Lecture 29: Greedy Algorithms II

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Huffman encoding

- Suppose we wish to compress a file. There are n kinds of characters in the file, each character has a given frequency.
- For example: there are 6 kinds of characters labeled $\mathrm{a, b, c, d, e, f}$. They have frequency 45, 13, 12, 16, 9, 5, respectively.
- Assign a binary code (a unique binary string) for each character, in order to minimize the total size of coding for the file.

Huffman encoding: example

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

- **Fixed-length codeword:** assign each letter to the same number of bits.
  - With the fixed-length encoding above, the string $\mathbf{abd}$ is encoded as $\mathbf{000001011}$.
- **Variable-length codeword:** assign each letter a codeword with possibly different number of bits.
  - With the variable-length encoding above, the string $\mathbf{abd}$ is encoded as $\mathbf{0101111}$.

Prefix code

- **Prefix code:** a code is a prefix code if no codeword is a prefix of another. This simplifies decoding of a file.
- Example: given the prefix code

$$\mathrm{a} \rightarrow 0, \mathrm{b} \rightarrow 101, \mathrm{c} \rightarrow 100, \mathrm{d} \rightarrow 111, \mathrm{e} \rightarrow 1101, \mathrm{f} \rightarrow 1100$$

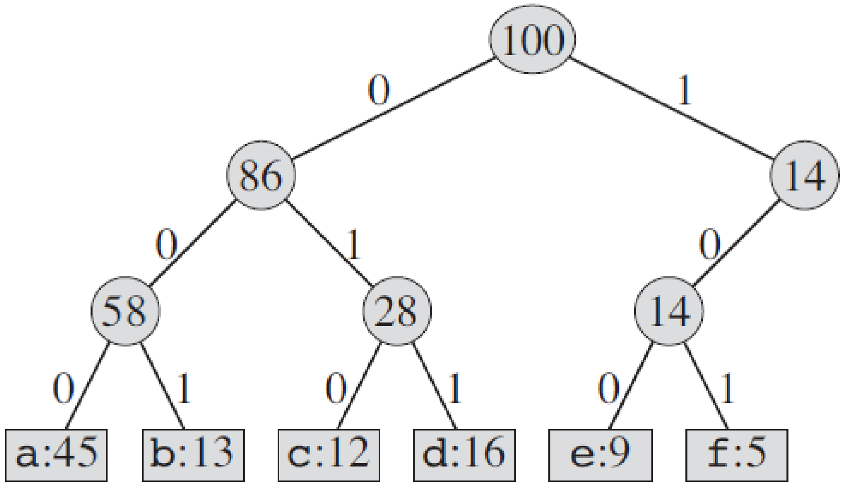
- The codeword

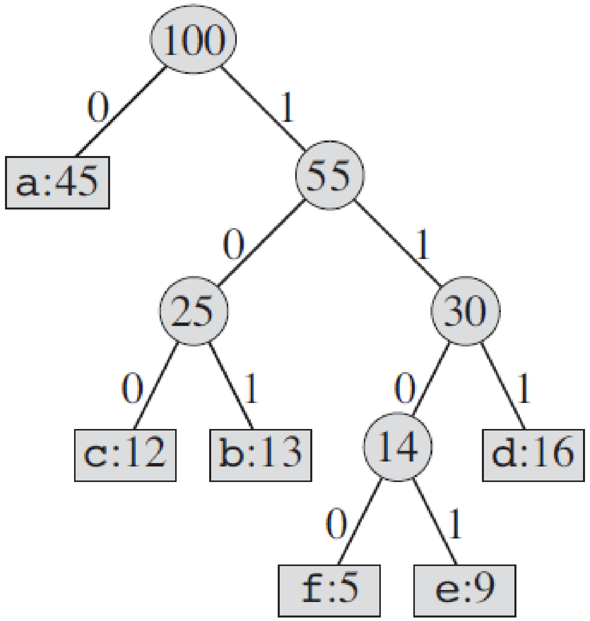
$$001011101$$

can be easily decoded as aabe.

