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FUNDAMENTAL THEOREM FOR INFINITE CASE (Final Algebra Note by Shun :))

L/k: Galois, $G=Gal(L/k)$, $\{E_i \mid i\in I\} = \{k\subseteq E\subseteq L \mid E/k: \text{finite Galois}\}$

$H_i = Gal(L/E_i)$, $G_i = Gal(E_i/k) \cong G/H_i$: finite group ⇒ Result: $G\cong \varprojlim G_i$ ($i\le j \Leftrightarrow E_i\subseteq E_j$, $\varphi_{ji}: G_j\to G_i$)

THE NATURAL TOPOLOGY

- G_i: finite ⇒ discrete topology (discrete points that are open and closed)

↪ $\prod_i G_i$: the product topology

$$\begin{tikzcd} \prod_i G_i \arrow[d, "p_i\ (\text{proj})"] & \supseteq & p_i^{-1}(g_i): \text{open basis} \\ G_i & \ni & g_i \arrow[u] \end{tikzcd}$$

So, $\underbrace{\varphi_i^{-1}(g_i)}_{\text{open basis}} \in \varprojlim G_i \subseteq \prod_i G_i$

$$\begin{tikzcd} \varprojlim G_i \arrow[dr, "\varphi_i"'] & & \prod_i G_i \arrow[dl, "p_i"] \\ & g_i \in G_i & \end{tikzcd}$$

DEFINITION (ぼりは全全分んない...)

G is called a **topological group** if G is both a topological space and group s.t. $G\times G \xrightarrow{m} G$, $(x,y)\mapsto xy$ and $G\xrightarrow{i} G$, $g\mapsto g^{-1}$ are **continuous**

CLAIM

$\varprojlim G_i$ is a topological group

Proof

- m is cont:

$$\begin{tikzcd} \varprojlim G_i \times \varprojlim G_i \arrow[r, "m"] \arrow[d, "\varphi_i \times \varphi_i"] & \varprojlim G_i \supseteq \varphi_i^{-1}(g_i) \arrow[d, "\varphi_i"] \\ G_i \times G_i \arrow[r] & G_i \ni g_i \end{tikzcd}$$

g_ih^{-1} (deduced), h

∴ $m^{-1}(\varphi_i^{-1}(g_i)) = \bigcup_{h\in G_i} \varphi_i^{-1}(g_ih^{-1})\times\varphi_i^{-1}(h)$ is open in the product space $\varprojlim G_i \times \varprojlim G_i$

- $\mathbf{i}$ is cont:

$$\begin{tikzcd} \varprojlim G_i \arrow[r, "\mathbf{i}"] \arrow[d, "\varphi_i"] & \varprojlim G_i \arrow[d, "\varphi_i"] \\ g_i^{-1}\in G_i \arrow[r] & G_i \ni g_i \end{tikzcd}$$

∴ $\mathbf{i}^{-1}(\varphi_i^{-1}(g_i)) = \varphi_i^{-1}(g_i)$ is open

OBSERVE

As G_i has autom, we write σ_i here for its elems, where $\underbrace{G}_{\text{want to be open}} \overset{f}{\cong} \underbrace{\varprojlim G_i}_{\text{open}}$

$f^{-1}(\underbrace{\varphi_i^{-1}(\sigma_i)}_{\subset \varprojlim G_i}) = \{\sigma\in G \mid \sigma|_{E_i} = \sigma_i\} = \bigcup_{\sigma\in G,\ \sigma|_{E_i}=\sigma_i} \sigma \underbrace{Gal(L/E_i)}_{= H_i} \rightsquigarrow \sigma H_i$: open set

DEFINITION

Krull topology on G is the topology with basis consisting of all left cosets σH_i, $\sigma\in G$, $i\in I$

