Algebra II Theorems

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Statements

Notice: I have briefly mentioned this in my README.md document, but by "Theorems", I refer to not just Theorems, but also Lemmas, Propositions, Corollaries and other things of the sort.

2-19-25 (Week 1): Rings and Modules (Quotient)

Fact 1.1. For the relationship between I and R/I,

(1) I is **max** $\Leftrightarrow$ R/I is a **field**

(2) I is **prime** $\Leftrightarrow$ R/I is an **integral domain**

Fact 1.2. $n_R \in {}_Rm$, i.e. it is an **ideal**, and $\underline{R/n_R}$ is **reduced**

Proposition 1.1. $\boxed{n_R = \cap_{P \in \text{Spec}R} P}$

Corollary 1.1. $\sqrt{I} = \bigcap_{\text{Spec}R \ni P \supseteq} P$

Example 1.1. Usually $\sqrt{I^n} \neq I$, but if $\underline{P' \in \text{Spec}R}$, then $\boxed{\sqrt{(P')^n} = P'}$

Fact 1.3. The following are true about **primary ideals**

(1) Q is **primary** $\Leftrightarrow$ $R/Q \neq 0$ and the **zero-divisors** in R/Q are **nilpotent**

(2) If Q is **primary**, then $\sqrt{Q}$ is the **smallest prime ideal** containing Q

Example 1.2. $\boxed{\sqrt{I} \in \text{Spec}R \not\Leftarrow I \text{ is } \textbf{primary}}$ *(Key example)*

Statements and Proof Outlines

Notice: I would give enough detail but not write my proofs formally here. They are proof **outlines** so it is easier to recall **for me**, not actual proofs nor in complete English. It may look messy to you, that's understandable. I have a certain chronic eye condition and am dyslexic, so I need such "messiness" to understand what I write. When I can't color-code stuff like in my notes, this is the best alternative.

2-19-25 (Week 1): Rings and Modules (Quotient)

Fact 1.1. For the relationship between I and R/I,

(1) I is **max** $\Leftrightarrow$ R/I is a **field**

(2) I is **prime** $\Leftrightarrow$ R/I is an **integral domain**

Proof.

(1) "$\Rightarrow$": $\forall\, \overline{0} \neq \overline{x} \in R/I,\ x \notin I \Rightarrow \langle x \rangle + I \supsetneq I \Rightarrow \langle x \rangle + I = R$. In particular, $1 \in R \Rightarrow \exists\, a \in I$, s.t. $yx + a = 1 \Rightarrow \overline{yx} = \overline{1} \Rightarrow \overline{y} = \overline{x}^{-1}$
"$\Leftarrow$": Let $I \subsetneq J$, pick x in $J \setminus I$, $\overline{x} \neq \overline{0}$ in R/I. Let $\overline{y} \in R/I$, st. $\overline{yx} = \overline{1} \Rightarrow yx + a = 1,\ a \in J$. In particular, $1 \in J \Rightarrow \forall\, r \in R, 1(r) = r \in J \therefore R = J$, and hence is **max** $\square$

(2) "$\Rightarrow$": $\overline{xy} = \overline{0}$ and $\overline{x} \neq \overline{0} \Rightarrow xy \in I$ and $x \notin I \Rightarrow y \in I \Rightarrow \overline{y} = \overline{0}$, by def, OK
"$\Leftarrow$": $xy \in I$ and $x \notin I \Rightarrow \overline{xy} = \overline{0}$ and $\overline{x} \neq \overline{0} \Rightarrow \overline{y} = \overline{0} \Rightarrow y \in I$, by def, OK $\square$

Fact 1.2. $\mathcal{n}_R \in {}_R\mathcal{m}$, i.e. it is an **ideal**, and $\underline{R/\mathcal{n}_R}$ is **reduced**

Proof. $a, b \in \mathcal{n}_R$, say $a^n = 0,\ b^m = 0$ and $r \in R \Rightarrow \underline{(ra)^n} = r^n a^n = 0 \Rightarrow \underline{ra \in \mathcal{n}_R}$ and $\underline{(a+b)^{n+m}} = \sum_{i=0}^{n+m} \binom{n+m}{i}\ a^i\ b^{m+n-i} = 0 \Rightarrow \underline{a + b \in \mathcal{n}_R}$. For **reduced**, of course, quotient $\mathcal{n}_R$ means no more $\underline{\text{non-zero nilpotent}}$ $\square$

Proposition 1.1. $\boxed{\mathcal{n}_R = \cap_{P \in \text{Spec}R} P}$

Proof.

- "$\subseteq$": For $a \in \mathcal{n}_R$, say $\underline{a^n = 0 \in P}\ \forall P \in \text{Spec}R$. By def of P, $a \in P\ \forall P \Rightarrow a \in$ RHS
- "$\supseteq$": Use contraposition and Zorn's Lemma. Let $a \notin \mathcal{n}_R,\ S = \{{}_R\mathcal{m} \ni I \subseteq R \mid \underline{a^n \notin I}\ \forall n \in \mathbb{N}\}$. Note $\underline{S \neq \emptyset}$ since $\underline{\{0\} \in S}$ ($a \notin \mathcal{n}_R \Rightarrow a^k \neq 0\ \forall k \in \mathbb{N}$)