Prefix code as binary trees

- Represent each codeword as a path from root to leaf of the tree.
- Example: the fixed-length code $\mathrm{a} \rightarrow \mathbf{000}, \mathrm{b} \rightarrow \mathbf{001} \ldots$ on the left, the variable-length code $\mathrm{a} \rightarrow \mathbf{0}, \mathrm{b} \rightarrow \mathbf{101} \ldots$ on the right.





Cost of a binary tree

- Cost of a binary tree computes the average depth weighted by frequency:

$$B(T) = \sum_{c \in C} c.freq \cdot d_T(c)$$

- This corresponds to the size of the compressed file.

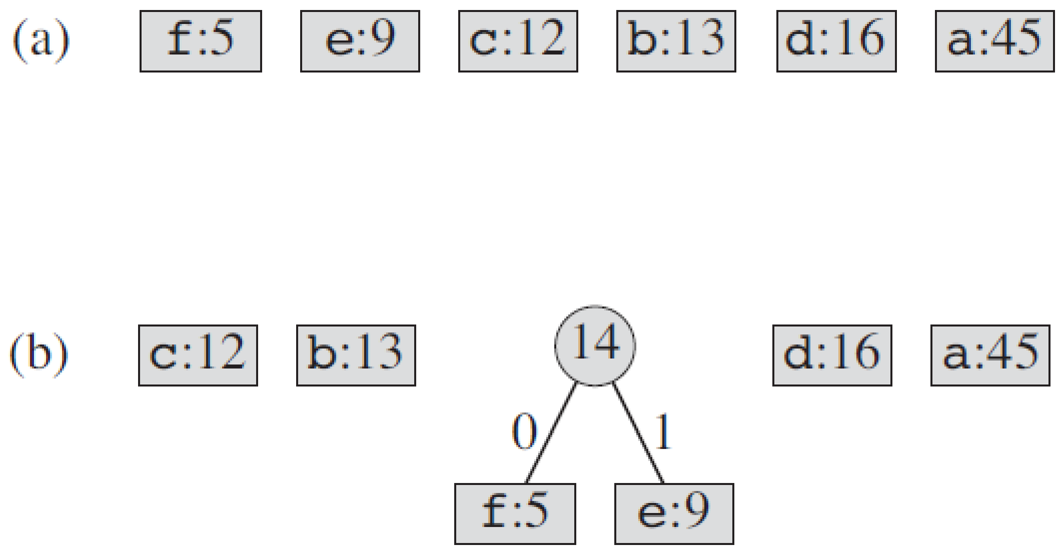
Greedy algorithm for optimal encoding

- **Huffman encoding:** start from the leaves of the tree labeled by frequencies, gradually build the tree by combining nodes. Each time, combine the two nodes with smallest total frequency.

```
HUFFMAN(C)
n = |C|
Q = C
for i = 1 to n - 1
    allocate a new node z
    z.left = x = EXTRACT-MIN(Q)
    z.right = y = EXTRACT-MIN(Q)
    z.freq = x.freq + y.freq
    INSERT(Q, z)
return EXTRACT-MIN(Q)      // return the root of the tree
```