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CHECK IT IS A TOPOLOGY

- $\emptyset$ is open: For some $H_i \neq G$ (ok since infinite), take $\sigma_1 H_i \neq \sigma_2 H_i \Rightarrow \sigma_1 H_i \cap \sigma_2 H_i = \emptyset$
- G is open: $G = Gal(L/K)$, $[K:K] = 1$ is open
- An arbitrary union sets is open
- $\underbrace{\sigma_i H_i \cap \sigma_j H_j}_{\neq \emptyset}$ is open $\Rightarrow \sigma H_i = \sigma_i H_i$, $\sigma H_j = \sigma_j H_j \Rightarrow \sigma_i H_i \cap \sigma_j H_j = \sigma(H_i \cap H_j)$, $H_i \cap H_j = Gal(L/E_i E_j)$ ($\ni \sigma$)

PROPOSITION

G is a topological group with the Krull topology

- f is conti:
- f^{-1} is conti: $f(\sigma H_i) = \{((\sigma\tau_i)|_{E_j})_{j\in I} \mid \tau_i \in H_i\} = \{\sigma|_{E_j} w_j \mid w_j|_{E_j} = id_{E_i \cap E_j}\} = \{(v_j)_{j\in I} \mid (\sigma|_{E_j})^{-1} v_j|_{E_i} = id_{E_i\cap E_j}\} = \{(v_j)_{j\in I} \mid v_j|_{E_i} = \sigma|_{E_i\cap E_j}\} = \varphi_i^{-1}(\sigma|_{E_i})$

  ($\sigma|_{E_j} w_j = v_j$, $G_j = Gal(E_j/K)$)

  $\tau_i \in Gal(L/K_i)$

  $\downarrow$

  $\tau_i|_{E_j} \in Gal(E_j/E_i\cap E_j)$

  $\wr$

  $w_j \in Gal(E_j/K)$ □

FUNDAMENTAL THEOREM

$\mathcal{F} = \{E \mid K \le E \le L\} \longleftrightarrow \mathcal{G}_0 = \{H \mid H: \text{closed subgroup of } G\}$

KEY LEMMA

If $H \le G$, $E = L^H$ and $H \le H' = Gal(L/E)$, then $H' = \overline{H}$ (if H is closed, $H' = \overline{H} = H$) is the closure of H in the Krull topology on G

Proof ($H' \supseteq \overline{H}$)

- H' is closed, i.e. $G\setminus H'$ is open: For $\sigma \in G\setminus H'$, by def, $\exists \alpha \in E$, s.t. $\sigma(\alpha) \neq \alpha$
  We can choose E_i, $i \in I$, s.t. $\alpha \in E_i$ (fix E_i). Now, $\forall \tau \in H_i := Gal(L/E_i) \Rightarrow \sigma\tau(\alpha) = \sigma(\alpha) \neq \alpha \Rightarrow \sigma\tau \in G\setminus H'$
  $\therefore \sigma H_i \subseteq G\setminus H'$ and σH_i can be regarded as an open neighborhood of σ, so $G\setminus H'$ is open.
- $H' \subseteq \overline{H}$: "$\forall \sigma \in H'\setminus H$, $\forall i \in I$, $(\sigma H_i \setminus \{\sigma\}) \cap H \neq \emptyset$"
  Fix $i \in I$. Let $N = \{\rho|_{E_i} \mid \rho \in H\} \le Gal(E_i/K)$ (finite)
  Note that E_i/K is finite Galois, so the fundamental thm holds for $Gal(E_i/K)$
  $\therefore$ We have $N = Gal(E_i/E_i^N) = Gal(E_i/E \cap E_i)$. For any $\sigma \in H'\setminus H$, $\sigma|_{E_i} \in Gal(E_i/E_i \cap E) = N = \{\rho|_{E_i} \mid \rho \in H\}$, say $\sigma|_{E_i} = \rho|_{E_i}$ for some $\rho \in H$.
  $\Rightarrow \sigma^{-1}\rho|_{E_i} = Id_{E_i} \Rightarrow \sigma^{-1}\rho \in Gal(L/E_i) = H_i \Rightarrow \rho \in \sigma H_i \cap H \Rightarrow \rho \in (\sigma H_i \setminus \{\sigma\}) H$ □

