With perturbed metric, geodesic equations are:

$$\begin{cases} \varphi_2\ddot{\varphi}_1 - 2\dot{\varphi}_1\dot{\varphi}_2 = \dfrac{\epsilon}{8\pi^2}\dfrac{\varphi_2^2}{k^2}(m\dot{\varphi}_1^2 + 2M_2\dot{\varphi}_1\dot{\varphi}_2 - m\dot{\varphi}_2^2) & (1) \\ \varphi_2\ddot{\varphi}_2 + \dot{\varphi}_1^2 - \dot{\varphi}_2^2 = \dfrac{\epsilon}{8\pi^2}\dfrac{\varphi_2^2}{k^2}(-M_1\dot{\varphi}_1^2 - 2m\dot{\varphi}_1\dot{\varphi}_2 + M_1\dot{\varphi}_2^2) & (2) \end{cases}$$

Now let us have an ansatz such that the solutions for those equations (φ_1, φ_2) are in the forms of $(\tilde{\varphi}_1 + \epsilon f(s), \tilde{\varphi}_2 + \epsilon g(s))$, where tildes stand for the unperturbed solutions, with the constraints $(\varphi_1, \varphi_2)\big|_{s=0} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ and $(\dot{\varphi}_1, \dot{\varphi}_2)\big|_{s=0} = (\dot{\tilde{\varphi}}_1, \dot{\tilde{\varphi}}_2)$. Knowing $\tilde{\varphi}_1 = \varphi_1^0 =$ constant, and $\tilde{\varphi}_2 = \varphi_2^0 e^{\frac{s}{\sqrt{k}}}$, 1 leads:

$$\begin{aligned} &(\tilde{\varphi}_2 + \epsilon g(s))(\ddot{\tilde{\varphi}}_1 + \epsilon\ddot{f}(s)) - 2(\dot{\tilde{\varphi}}_1 + \epsilon\dot{f}(s))(\dot{\tilde{\varphi}}_2 + \epsilon\dot{g}(s)) \\ &= \frac{\epsilon}{8\pi^2}\frac{1}{k^2}(\tilde{\varphi}_2 + \epsilon g(s))^2\{m(\dot{\tilde{\varphi}}_2 + \epsilon\dot{f}(s))^2 + 2M_2(\dot{\tilde{\varphi}}_1 + \epsilon\dot{f}(s))(\dot{\tilde{\varphi}}_2 + \epsilon\dot{g}(s)) - m(\dot{\tilde{\varphi}}_2 + \epsilon\dot{g}(s))^2\} \end{aligned} \quad (3)$$

$$\Rightarrow \begin{aligned} &\tilde{\varphi}_2\ddot{\tilde{\varphi}}_1 - 2\dot{\tilde{\varphi}}_1\dot{\tilde{\varphi}}_2 + \epsilon\{\tilde{\varphi}_2\ddot{f}(s) + \ddot{\tilde{\varphi}}_1 g(s) - 2(\dot{\tilde{\varphi}}_1\dot{g}(s) + \dot{\tilde{\varphi}}_2\dot{f}(s))\} + \mathcal{O}(\epsilon^2) \\ &= \frac{\epsilon}{8\pi^2}\frac{1}{k^2}\tilde{\varphi}_2{}^2(m\dot{\tilde{\varphi}}_2^2 + 2M_2\dot{\tilde{\varphi}}_1\dot{\tilde{\varphi}}_2 - m\dot{\tilde{\varphi}}_2^2) + \mathcal{O}(\epsilon^2) \end{aligned} \quad (4)$$

$$\Rightarrow \ddot{f}(s) - \frac{2}{\sqrt{k}}\dot{f}(s) + \frac{1}{8\pi^2}\frac{m}{k^3}\tilde{\varphi}_2{}^3 = 0 \quad (5)$$

Solution to the homogeneous equation is;

$$f_h(s) = C_1 + C_2 e^{2s/\sqrt{k}} \quad (6)$$

and let $f_p(s) = Ae^{3s/\sqrt{k}}$ be the particular solution, then since $\ddot{f}_p(s) = \frac{9}{k}Ae^{3s/\sqrt{k}}$, $\dot{f}_p(s) = \frac{3}{\sqrt{k}}$, $A = -\frac{k}{3}\frac{1}{8\pi^2}\frac{m}{k^3}(\varphi_2^0)^3$. Therefore:

$$\begin{aligned} f(s) &= f_h(s) + f_p(s) \\ &= C_1 + C_2 e^{2s/\sqrt{k}} - \frac{k}{3}\frac{1}{8\pi^2}\frac{m}{k^3}(\varphi_2^0)^3 e^{3s/\sqrt{k}} \end{aligned} \quad (7)$$

According to the initial conditions:

$$f(0) = C_1 + C_2 - \frac{1}{3}\frac{m}{8\pi^2 k^2}(\varphi_2^0)^3 \quad (8)$$

$$\dot{f}(0) = \frac{2}{\sqrt{k}}C_2 - \frac{1}{8\pi^2}\frac{m}{k^2\sqrt{k}}(\varphi_2^0)^3 \quad (9)$$

Eventually, the correction for φ_1 is:

$$f(s) = -\frac{1}{8\pi^2}\frac{m}{6k^2}(\varphi_2^0)^3(1 - 3e^{\frac{2s}{\sqrt{k}}} + 2e^{\frac{3s}{\sqrt{k}}}) \quad (10)$$

