Math for CS 2015/2019 Problem Set 4 solutions

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1 Problem 1

A robot moves on the two-dimensional integer grid. It starts out at $(0, 0)$ and is allowed to move in any of these four ways:

1. $(+2, -1)$: right 2, down 1

2. $(-2, +1)$: left 2, up 1

3. $(+1, +3)$

4. $(-1, -3)$

Prove that this robot can never reach $(1, 1)$.

Proof. 1. Argue by contradiction and assume that there is a sequence of moves that starts at $(0, 0)$ and ends at $(1, 1)$.

2. Let's say that, in this sequence of moves, the move $(+2, -1)$ was used a total number of a times, $(-2, +1)$ b times, $(+1, +3)$ c times and $(-1, -3)$ d times. (Here a, b, c, d are natural numbers.)

3. Then the final grid location of the robot is:

$$a(+2,-1)+b(-2,+1)+c(+1,+3)+d(-1,-3)=(2a-2b+c-d, b-a+3c-3d)$$

4. Since the final location is $(1,1)$, we have

$$(2a-2b+c-d, b-a+3c-3d)=(1,1)$$

So $2(a-b)+(c-d)=1$, and $-(a-b)+3(c-d)=1$.

5. Consider the 2x2 system of linear equations: $2x+y=1$, $-x+3y=1$.

6. Solving for y in the first equation we get $y=-2x+1$.

7. Plugging this into the second equation we get $-x+3(-2x+1)=1$.

8. So $-7x+3=1$, and solving for x we get that the unique solution is $x=2/7$ and $y=3/7$.

9. So the only possible solution to the equations in step (4) is: $a-b=2/7$ and $c-d=3/7$. This is a contradiction to the fact that a,b,c,d are natural numbers.

10. Therefore our initial assumption was false, so it's impossible for the robot to reach $(1,1)$. $\square$

2 Problem 2

Let L be some convenient set whose elements will be called *labels*. The labeled binary trees, LBT's, are defined recursively as follows:

Definition. Base case: if l is a label, then $\langle l, \mathsf{leaf}\rangle$ is an LBT, and

Constructor case: if B and C are LBT's, then $\langle l, B, C\rangle$ is an LBT.

The leaf-labels and internal-labels of an LBT are defined recursively in the obvious way:

Definition. Base case: The set of leaf-labels of the LBT $\langle l, \mathsf{leaf}\rangle$ is $\{l\}$, and its set of internal-labels is the empty set.

Constructor case: The set of leaf labels of the LBT $\langle l, B, C\rangle$ is the union of the leaf-labels of B and of C, the set of internal-labels is the union of $\{l\}$ and the sets of internal-labels of B and of C.

The set of labels of an LBT is the union of its leaf- and internal-labels.

The LBT's with unique labels are also defined recursively:

Definition. Base case: The LBT $\langle l, \mathsf{leaf}\rangle$ has unique labels.

Constructor case: If B and C are LBT's with unique labels, no label of B is a label C and vice-versa, and l is not a label of B or C, then $\langle l, B, C\rangle$ has unique labels.

If B is an LBT, let n_B be the number of distinct internal-labels appearing in B and f_B be the number of distinct leaf labels of B. Prove by structural induction that

$$f_B = n_B + 1$$

for all LBT's B with unique labels. This equation can obviously fail if labels are not unique, so your proof had better use uniqueness of labels at some point; be sure to indicate where.

Proof. We want to prove $f_B = n_B + 1$ for all LBT's B with unique labels. So assume that B is an LBT with unique labels. Argue by Structural Induction.

Base Case. B is of the form $\langle l, \mathsf{leaf} \rangle$.

In this case B has no internal-labels, and only one leaf-label. So $f_B = 1$ and $n_B = 0$, so the equation $f_B = n_B + 1$ is true in this case.

Constructor case. B is of the form $\langle l, C, D \rangle$ where:

C and D are LBT's with unique labels,

no label of C is a label D and vice-versa, and

l is not a label of C or D.

Remember that we want to prove $f_B = n_B + 1$.

1. By the induction hypothesis, $f_C = n_C + 1$ and $f_D = n_D + 1$.

2. First let's compute f_B. f_B is defined as the number of distinct leaf-labels of B. Since $B = \langle l, C, D \rangle$, and l is an internal-label, the leaf-labels of B come from C and D only.

