3-26-25 (WEEK 6) Shun/翔海 (@shun4midx)

PRIMARY DECOMPOSITION

For this section, let R be a commutative ring

DEFINITION

- An ideal I of R is irreducible if $I=q_1\cap q_2 \Rightarrow I=q_1$ or $I=q_2$ (i.e. int of proper ideals)
- We define the quotient ideal $(I:x)=\{r\in R \mid rx\in I\}$, which is also an ideal

PROPOSITION

In a Noetherian R, each irr ideal is primary

Proof

Let $xy\in I$, and $x\notin I$ (Hope for "$y\in I$")

Consider the ascending chain $(I:y)\subseteq(I:y^2)\subseteq(I:y^3)\subseteq\dots$

As R is Noeth, thus $\exists n$, s.t. $(I:y^n)=(I:y^{n+1})=\dots$

Claim: $(\langle y^n\rangle_R+I)\cap(\langle x\rangle+I)=I$ ($\Rightarrow \langle y^n\rangle+I=I\Rightarrow y^n\in I$)

Proof

Let $b=r_1y^n+\underset{\in I}{a_1}=r_2x+\underset{\in I}{a_2}\Rightarrow r_1y^{n+1}=r_2\underset{\in I}{xy}+\underset{\in I}{a_2y}-\underset{\in I}{a_1y}\in I$

$\therefore r\in(I:y^{n+1})=(I:y^n)\Rightarrow r_1y^n\in I\Rightarrow b\in I$ □

PROPOSITION

In a Noeth ring R, each I is a finite intersection of irr ideals

Proof (Proof by contradiction)

If not, $\emptyset\neq S=\{I\subseteq R \mid I$ is not a finite intersection of irr ideals$\}$

$\because R$ is Noeth

$\therefore \exists$ a max element $I_0\in S$

We find that I_0 must be reducible, say $I_0=I_1\cap I_2$ with $\underbrace{I_0\subsetneq I_1,\ I_0\subsetneq I_2}_{\text{intersection of proper ideals}}\Rightarrow I_1, I_2\notin S$

$\therefore I_1$ is a finite intersection of irr ideals, I_2 is a finite intersection of irr ideals

$\therefore$ So is I_0. ⟶※

SUMMARY

Prop 1 + Prop 2 $\Rightarrow \left\{\begin{smallmatrix}R:\text{ Noeth}\\ I\subseteq R\end{smallmatrix}\right.$, $I=q_1\cap\dots\cap q_n$, q_i: primary (∴ we have the existence of decomposition, how about uniqueness?)

UNIQUENESS THEOREM

FACT 1

If q_i is P-primary $\forall i=1,\dots,n$, then $q=\bigcap_{i=1}^n q_i$ is also P-primary

Proof

- $\sqrt{q}=\bigcap_{i=1}^n\sqrt{q_i}=P$
- $xy\in q$ and $x\notin q\Rightarrow xy\in q_i\ \forall i$ and $x\notin q_j$ for some $j\Rightarrow y^n\in q_j\Rightarrow y\in\sqrt{q_j}\subseteq\sqrt{q}$, i.e. $y^m\in q$ for some m. □

FACT 2

Let q be P-primary and $x\in R$

(1) If $x\in q$, then $(q:x)=R$

(2) If $x\notin q$, then $(q:x)$ is a P-primary ideal

(3) If $x\notin P$, then $(q:x)=q$ (Proof: $y\in(q:x)\Rightarrow xy\in q$, $x\notin P\Rightarrow xy\in q$, $x^n\notin q\ \forall n\Rightarrow y\in q$)

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Proof

(1) $1\cdot x\in q \Rightarrow 1\in(q:x)\Rightarrow (q:x)=R$

(2) "$\sqrt{(q:x)}=p$": For $y\in(q:x)$, $xy\in q$ and $x\notin q\Rightarrow y^n\in q\Rightarrow y\in\sqrt{q}=p$, so $q\subseteq(q:x)\subseteq p\Rightarrow \overbrace{p=\sqrt{q}\subseteq\sqrt{(q:x)}\subseteq\sqrt{p}=p}^{\text{sandwich}}$

