Math for CS 2015/2019 Problem Set 7 solutions

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1 Problem 1

Let R and S be transitive binary relations on the same set, A. Which of the following new relations must also be transitive? For each part, justify your answer with a brief argument if the new relation is transitive and a counterexample if it is not.

(I will think of binary relations in terms of sets, like in Chapter 4.4 "Binary Relations". So a binary relation R on A is a subset of $A \times A$. A binary relation is very similar to a function, but not necessarily single-valued. Same input can have multiple different outputs.)

1.1 (a) R^{-1}

Let's clarify: R^{-1} is just like function inversion. So the meaning of $aR^{-1}b$ is that bRa, in other words the pair $(a, b) \in A \times A$ is a member of $R^{-1} \subseteq A \times A$ iff the pair (b, a) is a member of $R \subseteq A \times A$. (Defined on page 92, Chapter 4.4.5)

Proof. 1. Assume R is transitive. We want to prove that R^{-1} is transitive.

2. Assume $a, b, c \in A$ and assume $aR^{-1}b$ and $bR^{-1}c$. We want to prove $aR^{-1}c$.

3. By definition of R^{-1}, from (2) we have bRa and cRb.

4. Since R is transitive, by (3) we have cRa.

5. By definition of R^{-1} we have $aR^{-1}c$. So R^{-1} is transitive. □

1.2 (b) $R \cap S$

Let's clarify: relations are similar to functions, but they are sets, so here we have set intersection. Both R and S are subsets of $A \times A$. So $R \cap S \subseteq A \times A$. This means that the pair $(a, b) \in A \times A$ is a member of $R \cap S$ iff it is a member of both R and of S. We will switch back and forth between the two notations aRb and $(a, b) \in R$ for convenience, they mean the same thing.

Proof. 1. Assume R, S are transitive. We want to prove that $R \cap S$ is transitive.

2. Assume $a, b, c \in A$ and assume $(a, b) \in (R \cap S)$ and $(b, c) \in (R \cap S)$. We want to prove $(a, c) \in (R \cap S)$.

3. By definition of $R \cap S$, from (2) we have $(a, b) \in R$, $(a, b) \in S$, $(b, c) \in R$ and $(b, c) \in S$.

4. By (3) we have aRb and bRc; and aSb and bSc.

5. Since R is transitive, by (4) we have aRc.

6. Since S is transitive, by (4) we have aSc.

7. By (5) and (6) we have $(a, c) \in (R \cap S)$. So $R \cap S$ is transitive. □

1.3 (c) $R \circ R$

Let's clarify: here $\circ$ is function composition. So the meaning of $a(R \circ R)b$ is that there exists some $c \in A$ such that aRc and cRb.

Proof. 1. Assume R is transitive. We want to prove that $R \circ R$ is transitive.

2. Assume $a, b, c \in A$ and assume $a(R \circ R)b$ and $b(R \circ R)c$. We want to prove $a(R \circ R)c$.

3. By definition of $R \circ R$, from (2) we have:

there exists $d_1 \in A$ such that aRd_1 and d_1Rb, and

there exists $d_2 \in A$ such that bRd_2 and d_2Rc.

4. By (3) aRd_1 and d_1Rb, and since R is transitive, we have aRb.

5. By (3) bRd_2 and d_2Rc, and since R is transitive, we have bRc.

6. So there exists $b \in A$ such that aRb and bRc, which proves $a(R \circ R)c$. So $R \circ R$ is transitive. □

1.4 (d) $R \circ S$

Let's clarify: here $\circ$ is function composition. So the meaning of $a(R \circ S)b$ is that there exists some $c \in A$ such that aRc and cSb.

Proof. This one is false. We will give a counterexample of the set A and two transitive binary relations R, S on A, where $R \circ S$ is not transitive. (Drawing a picture would be much easier and clearer, but it's too difficult to do that here in LaTeX.)

1. Define $A = \{a, b, c, d\}$.

2. Define a binary relation R on A by $R = \{(a, c), (b, d), (c, c), (d, d)\}$ (or, in other words, aRc, bRd, cRc, dRd).

3. Define a binary relation S on A by $S = \{(a, a), (b, b), (c, b), (d, b), (d, c)\}$ (or, in other words, aSa, bSb, cSb, dSb, dSc).

