

RINGS AND MODULES

For $M \in \mathcal{R}M$, $R \times M \rightarrow M \Leftrightarrow \exists R \xrightarrow{\text{ring hom}} \text{End}(M)$ (representation)

QUOTIENT

In fact, $R \in \mathcal{R}M$, and for $I \subseteq R$, $I = \text{left ideal of } R \Rightarrow R/I$ is a left R -module

If $I \subseteq \mathcal{R}M$, then I is called an ideal and R/I is a ring
 \Rightarrow The most important structure to investigate rings aren't subrings but are ideals

Today's notes have R as commutative.

DEFINITION

Let $I \subseteq R$ be an ideal

- I is **maximum** if $\forall \mathcal{R}M \ni J \subseteq R$, $I \subseteq J \Rightarrow J = R$ (not "biggest" but not comparable to anything bigger)
- I is **prime** if $\forall x, y \in R$, $xy \in I \Rightarrow x \in I$ or $y \in I$ or $x \notin I, y \notin I \Rightarrow xy \notin I$

FACT 1

(1) I is max $\Leftrightarrow R/I$ is a field

Proof

" \Rightarrow ": $\forall \bar{0} \neq \bar{x} \in R/I$, $x \notin I \Rightarrow (x) + I \not\subseteq I \Rightarrow (x) + I = R \Rightarrow \bar{y}x = \bar{1} \Rightarrow \bar{y} = \bar{x}^{-1}$ ✓
 $yx + a = 1$

" \Leftarrow ": Let $I \subseteq J$, pick $x \in J \setminus I$, then $\bar{x} \neq \bar{0}$ in R/I

Let $\bar{y} \in R/I$, s.t. $\bar{y}\bar{x} = \bar{1}$, i.e. $yxt + a = 1 \Rightarrow 1 \in J \Rightarrow \forall r \in R, 1 \cdot r = r \in J \therefore R = J$ (of course max) ✓

(2) I is prime $\Leftrightarrow R/I$ is an integral domain

Proof

" \Rightarrow ": $\begin{cases} \bar{x}\bar{y} = \bar{0} \\ \bar{x} \neq \bar{0} \end{cases} \Rightarrow \begin{cases} xy \in I \\ x \notin I \end{cases} \Rightarrow y \in I \Rightarrow \bar{y} = \bar{0} \therefore$ By def, R/I is an integral domain ✓
 "Can divide"

" \Leftarrow ": $\begin{cases} xy \in I \\ x \notin I \end{cases} \Rightarrow \begin{cases} \bar{x}\bar{y} = \bar{0} \\ \bar{x} \neq \bar{0} \end{cases} \Rightarrow \bar{y} = \bar{0} \Rightarrow y \in I \therefore$ By def, I is prime ✓

DEFINITION

$a \in R$ is **nilpotent** if $\exists n \in \mathbb{N}$, s.t. $a^n = 0$
 ↳ special type of zero divisors

FACT 2

- $\mathcal{N}_R = \{\text{nilpotent elements of } R\} \subseteq \mathcal{R}M$, i.e. it is an **ideal** (good thing, means we can quotient it)
- R/\mathcal{N}_R has no non-zero nilpotent elements, which is said to be **reduced**

Proof

for $a \in \mathcal{N}_R$, say $a^n = 0$, for $r \in R$, $(ra)^n = r^n a^n = 0 \Rightarrow ra \in \mathcal{N}_R$

for $b \in \mathcal{N}_R$, say $b^m = 0$, then $(a+b)^{n+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} a^i b^{n+m-i} = 0 \Rightarrow a+b \in \mathcal{N}_R \quad \square$

DEFINITION

- \mathcal{N}_R is called the **nilradical** of R (root)
- $\text{Max } R = \{\text{max ideals of } R\}$
- $\text{Spec } R = \{\text{prime ideals of } R\}$
- $\sqrt{I} := \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$
 ↳ "the radical of I "

