Generic Potential

April 2, 2023

Let us work on the case for a generic potential $V(\phi)$ for scalars ϕ. For the simplicity, start with the single scalar action:

$$S = \int d^dx\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\} \tag{1}$$

Then, the quadratic term is:

$$S^{(2)} = \frac{1}{2}\int d^dx\eta(-\Box + V''(\phi))\eta \tag{2}$$

So one may obtain:

$$\frac{\delta^2 S}{\delta\phi\delta\phi} = -\Box + V''(\phi) := D \tag{3}$$

Therefore:

$$\begin{aligned}\mathcal{W} &= \frac{1}{2}\mathrm{Tr}\ln D \\ &= -\frac{1}{2}\int_0^\infty \frac{dt}{t}\mathrm{Tr}e^{-tD}\end{aligned} \tag{4}$$

But, now one can use the expansion for the trace of the heat kernel:

$$\mathrm{Tr}e^{-tD} = \sum_{k\geq 0} t^{(k-d)/2}a_k \tag{5}$$

where a_k are heat kernel coefficients which are, for instance, given in Vassilevich.
Thus, introducing a cut-off at $t = \Lambda^{-2}$:

$$\begin{aligned}\mathcal{W} &= -\frac{1}{2}\int_{\Lambda^{-2}}^\infty dt\sum t^{\frac{k-d}{2}-1}a_k \\ &= -\frac{1}{2}\int_{\Lambda^{-2}}^\infty dt\{t^{-d/2-1}a_0 + t^{-d/2}a_2 + t^{-d/2+1}a_4\cdots\}\end{aligned} \tag{6}$$

Notice that only finitely many terms involve the divergent one would need to care, i.e. only the coefficients with $k \leq d$ are necessary. Therefore, defining $\tau \equiv \ln\frac{\Lambda}{\Lambda_0}$:

$$\partial_\tau V_\Lambda = -\frac{1}{2}\partial_\tau\{\int_{\Lambda^{-2}}^\infty dt\sum_{k\leq d} t^{\frac{k-d}{2}-1}a_k\} \tag{7}$$

From this "beta functional", each beta-function corresponding to each coupling can be extracted.
For a theory with multiple scalars ϕ^i, this could be generalized. In this case, the second functional derivative is in the form:

$$\frac{\delta S}{\delta\phi^i\delta\phi^j} = -\Box\delta^{ij} + U(\phi) \tag{8}$$

where $U(\phi)$ includes the additional terms in its potential which involve $i \neq j$.
Thus, specifying terms involving the divergence with heat kernel coefficients with same procedure, eventually one may obtain a "beta-functional" depending on scalars from which all beta-functions of all couplings are extracted. Ultimately, once the form of the potential of the theory is determined, plugging them into the heat kernel coefficients, all beta-functions can be specified.