4. By (3) and (4), we have the composed relation

$$R \circ S = \{(a, b), (b, b), (b, c), (c, b), (d, b), (d, c)\}$$

(or, in other words, $a(R \circ S)b, b(R \circ S)b, b(R \circ S)c, c(R \circ S)b, d(R \circ S)b, d(R \circ S)c$).

5. Notice that R is transitive: we have aRc and cRc and aRc; bRd and dRd and dRd.

6. Notice that S is transitive: we have cSb and bSb and cSb; dSb and bSb and dSb; dSc and cSb and dSb.

7. Notice that $R \circ S$ is not transitive. We have $a(R \circ S)b$ and $b(R \circ S)c$ but not $a(R \circ S)c$. □

2 Problem 2

Let R_1 and R_2 be two equivalence relations on a set, A. Prove or give a counterexample to the claims that the following are also equivalence relations:

2.1 (a) $R_1 \cap R_2$

Proof. 1. Assume R_1 and R_2 are equiv. rel.s on A. Want to prove $R_1 \cap R_2$ is an equiv. rel. on A.

2. To prove reflexivity, assume $a \in A$. We want to show $(a, a) \in R_1 \cap R_2$. Since R_1 is reflexive, $(a, a) \in R_1$. Since R_2 is reflexive, $(a, a) \in R_2$. Therefore $(a, a) \in R_1 \cap R_2$. So $R_1 \cap R_2$ is reflexive.

3. To prove symmetry, assume $a, b \in A$ and assume $(a, b) \in R_1 \cap R_2$. We want to prove $(b, a) \in R_1 \cap R_2$. Since R_1 is symmetric, $(b, a) \in R_1$. Since R_2 is symmetric, $(b, a) \in R_2$. Therefore $(b, a) \in R_1 \cap R_2$. So $R_1 \cap R_2$ is symmetric.

4. To prove transitivity, assume $a, b, c \in A$ and assume $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$. We want to prove $(a, c) \in R_1 \cap R_2$. Since R_1 is transitive, $(a, c) \in R_1$. Since R_2 is transitive, $(a, c) \in R_2$. Therefore $(a, c) \in R_1 \cap R_2$. So $R_1 \cap R_2$ is transitive.

5. By (2), (3) and (4) $R_1 \cap R_2$ is an equivalence relation. □

2.2 (b) $R_1 \cup R_2$

Proof. This one is false. We will give a counterexample of a set A with two equivalence relations R_1, R_2 on A where $R_1 \cup R_2$ is not an equivalence relation. (Drawing a picture would be much easier and clearer, but it's too difficult to do that here in LaTeX.)

1. Define $A = \{a, b, c\}$.

2. Define a binary relation R_1 on A by $R_1 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$.

3. Define a binary relation R_2 on A by $R_2 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$.

4. By (3) and (4), we have the union relation

$$R_1 \cup R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$$

5. Notice that R_1 is an equivalence relation:

it is reflexive since it contains all 3 of $(a, a), (b, b), (c, c)$;

it is symmetric because it contains (a, b) and (b, a);

it is transitive because it contains $(a, b), (b, a), (a, a)$, and $(b, a), (a, b), (b, b)$.

6. Notice that R_2 is an equivalence relation:

it is reflexive since it contains all 3 of $(a, a), (b, b), (c, c)$;

it is symmetric because it contains (b, c) and (c, b);

it is transitive because it contains $(b, c), (c, b), (b, b)$, and $(c, b), (b, c), (c, c)$.

7. Notice that $R \cup S$ is not transitive. We have $(a, b) \in R \cup S$ and $(b, c) \in R \cup S$ but not $(a, c) \in R \cup S$. □

3 Problem 3

Determine which among the four graphs pictured in Figure 1 are isomorphic. For each pair of isomorphic graphs, describe an isomorphism between them. For each pair of graphs that are not isomorphic, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, prove that it is indeed preserved under isomorphism (you only need prove one of them).

