

The renormalization group equation for the metric has been given:

$$g_{ab}^\mu = g_{ab}^\Lambda + \frac{\epsilon}{8\pi^2} \{ \text{Tr}(\text{Hess}(V) R_{acbd}) - \Lambda^2 R_{ab} \} \quad (1)$$

Since

$$\text{Tr}(\text{Hess}(V) R_{acbd}) = \frac{1}{k} \begin{bmatrix} -M_2 & m \\ m & -M_1 \end{bmatrix} \quad (2)$$

$$R_{ab} = -\frac{1}{\varphi_2^2} \mathbb{1} \quad (3)$$

the metric at a scale μ can be written:

$$\tilde{g}_{ab} := g_{ab}^\mu = \begin{bmatrix} \frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2} \left\{ \frac{M_2}{k} - \left(\frac{\Lambda}{\varphi_2} \right)^2 \right\} & \frac{\epsilon}{8\pi^2} \frac{m}{k} \\ \frac{\epsilon}{8\pi^2} \frac{m}{k} & \frac{k}{\varphi_2^2} - \frac{\epsilon}{8\pi^2} \left\{ \frac{M_1}{k} - \left(\frac{\Lambda}{\varphi_2} \right)^2 \right\} \end{bmatrix} \quad (4)$$

Then inverse metric is:

$$\tilde{g}^{ab} = \begin{bmatrix} \frac{\varphi_2^2}{k} + \frac{\epsilon}{8\pi^2} \left(\frac{\varphi_2^2}{k} \right)^2 \left(\frac{M_2}{k} - \left(\frac{\Lambda}{\varphi_2} \right)^2 \right) & -\frac{\epsilon}{8\pi^2} \frac{m}{k} \left(\frac{\varphi_2^2}{k} \right)^2 \\ -\frac{\epsilon}{8\pi^2} \frac{m}{k} \left(\frac{\varphi_2^2}{k} \right)^2 & \frac{\varphi_2^2}{k} + \frac{\epsilon}{8\pi^2} \left(\frac{\varphi_2^2}{k} \right)^2 \left(\frac{M_1}{k} - \left(\frac{\Lambda}{\varphi_2} \right)^2 \right) \end{bmatrix} \quad (5)$$

and one may compute:

$$\tilde{\Gamma}_{11}^1 = -\frac{\epsilon}{8\pi^2} \frac{m}{k^2} \varphi_2 \quad (6)$$

$$\tilde{\Gamma}_{12}^1 = -\frac{1}{\varphi_2} - \frac{\epsilon}{8\pi^2} \frac{M_2}{k^2} \varphi_2 \quad (7)$$

$$\tilde{\Gamma}_{22}^1 = \frac{\epsilon}{8\pi^2} \frac{m}{k^2} \varphi_2 \quad (8)$$

$$\tilde{\Gamma}_{11}^2 = \frac{1}{\varphi_2} + \frac{\epsilon}{8\pi^2} \frac{M_1}{k^2} \varphi_2 \quad (9)$$

$$\tilde{\Gamma}_{12}^2 = \frac{\epsilon}{8\pi^2} \frac{m}{k^2} \varphi_2 \quad (10)$$

$$\tilde{\Gamma}_{22}^2 = -\frac{1}{\varphi_2} - \frac{\epsilon}{8\pi^2} \frac{M_1}{k^2} \varphi_2 \quad (11)$$

Therefore, the new geodesic equations are:

$$\begin{cases} \ddot{\varphi}_1 - \frac{2}{\varphi_2} \dot{\varphi}_1 \dot{\varphi}_2 = \frac{\epsilon}{8\pi^2} \frac{\varphi_2}{k^2} (m \dot{\varphi}_1^2 + 2M_2 \dot{\varphi}_1 \dot{\varphi}_2 - m \dot{\varphi}_2^2) \end{cases} \quad (12)$$

$$\begin{cases} \ddot{\varphi}_2 + \frac{1}{\varphi_2} (\dot{\varphi}_1^2 - \dot{\varphi}_2^2) = \frac{\epsilon}{8\pi^2} \frac{\varphi_2}{k^2} (-M_1 \dot{\varphi}_1^2 - 2m \dot{\varphi}_1 \dot{\varphi}_2 + M_1 \dot{\varphi}_2^2) \end{cases} \quad (13)$$

Again, from (12), it can be seen that the vertical lines are no longer the geodesics, and there need some correction terms if $m \neq 0$. Restricting corrected geodesics with the initial conditions that the initial values and initial derivatives are equal to the ones for uncorrected geodesics, which is written as $(\varphi_1, \varphi_2) = (\varphi_1^0, \varphi_2^0 e^{\pm \frac{s}{\sqrt{k}}})$, find the new geodesics. What I did tried was as following: First, as I did, plugged the original solution $\varphi_2 = \varphi_2^0 e^{\pm \frac{s}{\sqrt{k}}}$ into eq. (13) and solved for φ_1 . This ended up with:

$$\varphi_1 = \varphi_1^0 - \frac{\epsilon}{24\pi^2} \frac{m}{k} (\varphi_2^3 - (\varphi_2^0)^3) \quad (14)$$

And then from this, plug $\dot{\varphi}_1 = -\frac{\epsilon}{8\pi^2} \frac{m}{k} \varphi_2^2 \dot{\varphi}_2$ into eq.(12) back to back. This gives:

$$\begin{aligned} & \varphi_2 \left\{ -\frac{\epsilon}{8\pi^2} \frac{m}{k} (2\varphi_2 \dot{\varphi}_2^2 + \varphi_2^2 \ddot{\varphi}_2) \right\} + 2 \frac{\epsilon}{8\pi^2} \frac{m}{k} \varphi_2^2 \dot{\varphi}_2^2 = \frac{\epsilon}{8\pi^2} \frac{\varphi_2^2}{k^2} \{ 2M_2 \left(-\frac{\epsilon}{8\pi^2} \frac{m}{k} \right) \varphi_2^2 \dot{\varphi}_2^2 - m \dot{\varphi}_2^2 \} \\ \Rightarrow & \varphi_2 \ddot{\varphi}_2 = \frac{1}{k} (\dot{\varphi}_2^2 + \frac{\epsilon}{8\pi^2} \frac{2M_2}{k^2} \varphi_2^2 \dot{\varphi}_2^2) \end{aligned} \quad (15)$$

However, by solving this differential equation, I stucked with:

$$\int \exp\left(-\frac{\ln \varphi_2}{k} - \frac{\epsilon}{8\pi^2} \frac{M_2}{k^3} \varphi_2^2\right) d\varphi_2 = \int C ds \quad (16)$$

here C is a constant which can be determined with initial conditions.

So I tried another way. Instead of $\varphi_2 = \varphi_2^0 e^{\pm \frac{s}{\sqrt{k}}}$, plugging $\varphi_2 = \varphi_2^0 e^{\pm \frac{s}{\sqrt{k}}} + \epsilon \delta \varphi_2$ with the constraints stated above into eq. (13), then solve for φ_1 , but it didn't work well too.

I had been suspicious if I did my calculation correctly and so checked from the beginning, but it seemed fine to me. So I suppose I have had mistakes and/or misunderstandings with computations of geodesic equations with corrections.