Assignment 5
Optimization methods

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Problem 3.

$$\begin{aligned} &\text{Minimize}_{y\in\mathbb{R}^2}\ 2y_1 \\ &\text{subject to } \begin{bmatrix} 0 & 5y_1 & 0 \\ 5y_1 & 3y_2 & 0 \\ 0 & 0 & 4y_1+2 \end{bmatrix} \succeq 0 \end{aligned} \qquad .$$

Solution:

First, let's solve initial problem and find the primal solution p^*.
Let's use Sylvester's criterion to find y_1 such that positive-definiteness holds.
$\det\begin{bmatrix} 0 & 5y_1 \\ 5y_1 & 3y_2 \end{bmatrix} = -25y_1^2 = 0$. Consequently, $y_1 = 0$.
Thus, solution for conditions is $(0, y_2)$.

$$p^* = \min_{y^2_{\mathbb{R}}} 2y_1 = \min_{y^2_{\mathbb{R}}} 0 = 0$$

Now let's define a dual problem and solve it. First, I want to rewrite initial conditions in the following way:

$$\begin{bmatrix} 0 & 5y_1 & 0 \\ 5y_1 & 3y_2 & 0 \\ 0 & 0 & 4y_1+2 \end{bmatrix} = F_1y_1 + F_2y_2 + G \succeq 0,$$

where

$$F_1 = \begin{bmatrix} 0 & 5 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}, F_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

As it was at seminar, dual problem is:

$$
\begin{aligned}
&\text{Maximize}_{Z\in\mathbb{R}^{3\times 3}}\langle -Z, G\rangle\\
&\text{subject to}\\
&\langle Z, F_1\rangle = 2,\\
&\langle Z, F_2\rangle = 0,\\
&Z \succeq 0
\end{aligned}
$$

Let's solve it.

$$Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} & z_{23} \\ z_{13} & z_{23} & z_{33} \end{bmatrix}$$

1. $\langle Z, F_1\rangle = 5z_{12} + 5z_{12} + 4z_{33} = 2 \Rightarrow z_{12} = \frac{1}{5} - \frac{2}{5}z_{33}$.
2. $\langle Z, F_2\rangle = 3z_{22} = 0 \Rightarrow z_{22} = 0$.
3. $\langle -Z, G\rangle = -2z_{33}$.

$$Z = \begin{bmatrix} z_{11} & \frac{1}{5} - \frac{2}{5}z_{33} & z_{13} \\ \frac{1}{5} - \frac{2}{5}z_{33} & 0 & z_{23} \\ z_{13} & z_{23} & z_{33} \end{bmatrix}$$

Let's use Sylvester's criterion to find z_{33} such that positive-definiteness holds.
$\det\begin{bmatrix} z_{11} & \frac{1}{5} - \frac{2}{5}z_{33} \\ \frac{1}{5} - \frac{2}{5}z_{33} & z_{13} \end{bmatrix} = -(\frac{1}{5} - \frac{2}{3}z_{33})^2 \geq 0 \Rightarrow z_{33} = \frac{1}{2}$.

$$d^* = \max_{Z \in \mathbb{R}^{3\times 3}} -2z_{33} = -2 \cdot \frac{1}{2} = -1.$$

You see that strong duality doesn't hold since $d^* \neq p^*$

Answer: $p^* = 0, d^* = -1$.

Problem 4.

Solution:

1. $f_0(x) = \max_{i=1,\ldots,m} a_i^\top x + b$, where $x \in \mathbb{R}^n, a_i \in \mathbb{R}^n, b_i \in \mathbb{R}, \forall i$.

