NP Completeness IV

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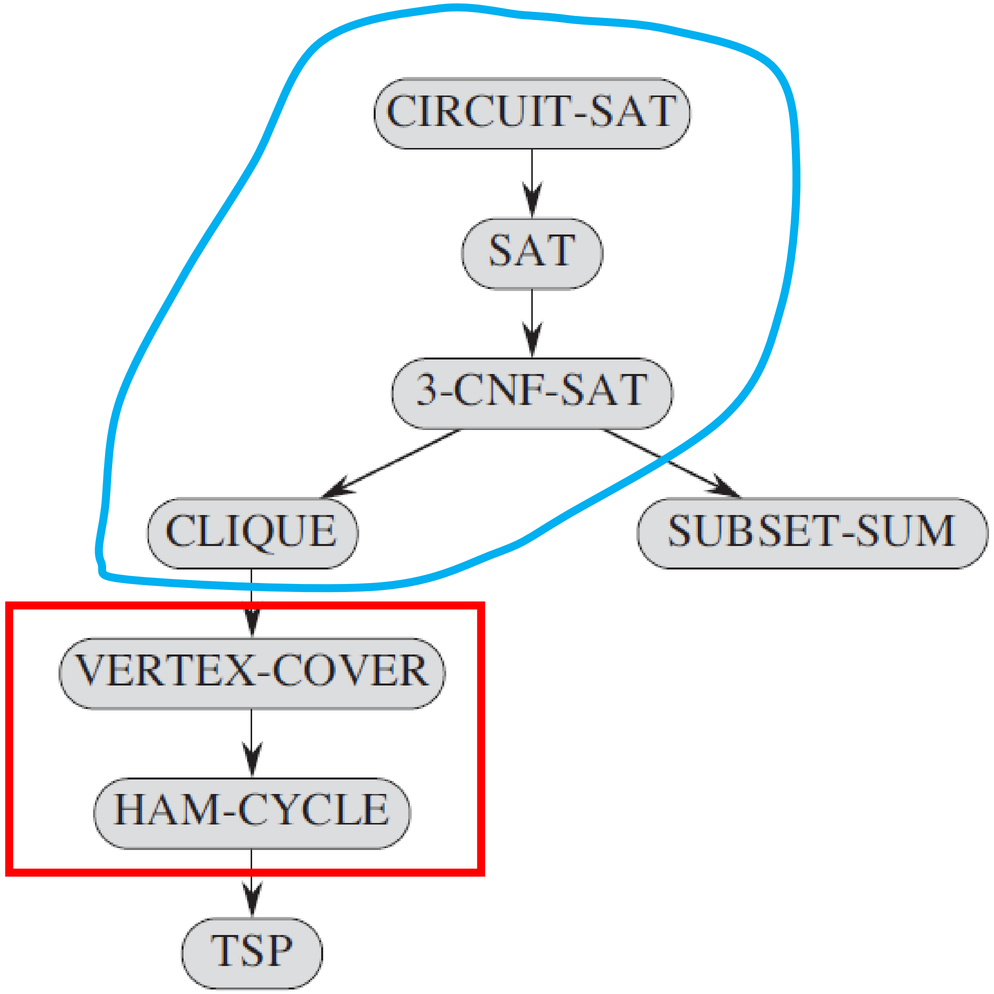
Review: NP-completeness

- We defined complexity classes P, NP, and NP-complete:
  - **Class P:** problems that can be solved in polynomial time.
  - **Class NP:** problems whose (positive) solutions can be verified in polynomial time.
  - **NP-complete:** problems that are the hardest in NP (according to the reduction relation). These problems will not have polynomial solutions unless P = NP.
- We sketched the proof that circuit-satisfiability is NP-complete.
- We showed use of reduction to prove boolean satisfiability, 3-CNF satisfiability, and clique problems are NP-complete.

More NP-complete problems

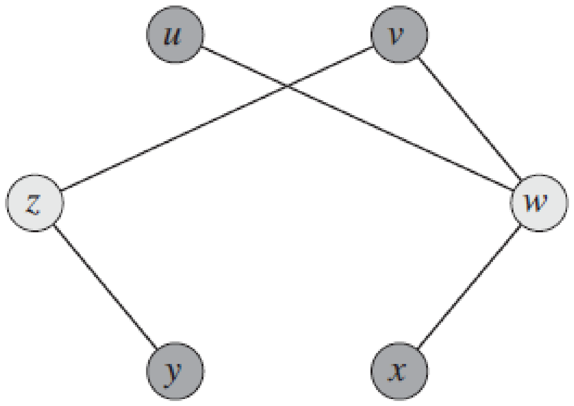
In this lecture, we show:

- **Vertex cover** is NP-complete, by reducing from clique problem.
- **Hamiltonian cycle** is NP-complete, by reducing from vertex cover.



Vertex cover problem

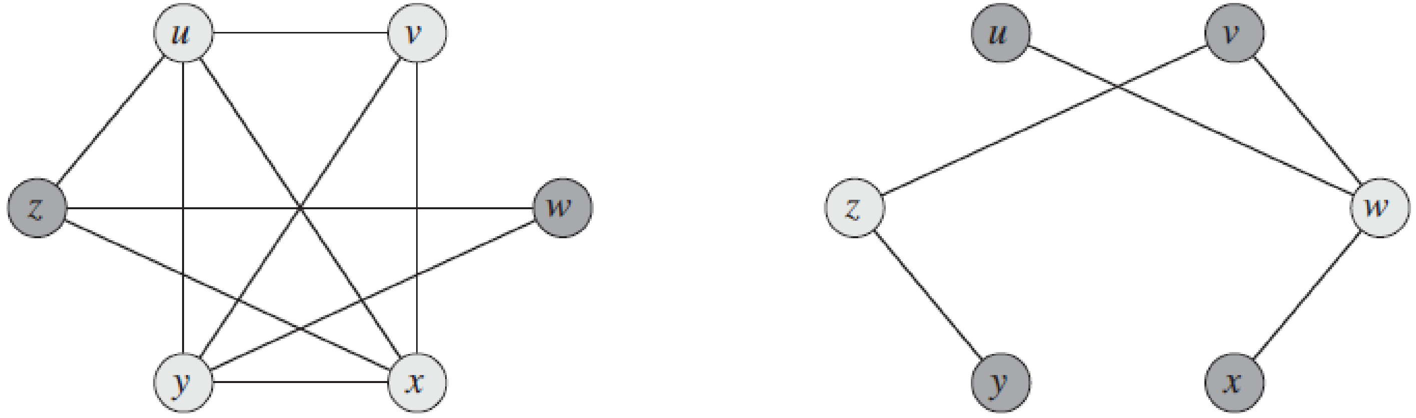
- A vertex cover of an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ such that each edge in E has at least one vertex in V'.
- In the graph below, there is a vertex cover of size 2 ($V' = \{z, w\}$).



