Beta Function

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1 Beta Functions for ϕ^4 theory

1.1 Setting

Let us start with the bare Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{4!}\lambda_0\phi^4 \tag{1}$$

Then, with the renormalized field,

$$\phi = Z^{1/2}\phi_r \tag{2}$$

the renormalized Lagrangian takes tthe form;

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_r + \Delta\mathcal{L} \\ &= (\frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}m^2\phi_r^2 - \frac{1}{4!}\lambda\phi_r^4) + (\frac{1}{2}\Delta_Z\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}\Delta_m\phi_r^2 - \frac{1}{4!}\Delta_\lambda\phi_r)\end{aligned} \tag{3}$$

where the counter terms are denoted as;

$$\Delta_Z = Z - 1, \Delta_m = m_0^2 Z - m^2, \Delta_\lambda = \lambda_0 Z^2 - \lambda \tag{4}$$

Also consider the renormalized Green's functions defined with the renormalized fields and depend on the scale μ, the coupling λ and the mass m^2;

$$G_r(x_1, x_2, ..., x_n : \mu, \lambda, m^2) = \langle\phi_r(x_1)...\phi_r(x_n)\rangle \tag{5}$$

On the other hand, the bare Green's functions are defined with the bare fields and depend on the bare coupling λ_0 and the bare mass m_0^2:

$$G(x_1, x_2, ..., x_n : \lambda_0, m_0^2) = \langle\phi(x_1)...\phi(x_n)\rangle \tag{6}$$

Recalling that $\phi(x) = Z^{1/2}\phi_r(x)$, where Z is depending on the scaling μ, one may deduce that:

$$G(x_1, x_2, ..., x_n : \lambda_0, m_0^2) = Z^{1/2}G_r(x_1, x_2, ..., x_n : \mu, \lambda, m^2) \tag{7}$$

under the infinitesimal transformations $\mu \to \mu + \delta\mu$, $\lambda \to \lambda + \delta\lambda$, $m^2 \to m^2 + \delta m^2$ combined with $Z \to Z + \delta Z$:

$$\begin{aligned}&(\delta\mu\frac{\partial}{\partial\mu} + \delta\lambda\frac{\partial}{\partial\lambda} + \delta Z\frac{\partial}{\partial Z} + \delta m^2\frac{\partial}{\partial m^2})Z^{n/2}G_r(x_1, x_2, ..., x_n : \mu, \lambda, m^2) = 0 \\ &\Leftrightarrow (\delta\mu\frac{\partial}{\partial\mu} + \delta\lambda\frac{\partial}{\partial\lambda} + \frac{n}{2}\frac{\delta Z}{Z} + \delta m^2\frac{\partial}{\partial m^2})G_r(x_1, x_2, ..., x_n : \mu, \lambda, m^2) = 0\end{aligned} \tag{8}$$

Eventually, multiplying by $\frac{\mu}{\delta\mu}$, and rewriting the differentials as derivatives taken at constant value for the bare coupling, it is in the form:

$$(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + n\gamma + \beta_m m^2\frac{\partial}{\partial m^2})G_r = 0 \tag{9}$$

where:

$$\beta = \mu \frac{\partial \lambda}{\partial \mu} = \frac{\partial \lambda}{\partial \ln \mu} \tag{10}$$

$$\gamma = \frac{1}{2} \frac{\mu}{Z} \frac{\partial Z}{\partial \mu} = \frac{\partial \ln Z}{\partial \ln \mu} \tag{11}$$

$$\beta_m = \frac{\mu}{m^2} \frac{\partial m^2}{\partial \mu} = \frac{\partial \ln m^2}{\partial \ln \mu} \tag{12}$$

