4-2-25 (WEEK 7)

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HILBERT POLYNOMIAL

MOTIVATION

For a Noeth local R and a fin gen R-module M, $\{m\} = \mathrm{Max}\, R$, how do we study m?

KRULL'S THEOREM (RECALL)

$\bigcap_{i=0}^{\infty} m^i M = \{0\} \Rightarrow \forall v \in M, \exists k \in \mathbb{N} \cup \{0\}$, s.t. $v \in m^k M$ but $v \notin m^{k+1} M$ with $m^0 := R$. Here, we define the order $o(v) := k$

Now, how do we investigate R-module $M/\ell M$?

Notice, if we separate based on order: $M/mM, \ mM/m^2M, \dots, \ m^{\ell-1}M/m^{\ell}M$

As $\forall k$, $m(m^{k-1}M/m^kM) = 0$, thus $m^{k-1}M/m^kM$ is an R/m-module, i.e. a field. (vector space!)

How do we consider its dimension then? Consider $\sum_{\ell=0}^{\infty} \dim m^{\ell}M/m^{\ell+1}M \, t^{\ell}$

DEFINITION

"generating function" to calculate dim

Let G be an abelian group and $\varphi: {}_R\mathcal{M} \longrightarrow G$, $M \longmapsto \varphi(M)$. We say φ is an Euler-Poincaré mapping if $\varphi(0) = 0$ and $\forall\, 0 \to M_1 \to M_2 \to M_3 \to 0$ in ${}_R\mathcal{M}$, $\varphi(M_3) = \varphi(M_1) + \varphi(M_2)$

DEFINITION

$R = \bigoplus_{i=0}^{\infty} R_i$ is a graded Noeth, and $M = \bigoplus_{i=0}^{\infty} M_i$ is a fin gen R-module, where R_0 is Noeth, $R = R_0[a_1, \dots, a_n]$ with $a_i \in R_{d_i}$, $d_i > 0$, $M = \langle x_1, \dots, x_m \rangle_R$ with $x_i \in M_{\ell_i}$ and M_i are fin gen R_0-modules. Then, for Euler-Poincaré mapping $\varphi: {}_{R_0}\mathcal{M}^{<\infty} \xrightarrow{\text{fin gen}} \mathbb{Z}$, we define:

$P_\varphi(M,t) := \sum_{i=0}^{\infty} \varphi(M_i) t^i \in \mathbb{Z}[[t]]$ is called a Poincaré series

DEFINITION

$p(z) \in \mathbb{Q}[z]$ is called a numerical polynomial if $p(n) \in \mathbb{Z}$ $\forall n \gg 0$, $n \in \mathbb{Z}$

PROPOSITION

If $p(z)$ is numerical, then $\exists\, c_0, \dots, c_r \in \mathbb{Z}$, s.t. $p(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_{r-1}\binom{z}{1} + c_r$, where even for $z \in \mathbb{R}$, $\binom{z}{r} = \frac{z(z-1)\cdots(z-r+1)}{r!}$

In particular, $p(n) \in \mathbb{Z}$ $\forall n \in \mathbb{Z}$

<u>Proof</u>

Since $\binom{z}{r} = \frac{z^r}{r!} + \dots$, $\binom{z}{0} := 1$, thus by viewing $\binom{z}{r}$ as a z-polynomial, $\{\binom{z}{r} \mid r \in \mathbb{N} \cup \{0\}\}$ forms a basis for $\mathbb{Q}[z]$ over $\mathbb{Q}$.

Then, we can write $p(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_0(1)$ with $c_i \in \mathbb{Q}$

By induction on $\deg p$,

- $\deg p = 0$: $p(z) = c \in \mathbb{Z}$, ok ✓
- Recall: $\binom{z+1}{r} - \binom{z}{r} = \binom{z}{r-1}$
- $\because \deg(p(z+1) - p(z)) < \deg(p(z))$ and "numerical" still is true

  $\therefore$ By induction hypothesis, $\exists\, c_0', \dots, c_{r-1}' \in \mathbb{Z}$, s.t. $p(z+1) - p(z) = c_0'\binom{z}{r-1} + \dots + c_{r-1}'$

  Notice, $\binom{z}{r-1}, \dots, \binom{z}{0}$ are lin indep, i.e. $c_i' = c_i$ $\forall i$, so $c_r = \underbrace{p(n)}_{\in\mathbb{Z}} - \underbrace{(c_0\binom{n}{r} + c_1\binom{n}{r-1} + \dots + c_{r-1}\binom{n}{1})}_{\in\mathbb{Z}}$ for some $n \gg 0$, i.e. $c_r \in \mathbb{Z}$. □

PROPOSITION 2

If $f: \mathbb{Z} \to \mathbb{Z}$, s.t. $f(n+1) - f(n) = Q(n)$ $\forall n \gg 0$ with numerical $Q(z)$, then $f(n) = p(n)$ $\forall n \gg 0$ for some numerical poly $p(z)$