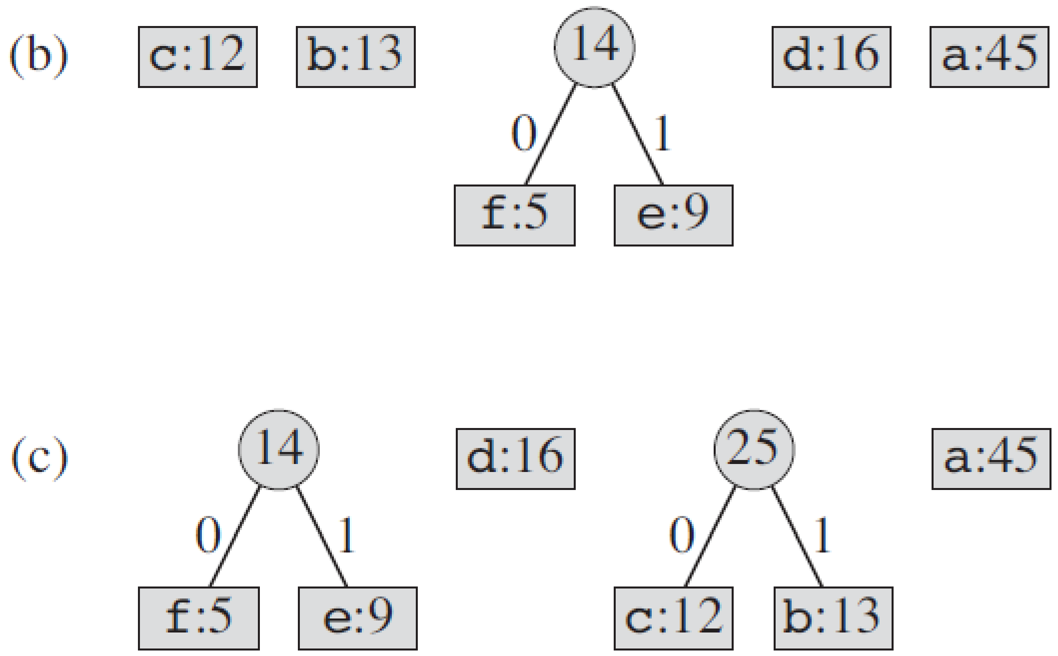
Huffman encoding: example

- **Step 1:** $\{f\}$ and $\{e\}$ are the smallest, with frequency 5 and 9, respectively. They are combined to form node with frequency 14.



Huffman encoding: example

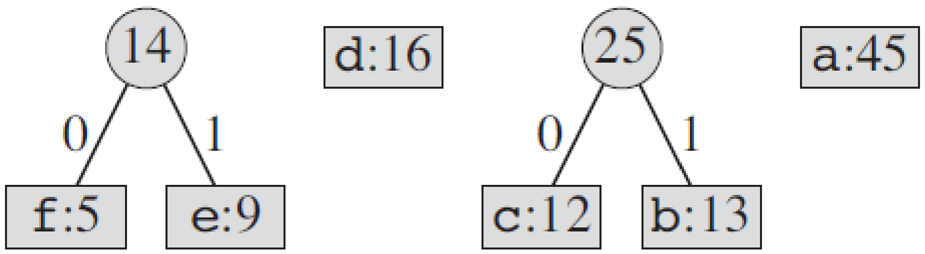
- **Step 2:** $\{b\}$ and $\{c\}$ are now the smallest, with weight 12 and 13, respectively. They are combined to form node with frequency 25.



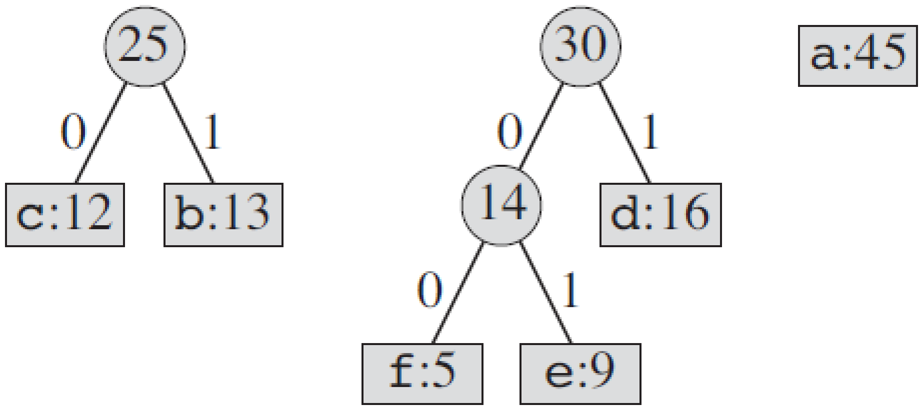
Huffman encoding: example

- **Step 3:** $\{e, f\}$ and $\{d\}$ are now the smallest, with frequency 14 and 16, respectively.

