3-21-25 (WEEK 5)

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# NOETHERIAN AND ARTINIAN

## DEFINITION

Say $M \in {}_R\mathfrak{M}$.

- $M$ is Noetherian if it satisfies ACC on submodules
- $M$ is Artinian if it satisfies DCC on submodules

## FACTS

1. TFAE: (a) $M$ is Noetherian
   (b) Each submodule of $M$ is finitely generated
   (c) Any nonempty collection of submodules of $M$ has a max member
2. $M$ is Artinian $\Leftrightarrow$ Any nonempty collection of submodules of $M$ has a min member

Note: If $R$ is not commutative, we would say it's left-Noetherian/left-Artinian instead
For simplicity, assume $R$ is commutative, then $R$ is a Noetherian/Artinian ring and $M$ is a Noetherian/Artinian $R$-module.

## QUESTION

How do we define a reasonable notation for the size of a module?

## DEFINITION

A chain $M = C_0 \supseteq C_1 \supseteq C_{r-1} \supseteq C_r = 0$ is called a composition series if each factor $C_{i-1}/C_i$ is simple, i.e. $\neq 0$ and has no submodule other than $0$ and itself.

Here, $r$ is called the composition length

## MAIN THEOREM

If $M$ has a composition series, then all its composition series have the same length, denoted by $\ell(M)$ ($\ell(M) := \infty$ if $M$ has no composition series)

Strategy

Apply Schrier refinement thm and Jordan-Hölder thm

## SCHRIER REFINEMENT THEOREM

For any two chains in $M$, $M = C_0 \supseteq C_1 \supseteq \dots \supseteq C_r = 0$ as chain $C$ and $M = D_0 \supseteq D_1 \supseteq \dots D_s = 0$ as chain $D$, they have isomorphic refinements $\tilde{C} \cong \tilde{D}$ ($\Leftarrow \tilde{r} = \tilde{s}$, $\tilde{C}_{i-1}/\tilde{C}_i \cong \tilde{D}_{i'-1}/\tilde{D}_{i'}$, $i \mapsto i'$ is a permutation of $\{1, \dots, \tilde{r}\}$)

## JORDAN-HÖLDER THEOREM

Any two composition series $C$ and $D$ are isomorphic

Proof

If we have Schrier refinement thm, then $C \to \tilde{C}$, $D \to \tilde{D} \Rightarrow \because \tilde{C} \cong \tilde{D} \therefore C \cong D$ $\square$

## PROOF OF SCHRIER REFINEMENT THEOREM

Define $C_{ij} = (C_{i-1} \cap D_j) + C_i$ for $i = 1, \dots, r$, $j = 0, \dots, s$
$D_{ji} = (D_{j-1} \cap C_i) + D_j$ for $i = 0, \dots, r$, $j = 1, \dots, s$

$\therefore$ We construct $M = C_0 = C_{10} \supseteq C_{11} \supseteq C_{12} \supseteq \dots \supseteq C_{1s} = C_1 = C_{20} \supseteq C_{21} \supseteq \dots \supseteq C_{rs} = 0$
and also $M = D_0 = D_{10} \supseteq D_{11} \supseteq D_{12} \supseteq \dots \supseteq D_{1r} = D_1 = D_{20} \supseteq D_{21} \supseteq \dots \supseteq D_{rs} = 0$

Assuming none of the elements overlap, then we have both chains of length $rs$

Notice, (Butterfly lemma)

$$C_{i,j-1}/C_{ij} = \frac{(C_{i-1} \cap D_{j-1}) + C_i}{(C_{i-1} \cap D_j) + C_i} \cong \frac{C_{i-1} \cap D_{j-1}}{C_i \cap D_{j-1} + C_{i-1} \cap D_j} \cong \frac{(D_{j-1} \cap C_{i-1}) + D_j}{(D_{j-1} \cap C_i) + D_j} \cong D_{j,i-1}/D_{ji}$$

If $C_{i,j-1} = C_{ij}$, then we get $D_{j,i-1}/D_{ji} \cong C_{i,j-1}/C_{ij} = 0 \Rightarrow D_{j,i-1} = D_{ji}$, so ommit $C_{i,j-1}$ and $D_{j,i-1}$, which means our length is still preserved and equal. □