<u>Proof</u>

Write $Q(z) = c_0\binom{z}{r} + \dots + c_r$ with $c_i \in \mathbb{Z}$. Let $\tilde{p}(z) = c_0\binom{z}{r+1} + \dots + c_r\binom{z}{1}$ (rewrite r only)

Then, $\tilde{p}(z+1) - \tilde{p}(z) = Q(z) \Rightarrow \tilde{p}(n+1) - \tilde{p}(n) = f(n+1) - f(n)$ $\forall n \gg 0$

$\therefore f(n+1) - \tilde{p}(n+1) = f(n) - \tilde{p}(n)$ $\forall n \gg 0$ (i.e. constant)

Say $f(n)-\tilde{p}(n)=C_{r+1}$ $\forall n\gg 0$. Let $p(z)=\tilde{p}(z)+C_{r+1}$. Then, $f(n)=p(n)$ $\forall n\gg 0$.

THEOREM (HILBERT-SERRE)

(1) $P_\varphi(M,t)=\dfrac{f(t)}{\prod_{i=1}^{n}(1-t^{d_i})}$ for some $f(t)\in\mathbb{Z}[t]$, $d_i\in\mathbb{N}$

(2) If $d_i=1$ $\forall i=1,\dots,n$ and $P_\varphi(M,t)=\frac{h(t)}{(1-t)^d}$, $(1-t)\nmid h(t)$, then $\exists!$ $p(z)\in\mathbb{Q}[z]$ of deg $d-1$, s.t. $\varphi(M_n)=p(n)$ $\forall n\gg 0$

Proof

(1) By induction on n,

- $n=0$: $R=R_0$, M is a fin gen R-module, say $M=\langle \underset{\substack{\cap\\ M_{\ell_1}}}{x_1},\dots,\underset{\substack{\cap\\ M_{\ell_m}}}{x_m}\rangle_R$

  $\therefore M_i=0$ $\forall i>\max\{\ell_1,\dots,\ell_m\}\Rightarrow \varphi(M_i)=0$ $\forall i\gg 0\Rightarrow P_\varphi(M,t)$ is a polynomial

- $n>0$: Consider $0\to \underset{\substack{\|\\ \mathrm{Ker}(\cdot a_n)}}{K_i}\to M_i\xrightarrow{\cdot a_n}M_{i+d_n}\to \underset{\substack{\|\\ \mathrm{Coker}(\cdot a_n)}}{L_{i+d_n}}\to 0$

  Let $K=\bigoplus_{i=0}^{\infty}K_i\subseteq M$, $L=\bigoplus_{i=0}^{\infty}L_i=M/\sim$, which are fin gen R-modules and annihilated by a_n

  Also, we have $0\to K_i\to M_i\to \mathrm{Im}(\cdot a_n)\to 0$ and $0\to \mathrm{Im}(\cdot a_n)\to M_{i+d_n}\to L_{i+d_n}\to 0$

  $$\Rightarrow\begin{cases}\varphi(K_i)+\varphi(\mathrm{Im}(\cdot a_n))=\varphi(M_i)\\ \varphi(\mathrm{Im}(\cdot a_n))+\varphi(L_{i+d_n})=\varphi(M_{i+d_n})\end{cases}\Rightarrow \varphi(K_i)-\varphi(M_i)+\varphi(M_{i+d_n})-\varphi(L_{i+d_n})=0$$

  Multiply by t^{i+d_n}, we get $t^{d_n}(\varphi(K_i)t^i-\varphi(M_i)t^i)+\varphi(M_{i+d_n})t^{i+d_n}-\varphi(L_{i+d_n})t^{i+d_n}=0$

  Take summation from $i=0$ to ∞, $t^{d_n}(P_\varphi(K,t)-P_\varphi(M,t))+P_\varphi(M,t)-P_\varphi(L,t)-\underbrace{\left(\sum_{i=0}^{d_n-1}\varphi(M_i)t^i-\sum_{i=0}^{d_n-1}\varphi(L_i)t^i\right)}_{=g(t)\in\mathbb{Z}[t]}$

  $\therefore (1-t^{d_n})P_\varphi(M,t)=P_\varphi(L,t)-t^{d_n}P_\varphi(K,t)+g(t)$

  As L,K are $R_0[a_1,\dots,a_{n-1}]$-modules,

  $$P_\varphi(L,t)=\frac{f_1(t)}{\prod_{i=1}^{n-1}(1-t^{d_i})},\quad P_\varphi(K,t)=\frac{f_2(t)}{\prod_{i=1}^{n-1}(1-t^{d_i})}$$

  $$\therefore P_\varphi(M,t)=\frac{f(t)}{\prod_{i=1}^{n}(1-t^{d_i})}\quad\square$$

