2-19-25 (WEEK 1)

Shun/羊羽海 (@shun4midx)

RINGS AND MODULES

For $M \in {}_R\mathfrak{M}$, $R \times M \to M \iff \exists R \xrightarrow{\text{ring homo}} \mathrm{End}(M)$ (representation)

QUOTIENT

In fact, $R \in {}_R\mathfrak{M}$, and for $I \subseteq R$ (submodule), I: left **ideal** of $R \Rightarrow R/I$ is a left R-module

If $I \in {}_R\mathfrak{M}, \mathfrak{M}_R$, then I is called an ideal and R/I is a ring

$\Rightarrow$ The most important structure to investigate rings aren't subrings but are **ideals**

Today's notes have R as commutative.

DEFINITION

Let $I \subsetneq R$ be an ideal

- I is **maximum** if $\forall\, {}_R\mathfrak{M} \ni J \subsetneq R$, $I \subsetneq J \Rightarrow J = R$ (not "biggest" but not comparable to anything bigger)
- I is **prime** if $\forall x, y \in R$, $xy \in I \Rightarrow x \in I$ or $y \in I$ or $x \notin I, y \notin I \Rightarrow xy \notin I$

FACT 1

(1) I is max $\iff R/I$ is a field

Proof

"$\Rightarrow$": $\forall \bar{0} \neq \bar{x} \in R/I$, $x \notin I \Rightarrow (x) + I \supsetneq I \Rightarrow \underbrace{(x)+I}_{yx+a} = \underbrace{R}_{=1} \Rightarrow \overline{yx} = \bar{1} \Rightarrow \bar{y} = \bar{x}^{-1}$ ✓

"$\Leftarrow$": let $I \subsetneq J$, pick $x \in J \setminus I$, then $\bar{x} \neq \bar{0}$ in R/I

let $\bar{y} \in R/I$, s.t. $\bar{y}\bar{x} = \bar{1}$, i.e. $\overset{\in J}{yx} + \overset{\in J}{a} = 1 \Rightarrow$ **$1 \in J$** $\Rightarrow \forall r \in R$, $1 \cdot r = r \in J$ $\therefore R = J$ (of course max) ✓

(2) I is prime $\iff R/I$ is an integral domain

Proof

"$\Rightarrow$": $\begin{cases} \bar{x}\bar{y} = \bar{0} \\ \bar{x} \neq \bar{0} \end{cases} \Rightarrow \begin{cases} xy \in I \\ x \notin I \end{cases} \Rightarrow y \in I \Rightarrow \bar{y} = 0$ $\therefore$ By def, R/I is an integral domain ("Can divide") ✓

"$\Leftarrow$": $\begin{cases} xy \in I \\ x \notin I \end{cases} \Rightarrow \begin{cases} \bar{x}\bar{y} = \bar{0} \\ \bar{x} \neq \bar{0} \end{cases} \Rightarrow \bar{y} = \bar{0} \Rightarrow y \in I$ $\therefore$ By def, I is prime ✓

DEFINITION

$a \in R$ is **nilpotent** if $\exists n \in \mathbb{N}$, s.t. $a^n = 0$

special type of zero divisors

FACT 2

- $\mathfrak{n}_R := \{\text{nilpotent elements of } R\} \in {}_R\mathfrak{M}$, i.e. it is an **ideal** (good thing, means we can quotient it)
- **$R/\mathfrak{n}_R$** has no non-zero nilpotent elements, which is said to be **reduced**

Proof

For $a \in \mathfrak{n}_R$, say $a^n = 0$, for $r \in R$, $(ra)^n = r^n a^n = 0 \Rightarrow$ **$ra \in \mathfrak{n}_R$**

For $b \in \mathfrak{n}_R$, say $b^m = 0$, then $(a+b)^{n+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} a^i b^{n+m-i} = 0 \Rightarrow$ **$a+b \in \mathfrak{n}_R$** □

DEFINITION

- $\mathfrak{n}_R$ is called the **nilradical** ($\sqrt{\{0\}}$) of R
- $\mathrm{Max}\, R = \{\text{max ideals of } R\}$
- $\mathrm{Spec}\, R = \{\text{prime ideals of } R\}$
- $\sqrt{I} := \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$

