UNIVERSITY NAME

MASTER THESIS

Constraints on The Infinite Distance Limit Under Renormalization Group

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A thesis submitted in fulfillment of the requirements for the degree of Master of Physics

in the

Research Group Name
Department or School Name

August 31, 2023

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Chapter 1

Introduction

One of the biggest goal particle physics has been trying to achieve is find the most elementary theory of this world. Recently finding the Higgs particle, researches has constructed successfully the Standard model (SM). However, this is believed as not a 'theory of everything', but a good approximation which is valid under a certain energy scale. In fact, there are a few phenomenological flaws in the SM such as dark matters. In addition, the SM is the quantum theory of interactions except gravity, and therefore it is natural to assume that it would collapse at Planck scale M_p, where the effects of gravity is needed to take into account. Practically, it had been believed that such phenomena at extremely high energy can be ignored when we were dealing with characteristic energy scales much smaller than M_p according to the idea of effective field theories (EFTs).

Nevertheless, recent researches on the string theory and discussions about black holes suggest that requirements of consistency of the physics at Planck scale M_p, where quatum gravity is needed, likely impose constraints on its low-energy EFT. In other words, it is expected that not every theory which is consistent from the perspective of quantum field theory will be also consistent when quantum gravity is considered. The swampland is the set of the low-energy theories which cannot be compatible with quantum gravity, while subsets of entire low-energy theories coupled to gravity are called the landscape if they are compatible. Those constraints are called the swampland constraints, and lead to possibilities that we should not consider the phenomenological problems at low-energy scales and quantum gravity at ultraviolet scale separately, but think of them as they are related each other. Attempts to discover such constraints, prove (or disprove) and improve them are called the Swampland program of quantum gravity.

Since we haven't discovered apparent suggenstions for novel physics while SM has been established experimentally, it has been considered to be useful that we reconsider the belief that SM is the low-energy EFT coupled to gravity, and examine chances if the Swampland program may offer certain hints for particle phenomenology. For this reason, some statements have been proposed to clarify the border separating the swampland and landscape in the space of consistent low-energy EFTs. They are called the Swampland conjectures, because they are still being studied their validities, while some of them are widely accepted as most likely correct. Those conjectures give certain properties qualitatively or quantitatively which the EFTs must obey or avoid in order to realize a consistent completion to quantum gravity. For instance, almost all literatures about the Swampland conjectures start from a claim that there cannot be global symmetries in quantum gravity, that is, any symmetries must be either broken or gauged at high-energy scale. Other examples are such as Weak Gravity Conjecture, Swampland Distance Conjecture, and de Sitter Conjecture. Those conjectures are often supported and motivated string theory arguments, while the concepts of the swampland is originally not restricted to string theory. Interestingly, it has seemed from recent studies that some of those conjecures are closely related. It suggests that they are perhaps alternative aspects of certain unknown properties of quantum gravity.

Our interest in this thesis will be on the Swampland Distance Conjecture (SDC). SDC gives a statement about the moduli space, which is controlled by the vacuum expectation value of scalar fields. It claims that in such a space there will be an infinite tower of states which becomes exponentially light at any infinite field distance limit. This is equivalent to say that the cutoff scale for the EFTs decays exponentially. Due to the emergence of an infinite tower of light states, an EFT will collapse, and need to be modified. What we will study about SDC is whether it is valid under the renormalization group (RG). By the computation of RG, the metric of the moduli space will vary, and consequently we have different space. The question we will try to achieve is: Does SDC still hold as a swampland constraint for a new moduli space? Assuming the answer to this question be yes, we will try to find new constraints that theories must satisfies in addition to SDC.

The structure of the thesis is as follows: In chapter 2, we will introduce the notion of the Swampland programs. Starting from a preliminary of EFTs, more detailed discussions about the swampland and SDC will be provided. Moreover, a brief note on the moduli space, and a focused review of string theory will be presented for the realization of SDC. Chapter 3 will be devoted to another preliminary; renormalization group and EFT. In this chapter, we will review the basic notions of RG, especially Wilsonian renormalization, as well as the RG method we will adopt for the following discussions. In chapter 4, we will compute RG with some simple but non-trivial examples, and study constraints. Finally, we will conclude the discussions.

Chapter 2

Swampland Programs

This chapter will be devoted to the introductions of the Swampland programs. In the first section, we will review some basics of the effective field theory. Then, with certain understandings of EFTs, the following sections will introduce concepts of the Swampland, and the Swampland programs. To keep things focused for the purpose of this thesis, stresses are mainly put on the Swampland Distance Conjecture. For statements and discussions about other conjectures and relations between them, we will provide useful references.

2.1 Effective Field Theory

Nature around us comes from a lot of factors. From the perspective of physics, physical phenomena occur in a variety of scales of size: from electrons, nuclei, atoms, molecules to planets, stars, glaxies, and so on. Physics have benn progressing because scientists could undertand them independently without necessity of understanding all of them at once. What made it possible is a property of nature called decoupling, which claims that most of information regarding physics at short distance scale has nothing to do with ones at long-distance scale. For instance, suppose a grass of water which consists of numerous H_2O molecules. Each molecule is made from atoms which involve electrons executing orbits around a nucleus. A nucleus itself is consisting of protons and neutrons, and there are more complicated structures as we inquire into it deeper and deeper. However, we are accustomed to be able to know the dynamics of water without knowing such complex structure. The notion of decoupling is important because in some case it is not necessary to know about the detailed properties of nuclei in order to unravell the atomic physics, just as ignoracne about atoms does not keep us from understanding of laws of nature at larger scales such as fluid dynamics and motions of springs. Laws of nature are also described by quantum field theories (QFTs), which shares the property of decoupling. When a particular QFT for short-distance physics is investigated, it is apparently useful if we are able to identify which observables are relevant for a description of the theory and those from which they decouple. A mathematical tool for exploiting the property of decouple is effective field theories (EFTs). Since the Swampland program involves EFT coupled to gravity, in this chapter, we will review some basics of EFTs. Useful reference for this section might be as (Burgess, 2020).

2.1.1 Basic Ideas

As stated above, the key idea behind EFTs is that physics is symplified when viewed from a distance. For instance, suppose a system of electric charges within a region of a space of characteristic size $\sim l$. If one may want to study their effects at a distance much larger than l, $r \gg l$, detailed knowledge of the charge distribution is not required. Instead, one can approximate the system by a few terms in the multipole expansion, such as the total charge, the dipole moment, and so on. This is the same for particle physics. However, it is more common to use the energy scale rather than the length scale $E = l^{-1}$ to distinguish different regimes. Thus, long distances can be interpreted as low energies. In this sense, experiments with the characteristic energy of order E do not require detailed knowledge of physics at higher energy scales $\Lambda \gg E$. The property of decoupling enable us to pursue particular physics by separating theoretically the system with various energy scales when entire theories (full theories) are still unknown. In addition, even if full theories or other EFTs at higher energy scales are known, EFTs are great tools to symplify their computations by lowering scales.

2.2 The Swampland Program

In the previous section, the concepts of EFTs were reviewed. Thanks to the property of the decoupling, we are able to study physics starting from low-energy regimes where we are supposed to have enough knowledge without knowing anything about high-energy scale physics. This also enables us to scrutinize physics in higher energy scales beginning from a low energy EFTs whose structures are well-understood. However, how far is this procedure valid? In other words, is there a specific energy scale above which this "bottom-up" study ceases to work? Let us think about the black hole and the Schwarzschild radius R_s. As the mass increases (i.e. the energy increases), and finally becomes Planck scale $M \sim m_p \sim 10^{28}$eV, the Compton wavelength is shorter than the Schwarzschild radius, and therefore the more detailed structures are hiding behind the event horizon. In this sense, Planck scale might be where the structure of EFTs will be collapsing. In addition, in order to describe physics at Planck scale, it is believed that the theory of qunatum gravity is needed. Hence, if we take quantum gravity into acccount, not all low-energy (IR) EFTs which seem to be consistent are valid at Planck scale. That is, there are supposed to be subsets of EFTs that are inconsistent in consideration of quantum gravity inside the entire set of consistent effective QFTs. This is the emergence of the idea of the Swampland, and necessity of the Swampland program. The aim is, roughly speaking, to determine conditions which distinguish such subsets. In this section, we will introduce the concept of the Swampland, and provide few conjectures needed for the purpose of this thesis. Possible references for this section are (Agmon et al., 2022).

2.2.1 The Swampland and The Swampland Program

From the EFTs perspective, at an energy scale much smaller than Planck scale, we can exploit the decoupling property to study low-energy physics. And once a theory below a certain energy is well understood, we proceed to investigate a new physics whose scale is a little higher than former one's. By repeating this process, EFTs have been thought of as approaches to unknown ultraviolet physics. Nevertheless, at Planck scale, where it is believed that quantum gravity has to be taken into account, it is predicted that the structure of EFTs will be problematic. Normally, when phenomenological problems are considered, since the energy scale is typically much smaller than Planck scale, the effect of the gravity can be ignored or considered classically. While in the case when the energy scale is extremely high, we have to think about the quantization of gravity, it has been believed as an independent problem at a UV scale. Recent researches, however, have been revealing the possibilities that requirements of consistency by quantum gravity are expected to have certain restrictions on its infrared (IR)

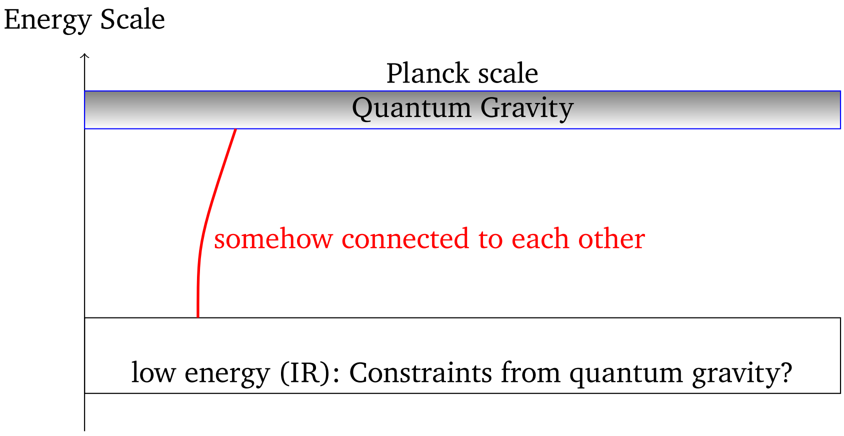


FIGURE 2.1: On the contrary to the EFTs feature of UV/IR decoupling, it is suggested that low-energy physics is not independent of extremely high-energy theories (quantum gravity).

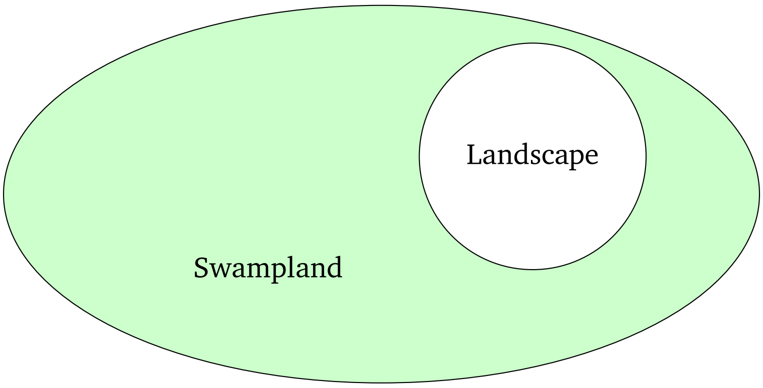


FIGURE 2.2: The entire set of EFTs coupled to gravity is separated into subsets: Landscape and Swampland. Green region is Swampland, and white circle indicates Landscape.

low-energy physics. This observation suggests that, on the contrary to the EFTs property of decoupling, low-energy phenomenological problems and UV quantum gravity are closely related to each other, and should not be considered independently. Below Planck scale, M_p, theories can be described by an EFT coupled to classical gravity, since effect of quantum gravity is ignored. Picking a theory of quantum gravity, we can find one low-energy EFT corresponding to such theory at Planck scale. Now let us consider oppositely. That is, if we consider an EFT, can we obtain a theory of quantum gravity which is compatible with it? The answer to this question separates the entire set of EFTs coupled to gravity into two subsets. One is the set of EFTs obtained as consistent low-energy theories of quantum gravity, and the other is its complementary which cannot be valid ones. Former is called the landscape, and latter is the Swampland.

The Swampland

The set of consistent low-energy EFTs that seem to be consistent, but cannot be compatible in consideration of quantum gravity.

Figure 2.2 shows the entire set of EFTs coupled to gravity divided into landscape

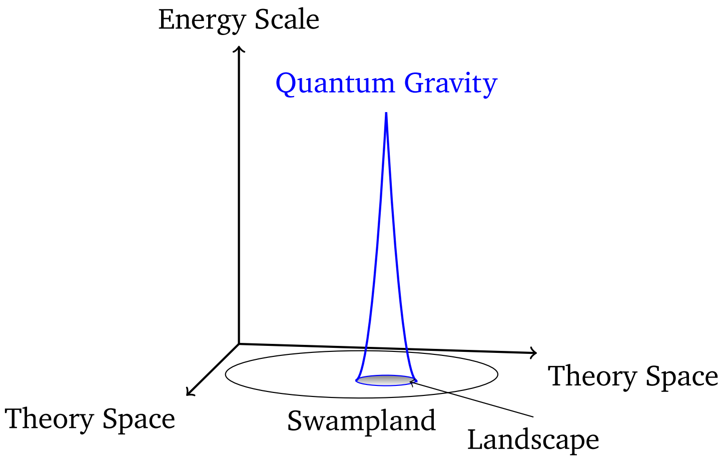


FIGURE 2.3: EFTs on the theory space. There is a landscape inside the Swampland of EFTs. Since the constraints get more strict at the higher energy scale, the space consistent with QG forms a cone shape.

and the Swampland. The attempt to find out the boundary which divides the landscape and the Swampland, and reveal constraints from quantum gravity is the Swampland program.

Notice that constraints which define those boundaries should vanish as we decouple gravity since once we no longer consider gravity all EFTs are consistent. This implies that the higher the energy scale at which the EFT is supposed to be valid is, the more strict the constraints are. It is shown in figure 2.3, where the landscape at various energy scales is depicted as the blue cone. The area of landscape becomes smaller and smalelr as the energy increases, and we get closer and closer to Planck scale where we consider quantum gravity, and ends up with a consistent theory of quantum gravity.

As repeatedly stated, there is a specific cut-off, say, Λ above which the EFT is no longer valid. It is necessary for the EFT to be modified so that the new EFT is valid above the cut-off Λ with some new cut-off Λ' by integrating in new degrees of freedom, as opposed to integrating out when obtaining an EFT given some UV theory. However, this process cannot be proceeded indefinitely: no EFT cannot deal with infinitely many degrees of freedom. Therefore, any EFT has a certain cut-off where it cannot be modified anymore to obtain a consistent theory coupled to Einstein gravity. This leads to an important notion. An EFT is determined if it is on the Swampland or the landscape not only by whether it is possible to be UV completed in QG, but also by whether it is not broken down, that is, it is valid or not up to such a new cut-off. This fact can be seen from figure 2.3 and is shown more clearly in figure 2.4. Given an EFT belonging to the landscape at some lower energy scale, it belongs to the Swampland once the energy scale is greater than specific value.

Thus, whether an EFT belongs to the Swampland or the landscape is depending on at which energy scale it is used for an illustration of a physical system. As such energy increases, there are more strict constraints, and eventually the landscape gets smaller and smaller.

To achieve concrete conditions which determine the boundaries between the Swampland and the landscape, phenomenological consequences, a lot of statements have been proposed. Those are called the Swampland conjectures.

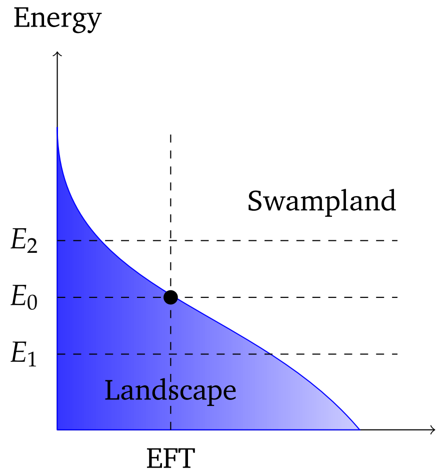


FIGURE 2.4: This shows the cross section of figure 2.3. Given an EFT at some lower energy scale, it belongs to the landscape as long as its characteristic energy is below E_0. However, the same EFT turns to be in the Swampland if the energy scale is above it. For instance, this EFT is in the landscape if it provides a description of physics of energy E_1, while it is in the Swampland if it does of E_2.