   $f_0(x) \to \min_{x \in \mathbb{R}^n}$

   Equivalent LP problem:

   $$\begin{aligned}&\text{Minimize}_{x\in\mathbb{R}^n, t\in\mathbb{R}} t\\ &\text{subject to } t - a_i^\top \leq b_i, i = 1, \ldots, m.\end{aligned}$$

   Let's construct a dual problem.

   $$L(t, x, \lambda) = t - \sum_{i=1}^m \lambda_i (t - a_i^\top x - b_i), \lambda \in \mathbb{R}^m.$$

   $$\nabla_{t,x} L(t, x, \lambda) = (1 - \sum_{i=1}^m \lambda_i, \sum_{i=1}^m a_i^{(1)} \lambda_i, \ldots, \sum_{i=1}^m a_i^{(n)} \lambda_i).$$

Consequently, the dual problem is:

$$
\begin{aligned}
&\text{Maximize}_{\lambda\in\mathbb{R}^m} \sum_{i=1}^m \lambda_i b_i,\\
&\text{subject to}\\
&\lambda \geq 0,\\
&\sum_{i=1}^m \lambda_i = 1,\\
&\sum_{i=1}^m \lambda_i a_i^{(j)} = 0, j = 1, \ldots, n.
\end{aligned}
$$

2. $f_2(x) = \frac{1}{\alpha}\log\big(\sum_{i=1}^m e^{\alpha(a_i^\top x + b_i)}\big), \alpha > 0.$

   $f_2(x) \to \min_{x\in\mathbb{R}^n}$

   Problem:

$$
\begin{aligned}
&\text{Minimize}_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \frac{1}{\alpha}\log\big(\sum_{i=1}^m e^{\alpha y_i}\big)\\
&\text{subject to } y_i = a_i^\top x + b_i, i = 1, \ldots, m.
\end{aligned}
$$

   Let's construct a dual problem.

$$
L(y, x, \lambda) = \frac{1}{\alpha}\log\big(\sum_{i=1}^m e^{\alpha y_i}\big) - \sum_{i=1}^m \lambda_i(y_i - a_i^\top x - b_i), \lambda \in \mathbb{R}^m.
$$

$$
\nabla_{y,x} L(t, x, \lambda) = \Big(\frac{e^{\alpha y_1}}{\sum_{i=1}^m e^{\alpha y_i}} - \lambda_1, \ldots, \frac{e^{\alpha y_m}}{\sum_{i=1}^m e^{\alpha y_i}} - \lambda_m, \sum_{i=1}^m a_i^{(1)}\lambda_i, \ldots, \sum_{i=1}^m a_i^{(n)}\lambda_i\Big).
$$

Consequently, the dual problem is:

$$\text{Maximize}_{\lambda \in \mathbb{R}^m} - \sum_{i=1}^{m} \lambda_i \Big(\frac{\log \lambda_i}{\alpha} - b_i \Big),$$

subject to

$$\lambda \geq 0,$$

$$\sum_{i=1}^{m} \lambda_i = 1,$$

$$\sum_{i=1}^{m} \lambda_i a_i^{(j)} = 0, j = 1, \ldots, n.$$

3. (a) $f_0(x) = \max_i(a_i^\top x + b_i) = \frac{1}{\alpha} \log \left(e^{\alpha \max_i (a_i^\top x + b_i)} \right) = \frac{1}{\alpha} \log \left(\max_i e^{\alpha(a_i^\top x + b_i)} \right) \leq \frac{1}{\alpha} \log \left(\sum_{i=1}^{n} e^{\alpha(a_i^\top x + b_i)} \right) \leq f_2(x) \Rightarrow f_0(x_0^*) \leq f_0(x_2^*) \leq f_2(x_2^*) \Rightarrow f_2(x_2^*) - f_0(x_0^*) \geq 0.$

   (b) $f_2(x_2^*) - f_0(x_0^*) = f_2^{dual}(\lambda_2^*) - f_0^{dual}(\lambda_0^*) \leq f_2^{dual}(\lambda_2^*) - f_0^{dual}(\lambda_2^*) \leq \sum_{i=1}^{m} \lambda_{2_i}^* \big(\frac{\log \lambda_{2_i}^*}{\alpha} \big) \leq \frac{1}{\alpha} \log m.$

Problem 5.

$$\text{Minimize}_{x \in \mathbb{R}^n} \ \|Ax - b\|_2^2$$
$$\text{subject to } Cx = d,$$

where $A \in \mathbb{R}^{m \times n} : \mathrm{rk}(A) = n, C \in \mathbb{R}^{p \times n} : \mathrm{rk}(A) = p.$

Solution:

Denote: $f_0(x) = \|Ax - b\|_2^2, h_1(x) = Cx - d.$

KKT:

1. $h_1(x^*) = 0 \to Cx^* = d$.
2. $\nabla f_0(x^*) + \nu^* \nabla h_1(x^*) = 0 \to 2A^\top(Ax^* - b) + C^\top \nu^* = 0$.