PROPOSITION 1

Shun/翔海 (@shun4mide)

$$N_R = \bigcap_{P \in \text{Spec } R} P$$

Proof

" \subseteq ": For $a \in N_R$, say $a^n = 0 \in P \forall P \in \text{Spec } R$, so by def of P , $a \in P \forall P \Rightarrow a \in \text{RHS} \checkmark$

" \supseteq ": Use contraposition, and Zorn's Lemma. $I \cap \{a, a^2, a^3, \dots\} = \emptyset$ (Goal to create \cap by replacing \cap with \cup)

Let $a \notin N_R$, for $S = \{a^n \in I \mid I \cap \{a, a^2, a^3, \dots\} = \emptyset\}$. We know $S \neq \emptyset$ since $\{0\} \in S$ ($a \notin N_R \Rightarrow a^k \neq 0 \forall k \in \mathbb{N}$)

Define partial order " \leq " in S as " $I \leq J$ " \Leftrightarrow " $I \subseteq J$ "

Let $\{I_i \mid i \in \Lambda\}$ be a chain in S . Then, $a^n \in I = \bigcup_{i \in \Lambda} I_i$ is a least upper bound of $\{I_i \mid i \in \Lambda\}$.
 $\neg (a, b \in I \Rightarrow a \in I_i, b \in I_j \Rightarrow I_i \subseteq I_j \text{ or } I_j \subseteq I_i \Rightarrow a + b \in I_i \text{ or } I_j \subseteq I_i)$

By Zorn's Lemma, \exists a max element Q in S .

Claim: $Q \in \text{Spec } R$ ($\Rightarrow a \notin Q \Rightarrow a \notin \text{RHS}$)

Proof

If $x \notin Q$, then $\langle x \rangle + Q \not\subseteq Q \Rightarrow \langle x \rangle + Q \in S \Rightarrow a^n \in \langle x \rangle + Q$

Similarly, if $y \notin Q$, $a^n \in \langle y \rangle + Q - b \in U$

$\therefore a^{n+m} \in \langle xy \rangle + Q \Rightarrow \langle xy \rangle + Q \not\subseteq Q$, i.e. $xy \notin Q \checkmark$ (Just proved $x \notin Q$ and $y \notin Q \Rightarrow xy \notin Q$)

COROLLARY

$$\sqrt{I} = \bigcap_{P \in \text{Spec } R} P$$

Proof

Let $\phi: R \rightarrow R/I$. Then, $\sqrt{I} = \phi^{-1}(\bigcap_{P \in \text{Spec } R/I} P) = \bigcap_{P \in \text{Spec } R/I} \phi^{-1}(P) = \bigcap_{P \in \text{Spec } R} P$ \square

OBSERVE

$$\begin{array}{ccc} R & \xrightarrow{\quad} & R/I \\ \downarrow & & \downarrow \\ U & & U \\ \downarrow & & \downarrow \\ P & \xrightarrow{\quad} & P/I \\ \downarrow & & \downarrow \\ I & & I \end{array}$$

By 3rd isom thm, $R/I/P/I \cong P/P$

$$R_1 \xrightarrow{\text{ring homo}} R_2 \text{ in Ring}$$

$$P \in \text{Spec } R_2 \Rightarrow \psi^{-1}(P) \in \text{Spec } R_1 \quad (xy \in \psi^{-1}(P) \Rightarrow \psi(xy) = \psi(x)\psi(y) \in P \Rightarrow \psi(x) \in P \text{ or } \psi(y) \in P \Rightarrow x \in \psi^{-1}(P) \text{ or } y \in \psi^{-1}(P))$$

EXAMPLE

We know usually $\sqrt{I} \neq I$, but if $P \in \text{Spec } R$, then $\sqrt{(P)^n} = P$.