1.2 Computation

First, let us define the renormalization conditions as following:

$$G_r^{(2)}(p^2 = m^2) = \frac{i}{p^2 - m^2} + (reg.) \tag{13}$$

$$i\,\mathcal{A}(s = 4m^2, t = u = 0) = -i\lambda \tag{14}$$

where G_r^2 is a connected 2-point function, and $i\mathcal{A}$ stands for the amplitude for the scattering two scalars into two scalars. Also, the Mandelstam variables are $s = (p_1 + p_2)^2$, $t = (p_3 - p_1)^2$, $u = (p_4 - p_1)^2$.
Start with the 4-point function on 4-dimension and up to 1-loop correction. The 1-loop contribution to s-channel is given by:

$$\Sigma(p^2) = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p-k)^2 - m^2 + i\epsilon} \tag{15}$$

where $p = p_1 + p_2 = p_3 + p_4$ is the external momentum. Suppose that one analytically continues to a space of $d-1$ spatial and 1 time dimensions, then the momentum in the integral can be written in;

$$k^\mu = (k_0, k_1, ..., k_{d-1}) \tag{16}$$

But the external momenta are:

$$p^\mu = (p_0, p_1, p_2, p_4, 0, ..., 0) \tag{17}$$

in d-dimensional Minkowski space.
With this note, one may have a d-dimensional integral:

$$I = \int \frac{d^dk}{(2\pi)^d} \mu^{4-d} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon} \tag{18}$$

Here, the scale μ is introduced to compensate the change in the dimensions of fields and couplings. Notice that this integral is convergent for $d < 4$. In oder to continue, it is useful to apply the Feynman parametrization:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \tag{19}$$

Then with writing $\epsilon \equiv 4 - d$, the integral would be:

$$\begin{aligned}
I &= \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \mu^\epsilon \frac{1}{[x((p-k)^2 - m^2) + (1-x)(k^2 - m^2)]^2} \\
&= \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \mu^\epsilon \frac{1}{[k^2 - 2xk \cdot p + xp^2 - m^2]^2} \\
&= \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \mu^\epsilon \frac{1}{[(k - xp)^2 + x(1-x)p^2 - m^2]^2} \\
&= \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \mu^\epsilon \frac{1}{[l^2 - \Delta^2 + i\epsilon]^2} \qquad (l^\mu \equiv k^\mu - xp^\mu,\ \Delta^2 \equiv m^2 - x(1-x)p^2) \\
&= i\mu^\epsilon \int_0^1 dx \int \frac{d^dl_E}{(2\pi)^d} \mu^\epsilon \frac{1}{[l_E^2 - \Delta^2 + i\epsilon]^2} \qquad \text{(Wick rotated)} \\
&= \frac{1}{(2\pi)^d} i\mu^\epsilon \int_0^1 dx \int d\Omega_d \int dl_E \frac{l_E^{d-1}}{[l_E^2 - \Delta^2 + i\epsilon]^2}
\end{aligned} \tag{20}$$

But notice the Gaussian integral is:

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi} \tag{21}$$

Then,

$$\begin{aligned}
(\sqrt{\pi})^d = (\int_{-\infty}^{\infty} dx e^{-x^2})^d &= \int d^d x e^{-\Sigma_{i=1}^d x_i^2} \\
&= \int d\Omega_d \int dx x^{d-1} e^{-x^2} \\
&= \int d\Omega_d \int_0^\infty \frac{dx^2}{2} (x^2)^{d/2-1} e^{-x^2} \\
&= \int d\Omega_d \frac{1}{2} \Gamma(d/2)
\end{aligned} \tag{22}$$