2.2.2 Swampland Conjectures

At the moment, the Swampland conjectures are still proposals, not theorems because they have not been fully proven. That is why they are named conjectures. Yet, researches in recent years have found a large amount of evidence for some of them, and expectations that those conjectures might open the doors to the novel features of quantum gravity are increasing. Figure 2.5 shows a map of conjectures as well as connections between them. In the following sections, only selected conjectures will be discussed in detail, which are most relevant for the purpose of the thesis. The connections between conjectures might indicate that some of them are equivalent statements with different appearances. Unifying them by better understanding of QG in the future is one of the goal of this area.

2.3 No Global Symmetries in Quantum Gravity

The absence of global symmetries is considered as the first Swampland conjecture, and widely accepted. Also, many of the Swampland conjectures can be considered as attempts to expand the idea that quantum gravity should have no global symmetries. The statement of the conjecture is as follows:

No Global Symmetries Conjecture

There are no exact global symtries in quantum gravity.

First of all, it is worth to recall briefly that a global symmetry is defined as an invariance under transformations described by a unitary operator commuting with the Hamiltonian. According to Noether's theorem, the continuous symmetry of a system indicates the existence of the conserved quantity, or charge, in it. For instance, the spatial translational symmetry leads to the conservation of momentum. There are a few kinds of arguments which support this conjecture. One motivation is from black hole physics. In order to illustrate what would make it invalid if a global symmetry

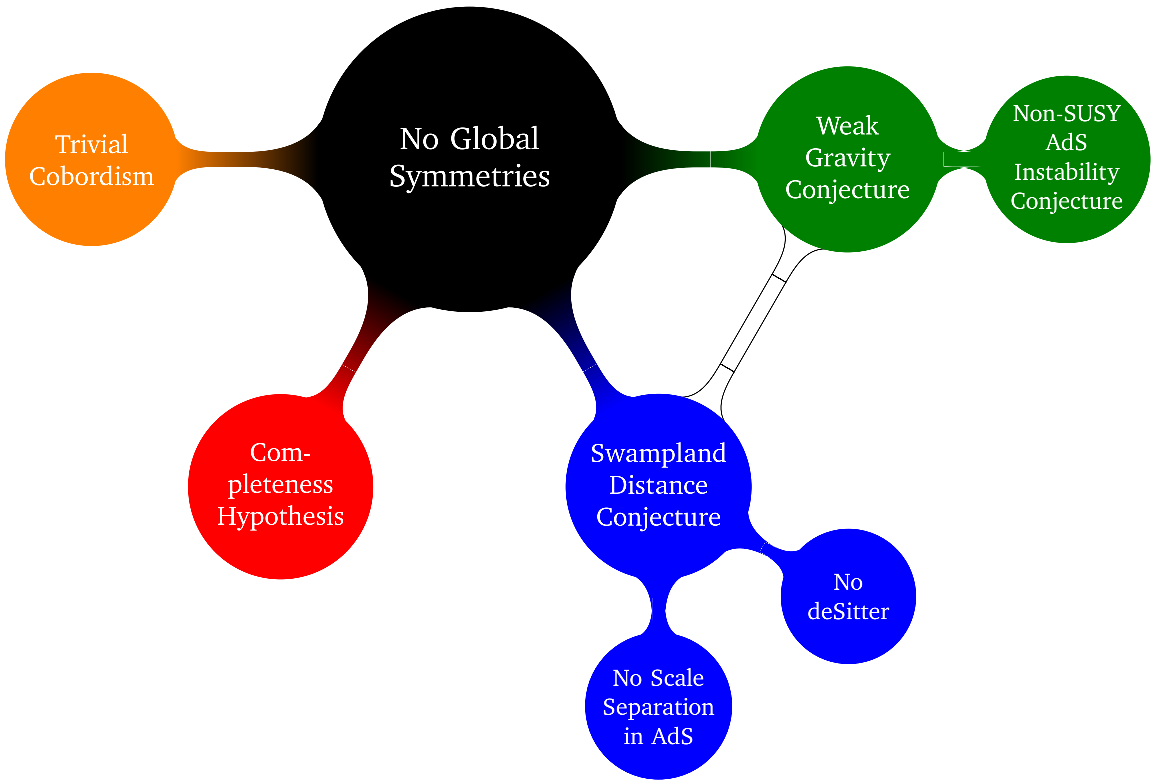


FIGURE 2.5: A schematic map of the Swampland conjectures

is allowed in quantum gravity, assume that there *is* an EFT coupled to gravity with a global symmetry. In the spectrum of EFT there are supposed to be states charged under such global symmetry. Suppose a black hole is charged by sending a charged state into it, it will evaporate decreasing its mass but maintaining its charge because Hawking radiation contains same number of positive/negative charged particles. Letting it evaporate completely, charge is supposed to vanish. Thus, it leads to a violation of the charge conservation. Also, even if such evaporation stops , and ends up with a remnant of mass $M \sim M_P$, since, according to the No-hair theorem, stable black holes can be distinguished from outside only by its mass, angular momentum, and gauge charge, it is impossible to determine its global charge given a black hole of a specific mass. Therefore, this process would violate charge conservations in a global symmetry as we started with non-zero charge and all the charge has disappeared after the evaporation of the black hole and no net charge has come out. Hence, in order to avoid this contradiction, it is necessary that global symmetries are forbidden in any theory of quantum gravity. This discussion can be extended to discrete symmetries and also to more generic global symmetries. Those are followed by the proposal of the Cobordism conjecture, which will be presented later.

The arguments above are rather motivation that proofs to grasp good intuitions for the conjecture. There are more formal evidences mainly from string theory. In particular, it is stated that there are no continuous global symmetry in the target space of perturbative string theory. Here we will briefly provide the proof for the case of bosonic string theory. For any global symmetry in the world sheet, we have a worldsheet charge given:

$$Q = \frac{1}{2\pi i} \oint (dz j_z - d\bar{z} j_{\bar{z}}) \tag{2.1}$$

where j_z and $j_{\bar{z}}$ are the symmetry currents. These char must be conformally invariant, and thus they transform as $(1,0)$, $(0,1)$ tensors respectively. Then we can build vertex operators in the forms:

$$j_z \bar{\partial} X^\mu e^{ikX}, \qquad \partial X^\mu j_{\bar{z}} e^{ikX} \tag{2.2}$$

They create massless gauge vectors in the target space coupling to the left and right moving parts of the charge Q. Therefore, any global symmetry in the worldsheet turns into a gauge symmetry in the spacetime, and not to a global symmetry. So far, there is no counterexample of this conjecture. There is also a discussion through AdS/CFT and holography. A good reference for this evidence is .

It is important to note that it is permitted that there is a low-energy theory with a global symmetry, as long as at higher energies such symmetry is gauged or broken. While it still gives understanding of the principles of quantum gravity, the No-global symmetries conjecture doesn't provide meaningful phenomenological constraints since nothing about at which scale such a global symmetry gets forbidden is stated. However, it is followed by other conjectures such as the Weak Gravity Conjectures and the Swampland Distance Conjecture, which give more concrete constraints, as expansions and refinements of the concept of the impossibility of exact global symmetries in quantum gravity.

2.4 Swampland Distance Conjecture

The Swampland Distance Conjecture (SDC) is the one of the Swampland conjectures which give certain quantitative constraints. It provides a restriction on an EFT of quantum gravity when it is getting closer to where a global symmetry is restored.

The motivation for the conjecture comes from the investigation in a moduli space, which is a space parametrized by the vacuum expectation value of some scalar fields. It is particular interest in that space to observe how an EFT changes as moving toward a certain direction where a global symmetry gradually is recovered, that is, some gauge coupling decreases to zero, because an exact global symmetry is not acceptable in quantum gravity. Therefore it is expected there is some phenomena in such limits in order to protect a theory to get back a global symmetry. One natural assumption to do so is that such poins are infinitely far away in a space. In addition, as an approximate global symmetry is getting exact as moving towards those limits, it is reasonable that an EFT is supposed to become an invalid description continuously. That is also predicted by the consequence of the Weak Gravity Conjecture.

One important key point is that it is characteristic for a quantum gravity. Without considerring the quantum gravity, it is perfectly fine to have a point where a gauge coupling vanishes at infinite distance, and being close to it with maintaining a valid theory. Once the quantum gravity is taken into account, since taking a exact global symmetry strictly forbidden, an EFT should break down continuously as being closing to those limits. This can be generalized to other types of global symmetries. The Swampland Distance Conjecture provides information how EFT behaves when approaching infinite distance in a moduli space, and what features of quantum gravity makes it happen.

2.4.1 Moduli Space

To begin with, let us clarify the notion of moduli space. For our purpose, the notion of moduli space denotes the space of different vacua. Suppose we have scalar fields ϕ^i, $i = 1, \ldots, n$, with a background and an effective action Γ. The expectation value of the scalar fields extremized this effective action, that is:

$$\frac{\delta}{\delta\phi^i}\Gamma(\left\langle\phi^i\right\rangle) = 0 \tag{2.3}$$

Once we find a quantum mechanically stable background, wee can study perturvation around such a background. Assume that the an effective action at low-energy regimes can be approximated with the action:

$$\Gamma = \int d^d x\{\frac{1}{2}g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - V_{\text{eff}}(\phi^i)\} \tag{2.4}$$

plus terms with fields other than scalars. Now we want to find values of scalar fields which make the potential minimum. This is not necessarily a single point, but can be a manifold. Because the potential remains constant in the direction of field space that potential stays minimum, they are called the flat directions of the field space. Let us assume the minimum value of the potential is zero.

The subspace where $V_{\text{eff}} = 0$ of the field space is called the moduli space and it represents the space of different vacua. Assuming the moduli space is parametrized by the vacuum expectation values of scalar fields ϕ^I. The kinetic term for those scalar fields takes the form as following:

$$\int d^d x\frac{1}{2}g_{IJ}(\phi)\partial_\mu\phi^I\partial^\mu\phi^J \tag{2.5}$$

where g_{IJ} can be thought of as a metric on the field space. This action is apparently the non-linear sigma model. Using this metric, we can compute geometric quantities associated with this moduli space such as the geodesic distance and volume.

2.4.2 preliminary Introduction to String Theory

Since the Distance conjecture is closely related to the string theory, it is essential to provide the basic introduction to it. It will be based on familiar discussions covered by standard textbooks for the string theory.

The Nambu-Goto Action

Consider a point particle and assume it propagates in D-dimensional spacetime $\mathbb{R}^{1,D-1}$. We can assign the coordinates as $X^0 = t$ and X^i, with $i = 1,\ldots,D-1$ for the description of the motion of the particle. Then its motion is associated to a world-line γ which identify $X^i(X^0)$. Also the particle world-line in relativistic coordinates can be described by a world-line parameter τ so that γ specifies $X^\mu(\tau)$, where $\mu = 0,1,\ldots,D-1$:

$$\gamma : \tau \hookrightarrow X^\mu(\tau) \in \mathbb{R}^{1,D-1} \tag{2.6}$$

Figure 2.6 shows the description of the particle motion.

In Minkowski space, the line element is given:

$$ds^2 = \eta_{\mu\nu}dX^\mu dX^\nu \tag{2.7}$$

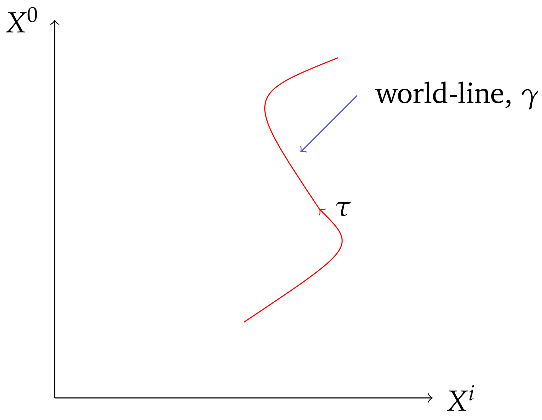


FIGURE 2.6: Illustration of the world-line of a moving particle.

Then we can write an action for a particle, which can be simply written as the length of time-like world-line:

$$S_{\text{NG}} = -m \int_\gamma \sqrt{-ds^2} = -m \int_\gamma (-\dot{X}^2)^{1/2} d\tau \tag{2.8}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$, $\dot{X}^2 = \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu$, and m is a constant related to the mass of the particle. This action is called the Nambu-Goto action. Consider a Lagrangian:

$$S_{\text{NG}} = -m \int_\gamma d\tau L(\tau) \tag{2.9}$$

The canonical momentum is:

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m \dot{X}_\mu}{(-\dot{X}^2)^1/2} \tag{2.10}$$

Then it ends up with a constraint:

$$p^2 + m^2 = 0 \tag{2.11}$$

which illustrates that m is actually the mass of the particle. The equation of motion for X^μ gives:

$$m\ddot{X}^\mu = 0 \tag{2.12}$$

then we can see that the action is correctly describing the free propagation of the particle.

The Polyakov Action

We can suppose another action for a particle which is called the Polyakov action given:

$$S_P = \frac{1}{2} \int_\gamma d\tau e(\tau) \{ \frac{1}{e(\tau)^2} \dot{X}^2 - m^2 \} \tag{2.13}$$

$e(\tau)$ is named the world-line metric. The equation of motion corresponding to this action is:

$$\dot{X}^2 + m^2 e(\tau)^2 = 0 \tag{2.14}$$

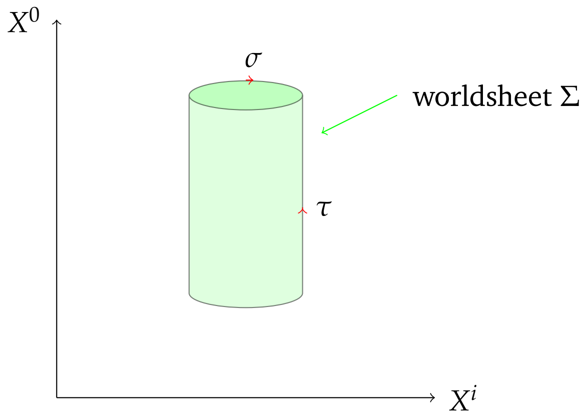


FIGURE 2.7: The worldsheet of a string parametrized by two coordinates (σ, τ)

Using it, we can deform the Polyakov action by eliminating the world-line metric:

$$S_P = \frac{1}{2}\int_\gamma d\tau e(\tau)(-2m^2) = -m\int_\gamma d\tau(-\dot{X}^2)^{1/2} = S_{\text{NG}} \tag{2.15}$$

Therefore, the Nambu-Goto action and the Polyakov action are equivalent to each other. The reason we use the polyakov action is that it is much easier to quantize.

The String Worldsheet

Let us do the same computations, but now we are dealing with a string, not a particle. Considering the motion of a string, it sweeps out a 2-dimensional worldsheet Σ parametrized by two coordinates (σ, τ):

$$\Sigma : (\sigma, \tau) \hookrightarrow X^\mu(\sigma, \tau) \in \mathbb{R}^{1,D-1} \tag{2.16}$$

Define the ranges:

$$0 \leq \sigma \leq 2\pi, \qquad \tau \in \mathbb{R} \tag{2.17}$$

SInce we are mostly interested in closed strings, rather than open ones, we impose the boundary condition given:

$$\sigma \simeq \sigma + 2\pi \tag{2.18}$$

So the string worldsheet parametrized by such coordinates forms a cylinder as shown in figure 2.7. It is useful to denote the coordinates as:

$$\{\sigma, \tau\} \equiv \xi^a, \qquad a = 0, 1 \tag{2.19}$$

With the worldsheet metric $h_{ab}(\xi)$, the string tension T which can be written in terms of a parameter α':

$$T \equiv \frac{1}{2\pi\alpha'} \tag{2.20}$$

the Polyakov actions for the string is given:

$$S_P = -\frac{T}{2}\int_\Sigma d^2\xi(-\det h)^{1/2} h^{ab}(\xi)\eta_{\mu\nu}\partial_a X^\mu(\xi)\partial_b X^\nu \tag{2.21}$$

Besides, there are also the string length l_s and the string scale M_s, given:

$$l_s \equiv \sqrt{\alpha'}, \qquad M_s \equiv \frac{1}{2\pi\sqrt{\alpha'}} \tag{2.22}$$

The worldsheet action eq2.21 should be viewed as a description of a two-dimensional theory with scalars $X^\mu(\xi)$, which is the sigma model. The space-time parametrized by $X^\mu(\xi)$ is known as the target space of the theory. The metric $\eta_{\mu\nu}$ on the space-time is the metric on the field space of the scalar fields X^μ. Therefore, there are supposed to be different metrics on the field spaces for different target spaces in which strings propagate.