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## THEOREM

TFAE

(a) $M$ has a composition series

(b) $M$ is both Noetherian and Artinian

Proof

(a) $\Rightarrow$ (b): Suppose $\ell(M) = n$ and $M = D_0 \supseteq D_1 \supseteq \dots \supseteq D_n = 0$ as a composition series

Assume $M$ is not Noetherian, i.e. $\exists\, 0 = N_1 \subsetneq N_2 \subsetneq \dots \subsetneq N_n \subsetneq \dots$

Define a chain $C$: $M = C_0 \supsetneq C_1 = N_n \supsetneq \dots \supsetneq C_n = N_1 = 0$

$\therefore$ By Schrier's thm, $\exists\, \tilde{C} \cong \tilde{D} = D \Rightarrow \tilde{C}$ is a composition series $\rightarrow\!\!\times\!\!\rightarrow$

(b) $\Rightarrow$ (a): $\because$ $M$ is Noetherian

$\therefore$ $\exists$ a max submodule $C_1$ of $M \Rightarrow \exists$ a max submodule $C_2$ of $C_1 \Rightarrow \dots$

In other words, $M = C_0 \supsetneq C_1 \supsetneq C_2 \supsetneq \dots$

$\because$ $M$ is Artinian

$\therefore$ $\exists\, n$, s.t. $C_n = 0$

$\therefore$ We have constructed $M$ as a composition series (of finite length). □

## EXAMPLE OF WHY NOT ALL RINGS ARE ARTINIAN

$\ell(\mathbb{Z}) = \infty$

- $\mathbb{Z}$ is Noetherian since it is a PID
- However, $\mathbb{Z}$ is not Artinian! Consider the following infinitely long DCC: $\langle 2 \rangle \supsetneq \langle 2^2 \rangle \supsetneq \langle 2^3 \rangle \supsetneq \dots$

## EXAMPLE OF AN ARTINIAN RING

Say $m = p_1 \cdots p_r$.

Claim: $\mathbb{Z}/\langle m \rangle \supsetneq \langle p_1 \rangle/\langle m \rangle \supsetneq \langle p_1 p_2 \rangle/\langle m \rangle \supsetneq \dots \supsetneq \langle p_1 p_2 \cdots p_r \rangle/\langle m \rangle = 0$ is a composition series

Proof

Notice, $\dfrac{\mathbb{Z}/\langle m \rangle}{\langle p_1 \rangle/\langle m \rangle} \cong \mathbb{Z}/\langle p_1 \rangle$ is a field and is hence simple

Then, $\dfrac{\langle p_1 \rangle/\langle m \rangle}{\langle p_1 p_2 \rangle/\langle m \rangle} \cong \langle p_1 \rangle/\langle p_1 p_2 \rangle$ is simple since $\langle p_1 \rangle \supseteq I \supseteq \langle p_1 p_2 \rangle$

$\therefore$ $\ell(\mathbb{Z}/\langle m \rangle) = r$

# ARTINIAN RING

## PROPOSITION

If $R$ is Artinian, then $\#\mathrm{Max}R < \infty$

Proof

Define $S := \{\bigcap_{\text{finite}} m \mid m \in \mathrm{Max}R\} \neq \emptyset \Rightarrow \exists$ a minimal member, say $m_1 \cap \dots \cap m_k$ (has a "stopping condition" for DCC)

Now, for $m \in \mathrm{Max}R$, $m \cap (m_1 \cap \dots \cap m_k) = m_1 \cap \dots \cap m_k \Rightarrow m \supseteq m_1 \cap \dots \cap m_k \Rightarrow m \supseteq m_\ell$ for some $\ell \Rightarrow m = m_\ell$ (both are max ideals)

Otherwise, $\exists\, x_i \in m_i \setminus m$, $x_1 \cdots x_k \in m_1 \cap \dots \cap m_k \subseteq m$

## PROPOSITION 2

If $R$ is Artinian and $\mathrm{Max}\,R = \{m_1, \dots, m_k\}$, then $\exists\, n_1, \dots, n_k \in \mathbb{N}$, s.t. $\{0\} = m_1^{n_1} \cdots m_k^{n_k} = \bigcap_{i=1}^{k} m_i^{n_i}$

$\sqrt{m_i^{n_i} + m_j^{n_j}} = \sqrt{\sqrt{m_i^{n_i}} + \sqrt{m_j^{n_j}}} = \sqrt{m_i + m_j} = \sqrt{R} = R$