"the radical of I"

Shun/翔海(@shun4midx)

PROPOSITION 1

$n_R = \bigcap_{P\in \operatorname{Spec}R} P$

Proof

"$\subseteq$": For $a\in n_R$, say $a^n=0\in P\ \forall P\in\operatorname{Spec}R$, so by def of P, $a\in P\ \forall P \Rightarrow a\in RHS$ ✓

"$\supseteq$": Use contrapositive, and Zorn's lemma. $I\cap\underbrace{\{a,a^2,a^3,\dots\}}_{=U}=\emptyset$ (Goal to create is by replacing ___ $=U$)

Let $a\notin n_R$, for $S=\{{}_R m\ni I\subseteq R \mid a^n\notin I\ \forall n\in\mathbb{N}\}$. We know $S\neq\emptyset$ since $\{0\}\in S$ ($a\notin n_R\Rightarrow a^k\neq 0\ \forall k\in\mathbb{N}$)

Define partial order "$\leq$" in S as "$I\leq J$"$\Leftrightarrow$"$I\subseteq J$"

($a,b\in I\Rightarrow a\in I_i,\ b\in I_j \Rightarrow I_i\subseteq I_j$ or $I_j\subseteq I_i \Rightarrow a+b\in I_i$ or $I_j\subseteq I$)

Let $\{I_i \mid i\in\Lambda\}$ be a chain in S. Then, ${}_R m\ni I=\bigcup_{i\in\Lambda} I_i$ is a least upper bound of $\{I_i\mid i\in\Lambda\}$.

By Zorn's Lemma, $\exists$ a max element Q in S.

Claim: $Q\in\operatorname{Spec}R$ ($\Rightarrow a\notin Q\Rightarrow a\notin RHS$)

Proof

$a\in U$ ($\because a\notin I$, $I\cup U=\emptyset$, $0\notin U$ is max)

If $x\notin Q$, then $\langle x\rangle+Q\supsetneq Q\Rightarrow\langle x\rangle+Q\notin S\Rightarrow a^n\in\langle x\rangle+Q$

Similarly, if $y\notin Q$, $a^m\in\langle y\rangle+Q$ — $b\in U$

$\therefore$ $a^{n+m}\in\langle xy\rangle+Q\Rightarrow\langle xy\rangle+Q\supsetneq Q$, i.e. $xy\notin Q$ ✓ (Just proved $x\notin Q$ and $y\notin Q\Rightarrow xy\notin Q$)

$ab\in U$

COROLLARY

$\sqrt{I} = \bigcap_{\operatorname{Spec}R\ni P\supseteq I} P$

Proof

Let $\phi: R\longrightarrow R/I$, $r\longmapsto\bar{r}$. Then, $\sqrt{I}=\phi^{-1}(n_{R/I})=\phi^{-1}\big(\bigcap_{\bar P\in\operatorname{Spec}R/I}\underbrace{\bar P}_{P/I}\big)=\bigcap_{\operatorname{Spec}R\ni P\supseteq I}P$ □

OBSERVE

$$\begin{array}{ccc} R & \twoheadrightarrow & R/I\\ \cup & & \cup\\ P & \longleftrightarrow & P/I\\ \cup & & \\ I & & \end{array}$$

By 3rd isom thm, $(R/I)/(P/I)\cong R/P$

$R_1\xrightarrow[\psi]{\text{ring hom}}R_2$ in Ring

$P\in\operatorname{Spec}R_2\Rightarrow\psi^{-1}(P)\in\operatorname{Spec}R_1$ ($xy\in\psi^{-1}(P)\Rightarrow\psi(xy)=\psi(x)\psi(y)\in P\Rightarrow\psi(x)\in P$ or $\psi(y)\in P\Rightarrow x\in\psi^{-1}(P)$ or $y\in\psi^{-1}(P)$)

EXAMPLE

We know usually $\sqrt{I^n}\neq I$, but if $P'\in\operatorname{Spec}R$, then $\sqrt{(P')^n}=P'$.