Symmetries on the Worldsheet

Eq 2.21 is invariant under:

$$\xi^a \to \tilde{\xi}^a(\xi) \tag{2.23}$$

In addition, it is invariant under the Weyl transformation given:

$$\delta X^\mu = 0, \qquad h_{ab} \to \tilde{h}_{ab} = e^{2\Lambda(\xi)} h_{ab} \tag{2.24}$$

The symmetries are useful for fixing the worldsheet metric h_{ab}. For D-dimensional theory, each number of degrees of freedom in the metric, a symmetric tensor, and so on. Particulaely, the number of degrees of freedom in h_{ab} is $\frac{1}{2}D(D+1)$, and for a string, $D = 2$, the number of symmetry parameters is the same as the degrees of freedom in the metric. With symmetries, we can after all get:

$$h_{ab} = \eta_{ab} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.25}$$

which is called flat gauge. Notice that it is still necessary to have equations of motion while the metric can be gaged. The equation of motion for the metric corresponding to the vanishing of the energy momentum tensor is:

$$T_{ab} = 0 \tag{2.26}$$

where:

$$T_{ab} \equiv \frac{4\pi}{\sqrt{-\det h}} \frac{\delta S_P}{\delta h^{ab}} \tag{2.27}$$

This is called a Virasoro constraint, and plays important role for the quantization of the string.

In flat gauge, the reduced Polyakov action is:

$$S_P = \frac{T}{2} \int_\Sigma d\sigma d\tau \{(\partial_\tau X)^2 - (\partial_\sigma X)^2 \tag{2.28}$$

Now we can go to light-cone coordinates:

$$\xi^\pm \equiv \tau \pm \sigma, \qquad \partial_\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{2.29}$$

Then the action deforms:

$$S_P = T \int_\Sigma d\xi^+ d\xi^- \partial_+ X \partial_- X \tag{2.30}$$

The equation of motion therefore for X^μ are:

$$\partial_+ \partial_- X^\mu = 0 \tag{2.31}$$

Thus X^μ can be thought of as the sum of left- and right-mooving parts of the strings as:

$$X^\mu = X^\mu_L(\xi^+) + X^\mu_R(\xi^-) \tag{2.32}$$

Also considering the periodic boundary conditions $X^\mu(\tau, \sigma = 0) = X^\mu(\tau, \sigma = 2n\pi)$, the most general solution is:

$$\begin{aligned} X^\mu_R(\xi^-) &= \frac{1}{2}(x^\mu) + c^\mu) + \frac{1}{2}\alpha' p^\mu_R \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-in\xi^-} \\ X^\mu_L(\xi^+) &= \frac{1}{2}(x^\mu) - c^\mu) + \frac{1}{2}\alpha' p^\mu_L \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-in\xi^+} \end{aligned} \tag{2.33}$$

where x^μ, c^μ, p^μ_R, p^μ_L, α^μ_n, $\tilde{\alpha}^\mu_n$ are constants. Due to the periodicity:

$$p^\mu_R = p^\mu_L \equiv p^\mu \tag{2.34}$$

Averaging over the string, one may find:

$$q^\mu \equiv \frac{1}{2\pi} \int_0^{2\pi} d\sigma X^\mu = x^\mu + \alpha' p^\mu \tau \tag{2.35}$$

Thus we can say that x^μ is the center of mass position, and p^μ is the target space momentum.

Spectrum

Considerations so far have been in classical manner, but it is needed to quantize strings in order to investigate the spectrum of excitations. In the following, we will manipulate so-called light-cone quantization. Introduce target-space light-cone coordinates:

$$X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}), \qquad X^i, \qquad\qquad i = 1, \dots, D-2 \tag{2.36}$$

Then the metric takes the form:

$$\eta_{+-} = \eta{-+} = -1, \qquad \eta_{oj} = \delta_{ij} \tag{2.37}$$

With this, an inner product is defined:

$$X^2 = -2X^+X^- + \dot{X}^i \dot{X}^i \tag{2.38}$$

The expansion of X^+ is:

$$X^+(\tau,\sigma) = x^+ + \alpha' p^+ \tau + i\sqrt{\frac{\alpha'}{1}} \sum_{n\in\mathbb{Z},n\neq 0} \frac{1}{n}\alpha_n^+ e^{-in\xi^-} + i\sqrt{\frac{\alpha'}{1}} \sum_{n\in\mathbb{Z},n\neq 0} \frac{1}{n}\tilde{\alpha}_n^+ e^{-in\xi^+} \tag{2.39}$$

Since there are residual symmetries even if we choose the light-cone gauge, we can set all the oscillator modes of X^+ to zero. Then:

$$X^+(\tau,\sigma) = x^+ + \alpha' p^+ \tau \tag{2.40}$$

Imposing the Virasoro constraints as well:

$$\partial_\pm X^- = \frac{1}{\alpha' p^+}(\partial_\pm X^i)^2 \tag{2.41}$$

We can observe that the X^- oscillators are written in the transverse oscillators in X^i too. Therefore, only the transverse oscillators are independent degrees of freedom. This is very useful because only X^i contains physically independent oscillators and rules out tow unphysical polarizations of the string. The action in the light-cone gauge becomes:

$$\begin{aligned} S_{LC} &= \frac{1}{4\pi\alpha'}\int_\sigma d\tau d\sigma\{(\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2 + 2(\partial_\sigma X^+ \partial_\sigma X^- - \partial_\tau X^+ \partial_\tau X^-)\} \\ &= \frac{1}{4\pi\alpha'}\int_\sigma d\tau d\sigma\{(\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2 - \int d\tau p^+ \partial_\tau q^- \\ &\equiv \int d\tau L \end{aligned} \tag{2.42}$$

where defined as:

$$q^- \equiv \frac{1}{2\pi}\int_0^{2\pi} d\sigma X^- \tag{2.43}$$

From this, canonical momenta are given:

$$p_- \equiv \frac{\partial L}{\partial \dot{q}^-} = -p^+, \qquad \Pi_i \equiv \frac{\partial L}{\partial \dot{X}^i} = \frac{\dot{X}_i}{2\pi\alpha'} \tag{2.44}$$

Then quantize the theory by introducing the canonical commutaion relations:

$$[X^\mu(\tau,\sigma), \Pi^\mu(\tau,\sigma;)] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{2.45}$$

followed by:

$$[x^i, p^i] = i\delta_{ij}, \tag{2.46}$$

$$[p^+, q^-] = i, \tag{2.47}$$

$$[\alpha_m^i, \alpha_n^j] = m\delta_{m+n,0}\delta_{ij}, \tag{2.48}$$

$$[\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta_{m+n,0}\delta_{ij} \tag{2.49}$$

Then we can adopt the standard procedure of quantization by promoting the oscillator modes to operators acting on a Hilbert space as in QFT. In this case, α_{-n}^i with $n > 0$ are creation operators acting on a vacuum state $|0,p\rangle$. On the other hand α_n^i with $n > 0$ represent annihilation operators. Recall that there are no oscillators to quantize for

X^+, while the X^- oscillators are given in terms of X^i. Explicitly:

$$\alpha_n^- = \frac{1}{2p^+\sqrt{2\alpha'}} \sum_{m=-\infty}^{m=\infty} \alpha_{n-m}^i \alpha_m^i \tag{2.50}$$

Since the ordering of the α matters, we need the notions of normal ordered products and a normal ordering constant a where:

$$\alpha_n^- = \frac{1}{2p^+\sqrt{2\alpha'}} \left(\sum_{m=-\infty}^{m=\infty} : \alpha_{n-m}^i \alpha_m^i : - a\delta_{n,0} \right) \tag{2.51}$$

with the definition:

$$: \alpha_m^i \alpha_n^i :\equiv \begin{cases} \alpha_m^i \alpha_n^i & \text{for } m \leq n \\ \alpha_n^i \alpha_m^i & \text{for } n < m \end{cases} \tag{2.52}$$

Lorentz Invariance

So far we have performed quantization in special target-space light-cone coordinates. So we should also think about Lorentz invariance in the theory. Generally, the generators of the Lorentz transformations are given:

$$J^{\mu\nu} = \int_0^{2\pi} d\sigma (X^\mu \Pi^\nu - X^\nu \Pi^\mu) \equiv l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} \tag{2.53}$$

where

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{2.54}$$

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu) \tag{2.55}$$

$$\tilde{E^{\mu\nu}} = -i \sum_{n=1}^{\infty} \frac{1}{n} (\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu) \tag{2.56}$$

Lorentz algebra reads:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\mu\rho} J^{\nu\sigma} + i\eta^{\nu\sigma} J^{\mu\rho} - i\eta^{\mu\sigma} J^{\nu\rho} - i\eta^{\nu\rho} J^{\mu\sigma} \tag{2.57}$$

Particularly:

$$[J^{-i}, J^j] = 0 \tag{2.58}$$

but at the same time, this calculation followed by:

$$[J^{-i}, J^{-j}] = -\frac{1}{(p^+)^2} \sum_{m=1}^{\infty} \{ \frac{26-D}{12} m + \frac{1}{m} (\frac{D-26}{12} + 2(1-a)) (\alpha_{-m}^i \alpha_m^j - \alpha_{-m}^j \alpha_m^i) + (\text{tilde terms}) \tag{2.59}$$

So for the Lorentz invariance at quantum level, there are constraints such as:

$$D = 26, \qquad a = 1 \tag{2.60}$$

It claims that the quantum bosonic string is only consistent in 26-dimensions. Such a restricion on the dimensionality is named criticality. Note that this is the case for the bosonic string, and for the superstring, the number of dimensions is 10.

Quantum String Spectrum

Let us study the spectrum of the quantized string. The Hamiltonian is:

$$\begin{aligned} H &= p_{-}\dot{q}^{-} + \int_0^{2\pi} d\sigma \Pi_i \dot{X}^i - L \\ &= \frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \left[(\partial_\tau X^i)^2 + (\partial_\sigma X^i) \right] \\ &= \frac{\alpha'}{2} p^i p^i + \frac{1}{2} \sum_{n=-\infty}^{\infty} (\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n) \end{aligned} \tag{2.61}$$

From the Virasoro constraint:

$$\partial_\tau X^- = \frac{1}{2p^+\alpha'} \left[(\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2 \right] \tag{2.62}$$

we may have:

$$p^- = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \partial_\tau X^- = \frac{H}{\alpha' p^+} \tag{2.63}$$

Considering normal ordering of α:

$$p^+ p^- = \frac{1}{\alpha'} \left[\sum_{n=1}^{\infty} : \alpha^i_{-n}\alpha^i_n : + \sum_{n=1}^{\infty} : \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n : -2a + \frac{\alpha'}{2} p^i p^i \right] \tag{2.64}$$

The mass in the target space is:

$$M^2 = -p^2 = 2p^+p^- - p^i p^i = \frac{2}{\alpha'} \left[\sum_{n=1}^{\infty} : \alpha^i_{-n}\alpha^i_n : + \sum_{n=1}^{\infty} : \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n : -2a \right] \tag{2.65}$$

But for the closed string there is a symmetry regarding translations along σ implying the level matchong condition:

$$\sum_{n=1}^{\infty} : \alpha^i_{-n}\alpha^i_n := \sum_{n=1}^{\infty} : \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n : \tag{2.66}$$

this leads to the mass expression in:

$$M^2 = \frac{4}{\alpha'} (\sum_{n=1}^{\infty} : \alpha^i_{-n}\alpha^i_n : -1) \tag{2.67}$$

applying the criticality to fix $a = 1$. Now we can start to study the spectrum. Let us define $\sum_{n=1}^{\infty} : \alpha^i_{-n}\alpha^i_n := N$.
For $N = 0$ case, we have:

$$M^2 = -\frac{4}{\alpha'} \tag{2.68}$$

This is a tachyonic mode and suggesting an instability in the bosonic string.
For $N = 1$:

$$M^2 = 0 \tag{2.69}$$

So it is the massless spectrim, and given by:

$$\xi_{ij} \tilde{\alpha}^i_{-1} \alpha^j_{-1} \left| 0, p \right\rangle , \qquad i, j = 1. \ldots , 24 \tag{2.70}$$

The tensor ξ_{ij} can be decomposed into irreducible representations of SO(24):

$$\xi_{ij} = g_{(ij)} + B_{[ij]} + \Phi \tag{2.71}$$

where $g_{(ij)}$ and $B_{[ij]}$ are traceless symmetric and anti-symmetric respectively as well as Φ represents a scalar. Thus we obtain a massless, transversely polarized symmetric tensor fields g_{ij}, which is a graviton. Therefore the quantum string contains gravitational modes in spectrum.
For $N > 1$, we may have massive oscillator string modes.

2.4.3 Realization of the Distance Conjecture

We have found that the bosonic string lives in 26-dimensions, and mentioned the superstring does in 10-dimension. Those facts seem to make them incompatible directly with the observed universe. Nevertheless, they can be observed yet since the addtional dimensions might be compact and small. In this language, it is naturall to thinka about string theory in a spacetime which has a compact direction. One of possible settings is the case where one of the dimensions is in the form of a circle. Studying this, we will see a realization of the Distance conjecture.

Compactification on a Circle

Consider $D = d + 1$ dimensional spacetime. Let the spatial direction X^d to be compact in the shape of a circle with periodicity:

$$X^d \simeq X^d + 1 \tag{2.72}$$

notice that we are working in Planck units, so the periodicity is set to one.

The metric on the D-dimensional space is given:

$$ds^2 \equiv G_{MN} dX^M dX^N = e^{2\alpha\phi} g_{\mu\nu} dX^\mu dX^\nu + e^{\beta\phi} (dX^d)^2 \tag{2.73}$$

There some coordinates and parameters are introduced. Coordinates X^M are D-dimensional, $M = 0, \ldots, d$, while $\mu = 0, \ldots, d-1$. The D-dimensional metric is G_{MN}, and d-dimensional one is $g_{\mu\nu}$. The metric has a parameter ϕ which represents a d-dimensional scalar field. In addition, the constants α and β have been defined:

$$\alpha^2 = \frac{1}{2(d-1)(d-2)}, \qquad \beta = -(d-2)\alpha \tag{2.74}$$

The circumference of the circle denoted $2\pi R$ is:

$$2\pi R \equiv \int_0^1 \sqrt{G_{dd}} dX^d = e^{\beta\phi} \tag{2.75}$$

Then the radius is a dynamical field in d-dimensions. Our interest will be in the behaviors of the d-dimensional theory under variations of the expectation value of the field ϕ, which amounts to variations of the size of the circle.

