Heat Kernel Expansion

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1 Method

1.1 Effective Action

Let us start from the computation of the effective action in general. Given the classical Lagrangian, as we have seen it can be rewritten in terms of the renormalized field ϕ_r:

$$\mathcal{L}[\phi_r] = \mathcal{L}_r[\phi_r] + \Delta\mathcal{L}[\phi_r] \tag{1}$$

Then, introduce the external source J and also split it as:

$$J(x) = J_r(x) + \Delta J(x) \tag{2}$$

where, with the expansion $\phi_r(x) = \phi_{cl}(x) + \eta(x)$:

$$\left.\frac{\delta S_r[\phi_r]}{\delta\phi_r(x)}\right|_{\phi_r=\phi_{cl}} = -J_r(x) \tag{3}$$

$$\phi_{cl} = \langle\phi(x)\rangle|_{J_r+\Delta J} \tag{4}$$

To proceed, one may consider the functional $Z[J]$ in the following form:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \exp\{i\int d^dx(\mathcal{L}_r[\phi_r] + J(x)\phi_r(x))\} \\ &= \int \mathcal{D}\phi \exp\{i\int d^dx(\mathcal{L}_r[\phi_r] + J_r(x)\phi_r(x) + \Delta\mathcal{L}[\phi_r] + \Delta J(x)\phi_r(x))\} \end{aligned} \tag{5}$$

The action can be expanded around the background ϕ_{cl}. The first two terms are:

$$\begin{aligned} \int d^dx(\mathcal{L}_r[\phi_r] + J_r(x)\phi_r(x)) = &\int d^dx(\mathcal{L}_r[\phi_{cl}] + J_r(x)\phi_{cl}(x)) \\ &+ \int d^dx\eta(x)(\left.\frac{\delta S_r[\phi_r]}{\delta\phi_r(x)}\right|_{\phi_r=\phi_{cl}} + J_r(x)) \\ &+ \frac{1}{2}\int d^dxd^dy\eta(x)\eta(y)\left.\frac{\delta^2 S_r[\phi_r]}{\delta\phi_r(x)\delta\phi_r(y)}\right|_{\phi_r=\phi_{cl}} \\ &+ \cdots \end{aligned} \tag{6}$$

The second term vanishes by the classical equation of motion. The last two terms of the action is also expanded:

$$\begin{aligned} \int d^dx(\Delta\mathcal{L}_r[\phi_r] + \Delta J_r(x)\phi_r(x)) = &\int d^dx(\Delta\mathcal{L}_r[\phi_{cl}] + \Delta J_r(x)\phi_{cl}(x)) \\ &+ \int d^dx\eta(x)(\left.\frac{\delta\Delta S_r[\phi_r]}{\delta\phi_r(x)}\right|_{\phi_r=\phi_{cl}} + \Delta J_r(x)) \\ &+ \frac{1}{2}\int d^dxd^dy\eta(x)\eta(y)\left.\frac{\delta^2\Delta S_r[\phi_r]}{\delta\phi_r(x)\delta\phi_r(y)}\right|_{\phi_r=\phi_{cl}} \\ &+ \cdots \end{aligned} \tag{7}$$

The second term stands for a tadpole and must be canceled in such a way that $\langle\eta(x)\rangle_J = 0$, i.e, $\langle\phi_r\rangle_J = \phi_{cl}$. The other terms represent as the counter-terms for the self-interaction vertices. In total:

$$\begin{aligned} Z[J] =&\text{exp}\{i\int d^dx(\mathcal{L}_r[\phi_{cl}] + J_r(x)\phi_{cl}(x) + \Delta\mathcal{L}_r[\phi_{cl}] + \Delta J_r(x)\phi_{cl}(x))\} \\ &\int \mathcal{D}\eta\text{exp}\{i\tilde{S}[\eta] + i\Delta\tilde{S}[\eta]\} \end{aligned} \tag{8}$$

where:

$$\tilde{S}[\eta] = \frac{1}{2}\int d^dxd^dy\eta(x)(\frac{\delta^2 S_r}{\delta\phi_r^2}[\phi_{cl}](x,y))\eta(y) + \text{vertices.} \tag{9}$$

