Math for CS 2015/2019 solutions to “In-Class Problems Week 2, Fri. (Session 4)”

https://github.com/spamegg1

October 20, 2022

Contents

1 Problem 1

The inverse, R^{-1} , of a binary relation, R, from A to B, is the relation from B to A defined by:

$$bR^{-1}a \text{ iff } aRb$$

In other words, you get the diagram for R^{-1} from R by “reversing the arrows” in the diagram describing R. Now many of the relational properties of R correspond to different properties of R^{-1} . For example, R is total iff R^{-1} is a surjection.

Fill in the remaining entries is this table:

R is	iff	R^{-1} is
total		a surjection
a function		
a surjection		
an injection		
a bijection		

Hint: Explain what's going on in terms of "arrows" from A to B in the diagram for R.

Arrow Properties

Definition. A binary relation R:

is a function when it has the ≤ 1 arrow-out property.

is surjective when it has the ≥ 1 arrows-in property, that is, for every point in the right hand, codomain column has at least one arrow pointing to it.

is total when it has the ≥ 1 arrows-out property.

is injective when it has the ≤ 1 arrow-in property.

is bijective when it has both the $= 1$ arrow-out property and the $= 1$ arrow-in property.

Proof.

R is	iff	R^{-1} is
total		a surjection
a function		an injection
a surjection		total
an injection		a function
a bijection		a bijection

□

2 Problem 2

Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of size n, and $B = \{b_0, b_1, \ldots, b_{m-1}\}$ be a set of size m. Prove that $|A \times B| = mn$ by defining a simple bijection from $A \times B$ to the nonnegative integers from 0 to $mn - 1$.

Proof. A bijection $f : A \times B \to \{0, 1, \ldots, mn - 1\}$ can be defined by the rule

$$f(a_k, b_j) ::= jn + k$$

□

3 Problem 3

Assume $f : A \to B$ is a total function, and A is finite. Replace the $*$ with one of $\leq, =, \geq$ to produce the strongest correct version of the following statements:

3.1 (a)

$|f(A)| * |B|$.

Proof. $|f(A)| \leq |B|$. $\square$

3.2 (b)

If f is a surjection, then $|A| * |B|$.

Proof. $|A| \geq |B|$. $\square$

3.3 (c)

If f is a surjection, then $|f(A)| * |B|$.

Proof. $|f(A)| = |B|$. $\square$

3.4 (d)

If f is an injection, then $|f(A)| * |A|$.

Proof. $|f(A)| = |A|$. $\square$

3.5 (e)

If f is a bijection, then $|A| * |B|$.

Proof. $|A| = |B|$. $\square$

4 Problem 4

Let $R : A \to B$ be a binary relation. Use an arrow counting argument to prove the following generalization of the Mapping Rule 1 in the course textbook.

Lemma 1. *If R is a function, and $X \subseteq A$, then*

$$|X| \geq |R(X)|$$

Proof. 1. Assume $R : A \to B$ is a function and $X \subseteq A$.

2. Since R is a function, it has the ≤ 1 arrow-out property.

3. So by (2), we have $|X| \geq$ (# arrows from X).

4. By definition of $R(X)$, each element of $R(X)$ is the endpoint of an arrow going out from X.

5. So by (4) we have (# arrows from X) $\geq |R(X)|$.

6. Combining (3) and (5) we get $|X| \geq |R(X)|$. □

5 Problem 5

5.1 (a)

Prove that if A surj B and B surj C, then A surj C.

Proof. By definition of surj, there are surjective functions, $F : A \to B$ and $G : B \to C$.

Let $H ::= G \circ F$ be the function equal to the composition of G and F, that is

$$H(a) ::= G(F(a))$$

We show that H is surjective, which will complete the proof.

So suppose $c \in C$. Then since G is a surjection, $c = G(b)$ for some $b \in B$. Likewise, $b = F(a)$ for some $a \in A$. Hence $c = G(F(a)) = H(a)$, proving that c is in the range of H, as required. □

5.2 (b)

Explain why A surj B iff B inj A.

Proof. (right to left): By definition of inj, there is a total injective relation, $R : B \to A$.

But this implies that R^{-1} is a surjective function from A to B.

(left to right): By definition of surj, there is a surjective function, $F : A \to B$. But this implies that F^{-1} is a total injective relation from A to B. □

5.3 (c)

Conclude from (a) and (b) that if A inj B and B inj C, then A inj C.

Proof. From (b) and (a) we have that if C inj B and B inj A, then C inj A, so just switch the names A and C. □

5.4 (d)

Explain why A inj B iff there is a total injective function ($= 1$ out, ≤ 1 in) from A to B.

Proof. (left to right) Assume A inj B. By definition of inj, there is a total injective relation $R : A \to B$.

So R has the ≥ 1 arrows-out property and the ≤ 1 arrow-in property. We can modify R into a total injective function F ($= 1$ out, ≤ 1 in) as follows.

For every $a \in A$ such that R has more than 1 arrows going out from a, remove all but 1 of those arrows. This way the ≥ 1 arrows-out property turns into the $= 1$ arrow-out property, and we still have the ≤ 1 arrow-in property.

(right to left) Assume there is a total injective function ($= 1$ out, ≤ 1 in) F from A to B. Since every function is also a relation, F is a total injective relation from A to B. This is the definition of inj, therefore A inj B. $\square$