Consider the decomposition of the D-dimensional Ricci scalar R^D for the metric 2.73. We have:

$$\int d^D X \sqrt{-G} R^D = \int d^d X \sqrt{-g} \left[R^d - \frac{1}{2} (\partial\phi)^2 \right] \tag{2.76}$$

Introduce a massless D-dimensional scalar field Ψ. Due to the periodicity:

$$\Psi(X^M) = \sum_{n=-\infty}^{\infty} \psi_n(X^\mu) e^{2\pi i n X^d} \tag{2.77}$$

The modes ψ_n are d-dimensional scalar fields caleld Kaluza-Klein (KK) modes, and ψ_0 is the zero-mode of Ψ. The momentum is quantized along the compact direction:

$$-i\frac{\partial}{\partial X^d}\Psi = 2\pi n \Psi \tag{2.78}$$

Now to make things simple, assume $g_{\mu\nu} = \eta_{\mu\nu}$. The equation of motion for Ψ is:

$$\partial^M \partial_M \Psi = (e^{-2\alpha\phi}\partial^\mu\partial_\mu + e^{-2\beta\phi}\partial^2_{X^d})\Psi = 0 \tag{2.79}$$

since Ψ is massless. Then this leads the equation of motion for ψ_n modes:

$$\left[\partial^\mu\partial_\mu - (\frac{1}{2\pi R})^2 (\frac{1}{2\pi R})^{\frac{2}{d-2}} (2\pi n)^2\right]\psi_n = 0 \tag{2.80}$$

Therefore, the mass of the KK modes can be read off as:

$$M_n^2 = (\frac{n}{R})^2 (\frac{1}{2\pi R})^{\frac{2}{d-2}} \tag{2.81}$$

Compactification of String on a Circle

Now let us do the same procedure in the string theory considering strings on a circle of radius R. A metric is given:

$$ds^2 = \eta_{MN} dX^M dX^N \tag{2.82}$$

taking the X^d direction as R-periodic:

$$X^s \simeq X^d + 2\pi R \tag{2.83}$$

Consider the bosonic mode expansion as before but for now, we will not impose $X^M(\sigma + 2\pi, \tau) = X^M(\sigma, \tau)$. Thus:

$$X^M(\tau,\sigma) = x^\mu + \alpha' p^M \tau + \frac{\alpha'}{2}(p_L^M - p_R^M)\sigma + \text{oscillator} \tag{2.84}$$

Here left- and right mooving momenta are considered independently, and the overall momentum is:

$$\frac{1}{2}(p_R^M + p_L^M) \tag{2.85}$$

In the non-compact space, imposing $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$, we have $p_R^\mu = p_L^\mu$. However, for a circle we have a winding:

$$X^d(\sigma + 2\pi, \tau) = X^d(\sigma, \tau) + w 2\pi R, \qquad w \in \mathbb{Z} \tag{2.86}$$

The string is wrapping around a circle w times. Therefore, we may have:

$$\frac{\alpha'}{2}(p_L^d - p_R^d) = wR \tag{2.87}$$

Consider the mass spectrum for the string by going to target space light-cone gauge again. The hamiltonian reads:

$$H = \frac{\alpha'}{2}\left[\frac{1}{4}(p_L^d - p_R^d)^2 + p^\alpha p^\alpha + (p^d)^2\right] + (N + \tilde{N} - 2) \tag{2.88}$$

here we split the index $i = \{\alpha, d\}$. Notice in this case we no longer have the level matching condition, but instead:

$$N - \tilde{N} = nw \tag{2.89}$$

Then the d-dimensional mass is given by $-p_\mu p^\mu = 2p^+p^- - p^\alpha p^\alpha$ which leads:

$$(M_{n,w})^2 = (\frac{n}{R})^2 + (\frac{wR}{\alpha'})^2 \tag{2.90}$$

We want to make it related to the effectiveaction 2.76. Nevertheless, we need to change form the metric 2.82 to the one 2.73. This procedure is called going from the string frame to the Einstein frame. The difference is the factor of $E^{2\alpha\phi}$ multiplying the $g_{\mu\nu}$ directions. An important object is the d-dimensional scalar Ψ, called dilaton, defined as:

$$\Psi^d \equiv \Psi - \frac{1}{2}\log(2\pi RM) \tag{2.91}$$

We would like to observe variations of R which maintain Ψ^d fixed. That is, we must vary:

$$e^{2\Psi} \sim \frac{1}{2\pi RM} \tag{2.92}$$

But considering the definition of the d-dimensional Planck mass M_p^d:

$$\frac{(M_p^d)^{d-2}}{2} \equiv 2\pi M^{D-2} e^{-2\Psi} \tag{2.93}$$

From this we can see that in order to remain the Einstein frame $M_p^d = 1$, we must choose our units so that:

$$M \sim (2\pi R)^{\frac{1}{d-2}} \tag{2.94}$$

This affects the mass of the winding modes in 2.90.

Finally, performing the change of frames leads to the Einstein frame mass:

$$(M_{n,w})^2 = (\frac{1}{2\pi R})^{\frac{2}{d-2}}(\frac{n}{R})^2 + (2\pi R)^{\frac{2}{d-2}}(\frac{wR}{\alpha'_0})^2 \tag{2.95}$$

Distance Conjecture

We will now study the d-dimensional effective theory. The action is given in 2.76 and this has to be supplemented by the spectrum 2.95. Our particular interest is how the spectrum of states behaves under variations of the expectation value of the field ϕ. This can be determined easily from the relation 2.75. The possible expectation values

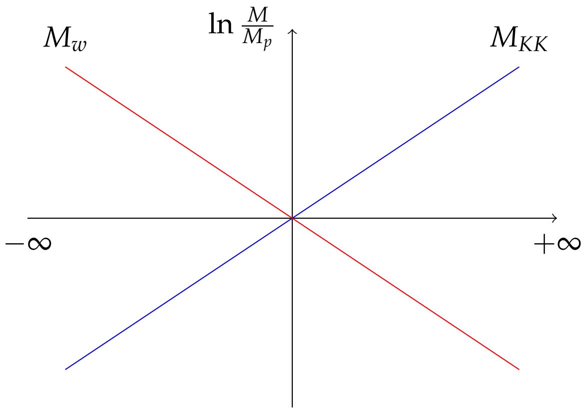


FIGURE 2.8: Illustration of the mass scales on a log plot. For the KK and winding towers as a function of the scalar field ϕ expectation value.

of the field ϕ define a field space $\mathcal{M}_\phi$, which has now one infinite real dimension. Thus consider:

$$\mathcal{M}_\phi : -\infty < \phi < \infty \tag{2.96}$$

We are now having two infinite towers of massive states in this theory, in eq 2.95, the tower of KK modes with masses $M_{n,0}$, and of winding modes with $M_{0,m}$. Using the relation 2.75, those mass scales can be written as:

$$M_{\text{KK}} \sim E^{\alpha\phi}, \qquad M_w \sim e^{-\alpha\phi} \tag{2.97}$$

where

$$\alpha = \sqrt{2}(\frac{d-1}{d-2})^{1/2} > 0 \tag{2.98}$$

For any variation $\Delta\phi$ there exists an infinite tower of states with associated mass scale M. We can also see that such a tower becomes light at an exponential rate in $\Delta\phi$:

$$M(\phi_i + \Delta\phi) \sim M(\phi_i)e^{-\alpha|\Delta\phi|} \tag{2.99}$$

as illustrated in figure 2.8. There are a few things to note:

- The tower of states becoming light is the KK tower if $\Delta\phi < 0$ while it is the winding tower when $\Delta\phi > 0$. That is, either tower of states always becomes light regardless of the sign of $\Delta\phi$ is.
- This is a characteristic feature of the string theory. It is not true in QFT since winding states are absent in field theories.
- The product of the mass scales of the two towers is independent of ϕ.
- The exponent α is a constant of order one $\mathcal{O}(1)$.
- When $|\Delta\phi| \to \infty$, an infinite number of states get massless.

The last point has a discussion. If we consider an effective field theory which has a cut-off Λ below the mass scale of an infinite tower of states. Thus, this teory can be valid only for a finite range of expectation values of ϕ.

The exponential behavior is particularly interesting. Those towers are deeply related to each other as there is a $\mathbb{Z}_2$ symmetry which interchanges them. This relation is called T-duality, and it is able to be seen directly in the string frame where the mass spectrum is invariant under the action.

$$\text{T-duality} : R \Leftrightarrow \frac{\sqrt{\alpha'}}{R} \tag{2.100}$$

Actually, it is a symmetry not only for the mass, but also for the full string theory. There are many other kinds of dualities other than T-duality. Then it is natural to think that there are many different towers of states which are dual so that as one moves in a space of the theory, the product of the mass scale of the dual towers remains constant and thus one must become light in any direction. Then as we move an infinite distance in parameter space, the tower must be getting light.

Those discussions and examples realize the Swampland Distance Conjecture, which is similar to 2.99, but more generally stated in the next section.

2.4.4 Swampland Distance Conjecture

Consider an EFT coupled to Einstein gravity with moduli space $\mathcal{M}$ parametrized by scalar fields, and whose metric g_{ij} is given by its kintic term. The first statement of the conjecture is as follows: Starting from a point $P \in \mathcal{M}$, there always exits a point $Q \in \mathcal{M}$ at infinite geodesic distance $d(P,Q)$. The second part of the conjecture describes a phenomena supposed to happen as approaching a point at infinite field distance:

Swampland Distance Conjecture

There happens an infinite tower of states which becomes exponentially light with the geodesic distance such as

$$M(Q) \sim M(P)e^{-\lambda d(P,Q)} \tag{2.101}$$

with λ an $\mathcal{O}(1)$ constant in Planck units.

It is abstractly illustrated in figure 2.9. Since an EFT cannot deal with infinitely many degrees of freedom under its cut-off, the emergence of the infinite tower of states indicates the breakdown of EFT. Thus, there supposed to be a quantum gravity cut-off associated to the infinite tower of states, which decreases exponentially with the geodesic distance:

$$\Lambda_{\text{QG}} = \Lambda e^{-\lambda d(P,Q)} \tag{2.102}$$

Note that since this is an asymmptotic statement about infinite distance $d(R_i, R_f) \to \infty$, the mass scale value at R_i is not that important. From the perspective of QFT, this is really surprising. As it has been seen, the conjecture can holds only due to the string theory, more generally, quantum gravity. The typical mass scale for such physics is M_p. However, this conjecture claims that if in the bulk of moduli space, the tower of states has a mass scale around the Planck mass, and then at large expectation values, this is getting lower than M_p. Thus, it states that the naive application of decoupling on effective QFT breaks down at an exponentially lower energy scale than expected whenever a field develops a large expectation value.

To sum up, the SDC can be thought of as a constraint on the validity for finite field variations, as an EFT at a point in moduli space cannot be extended to an arbitrary far point from initial one. By approaching such a point, an infinite number of degrees

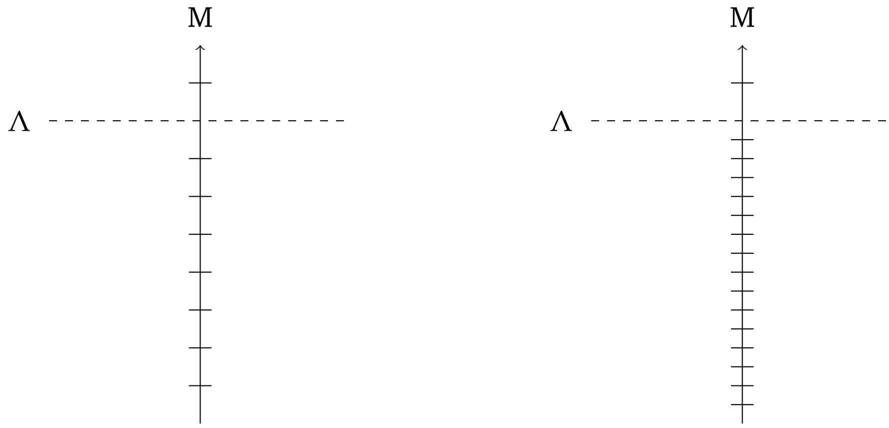


FIGURE 2.9: Left picture represents an EFT with a cut-off Λ. There are finitely many states below this cut-off. As moving toward a point at infinite geodesic distance, it becomes as shown right. Continuously, the closer it is to such a point, the more degrees of freedom coming in the EFT description. After all, in order to try to avoid infinitely many degrees of freedom for an EFT, a cut-off is supposed to fall down to zero.

of freedom would become exponentially light and eventually an EFT description broken. So far computations for the distance have been considered on the moduli space. Nevertheless, this notion of the distance can be generalized to the one between more generic field with a generic metric given also by the kinetic term. It will be investigated how the renoemalization group affects on the statement of this conjecture later.

Chapter 3

Renormalization Group

Since it is central for this thesis to see how the renormalization group (RG) affects on the SDC, another preliminary we need before the computaion is the concept of renormalization group and how we can carry out the calculations. In this chapter, first we will have introductions to the renormalization group (RG) starting from a review of the generating functional to the beta function, or the RG equations. Understanding the basics of RG, we will provide a specific method for our later computation, named Heat Kernel expansion.

3.1 Focused Review of Generating Functional

Consider a theory with n fields $\phi^i(x)$, $i = 1, \dots, n$, governed by a classical action $S[\phi]$, and suppose computing its correlation functions:

$$G^{i_1,\cdots,i_n}(x_1, \dots, x_n) := \left\langle \Omega_f \middle| T[\phi^{i_1}(x_1)\dots\phi^{i_n}(x_n)] \middle| \Omega_i \right\rangle \tag{3.1}$$

where $|\Omega\rangle$ stands for the ground state in the past or future, and T indicates time ordering. We can extract general observables of the system such as scattering amplitudes from them. By introducing the generating functional $Z[J]$, we can deal with such correlators at onece. Definition of $Z[J]$ is:

$$Z[J] := \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n G^{i_1,\cdots,i_n}(x_1, \dots, x_n) J_{i_1}(x_1) \dots J_{i_n}(x_n) \tag{3.2}$$

then each correlation function can be in a form by functional differentiation:

$$G^{i_1,\cdots,i_n}(x_1, \dots, x_n) = (-i)^n \left(\frac{\delta^n Z[J]}{\delta J_{i_1}(x_1)\dots\delta J_{i_n}(x_n)}\right)_{J=0} \tag{3.3}$$

The generating functional is useful because it is a simple expression in terms of path integral formalism. Such simplicity means not only the easiness to be computed but also practically that it is relatively straightforward to compute perturbatively in terms of Feynman diagrams.

$$G^{i_1,\cdots,i_n}(x_1, \dots, x_n) = \int \mathcal{D}\phi e^{iS[\phi]} \phi^{i_1}(x_1) \dots \phi^{i_n}(x_n) \tag{3.4}$$

where the integration measure is:

$$\int \mathcal{D}\phi = \prod_{x^\mu} \int d\phi(x) \tag{3.5}$$

The special case $n = 0$ is the typical example for which

$$\langle \Omega_f | \Omega_i \rangle = \int \mathcal{D}\phi e^{iS[\phi]} \tag{3.6}$$

Using the definition, we can obtain the following expression for the generating functional:

$$Z[J] = \int \mathcal{D}\phi exp\{iS[\phi] + i \int d^4x \phi^i(x) J_i(x)\} \tag{3.7}$$

and immediately find:

$$Z[J = 0] = \langle \Omega_f | \Omega_i \rangle \tag{3.8}$$

Semiclassical Evaluation

Expand the action about a classical background so that:

$$\phi(x) = \varphi_{\text{cl}}(x) + \tilde{\phi}(x) \tag{3.9}$$

where φ_{cl} fulfills:

$$\left.\frac{\delta S}{\delta \phi}\right|_{\phi = \varphi_{\text{cl}}} + J = 0 \tag{3.10}$$

Suppose we have an action $S_j[\phi] := S[\phi] + \int d^4x (\phi^i J_i)$, and want to write it in a form:

$$S_j[\varphi_{\text{cl}} + \tilde{\phi}] = S_J[\varphi_{\text{cl}}] + S_2[\varphi_{\text{cl}}, \tilde{\phi}] + S_{\text{int}}[\varphi_{\text{cl}}, \tilde{\phi}] \tag{3.11}$$

where:

$$S_2 = -\int d^4x \tilde{\phi}^i \Delta_{ij}(\varphi_{\text{cl}}) \tilde{\phi}^j \tag{3.12}$$

being the quadratic part in the expansion for some differential operator Δ_{ij}. S_{int}, namely interaction term, contains higher order terms in $\tilde{\phi}^i$. Notice that since the background field satisfies eq.3.10, there will be no linear terms.

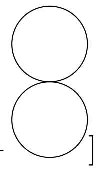
Then the expression inside the above path integral can be expanded as:

$$\exp\{iS[\phi] + i \int d^4x \phi^i J_i\} = \exp\{iS_J[\varphi_{\text{cl}}] + iS_2[\varphi_{\text{cl}}, \tilde{\phi}]\} \sum_{r=0}^{\infty} \frac{1}{r!} (iS_{\text{int}}[\varphi_{\text{cl}}, \tilde{\phi}])^r \tag{3.13}$$

This will be followed by the graphical representation of any correlators. Gaussian integrals involve integrands which are powers of fields:

$$\int \mathcal{D}\tilde{\phi} e^{i\tilde{\phi}^i \Delta_{ij} \tilde{\phi}^j} \tilde{\phi}^{k_1}(x_1) \ldots \tilde{\phi}^{k_n}(x_n) \propto (\det{}^{-1/2}\Delta)[(\Delta^{-1})^{k_1 k_2} \ldots (\Delta^{-1})^{k_{n-1} k_n} + (\text{permutations})] \tag{3.14}$$

for even n, while the integrand vanishes if n is odd. The combinatorics in an integral correspond to the combinatorics of all posiible ways to connect graphs whose internal lines represent factors of Δ^{-1} and vertices to interactions within S_{int}. In this language, $Z[J]$ denotes the sum over all vacuum graphs with no external lines. Some leading perturbative contributions to the generating functional are:

$Z[J] = N(\det^{-1/2}\Delta)[1 +$  $]$ (3.15)

where solid lines are propagators and black dots represent interactions appear in S_{int}. This indeed includes all graphs containing those that are disconnected. Such graphs are disconnected in the sense that it is not possible to get between any pair of vertices along some sequence of internal lines.