- **Vertex cover problem:** given graph G and integer k, determine whether there exists a vertex cover of G with size k.

Vertex cover problem is NP-complete

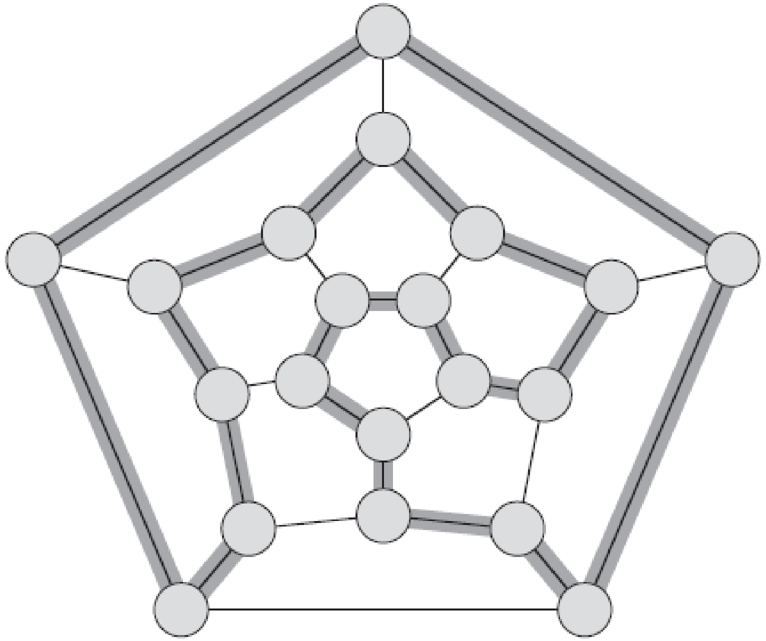
- We show this by reduction from the clique problem.



- Given an instance of clique problem (left), obtain its complement (right). A set of vertices $V' \subseteq V$ is a vertex cover if and only if $V - V'$ is a clique in the original graph.
- In this example, $\{z, w\}$ is a vertex cover, and $\{u, v, y, x\}$ is a clique in the original graph.

Hamiltonian cycle

- Recall the **Hamiltonian cycle** problem: given graph G, does there exist a cycle in G that passes through each vertex exactly once?
- Proof of NP-completeness of Hamiltonian cycle is more complex than what we have seen before.
- Reduction from vertex cover.



Outline of proof

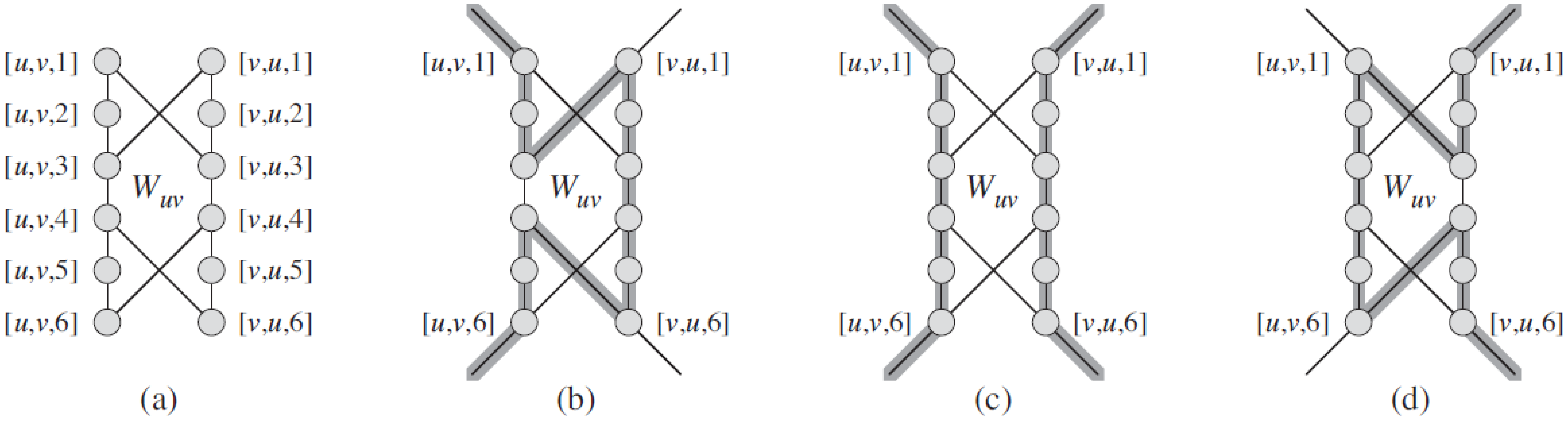
Recall what has to be done to reduce vertex cover to Hamiltonian cycle:

- Given any graph G and integer k, construct a new graph G' such that G' has a Hamiltonian cycle if and only if G has a vertex cover of size k.
- We make use of **widgets**: each edge in G is translated into a widget, satisfying certain properties related to Hamiltonian cycles.

