

LOCALIZATION

Recall: "units" = elements that exist multiplicative inverse

Today, R is assumed to be commutative

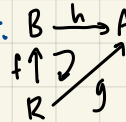
Let S be a multiplicatively closed set with $1 \in S$, $0 \notin S$

DEFINITION

Suppose that there is a ring B and a ring homo $f: R \rightarrow B$ s.t.

(1) $f(x)$ is a unit of $B \forall x \in S$

(2) If $g: R \rightarrow A$ is another ring homo s.t. $g(x)$ is a unit of $A \forall x \in S$, then $\exists!$ ring homo $h: B \rightarrow A$ s.t. $B \xrightarrow{h} A$ (Universal Property)



Such B , if it exists, is unique up to isom and is called the localization of R w.r.t. S , denoted by R_S

THEOREM

R_S exists

Proof

Set $R_S = R \times S / \sim$ where $(a, s) \sim (b, t) \Leftrightarrow \exists u \in S$ s.t. $(at - bs)u = 0$

Step 1: " \sim " is an equivalence relation

- $(a, s) \sim (a, s)$ since $(as - as)1 = 0$
- " $(a, s) \sim (b, t) \Rightarrow (b, t) \sim (a, s)$ " since $(at - bs)u = 0 \Rightarrow (bs - at)u = 0$
- $(a, s) \sim (b, t), (b, t) \sim (c, u) \Rightarrow (a, s) \sim (c, u)$ since $(at - bs)v = 0, (bu - ct)w = 0 \Rightarrow (at - bs)vwu = 0, (bu - ct)wvs = 0 \Rightarrow (au - cs)vwu = 0$

Define $\frac{a}{s} := (a, s)$

Step 2: R_S has a ring structure: $\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$

Well-defined:

$$\frac{a}{s} = \frac{a'}{s'} \Rightarrow (as' - a's)v = 0 \Rightarrow (as' - a's)vw + t't' = 0$$

$$\frac{b}{t} = \frac{b'}{t'} \Rightarrow (bt' - b't)w = 0 \Rightarrow (bt' - b't)wvs' = 0$$

$$+ [(at + bs)s't' - (a't + b's')st]vw = 0$$

$$\therefore as'v = a'sv, bt'w = b'tw \Rightarrow (abs't' - a'b'st)vw = 0$$

Actually, $(R_S, +, \cdot)$ forms a ring

Step 3: $f: R \rightarrow R_S$ satisfies the universal property

$$a \mapsto \frac{a}{1}$$

(1) $\forall x \in S, f(x) = \frac{x}{1} \because \frac{1}{x} \cdot \frac{x}{1} = \frac{x}{x} = \frac{1}{1} \therefore \frac{x}{1}$ is a unit in R_S

(2) Let $g: R \rightarrow A$ with $g(x)$ being a unit of $A \forall x \in S$

If \exists a ring homo $R_S \rightarrow A$ with $g = hf$, then $h(\frac{a}{s}) = h(\frac{a}{1} \cdot \frac{1}{s}) = h(\frac{a}{1})h(\frac{1}{s}) = h(\frac{a}{1})(h(\frac{1}{s}))^{-1} = hf(a)(hf(s))^{-1} = g(a)g(s)^{-1} = \frac{g(a)}{g(s)}$

So, we define $h(\frac{a}{s}) = g(a)g(s)^{-1}$

It is well-defined as follows: $\frac{a}{s} = \frac{b}{t} \Rightarrow (at - bs)u = 0 \Rightarrow (g(a)g(t) - g(b)g(s))g(u) = 0 \Rightarrow g(a)g(s)^{-1} = g(b)g(t)^{-1}$

PROPERTIES

If S contains no zero divisor, then $f: R \rightarrow R_S$ is injective.

prime ideal

$x \mapsto \frac{x}{1} = \frac{0}{s}$, i.e. $\exists u \in S$ s.t. $ux = 0 \Rightarrow ux$ is not a zero divisor, so $x = 0$

If R is an integral domain and $S = R \setminus \{0\}$, then R_S is called the quotient field of R which is the smallest field containing R .

If for a field F , say $g: R \hookrightarrow F$, with $g(a) \neq 0$ being a unit, by universal property, $\exists! h: R_S \rightarrow F$ which is injective since

$$h(\frac{a}{s}) = g(a)g(s)^{-1} = 0 \Rightarrow g(a) = 0 \Rightarrow a = 0 \Rightarrow \frac{a}{s} = 0$$

Pick $0 \neq f \in R$, consider $S = \{1, f, f^2, \dots\} \Rightarrow R_S = R_f$

non-nilpotent

• $S=R/p$ for some $p \in \text{Spec } R$, $R_S=R_p$

Shun/翔海 (@shun4mide)

REMARK

$S \subseteq T$ with $1 \in S$, $T \neq \emptyset$, S, T are mcs (multiplicatively closed sets)

When will $R_S \subseteq R_T$?

