of the primal and of the dual LP with the same objective function value, so they both are optimal.

Initializing a LP

- If the initial objective function needs to be minimized, one can instead maximize the negative of the function.
- If some variables do not need to be nonnegative, e.g. $x_2 \in \mathbb{R}$, then replace x_2 by $(x_{2a} - x_{2b})$. The new variables x_{2a} and x_{2b} may be restricted to be nonnegative.
- If there are equality constraints, replace them by the two constraints $\geq$ and $\leq$.
- If some constraint has a $\geq$ instead of $\leq$, multiply the constraint with -1.

Further Example

minimize 最小化

$$x_1 + x_2 + x_3 + x_4$$

subject to 满足约束

$$\begin{aligned} -2x_1 + 8x_2 \qquad\qquad + 10x_4 &\geq 50 \\ 5x_1 + 2x_2 \qquad\qquad\qquad\qquad &\geq 100 \\ 3x_1 - 5x_2 + 10x_3 - 2x_4 &\geq 25 \end{aligned}$$

maximize 最大化

$$-x_1 - x_2 - x_3 - x_4$$

subject to 满足约束

$$\begin{aligned} 2x_1 - 8x_2 \qquad\qquad - 10x_4 &\leq -50 \\ -5x_1 - 2x_2 \qquad\qquad\qquad\qquad &\leq -100 \\ -3x_1 + 5x_2 - 10x_3 + 2x_4 &\leq -25 \end{aligned}$$

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Further Example in Standard Form

$$
\begin{array}{ll}
\text{maximize 最大化} & -x_1 - x_2 - x_3 - x_4 \\
\text{subject to 满足约束} & 2x_1 - 8x_2 \qquad\qquad - 10x_4 \leq -50 \\
& -5x_1 - 2x_2 \qquad\qquad\qquad\quad \leq -100 \\
& -3x_1 + 5x_2 - 10x_3 + 2x_4 \leq -25 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0 \\
& x_4 \geq 0
\end{array}
$$

Initializing a LP 初始

- We assumed that we start the SIMPLEX algorithm with an initial feasible solution. What if the first basic solution is not feasible?

original LP 原来线性规划

$$\begin{array}{ll} \text{maximize} & 2x_1 - x_2 \\ \text{subject to} & 2x_1 - x_2 \leq 2 \\ & x_1 - 5x_2 \leq -4 \end{array}$$

auxiliary LP 辅助线性规划

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & 2x_1 - x_2 - x_0 \leq 2 \\ & x_1 - 5x_2 - x_0 \leq -4 \end{array}$$

- The auxiliary LP is always feasible.
  The original LP is feasible iff the auxiliary LP has optimal value $\bar{x}_0 = 0$.

Solving the Auxiliary LP

- Pivot x_0 with the variable x_i that has the smallest (most negative) b_i. The resulting basic solution is feasible (but may have $x_0 \neq 0$).
- Solve the auxiliary LP with the SIMPLEX method.
- If the resulting $x_0 = 0$, the original LP is feasible. (If necessary, change x_0 to a nonbasic variable.)
- Remove all occurrences of x_0 from the auxiliary LP and restore the original objective function
  ➥ original LP is now in a form that has a feasible basic solution!

Further Example in Standard Form

maximize 最大化 $-x_1 - x_2 - x_3 - x_4$

subject to 满足约束

$$
\begin{aligned}
2x_1 - 8x_2 \phantom{{}-10x_3} - 10x_4 &\le -50 \\
-5x_1 - 2x_2 &\le -100 \\
-3x_1 + 5x_2 - 10x_3 + 2x_4 &\le -25
\end{aligned}
$$

maximize 最大化 $-x_0$

subject to 满足约束

$$
\begin{aligned}
2x_1 - 8x_2 - 10x_4 - x_0 &\le -50 \\
-5x_1 - 2x_2 - x_0 &\le -100 \\
-3x_1 + 5x_2 - 10x_3 + 2x_4 - x_0 &\le -25 \\
x_1 &\ge 0 \\
x_2 &\ge 0
\end{aligned}
$$

Further Example: Auxiliary LP

maximize 最大化

$$-x_0$$

subject to 满足约束

$$
\begin{aligned}
2x_1 - 8x_2 \phantom{{}-10x_3} - 10x_4 - x_0 &\le -50 \\
-5x_1 - 2x_2 - x_0 &\le -100 \\
-3x_1 + 5x_2 - 10x_3 + 2x_4 - x_0 &\le -25 \\
x_1 &\ge 0 \\
x_2 &\ge 0 \\
x_3 &\ge 0 \\
x_4 &\ge 0 \\
x_0 &\ge 0
\end{aligned}
$$

