Math for CS 2015/2019 solutions to “In-Class Problems Week 9, Fri. (Session 23)”

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1 Problem 1

We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of n times.

1.1 (a)

Describe a closed-form formula for the amount of wine in the first glass after n back-and-forth pourings.

Proof. Denote the glasses by G and H. Denote the amounts of water and wine in them by $G(water), H(water), G(wine), H(wine)$. Denote the states of G and H after the ith back-and-forth pouring by G_i and H_i.

Initially

$$\begin{aligned} G_0(water) &= 1 \\ G_0(wine) &= 0 \\ H_0(water) &= 0 \\ H_0(wine) &= 1 \end{aligned}$$

Let's think about moving from ith back-and-forth pouring to the $i+1$st back-and-forth pouring.

At the ith stage, we have: $G_i(water) + G_i(wine) = 1$ and $H_i(water) + H_i(wine) = 1$.

When 1/3 pint of the mix from G is poured into H, that 1/3 pint contains $(1/3) \cdot G_i(water)$ pint of water, and $(1/3) \cdot G_i(wine)$ pint of wine.

There is $(2/3) \cdot G_i(water)$ pint of water and $(2/3) \cdot G_i(wine)$ pint of wine remaining in G.

Now H has $H_i(water) + (1/3) \cdot G_i(water)$ pint of water, and $H_i(wine) + (1/3) \cdot G_i(wine)$ pint of wine, for a total of 4/3 pint of liquid.

1/3 pint of this will be poured back into G. Since there is 4/3 pints in total, we are pouring 1/4th of it. So $(1/4) \cdot H_i(water) + (1/12) \cdot G_i(water)$ pint of water and $(1/4) \cdot H_i(wine) + (1/12) \cdot G_i(wine)$ pint of wine is poured back into G.

This leaves $(3/4) \cdot H_i(water) + (3/12) \cdot G_i(water)$ pint of water and $(3/4) \cdot H_i(wine) + (3/12) \cdot G_i(wine)$ pint of wine in H.

With the pouring, now G has $(2/3) \cdot G_i(water) + (1/4) \cdot H_i(water) + (1/12) \cdot G_i(water)$ pint of water and $(2/3) \cdot G_i(wine) + (1/4) \cdot H_i(wine) + (1/12) \cdot G_i(wine)$ pint of wine.

Now we can write down the relationships between step i and step $i+1$ (we are simplifying $2/3 + 1/12 = 9/12 = 3/4$, and $3/12 = 1/4$):

$$\begin{aligned} G_{i+1}(water) &= (3/4) \cdot G_i(water) &+& (1/4) \cdot H_i(water) \\ G_{i+1}(wine) &= (3/4) \cdot G_i(wine) &+& (1/4) \cdot H_i(wine) \\ H_{i+1}(water) &= (3/4) \cdot H_i(water) &+& (1/4) \cdot G_i(water) \\ H_{i+1}(wine) &= (3/4) \cdot H_i(wine) &+& (1/4) \cdot G_i(wine) \end{aligned}$$

Let's forget about water amounts for now and focus on the equation involving $G(wine)$ only:

$$G_{i+1}(wine) = (3/4) \cdot G_i(wine) + (1/4) \cdot H_i(wine)$$

Notice that the total amount of wine in two glasses is always 1 pint, so $G_i(wine) + H_i(wine) = 1$. So $H_i(wine) = 1 - G_i(wine)$; substituting this we get

$$G_{i+1}(wine) = (3/4) \cdot G_i(wine) + (1/4) \cdot (1 - G_i(wine)) = 1/4 + (1/2) \cdot G_i(wine)$$

Now we can try to guess a closed formula for $G_n(wine)$ by looking at the first few values. Later we can prove it by induction:

$$\begin{aligned}
G_0(wine) &= 0 \\
G_1(wine) &= 1/4 + (1/2) \cdot G_0(wine) = 1/4 \\
G_2(wine) &= 1/4 + (1/2) \cdot G_1(wine) = 3/8 \\
G_3(wine) &= 1/4 + (1/2) \cdot G_2(wine) = 7/16 \\
G_4(wine) &= 1/4 + (1/2) \cdot G_3(wine) = 15/32
\end{aligned}$$

The general formula seems to be: $G_n(wine) = \dfrac{2^n - 1}{2^{n+1}}$. Let's prove it by induction. We are using $P(n) ::= G_n(wine) = \dfrac{2^n - 1}{2^{n+1}}$.