Those expansions here are not practical yet in order to carry out the explicit computaions because of the appearance of the background field in the propagator $\Delta_{ij}(x,y) = \frac{\delta^2 S}{\delta\phi^i(x)\delta\phi^j(y)}$. For example, for a scalar field wit a scalar potential $U(\phi)$, we have $\Delta(x,y) = [-\Box + U''(\varphi)]\delta^4(x-y)$. This is easily inverted in momentum space if φ is constant, but is not straightforward to invert for arbitrary $\varphi(x)$. Such a problem is often addressed by expanding in powers of $J_i(x)$ so that path integral is evaluated as a semiclassical expansion about a simple backgroud configuration φ^i_{cl} which satisfies:

$$\left.\frac{\delta S}{\delta\phi^i}\right|_{\phi=\varphi_{\text{cl}}} = 0 \tag{3.16}$$

By doing this, for the modified expansion the term $\phi^i J_i$ in the exponent of the integrand is viewed as an interaction. This is linear in ϕ^i, so it corresponds to a tadpole contribution, with the line ending in a cross whose Feynman rule is $J_i(x)$.

3.1.1 Connected Correlation

Instead of dealing with all graphs including both connected and disconnected, it is usually useful to work with a generating functional $W[J]$ whose graphical representaion contains only connected ones. This is achieved by defining as $Z[J] := \exp\{iW[J]\}$, because taking logarithm means subtracting the disconnected graphs. This leads:

$$\exp\{iW[J]\} = \int \mathcal{D}\phi \exp\{iS[\phi] + i\int d^4x\phi^i J_i\} \tag{3.17}$$

Then corresponding correlation functions are given:

$$\left\langle T[\phi^{i_1}(x_1)\ldots\phi^{i_n}(x_n)]\right\rangle := (-i)^{n-1} \left.\frac{\delta^n W[J]}{\delta J_{i_1}(x_1)\ldots\delta J_{i_n}(x_n)}\right|_{J=0} \tag{3.18}$$

3.1.2 The 1-Particle Irreducible Action

As discussed, $Z[J] = \exp\{iW[J]\}$ can be thought of as the 'in-out' vacuum amplitude with a current J_i. In addition, the current can be interpreted as being responsible for varying the expectation value of the field since at $J_i = 0$:

$$\varphi^i(x) := \left\langle\phi^i(x)\right\rangle = \frac{\delta W}{\delta J_i(x)} \tag{3.19}$$

But it is often more simple to have the vacuum-to-vacuum amplitude expressed as in terms of a functional of the expectation value φ^i. This can be done through a Legendre transform. In order to perform Legendre transform, we may define:

$$\Gamma[\varphi] := W[J] - \int d^4x \varphi^i J_i \tag{3.20}$$

here $J_i(x)$ is a functional of $\varphi^i(x)$. Then we have a relation:

$$\frac{\delta\Gamma}{\delta\varphi^i(x)} = -J_i(x) \tag{3.21}$$

It shows that the expectation value for the system with $J_i = 0$ is a stationary point of $\Gamma[\varphi]$. Therefore, the relation between $\Gamma[\varphi]$ and $\langle\phi^i\rangle$ is to the one between the classical action and a classical background configuration.

Semiclassical Expansion

Now consider about the computation of $\Gamma[\varphi]$ within perturbation theory. From eq.3.17, we may have:

$$\begin{aligned}
\exp\{i\Gamma[\varphi]\} &= \exp\{iW[J] - i\int d^4x\varphi^i J_i\} \\
&= \int \mathcal{D}\phi \exp\{iS[\phi] + i\int d^4x(\phi^i - \varphi^i)J_i\} \\
&= \int \mathcal{D}\tilde{\phi} \exp\{iS[\varphi + \tilde{\phi}] + i\int d^4x\tilde{\phi}^i J_i
\end{aligned} \tag{3.22}$$

in the last line we changed the integration variable as $\phi^i \to \tilde{\phi}^i := \phi^i - \varphi^i$. Expanding the action inside the path integral about $\phi^i = \varphi^i$:

$$S[\varphi + \tilde{\phi}] = S[\varphi] + S_2[\varphi, \tilde{\phi}] + S_{\text{int}}[\varphi, \tilde{\phi}] \tag{3.23}$$

This is a similar expression we had before except the term linear in $\tilde{\phi}^i$:

$$S_{\text{int}}[\varphi, \tilde{\phi}] = \int d^4x[\left.\frac{\delta S}{\delta\phi^i}\right|_{\phi=\varphi} + J_i(x)]\tilde{\phi}^i(x) \tag{3.24}$$

which does not vanish since $\delta\varphi^i := \varphi^i - \varphi^i_{\text{cl}} \neq 0$. However, $\delta\varphi^i$ is perturbatively so small that it would be within S_{int}, expanded within the integrand, and not kept in the exponential. The expantion eventually becomes:

$$e^{i\Gamma[\varphi]} = e^{iS[\varphi]} \int \mathcal{D}\tilde{\phi} e^{iS_2[\varphi,\tilde{\phi}]} \sum_{r=0}^{\infty} \frac{1}{r!}(iS_{\text{int}} + iS_{\text{lin}})^r \tag{3.25}$$

and then:

$$\Gamma[\varphi] = S[\varphi] + \frac{i}{2}\ln\det\Delta + (\text{more than 1-loops}) \tag{3.26}$$

The first two terms in eq.3.26 are the classical and one-loop contributions respectively, though the last contains the sum over all Feynman graphs with two or more loops.

Since S_{lin} is linear term in $\tilde{\phi}^i$, its Feynman rule is as seen before, which inserts a tadpole contribution proportional to $\frac{\delta S}{\delta\varphi^i} + J_i$. Notice that evaluating at $J_i = -\frac{\delta\Gamma}{\delta\varphi^i}$, eq.3.26 gives:

$$\begin{aligned}\frac{\delta S}{\delta\varphi^i(x)} + J_i(x) &= \frac{\delta}{\delta\varphi^i(x)}(S[\varphi] - \Gamma[\varphi]) \\ &= -\frac{\delta}{\delta\varphi^i(x)}[\frac{i}{2}\ln\det\Delta + (\text{more than 1-loops})] \\ &= -(\text{sum of tadpole graphs})\end{aligned} \tag{3.27}$$

Therefore, we can manipulate without knowing expression for $\Gamma[\varphi]$. $J_i = -\frac{\delta\Gamma}{\delta\varphi}$ imdicates that all graphs involving explicit dependence on J_i cancel all graphs which can be split into two disconnected graphs. In this language, a graph which cannot be separated into tow pieces is called 1-particle irreducible (1PI). Thus, $\Gamma[\varphi]$ can be obtained by evaluating only 1PI graphs. Therefore, $\Gamma[\varphi]$ is often called the generator of 1PI correlations, or 1PI action. In semiclassical expansion, eq.3.26 is computed as a sum of 1PI connected graphs without external lines.

Viewing $\Gamma[\varphi]$ only involves 1PI suggests alternative way in which $\Gamma[\varphi]$ generalizes the classical action. In a perturbative expansion the leading approximation is the leading term in $\Gamma[\varphi] \approx S[\varphi]$. Considering scattering amplitudes, this amounts to summing the tree level graphs from vertices from the classical interactions in S_{int}. On the other hand, suppose computing Feynman graphs by the expansion of $\Gamma[\varphi + \tilde{\phi}]$ to obtain the propagators and vertices. For this, loops for any correlators with Feynman rules for $S[\varphi]$ is same as the quantities given by summing all tree graphs made from the Feynman rules for $\Gamma[\varphi]$. Thus, the full quantum amplitude can be found by computing with $\Gamma[\varphi]$ within the classical approxiamtion.

3.2 Wilsonian Effective Action

Now let us consider a field theory with characteristic (large) energy scale M, and suppose we want to study physics at a lower energy regime namely $E \ll M$. This is precisely the situation the EFTs are useful. The full theory is defined in terms of the path integral, and we can extract observables from calculating the correlator:

$$\langle 0|T[\phi(x_1)\ldots\phi(x_n)]|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi e^{iS[\phi]}\phi(x_1)\ldots\phi(x_n) \tag{3.28}$$

with the integration measure is:

$$\int \mathcal{D}\phi = \prod_{x^\mu}\int d\phi(x) \tag{3.29}$$

and

$$Z = \int \mathcal{D}\phi e^{iS[\phi]} \tag{3.30}$$

Then consider what happend as we try to perform the path integral by integrating those modes first with energy between Λ_0 and $\Lambda < \Lambda_0$. We can split a generic field

$\phi(x)$ as:

$$
\begin{aligned}
\phi(x) &= \int_{|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ip\dot{x}} \tilde{\phi}(p) \\
&= \int_{|p|\leq\Lambda} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x} \tilde{\phi}(p) + \int_{\Lambda<|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x} \tilde{\phi}(p) \\
&:= \varphi(x) + \chi(x)
\end{aligned}
\tag{3.31}
$$

where $\varphi(x)$ and $\chi(x)$ are low- and high-energy part of the field respectively. Thus the path integral measure can be rewritten as:

$$\mathcal{D}\phi = \mathcal{D}\varphi\mathcal{D}\chi \tag{3.32}$$

into a product of measures over low- and high-energy modes. Since we are now interested in low-energy physics, we just need to compute the correlation function:

$$\langle 0|T[\varphi(x_1)\ldots\varphi(x_n)]|0\rangle = \frac{1}{Z}\int \mathcal{D}\varphi \int \mathcal{D}\chi e^{iS[\varphi+\chi]}\varphi(x_1)\ldots\varphi(x_n) \tag{3.33}$$

Then we define a Wilsonian effective action at scale Λ $S_\Lambda[\varphi]$ as follows:

$$e^{iS_\Lambda[\varphi]} := \int \mathcal{D}\chi e^{iS[\varphi+\chi]} \tag{3.34}$$

and we have chosen $\Lambda < M$ to integrate out the physics associated with M. $S_\Lambda[\varphi]$ is non-local n scales $\Delta x^\mu \gtrsim \frac{1}{\Lambda}$, that is, the Lagrangian is not just a polynomial of the fields or their derivatives evaluated at a single point in spacetime, since high-energy fluctuations have been integrated out. Expanding the non-local action a series of local operators:

$$S_\Lambda[\varphi] = \int d^d x \mathcal{L}_\Lambda^{\text{eff}}(x) \tag{3.35}$$

$$\mathcal{L}_\Lambda^{\text{eff}(x)} = \sum_i g_i \mathcal{O}_i(x) \tag{3.36}$$

here $\mathcal{L}_\Lambda^{\text{eff}}(x)$ is an effective Lagrangian at scale Λ. It is an infinite sum over local operators $\mathcal{O}_i$ allowed by symmetries. The coefficients gi are referred to as Wilson coefficients.

3.2.1 Running Couplings and Their Beta-functions

It is important to notice that the partition function:

$$Z_\Lambda(g_i(\Lambda)) = \int_{\leq\Lambda} \mathcal{D}\phi e^{-iS_\Lambda[\phi]} \tag{3.37}$$

obtained from the effective action at scale Λ is exactly the same as the one we started with:

$$Z_\Lambda(g_i(\Lambda)) = Z_{\Lambda_0}(g_{i0};\Lambda_0) \tag{3.38}$$

because we are just computing the remaining integrals over the low-energy modes. Particularly, as the scale is lowered infinitesimally, this becomes:

$$\Lambda\frac{\mathrm{d}Z_\Lambda(g)}{\mathrm{d}\Lambda} = (\Lambda\frac{\partial}{\partial\Lambda}\bigg|_{g_i} + \Lambda\frac{\partial g_i(\Lambda)}{\partial\Lambda}\frac{\partial}{\partial g_i}\bigg|_\Lambda)Z_\Lambda(g) = 0 \tag{3.39}$$

Eq.3.39 is known as the renormalization group equation for the partition function. It is claiming that as we change the scale by integrating out modes, the couplings in the effective action S_Λ vary to account for the change in the degrees of freedom over which we take the path integral so that the partition function is indeed independent of the scale at which we define our theory, provided this scale is below our initial cut-off scale Λ_0. Since the running of couplings is so important, it has a specific name and we can define the beta-function β_i of the coupling g_i to be the derivative with respective to the logarithm of the scale:

$$\beta_i := \Lambda \frac{\partial g_i}{\partial \Lambda} = \frac{\partial g_i}{\partial \ln \Lambda} \tag{3.40}$$

3.2.2 1PI for Wilsonian Action

Here let us consider the 1PI for a Wilsonian effective action $S_\Lambda[\varphi]$ following the procedure in previous section. To begin with, for our convinience by the Wick rotation we may have:

$$S_\Lambda[\varphi] := iS_{E,\Lambda}[\varphi] \tag{3.41}$$

and from now on we will just write this as $S_\Lambda[\varphi]$ unless it makes serious confusions. Then the generating functional for the Green functions of a field φ in the path integral representaion is:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\varphi \exp\left(-S_\Lambda[\varphi] - \int d^d x J(x)\varphi(x)\right) \\ &= \int \mathcal{D}\varphi \exp\left(-\int d^d x (\mathcal{L}[\varphi(x)] + J(x)\varphi(x))\right) \end{aligned} \tag{3.42}$$

with an external source $J(x)$. Suppose then that a field $\varphi(x)$ can be splitted into two pieces: a classical background $\zeta(x)$ and a quantum fluctuations $\eta(x)$ so that:

$$\varphi(x) = \zeta(x) + \eta(x) \tag{3.43}$$

with a condition $\frac{\delta S}{\delta \varphi}\Big|_{\varphi=\zeta} + J(x) = 0$ is satisfied. Then the expansion of the action around the backgroud gives:

$$\begin{aligned} \int d^d x (\mathcal{L}[\varphi(x)] + J(x)\varphi(x)) &= \int d^d x (\mathcal{L}[\zeta(x)] + J(x)\zeta(x)) \\ &+ \int d^d x (\frac{\delta \mathcal{L}}{\delta \varphi}\Big|_{\phi=\eta} + J(x))\eta(x) \\ &+ \frac{1}{2}\int d^d x d^d y \frac{\delta^2 \mathcal{L}}{\delta\varphi(x)\delta\varphi(y)}\Big|_{\varphi=\zeta} \eta(x)\eta(y)\cdots \end{aligned} \tag{3.44}$$

The second term of eq.3.44 is identically zero due to the equation of motion. Then the generating functional can be in the form:

$$Z[J] = e^{(-\int d^d x(\mathcal{L}[\zeta(x)] + J(x)\zeta(x)))} \int \mathcal{D}\eta e^{-\frac{1}{2}\int d^d x d^d y \frac{\delta^2 \mathcal{L}}{\delta\varphi(x)\delta\varphi(y)}\big|_{\phi=\zeta}\eta(x)\eta(y)\cdots} \tag{3.45}$$

because the first integral in eq.3.44 has nothing to do with the fluctuation $\eta(x)$. Notice that the remaining path integral over η is of Gaussian form and can be computed explicitly:

$$Z[J] = e^{\left(-\int d^dx(\mathcal{L}[\zeta(x)]+J(x)\zeta(x))\right)}\left(\det\frac{\delta^2\mathcal{L}}{\delta\varphi(x)\delta\varphi(y)}\right)^{-1/2} + \cdots \tag{3.46}$$

A generating functional $Z[J]$ takes the form $Z[J] = e^{-W[J]}$, then, with an identity $\det\mathcal{A} = e^{\mathrm{Tr}\ln\mathcal{A}}$:

$$W[J] = \int d^dx(\mathcal{L}[\zeta(x)] + J(x)\zeta(x)) + \frac{1}{2}\mathrm{Tr}\ln\frac{\delta^2\mathcal{L}}{\delta\varphi(x)\delta\varphi(y)} + \cdots \tag{3.47}$$

Performing a Legendre transformation in order to compute the effective action $\Gamma[\varphi]$:

$$\begin{aligned}\Gamma[\zeta] &= W[J] - \int d^dxJ(x)\zeta(x)\\ &= \int d^dx\mathcal{L}[\zeta(x)] + \frac{1}{2}\mathrm{Tr}\left[\ln\frac{\delta^2\mathcal{L}}{\delta\varphi(x)\delta\varphi(y)}\right] + \cdots\end{aligned} \tag{3.48}$$

The first term is apparently the classical action. Thus the one-loop effective action is given:

$$\Gamma_{\text{1-loop}}[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\ln\frac{\delta^2\mathcal{L}}{\delta\varphi(x)\delta\varphi(y)}\right] \tag{3.49}$$

The divergent part of eq.3.49 is the same as the counter-terms, i.e. the action at a renormalized scale. Therefore, it is necessary to be able to carry out this trace in some ways in order to find the beta functions. Here we will adopt the method of Heat Kernel expansion.