FUNDAMENTAL THEOREM

(1) $Gal(L/E) = H$ is closed: We have known $E = L^H$, so $Gal(L/E) = H = \overline{H} \Rightarrow H$ is closed ✓

(2) H: closed $\longrightarrow L^H \longrightarrow Gal(L/L^H) = \overline{H} = H$

$\therefore$ We proved the 1-1 corr of $\mathcal{F}$ and $\mathcal{G}_0$

NOTE

$\forall i \in I$, H_i is open (by def) and closed ($\because [G:H_i] = n_i \therefore G = \underbrace{H_i}_{\therefore \text{closed}} \cup \underbrace{\sigma_1 H_i \cup \dots \cup \sigma_{n_i - 1} H_i}_{\text{open}}$)

FUNDAMENTAL THEOREM CONTINUED

(3) If E corresponds to H, then $[E:K] < \infty \Leftrightarrow H$: open

"$\Rightarrow$": Write $E = K(\alpha_1, \dots, \alpha_n)$. Consider E' as a splitting field of $m_{\alpha_1,K}, m_{\alpha_2,K}, \dots, m_{\alpha_n,K} \Rightarrow E \subseteq E' = E_i$ is finite + Galois for some $i \in I$
$E \subseteq E_i \Rightarrow H_i \le H$ and $[G:H_i] < \infty \Rightarrow [G:H] < \infty$, say

Claim: $\sigma_\ell H$: closed $\forall \ell = 1, \dots, m$

Proof

$\forall \tau \in G\setminus\sigma_\ell H$, $\tau \notin \sigma_\ell H \Rightarrow \sigma_\ell^{-1}\tau \notin H$ $\because H$ is closed $\therefore \exists i$, s.t. $\sigma_\ell^{-1}\tau H_i \le G\setminus H \Rightarrow \tau H_i \subseteq G\setminus\sigma_\ell H \Rightarrow \sigma_\ell H$: closed ✓

"$\Leftarrow$": $\overset{\text{identity}}{e}\in H$ and H: open $\Rightarrow \exists i\in I$, s.t. $H_i = eH_i \subseteq H \Rightarrow E\subseteq E_i$. E_i is a finite extension $\Rightarrow$ E is a finite extension ✓

(4) E/K: normal $\Leftrightarrow H \trianglelefteq G$

"$\Rightarrow$": ok

"$\Leftarrow$": $\forall \alpha\in E$, β is a root of $m_{\alpha,K} \Rightarrow \exists \sigma\in G$, s.t. $\sigma(\alpha)=\beta$

If $\tau\in H$, then $\tau(\beta)=\tau\sigma(\alpha)=\sigma(\underbrace{\sigma^{-1}\tau\sigma}_{\in H})(\alpha)=\sigma(\alpha)=\beta$

$\therefore \beta\in E$. □

EXAMPLE

$Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p)=?$

Notice, $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)\cong \mathbb{Z}/n\mathbb{Z}$. $\therefore Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p)\cong \varprojlim \mathbb{Z}/n\mathbb{Z}$

(Because $\mathbb{F}_{p^m}\subseteq \mathbb{F}_{p^n} \Leftrightarrow m|n$, so $m\leq n \Leftrightarrow m|n$ for inverse limit)

Actually, $m=pq \Rightarrow \mathbb{Z}/m\mathbb{Z}\cong \mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/q\mathbb{Z}$

$m'=p^2q \Rightarrow \mathbb{Z}/m'\mathbb{Z}\cong \mathbb{Z}/p^2\mathbb{Z}\times \mathbb{Z}/q\mathbb{Z}$

$\downarrow p^2q^2r^5$

$\vdots$

$\Rightarrow$ Here, $\mathbb{Z}/p^k\mathbb{Z}$, $\mathbb{Z}/q^\ell\mathbb{Z}$ are unrelated, so it is a direct product of inverse limit

$\therefore \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_{p:\text{prime}} \underbrace{\mathbb{Z}_p}_{p\text{-adic}} = \hat{\mathbb{Z}}$