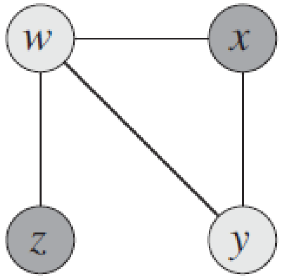
Widget

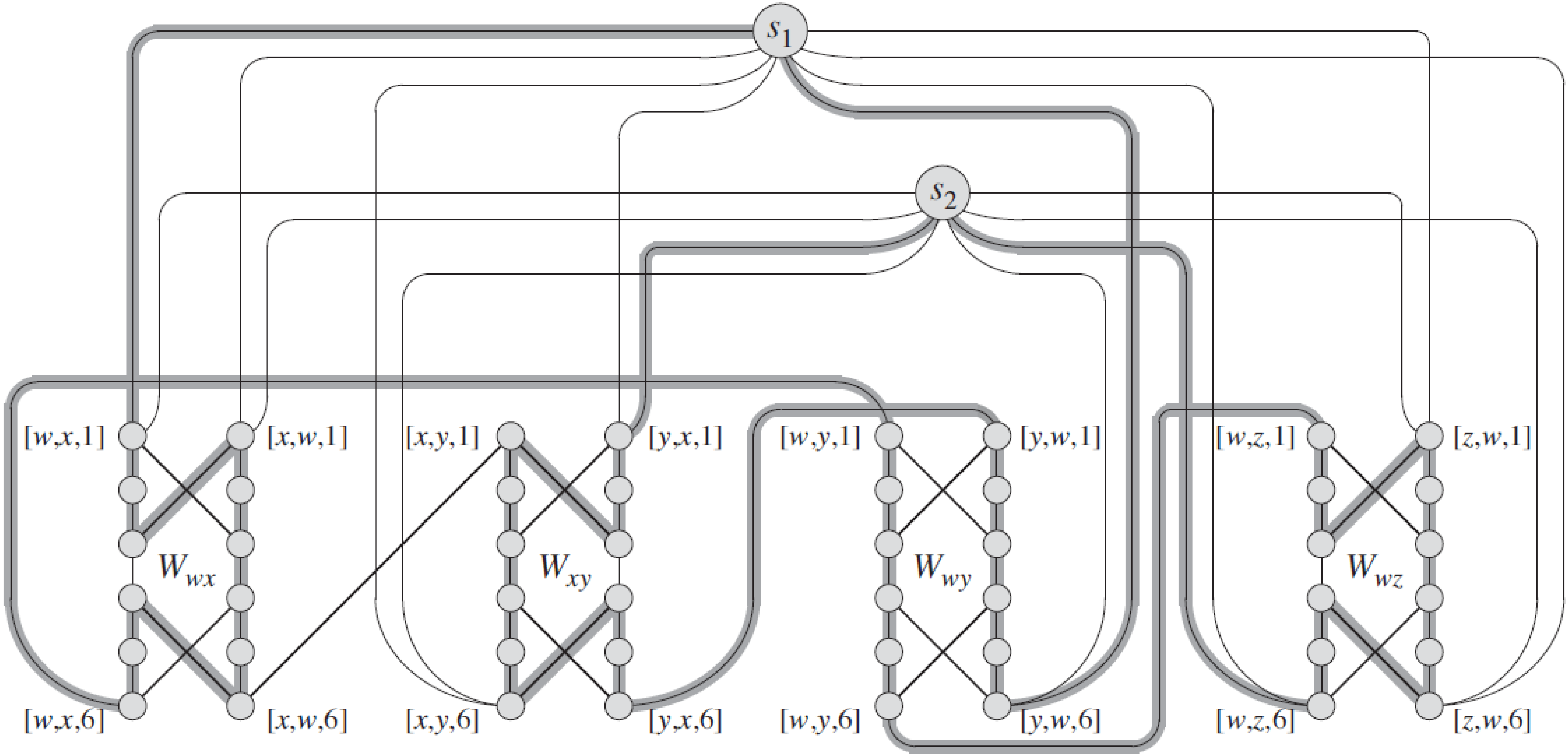
Small piece of graph that is used in the reduction.

- (a): the widget corresponding to edge (u, v).
- (b, c, d): three ways to traverse the widget in a Hamiltonian cycle.