3. Since C and D have no labels in common, all distinct leaf-labels of B are all distinct leaf-labels of C together with all distinct leaf-labels of D.

4. So $f_B = f_C + f_D$.

5. Using the induction hypothesis on (4) we get $f_B = f_C + f_D = (n_C + 1) + (n_D + 1) = n_C + n_D + 2$.

6. Now let's calculate n_B. n_B is defined as the number of distinct internal-labels of B. Since $B = \langle l, C, D \rangle$, and l is an internal-label, the internal-labels of B come from C and D, together with l.

7. Since C and D have no labels in common, and l does not appear in C or D, we have $n_B = n_C + n_D + 1$.

8. Using (5) and (7) together we see that $f_B = n_C + n_D + 2 = (n_C + n_D + 1) + 1 = n_B + 1$.

Therefore by the Principle of Structural Induction, $f_B = n_B + 1$ for all LBT's B with unique labels. $\square$

3 Problem 3

In this problem you will prove a fact that may surprise you—or make you even more convinced that set theory is nonsense: the half-open unit interval is actually the "same size" as the nonnegative quadrant of the real plane! Namely, there is a bijection from $(0, 1]$ to $[0, \infty) \times [0, \infty)$.

3.1 (a)

Describe a bijection from $(0, 1]$ to $[0, \infty)$.

Hint: $1/x$ almost works.

Proof. Define $f : (0, 1] \to [0, \infty)$ by $f(x) = \dfrac{1}{x} - 1$ for all $x \in (0, 1]$.

This is a bijection, because $y = 1/x$ is a bijection from $(0, 1]$ to $[1, \infty)$, so subtracting 1 from the output makes it a bijection from $(0, 1]$ to $[0, \infty)$. □

3.2 (b)

An infinite sequence of the decimal digits $\{0, 1, \ldots, 9\}$ will be called *long* if it does not end with all 0's. An equivalent way to say this is that a long sequence is one that has infinitely many occurrences of nonzero digits. Let L be the set of all such long sequences. Describe a bijection from L to the half-open real interval $(0, 1]$.

Hint: Put a decimal point at the beginning of the sequence.

Proof. Define $f : L \to (0, 1]$ as follows: given an infinite sequence $x = a_1, a_2, a_3, \ldots$ in L, define $f(x)$ to be the real number in $(0, 1]$ with the decimal representation $0.a_1a_2a_3\ldots$

f is injective, because... ??? (not entirely sure about this. These are basically infinite series. There's the issue of convergence, maybe two different sequences can converge to the same real number? Use a theorem about real numbers? Have to use the fact that these are LONG sequences.)

f is surjective because... ??? (use a theorem about the real numbers? Have to use the fact that these are LONG sequences.) □

3.3 (c)

Describe a surjective function from L to L^2 that involves alternating digits from two long sequences.

Hint: The surjection need not be total.

Proof. We have to define a function $f : L \to L \times L$ in a backwards way.

Suppose we are given $(y, z) \in L^2$, in other words we are given two long sequences $y = a_1, a_2, a_3, \ldots$ and $z = b_1, b_2, b_3 \ldots$ which do not end with all 0's.

Then we can construct the infinite sequence $x = a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ Notice that this sequence is in L, because it does not end with all 0's (otherwise y and z would end with all 0's).

This gives us an injection $g : L^2 \to L$ defined by $g(y, z) = x$ where x, y, z are as defined above. (Notice g is total.)

Consider the inverse of this function $f : L \to L^2$ defined by $f(x) = (y, z)$ where x, y, z are as above. (The inverse of g exists, because g is injective.)

Notice that f may not be total, because not all long sequences in L have to be of the form $x = a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ that is obtained by interleaving two other long sequences. So we are defining f on a subset $L' \subset L$ of sequences that have this form. (L' is the "forward-image" of g, denoted by $g(L^2)$, the set of outputs of g obtained by applying g to all possible inputs in its domain.)

But that's OK. We just need a surjection $L \to L^2$, it does not have to be total.

However f is surjective, because it's the inverse of g which was injective. g was defined on every input in L^2, so the output of f covers all of L^2. □

3.4 (d)

Prove the following lemma and use it to conclude that there is a bijection from L^2 to $(0, 1]^2$.

