# Math for CS 2015/2019 solutions to "In-Class Problems Week 3, Wed. (Session 6)"

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# 1 Problem 1

## 1.1 (a)

Verify that the propositional formula ($P$ AND $\overline{Q}$) OR ($P$ AND $Q$) is equivalent to $P$.

*Proof.* There is a simple verification by truth table with 4 rows which we omit.

There is also a simple cases argument: if $Q$ is T, then the formula simplifies to

$$(P \text{ AND F}) \text{ OR } (P \text{ AND T})$$

which further simplifies to

$$\text{F OR } (P \text{ AND } T)$$

which is $P$ AND T, which is equivalent to $P$.

Otherwise, if $Q$ is F, then the formula simplifies to

$$(P \text{ AND } T) \text{ OR } (P \text{ AND } F)$$

which is

$$(P \text{ AND } T) \text{ OR } F$$

which is $P$ AND T, which is likewise equivalent to $P$. □

### 1.2 (b)

The set difference, $A - B$, of sets $A$ and $B$ is defined as:

$$A - B ::= \{a \in A \mid a \notin B\}$$

Prove that

$$A = (A - B) \cup (A \cap B)$$

for all sets $A, B$, by showing

$$x \in A \text{ IFF } x \in (A - B) \cup (A \cap B)$$

for all elements $x$, using the equivalence of part (a) in a chain of IFF's.

*Proof.* Let $P$ be: $x \in A$ and let $Q$ be: $x \in B$. Then:

$$\begin{aligned}
&x \in (A - B) \cup (A \cap B) \\
\text{iff } &x \in (A - B) \text{ OR } (A \cap B) && \text{(by definition of } \cup) \\
\text{iff } &(x \in A \text{ AND NOT}(x \in B)) \text{ OR } (x \in A \text{ AND } x \in B) && \text{(by definition of } -, \cap) \\
\text{iff } &(P \text{ AND NOT}(Q)) \text{ OR } (P \text{ AND } Q) && \text{(by rewriting)} \\
\text{iff } &P && \text{(by part (a))} \\
\text{iff } &x \in A && \text{(by rewriting)}
\end{aligned}$$

□

# 2 Problem 2

A formula of set theory is a predicate formula that only uses the predicate "$x \in y$". The domain of discourse is the collection of sets, and "$x \in y$" is interpreted to mean the set $x$ is one of the elements in the set $y$.

For example, since $x$ and $y$ are the same set iff they have the same members, here's how we can express equality of $x$ and $y$ with a formula of set theory:

$$(x = y) ::= \forall z (z \in x \text{ IFF } z \in y)$$

Express each of the following assertions about sets by a formula of set theory.

## 2.1 (a)

$x = \emptyset$

*Proof.* $(x = \emptyset) ::= \forall z (z \notin x)$ □

## 2.2 (b)

$x = \{y, z\}$

*Proof.* $x = \{y, z\} ::= \forall w (w \in x \text{ IFF } (w = y \text{ OR } w = z))$ □

## 2.3 (c)

$x \subseteq y$ ($x$ is a subset of $y$ that might equal $y$.)

*Proof.* $x \subseteq y ::= \forall z (z \in x \text{ IMPLIES } z \in y)$ □

Now we can explain how to express "$x$ is a proper subset of $y$" as a set theory formula using things we already know how to express. Namely, letting "$x \neq y$" abbreviate NOT$(x = y)$, the expression

$$(x \subseteq y) \text{ AND } x \neq y$$

describes a formula of set theory that means $x \subset y$.

From here on, feel free to use any previously expressed property in describing formulas for the following:

## 2.4 (d)

$x = y \cup z$

*Proof.* $x = y \cup z ::= \forall w (w \in x \text{ IFF } (w \in y \text{ OR } w \in z))$ □

## 2.5 (e)

$x = y - z$

*Proof.* $x = y - z ::= \forall w (w \in x \text{ IFF } (w \in y \text{ AND } w \notin z))$ □

### 2.6 (f)

$x = pow(y)$

*Proof.* Remember "pow" stands for "power set", which is "the set of all subsets".

$x = pow(y) ::= \forall w(w \in x \text{ IFF } w \subseteq y)$ □

### 2.7 (g)

$x = \bigcup_{z \in y} z$

This means that $y$ is supposed to be a collection of sets, and $x$ is the union of all them. For example if $y = \{a, b, c, d\}$ then $x = a \cup b \cup c \cup d$.

A more concise notation for $\bigcup_{z \in y} z$ is simply $\bigcup y$.

*Proof.* $x = \bigcup_{z \in y} z ::= \forall w(w \in x \text{ IFF } \exists z(z \in y \text{ AND } w \in z))$ □

## 3 Problem 3

Forming a pair $(a, b)$ of items $a$ and $b$ is a mathematical operation that we can safely take for granted. But when we're trying to show how all of mathematics can be reduced to set theory, we need a way to represent the pair $(a, b)$ as a set.

(The property we want from ordered pairs $(a, b)$ is that, two ordered pairs are equal iff their "first coordinates" are equal to each other, and their "second coordinates" are equal to each other. That's how we intuitively understand ordered pairs in, say, analytic geometry or Calculus.)

### 3.1 (a)

Explain why representing $(a, b)$ by $\{a, b\}$ won't work.