(c)

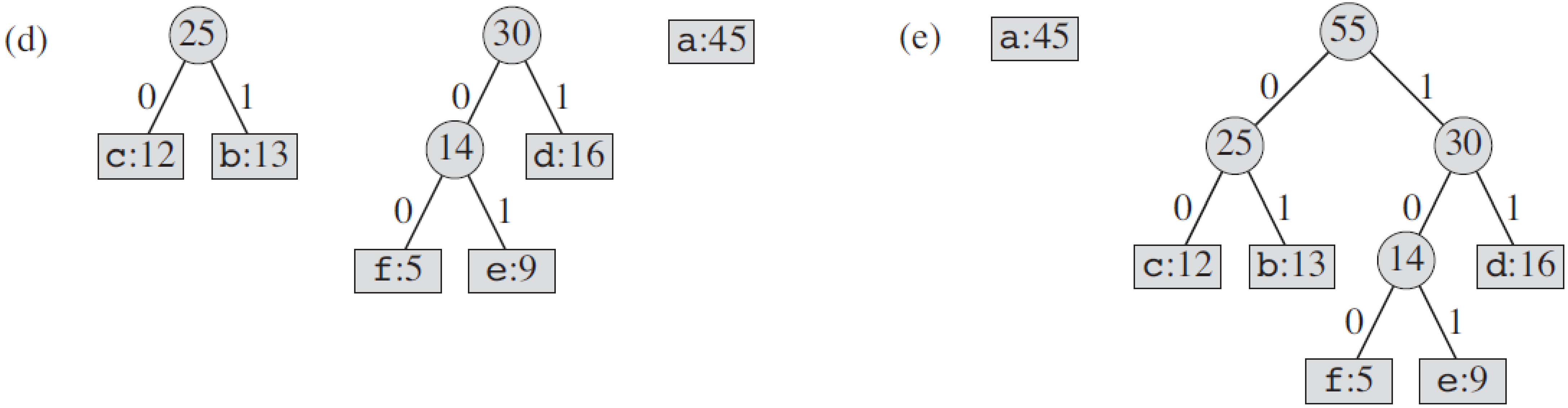


(d)