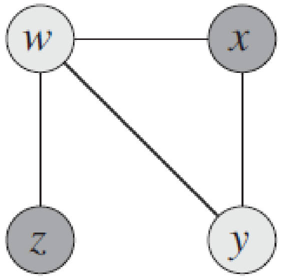


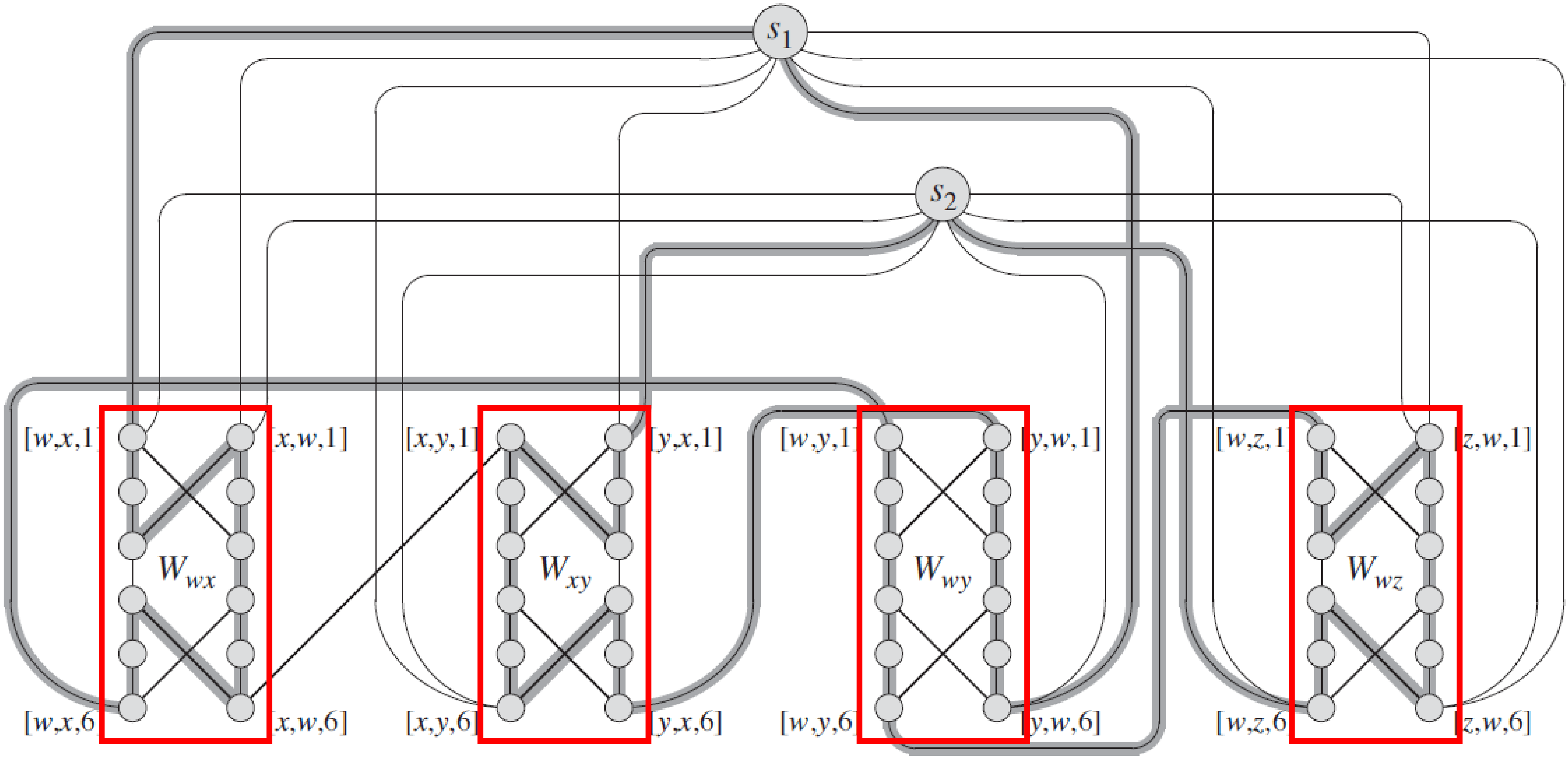
Full translation of graph



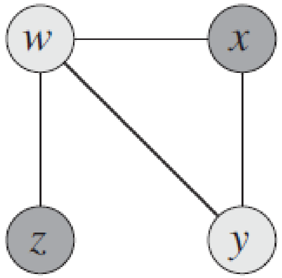


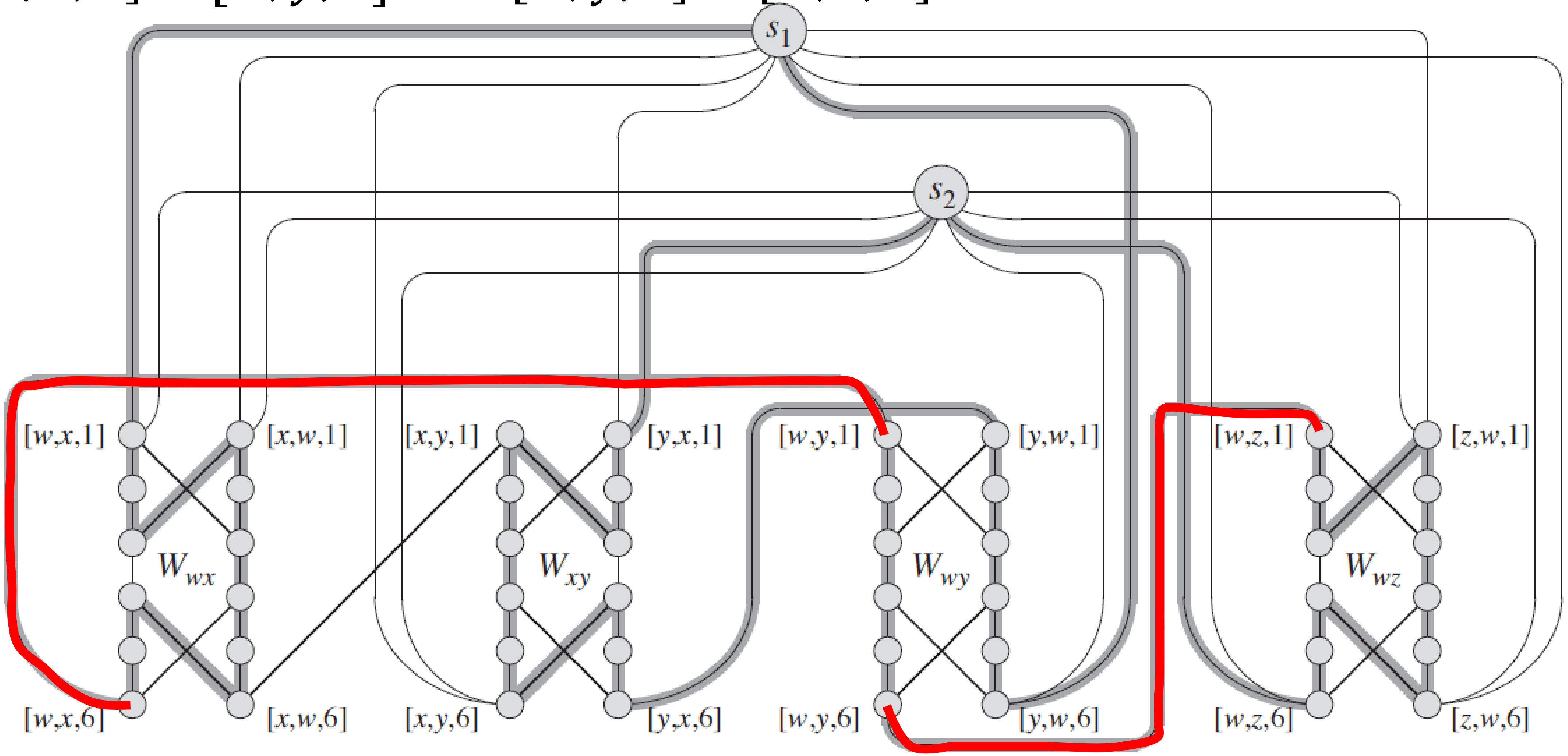
Step 1: Translate each edge into a widget. Here we have four edges: (w, x), (x, y), (w, y), (w, z).