"$\subseteq$": $\sqrt{(P')^n}=\bigcap_{P\supseteq(P')^n}P\subseteq P'$ ✓

"$\supseteq$": $\forall x\in P'$, $x^n\in(P')^n\Rightarrow x\in\sqrt{(P')^n}$ ✓

DEFINITION

An ideal Q of R is primary if $Q\neq R$ and "$xy\in Q$, $x\notin Q\Rightarrow y^n\in Q$ for some $n\in\mathbb{N}$"

FACT 3

(1) Q is primary $\Leftrightarrow R/Q\neq 0$ and the zero-divisors in R/Q are nilpotent

Proof

"$\Rightarrow$": $\begin{cases}\bar x\bar y=\bar 0\\ \bar x\neq\bar 0\end{cases}\Rightarrow\begin{cases}xy\in Q\\ x\notin Q\end{cases}\Rightarrow y^n\in Q\Rightarrow\bar y^n=\bar 0$ ✓

"$\Leftarrow$": $\begin{cases}xy\in Q\\ x\notin Q\end{cases}\Rightarrow\begin{cases}\bar x\bar y=\bar 0\\ \bar x\neq\bar 0\end{cases}$ in $R/Q\Rightarrow\bar y^n=\bar 0$ for some $n\in\mathbb{N}\Rightarrow y^n\in Q$. ✓

(2) If Q is primary, then $\sqrt{Q}$ is the smallest prime ideal containing Q.

Proof

- "$\sqrt{Q} \in \operatorname{Spec} R$": $\begin{cases} xy \in \sqrt{Q} \Rightarrow (xy)^n = x^n y^n \in Q \\ x \notin \sqrt{Q} \Rightarrow x^m \notin Q \ \forall m \Rightarrow x^n \notin Q \end{cases} \Rightarrow (y^n)^1 \in Q \Rightarrow y \in \sqrt{Q}$
- $\sqrt{Q} = \bigcap_{P \supseteq Q} P \Rightarrow \sqrt{Q} \subseteq P \ \forall P \in \operatorname{Spec} R,\ P \supseteq Q$ □

DEFINITION

Q is P-primary if Q is primary and $\sqrt{Q} = P$

EXAMPLES

1. $R = \mathbb{Z}$ PID
   - $\operatorname{Max} R = \{n\mathbb{Z} \mid n \text{ is prime}\}$ ($\because R/\operatorname{Max} R$ must be a field when R is a PID)
   - $\operatorname{Spec} R = \operatorname{Max} R \cup \{\{0\}\}$
   - The primary ideals of R: Either $Q = \langle 0 \rangle$ or if $Q \neq \langle 0 \rangle$, say $Q = \langle s \rangle$ (ideals over PID are generated by 1 element only) and for some prime p, $\sqrt{Q} = \langle p \rangle$

     $\therefore p^n \in Q = \langle s \rangle$, say $p^n = st$ in $\mathbb{Z} \Rightarrow s = p^m \Rightarrow Q = \langle p^m \rangle$ □
2. $\sqrt{I} \in \operatorname{Spec} R \not\Rightarrow I$ is primary (key example)

   For $R = \mathbb{R}[x,y]$, we need $xy \in I$ and $x \notin I \Rightarrow y^n \in I$

   ↪ $I = \langle x^2, xy \rangle$ is not primary

   Notice, $I = \langle x \rangle \cap \langle x^2, xy, y^2 \rangle = \langle x \rangle \cap \langle x, y \rangle^2$.

   Now, $R/\langle x \rangle = \mathbb{R}[x,y]/\langle x \rangle \cong \mathbb{R}[y]$, which is not a field $\therefore \langle x \rangle$ is not a maximal ideal

   $R/\langle x,y \rangle \cong \mathbb{R}$, which is a field $\Rightarrow \langle x, y \rangle$ is a max ideal

   Now, we know $\sqrt{I} = \sqrt{\langle x \rangle} \cap \sqrt{\langle x,y \rangle^2} = \langle x \rangle \cap \langle x, y \rangle = \langle x \rangle$, which is primary ✓