$$\Delta\tilde{S}[\eta] = \text{counter-terms} \tag{10}$$

Neglecting the interactions, the path integral over η is of Gaussian form and can be integrated explicitly. The generating functionals $Z[J]$ takes the form $Z[J] = \text{exp}[iW[J]]$, and $W[J]$ can be written in:

$$\begin{aligned} W[J] = &\int d^dx\{\mathcal{L}_r[\phi_{cl}(x)] + J_r(x)\phi_r(x) + \Delta\mathcal{L}_r[\phi_{cl}] + \Delta J_r(x)\phi_{cl}(x)\} \\ &+ \frac{1}{2}Tr\ln\frac{\delta^2 S_r}{\delta\phi_r\delta\phi_r}[\phi_{cl}] - i(\text{connected diagram}) \end{aligned} \tag{11}$$

One may need to perform the Legendre transform in order to compute the effective action $\Gamma[\phi_{cl}]$:

$$\Gamma[\phi_{cl}] = W[J] - \int d^dxJ(x)\phi_{cl}(x) \tag{12}$$

$$= S_r[\phi_{cl}] + \frac{1}{2}Tr\ln\frac{\delta^2 S_r}{\delta\phi_r\delta_r}[\phi_{cl}] - i(\text{connected diagram}) + \Delta S[\phi_{cl}] \tag{13}$$

Our particular interest is in the 1-loop corrections:

$$\Gamma^{\text{1-loop}}[\phi_{cl}] = \frac{1}{2}Tr\ln\frac{\delta^2 S_r}{\delta\phi_r\delta\phi_r} + \Delta^1 S \tag{14}$$

So it is necessary to evaluate the trace in some way to know the 1-loop corrections to the effective action.

1.2 Heat Kernel Expansion

In order to compute the trace given above, introduce the heat kernel:

$$K(t;x,y;D) = \langle x|\text{exp}(-tD)|\rangle \tag{15}$$

which should satisfy the heat conduction equation:

$$(\partial_t + D_x)K(t;x,y;D) = 0 \tag{16}$$

with the initial condition:

$$K(0;x,y;D) = \delta(x-y) \tag{17}$$

For instance, the kernel for $-\Delta$ is:

$$K(t;,x,y;-\Delta) = (4\pi t)^{-d/2}\text{exp}(-\frac{(x-y)^2}{4t}) \tag{18}$$

and for $D = D_0 = -\Delta + m^2$:

$$K(t;x,y;D_0) = (4\pi t)^{-d/2}\text{exp}(-\frac{(x-y)^2}{4t} - m^2t) \tag{19}$$

For a general D, $K(t;x,y;D_0)$ still describes the leading singularity as $t \to 0$ as in the form:

$$K(t;x,y;D) = K(t;,x,y;D_0)(1 + tb_2(x,y) + t^2b_4(x,y) + \cdots) \tag{20}$$

The heat kernel coefficients $b_{2k}(x,y)$ are regular n the limit $y \to x$.
Then we need to compute the functional:

$$\mathcal{W} = \frac{1}{2}\mathrm{Tr}\ln D \tag{21}$$

But for each positive eigenvalue λ of the operator D, one may have:

$$\ln\lambda = -\int_0^\infty \frac{dt}{t}e^{-t\lambda} \tag{22}$$

Then,

$$\begin{aligned}\mathcal{W} &= -\frac{1}{2}\int_0^\infty \frac{dt}{t}\mathrm{Tr}e^{-tD} \\ &= -\frac{1}{2}\int_0^\infty \frac{dt}{t}\int d^dx\sqrt{g}K(t;x,x;D)\end{aligned} \tag{23}$$