Also, for 2:

$$\tilde{\varphi}_2\ddot{\tilde{\varphi}}_2 - \dot{\tilde{\varphi}}_2^2 + \epsilon(\varphi_2\ddot{g}(s) + \ddot{\tilde{\varphi}}_2 g(s) - 2\dot{\tilde{\varphi}}_2\dot{g}(s)) = \frac{\epsilon}{8\pi^2}\frac{M_1}{k^2}\tilde{\varphi}_2{}^2\dot{\tilde{\varphi}}_2^2 \quad (11)$$

$$\Rightarrow \ddot{g}(s) - \frac{2}{\sqrt{k}}\dot{g}(s) + \frac{1}{k}g(s) - \frac{1}{8\pi^2}\frac{M_1}{k^3}(\varphi_2^0)^3 e^{\frac{3s}{\sqrt{k}}} = 0 \quad (12)$$

Similarly to $f(s)$:

$$g(s) = \frac{1}{8\pi^2}\frac{M_1}{4k^{5/2}}(\varphi_2^0)^3 e^{s/\sqrt{k}}(-\sqrt{k} - 2s + \sqrt{k}e^{2s/\sqrt{k}}) \quad (13)$$

Thus, corrected geodesics, up to $\mathcal{O}(\epsilon)$, are:

$$\begin{cases} \varphi_1 = \varphi_1^0 - \dfrac{\epsilon}{8\pi^2}\dfrac{m}{6k^2}(\varphi_2^0)^3(1 - 3e^{\frac{2s}{\sqrt{k}}} + 2e^{\frac{3s}{\sqrt{k}}}) & (14) \\ \varphi_2 = \varphi_2^0 e^{\frac{s}{\sqrt{k}}} + \dfrac{\epsilon}{8\pi^2}\dfrac{M_1}{4k^{5/2}}(\varphi_2^0)^3 e^{\frac{s}{\sqrt{k}}}(-\sqrt{k} - 2s + \sqrt{k}e^{\frac{2s}{\sqrt{k}}}) & (15) \end{cases}$$

Hence:

$$
\begin{aligned}
\frac{\mathrm{d}\varphi_1}{\mathrm{d}\varphi_2} &= \frac{\mathrm{d}\varphi_1}{\mathrm{d}\tilde{\varphi_2}}\frac{\mathrm{d}\tilde{\varphi_2}}{\mathrm{d}\varphi_2} \\
&= -\frac{\epsilon}{8\pi^2}\frac{m}{6k^2}(-6\varphi_2^0\tilde{\varphi_2}+6\tilde{\varphi_2}^2)(1+\frac{\epsilon}{8\pi^2}\frac{M_1}{4k^{5/2}}(-\sqrt{k}(\varphi_2^0)^2-2s(\varphi_2^0)^2+3\sqrt{k}\tilde{\varphi_2}^2))^{-1} \\
&= -\frac{\epsilon}{8\pi^2}\frac{m}{k^2}(\tilde{\varphi_2}-\varphi_2^0)\tilde{\varphi_2}+\mathcal{O}(\epsilon^2) \qquad (16)
\end{aligned}
$$

With this aid, the distance between two points on the corrected geodesic is computated:

$$
\begin{aligned}
s &= \int ds = \int \sqrt{g_{ij}\mathrm{d}\varphi_i\mathrm{d}\varphi_j} \\
&= \int \sqrt{[\frac{k}{\varphi_2^2}-\frac{\epsilon}{8\pi^2}\{\frac{M_2}{k}-(\frac{\Lambda}{\varphi_2})^2\}]\mathrm{d}\varphi_1^2+\frac{\epsilon}{4\pi^2}\frac{m}{k}\mathrm{d}\varphi_1\mathrm{d}\varphi_2+[\frac{k}{\varphi_2^2}-\frac{\epsilon}{8\pi^2}\{\frac{M_1}{k}-(\frac{\Lambda}{\varphi_2})^2\}]\mathrm{d}\varphi_2^2} \\
&= \int [[\frac{k}{\varphi_2^2}-\frac{\epsilon}{8\pi^2}\{\frac{M_2}{k}-(\frac{\Lambda}{\varphi_2})^2\}]\{-\frac{\epsilon}{8\pi^2}\frac{m}{k^2}\tilde{\varphi_2}(\tilde{\varphi_2}-\varphi_2^0)\}^2+\frac{\epsilon}{4\pi^2}\frac{m}{k}\{-\frac{\epsilon}{8\pi^2}\frac{m}{k^2}\tilde{\varphi_2}(\tilde{\varphi_2}-\varphi_2^0)\} \\
&+[\frac{k}{\varphi_2^2}-\frac{\epsilon}{8\pi^2}\{\frac{M_1}{k}-(\frac{\Lambda}{\varphi_2})^2\}]]^{1/2}\,\mathrm{d}\varphi_2 \\
&\sim \int \frac{\sqrt{k}}{\varphi_2}[1-\frac{\epsilon}{16\pi^2}\frac{\varphi_2^2}{k}\{\frac{M_1}{k}-(\frac{\Lambda}{\varphi_2})^2\}]+\mathcal{O}(\epsilon^2)\,\mathrm{d}\varphi_2 \qquad (17) \\
&= \sqrt{k}\ln|\frac{\varphi_2}{\varphi_2^0}|-\frac{\epsilon}{16\pi^2}\{\frac{M_1}{2k\sqrt{k}}(\varphi_2^2-(\varphi_2^0)^2)-\frac{\Lambda^2}{\sqrt{k}}\ln|\frac{\varphi_2}{\varphi_2^0}|\}+\mathcal{O}(\epsilon^2) \qquad (18)
\end{aligned}
$$

Apparently in the limit $\epsilon \to 0$, $\varphi_2 \to \tilde{\varphi_2}$ and therefore the distance of the original geodesic is recovered.