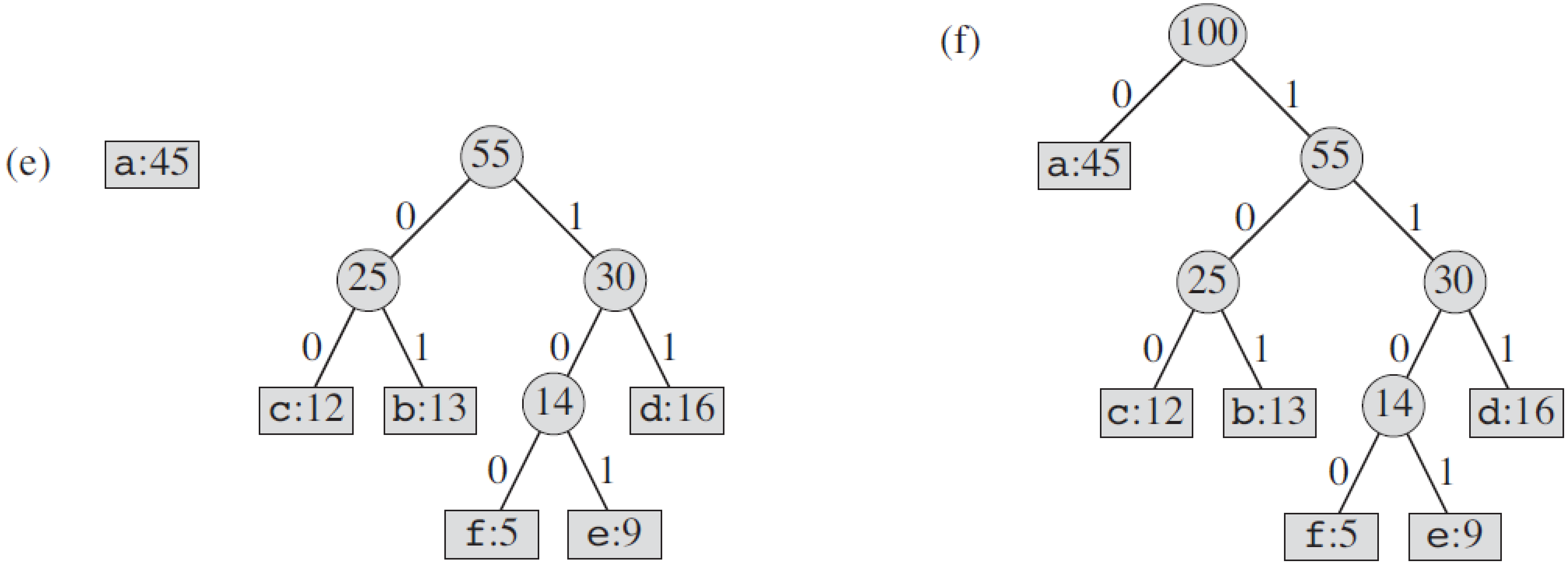
Huffman encoding: example

- **Step 4:** The nodes $\{b, c\}$ and $\{d, e, f\}$ are the smallest, with frequency 25 and 30, respectively.



Huffman encoding: example

- **Step 5:** finally, combine the node with {a}.



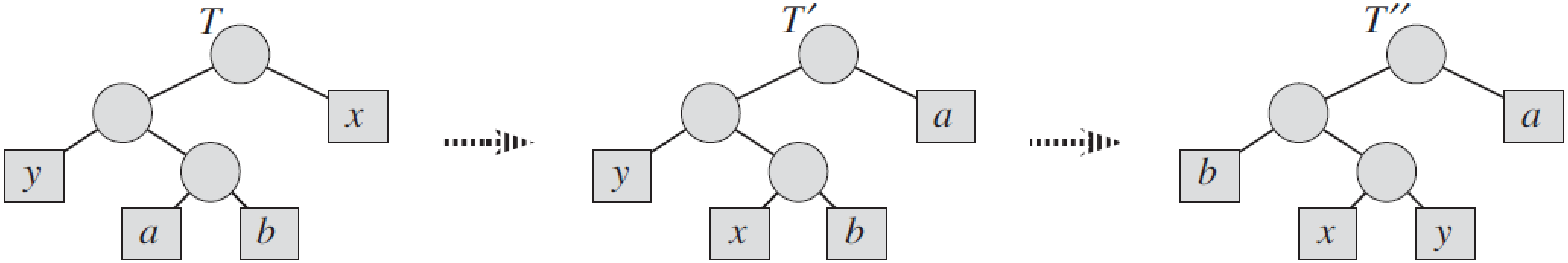
Why is greedy correct?

The proof proceeds by the following two steps:

1. **Greedy-choice property:** it is always optimal to put the two nodes with lowest frequencies together in a single subtree with maximum depth. (Lemma 16.2)
2. **Optimal substructure:** the problem can be reduced to the case after merging the two nodes with lowest frequency into a single node. (Lemma 16.3).

Greedy-choice property (Lemma 16.2)

- Suppose a and b are nodes with maximum depth, x and y are nodes with lowest frequencies. Assuming $x \neq b$, swap a with x and b with y to yield a tree that has equal or lower cost.



- We get:

$$B(T) - B(T') = (a.\text{freq} - x.\text{freq})\big(d_T(a) - d_T(x)\big) \geq 0,$$

and similarly $B(T') - B(T'') \geq 0$.

Optimal substructure (Lemma 16.3)

- Consider removing the two characters with lowest frequency x and y, replacing with a new character z:

$$C' = C - \{x, y\} \cup \{z\},$$

$$z.\mathrm{freq} = x.\mathrm{freq} + y.\mathrm{freq}.$$

- Let T' be any tree representing an optimal prefix code for the alphabet C'. Then the tree T, obtained from T' by replacing the leaf node for z with a subtree containing x and y represents an optimal prefix code for the alphabet C.
- Key idea: $B(T)$ and $B(T')$ are related by

$$B(T) = B(T') + x.\mathrm{freq} + y.\mathrm{freq}.$$

Final Theorem

- Huffman encoding algorithm produces an optimal prefix code.
- **Proof by induction:** the first step is optimal (greedy-choice property). The remaining steps can be viewed as carried out on the reduced tree T' and alphabet C', and are optimal by induction hypothesis.