Algebra II Definitions

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Definitions

2-19-25 (Week 1): Rings and Modules (Quotient)

Today's notes assume R to be commutative

Statement 1.1. Given $R \in {}_R\mathcal{M}$, and $I \subseteq R$, we say I is a left **ideal** of $R \Rightarrow \boxed{R/I}$ is a left R-module

Definition 1.1. Let $I \subsetneq R$ be an **ideal**

- I is **maximum** if $\forall\ {}_R\mathcal{M} \ni J \subsetneq R,\ I \subsetneq J \Rightarrow J = R$ *(It's not the "biggest", it just is not comparable to anything bigger)*
- I is **prime** is $\forall\, x, y \in R$, we have "$xy \in I \Rightarrow x \in I$ or $y \in I$", i.e. "$x \notin I,\ y \notin I \Rightarrow xy \notin I$"

Definition 1.2. Here are some special terms:

- $a \in R$ is **nilpotent** if $\exists\, n \in \mathbb{N}$, s.t. $a^n = 0$ (i.e. a special type of zero divisor)
- $\mathcal{n}_R$ is called the **nilradical** of R (because it is $\sqrt{\{0\}}$)
- Max$R$ = **max ideals** of R
- Spec$R$ = **prime ideals** of R
- $\sqrt{I} := \{a \in R \mid \underline{a^n \in I}$ for some $n \in \mathbb{N}\}$, it is called the **radical** of I

Definition 1.3. An **ideal** Q of R is **primary** if $Q \neq R$ and "$\boxed{xy \in Q, x \notin Q \Rightarrow y^n \in Q}$ for some $n \in \mathbb{N}$"

Definition 1.4. Q is **P-primary** if Q is **primary** and $\underline{\sqrt{Q} = P}$