Ans: When $T \subseteq S = R \setminus \bigcup_{p \in \text{Spec } R} p$ ($\frac{1}{s} \cdot \frac{a}{t} = \frac{1}{t} \Rightarrow \exists u \in S, s \cdot t \cdot \frac{1}{u} = \frac{a}{t} \Rightarrow \frac{1}{u} = \frac{a}{s} \Rightarrow \frac{1}{u} \in S$, $t \in S \Rightarrow \langle t \rangle \cap S \neq \emptyset \Rightarrow s = ta$)

CONSTRUCTION FOR MODULES

For an R -module M , $1 \in S \neq \emptyset$ mcs in R , define $M_S := \{(m, s) \mid m \in M, s \in S\} / \sim$, $(m, s) \sim (n, t) \Leftrightarrow \exists u \in S, s \cdot t \cdot u \cdot (tm - sn) = 0$

$\Rightarrow \sim$ is an equivalence relation, $\frac{m}{s} := [(m, s)]$ in M_S

Notice: M_S is an R_S -module

PROPOSITION

If $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is exact in $\mathcal{R}M$, then $0 \rightarrow M'_S \xrightarrow{f_S} M_S \xrightarrow{g_S} M''_S \rightarrow 0$ is exact

Proof

- f_S is 1-1: $f_S(\frac{a}{s}) = f_S(\frac{a'}{s'}) \Rightarrow \frac{f(a)}{s} = \frac{f(a')}{s'} \Rightarrow \exists u \in S, s \cdot t \cdot u \cdot (f(a) - f(a')) = 0 \Rightarrow f(ut'a - ut'a') = 0 \Rightarrow ut'a - ut'a' = 0 \Rightarrow \frac{a}{s} = \frac{a'}{s'}$ ✓
- g_S is onto: $\forall \frac{b}{t} \in M''_S$, let $g(a) = b \Rightarrow g_S(\frac{a}{s}) = \frac{b}{t}$ ✓
- $\text{Im } f_S \subseteq \text{Ker } g_S$: $g_S f_S(\frac{a}{s}) = \frac{g(f(a))}{s} = \frac{0}{s} = 0$ ✓
- $\text{Ker } g_S \subseteq \text{Im } f_S$: Let $\frac{a}{s} \in \text{Ker } g_S$, i.e. $g_S(\frac{a}{s}) = \frac{g(a)}{s} = 0 \Rightarrow \exists u \in S, s \cdot t \cdot u \cdot g(a) = 0 \Rightarrow g(ua) = 0 \Rightarrow f(b) = a \Rightarrow \frac{a}{s} = \frac{ua}{st} = \frac{f(b)}{st} = f_S(\frac{b}{st})$ ✓

FACT

$R_S \otimes_R M \cong M_S$

Proof

$f: R_S \times M \rightarrow M_S$ is bilinear $\Rightarrow \exists R$ -module homo $\bar{f}: R_S \otimes_R M \rightarrow M_S$

- \bar{f} is onto: $\forall \frac{a}{s} \in M_S$, $\bar{f}(\frac{a}{s} \otimes 1) = \frac{a}{s}$ ✓
- \bar{f} is 1-1: Let $\sum_i (\frac{a_i}{s_i}) \otimes m_i \in R_S \otimes_R M$

Set $t = \prod_i s_i$, $\bar{f}_i = \frac{t}{s_i} \cdot t_j$, then $\sum_i (\frac{a_i}{s_i}) \otimes m_i = \sum_i (\frac{a_i \bar{f}_i}{t}) \otimes m_i = \frac{1}{t} \otimes \sum_i \bar{f}_i a_i m_i$

If $\frac{1}{t} \otimes m \in \text{Ker } \bar{f}$, i.e. $\bar{f}(\frac{1}{t} \otimes m) = \frac{m}{t} = 0$ in M_S , i.e. $\exists u \in S, s \cdot t \cdot u \cdot m = 0$.

Then, $\frac{1}{t} \otimes m = \frac{1}{tu} \otimes m = \frac{1}{tu} \otimes um = 0$ ✓

THEOREM

R_S is a flat R -module

Proof

Given $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ in $\mathcal{R}M$, by prop, $0 \rightarrow M'_S \rightarrow M_S \rightarrow M''_S \rightarrow 0$ is exact \square

REMARK

Given $M \in \mathcal{R}M$, TFAE:

- (1) $M=0$
- (2) $M_p=0 \quad \forall p \in \text{Spec } R$
- (3) $M_{\mathfrak{a}}=0 \quad \forall \mathfrak{a} \in \text{Max } R$

Proof

(1) \Rightarrow (2) \Rightarrow (3) is straightforward.

Consider proving (3) \Rightarrow (1),

Assume $\exists 0 \neq z \in M$

\mathbb{G}^{RM}

Define $\text{ann}(z) := \{r \in R \mid rz = 0\} \subseteq R \Rightarrow \exists Q \in \text{Max } R \text{ s.t. } \text{ann}(z) \subseteq Q$

By assumption, $\frac{z}{1} = \frac{0}{1}$ in $M_a \Rightarrow \exists u \notin Q, \text{ s.t. } uz = 0 \Rightarrow u \in \text{ann}(z) \subseteq Q \rightarrow \times$

Shun/羊羽海 (@shun4midx)