Base Case. $n = 0$. In this case $G_0(wine) = 0$, and $\dfrac{2^0 - 1}{2^{0+1}} = \dfrac{0}{2} = 0$. So $P(0)$ is true.

Induction step. Assume $n \geq 0$ and $P(n)$ is true, so $G_n(wine) = \dfrac{2^n - 1}{2^{n+1}}$. Want to prove $P(n+1)$, in other words $G_{n+1}(wine) = \dfrac{2^{n+1} - 1}{2^{n+2}}$.

By the relation we found above,

$$\begin{aligned}
G_{n+1}(wine) &= 1/4 + (1/2) \cdot G_n(wine) \\
&= 1/4 + (1/2) \cdot \frac{2^n - 1}{2^{n+1}} \qquad \text{(by IH)} \\
&= \frac{1}{2^2} + \frac{2^n - 1}{2^{n+2}} \\
&= \frac{2^n}{2^{n+2}} + \frac{2^n - 1}{2^{n+2}} \\
&= \frac{2^n + 2^n - 1}{2^{n+2}} \\
&= \frac{2^{n+1} - 1}{2^{n+2}}
\end{aligned}$$

so $P(n+1)$ is also true. By the Induction Principle, $P(n)$ is true for all $n \geq 0$. $\square$

1.2 (b)

What is the limit of the amount of wine in each glass as n approaches infinity?

Proof. In part (a) we found

$$G_n(wine) = \frac{2^n - 1}{2^{n+1}} = \frac{2^n}{2^{n+1}} - \frac{1}{2^{n+1}} = 1/2 - \frac{1}{2^{n+1}}$$

So

$$\lim_{n\to\infty} G_n(wine) = \lim_{n\to\infty} (1/2 - \frac{1}{2^{n+1}}) = 1/2 - 0 = 1/2$$

Since $G(wine) + H(wine)$ is always equal to 1,

$$\lim_{n\to\infty} H_n(wine) = \lim_{n\to\infty} (1 - G_n(wine)) = 1 - 1/2 = 1/2$$

□

2 Problem 2

An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to make multiple trips carrying up to a gallon each time to create water caches out in the desert.

For example, if the shrine were 2/3 of a day's walk into the desert, then she could recover the Holy Grail after two days using the following strategy. She leaves the oasis with 1 gallon of water, travels 1/3 day into the desert, caches 1/3 gallon, and then walks back to the oasis - arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks 1/3 day into the desert, tops off her water supply by taking the 1/3 gallon in her cache, walks the remaining 1/3 day to the shrine, grabs the Holy Grail, and then walks for 2/3 of a day back to the oasis - again arriving with no water to spare.

But what if the shrine were located farther away?

2.1 (a)

What is the most distant point that the explorer can reach and then return to the oasis, with no water precached in the desert, if she takes a total of only 1 gallon from the oasis?

Proof. At best she can walk 1/2 day into the desert and then walk back. □

2.2 (b)

What is the most distant point the explorer can reach and still return to the oasis if she takes a total of only 2 gallons from the oasis? No proof is required; just do the best you can.

Proof. The explorer walks 1/4 day into the desert, drops 1/2 gallon, then walks home. Next, she walks 1/4 day into the desert, picks up 1/4 gallon from her cache, walks an additional 1/2 day out and back, then picks up another 1/4 gallon from her cache and walks home. Thus, her maximum distance from the oasis is 3/4 of a day's walk. □

2.3 (c)

The explorer will travel using a recursive strategy to go far into the desert and back, drawing a total of n gallons of water from the oasis. Her strategy is to build up a cache of $n-1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n-1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with n gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where H_n is the nth Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$$

Conclude that she can reach the shrine, however far it is from the oasis.

Proof. To build up the first cache of $n-1$ gallons, she should make n trips $1/(2n)$ days into the desert, dropping off $(n-1)/n$ gallons each time. Before she leaves the cache for the last time, she has $n-1$ gallons plus enough for the walk home. Then she applies her $(n-1)$-day strategy. So letting D_n be her maximum distance into the desert and back, we have

$$D_n = \frac{1}{2n} + D_{n-1}$$

So

$$\begin{aligned} D_n &= \frac{1}{2n} + \frac{1}{2(n-1)} + \ldots + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 1} \\ &= \frac{1}{2}\left(\frac{1}{n} + \frac{1}{(n-1)} + \ldots + \frac{1}{2} + \frac{1}{1}\right) \\ &= \frac{H_n}{2} \end{aligned}$$