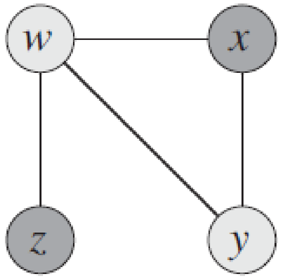


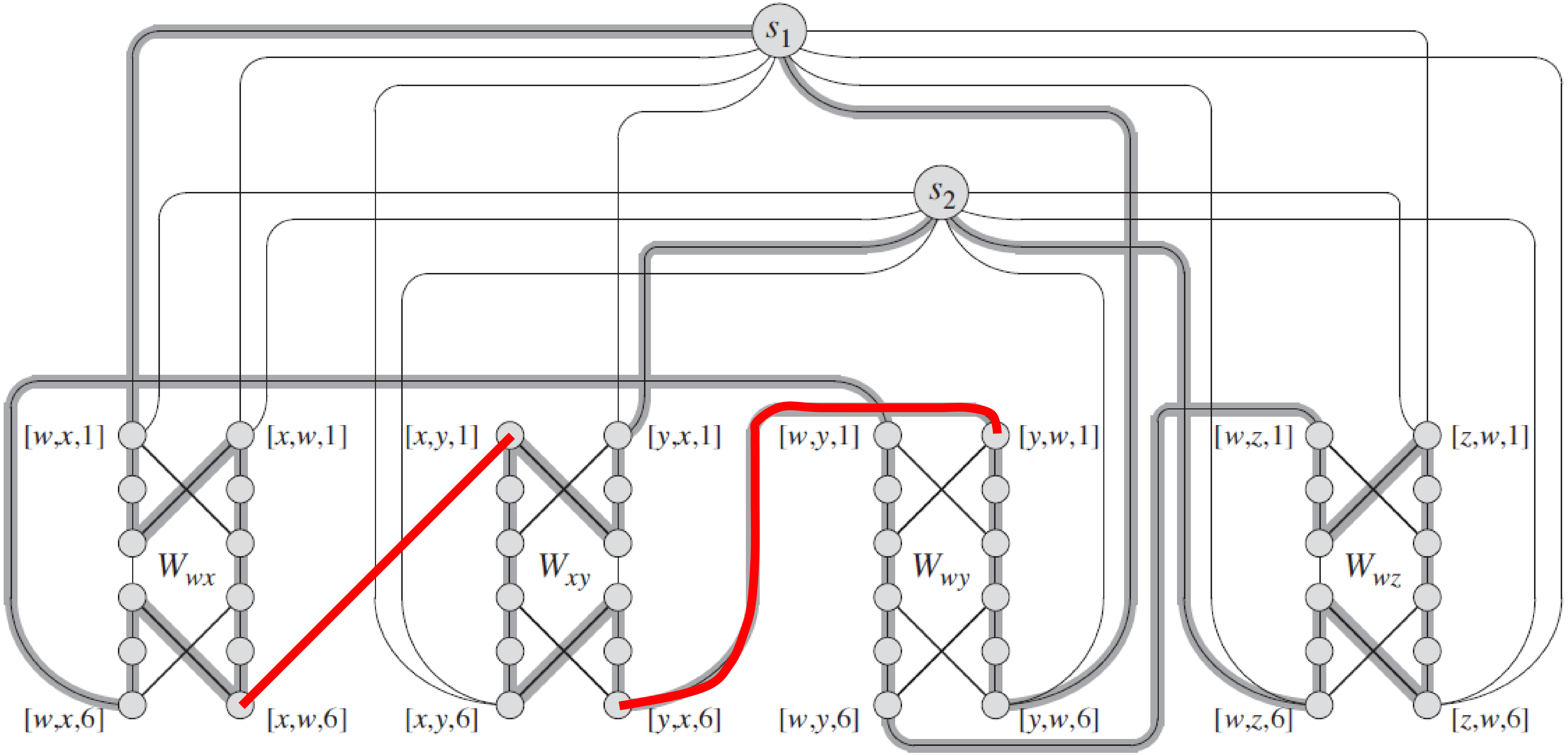
Step 2: For each vertex, order its neighbors and connect according to this order. For example: w has neighbors x, y, z. So connect vertices $[w, x, 6] - [w, y, 1]$ and $[w, y, 6] - [w, z, 1]$.