  Define **partial order** "$\leq$" in S as "$I \leq J \Leftrightarrow I \subseteq J$". Let $\underline{\{I_i \mid i \in \Lambda\}}$ be a **chain** in S. Then, ${}_R\mathcal{m} \ni I := \underline{\cup_{i \in \Lambda} I_i}$ is a **least upper bound** of $\{I_i \mid i \in \Lambda\}$. (Module because $a, b \in I \Rightarrow a \in I_i,\ b \in I_j \Rightarrow I_i \subseteq I_j$ or $I_j \subseteq I_i \Rightarrow a + b \subseteq I_i$ or $I_j \subseteq I$). By **Zorn's Lemma**, $\exists$ a **max element** Q in S

  Claim 1.1. $\boxed{Q \in \text{Spec}R}$ (Then $a \notin Q \Rightarrow a \notin$ RHS)
  Proof. $x \notin Q \Rightarrow \underline{\langle x \rangle + Q \supsetneq Q} \Rightarrow \underline{\langle x \rangle + Q \notin S} \Rightarrow \underline{a^n \in \langle x \rangle + Q}$. Similarly, $y \notin Q \Rightarrow a^m \in \langle y \rangle + Q$. Thus, $x \notin Q,\ y \notin Q \Rightarrow \underline{a^{m+n} \in \langle xy \rangle + Q} \Rightarrow \underline{\langle xy \rangle + Q \supsetneq Q}$, i.e. $\underline{xy \notin Q}$ (def of prime ideal) $\square$

Corollary 1.1. $\sqrt{I} = \bigcap_{\text{Spec}R \ni P \supseteq I} P$

Proof. Let $\begin{aligned} \phi : R &\longrightarrow R/I \\ r &\longmapsto \overline{r} \end{aligned}$. Then, $\sqrt{I} = \phi^{-1}(\mathcal{n}_{R/I}) = \phi^{-1}(\bigcap_{\overline{P} \in \text{Spec}R/I} \overline{P}) = \bigcap_{\text{Spec}R \ni P \supseteq I} P$, $\overline{P} = P/I$

$\square$

Example 1.1. Usually $\sqrt{I^n} \neq I$, but if $\underline{P' \in \text{Spec}R}$, then $\boxed{\sqrt{(P')^n} = P'}$

Proof. "$\subseteq$": By Prop 1.1, $\boxed{\sqrt{(P')^n} = \cap_{P \subseteq (P')^n} P} \subseteq P'$
"$\supseteq$": $\forall\, x \in P',\ \underline{x^n \in (P')^n} \Rightarrow x \in \sqrt{(P')^n}$ $\square$

Fact 1.3. The following are true about **primary ideals**

(1) Q is **primary** $\Leftrightarrow$ $R/Q \neq 0$ and the **zero-divisors** in R/Q are **nilpotent**

(2) If Q is **primary**, then $\sqrt{Q}$ is the **smallest prime ideal** containing Q

Proof.

(1) "$\Rightarrow$": $\underline{\overline{xy} = \overline{0},\ \overline{x} \neq 0} \Rightarrow xy \in Q,\ x \notin Q \Rightarrow y^n \in Q \Rightarrow \boxed{(\overline{y})^n = \overline{0}}$
"$\Leftarrow$": $\underline{xy \in Q,\ x \notin Q} \Rightarrow \overline{xy} = \overline{0},\ \overline{x} \neq \overline{0}$ in $R/Q \Rightarrow \underline{(\overline{y})^n = \overline{0}}$ for some $n \in \mathbb{N} \Rightarrow \boxed{y^n \in Q}$

(2) "$\sqrt{Q} \in \text{Spec}R$": We know $\underline{xy \in \sqrt{Q}} \Rightarrow (xy)^n = \underline{x^n y^n \in Q}$. $\underline{x \notin \sqrt{Q}} \Rightarrow x^m \notin Q \forall m \Rightarrow \underline{x^n \notin Q}$. By def, $\underline{(y^n)^l \in Q} \Rightarrow y \in \sqrt{Q}$

"Smallest": By Prop 1.1, $\sqrt{Q} = \bigcap_{P \supseteq Q} P \Rightarrow \underline{\sqrt{Q} \subset P}$, so $\forall P \in \text{Spec}R,\ P \supseteq Q$ $\square$

Example 1.2. $\boxed{\sqrt{I} \in \text{Spec}R \not\Leftarrow I \text{ is } \textbf{primary}}$

Proof. For $R = \mathbb{R}[x, y]$, we need $xy \in I$ and $x \notin I \Rightarrow y^n \in I$, so $\boxed{I = \langle x^2, xy\rangle}$ is **not primary**. Notice, $I = \langle x\rangle \cap \langle x^2, xy, y^2\rangle = \langle x\rangle \cap \langle x, y\rangle^2$.

Now, $R/\langle x\rangle = \mathbb{R}[x, y]/\langle x\rangle \cong \mathbb{R}[y]$, which is **not a field** $\therefore$ $\underline{\langle x\rangle \text{ is } \textbf{not a maximal ideal}}$. $R/\langle x, y\rangle \cong \mathbb{R}$, which is a field $\Rightarrow$ $\underline{\langle x, y\rangle \text{ is a } \textbf{max ideal}}$

Now, we know $\boxed{\sqrt{I}} = \sqrt{\langle x\rangle} \cap \sqrt{\langle x, y\rangle^2} = \langle x\rangle \cap \langle x, y\rangle = \boxed{\langle x\rangle}$, which is **primary** $\square$