□

2.4 (d)

Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \approx \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

Proof. She obtains the Grail when:

$$\frac{H_n}{2} \approx \frac{\ln n}{2} \geq 10$$

This requires $n \geq e^{20} = 4.8 \cdot 10^8$ days > 1.329 million years. □

3 Problem 3

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a weakly decreasing function. Define

$$S ::= \sum_{i=1}^{n} f(i)$$

and

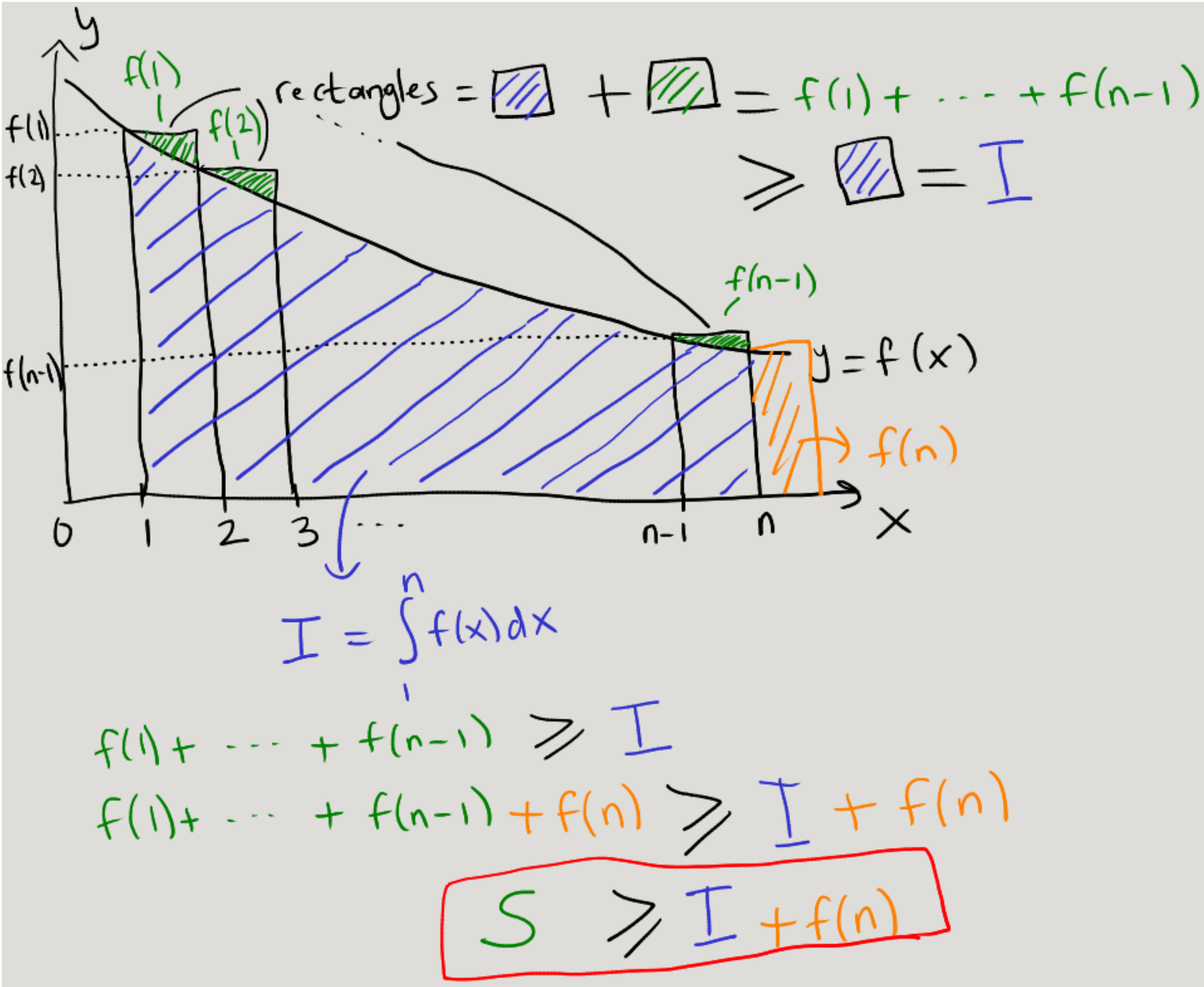
$$I ::= \int_1^n f(x)\, dx$$

Prove that

$$I + f(n) \leq S \leq I + f(1)$$

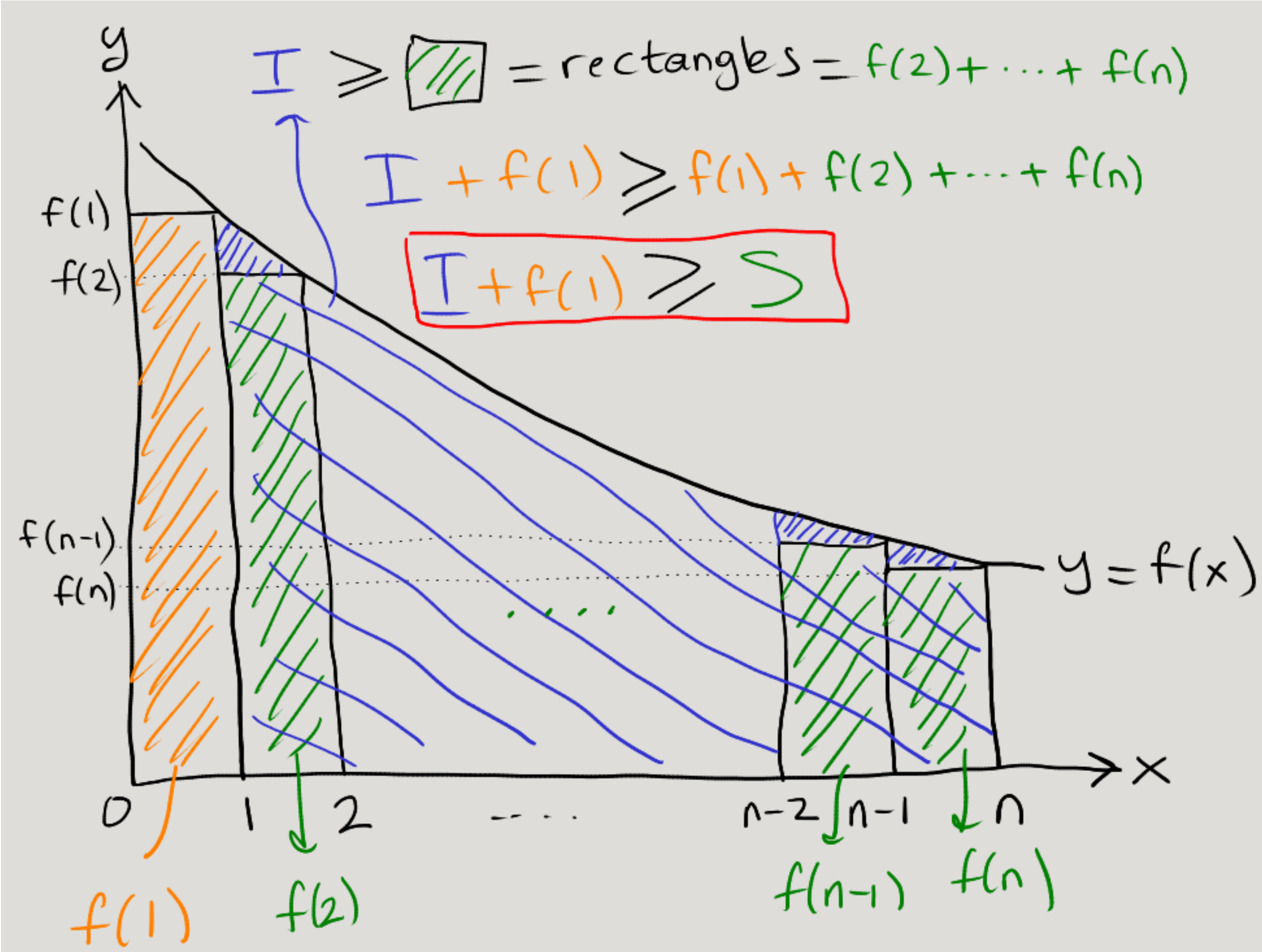
(Proof by very clear picture is OK.)

Proof. See below, and also the next page. $\square$



4 Problem 4

Sammy the Shark is a financial service provider who offers loans on the following terms.



Sammy loans a client m dollars in the morning. This puts the client m dollars in debt to Sammy.

Each evening, Sammy first charges a service fee which increases the client's debt by f dollars, and then Sammy charges interest, which multiplies the debt by a factor of p. For example, Sammy might charge a "modest" ten cent service fee and 1% interest rate per day, and then f would be 0.1 and p would be 1.01.

