

# Algebra II Definitions

By Shun (@shun4midx)

## Definitions

### 2-19-25 (Week 1): Rings and Modules (Quotient)

Today's notes assume  $R$  to be commutative

**Statement 1.1.** Given  $R \in {}_R\mathcal{M}$ , and  $I \subseteq R$ , we say  $I$  is a left **ideal** of  $R \Rightarrow \boxed{R/I}$  is a left  $R$ -module

**Definition 1.1.** Let  $I \subsetneq R$  be an **ideal**

- $I$  is **maximum** if  $\forall {}_R\mathcal{M} \ni J \subsetneq R, I \subsetneq J \Rightarrow J = R$  (It's not the "biggest", it just is not comparable to anything bigger)
- $I$  is **prime** is  $\forall x, y \in R$ , we have " $xy \in I \Rightarrow x \in I$  or  $y \in I$ ", i.e. " $x \notin I, y \notin I \Rightarrow xy \notin I$ "

**Definition 1.2.** Here are some special terms:

- $a \in R$  is **nilpotent** if  $\exists n \in \mathbb{N}$ , s.t.  $a^n = 0$  (i.e. a special type of zero divisor)
- $\mathcal{N}_R = \{\text{nilpotent elements of } R\}$  is called the **nilradical** of  $R$  (because it is  $\sqrt{\{0\}}$ )
- $\text{Max}R = \text{max ideals of } R$
- $\text{Spec}R = \text{prime ideals of } R$
- $\sqrt{I} := \{a \in R \mid \underline{a^n \in I} \text{ for some } n \in \mathbb{N}\}$ , it is called the **radical** of  $I$

**Definition 1.3.** An ideal  $Q$  of  $R$  is **primary** if  $Q \neq R$  and " $xy \in Q, x \notin Q \Rightarrow \boxed{y^n \in Q}$  for some  $n \in \mathbb{N}$ "

**Definition 1.4.**  $Q$  is **P-primary** if  $Q$  is **primary** and  $\underline{\sqrt{Q} = P}$