3.3 Heat Kernel Expansion

3.3.1 General Formalism

For the manipulation of the trace, introduce the heat kernel:

$$K(t;x,y;D) = \langle x|e^{-tD}|y\rangle \tag{3.50}$$

such that it satisfies the heat conduction equation:

$$(\partial_t + D_x)K(t;x,y;D) = 0 \tag{3.51}$$

with an initial condition:

$$K(0;x,y;D) = \delta(x-y) \tag{3.52}$$

For instance, for a kernel for $D = -\Delta$:

$$K(t;x,y;-\Delta) = (4\pi t)^{-d/2}\mathrm{exp}^{\left(-\frac{(x-y)^2}{4t}\right)} \tag{3.53}$$

and for $D = D_0 = -\Delta + m^2$:

$$K(t;x,y;D) = (4\pi t)^{-d/2}\mathrm{exp}^{\left(-\frac{(x-y)^2}{4t}-tm^2\right)} \tag{3.54}$$

For a general D, $K(t;x,y;D_0)$ still describes the leading singularity at $t \to 0$:

$$K(t;x,y;D) = K(t;x,y;D_0)(1 + tb_2(x,y) + t^2 b_4(x,y) + \cdots) \tag{3.55}$$

Coefficients $b_k(x,y)$ are regular in the limit $y \to x$, and called the heat kernel coefficients.

It is needed to compute the functional:

$$\Gamma_{\text{1-loop}} = \frac{1}{2} \operatorname{Tr} \ln D \tag{3.56}$$

where;

$$D = \frac{\delta^2 \mathcal{L}}{\delta\phi\delta\phi} \tag{3.57}$$

But for each positive eignevalue λ of the operator D, an identity is:

$$\ln\lambda = -\int_0^\infty \frac{dt}{t} e^{-t\lambda} \tag{3.58}$$

Then eq.3.49 can be rewritten:

$$\Gamma_{\text{1-loop}} = -\frac{1}{2}\int_0^\infty \frac{dt}{t} K(t,D) \tag{3.59}$$

where

$$K(t,D) = \operatorname{Tr} e^{-tD} = \int d^d x \sqrt{g} K(t;x,x;D) \tag{3.60}$$

To see a UV divergence, introduce a cut-off at $t = \Lambda^{-2}$, and compute a part of $\Gamma_{\text{1-loop}}$ which diverges in the limit $\Lambda \to 0$:

$$\begin{aligned}\Gamma^{\text{div}}_{\text{1-loop}} = &-(4\pi)^{-d/2} \int d^d x \sqrt{g} [\sum_{2(j+l)<d} \Lambda^{d-2j-2l} b_{2j}(x,x) \frac{(-m^2)^l l!}{d-2j-2l} \\ &+ \sum_{2(j+l)=d} \ln\Lambda (-m^2)^l l! b_{2j}(x,x) + \mathcal{O}(\Lambda^0)]\end{aligned} \tag{3.61}$$

It can be seen that the UV divergence in the one-loop effective action are defined by finitely many heat kernel coefficients $b_k(x,x)$ with $k < d$.

Moreover, suppose the operator D is in a form:

$$D = -(g^{\mu\nu}\Delta_\mu\Delta_\nu + E) \tag{3.62}$$

according to the trace can be expanded as:

$$\operatorname{Tr} e^{-tD} = \sum_{k\geq 0} t^{(k-d)/2} a_k \tag{3.63}$$

where

$$a_k = (4\pi)^{-d/2} \int d^d x \sqrt{g} b_k(x,x) \tag{3.64}$$

and not showing in detail, but just borrowing the results:

$$a_0 = (4\pi)^{-d/2} \int d^d x \sqrt{g} \tag{3.65}$$

$$a_2 = (4\pi)^{-d/2} 6^{-1} \int d^d x \sqrt{g}\, \mathrm{Tr}[6E + R] \tag{3.66}$$

$$\begin{aligned} a_4 = (4\pi)^{-d/2} 360^{-1} \int d^d x \sqrt{g}\, \mathrm{Tr}\{&60E_{;kk} + 60RE + 180E^2 \\ &+ 12R_{;kk} + 5R^2 - 2R_{ij}R_{ij} + 2R_{ijkl}R_{ijkl} + 30\Omega_{ij}\Omega_{ij}\} \end{aligned} \tag{3.67}$$

and so on. R stands for curvetures, and $\Omega_{\mu\nu}$ represents the field strength of the connection ω.

3.3.2 Example

Let us see how heat kernel expansion works through a simple example. Consider the ϕ^4 theory in 4d case. The Laagrangian density is given:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_0^2 \phi^2 - \frac{1}{4!}\lambda_0 \phi^4 \tag{3.68}$$

Therefore, the operator D is:

$$D = -(\Delta + m_0^2 + \frac{\lambda_0}{2}\phi^2) \tag{3.69}$$

Introducing UV and IR cut-off, a 1-loop effective action in terms of coefficients a_k is:

$$\Gamma_{\text{1-loop}} = -\frac{1}{2}\int_{\Lambda^{-2}}^{\mu^{-2}} dt \sum_{k\geq 0} t^{(k-d-2)/2} a_k \tag{3.70}$$

Thus the divergent part is given by only three terms:

$$\begin{aligned} \Gamma^{\text{div}}_{\text{1-loop}} &= -\frac{1}{2}\int_{\Lambda^{-2}}^{\mu^{-2}} dt (t^{-3}a_0 + t^{-2}a_2 + t^{-1}a_4) \\ &= -\frac{1}{2}[\frac{1}{2}(\Lambda^4 - \mu^4)a_0 + (\Lambda^2 - \mu^2)a_2 + \ln(\Lambda^2/\mu^2)a_4] \\ &= -\frac{1}{2}\frac{1}{(4\pi)^2}\{\frac{1}{2}(\Lambda^4 - \mu^4)\int d^4x + (\Lambda^2 - \mu^2)\int d^4x E \\ &+ \frac{1}{2}\ln(\Lambda^2/\mu^2)\int d^4 x E^2\} \end{aligned} \tag{3.71}$$

This shoud agree with:

$$S = \int d^4x (\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4) \tag{3.72}$$

where m and λ are couplings at a scale μ. Thus comparing only the relevant coefficients:

$$\Gamma^{\text{div}}_{\text{1-loop}} = -\frac{1}{2}\frac{1}{(4\pi)^2}\{\ln(\Lambda/\mu)\int d^4x(\lambda_0 m_0^2\phi^2 + \frac{\lambda_0^2}{4}\phi^4) + (\Lambda^2 - \mu^2)\int d^4x \frac{\lambda_0}{2}\phi^2\} \tag{3.73}$$

gives equations:

$$-\frac{1}{2}\frac{1}{(4\pi)^2}\ln(\Lambda/\mu)\frac{\lambda_0^2}{4} = \frac{\lambda}{4!} \tag{3.74}$$

$$-\frac{1}{2}\frac{1}{(4\pi)^2}(\ln(\Lambda/\mu)\lambda_0 m_0^2 + (\Lambda^2 - \mu^2)\frac{\lambda_0}{2}) = \frac{1}{2}m^2 \tag{3.75}$$

To this end, beta functions for those couplings are:

$$\mu\frac{\partial\lambda}{\partial\mu} = \frac{3}{(4\pi)^2}\lambda_0^2 \tag{3.76}$$

$$\mu\frac{\partial m^2}{\partial\mu} = \frac{\lambda_0}{(4\pi)^2}(m_0^2 + \mu^2) \tag{3.77}$$

They are matching to the known results indeed.

Chapter 4

Computations

In previous chapters, we have all tools for the computaions. In this chapter, we will first provide a simple but non-trivial model, non-linear sigma model. We will also see SDC is satisfied onsuch a model. Then adding various potential terms, the RG computations will be carried out, and by finding the variations of the geodesic distances and assuming SDC will be still valid at varied scale, we may observe new kinds of constraints.

4.1 Non-Linear Sigma Model

Consider an effective theory with two real scalars φ_1 and φ_2 with its kinetic term given:

$$\frac{k}{\varphi_2^2} \tag{4.1}$$

where $\sqrt{k}$ is a free parameter. That is, we have an action defiend as:

$$S[\varphi] = \frac{1}{2}\int d^dx\sqrt{g}h_{ab}(\varphi)\partial_\mu\varphi^a\partial^\mu\varphi^b \tag{4.2}$$

with metric:

$$\mathrm{d}s^2 = \frac{k}{\varphi_2^2}(\mathrm{d}\varphi_1^2\mathrm{d}\varphi_2^2) \tag{4.3}$$

Notice that k determines the Ricci scalar curvature:

$$R = -\frac{2}{k} \tag{4.4}$$

Now we consider that SDC is satisfied on this moduli space. In other words, we assume there exists a tower of states with mass scale:

$$M \sim \varphi_2^{-a}, \qquad a > 0 \tag{4.5}$$

Besides, curvatures can be computed with formulae:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}h^{\lambda\alpha}(h_{\alpha\nu,\mu} + h_{\mu\alpha,\nu} - h_{\mu\nu,\alpha}) \tag{4.6}$$

$$R^\sigma_{\alpha\beta\gamma} = \Gamma^\sigma_{\alpha\gamma,\beta} - \Gamma^\sigma_{\beta\gamma,\alpha} + \Gamma^\tau_{\alpha\gamma}\Gamma^\sigma_{\tau\beta} - \Gamma^\tau_{\alpha\beta}\Gamma^\sigma_{\tau\gamma} \tag{4.7}$$

here the summation convention is applied. Non-zero terms are:

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{\varphi_2} \tag{4.8}$$

$$\Gamma^2_{11} = \frac{1}{\varphi_2} \tag{4.9}$$

$$\Gamma^2_{22} = -\frac{1}{\varphi_2} \tag{4.10}$$

from which

$$R^1_{212} = R^2_{121} = -R^1_{221} = -R^2_{112} = -\frac{1}{\varphi_2^2} \tag{4.11}$$

These read:

$$R_{1212} = R_{2121} = -\frac{k}{\varphi_2^4}, \qquad R_{1221} = R_{2112} = +\frac{k}{\varphi_2^4} \tag{4.12}$$

Furthermore, the non-vanishing components of Ricci tensor are:

$$R_{11} = R_{22} = -\frac{1}{\varphi_2^2} \tag{4.13}$$

Also, its geodesic equations are:

$$\begin{cases} \ddot{\varphi}_1 - \dfrac{2}{\varphi_2}\dot{\varphi}_1\dot{\varphi}_2 = 0 \\ \ddot{\varphi}_2 + \dfrac{1}{\varphi_2}(\varphi_1^2 - \varphi_2^2) = 0 \end{cases} \tag{4.14}$$

Thus it can be easily seen that on the (φ_1, φ_2) space, geodesics are either vertical lines or upper-half circles with centers on the $\varphi_2 = 0$ line. Therefore, the only geodesics approaching the infinite distance limits are vertical lines with $\varphi_1 =$ constant. Such geodesics are $(\varphi_1, \varphi_2) = (\varphi_1^0, \varphi_2^0 e^{\frac{s}{\sqrt{k}}})$, and thus the distance is:

$$\begin{aligned} s &= \int \sqrt{h_{11}\mathrm{d}\varphi_1^2 + h_{22}\mathrm{d}\varphi_2^2} \\ &= \int \sqrt{k}\frac{\mathrm{d}\varphi_2}{\varphi_2} \\ &= \sqrt{k}\ln\left|\frac{\varphi_2}{\varphi_2^2}\right| \end{aligned} \tag{4.15}$$

The mass scale of the tower is now:

$$M \sim \exp\left(-\frac{a}{\sqrt{k}}s\right) \sim \exp(-\omega s) \tag{4.16}$$

then leading to SDC with decay rate $\omega = \frac{a}{\sqrt{k}}$. Indeed, this is $\mathcal{O}(1)$.

As before, consider writing the field $\varphi(x)$ in terms of a bckground $\bar{\varphi}(x)$ and a corresponding quantum fluctuation ξ (x). Assume that there is a smooth map $\varphi_s(x)$ such that $\varphi_0(x) = \bar{\varphi}(x)$, $\varphi_1 = \varphi(x)$ with $\dot{\varphi}_0 = \xi$. Consider a curve φ_s in the target space which represents the geodesic between initial and final points, i.e. φ_0 and φ_1:

$$\ddot{\varphi}^a_s(x) + \Gamma^a_{bc}\dot{\varphi}^b_s(x)\dot{\varphi}^c_s(x) \tag{4.17}$$

where Γ denotes the Christoffel symbol corresponding to the target space metric h_{ab}. Then the expansion of the action around the background $\bar{\varphi}$ is given:

$$\begin{aligned} S[\varphi] &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}s^n} S[\varphi_s] \Big|_{s=0} \\ &= \sum_{n=0}^{\infty} (\nabla_s)^n S[\varphi_s] \Big|_{s=0} \end{aligned} \tag{4.18}$$

where ∇_s is the covariant derivative along the curve φ_s^a. Using identities:

$$\nabla_s \partial_\mu \varphi^i = \partial_\mu \frac{\mathrm{d}\varphi^i}{\mathrm{d}s} + \partial_\mu \varphi^k \Gamma^i_{kj} \frac{\mathrm{d}\varphi^j}{\mathrm{d}s} = \nabla_\mu \xi^i \qquad \nabla_s h_{ab} = 0 \tag{4.19}$$

$$\nabla_s \frac{\mathrm{d}\varphi^i}{\mathrm{d}s} = 0 \tag{4.20}$$

$$[\nabla_s, \nabla_\mu] Z^k = \frac{\mathrm{d}\varphi^i}{\mathrm{d}s} \partial_\mu \varphi^j R^k_{lij} Z^l \tag{4.21}$$

with Z^i is an arbitrary vector, it is followed by:

$$\begin{aligned} S[\varphi] =& \frac{1}{2} \int d^d x \sqrt{g} h_{ab}(\bar{\varphi}) \partial_\mu \bar{\varphi}^a \partial^\mu \bar{\varphi}^b + \int d^d x \sqrt{g} h_{ab} \partial_\mu \xi^a \partial^\mu \xi^b \\ &+ \frac{1}{2} \int d^d x \sqrt{g} \{ h_{ab} \nabla_\mu \xi^a \nabla^\mu \xi^b + R_{acdb} \xi^c \xi^d \partial_\mu \bar{\varphi}^a \partial^\mu \bar{\varphi}^b + \cdots \end{aligned} \tag{4.22}$$

Looking at the quadratic part, it lead to an operator:

$$D = -(h_{ab} \Box + R_{acbd} \partial_\mu \bar{\varphi}^c \partial^\mu \bar{\varphi}^d) \tag{4.23}$$

Therefore, for instance in $d = 4$ case, the 1-loop effective action is given:

$$\begin{aligned} \Gamma^{\text{div}}_{\text{1-loop}} =& -\frac{1}{2} \frac{1}{(4\pi)^2} \{ \frac{1}{2} (\Lambda^4 - \mu^4) \int d^4 x \sqrt{g} + (\Lambda^2 - \mu^2) \int d^4 x \sqrt{g} \, \mathrm{Tr} \, R_{acbd} \partial_\mu \bar{\varphi}^c \partial^\mu \bar{\varphi}^d \\ &+ \frac{1}{2} \ln(\Lambda^2/\mu^2) \int d^4 x \sqrt{g} \, \mathrm{Tr} \Big(R_{acbd} \partial_\mu \bar{\varphi}^c \partial^\mu \bar{\varphi}^d \Big)^2 \} \end{aligned} \tag{4.24}$$

Thus, comparing only relevant coefficients:

$$\mu \frac{\partial}{\partial \mu} g_{ab} = \frac{\mu^2}{(4\pi)^2} R_{ab} \tag{4.25}$$

Adding a generic potential term V, the operator reads:

$$D = -(h_{ab} \Box + R_{acbd} \partial_\mu \bar{\varphi}^a \partial^\mu \bar{\varphi}^b - \mathrm{Hess}(V)) \tag{4.26}$$

where Hess(V) is a Hessian matrix of V. Then the divergent part of 1-loop effective action in 4d becomes:

$$\begin{aligned}\Gamma^{\text{div}}_{\text{1-loop}} = &-\frac{1}{2}\frac{1}{(4\pi)^2}\{(\Lambda^2-\mu^2)\int d^4x\sqrt{g}\,\text{Tr}\Big(R_{acbd}\partial_\mu\bar{\varphi}^a\partial^\mu\bar{\varphi}^b - \text{Hess}(V)\Big)\\ &+\ln(\Lambda/\mu)\int d^4x\sqrt{g}\,\text{Tr}\Big(R_{acbd}\partial_\mu\bar{\varphi}^a\partial^\mu\bar{\varphi}^b - \text{Hess}(V)\Big)^2\end{aligned} \tag{4.27}$$

By comparing coefficients with an action at scale μ:

$$S[\varphi] = \int d^4x\sqrt{g}(\frac{1}{2}\tilde{h}_{ab}\partial_\mu\varphi^a\partial^\mu\varphi^b + V) \tag{4.28}$$

flow equations can be obtained as did above.