Thus, the integral being discussed is:

$$\begin{aligned}
I &= i\mu^\epsilon \int_0^1 dx \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{(2\pi)^d} \int_0^\infty dl_E \frac{l_E^{d-1}}{[l_E^2 - \Delta^2 + i\epsilon]^2} \\
&= \frac{i\mu^\epsilon \pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_0^1 dx \int_0^\infty dl_E^2 \frac{(l_E^2)^{d/2-1}}{[l_E^2 - \Delta^2 + i\epsilon]^2} \\
&= \frac{i\mu^\epsilon \pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_0^1 dx \frac{1}{[\Delta^2 - i\epsilon]^{2-d/2}} \frac{\Gamma(d/2)\Gamma(2-d/2)}{\Gamma(2)} \\
&= \frac{i\mu^\epsilon \pi^{d/2}}{(2\pi)^d} \Gamma(2 - \frac{d}{2}) \int_0^1 dx \frac{1}{(\Delta^2)^{2-d/2}} \\
&= \frac{i\mu^\epsilon \pi^{d/2}}{(2\pi)^d} \Gamma(\frac{\epsilon}{2}) \int_0^1 dx \frac{1}{(\Delta^2)^{\epsilon/2}}
\end{aligned} \tag{23}$$

On the third line, the relation between the beta and gamma functions was used:

$$\int_0^\infty dt \frac{t^{m-1}}{(t^2 + \Delta^2)^n} = \frac{1}{(\Delta^2)^{n-m}} \frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)} \tag{24}$$

Notice that $\Gamma(z)$ has poles at $z \in \mathbb{Z}_{\leq 0}$. Therefore, the integral I has poles for $d = 4, 6, 8, \cdots$. As mentioned above, our particular interest is in the limit $d \to 4$, that is, $\epsilon \to 0$. Then one may have expansions:

$$\Gamma(\frac{\epsilon}{2}) \simeq \frac{2}{\epsilon} - \gamma_E - \mathcal{O}(\epsilon) \tag{25}$$

$$\frac{1}{(\Delta^2)^{\epsilon/2}} \simeq 1 - \frac{\epsilon}{2} \ln \Delta^2 + \mathcal{O}(\epsilon^2) \tag{26}$$

$$\mu^\epsilon = (\mu^2)^{\epsilon/2} \simeq 1 + \frac{2}{\epsilon} \ln \mu^2 + \cdots \tag{27}$$

where γ_E is the Eüler-Mascheroni constant. And therefore,

$$\mu^\epsilon \Gamma(\frac{\epsilon}{2}) \frac{1}{(\Delta^2)^{\epsilon/2}} \simeq \frac{2}{\epsilon} - \gamma_E - \ln \frac{\Delta^2}{\mu^2} + \mathcal{O}(\epsilon) \tag{28}$$

This reads:

$$I = \frac{i}{16\pi^2} \int_0^1 dx (\frac{2}{\epsilon} - \gamma_E - \ln \frac{m^2 - x(1-x)p^2}{\mu^2} + \mathcal{O}(\epsilon)) \tag{29}$$

The contribution is:

$$\begin{aligned}
\Sigma(p^2) &= -\frac{(-i\lambda)^2}{2} I \\
&= \frac{i\lambda^2}{32\pi^2} \int_0^1 dx (\frac{2}{\epsilon} - \gamma_E - \ln \frac{m^2 - x(1-x)p^2}{\mu^2} + \mathcal{O}(\epsilon))
\end{aligned} \tag{30}$$

According to the renormalization condition:

$$\begin{aligned}
i\mathcal{A}(s=4m^2, t=u=0) &= -i\lambda + \Sigma(4m^2) + 2\Sigma(0) - i\delta_\lambda \\
&= -i\lambda \\
\Leftrightarrow \Sigma(4m^2) + 2\Sigma(0) - i\delta_\lambda = 0 & \\
\Leftrightarrow \delta_\lambda &= \frac{\lambda^2}{32\pi^2}\int_0^1 dx\Big(\frac{6}{\epsilon} - 3\gamma_E - 3\ln\frac{m^2}{\mu^2} - \ln 4(1-x(1-x))\Big) \\
&= \frac{3\lambda^2}{32\pi^2}\ln\frac{\zeta}{\mu^2} + \text{finite term}
\end{aligned} \tag{31}$$