Further Example: Auxiliary LP

$$
\begin{array}{llrrrrrrl}
\text{maximize 最大化} & & & & & & -x_0 & & \\
\text{subject to 满足约束} & 2x_1 & - 8x_2 & & - 10x_4 & - x_0 & \leq & -50 \\
& -5x_1 & - 2x_2 & & & - x_0 & \leq & -100 \\
& -3x_1 & + 5x_2 & - 10x_3 & + 2x_4 & - x_0 & \leq & -25
\end{array}
$$

$$
\begin{array}{llrrrrrrr}
\text{maximize 最大化} & z & = & & & & & & -x_0 \\
\text{subject to 满足约束} & x_5 & = & -50 & - 2x_1 & + 8x_2 & & + 10x_4 & + x_0 \\
& x_6 & = & -100 & + 5x_1 & + 2x_2 & & & + x_0 \\
& x_7 & = & -25 & + 3x_1 & - 5x_2 & + 10x_3 & - 2x_4 & + x_0
\end{array}
$$

Further Example: Slack Form

$$
\begin{array}{llrl}
\text{maximize 最大化} & z & = & -x_0 \\
\text{subject to 满足约束} & x_5 & = & -50 - 2x_1 + 8x_2 + 10x_4 + x_0 \\
 & x_6 & = & -100 + 5x_1 + 2x_2 + x_0 \\
 & x_7 & = & -25 + 3x_1 - 5x_2 + 10x_3 - 2x_4 + x_0
\end{array}
$$

Further Example: First Pivot 转动

maximize 最大化 $z = -x_0$

subject to 满足约束

$$
\begin{aligned}
x_5 &= -50 - 2x_1 + 8x_2 + 10x_4 + x_0 \\
x_6 &= \boxed{-100} + 5x_1 + 2x_2 + x_0 \\
x_7 &= -25 + 3x_1 - 5x_2 + 10x_3 - 2x_4 + x_0
\end{aligned}
$$

Still infeasible. Start with a dummy pivot $x_0 \Longleftrightarrow x_6$ (smallest constant b_6)

$N = \{0, 1, 2, 3, 4\}$

The auxiliary LP has an additional variable x_0.

$B = \{5, 6, 7\}$

$c_0 = -1, c_1 = c_2 = c_3 = c_4 = 0$

$b_5 = -50, a_{51} = 2, a_{52} = -8, a_{53} = 0, a_{54} = -10, a_{50} = -1$

$b_6 = -100, a_{61} = -5, a_{62} = -2, a_{63} = 0, a_{64} = 0, a_{60} = -1$

$b_7 = -25, a_{71} = -3, a_{72} = 5, a_{73} = -10, a_{74} = 2, a_{70} = -1$

Further Example: First Pivot 转动

$$\begin{array}{llrl}
\text{maximize 最大化} & z & = & -x_0 \\
\text{subject to 满足约束} & x_5 & = & -50 - 2x_1 + 8x_2 + 10x_4 + x_0 \\
 & x_6 & = & \boxed{-100} + 5x_1 + 2x_2 + x_0 \\
 & x_7 & = & -25 + 3x_1 - 5x_2 + 10x_3 - 2x_4 + x_0
\end{array}$$

$$\begin{array}{llrl}
\text{maximize 最大化} & z & = & -100 + 5x_1 + 2x_2 - x_6 \\
\text{subject to 满足约束} & x_5 & = & 50 - 7x_1 + 6x_2 + 10x_4 + x_6 \\
 & x_0 & = & 100 - 5x_1 - 2x_2 + x_6 \\
 & x_7 & = & 75 - 2x_1 - 7x_2 + 10x_3 - 2x_4 + x_6
\end{array}$$

Further Example: Find x_e

maximize 最大化 $z = -100 \boxed{+ 5x_1 + 2x_2} - x_6$

subject to 满足约束

$$
\begin{aligned}
x_5 &= 50 - 7x_1 + 6x_2 + 10x_4 + x_6 \\
x_0 &= 100 - 5x_1 - 2x_2 + x_6 \\
x_7 &= 75 - 2x_1 - 7x_2 + 10x_3 - 2x_4 + x_6
\end{aligned}
$$