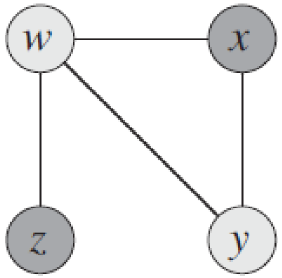


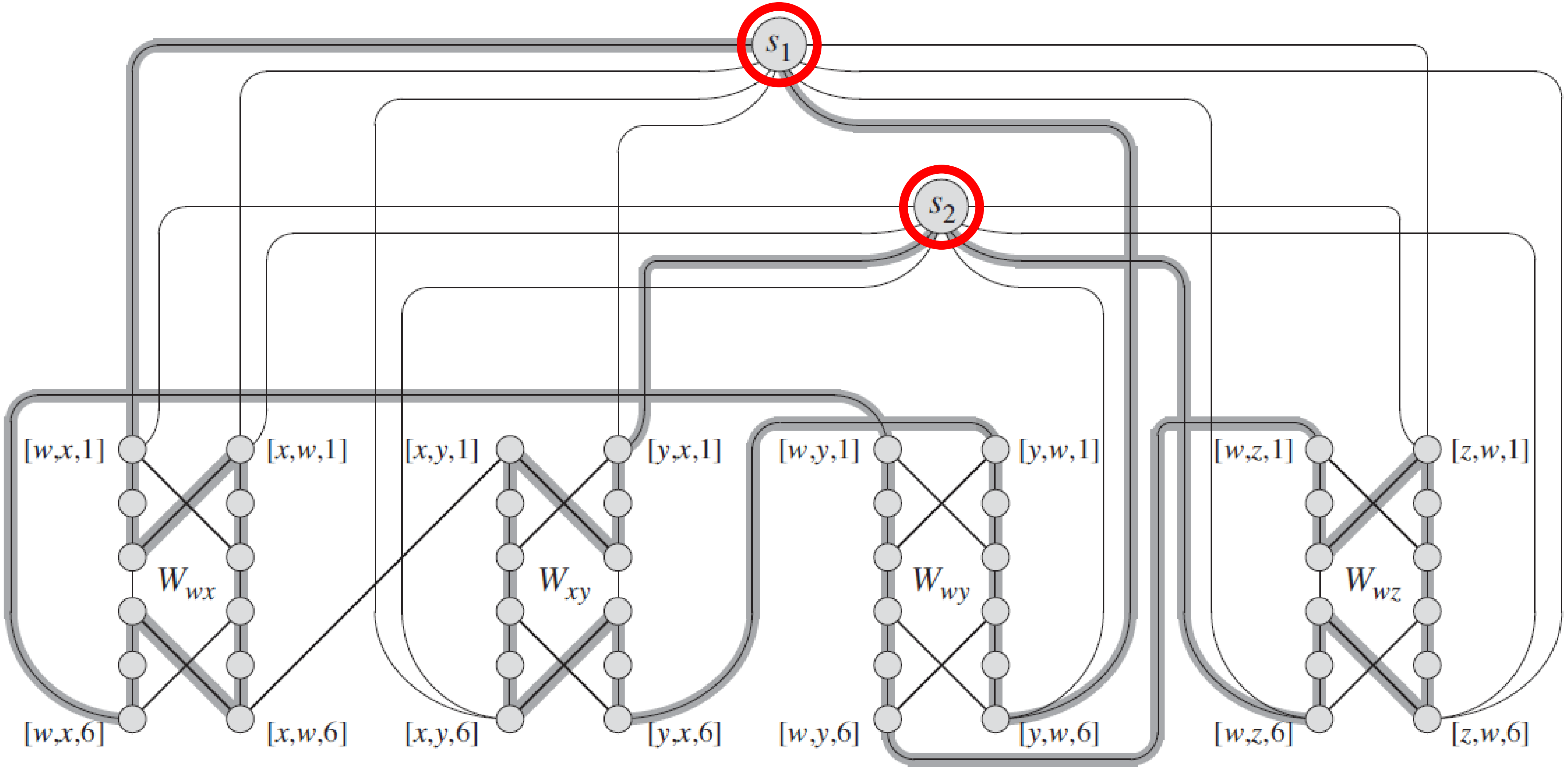
Step 2: Also, order neighbors of x as w, y, so connect $[x, w, 6] - [x, y, 1]$. Order neighbors of y as x, w, so connect $[y, x, 6] - [y, w, 1]$.