4.1.1 Generic Mass

Suppose that the potential is given in a form of the homogeneous quadratic function of φ_1 and φ_2:

$$V = \frac{1}{2}\sum_{i,j} M_{ij}\varphi^i\varphi^j \tag{4.29}$$

Hence, its Hessian matrix is:

$$Hess(V) = \begin{bmatrix} M_1 & m \\ m & M_2 \end{bmatrix} \tag{4.30}$$

here $M_{11} = M_1$, $M_{22} = M_2$ and $M_{12} = M_{21} = m$. Then The relevant divergence of 1-loop effective action is:

$$\begin{aligned}\Gamma^{\text{div}}_{\text{1-loop}} = &-\frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\{(\Lambda^2-\mu^2)(R_{cd}\partial_\mu\varphi^c\partial^\mu\varphi^d - \text{Tr}\,Hess(V))\\ &+\ln(\Lambda^2/\mu^2)(\frac{1}{2}\text{Tr}\,Hess(V)^2 - \text{Tr}\,Hess(V)R_{acbd}\partial_\mu\varphi^c\partial^\mu\varphi^d)\end{aligned} \tag{4.31}$$

Therefore, the metric h_{ab} flows as:

$$\mu\frac{\partial}{\partial\mu}h_{ab} = \frac{1}{8\pi^2}(\mu^2 R_{ab} - \text{Tr}\,Hess(V)R_{acbd}) \tag{4.32}$$

Now considering how the metric, and geodesics eventually, would be modified as the scale varies from Λ to μ, integrate again both side, it becomes:

$$h^\mu_{ab} - h^\Lambda_{ab} = \frac{1}{8\pi^2}\{\frac{1}{2}(\mu^2-\Lambda^2)R_{ab} - \ln\frac{\mu}{\Lambda}\,\text{Tr}\,Hess(V)R_{acbd}\} \tag{4.33}$$

Assume here that the variation of the scale is so small that the curvature on the right hand side can be thought of as the one corresponding to the initial metric at scale Λ, writing $\mu = \Lambda(1-\epsilon)$ with $\epsilon \ll 1$, eq.4.33 gives:

$$h^\mu_{ab} \approx h^\Lambda_{ab} + \frac{\epsilon}{8\pi^2}\big(\text{Tr}\,Hess(V)R_{acbd} - \Lambda^2 R_{ab}\big) \tag{4.34}$$

For given metric and a form of the potential, each term is:

$$\mathrm{Tr}\, Hess(V) R_{acbd} = \frac{1}{k}\begin{bmatrix} -M_2 & m \\ m & -M_1 \end{bmatrix} \tag{4.35}$$

$$R_{ab} = -\frac{1}{\varphi_2^2}\mathbb{1} \tag{4.36}$$

the metric at a scale μ is:

$$\tilde{h}_{ab} := h^{\mu}_{ab} = \begin{bmatrix} \frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}\{\frac{M_2}{k} - (\frac{\Lambda}{\varphi_2})^2\} & \frac{\epsilon}{8\pi^2}\frac{m}{k} \\ \frac{\epsilon}{8\pi^2}\frac{m}{k} & \frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}\{\frac{M_1}{k} - (\frac{\Lambda}{\varphi_2})^2\} \end{bmatrix} \tag{4.37}$$

and its inverse is:

$$\tilde{h}^{ab} = \begin{bmatrix} \frac{\varphi_2^2}{k} + \frac{\epsilon}{8\pi^2}(\frac{\varphi_2^2}{k})^2\{\frac{M_2}{k} - (\frac{\Lambda}{\varphi_2})^2\} & -\frac{\epsilon}{8\pi^2}\frac{m}{k}(\frac{\varphi_2^2}{k})^2 \\ -\frac{\epsilon}{8\pi^2}\frac{m}{k}(\frac{\varphi_2^2}{k})^2 & \frac{\varphi_2^2}{k} + \frac{\epsilon}{8\pi^2}(\frac{\varphi_2^2}{k})^2\{\frac{M_1}{k} - (\frac{\Lambda}{\varphi_2})^2\} \end{bmatrix} \tag{4.38}$$

Now in order to investigate how this geodesic varies as there is a tiny variation on the metric. From perturbed metric, one may find:

$$\tilde{\Gamma}^1_{11} = -\frac{\epsilon}{8\pi^2}\frac{m}{k^2}\varphi_2 \tag{4.39}$$

$$\tilde{\Gamma}^1_{12} = -\frac{1}{\varphi_2} - \frac{\epsilon}{8\pi^2}\frac{M_2}{k^2}\varphi_2 \tag{4.40}$$

$$\tilde{\Gamma}^1_{22} = \frac{\epsilon}{8\pi^2}\frac{m}{k^2}\varphi_2 \tag{4.41}$$

$$\tilde{\Gamma}^2_{11} = \frac{1}{\varphi_2} + \frac{\epsilon}{8\pi^2}\frac{M_1}{k^2}\varphi_2 \tag{4.42}$$

$$\tilde{\Gamma}^2_{12} = \frac{\epsilon}{8\pi^2}\frac{m}{k^2}\varphi_2 \tag{4.43}$$

$$\tilde{\Gamma}^2_{22} = -\frac{1}{\varphi_2} - \frac{\epsilon}{8\pi^2}\frac{M_1}{k^2}\varphi_2 \tag{4.44}$$

Therefore, the new geodesic equations are:

$$\begin{cases} \ddot{\varphi}_1 - \dfrac{2}{\varphi_2}\dot{\varphi}_1\dot{\varphi}_2 = \dfrac{\epsilon}{8\pi^2}\dfrac{\varphi_2}{k^2}(m\dot{\varphi}_1^2 + 2M_2\dot{\varphi}_1\dot{\varphi}_2 - m\dot{\varphi}_2^2) & (4.45) \\ \ddot{\varphi}_2 + \dfrac{1}{\varphi_2}(\varphi_1^2 - \varphi_2^2) = \dfrac{\epsilon}{8\pi^2}\dfrac{\varphi_2}{k^2}(-M_1\dot{\varphi}_1^2 - 2m\dot{\varphi}_1\dot{\varphi}_2 + M_1\dot{\varphi}_2^2) & (4.46) \end{cases}$$

Notice that $\dot{\varphi}_1 = 0$ is no longer a solution of the equations. Then assuming an ansatz such that the solutions for those geodesic equations $(\varphi_1, \varphi 2)$ are in the form:

$$\begin{cases} \varphi_1 = \tilde{\varphi}_1 + \epsilon f(s) & (4.47) \\ \varphi_2 = \tilde{\varphi}_2 + \epsilon g(s) & (4.48) \end{cases}$$

where tildes denote the unperturbed solutions, with constraints:

$$\varphi_i\Big|_{s=0} = \tilde{\varphi}_i, \qquad \dot{\varphi}_i\Big|_{s=0} = \dot{\tilde{\varphi}}_i \tag{4.49}$$

Knowing $\tilde{\varphi}_1 = \varphi_1^0 = \text{constant}$, and $\tilde{\varphi}_2 = \varphi_2^0 e^{s/\sqrt{k}}$, eq.4.45 leads:

$$
\begin{aligned}
&(\tilde{\varphi}_2 + \epsilon g(s))(\ddot{\tilde{\varphi}}_1 + \epsilon \ddot{f}(s)) - 2(\dot{\tilde{\varphi}}_1 + \epsilon \dot{f}(s))(\dot{\tilde{\varphi}}_2 + \epsilon \dot{g}(s)) \\
&= \frac{\epsilon}{8\pi^2}\frac{1}{k^2}(\tilde{\varphi}_2 + \epsilon g(s))^2\{m(\dot{\tilde{\varphi}}_2 + \epsilon \dot{f}(s))^2 + 2M_2(\dot{\tilde{\varphi}}_1 + \epsilon \dot{f}(s))(\dot{\tilde{\varphi}}_2 + \epsilon \dot{g}(s)) - m(\dot{\tilde{\varphi}}_2 + \epsilon \dot{g}(s))^2\} \\
\Rightarrow &\tilde{\varphi}_2\ddot{\tilde{\varphi}}_1 - 2\dot{\tilde{\varphi}}_1\dot{\tilde{\varphi}}_2 + \epsilon\{\tilde{\varphi}_2\ddot{f}(s) + \ddot{\tilde{\varphi}}_1 g(s) - 2(\dot{\tilde{\varphi}}_1\dot{g}(s) + \dot{\tilde{\varphi}}_2\dot{f}(s))\} + \mathcal{O}(\epsilon^2) \\
&= \frac{\epsilon}{8\pi^2}\frac{1}{k^2}\tilde{\varphi}_2{}^2(m\dot{\tilde{\varphi}}_2^2 + 2M_2\dot{\tilde{\varphi}}_1\dot{\tilde{\varphi}}_2 - m\dot{\tilde{\varphi}}_2^2) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

Then it ends up with a differential equation:

$$
\ddot{f}(s) - \frac{2}{\sqrt{k}}\dot{f}(s) + \frac{1}{8\pi^2}\frac{m}{k^3}\tilde{\varphi}_2^3 = 0 \tag{4.50}
$$

A solution to the homogeneous equation is:

$$
f_h(s) = C_1 + C_2 e^{2s/\sqrt{k}} \tag{4.51}
$$

and let $f_p = Ae^{3s/\sqrt{k}}$ be the particular solution, then since $\ddot{f}_p(s) = \frac{9}{k}Ae^{3s/\sqrt{k}}$, $\dot{f}_p(s) = \frac{3}{\sqrt{k}}$, $A = -\frac{k}{3}\frac{1}{8\pi^2}\frac{m}{k^3}(\varphi_2^0)^3$. Therefore:

$$
\begin{aligned}
f(s) &= f_h(s) + f_p(s) \\
&= C_1 + C_2 e^{2s/\sqrt{k}} - \frac{k}{3}\frac{1}{8\pi^2}\frac{m}{k^3}(\varphi_2^0)^3 e^{3s/\sqrt{k}}
\end{aligned} \tag{4.52}
$$

Considering its initial conditions:

$$
f(0) = C_1 + C_2 - \frac{1}{3}\frac{m}{8\pi^2 k^2}(\varphi_2^0)^3 \tag{4.53}
$$

$$
\dot{f}(0) = \frac{2}{\sqrt{k}}C_2 - \frac{1}{8\pi^2}\frac{m}{k^2\sqrt{k}}(\varphi_2^0)^3 \tag{4.54}
$$

the correction for φ_1 is given:

$$
f(s) = -\frac{1}{8\pi^2}\frac{m}{6k^2}(\varphi_2^0)^3(1 - 3e^{\frac{2s}{\sqrt{k}}} + 2e^{\frac{3s}{\sqrt{k}}}) \tag{4.55}
$$

Also, for eq.4.46:

$$
\tilde{\varphi}_2\ddot{\tilde{\varphi}}_2 - \dot{\tilde{\varphi}}_2^2 + \epsilon(\varphi_2\ddot{g}(s) + \ddot{\tilde{\varphi}}_2 g(s) - 2\dot{\tilde{\varphi}}_2\dot{g}(s)) = \frac{\epsilon}{8\pi^2}\frac{M_1}{k^2}\tilde{\varphi}_2{}^2\dot{\tilde{\varphi}}_2^2 \tag{4.56}
$$

$$
\Rightarrow \ddot{g}(s) - \frac{2}{\sqrt{k}}\dot{g}(s) + \frac{1}{k}g(s) - \frac{1}{8\pi^2}\frac{M_1}{k^3}(\varphi_2^0)^3 e^{\frac{3s}{\sqrt{k}}} = 0 \tag{4.57}
$$

Through the same procedure to $f(s)$, the correction for φ_2 is:

$$
g(s) = \frac{1}{8\pi^2}\frac{M_1}{4k^{5/2}}(\varphi_2^0)^3 e^{s/\sqrt{k}}(-\sqrt{k} - 2s + \sqrt{k}e^{2s/\sqrt{k}}) \tag{4.58}
$$

Therefore, up to $\mathcal{O}(\epsilon)$, corrected geodesics are:

$$\begin{cases} \varphi_1 = \varphi_1^0 - \frac{\epsilon}{8\pi^2}\frac{m}{6k^2}(\varphi_2^0)^3(1 - 3e^{\frac{2s}{\sqrt{k}}} + 2e^{\frac{3s}{\sqrt{k}}}) & (4.59) \\ \varphi_2 = \varphi_2^0 e^{\frac{s}{\sqrt{k}}} + \frac{\epsilon}{8\pi^2}\frac{M_1}{4k^{5/2}}(\varphi_2^0)^3 e^{\frac{s}{\sqrt{k}}}(-\sqrt{k} - 2s + \sqrt{k}e^{\frac{2s}{\sqrt{k}}}) & (4.60) \end{cases}$$

Finally, from those information, variations of a distance between two points as the scale variation can be also investigated. A distance for corrected geodesics is:

$$\begin{aligned} s &= \int \sqrt{\tilde{h}_{ij}\mathrm{d}\varphi_i\mathrm{d}\varphi_j} \\ &= \int \sqrt{[\frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}\{\frac{M_2}{k} - (\frac{\Lambda}{\varphi_2})^2\}]\mathrm{d}\varphi_1^2 + \frac{\epsilon}{4\pi^2}\frac{m}{k}\mathrm{d}\varphi_1\mathrm{d}\varphi_2 + [\frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}\{\frac{M_1}{k} - (\frac{\Lambda}{\varphi_2})^2\}]\mathrm{d}\varphi_2^2} \end{aligned}$$

However, from corrected geodesics, it can be deduced that:

$$\begin{aligned} \frac{\mathrm{d}\varphi_1}{\mathrm{d}\varphi_2} &= \frac{\mathrm{d}\varphi_1}{\mathrm{d}\tilde{\varphi}_2}\frac{\mathrm{d}\tilde{\varphi}_2}{\mathrm{d}\varphi_2} \\ &= -\frac{\epsilon}{8\pi^2}\frac{m}{6k^2}(-6\varphi_2^0\tilde{\varphi}_2 + 6\tilde{\varphi}_2{}^2)(1 + \frac{\epsilon}{8\pi^2}\frac{M_1}{4k^{5/2}} \\ &\quad (-\sqrt{k}(\varphi_2^0)^2 - 2s(\varphi_2^0)^2 + 3\sqrt{k}\tilde{\varphi}_2{}^2))^{-1} \\ &= -\frac{\epsilon}{8\pi^2}\frac{m}{k^2}(\tilde{\varphi}_2 - \varphi_2^0)\tilde{\varphi}_2 + \mathcal{O}(\epsilon^2) \end{aligned} \tag{4.61}$$

With this aid, the distance can be in the form:

$$\begin{aligned} s &= \int [[\frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}\{\frac{M_2}{k} - (\frac{\Lambda}{\varphi_2})^2\}]\{-\frac{\epsilon}{8\pi^2}\frac{m}{k^2}\tilde{\varphi}_2(\tilde{\varphi}_2 - \varphi_2^0)\}^2 \\ &+ \frac{\epsilon}{4\pi^2}\frac{m}{k}\{-\frac{\epsilon}{8\pi^2}\frac{m}{k^2}\tilde{\varphi}_2(\tilde{\varphi}_2 - \varphi_2^0)\} \\ &+ [\frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}\{\frac{M_1}{k} - (\frac{\Lambda}{\varphi_2})^2\}]]^{1/2}\,\mathrm{d}\varphi_2 \\ &\sim \int \frac{\sqrt{k}}{\varphi_2}[1 - \frac{\epsilon}{16\pi^2}\frac{\varphi_2^2}{k}\{\frac{M_1}{k} - (\frac{\Lambda}{\varphi_2})^2\}] + \mathcal{O}(\epsilon^2)\,\mathrm{d}\varphi_2 \\ &= \sqrt{k}\ln|\frac{\varphi_2}{\varphi_2^0}| - \frac{\epsilon}{16\pi^2}\{\frac{M_1}{2k\sqrt{k}}(\varphi_2^2 - (\varphi_2^0)^2) - \frac{\Lambda^2}{\sqrt{k}}\ln|\frac{\varphi_2}{\varphi_2^0}|\} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{4.62}$$

It can be easily noticed that in the limit $\epsilon \to 0$, $\varphi_2 \to \tilde{\varphi}_2$ and thus the distance for the original geodesic is restored. Also, notice that SDC is violated since for large enough φ_2:

$$\frac{\mathrm{d}\ln M}{\mathrm{d}s} = \frac{\mathrm{d}\ln M}{\mathrm{d}\varphi_2}\frac{\mathrm{d}\varphi_2}{\mathrm{d}s} = -\frac{\lambda}{\sqrt{k}}\left|\frac{\epsilon}{8\pi^2}\frac{M_1}{k^2}\varphi_2^2\right|^{-1/2} \to 0 \tag{4.63}$$

so the tower mass scale is no longer decaying exponentially with the geodesic distance. This is compatible with SDC if and only if the coefficient of the polynomial term vanishes. In this case, we have a new constraint $M_1 = 0$.