$yz\in(q:x)$ and $y\notin(q:x)\Rightarrow xyz\in q$, $xy\notin q\Rightarrow z^n\in q\subseteq(q:x)$

DEFINITION

$I=q_1\cap\dots\cap q_n$ is **minimal** if $\overbrace{\sqrt{q_1},\dots,\sqrt{q_n}}^{\text{radicals}}$ are distinct, and $q_i\not\supseteq\bigcap_{j\neq i}q_j$ $\forall i=1,\dots,n$

UNIQUENESS THEOREM

Let $I=\bigcap_{i=1}^n q_i$ be a minimal primary decomposition

If $p_i=\sqrt{q_i}$ $\forall i=1,\dots,n$, then $\{p_i\}=\{\sqrt{(I:x)}\mid x\in R,\ \sqrt{(I:x)}\in\operatorname{Spec}R\}$ which is indep of the particular decomp of p_i

Proof

If $x\in R\setminus I$, $(I:x)=(\bigcap_{i=1}^n q_i:x)=\bigcap_{i=1}^n(q_i:x)\overset{\text{sandwich}}{\Rightarrow}\sqrt{(I:x)}=\bigcap_{x\notin q_i}\sqrt{(q_i:x)}=\bigcap_{x\notin q_i}p_i$

"RHS $\subseteq$ LHS": $\sqrt{(I:x)}\in\operatorname{Spec}R\Rightarrow p_j\supseteq\sqrt{(I:x)}\supseteq p_j$ for some j, i.e. $\sqrt{(I:x)}=p_j$

"LHS $\subseteq$ RHS": $\because q_i\not\supseteq\bigcap_{j\neq i}q_j$ $\forall i=1,\dots,n$ $\therefore \exists x_i\in\bigcap_{j\neq i}q_j\setminus q_i$

$\Rightarrow p_i=\sqrt{(q_i:x_i)}=\bigcap_{j=1}^n\sqrt{(q_j:x_i)}=\sqrt{(I:x_i)}$ □

$(q_j:x_i)=R$

OBSERVE

For any $I=q_1\cap\dots\cap q_m$ in $F[x_1,\dots,x_n]$, $\sqrt{I}=P_1\cap\dots\cap P_m$

Consider the $\overset{\text{roots}}{\text{zero-locus}}$ Z, $\underline{Z(I)=Z(\sqrt{I})}=Z(P_1)\cup\dots\cup Z(P_m)$ ← just "$fg=0\Rightarrow f=0$ or $g=0$" このみたい

Why: $q_1\cap\dots\cap q_m\supseteq q_1\cdots q_m\Rightarrow \underset{Z(\sqrt{I})}{\underset{\shortparallel}{Z(q_1\cap\dots\cap q_m)}}\subseteq Z(q_1\cdots q_m)=Z(q_1)\cup\dots\cup Z(q_m)$

"$\supseteq$": $x\in Z(q_1)\Rightarrow f(x)=0$ $\forall x\in q_1$, so $g(x)=0$ $\forall x\in q_1\cap\dots\cap q_m$ ✓

EXAMPLE

$I=\langle x^2,xy\rangle=\underbrace{\langle x\rangle}_{P_1}\cap\underbrace{\langle x,y\rangle^2}_{P_2^2}$ ← we call P_1,P_2 its associated primes

THEOREM (RADICALS)

Let $I=\langle f_1,\dots,f_s\rangle\subsetneq F[x_1,\dots,x_n]$. Then, $f\in\sqrt{I}\Leftrightarrow\langle f_1,\dots,f_s,1-tf\rangle=F[x_1,\dots,x_n,t]$

Proof

"$\Rightarrow$": $f^m\in I\Rightarrow \overbrace{1=t^mf^m+(1-t^mf^m)}^{\text{rewriting identity}}=t^mf^m+(1-tf)(1+tf+t^2f^2+\dots+t^{m-1}f^{m-1})$ ✓ ($t^mf^m\in I$)