Proof. 1. Node degrees are preserved under isomorphism. G_3 is not isomorphic to G_1, G_2, G_4. Because G_3 has two nodes with degree 4 (namely, nodes number 8 and 10), but all nodes of G_1, G_2, G_4 have degree 3.

2. Cycle lengths are preserved under isomorphism. G_2 is not isomorphic to G_1 and G_4. Because G_2 has many cycles of length 4 (for example $1 \to 6 \to 9 \to 5 \to 1$), but C_1 and C_4 have no cycles of length 4.

3. So that leaves only G_1 and G_4. Here is an isomorphism $f : G_1 \to G_4$ (taken from "ps4-sol.pdf" in 2010 solutions, you should prove why it's an isomorphism):

$f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 8, f(5) = 9,$

$f(6) = 10, f(7) = 4, f(8) = 5, f(9) = 6, f(10) = 7.$ □

4 Problem 4

Let's say that a graph has "two ends" if it has exactly two vertices of degree 1 and all its other vertices have degree 2. For example, here is one such graph:

4.1 (a)

A line graph is a graph whose vertices can be listed in a sequence with edges between consecutive vertices only. So the two-ended graph above is also a line graph of length 4.

Prove that the following theorem is false by drawing a counterexample.

False Theorem. Every two-ended graph is a line graph.

Proof. I cannot draw pictures here, but think of a graph with 5 nodes. The first two nodes are connected to each other, so they are the "two ends" with degree 1. Separately from these two, the other 3 nodes form a triangle among themselves, so they all have degree 2. Clearly this is not a line graph. □

4.2 (b)

Point out the first erroneous statement in the following bogus proof of the false theorem and describe the error.

Bogus proof. We use induction. The induction hypothesis is that every two-ended graph with n edges is a path.

Base case ($n = 1$): The only two-ended graph with a single edge consists of two vertices joined by an edge: Sure enough, this is a line graph.

Inductive case: We assume that the induction hypothesis holds for some $n \geq 1$ and prove that it holds for $n+1$.

Let G_n be any two-ended graph with n edges. By the induction assumption, G_n is a line graph.

Now suppose that we create a two-ended graph G_{n+1} by adding one more edge to G_n. This can be done in only one way: the new edge must join an endpoint of G_n to a new vertex; otherwise, G_{n+1} would not be two-ended.

Clearly, G_{n+1} is also a line graph. Therefore, the induction hypothesis holds for all graphs with $n+1$ edges, which completes the proof by induction.

Proof. The error is in "Therefore, the induction hypothesis holds for all graphs with $n+1$ edges".

We haven't proved that the induction hypothesis holds for all graphs with $n+1$ edges. We only proved it for a specific graph with $n+1$ edges.

We started with an arbitrarily chosen two-ended graph G_n with n edges. By the induction hypothesis this is a line graph.

Then we added an edge to this graph to obtain a two-ended graph G_{n+1}. And we argued G_{n+1} must also be a line-graph (because there is no other way to add an edge to G_n to get a two-ended graph).

But G_{n+1} is not an arbitrarily chosen two-ended graph with $n+1$ edges!

There could be many other two-ended graphs with $n+1$ edges out there, which cannot be obtained by adding an edge to a two-ended graph with n edges.

First we would have to prove that every two-ended graph with $n+1$ edges HAS TO BE obtained by adding an edge to a two-ended graph with n edges. I don't think that's true though.

The statements leading up to "Therefore..." are technically correct, but they are not useful or relevant.

An attempt at a correct proof would go like this (even though we know a correct proof does not exist):

Inductive case: We assume that the induction hypothesis holds for some $n \geq 1$ and prove that it holds for $n+1$.

Let G_{n+1} be any two-ended graph with $n+1$ edges. (This is how you CORRECTLY use an arbitrarily chosen $n+1$-edge graph.)

(...remove one edge from G_{n+1} SOMEHOW, to obtain a two-ended graph G_n.)

(...use the induction hypothesis to show that G_n is a line graph.)

(...then add back the removed edge, use the fact that G_n is a line graph SOMEHOW to prove that G_{n+1} is also a line graph.) □