4.1.2 Quartic Coupling

Next, let us add a quartic coupling term to the action. Then the potential has a form:

$$V = \frac{1}{2}M_{ij}\varphi_i\varphi_j + \frac{\lambda}{4}(\varphi_1^2+\varphi_2^2)^2 \tag{4.64}$$

Then its Hessian matrix is:

$$Hess(V) = \begin{bmatrix} M_1 & m \\ m & M_2 \end{bmatrix} + \lambda \begin{bmatrix} 3\varphi_1^2+\varphi_2^2 & 2\varphi_1\varphi_2 \\ 2\varphi_1\varphi_2 & \varphi_1^2+3\varphi_2^2 \end{bmatrix} \tag{4.65}$$

Hence, the metric becomes:

$$\tilde{h}_{ab} = \begin{bmatrix} \frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}[\frac{1}{k}\{M_2+\lambda(\varphi_1^2+3\varphi_2^2)\} - (\frac{\Lambda}{\varphi_2})^2] & \frac{\epsilon}{8\pi^2}\frac{1}{k}(m+2\lambda\varphi_1\varphi_2) \\ \frac{\epsilon}{8\pi^2}\frac{1}{k}(m+2\lambda\varphi_1\varphi_2) & \frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}[\frac{1}{k}\{M_1+\lambda(3\varphi_1^2+\varphi_2^2)\} - (\frac{\Lambda}{\varphi_2})^2] \end{bmatrix} \tag{4.66}$$

Then we may have:

$$\Gamma^1_{11} = -\frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}(m+3\lambda\varphi_1\varphi_2) \tag{4.67}$$
$$\Gamma^1_{12} = -\frac{1}{\varphi_2} - \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}(M_2+\lambda(\varphi_1^2+6\varphi_2^2)) \tag{4.68}$$
$$\Gamma^1_{22} = \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}(m+6\lambda\varphi_1\varphi_2) \tag{4.69}$$
$$\Gamma^2_{11} = \frac{1}{\varphi_2} + \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}[M_1+\lambda(3\varphi_1^2+6\varphi_2^2)] \tag{4.70}$$
$$\Gamma^2_{12} = \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}(m-\lambda\varphi_1\varphi_2) \tag{4.71}$$
$$\Gamma^2_{22} = -\frac{1}{\varphi_2} - \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}[M_1+\lambda(3\varphi_1^2+2\varphi_2^2)] \tag{4.72}$$

With same procedure as before, we can obtain the geodesic equations:

$$\begin{aligned}\ddot{\varphi}_1 - \frac{2}{\varphi_2}\dot{\varphi}_1\dot{\varphi}_2 = \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}[&(m+3\lambda\varphi_1\varphi_2)\dot{\varphi}_1^2 + 2\{M_2+\lambda(\varphi_1^2+6\varphi_2^2)\}\dot{\varphi}_1\varphi_2 \\ &-(m+6\lambda\varphi_1\varphi_2)\dot{\varphi}_2^2]\end{aligned} \tag{4.73}$$

$$\begin{aligned}\ddot{\varphi}_2 + \frac{1}{\varphi_2}(\dot{\varphi}_1^2-\dot{\varphi}_2^2) = \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}[&-\{M_1+\lambda(3\varphi_1^2+6\varphi_2^2)\}\dot{\varphi}_1^2 - 2(m-\lambda\varphi_1\varphi_2)\dot{\varphi}_1\dot{\varphi}_2 \\ &+\{M_1+\lambda(3\varphi_1^2+2\varphi_2^2)\}\dot{\varphi}_2^2]\end{aligned} \tag{4.74}$$

Then, eq.4.73 gives:

$$\ddot{f}(s) - \frac{2}{\sqrt{k}}\dot{f}(s) = -\frac{1}{8\pi^2}\frac{1}{k^3}(m+6\lambda\tilde{\varphi}_1\tilde{\varphi}_2)\tilde{\varphi_2}^3 \tag{4.75}$$

where $f(s)$, and $\tilde{\varphi}_i$ are same as we did. Solving for $f(s)$, we end up with:

$$\varphi_1 = \varphi_1^0 - \frac{\epsilon}{8\pi^2}\frac{m}{6k^2}(\varphi_2^0)^3(1 - 3e^{\frac{2s}{\sqrt{k}}} + 2e^{\frac{3s}{\sqrt{k}}}) - \frac{\epsilon}{8\pi^2}\frac{3\lambda}{4k^2}\varphi_1^0(\varphi_2^0)^4(1 - 2e^{\frac{2s}{\sqrt{k}}} + e^{\frac{4s}{\sqrt{k}}}) \tag{4.76}$$

Similarly, eq.4.74 reads:

$$\ddot{g}(s) - \frac{2}{\sqrt{k}}\dot{g}(s) + \frac{1}{k}g(s) = \frac{1}{8\pi^2}\frac{1}{k^3}(M_1 + 3\lambda\tilde{\varphi}_1{}^2 + 2\lambda\tilde{\varphi}_2{}^2)\tilde{\varphi}_2{}^3 \tag{4.77}$$

and we eventually find:

$$\begin{aligned}\varphi_2 =& \varphi_2^0 e^{\frac{s}{\sqrt{k}}} + \frac{\epsilon}{8\pi^2}\frac{1}{4k^{5/2}}(\varphi_2^0)^3 e^{\frac{s}{\sqrt{k}}}(-\sqrt{k} - 2s + \sqrt{k}e^{\frac{2s}{\sqrt{k}}})(M_1 + 3\lambda(\varphi_1^0)) \\ &+ \frac{\epsilon}{8\pi^2}\frac{\lambda}{4k^{5/2}}(\varphi_2^0)^5(-\frac{\sqrt{k}}{2} - 2s + 2\sqrt{k}e^{\frac{4s}{\sqrt{k}}})\end{aligned} \tag{4.78}$$

In order to determine the geodesic distance, we again manipulate first:

$$\frac{d\varphi_1}{d\varphi_2} = \frac{d\varphi_1}{d\tilde{\varphi}_2}\frac{d\tilde{\varphi}_2}{d\varphi_2} = -\frac{\epsilon}{8\pi^2}[\frac{m}{k^2}(\tilde{\varphi}_2 - \varphi_2^0)\tilde{\varphi}_2 + \frac{3\lambda}{k^2}\varphi_1^0(\tilde{\varphi}_2{}^2 - (\varphi_2^0)^2)\tilde{\varphi}_2] + \mathcal{O}(\epsilon^2) \tag{4.79}$$

Thus, the distance is given:

$$\begin{aligned}s =& \int\sqrt{\tilde{h}_{ab}d\varphi_a d\varphi_b} \\ \sim& \sqrt{k}\ln\left|\frac{\varphi_2}{\varphi_2^0}\right| - \frac{\epsilon}{16\pi^2}\frac{1}{\sqrt{k}}[\frac{1}{2k}(M_1 + 3\lambda(\varphi_1^0)^2)(\varphi_2^2 - (\varphi_2^0)^2) + \frac{\lambda}{4k}(\varphi_2^4 - (\varphi_2^0)^4) + \Lambda^2\ln\left|\frac{\varphi_2}{\varphi_2^0}\right|]\end{aligned} \tag{4.80}$$

Notice that we have two polynomial terms here. Assuming SDC is always satisfied at different energy scales, we arrive the constraints:

$$\lambda = 0, \qquad M_1 = 0 \tag{4.81}$$

4.1.3 Exponential Mass

Now suppose $M_1 = Ae^{-\alpha\varphi_1}$. Then the Hessian is:

$$Hess(V) = \begin{bmatrix} Ae^{-\alpha\varphi_1}(\alpha^2\varphi_1^2 - 4\alpha\varphi_1 + 2) & m \\ m & M_2 \end{bmatrix} \tag{4.82}$$

and the perturved metric is:

$$\tilde{h}_{ab} = \begin{bmatrix} \frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}[\frac{M_2}{k} - (\frac{\Lambda}{\varphi_2})^2] & \frac{\epsilon}{8\pi^2}\frac{m}{k} \\ \frac{\epsilon}{8\pi^2}\frac{m}{k} & \frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2}[\frac{Ae^{-\alpha\varphi_1}}{k}(\alpha^2\varphi_1^2 - 4\alpha\varphi_1 + 2) - (\frac{\Lambda}{\varphi_1})^2] \end{bmatrix} \tag{4.83}$$

Then:

$$\Gamma^1_{11} = -\frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}m \tag{4.84}$$

$$\Gamma^1_{12} = -\frac{1}{\varphi_2} - \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}M_2 \tag{4.85}$$

$$\Gamma^1_{22} = \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}[\frac{A}{2}\varphi_2 e^{-\alpha\varphi_1}(-\alpha^3\tilde{\varphi}_1{}^2 + 6\alpha^2\tilde{\varphi}_1 - 6\alpha) + m] \tag{4.86}$$

$$\Gamma^2_{11} = \frac{1}{\varphi_2} + \frac{\epsilon}{8\pi^2}\frac{A\varphi_2}{k^2}e^{-\alpha\varphi_1}(\alpha^2\varphi_1^2 - 4\alpha\varphi_1 + 2) \tag{4.87}$$

$$\Gamma^2_{12} = \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}(m - \frac{A}{2}\varphi_2 e^{-\alpha\varphi_1}(-\alpha^3\varphi_1^2 + 6\alpha^2\varphi_1 - 6\alpha)) \tag{4.88}$$

$$\Gamma^2_{22} = -\frac{1}{\varphi_2} - \frac{\epsilon}{8\pi^2}\frac{A\varphi_2}{k^2}e^{-\alpha\varphi_1}(\alpha^2\varphi_1^2 - 4\alpha\varphi_1 + 2) \tag{4.89}$$

and geodesic equations are:

$$\begin{aligned}\ddot{\varphi}_1 - \frac{2}{\varphi_2}\dot{\varphi}_1\dot{\varphi}_2 = \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}[&m\dot{\varphi}_1^2 + 2M_2\dot{\varphi}_1\dot{\varphi}_2 - \frac{A}{2}\varphi_2 e^{-\alpha\varphi_1}(-\alpha^3\varphi_1^2 + 6\alpha^2\varphi_1 - 6\alpha)\dot{\varphi}_2^2 \\ &- m\dot{\varphi}_2^2]\end{aligned} \tag{4.90}$$

$$\begin{aligned}\ddot{\varphi}_2 + \frac{1}{\varphi_2}(\dot{\varphi}_1^2 - \dot{\varphi}_2)^2 = \frac{\epsilon}{8\pi^2}\frac{\varphi_2}{k^2}&(-A\varphi_2 e^{-\alpha\varphi_1(\alpha^2\varphi_1^2 - 4\alpha\varphi_1 + 2)\dot{\varphi}_1} \\ &+ A\varphi_2 e^{-\alpha\varphi_1}(-\alpha^3\varphi_1^2 + 6\alpha\varphi_1 - 6\alpha)\dot{\varphi}_1\dot{\varphi}_2 + Ae^{-\alpha\varphi_1}(\alpha^2\varphi_1^2 - 4\alpha\varphi_1 + 2)\dot{\varphi}_2^2)\end{aligned} \tag{4.91}$$

Repeating the same arguments, we arrive:

$$\begin{aligned}\varphi_1 =&\varphi_1^0 - \frac{\epsilon}{8\pi^2}\frac{m}{6k^2}(\varphi_2^0)^3[1 - 3e^{\frac{2s}{\sqrt{k}}} + 2e^{\frac{3s}{\sqrt{k}}}] \\ &- \frac{\epsilon}{8\pi^2}\frac{A}{16k^2}e^{-\alpha\tilde{\varphi}_1}(-\alpha^3\tilde{\varphi}_1{}^2 + 6\alpha^2\tilde{\varphi}_1 - 6\alpha)(\varphi_2^0)^4[1 - 2e^{\frac{2s}{\sqrt{k}}} + e^{\frac{4s}{\sqrt{k}}}]\end{aligned} \tag{4.92}$$

$$\varphi_2 =\varphi_2^0 e^{\frac{s}{\sqrt{k}}} + \frac{\epsilon}{8\pi^2}\frac{A}{4k^{5/2}}e^{-\alpha\tilde{\varphi}_1}(\alpha^2\tilde{\varphi}_1{}^2 - 4\alpha\tilde{\varphi}_1 + 2)(\varphi_2^0)^3 e^{\frac{s}{\sqrt{k}}}(-\sqrt{k} - 2s + \sqrt{k}e^{\frac{2s}{\sqrt{k}}}) \tag{4.93}$$

Also:

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}\varphi_2} = -\frac{\epsilon}{8\pi^2}\frac{m}{k^2}\tilde{\varphi}_2(\tilde{\varphi}_2 - \varphi_2^0) - \frac{\epsilon}{8\pi^2}\frac{A}{4k^2}e^{-\alpha\tilde{\varphi}_1}(-\alpha^3\tilde{\varphi}_1{}^2 + 6\alpha^2\tilde{\varphi}_1 - 6\alpha)\tilde{\varphi}_2(\tilde{\varphi}_2{}^2 - (\varphi_2^0)^2) + \mathcal{O}(\epsilon^2) \tag{4.94}$$

Then the distance is:

$$\begin{aligned} s =& \sqrt{\tilde{h}_{ab}\mathrm{d}\varphi_a\mathrm{d}\varphi_b} \\ \sim& \sqrt{k}\ln\left|\frac{\varphi_2}{\varphi_2^0}\right| - \frac{\epsilon}{16\pi^2}\frac{1}{\sqrt{k}}[\frac{A}{2k}e^{-\alpha\tilde{\varphi}_1}(\alpha^2\tilde{\varphi_1}^2 - 4\alpha\tilde{\varphi}_1 + 2)(\varphi_2^2 - (\varphi_2^0)^2) - \Lambda^2\ln\left|\frac{\varphi_2}{\varphi_2^0}\right|] \end{aligned} \tag{4.95}$$

Therefore, the non-trivial constraint which makes the theory satisfy SDC is:

$$\alpha = \frac{2 \pm \sqrt{2}}{\tilde{\varphi}_1} \tag{4.96}$$

Chapter 5

Conclusion

Bibliography

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