1.3 Generalization

With those coefficients, consider the functional $\mathcal{W}$, given the quadratic term of the Euclidean action:

$$S^{(2)} = \frac{1}{2}\int d^dx\Phi^T(-\nabla\nabla + Y)\Phi\ ,\ \Phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{pmatrix} \tag{24}$$

where $\nabla_\rho = \partial_\rho + X_\rho$, $X_\rho^T = -X_\rho$, and $Y^T = Y$. Then, one may have:

$$\begin{aligned}\mathcal{W} &= -\frac{1}{2}\int_0^\infty \frac{dt}{t}e^{-tm^2}\frac{1}{(4\pi t)^{-d/2}}\int d^dx\mathrm{Tr}(1 - tY + \frac{1}{2}t^2Y^2 + \frac{1}{12}t^2X_{\mu\nu}X_{\mu\nu} + \cdots) \\ &= \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int d^dx\{m^{d-2}\Gamma(1-\frac{d}{2})\mathrm{Tr}(Y) - m^{d-4}\Gamma(2-\frac{d}{2})\mathrm{Tr}(\frac{1}{2}Y^2 + \frac{1}{12}X_{\mu\nu}X_{\mu\nu}) + \cdots\}\end{aligned} \tag{25}$$

2 ϕ^4 case

Let us apply this method to the ϕ^4 theory, and reproduce its β functions. In this case, we have $X = 0$ and $Y = m^2 + \frac{\lambda}{2}\phi^2$. Then:

$$\mathcal{W} = \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int d^dx\{\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-\frac{d}{2}}}(m^2 + \frac{\lambda}{2}\phi^2) - \frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-\frac{d}{2}}}\frac{1}{2}(m^2 + \frac{\lambda}{2}\phi^2)^2 + \cdots\} \tag{26}$$

Then the counter-terms are:

$$\delta_{m^2} = -\frac{1}{2}\frac{\lambda}{(4\pi)^{d/2}}\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}} \tag{27}$$

$$\delta_z = 0 \tag{28}$$

$$\delta_\lambda = \frac{3}{2}\frac{\lambda^2}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-d/2}} \tag{29}$$

Consider the limit $d \to 4$ with $\epsilon \equiv d - 4$, then:

$$\begin{aligned}
\delta_{m^2} &= -\frac{1}{2}\mu^\epsilon \frac{\lambda}{(4\pi)^{2-\epsilon/2}} \frac{\Gamma(\frac{\epsilon}{2})}{-1+\frac{\epsilon}{2}} \frac{1}{(m^2)^{-1+\epsilon/2}} \\
&\xrightarrow[\epsilon \to 0]{} \frac{\lambda}{32\pi^2} m^2 (\frac{2}{\epsilon} - \gamma_E - \ln \frac{m^2}{\mu^2} + \mathcal{O}(\epsilon)) \\
&= \frac{\lambda}{32\pi^2} m^2 \ln \frac{\zeta}{\mu^2} \qquad (30) \\
\delta_\lambda &= \frac{3}{2} \frac{\mu^\epsilon \lambda^2}{(4\pi)^{2-\epsilon/2}} \frac{\Gamma(\frac{\epsilon}{2})}{(m^2)^{\epsilon/2}} \\
&\xrightarrow[d \to 4]{} \frac{3\lambda^2}{32\pi^2} \ln \frac{\zeta}{\mu^2} \qquad (31)
\end{aligned}$$

Therefore, β functions are:

$$\gamma = 0 \qquad (32)$$

$$\beta = \frac{3\lambda^2}{16\pi^2} \qquad (33)$$

$$\beta_{m^2} = \frac{\lambda}{16\pi^2} \qquad (34)$$

Now, let us try to reproduce the results for the ϕ-4 theory. Starting with the Euclidean bare action:

$$S[\phi] = \int d^d x \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4 \qquad (35)$$

Then, the Hessian is:

$$\begin{aligned}
\frac{\delta S}{\delta\phi\delta\phi} &= -\Box + m^2 + \frac{\lambda}{2}\phi^2 \\
&= \Delta + \frac{\lambda}{2}\phi^2 \qquad (36)
\end{aligned}$$

So the function to be traced is:

$$\mathrm{h}_\mu(\Delta, \omega) \frac{\partial_t R_\mu(\Delta)}{\Delta + \omega + R_\mu(\Delta)} \qquad (37)$$

and its Laplace transform:

$$\mathrm{h}_\mu(\Delta, \omega) = \int_0^\infty ds \tilde{\mathrm{h}}_\mu(s, \omega) e^{-s\Delta} \qquad (38)$$

where R_μ is a cutoff function. Thus, for an effective action Γ_μ with $t = \ln(\mu/\mu_0)$:

$$\begin{aligned}
\partial_t \Gamma_\mu[\phi] &= \frac{1}{2}\mathrm{Tr}\mathrm{h}_\mu(\Delta, m^2) \\
&= \frac{1}{2}\int_0^\infty ds \tilde{\mathrm{h}}_\mu(s, m^2) \mathrm{Tr} e^{-s\Delta} \qquad (39)
\end{aligned}$$