4.1 (a)

What is the client's debt at the end of the first day?

Proof. At the end of day 1, debt increases from m to $m + f$ then gets multiplied by a factor of p, so it's $p(m + f) = pm + pf$. □

4.2 (b)

What is the client's debt at the end of the second day?

Proof. Again we increase debt from day 1 by f, then multiply this new debt by p, so it's: $p(pm + pf + f) = p^2m + p^2f + pf$. □

4.3 (c)

Write a formula for the client's debt after d days and find an equivalent closed form.

Proof. Before we attempt the closed formula let's see the third day debt:

$$p(\text{2nd day debt} + f) = p(p^2m + p^2f + pf + f) = p^3m + p^3f + p^2f + pf$$

The general formula after d days seems to be: $p^dm + (p^df + p^{d-1}f + \ldots + p^2f + pf)$. Writing this more nicely, we get:

$$debt(d) = mp^d + fp(p^{d-1} + \ldots + 1) = mp^d + fp\frac{p^d - 1}{p - 1}$$

We should prove this by induction. We are working with:

$$P(d) ::= \text{the debt after } d \text{ days is: } mp^d + fp\frac{p^d - 1}{p - 1}.$$

Base Case. $d = 1$. By part (a) the debt at the end of day 1 is $pm + pf$. And the formula gives:

$$mp^1 + fp\frac{p^1 - 1}{p - 1} = mp + fp$$

which is the same. Therefore $P(1)$ is true.

Induction Step. Assume $d \geq 1$ and assume $P(d)$ is true. Want to show $P(d+1)$.

By induction hypothesis the debt after d days is

$$mp^d + fp\frac{p^d - 1}{p - 1}$$

At the end of day $d + 1$ this increases by f and gets multiplied by p. So the debt at the end of day $d + 1$ is

$$p(mp^d + fp\frac{p^d - 1}{p - 1} + f) = mp^{d+1} + fp\frac{p^{d+1} - p}{p - 1} + fp$$

If we factor out the fp in the last two terms and get a common denominator, we get

$$mp^{d+1} + fp(\frac{p^{d+1} - p}{p - 1} + 1) = mp^{d+1} + fp(\frac{p^{d+1} - p}{p - 1} + \frac{p - 1}{p - 1})$$

Finally simplifying, we get

$$mp^{d+1} + fp(\frac{p^{d+1} - p + p - 1}{p - 1}) = mp^{d+1} + fp\frac{p^{d+1} - 1}{p - 1}$$

which proves $P(d + 1)$. Therefore by Induction $P(d)$ is true for all $d \geq 1$. □

4.4 (d)

If you borrowed \$10 from Sammy for a year, how much would you owe him?

Proof. Here $m = 10, d = 365$ and we can use the values from the problem $f = 0.1, p = 1.01$ to get:

$$10 \cdot 1.01^{365} + 0.1 \cdot 1.01 \cdot \frac{1.01^{365} - 1}{1.01 - 1} = \$749.347030091$$

□

5 Problem 5 (Supplemental problem)

You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned}
S &= 1 + z + z^2 + \ldots + z^n \\
zS &= z + z^2 + \ldots + z^n + z^{n+1} \\
S - zS &= 1 - z^{n+1} \\
S &= \frac{1 - z^{n+1}}{1 - z}
\end{aligned}$$

Use the same approach to find a closed form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

Proof.

$$\begin{aligned}
T &= 1z + 2z^2 + 3z^3 + \ldots + (n-1)z^{n-1} + nz^n \\
zT &= 1z^2 + 2z^3 + \ldots + (n-1)z^n + nz^{n+1} \\
T - zT &= z + z^2 + z^3 + \ldots + z^n + nz^{n+1} \\
T - zT + 1 &= 1 + z + z^2 + z^3 + \ldots + z^n + nz^{n+1} \\
T - zT + 1 - nz^{n+1} &= 1 + z + z^2 + z^3 + \ldots + z^n \\
T - zT + 1 - nz^{n+1} &= S \\
T - zT + 1 - nz^{n+1} &= \frac{1 - z^{n+1}}{1 - z} \\
T - zT &= -1 + nz^{n+1} + \frac{1 - z^{n+1}}{1 - z} \\
T(1 - z) &= -1 + nz^{n+1} + \frac{1 - z^{n+1}}{1 - z} \\
T &= \frac{-1 + nz^{n+1}}{(1 - z)} + \frac{1 - z^{n+1}}{(1 - z)^2}
\end{aligned}$$

□