Start with the action of the non-linear sigma model (NLSM) on the target space $\mathcal{M}$ defined as:

$$S[X] = \frac{1}{2}\int_{\mathcal{M}} d^d x\sqrt{g}G_{AB}(X)\partial_\mu X^A\partial^\mu X^B \tag{1}$$

As before, one may need to obtain the 1-loop effective action for the observation of the renormalization equation for the theory. To do so, consider writing the field $X(x)$ in terms of a background $\bar{X}(x)$ and the corresponding quantum fluctuation $\xi(x)$. Assume that there is smooth map $X_s(x)$ such that $X_0 = \bar{X}(x)$, $X_1 = X(x)$ with $\dot{X}_0 = \xi$. Consider a curve X_s in $\mathcal{M}$ which represents the geodesic between initial and final points, i.e. X_0 and X_1,

$$\ddot{X}_s^A(x) + \Gamma^A_{BC}\dot{X}_s^B(x)\dot{X}_s^C(x) = 0 \tag{2}$$

here Γ^A_{BC} denotes the Christoffel symbol corresponding to the target space metirc G_{AB}. Then the expansion of the action around the background $\bar{X}$ is:

$$\begin{aligned} S[X] &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n}{ds^n}S[X_s]\bigg|_{s=0} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(\nabla_s)^n S[X_s]\bigg|_{s=0} \end{aligned} \tag{3}$$

where ∇_s is the covariant derivative along the curve X_s^A. Using the formulae:

$$\begin{aligned} \nabla_s\partial_\mu X^i = \partial_\mu\frac{dX^i}{ds} + \partial_\mu X^k\Gamma^i_{kj}\frac{dX^j}{ds} = \nabla_\mu\xi^i \ , \ \nabla_s G_{AB} = 0 \\ \nabla_s\frac{dX^i}{ds} = 0 \ , \ [\nabla_s, \nabla_\mu]Z^k = \frac{dX^i}{ds}\partial_\mu X^j R^k_{lij}Z^l \end{aligned} \tag{4}$$

where Z^i is an arbitrary vector, one finds:

$$\begin{aligned} S[X] =& \frac{1}{2}\sqrt{g}\int_{\mathcal{M}} d^d x G_{AB}(\bar{X})\partial_\mu\bar{X}^A\partial^\mu\bar{X}^B + \sqrt{g}\int_{\mathcal{M}} d^d x G_{AB}\partial_\mu\xi^A\partial^\mu\bar{X}^B + \\ &+ \frac{1}{2}\sqrt{g}\int_{\mathcal{M}} d^d x\{G_{AB}\nabla_\mu\xi^A\nabla^\mu\xi^B + R_{ACDB}\xi^C\xi^D\partial_\mu\bar{X}^A\partial^\mu\bar{X}^B\} + \cdots \end{aligned} \tag{5}$$

From the quadratic part:

$$D \equiv -G_{AB}\Box + R_{ACDB}\partial_\mu\bar{X}^C\partial^\mu\bar{X}^D \tag{6}$$

So our interest is to compute:

$$\begin{aligned} \Gamma_{\text{1-loop}} &= \frac{1}{2}\mathrm{Tr}\ln D \\ &= -\frac{1}{2}\int_0^\infty \frac{dt}{t}\mathrm{Tr}e^{-tD} \end{aligned} \tag{7}$$

As before, it can be expanded with the heat kernel coefficients a_k such that:

$$\Gamma_{\text{1-loop}} = -\frac{1}{2}\int\frac{dt}{t}\sum_{k\geq 0} t^{\frac{k-d}{2}}a_k(f, D) \tag{8}$$

where the first few terms are explicitly given in literatures. Introducing the UV and IR cutoff Λ and μ respectively on the proper time integral:

$$\Gamma_{\text{1-loop}} = -\frac{1}{2}\int_{\Lambda^{-2}}^{\mu^{-2}} dt\sum t^{\frac{k-d}{2}-1}a_k(f, D) \tag{9}$$

Again, there are finitely many terms which involve its divergence. For instance, for the case of $d = 2$:

$$\begin{aligned}
\Gamma^{div}_{\text{1-loop}} &= -\frac{1}{2}\int_{\Lambda^{-2}}^{\mu^{-2}} dt\{t^{-2}a_0(f,D) + t^{-1}a_2(f,D)\} \\
&= -\frac{1}{2}\{(\Lambda^2 - \mu^2)a_0 + \ln\frac{\Lambda^2}{\mu^2}a_2\} \\
&= -\frac{1}{2}\{(\Lambda^2 - \mu^2)(4\pi)^{-1}\int\sqrt{g} + \ln\frac{\Lambda^2}{\mu^2}(4\pi)^{-1}\int\sqrt{g}R_{DC}\partial_\alpha X^C\partial^\alpha X^D\}
\end{aligned} \tag{10}$$