Similarly for the case of the two-point function. Define $-iM^2(p^2)$ as the sum of all one-particle irreducible insertions (1PI) into the propagator, the full propagator is:

$$\Delta_F = \frac{i}{p^2 - m^2 - M^2(p^2)} \tag{32}$$

The renormalized condition imposed on this propagator requires that $p^2 = m^2$ should be the pole, and the residue is supposed to be 1. Expanding M^2 about $p^2 = m^2$, one may have:

$$\Delta_F = \frac{i}{p^2 - m^2 - (M^2(m^2) + (p^2 - m^2)\frac{d}{dp^2}M^2(m^2) + \cdots)} \tag{33}$$

Therefore, the renormalized condition can be reinterpreted into two conditions;

$$M^2(p^2)|_{p^2=m^2} = 0 \text{ and } \frac{d}{dp^2}M^2(p^2)|_{p^2=m^2} = 0 \tag{34}$$

Then, explicitly, again, to 1-loop order:

$$-iM^2(p^2) = -\frac{i\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2} + i(p^2\delta_z - \delta_m) \tag{35}$$

With the same method, regularizing the integral:

$$\begin{aligned}
J &= -\frac{i\lambda}{2}\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - m^2} \\
&= \frac{-i\lambda}{2}\frac{\mu^\epsilon}{(2\pi)^d}\int d\Omega_d \int_0^\infty dk_E \frac{k_E^{d-1}}{k_E^2 + m^2 - i\epsilon} \\
&= \frac{-i\lambda}{2}\frac{\mu^\epsilon}{(2\pi)^d}\frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty dk_E^2 \frac{(k_E^2)^{d/2-1}}{k_E^2 + m^2 - i\epsilon} \\
&= \frac{-i\lambda}{2}\frac{\mu^\epsilon}{(2\pi)^d}\frac{\pi^{d/2}}{\Gamma(d/2)}\frac{1}{(m^2)^{1-d/2}}\frac{\Gamma(\frac{d}{2})\Gamma(1-d/2)}{\Gamma(1)} \\
&= \frac{-i\lambda}{2}\frac{\mu^\epsilon}{(2\pi)^d}\pi^{d/2}\frac{1}{(m^2)^{1-d/2}}\Gamma(1-d/2)
\end{aligned} \tag{36}$$

Using the property of Gamma function:

$$z\Gamma(z) = \Gamma(z+1) \tag{37}$$

one may rewrite:

$$\begin{aligned}
(1-d/2)\Gamma(1-d/2) &= \Gamma(2-d/2) = \Gamma(\epsilon/2) \\
\Leftrightarrow \Gamma(1-d/2) &= \frac{\Gamma(\epsilon/2)}{1-d/2}
\end{aligned} \tag{38}$$

Also, notice that:

$$(m^2)^{1-d/2} = (m^2)^{-1}(m^2)^{\epsilon/2} \tag{39}$$

Finally, the expansion around the limit $d \to 4$ is given as following:

$$\begin{aligned}
J &= \frac{-i\lambda}{2} \frac{\pi^{d/2}}{(2\pi)^d} \frac{m^2}{1-d/2} \frac{\mu^\epsilon}{(m^2)^{\epsilon/2}} \Gamma(\epsilon/2) \\
&\xrightarrow[d\to 4]{} \frac{i\lambda}{32\pi^2} m^2 (\frac{2}{\epsilon} - \gamma_E - \ln\frac{m^2}{\mu^2} + \mathcal{O}(\epsilon)) \\
&= \frac{i\lambda}{32\pi^2} m^2 \ln\frac{\zeta}{\mu^2} + \text{finite terms.}
\end{aligned} \tag{40}$$