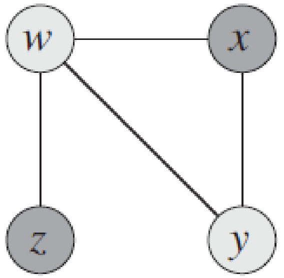


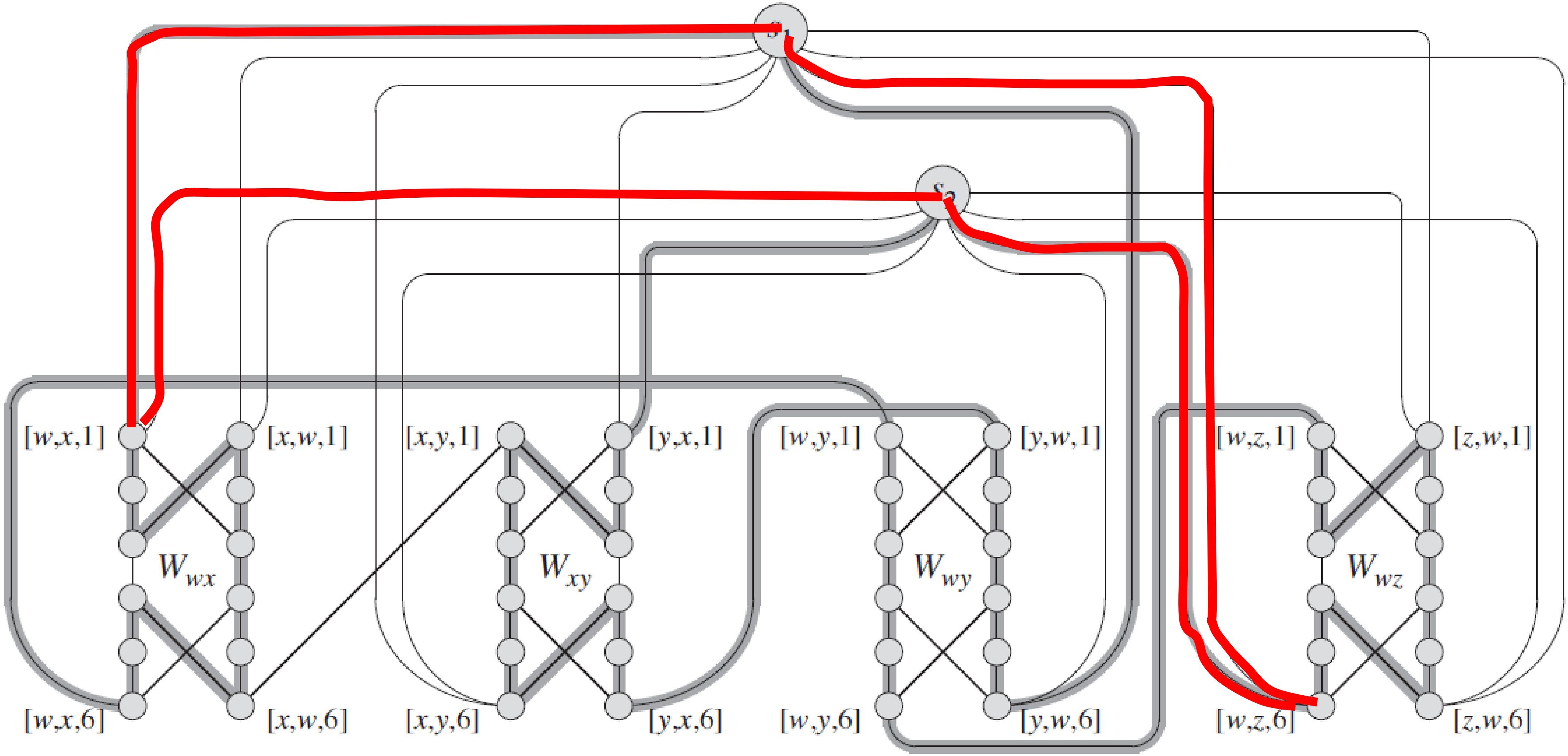
Step 3: Create vertices $s_1, \ldots, s_k$ (recall k is the required number of vertices in the cover).



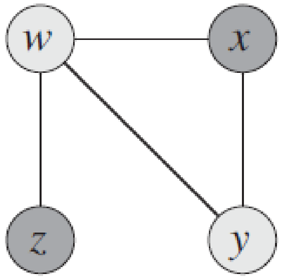


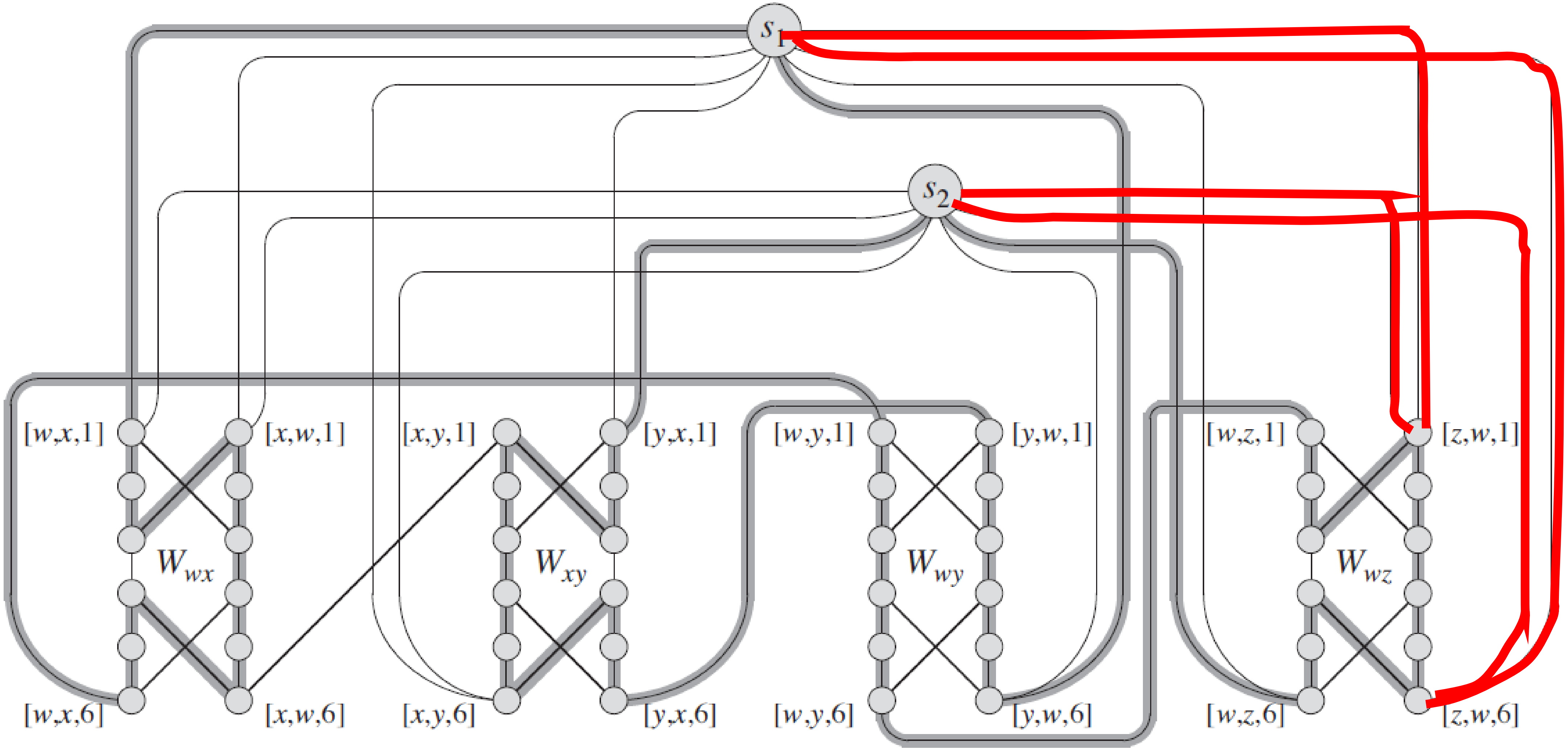
Step 4: For each s_i and each vertex in G, make connections to starting and ending neighbor. For example, w has neighbors x, y, z, so connect each s_i to $[w, x, 1]$ and $[w, z, 6]$.