(2) By (1), write $P_\varphi(M,t)=\frac{h(t)}{(1-t)^d}$, $(1-t)\nmid h(t)$, $h(t)=\sum_{i=0}^{N}a_it^i$

Since $(1-t)^{-d}=1-\binom{-d}{1}t+\binom{-d}{2}t^2-\dots+(-1)^{d-1}\binom{-d}{d-1}t^{d-1}+\dots$, notice $\binom{-d}{i}=(-1)^i\binom{d+i-1}{d-1}$

$=\sum_{i=0}^{\infty}\binom{d+i-1}{d-1}t^i$

$\therefore$ Comparing the coef of t^m in $P_\varphi(M,t)$, we get $\varphi(M_m)=\sum_{i=0}^{N}a_i\binom{d+(m-i)-1}{d-1}=\underbrace{\left(\sum_{i=0}^{N}a_i\right)}_{\neq 0 \text{ since } (1-t)\nmid h(t)\Rightarrow h(1)\neq 0\ (\therefore \text{degree is this val})}\frac{m^{d-1}}{(d-1)!}$ $\forall m\geq N$

THEOREM

For Noeth local (R,m), fin gen R-module M, and $F=R/m$, then:

(1) $\dim_F M/m^rM<\infty$ $\left(M/m^rM\leftrightarrow M/mM\oplus mM/m^2M\oplus\dots\oplus m^{r-1}M/m^rM\right)$

(2) If d is the least number of generators of m, then $\exists g(z)\in\mathbb{Q}[z]$ of deg $\leq d$, s.t. $g(n)=\dim_F M/m^nM$ $\forall n\gg 0$

Proof

Let $gr_m(R)=R/m\oplus m/m^2\oplus m^2/m^3\oplus\dots=\bigoplus_{i=1}^{\infty}m^i/m^{i+1}$, $m_0:=R$

Define $\forall x_i+m^{i+1}\in m^i/m^{i+1}$, $x_j+m_{j+1}\in m^j/m_{j+1}$, $(x_i+m^{i+1})(x_j+m^{j+1}):=x_ix_j+m^{i+j+1}$

- Well-defined: $x_i'-x_i\in m^{i+1}$, $x_j'-x_j\in m^{j+1}\Rightarrow x_i'x_j'-x_ix_j=\underset{\substack{\cap\\ m^i}}{x_i'}\underset{\substack{\cap\\ m^{j+1}}}{(x_j'-x_j)}+\underset{\substack{\cap\\ m^{i+1}}}{(x_i'-x_i)}\underset{\substack{\cap\\ m^j}}{x_j}\in m^{i+j+1}$

Define $gr_m(M)=\bigoplus_{i=0}^{\infty}m^iM/m^{i+1}M$:
$$\begin{aligned} gr_m(R)\times gr_m(M)&\longrightarrow gr_m(M)\\ (x_i+m^{i+1},\ a_j+m^{j+1}M)&\longmapsto x_ia_j+m^{i+j+1}M\end{aligned}$$

With Rees lemma, via $\varphi: S_{\underset{\text{max ideal}}{m}}(R)\longrightarrow gr_m(R)$ $\Longrightarrow$ a graded ring homo $\Rightarrow S/mS\cong \underbrace{gr_m(R)}_{\text{Noeth}}$

$S_m(R)=R\oplus mt\oplus m^2t^2\oplus\dots$, $gr_m(R)=R/m\oplus m/m^2\oplus\dots$

Similarly, $gr_m(M)=\tilde{M}/m\tilde{M}$ for some $\tilde{M}=M\oplus mMt\oplus m^2Mt^2\oplus\dots$, thus $\tilde{M}$ is Noeth $\Rightarrow gr_m(M)$ is a fin gen $gr_m(R)$-module

$\therefore m^iM/m^{i+1}M$ is a fin gen R/m-module. Also, $\dim_F M/m^rM=\sum_{i=1}^{r}\dim_F m^{i-1}M/m^iM<\infty$

(2) Let $\langle a_1, \dots, a_d \rangle_R = m$, then $gr_m(R) = R/m[\bar{a}_1, \dots, \bar{a}_d]$, $\bar{a}_i \in m/m^2$

By Hilbert-Serre (2), $\exists!\ p(z) \in \mathbb{Q}[z]$ of $\deg \leq d-1$, s.t. $p(n) = \dim_F m^n M/m^{n+1}M$ $\forall n \gg 0 \Rightarrow \dim_F M/m^{n+1}M - \dim_F M/m^n M = p(n)$

$\therefore$ By prop 2, $\exists\, g(z) \in \mathbb{Q}[z]$ with $\deg \leq d$, s.t. $g(n) = \dim M/m^n M$ $\forall n \gg 0$. $\square$