*Proof.* Since order does not matter for sets, we have $\{a, b\} = \{b, a\}$. This means that $(a, b) = (b, a)$. But obviously this is not what we want from an ordered pair. We want $(1, 2)$ to be different than $(2, 1)$. So this won't work. □

### 3.2 (b)

Explain why representing $(a, b)$ by $\{a, \{b\}\}$ won't work either. Hint: What pair does $\{\{1\}, \{2\}\}$ represent?

*Proof.* With this definition, $\{\{1\}, \{2\}\}$ represents (with $a = \{1\}$ and $b = 2$) the ordered pair $(\{1\}, 2)$.

But again, order does not matter for sets, so $\{\{1\}, \{2\}\} = \{\{2\}, \{1\}\}$, so the same set ALSO represents (with $a = \{2\}, b = 1$) the ordered pair $(\{2\}, 1)$.

So we end up with $(\{1\}, 2) = (\{2\}, 1)$. Obviously this is not the behavior we want from an ordered pair. We want these to be different pairs. $\square$

## 3.3 (c)

Define

$$pair(a, b) ::= \{a, \{a, b\}\}$$

Explain why representing $(a, b)$ as $pair(a, b)$ uniquely determines $a$ and $b$. Hint: Sets can't be indirect members of themselves: $a \in a$ never holds for any set $a$, and neither can $a \in b \in a$ hold for any $b$.

*Proof.* For uniqueness, we want to prove:

$$(a, b) = (c, d) \text{ IFF } a = c \text{ AND } b = d.$$

Here $(a, b)$ is represented by $pair(a, b)$ so we want to prove:

$$\{a, \{a, b\}\} = \{c, \{c, d\}\} \text{ IFF } a = c \text{ AND } b = d.$$

The right-to-left implication is obvious: if $a = c$ AND $b = d$ then it immediately follows that $\{a, \{a, b\}\} = \{c, \{c, d\}\}$. So we want to prove the left-to-right implication.

1. Assume $\{a, \{a, b\}\} = \{c, \{c, d\}\}$.

2. By definition of set equality, these two sets have the same elements.

3. There are 2 possibilities:

4. Case 1: $a = c$ and $\{a, b\} = \{c, d\}$.

4.1. Since $a = c$, we have $\{a, b\} = \{c, b\}$.

4.2. By (4) and (4.1) we have $\{c, d\} = \{c, b\}$.

4.3. By (4.2) and definition of set equality, we have $d = b$. So we proved what we wanted in this case.

5. Case 2: $a = \{c, d\}$ and $c = \{a, b\}$.

5.1. By (5) we get: $a = \{c, d\} = \{\{a, b\}, d\}$. So $a = \{\{a, b\}, d\}$. In particular $\{a, b\}$ is a member of $a$.

5.2. So by (5.2) we get a chain of membership relations:

$$a \in \{a, b\} \in a$$

which the Hint told us is impossible (this impossibility is actually a consequence of the ZFC axiom called "Axiom of Foundation", a.k.a. "Axiom of Regularity").

Contradiction! So Case 2 is impossible.

6. By (4) and (5) we see that $a = c$ and $b = d$. $\square$

# 4 Problem 4

Subset take-away is a two player game played with a finite set, A, of numbers. Players alternately choose nonempty subsets of A with the conditions that a player may not choose:

the whole set A, or

any set containing a set that was named earlier.

The first player who is unable to move loses the game.

For example, if the size of A is one, then there are no legal moves and the second player wins. If A has exactly two elements, then the only legal moves are the two one-element subsets of A. Each is a good reply to the other, and so once again the second player wins.

The first interesting case is when A has three elements. This time, if the first player picks a subset with one element, the second player picks the subset with the other two elements. If the first player picks a subset with two elements, the second player picks the subset whose sole member is the third element. In both cases, these moves lead to a situation that is the same as the start of a game on a set with two elements, and thus leads to a win for the second player.

Verify that when A has four elements, the second player still has a winning strategy.

*Proof.* There are way too many cases to work out by hand if we tried to list all possible games. But the elements of A all behave the same, so we can cut to a small number of cases using the fact that permuting around the elements of A in any game yields another possible game. We can do this by not mentioning specific elements of A, but instead using the variables $a, b, c, d$ whose values will be the four elements of A.

We consider two cases for the move of the Player 1 when the game starts:

1. Player 1 chooses a one element or a three element subset. Then Player 2 should choose the complement of Player one's choice. The game then becomes the same as playing the $n = 3$ game on the three element set chosen in this first round, where we know Player 2 has a winning strategy.

2. Player 1 chooses a subset of 2 elements. Let $a, b$ be these elements, that is, the first move is $\{a, b\}$. Player 2 should choose the complement, $\{c, d\}$, of Player 1's choice. We then have the following subcases:

(a) Player 1's second move is a one element subset, $\{a\}$. Player 2 should choose $\{b\}$. The game is then reduced to the two element game on $\{c, d\}$ where Player 2 has a winning strategy.

(b) Player 1's second move is a two element subset, $\{a, c\}$. Player 2 should choose its complement, $\{b, d\}$. This leads to two subsubcases:

i. Player 1's third move is one of the remaining sets of size two, $\{a, d\}$. Player 2 should choose its complement, $\{b, c\}$. The remaining possible moves are the four sets of size 1, where the Player 2 clearly wins after two more rounds.

ii. Player 1's third move is a one element set, $\{a\}$. Player 2 should choose $\{b\}$. The game is then reduced to the case two element game on $\{c, d\}$ where Player 2 has a winning strategy.

So in all cases, Player 2 has a winning strategy in the game for $n = 4$. $\square$