Step 4: Do the same for x, y, z. Since z has only one neighbor w, connect each s_i to $[z, w, 1]$ and $[z, w, 6]$.

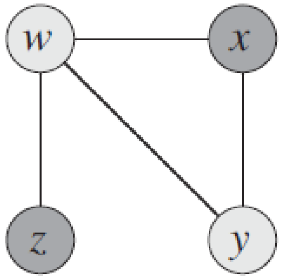


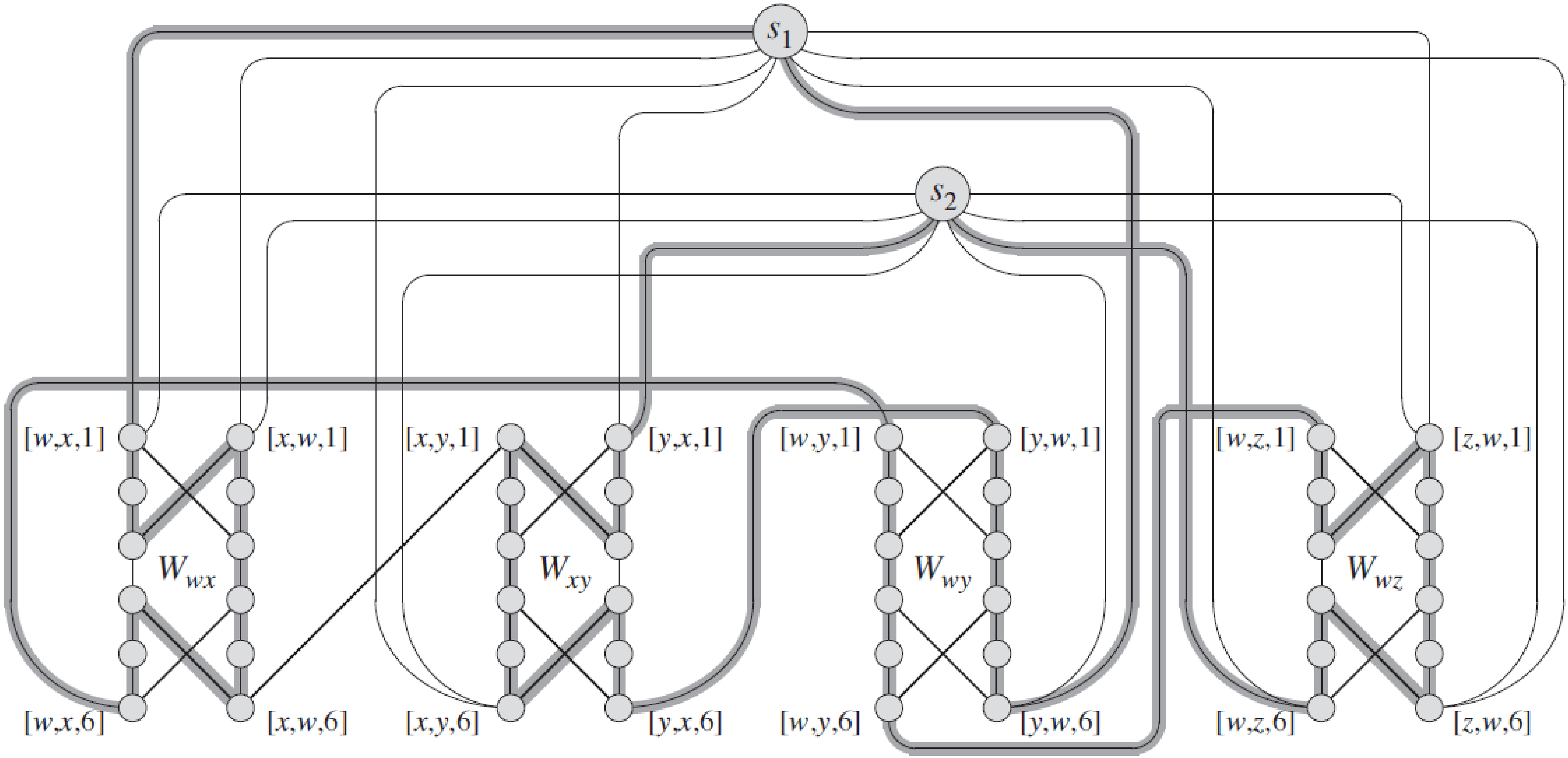


Intuition

- Each s_i corresponds to one of the vertices in the vertex cover.
- For each s_i, suppose it corresponds to vertex v, which has neighbors $w_1, \dots, w_j$, take the path $s_i - [v, w_1, 1] - [v, w_1, 6] - [v, w_2, 1] - [v, w_2, 6] - \cdots - [v, w_j, 1] - [v, w_j, 6] - s_{i+1}$.
- This ensures that the widgets corresponding to all edges (v, w_i) are reached in the Hamiltonian cycle.
- For each edge (v, w), depending on whether v, w or both are in the cover, use one of the three ways to traverse the widget.

The cycle below corresponds to choosing $\{w, y\}$ as the vertex cover. Let $s_1 \to w$ and $s_2 \to y$. Note (w, y) is covered twice, other edges once.





Remainder of proof

1. Show G' can be constructed in polynomial time from G and k (in particular size of G' is polynomial in size of G.
2. Show each vertex cover of G with size k corresponds to a Hamiltonian cycle in G'.
3. Show any Hamiltonian cycle of G' corresponds to a vertex cover of G with size k (start by analyzing different ways to traverse each widget).

Conclusion: Hamiltonian cycle is NP-complete.