"$\Leftarrow$": Let $1=\sum_{i=1}^s h_if_i+h(1-tf)$ (★), $h_i,h\in F[x_1,\dots,x_n,t]$

Consider the F-algebra homomorphism, $\varphi:F[x_1,\dots,x_n,t]\longrightarrow F(x_1,\dots,x_n)$ ← (·) denotes rational function, [·] is polynomial

$x_i\longmapsto x_i$

$t\longmapsto \frac{1}{f}$

Apply φ to (★), $1=\varphi(1)=\sum_{i=1}^s\varphi(h_i)\varphi(f_i)+\cancel{\varphi(h)(\varphi(1)-\varphi(t)\varphi(f))}\to 0$

$=\sum_{i=1}^s\frac{P_i}{f^{r_i}}(f_i)$, $P_i\in F[x_1,\dots,x_n]$

Let $\rho=\max\{r_i\}$, then $f^\rho\in I$ □ ← fraction addition denominator LCM

EXAMPLE 2

$I=\langle xy^2+2y^2,\ x^2-2x+1\rangle$, $f=y-x^2+1$, $f\overset{?}{\in}\sqrt{I}$

$J=\langle xy^2+2y^2,\ x^2-2x+1,\ 1-t(y-x^2+1)\rangle$ has the reduced Gröbner basis $\{1\}\Rightarrow f\in\sqrt{I}$

Alternate method:

I has Gröbner basis $G=\{x^4-2x^2+1,\ y^2\}$. $\overline{(y-x^2+1)^2}^G=-2x^2y+2y$, $\overline{(y-x^2+1)^3}^G=0$, so OK.

Basically... We need Gröbner basis no matter what. We can use the J shortcut, but if we need to find the power, we still need brute force.

EXAMPLE 3

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$I = \langle xz - y^2, x^3 - yz \rangle$ ... What are its associated primes?

For $I \subseteq F[x,y,z]$, we have:

- $(I:x)$: $I \cap \langle x \rangle \Rightarrow tI + (1-t)\langle x \rangle$ has the reduced Gröbner basis G and $G \cap F[x,y,z] = \{\overset{f_1}{\overset{\shortparallel}{x^2z - xy^2}}, \overset{f_2}{\overset{\shortparallel}{x^4 - xyz}}, \overset{f_3}{\overset{\shortparallel}{x^3y - xz^2}}\}$

  $\therefore I \cap \langle x \rangle = \langle f_1, f_2, f_3 \rangle$

  ofc just divide by x

  Then, $(I:x) = \langle \frac{f_1}{x}, \frac{f_2}{x}, \frac{f_3}{x} \rangle = \langle xz - y^2, x^3 - yz, x^2y - z^2 \rangle$

  Is $(I:x)$ an associated prime?

  Now, notice $(I:x)$ has the reduced Gröbner basis $G = \{x^3 - yz, x^2y - z^2, xy^3 - z^3, xz - y^2, y^5 - z^4\}$ (Notice with parametrization, $x = t^3$, $y = t^4$, $z = t^5$ makes this 0 xd)

  Define $\varphi: F[x,y,z] \longrightarrow F[t]$

  $$\begin{aligned} x &\longmapsto t^3 \\ y &\longmapsto t^4 \\ z &\longmapsto t^5 \end{aligned}$$

  Now, $\operatorname{Ker}\varphi = \langle G \rangle = \langle (I:x) \rangle \Rightarrow F[x,y,z]/\langle (I:x) \rangle \hookrightarrow F[t]$, which is an integral domain

  $\therefore$ $(I:x)$ is a prime ideal. $\square$

Remark: Even though this is still seemingly Algebra-heavy, this is quite a geometrical approach at the problem xddd

(Please don't get mad at me qwq~ I suck lol it's just some cool angle of interpretation imo, idrk Algebraic Geometry :-))