Further Example: Find x_e and x_ℓ

$$
\begin{array}{lrcl}
\text{maximize 最大化} & z & = & -100 \boxed{+\,5x_1} + 2x_2 \qquad\qquad\qquad\qquad - x_6 \\
\text{subject to 满足约束} & x_5 & = & 50 - 7x_1 + 6x_2 \qquad\qquad + 10x_4 + x_6 \qquad x_1 = \boxed{50/7} + \cdots \\
& x_0 & = & 100 - 5x_1 - 2x_2 \qquad\qquad\qquad\qquad + x_6 \qquad x_1 = 100/5 + \cdots \\
& x_7 & = & 75 - 2x_1 - 7x_2 + 10x_3 - \ 2x_4 + x_6 \qquad x_1 = 75/2 + \cdots
\end{array}
$$

$$
\begin{array}{lrcl}
\text{maximize 最大化} & z & = & -450/7 + 44/7 x_2 \qquad\qquad + 50/7 x_4 - 5/7 x_5 - 2/7 x_6 \\
\text{subject to 满足约束} & x_1 & = & 50/7 + 6/7 x_2 \qquad\qquad + 10/7 x_4 - 1/7 x_5 + 1/7 x_6 \\
& x_0 & = & 450/7 - 44/7 x_2 \qquad\qquad - 50/7 x_4 + 5/7 x_5 + 2/7 x_6 \\
& x_7 & = & 425/7 - 61/7 x_2 + 10x_3 - 34/7 x_4 + 2/7 x_5 + 5/7 x_6
\end{array}
$$

Further Example: Initial Feasible Solution

maximize 最大化 $z = -450/7 + 44/7\,x_2 \boxed{+ 50/7\,x_4} - 5/7\,x_5 - 2/7\,x_6$

subject to 满足约束

$$
\begin{aligned}
x_1 &= 50/7 + 6/7\,x_2 + 10/7\,x_4 - 1/7\,x_5 + 1/7\,x_6 & x_4 &= -50/10 + \cdots \\
x_0 &= 450/7 - 44/7\,x_2 - 50/7\,x_4 + 5/7\,x_5 + 2/7\,x_6 & x_4 &= \boxed{450/50} + \cdots \\
x_7 &= 425/7 - 61/7\,x_2 + 10x_3 - 34/7\,x_4 + 2/7\,x_5 + 5/7\,x_6 & x_4 &= 425/34 + \cdots
\end{aligned}
$$

maximize 最大化 $z = -x_0$

subject to 满足约束

$$
\begin{aligned}
x_1 &= 20 - 1/5\,x_0 - 2/5\,x_2 + 1/5\,x_6 \\
x_4 &= 9 - 7/50\,x_0 - 22/25\,x_2 + 1/10\,x_5 + 1/25\,x_6 \\
x_7 &= 17 + 17/25\,x_0 - 111/25\,x_2 + 10x_3 - 1/5\,x_5 - 13/25\,x_6
\end{aligned}
$$

Further Example: Initial Feasible Solution

maximize 最大化 $\quad z = -x_0$

subject to 满足约束

$$
\begin{aligned}
x_1 &= 20 - \tfrac{1}{5}x_0 - \tfrac{2}{5}x_2 + \tfrac{1}{5}x_6 \\
x_4 &= 9 - \tfrac{7}{50}x_0 - \tfrac{22}{25}x_2 + \tfrac{1}{10}x_5 + \tfrac{1}{25}x_6 \\
x_7 &= 17 + \tfrac{17}{25}x_0 - \tfrac{111}{25}x_2 + 10x_3 - \tfrac{1}{5}x_5 - \tfrac{13}{25}x_6
\end{aligned}
$$

The basic solution is optimal because all constants are ≤ 0.

$x_0 = 0$, $x_1 = 20$, $x_2 = 0$, $x_3 = 0$, $x_4 = 9$, $x_5 = 0$, $x_6 = 0$, $x_7 = 17$ is the found optimal solution of the auxiliary LP.

Because $x_0 = 0$, the original LP is feasible.