Then, given an arbitrary function $W(p^2)$ and its trace $\mathrm{Tr}[W(p^2)]$, with p^2 is the differential operator. Introduce the Laplace transform:

$$\mathrm{Tr}[W(p^2)] = \int_{-\infty}^\infty ds \tilde{W}(s) \mathrm{Tr}[e^{-sp^2}] \qquad (40)$$

the trace part on the r.h.s can be expanded:

$$\mathrm{Tr}[e^{-sp^2}] = \Sigma_{n=0} B_n(p^2) s^{-(d-n)/2} \qquad (41)$$

$$B_n(p^2) = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g}\mathrm{tr}[b_n] \qquad (42)$$

where b_n is heat kernel coefficients. Eventually:

$$\mathrm{Tr}[W(p^2)] = \frac{1}{(4\pi)^{d/2}}\Sigma_{n=0}(Q_{d/2-n}[W]\int d^dx\sqrt{g}\mathrm{tr}[b_n]) \tag{43}$$

Here, the Q-functionals are defined as:

$$Q_n[W] \equiv \int_0^\infty ds s^{-n}\tilde{W}(s) \tag{44}$$

$$Q_0[W] = W(0) \tag{45}$$

$$Q_n[W] = \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}W[z] \tag{46}$$

With those aids,

$$\begin{aligned}\partial_t\Gamma_\mu[\phi] &= \frac{1}{2(2\pi)^{d/2}}\int d^dx[Q_{d/2}[\mathrm{h}_\mu] - \frac{\lambda}{2}Q_{d/2-1}[\mathrm{h}_\mu] + \int_0^\infty \tilde{h}_\mu(s,m^2)s^{-d/2}\frac{\lambda^2}{4}s^2\phi^4(\frac{1}{2}+\frac{s\Box}{12}+\frac{s^2\Box^2}{120}+\cdots)] \\ &= \frac{1}{2(2\pi)^{d/2}}\int d^dx[Q_{d/2}[\mathrm{h}_\mu] - \frac{\lambda}{2}Q_{d/2-1}[\mathrm{h}_\mu] + \frac{1}{2}\frac{\lambda^2}{4}\phi^4\int_0^1 d\xi Q_{d/2-2}[h_\mu^{-\Box\xi(1-\xi)}]]\end{aligned} \tag{47}$$

where

$$h^a_{\mu,\omega}(z) \equiv h_\mu(z+a,\omega) \tag{48}$$

$$h_{\mu,\omega}(z) \equiv h_\mu(z,\omega) \tag{49}$$

Then with the ansatz;

$$\Gamma_\mu[\phi] = \int d^dx[\frac{Z_\mu}{2}\partial_\eta\phi\partial^\eta\phi + \frac{m_\mu^2}{2}\phi^2 + \frac{1}{4!}\phi^2F(-\Box)\phi^2] \tag{50}$$

with $F(0) = \lambda_\mu$. One may read off:

$$\partial_t Z_\mu = 0 \tag{51}$$

$$\partial_t m_\mu^2 = -\frac{1}{2}\frac{\lambda}{(4\pi)^{d/2}}Q_{d/2-1}[h_\mu] \tag{52}$$

$$\partial_t F(\Box) = \frac{3}{2}\frac{\lambda^2}{(4\pi)^{d/2}}\int_0^1 d\xi Q_{d/2-2}[h_\mu^{-\Box\xi(1-\xi)}] \tag{53}$$

$$\overrightarrow{\Box\to 0}\partial_t\lambda_\mu = \frac{3}{2}\frac{\lambda^2}{(4\pi)^{d/2}}Q_{d/2-2}[h_\mu] \tag{54}$$

If one sets $d=4$, the previous results would be recovered.