Math for CS 2015/2019 solutions to “In-Class Problems Week 8, Mon. (Session 18)”

https://github.com/spamegg1

October 31, 2022

Contents

1 Problem 1

For each of the binary relations below, state whether it is a strict partial order, a weak partial order, an equivalence relation, or none of these. If it is a partial order, state whether it is a linear order. If it is none, indicate which of the axioms for partial-order and equivalence relations it violates.

Weak partial order: reflexive, antisymmetric, transitive.

Strict partial order: irreflexive, asymmetric, transitive.

Equivalence relation: reflexive, symmetric, transitive.

Linear order: partial order (weak or strict) with trichotomy.

1.1 (a)

The superset relation $\supseteq$ on the power set $pow\{1,2,3,4,5\}$.

Proof. It is reflexive: $a \supseteq a$ is true for all $a \in pow\{1,2,3,4,5\}$.

It is transitive: for all $a, b, c \in pow\{1,2,3,4,5\}$, if $a \supseteq b$ and $b \supseteq c$ then $a \supseteq c$ is true.

It is antisymmetric: for all $a, b \in pow\{1,2,3,4,5\}$, if $a \neq b$ and $a \supseteq b$, then $a \supset b$ and therefore $b \not\supseteq a$.

This is a WEAK PARTIAL ORDER! $\square$

1.2 (b)

The relation between any two nonnegative integers a and b such that $a \equiv b \mod 8$.

Proof. It is reflexive: for all nonnegative integers a, $a \equiv a \mod 8$ is true.

It is symmetric: for all nonnegative integers a and b, if $a \equiv b \mod 8$, then 8 divides $(a-b)$, so 8 also divides $(b-a)$, therefore $b \equiv a \mod 8$.

It is transitive: for all nonnegative integers a, b and c, if $a \equiv b \mod 8$ and $b \equiv c \mod 8$, then 8 divides both $(a-b)$ and $(b-c)$, so 8 also divides their sum $(a-b)+(b-c)=(a-c)$, therefore $a \equiv c \mod 8$.

This is an EQUIVALENCE RELATION! $\square$

1.3 (c)

The relation between propositional formulas G and H such that [G IMPLIES H] is valid.

Proof. It is reflexive: for all propositional formulas G, G IMPLIES G is valid.

Below, let P, Q be propositional variables, and let TRUE denote the formula P IFF P, and let FALSE denote the formula P IFF NOT(P). So TRUE is true under every assignment, and FALSE is false under every assignment.

It is not symmetric: FALSE IMPLIES TRUE is valid, but TRUE IMPLIES FALSE is not valid.

It is not asymmetric because it's reflexive.

It is not antisymmetric: (P IFF P) IMPLIES (Q IFF Q) is valid, but (Q IFF Q) IMPLIES (P IFF P) is also valid!

It is transitive: for all propositional formulas F, G, H, if F IMPLIES G is valid, and G IMPLIES H is valid, then F IMPLIES H is also valid.

This is NOT A PARTIAL ORDER! □

1.4 (d)

The relation between propositional formulas G and H such that [G IFF H] is valid.

Proof. It is reflexive: for all propositional formulas G, G IFF G is valid.

It is symmetric: for all propositional formulas G, H, if G IFF H is valid, then H IFF G is also valid.

It is transitive: for all propositional formulas F, G, H, if F IFF G is valid, and G IFF H is valid, then F IFF H is also valid.

This is an EQUIVALENCE RELATION! □

1.5 (e)

The relation 'beats' on Rock, Paper, and Scissors (for those who don't know the game Rock, Paper, Scissors, Rock beats Scissors, Scissors beats Paper, and Paper beats Rock).

Proof. It is irreflexive: for all $x \in \{R, P, S\}$ x does not beat x.

It is antisymmetric: for all $x, y \in \{R, P, S\}$ if x beats y, then y does not beat x.

It is not transitive: Rock beats Scissors, and Scissors beats Paper, but Rock does not beat Paper.

This is NOT A PARTIAL ORDER! □

1.6 (f)

The empty relation on the set of real numbers.

Proof. It's irreflexive, asymmetric, transitive.

This is a STRICT PARTIAL ORDER! □

1.7 (g)

The identity relation on the set of integers.

Proof. It is reflexive: for all integers x, $x = x$ is true.

It is symmetric: for all integers x, y, if $x = y$ then $y = x$.

It is transitive: for all integers x, y, z, if $x = y$ and $y = z$ then $x = z$.