Lemma 3.1. Let A and B be nonempty sets. If there is a bijection from A to B, then there is also a bijection from $A \times A$ to $B \times B$.

Proof. 1. Assume there is a bijection $f : A \to B$.

2. Define $g : (A \times A) \to (B \times B)$ by $g(a_1, a_2) = (f(a_1), f(a_2))$ for all $(a_1, a_2) \in A \times A$.

3. We want to prove g is bijective.

4. To prove that g is injective, assume $a_1, a_2, a_3, a_4 \in A$ and assume $g(a_1, a_2) = g(a_3, a_4)$. We need to show that $a_1 = a_3$ and $a_2 = a_4$.

5.

$$\begin{aligned} g(a_1, a_2) &= g(a_3, a_4) \\ (f(a_1), f(a_2)) &= (f(a_3), f(a_4)) \end{aligned}$$

So $f(a_1) = f(a_3)$ and $f(a_2) = f(a_4)$.

6. Since f is injective, (5) implies that $a_1 = a_3$ and $a_2 = a_4$. So g is injective too.

7. To prove that g is surjective, assume $(b_1, b_2) \in B \times B$ is an arbitrary member of $B \times B$. We need to show that there exists $(a_1, a_2) \in A \times A$ such that $g(a_1, a_2) = (b_1, b_2)$.

8. Since f is surjective, there exists $a_1 \in A$ such that $b_1 = f(a_1)$.

9. Since f is surjective, there exists $a_2 \in A$ such that $b_2 = f(a_2)$.

10. Therefore $g(a_1, a_2) = (f(a_1), f(a_2)) = (b_1, b_2)$.

11. So there exists $(a_1, a_2) \in A \times A$ such that $g(a_1, a_2) = (b_1, b_2)$. Therefore g is surjective.

12. By (6) and (11) g is bijective. This finishes the proof of Lemma 3.1. $\square$

By part (b) there is a bijection from L to $(0, 1]$. Therefore by Lemma 3.1 there is a bijection from L^2 to $(0, 1]^2$.

3.5 (e)

Conclude from the previous parts that there is a surjection from $(0, 1]$ to $(0, 1]^2$. Then appeal to the Schröder-Bernstein Theorem to show that there is actually a bijection from $(0, 1]$ to $(0, 1]^2$.

Proof. 1. By part (b) there is a bijective function $f : (0, 1] \to L$.

2. By part (c) there is a surjective function $g : L \to L^2$.

3. By part (d) there is a bijective function $h : L^2 \to (0, 1]^2$.

4. Define $i : (0, 1] \to (0, 1]^2$ by $i(x) = h(g(f(x)))$. Let us prove that i is surjective too. We need to prove that for all $z \in (0, 1]^2$ there exists $w \in (0, 1]$ such that $i(w) = z$.

5. Assume $z \in (0, 1]^2$. Since h is surjective, there exists $y \in L^2$ such that $z = h(y)$.

6. Since g is surjective, there exists $x \in L$ such that $y = g(x)$.

7. Since f is surjective, there exists $w \in (0, 1]$ such that $x = f(w)$.

8. Therefore there exists $w \in (0, 1]$ such that $i(w) = h(g(f(w))) = h(g(x)) = h(y) = z$. So i is surjective.

9. Let us define an injective function $j : (0, 1] \to (0, 1]^2$. For all $x \in (0, 1]$ define $j(x) = (x, x)$. This function is clearly injective: if $(x, x) = (y, y)$ then $x = y$.

10. By (8) there exists a surjective function $(0, 1] \to (0, 1]^2$ and by (9) there exists an injective function $(0, 1] \to (0, 1]^2$. Therefore by the Schröder-Bernstein theorem, there exists a bijective function $(0, 1] \to (0, 1]^2$. $\square$

3.6 (f)

Complete the proof that there is a bijection from $(0, 1]$ to $[0, \infty)^2$.

Proof. 1. By part (a) there is a bijection from $(0, 1]$ to $[0, \infty)$.

2. By (1) and Lemma 3.1, there is a bijection $f : (0, 1]^2 \to [0, \infty)^2$.

3. By part (e) there is a bijection $g : (0, 1] \to (0, 1]^2$.

4. Define $h : (0,1] \to [0,\infty)^2$ by $h(x) = f(g(x))$, Then h is a bijection too (because it's the composition of two bijections). $\square$