Taking a derivative with respect to μ:

$$\begin{aligned}
\beta_{AB} &= \mu\frac{\partial}{\partial\mu}G_{AB,\mu} \\
&= \frac{1}{2\pi}R_{AB}
\end{aligned} \tag{11}$$

Adding a generic potential term to the action, it becomes:

$$S[X] = \frac{1}{2}\int_{\mathcal{M}} d^dx\sqrt{g}G_{AB}(X)\partial_\mu X^A\partial^\mu X^B + V(X) \tag{12}$$

Similarly to the previous computations:

$$D' = -G_{AB}\Box + R_{ACDB}\partial_\mu\bar{X}^C\partial^\mu\bar{X}^D + V^{(2)} \tag{13}$$

where $V^{(2)}$ is a Hessian matrix. Thus, in addition to the flow of the metric, there are flows of the couplings in the potential, which can be denoted as in this case:

$$\mu\frac{\partial}{\partial\mu}V_\mu = \frac{1}{4\pi}\text{Tr}V^{(2)} \tag{14}$$

and each beta function can be deduced.
For $4d$ case, the terms with coefficients a_4 must be included. Specifically:

$$\begin{aligned}
\Gamma^{\text{div}}_{\text{1-loop}} &= -\frac{1}{2}\int_{\Lambda^{-2}}^{\mu^{-2}} dt\{t^{-3}a_0 + t^{-2}a_2 + t^{-1}a_4\} \\
&= -\frac{1}{2}\{\frac{1}{2}(\Lambda^4 - \mu^4)a_0 + (\Lambda^2 - \mu^2)a_2 + \ln\frac{\Lambda^2}{\mu^2}a_4\} \\
&= -\frac{1}{2}\{\frac{1}{2}(\Lambda^4 - \mu^4)(4\pi)^{-2}\int\sqrt{g} + (\Lambda^2 - \mu^2)(4\pi)^{-2}\int\sqrt{g}(R_{DC}\partial_\alpha X^C\partial^\alpha X^D + \text{Tr}V^{(2)}) + \\
&+ \ln\frac{\Lambda^2}{\mu^2}(4\pi)^{-2}\int\sqrt{g}\text{Tr}(\frac{1}{2}P^2 + \frac{1}{6}\Box Q + \frac{1}{6}RO)\}
\end{aligned} \tag{15}$$

where P and Q represent the last two terms of the operator D' such that P^2 and $\Box Q$ include only relevant terms such as $\partial_\alpha X^A\partial^\alpha X^B$, and O denotes exactly those two terms in D'.
From this, taking the μ derivative,

$$\begin{aligned}
\mu\frac{\partial}{\partial\mu}\Gamma^{\text{div}}_{\text{1-loop}} &= \frac{1}{(4\pi)^2}\sqrt{g}\{\mu^4\int d^4x + \mu^2\int d^4x(R_{DC}\partial_\alpha X^C\partial^\alpha X^D + \text{Tr}V^{(2)}) + \\
&+ \int d^4x[\text{Tr}(\frac{1}{2}P^2 + \frac{1}{6}\Box Q) + \frac{1}{6}R(R_{EF}\partial_\alpha X^E\partial^\alpha X^F + \text{Tr}V^{(2)})]\}
\end{aligned} \tag{16}$$

$$= \int d^4x\sqrt{g}\{\frac{1}{2}\mu\frac{\partial G_{AB,\mu}}{\partial\mu}\partial_\alpha X^A\partial^\alpha X^B + \mu\frac{\partial V_\mu}{\partial\mu}\} \tag{17}$$

————-correction—————

In the equations above, $\Box Q$ does not have contributions because it is a total derivative and thus a boundary term. Also, we can neglect TrRO for 1-loop calculation of RG flow since it represents a non-minimal coupling between the gravity and the scalar. Eventually, only what we need is the term TrP^2.

$$\frac{1}{2}\mu\frac{\partial G_{AB,\mu}}{\partial\mu} + \mu\frac{\partial\tilde{V}_\mu}{\partial\mu} = \frac{1}{(4\pi)^2}\text{Tr}(\frac{1}{2}\tilde{P}^2) + \mu^2(R_{AB} + \text{Tr}\tilde{V}^{(2)}) \tag{18}$$

here tilde denotes the coefficients of only the relevant terms.