This is an EQUIVALENCE RELATION! □

1.8 (h)

The divisibility relation on the integers, $\mathbb{Z}$.

Proof. It is reflexive: for all $x \in \mathbb{Z}$, x divides x. (Yes this is true even when $x = 0$.)

It is not symmetric: 3 divides 6, but 6 does not divide 3.

It is not asymmetric because it's reflexive.

It is not antisymmetric: 2 divides -2, and -2 divides 2.

This is NOT A PARTIAL ORDER! □

2 Problem 2

The proper subset relation, $\subset$, defines a strict partial order on the subsets of $[1..6]$, that is, on $pow([1..6])$.

2.1 (a)

What is the size of a maximal chain in this partial order? Describe one.

Proof. It is 7:

$$\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \{1, 2, 3, 4\} \subset \{1, 2, 3, 4, 5\} \subset \{1, 2, 3, 4, 5, 6\}$$

□

2.2 (b)

Describe the largest antichain you can find in this partial order.

Proof. Subsets of [1..6] that have the same size form antichains. There are:

1 subset of [1..6] with 0 elements,

6 subsets of [1..6] with 1 elements,

15 subsets of [1..6] with 2 elements,

21 subsets of [1..6] with 3 elements,

15 subsets of [1..6] with 4 elements,

6 subsets of [1..6] with 5 elements,

1 subset of [1..6] with 6 elements.

So the largest antichain is formed by the 21 3-element subsets of [1..6]. □

2.3 (c)

What are the maximal and minimal elements? Are they maximum and minimum?

Proof. The maximal and maximum element is $\{1, 2, 3, 4, 5, 6\}$; the minimal and minimum element is $\emptyset$. □

2.4 (d)

Answer the previous part for the partial order on the set $pow([1..6]) - \emptyset$.

Proof. Now the maximal/maximum element is still $\{1, 2, 3, 4, 5, 6\}$; however there is no minimum element. Instead there are 6 minimal elements: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$. □

3 Problem 3

Let S be a sequence of n different numbers. A subsequence of S is a sequence that can be obtained by deleting elements of S.

For example, if

$$S = (6, 4, 7, 9, 1, 2, 5, 3, 8)$$

Then 647 and 7253 are both subsequences of S (for readability, we have dropped the parentheses and commas in sequences, so 647 abbreviates (6, 4, 7), for example).

An increasing subsequence of S is a subsequence of whose successive elements get larger. For example, 1238 is an increasing subsequence of S. Decreasing subsequences are defined similarly; 641 is a decreasing subsequence of S.

3.1 (a)

List all the maximum-length increasing subsequences of S, and all the maximum-length decreasing subsequences.

Proof. Maximum length increasing subsequences: 1258 1238

Maximum length decreasing subsequences: 641 642 643 653 753 953 $\square$

Now let A be the set of numbers in S. (So A is the integers [1..9] for the example above.) There are two straightforward linear orders for A. The first is numerical order where A is ordered by the $<$ relation. The second is to order the elements by which comes first in S; call this order $<_S$. So for the example above, we would have

$$6 <_S 4 <_S 7 <_S 9 <_S 1 <_S 2 <_S 5 <_S 3 <_S 8$$

Let $\prec$ be the product relation of the linear orders $<_s$ and $<$. That is, is defined by the rule

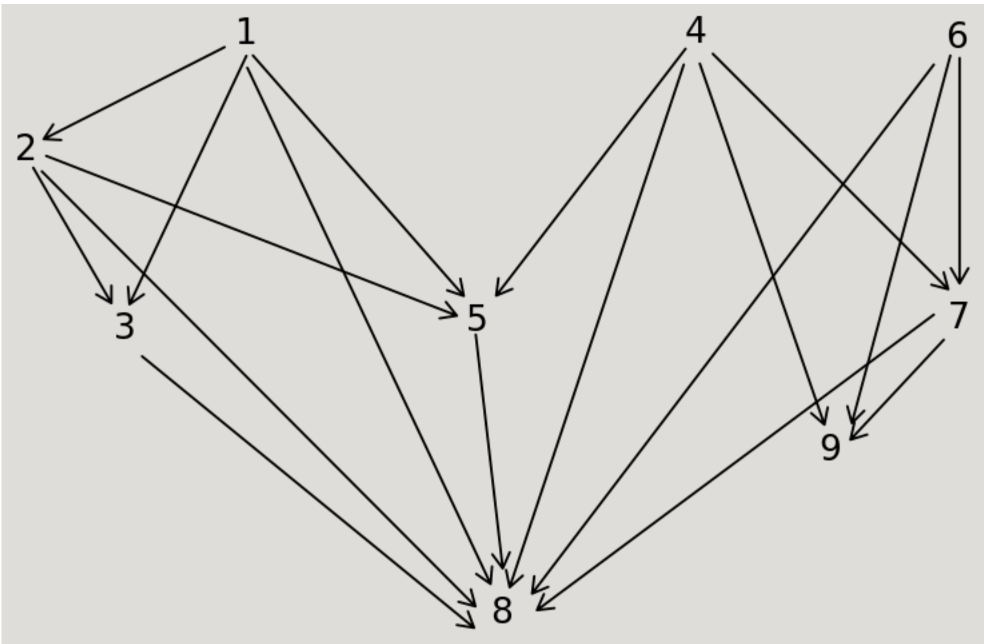
$$a \prec a' ::= a < a' \text{ AND } a <_S a'$$