Further Example: Restore Original Objective Function

$$
\begin{aligned}
\text{maximize 最大化} \quad & z = -x_0 \\
\text{subject to 满足约束} \quad & x_1 = 20 - \tfrac{1}{5}x_0 - \tfrac{2}{5}x_2 + \tfrac{1}{5}x_6 \\
& x_4 = 9 - \tfrac{7}{50}x_0 - \tfrac{22}{25}x_2 + \tfrac{1}{10}x_5 + \tfrac{1}{25}x_6 \\
& x_7 = 17 + \tfrac{17}{25}x_0 - \tfrac{111}{25}x_2 + 10x_3 - \tfrac{1}{5}x_5 - \tfrac{13}{25}x_6
\end{aligned}
$$

$$
\begin{aligned}
\text{maximize 最大化} \quad & z = -x_1 - x_2 - x_3 - x_4 \\
\text{subject to 满足约束} \quad & x_1 = 20 - \tfrac{2}{5}x_2 + \tfrac{1}{5}x_6 \\
& x_4 = 9 - \tfrac{22}{25}x_2 + \tfrac{1}{10}x_5 + \tfrac{1}{25}x_6 \\
& x_7 = 17 - \tfrac{111}{25}x_2 + 10x_3 - \tfrac{1}{5}x_5 - \tfrac{13}{25}x_6
\end{aligned}
$$

Further Example: Restore Original Objective Function

maximize 最大化 $z = -x_1 - x_2 - x_3 - x_4$

subject to 满足约束

$$
\begin{aligned}
x_1 &= 20 - \tfrac{2}{5}x_2 + \tfrac{1}{5}x_6 \\
x_4 &= 9 - \tfrac{22}{25}x_2 + \tfrac{1}{10}x_5 + \tfrac{1}{25}x_6 \\
x_7 &= 17 - \tfrac{111}{25}x_2 + 10x_3 - \tfrac{1}{5}x_5 - \tfrac{13}{25}x_6
\end{aligned}
$$

maximize 最大化 $z = -29 + \tfrac{7}{25}x_2 - x_3 - \tfrac{1}{10}x_5 - \tfrac{6}{25}x_6$

subject to 满足约束

$$
\begin{aligned}
x_1 &= 20 - \tfrac{2}{5}x_2 + \tfrac{1}{5}x_6 \\
x_4 &= 9 - \tfrac{22}{25}x_2 + \tfrac{1}{10}x_5 + \tfrac{1}{25}x_6 \\
x_7 &= 17 - \tfrac{111}{25}x_2 + 10x_3 - \tfrac{1}{5}x_5 - \tfrac{13}{25}x_6
\end{aligned}
$$

Further Example

maximize 最大化 $z = -29 \boxed{+ \ {}^{7}/_{25}x_2} - x_3 - {}^{1}/_{10}x_5 - {}^{6}/_{25}x_6$

subject to 满足约束

$$
\begin{aligned}
x_1 &= 20 - {}^{2}/_{5}x_2 + {}^{1}/_{5}x_6 & x_2 &= {}^{20\cdot 5}/_{2} + \cdots \\
x_4 &= 9 - {}^{22}/_{25}x_2 + {}^{1}/_{10}x_5 + {}^{1}/_{25}x_6 & x_2 &= {}^{9\cdot 25}/_{22} + \cdots \\
x_7 &= 17 - {}^{111}/_{25}x_2 + 10x_3 - {}^{1}/_{5}x_5 - {}^{13}/_{25}x_6 & x_2 &= \boxed{{}^{17\cdot 25}/_{111}} + \cdots
\end{aligned}
$$

maximize 最大化 $z = {}^{-3100}/_{111} - {}^{41}/_{111}x_3 - {}^{25}/_{222}x_5 - {}^{757}/_{2775}x_6 - {}^{7}/_{111}x_7$

subject to 满足约束

$$
\begin{aligned}
x_1 &= {}^{2050}/_{111} - {}^{100}/_{111}x_3 + {}^{2}/_{111}x_5 + {}^{137}/_{555}x_6 + {}^{10}/_{111}x_7 \\
x_4 &= {}^{625}/_{111} - {}^{220}/_{111}x_3 + {}^{31}/_{222}x_5 + {}^{397}/_{2775}x_6 + {}^{22}/_{111}x_7 \\
x_2 &= {}^{425}/_{111} + {}^{250}/_{111}x_3 - {}^{5}/_{111}x_5 - {}^{13}/_{111}x_6 - {}^{25}/_{111}x_7
\end{aligned}
$$