" \subseteq ": $\sqrt{(P)^n} = \bigcap_{P \in \text{Spec } R} P \subseteq P \checkmark$

" \supseteq ": $\forall x \in P$, $x^n \in (P)^n \Rightarrow x \in \sqrt{(P)^n} \checkmark$

DEFINITION

An ideal Q of R is primary if $Q \neq R$ and " $xy \in Q$, $x \notin Q \Rightarrow y^n \in Q$ for some $n \in \mathbb{N}$ "

FACT 3

(1) Q is primary $\Leftrightarrow R/Q \neq 0$ and the zero-divisors in R/Q are nilpotent

Proof

" \Rightarrow ": $\begin{cases} \bar{x}\bar{y} = \bar{0} \\ \bar{x} \neq \bar{0} \end{cases} \Rightarrow \begin{cases} xy \in Q \\ x \notin Q \end{cases} \Rightarrow y^n \in Q \Rightarrow \bar{y}^n = \bar{0} \checkmark$

" \Leftarrow ": $\begin{cases} xy \in Q \\ x \notin Q \end{cases} \Rightarrow \begin{cases} \bar{x}\bar{y} = \bar{0} \\ \bar{x} \neq \bar{0} \end{cases} \text{ in } R/Q \Rightarrow \bar{y}^n = \bar{0} \text{ for some } n \in \mathbb{N} \Rightarrow y^n \in Q. \checkmark$

(2) If \mathcal{Q} is primary, then $\sqrt{\mathcal{Q}}$ is the smallest prime ideal containing \mathcal{Q} .

Shun/翔海 (@shun4mide)

Proof

- " $\sqrt{\mathcal{Q}} \in \text{Spec } R$ ": $\begin{cases} xy \in \sqrt{\mathcal{Q}} \Rightarrow (xy)^n = x^n y^n \in \mathcal{Q} \\ x \notin \sqrt{\mathcal{Q}} \Rightarrow x^m \notin \mathcal{Q} \forall m \Rightarrow x^n \notin \mathcal{Q} \end{cases} \Rightarrow (y^n)^t \in \mathcal{Q} \Rightarrow y \in \sqrt{\mathcal{Q}}$
- $\sqrt{\mathcal{Q}} = \bigcap_{\mathcal{P} \supset \mathcal{Q}} \mathcal{P} \Rightarrow \sqrt{\mathcal{Q}} \subseteq \mathcal{P} \Rightarrow \forall \mathcal{P} \in \text{Spec } R, \mathcal{P} \supseteq \mathcal{Q} \quad \square$

DEFINITION

\mathcal{Q} is \mathcal{P} -primary if \mathcal{Q} is primary and $\sqrt{\mathcal{Q}} = \mathcal{P}$

EXAMPLES

1. $R = \mathbb{Z}$ ^{PID}

- $\text{Max } R = \{n\mathbb{Z} \mid n \text{ is prime}\}$ ($\because R/\text{Max } R$ must be a field when R is a PID)
- $\text{Spec } R = \text{Max } R \cup \{0\}$
- The primary ideals of R : Either $\mathcal{Q} = \langle 0 \rangle$ or if $\mathcal{Q} \neq \langle 0 \rangle$, say $\mathcal{Q} = \langle s \rangle$ and for some prime p , $\sqrt{\mathcal{Q}} = \langle p \rangle$
 $\therefore p^n \in \mathcal{Q} = \langle s \rangle$, say $p^n = st$ in $\mathbb{Z} \Rightarrow s = p^m \Rightarrow \mathcal{Q} = \langle p^m \rangle \quad \square$

ideals over PID are generated by 1 element only

2. $\sqrt{I} \in \text{Spec } R \nRightarrow I$ is primary (key example)

For $R = R[x, y]$, we need $xy \in I$ and $x \notin I \Rightarrow y^n \in I$

$\hookrightarrow I = \langle x^2, xy \rangle$ is not primary

Notice, $I = \langle x \rangle \cap \langle x^2, xy, y^2 \rangle = \langle x \rangle \cap \langle x, y \rangle^2$.

Now, $R/\langle x \rangle = R[x, y]/\langle x \rangle \cong R[y]$, which is not a field $\therefore \langle x \rangle$ is not a maximal ideal

$R/\langle x, y \rangle \cong R$, which is a field $\Rightarrow \langle x, y \rangle$ is a max ideal

Now, we know $\sqrt{I} = \sqrt{\langle x \rangle} \cap \sqrt{\langle x, y \rangle^2} = \langle x \rangle \cap \langle x, y \rangle = \langle x \rangle$, which is primary \checkmark