So $\prec$ is a partial order on A (Section 9.9 in the course textbook).

3.2 (b)

Draw a diagram of the partial order, $\prec$, on A. What are the maximal and minimal elements?

Proof. 8 and 9 are maximal elements. 1, 4, 6 are the minimal elements. $\square$



3.3 (c)

Explain the connection between increasing and decreasing subsequences of S, and chains and antichains under $\prec$.

Proof. An increasing subsequence of S is a chain of $\prec$.

A decreasing subsequence of S is an antichain of $\prec$. □

3.4 (d)

Prove that every sequence, S, of length n has an increasing subsequence of length greater than $\sqrt{n}$ or a decreasing subsequence of length at least $\sqrt{n}$.

Proof. Immediately follows from Dilworth's Lemma and part (c). □

4 Problem 4

For any total function $f : A \to B$ define a relation $\equiv_f$ by the rule:

$$a \equiv_f a' \text{ iff } f(a) = f(a')$$

4.1 (a)

Observe (and sketch a proof) that $\equiv_f$ is an equivalence relation on A.

Proof. 1. For all $a \in A$, we have $f(a) = f(a)$, therefore $a \equiv_f a$. So $\equiv_f$ is reflexive.

2. Assume $a, b \in A$ and $a \equiv_f b$. Then by definition $f(a) = f(b)$. Then $f(b) = f(a)$. Therefore $b \equiv_f a$. So $\equiv_f$ is symmetric.

3. Assume $a, b, c \in A$ and $a \equiv_f b$ and $b \equiv_f c$. Then $f(a) = f(b)$ and $f(b) = f(c)$. Therefore $f(a) = f(c)$. So $a \equiv_f c$. So $\equiv_f$ is transitive.

4. By (1), (2), (3) $\equiv_f$ is an equivalence relation. □

4.2 (b)

Prove that every equivalence relation, R, on a set, A, is equal to $\equiv_f$ for the function $f : A \to \mathit{pow}(A)$ defined as

$$f(a) ::= \{a' \in A \mid aRa'\}$$

That is, $f(a) = R(a)$.

Proof. 1. Assume R is an equivalence relation on A. We want to show R is equal to $\equiv_f$ for the function f defined in the problem.

2. Assume $a, b \in A$. We want to show aRb IFF $a \equiv_f b$.

3. Assume aRb. We want to show $f(a) = f(b)$.
4. To show $f(a) = f(b)$, we will show $f(a) \subseteq f(b)$ and $f(b) \subseteq f(a)$.
5. To show $f(a) \subseteq f(b)$, assume $x \in f(a)$. We want to show $x \in f(b)$.
6. By definition $f(a) = \{a' \in A \mid aRa'\}$. So aRx.
7. Since R is symmetric, xRa.
8. Since aRb and R is transitive, xRb.
9. Again by symmetry, bRx.
10. By definition $f(b) = \{a' \in A \mid bRa'\}$. So $x \in f(b)$. This proves $f(a) \subseteq f(b)$.
11. The proof of $f(b) \subseteq f(a)$ is very similar. Therefore we have shown $f(a) = f(b)$.
12. Therefore we have shown that aRb IMPLIES $a \equiv_f b$.
13. The proof of the converse $a \equiv_f b$ IMPLIES aRb is very similar.
14. So we have shown aRb IFF $a \equiv_f b$. Therefore R is equal to $\equiv_f$. $\square$