Write in the matrix form:

$$\begin{bmatrix} 2A^\top A & C^\top \\ C & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} 2A^\top b \\ d \end{bmatrix}$$
$$\Rightarrow$$
$$\begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} 2A^\top A & C^\top \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2A^\top b \\ d \end{bmatrix}$$

Thus,

$$p^* = \|Ax^* - b\|_2^2$$
$$d^* = L(x^*, \nu^*) = \|Ax^* - b\|_2^2 + \sum_{i=1}^{p} \nu_i^* (\langle C_i, x^* \rangle - d_i)$$

Problem 6.

Solution:

1. LP:

$$\begin{aligned} &\text{Minimize}_{x \in \mathbb{R}^n} \ c^\top x \\ &\text{subject to} \\ &Ax = b, \\ &x \geq 0. \end{aligned}$$

Denote: $f_0(x) = c^\top x, h_1(x) = Ax - b, f_1(x) = -x$.

$L(x, \lambda, \nu) = c^\top x - \lambda^\top x - \nu^\top (Ax - b), \lambda, \nu \in \mathbb{R}^n,$
$\nabla_x L(x, \lambda, \nu) = c - \lambda - A^\top \nu$
Consequently, the dual problem is:

$$\begin{aligned}
&\text{Maximize}_{\lambda, \nu \in \mathbb{R}^n} \nu^\top b, \\
&\text{subject to} \\
&\lambda \geq 0, \\
&c - \lambda - A^\top \nu = 0.
\end{aligned}$$

KKT:

(a) $f_1(x^*) \leq 0 \rightarrow x^* \geq 0, h_1(x^*) = 0 \rightarrow Ax^* = b.$
(b) $\lambda^* \geq 0.$
(c) $\nabla f_0(x^*) + \lambda^* \nabla f_1(x^*) + \nu^* \nabla h_1(x^*) = 0 \rightarrow c - \lambda^* + A^\top \nu^* = 0.$
(d) $\lambda^{*\top} f_1(x^*) = 0 \rightarrow \lambda^{*\top} x^* = 0.$

2. Problem:

$$\begin{aligned}
&\text{Minimize}_{x \in \mathbb{R}^n} \; c^\top x - \tau \sum_{i=1}^{n} \ln(x_i) \\
&\text{subject to } Ax = b.
\end{aligned}$$

Denote: $f_0(x) = c^\top x - \tau \sum_{i=1}^n \ln(x_i), h_1(x) = Ax - b.$

$L(x, \nu) = c^\top x - \tau \sum_{i=1}^n \ln(x_i) + \nu^\top (Ax - b), \nu \in \mathbb{R}^n,$
$\nabla_x L(x, \nu) = c - \tau(\frac{1}{x_1}, \ldots, \frac{1}{x_n})^\top + A^\top \nu.$

Consequently, the dual problem is:

$$\text{Maximize}_{\nu\in\mathbb{R}^n} n\tau - \tau\sum_{i=1}^{n}\log\big(\frac{\tau}{c_i+\sum_{j=1}^{n}a_{ji}\nu_i}\big) - \nu^\top b,$$

subject to

$c + A^\top \nu \geq 0.$

KKT:

(a) $h_1(x^*) = 0 \to Ax^* = b.$

(b) $\nabla f_0(x^*) + \nu^*\nabla h_1(x^*) = 0 \to c - \tau(\frac{1}{x_1}, \ldots, \frac{1}{x_n})^\top + A^\top\nu^* = 0.$

3. You see that if τ is small, then solution of the second problem approximates solution of the first problem. But if τ is large, solution of the second problem is tending to infinity.