Further Example: Solution found

maximize 最大化 $\quad z = {}^{-3100}/_{111} - {}^{41}/_{111}x_3 - {}^{25}/_{222}x_5 - {}^{757}/_{2775}x_6 - {}^{7}/_{111}x_7$

subject to 满足约束

$$
\begin{aligned}
x_1 &= {}^{2050}/_{111} - {}^{100}/_{111}x_3 + {}^{2}/_{111}x_5 + {}^{137}/_{555}x_6 + {}^{10}/_{111}x_7 \\
x_4 &= {}^{625}/_{111} - {}^{220}/_{111}x_3 + {}^{31}/_{222}x_5 + {}^{397}/_{2775}x_6 + {}^{22}/_{111}x_7 \\
x_2 &= {}^{425}/_{111} + {}^{250}/_{111}x_3 - {}^{5}/_{111}x_5 - {}^{13}/_{111}x_6 - {}^{25}/_{111}x_7
\end{aligned}
$$

The basic solution is optimal because all constants are ≤ 0.

$x_1 = {}^{2050}/_{111}$, $x_2 = {}^{425}/_{111}$, $x_3 = 0$, $x_4 = {}^{625}/_{111}$,
$x_5 = 0$, $x_6 = 0$, $x_7 = 0$
is the found optimal solution of the LP.

Further Example: Solution found

minimize 最小化 $x_1 + x_2 + x_3 + x_4$

subject to 满足约束

$$\begin{aligned}
-2x_1 + 8x_2 \qquad\qquad + 10x_4 &\geq 50 \\
5x_1 + 2x_2 \qquad\qquad\qquad\quad &\geq 100 \\
3x_1 - 5x_2 + 10x_3 - 2x_4 &\geq 25 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
x_3 &\geq 0 \\
x_4 &\geq 0
\end{aligned}$$

$x_1 = {}^{2050}/_{111}$, $x_2 = {}^{425}/_{111}$, $x_3 = 0$, $x_4 = {}^{625}/_{111}$,
$x_5 = 0$, $x_6 = 0$, $x_7 = 0$
is the found optimal solution of the LP.

Open Question: Exponential-Time?

- The known current worst-case time complexity of SIMPLEX is exponential.
- In practice, often SIMPLEX is efficient — much faster than other algorithms that have smaller worst-case time complexity.
- Sometimes, there are multiple possible choices for the leaving and entering variable. Which one is best?
➥ researchers try to find better ways to select variables,
that are more resistant against slow examples and do not allow cycles.

Quiz: How to solve a linear program

- First bring the LP into _________ form.
- Is the initial basic solution f_______?
  If no, solve an a_______ LP first, and then solve the original LP.
- To solve an LP, bring it into _____ form.
- Repeat _____ steps until the o_______ function has form: ______________.
- In every step, choose which variables to swap based on:
  - First choose the e_______ variable; it should satisfy: _________________.
  - Then choose the l_______ variable; it should satisfy: _________________.

Summary

- Linear Programs are a general class of optimization problems. (They include minimum spanning tree, shortest path and maximum flow.)
- The Simplex algorithm solves linear programs, i.e. finds an optimal solution.
- Its basic idea is to start at any corner of the feasible polytope and move along its edges until the best corner is found.
- The Simplex algorithm is exponential in the worst case but is often fast in practice.

Example: Break the cycle

maximize 最大化 $\boxed{x_2} - 5.5x_3 + 0.75x_4 - 5.75x_5$

subject to 满足约束

$$x_1 = \boxed{-0.5x_2} + 3.5x_3 + 0.5x_4 - 2.5x_5$$
$$x_6 = -2.5x_2 + 19.5x_3 + 3.5x_4 - 19.5x_5$$

maximize 最大化 $-2x_1 + 1.5x_3 + 1.75x_4 - 10.75x_5$

subject to 满足约束

$$x_2 = -2x_1 + 7x_3 + x_4 - 5x_5$$
$$x_6 = 5x_1 + 2x_3 + x_4 - 7x_5$$

Example: Break the cycle

maximize 最大化 $-2x_1 + 1.5x_3 \boxed{+ 1.75x_4} - 10.75x_5$

subject to 满足约束

$$x_2 = -2x_1 + 7x_3 + x_4 - 5x_5$$
$$x_6 = 5x_1 + 2x_3 + x_4 - 7x_5$$

All $a_{i4} \leq 0$, so the solution is unbounded.

$x_1 = 0$, $x_2 = \infty$, $x_3 = 0$, $x_4 = \infty$, $x_5 = 0$, $x_6 = \infty$ is the found optimal solution of the LP.