Therefore, comparing with the renormalization condition redefined above,

$$\delta_z = 0 \tag{41}$$

$$\delta_{m^2} = \frac{\lambda}{32\pi^2} m^2 \ln\frac{\zeta}{\mu^2} \tag{42}$$

The computations of the renormalization group functions are carried with the definitions of the counter-terms $\delta_Z = Z - 1$, $\delta_\lambda = \lambda_0 Z^2 - \lambda$, and $\delta_{m^2} = m_0^2 Z - m^2$:

$$\gamma = \frac{1}{2}\frac{\mu}{Z}\frac{\partial Z}{\partial \mu} = \frac{1}{2}\mu\frac{\partial \delta_Z}{\partial \mu} = 0 \tag{43}$$

$$\begin{aligned}
\beta &= \mu\frac{\partial \lambda}{\partial \mu} = \mu\frac{\partial}{\partial \mu}(\lambda_0 + 2\delta_Z\lambda_0 - \delta_\lambda + (\delta_Z)^2\lambda_0) \\
&= -\mu\frac{\partial \delta_\lambda}{\partial \mu} = \frac{3\lambda^2}{16\pi^2}
\end{aligned} \tag{44}$$

$$\begin{aligned}
\beta_m &= \frac{\mu}{m^2}\frac{\partial m^2}{\partial \mu} = -\frac{\mu}{m^2}\frac{\partial \delta_{m^2}}{\partial \mu} \\
&= \frac{\lambda}{16\pi^2}
\end{aligned} \tag{45}$$

Solving for λ and m^2:

$$\begin{aligned}
\ln\frac{\mu}{\mu_0} &= \frac{16\pi^2}{3}(\frac{1}{\lambda_0} - \frac{1}{\lambda}) \\
\Leftrightarrow \lambda(\mu) &= \frac{\lambda_0}{1 - \frac{3\lambda_0}{16\pi^2}\ln\frac{\mu}{\mu_0}}
\end{aligned} \tag{46}$$

$$\begin{aligned}
\ln\frac{m^2}{m_0^2} &= \frac{1}{3}\ln\frac{\lambda}{\lambda_0} \\
\Leftrightarrow m^2(\mu) &= [1 - \frac{3\lambda_0}{16\pi^2}\ln\frac{\mu}{\mu_0}]^{-1/3} m_0^2
\end{aligned} \tag{47}$$

A Feynman Parametrization

It is known the Feynman parametrization in general form:

$$\frac{1}{a_1 a_2 \cdots a_n} = \int_0^1 dx_1 \cdots \int_0^1 dx_n \frac{\Gamma(n)\delta(1 - x_1 - \cdots - x_n)}{(a_1 x_1 + \cdots + a_n x_n)^n} \tag{48}$$

Let us prove it briefly.

$$\begin{aligned}
I &= \frac{1}{a_1 \cdots a_n} \\
&= \int_0^\infty dt_1 \cdots \int_0^\infty dt_n e^{-(a_1 t_1 + \cdots + a_n t_n)} \\
&= \int_0^\infty dt \int_0^\infty dt_1 \cdots \int_0^\infty dt_n \delta(t - t_1 - \cdots - t_n) e^{-(a_1 t_1 + \cdots + a_n t_n)}
\end{aligned}$$

Substituting $t_i = tx_i$,

$$\begin{aligned}
I &= \int_0^\infty dt \int_0^\infty dx_1 \cdots \int_0^\infty dx_n t^n \delta(t(1 - x_1 - \cdots - t_n)) e^{-t(a_1 x_1 + \cdots + a_n x_n)} \\
&= \int_0^\infty dx_1 \cdots \int_0^\infty dx_n \delta(1 - x_1 - \cdots - x_n) \int_0^\infty dt t^{n-1} e^{-t(a_1 x_1 + \cdots + a_n x_n)} \\
&= \int_0^1 dx_1 \cdots \int_0^1 dx_n \frac{\Gamma(n)\delta(1 - x_1 - \cdots - x_n)}{(a_1 x_1 + \cdots + a_n x_n)}
\end{aligned} \tag{49}$$