Proof

Since $R$ is Artinian, it satisfies DCC, so $\exists\, n_i \in \mathbb{N}$, s.t. $m_i^{n_i} = m_i^{n_i+1}$ (何でも、そのdescending chainは終る事ができる)

If $m_1^{n_1} m_2^{n_2} \cdots m_k^{n_k} \neq 0$, then $S = \{J \subseteq R \mid J m_1^{n_1} \cdots m_k^{n_k} \neq 0\} \neq \emptyset$ (RES ✓)

Let $0 \neq J_0$ be a minimal member of $S$. Pick $0 \neq x \in J_0$, s.t. $\langle x \rangle \in S \Rightarrow \langle x \rangle = J_0$, since $\langle x \rangle \subseteq J_0$

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Observe that $x\mathfrak{m}_1^{n_1}\cdots\mathfrak{m}_k^{n_k} = (x\mathfrak{m}_1\cdots\mathfrak{m}_k)(\mathfrak{m}_1^{n_1}\cdots\mathfrak{m}_k^{n_k}) \neq 0$

$\Rightarrow x\mathfrak{m}_1\cdots\mathfrak{m}_k \in S \Rightarrow x\mathfrak{m}_1\cdots\mathfrak{m}_k = (\mathfrak{m}_1\cdots\mathfrak{m}_k)xR = \langle x\rangle = xR$. By Nakayama lemma, $xR=0 \Rightarrow x=0$ $\rightarrow\!\!\!\!\times$

$\subseteq$ $xR\cdots R = xR = \langle x\rangle$

## PROPOSITION 3

one maximal ideal

If $R$ is Artinian, then $R \cong R_1 \times \cdots \times R_k$, where $R_i$ is an Artinian local ring

Proof

critical race theory じゃないのwww, born to yap about gender, ended up as a STEM major fr

By CRT, $R = R/\langle 0\rangle = R/\mathfrak{m}_1^{n_1}\cdots\mathfrak{m}_k^{n_k} = R/\mathfrak{m}_1^{n_1}\cap\cdots\cap\mathfrak{m}_k^{n_k} \cong R/\mathfrak{m}_1^{n_1} \times\cdots\times R/\mathfrak{m}_k^{n_k}$

Write $R_i := R/\mathfrak{m}_i^{n_i}$. If $\bar{\mathfrak{m}} \in \operatorname{Max} R_i$, say $\bar{\mathfrak{m}} = \mathfrak{m}/\mathfrak{m}_i^{n_i}$, then $\mathfrak{m} \supseteq \mathfrak{m}_i^{n_i} \Rightarrow \mathfrak{m} \supseteq \mathfrak{m}_i \Rightarrow \mathfrak{m} = \mathfrak{m}_i$

That is, $\operatorname{Max} R_i = \{\bar{\mathfrak{m}}_i\}$ $\square$

## PROPOSITION 4

In a ring $R$, if we can find max ideals $\mathfrak{m}_1,\dots,\mathfrak{m}_n$ not necessarily in different $R$, s.t. $\mathfrak{m}_1\cdots\mathfrak{m}_n = 0$, then $R$ is Noetherian $\Leftrightarrow$ $R$ is Artinian

Proof

Consider $R \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \cdots \supseteq \mathfrak{m}_1\cdots\mathfrak{m}_n = 0$

a field

Let $M_i := \mathfrak{m}_1\cdots\mathfrak{m}_{i-1}/\mathfrak{m}_1\cdots\mathfrak{m}_i$, which is an $R/\mathfrak{m}_i$-module since $\mathfrak{m}_i M_i = 0$

Thus, $M_i$ is Artinian $\Leftrightarrow$ $M_i$ is Noetherian

Also, $0 \to \mathfrak{m}_1\cdots\mathfrak{m}_i \to \mathfrak{m}_1\cdots\mathfrak{m}_{i-1} \to M_i \to 0$

$\mathfrak{m}_0 = R$ is Artinian $\Leftrightarrow$ $\mathfrak{m}_1, M_1$ are Artinian $\Leftrightarrow$ $\mathfrak{m}_1\mathfrak{m}_2, M_1 M_2$ are Artinian $\Leftrightarrow \cdots\cdots \Leftrightarrow$ $0 = \mathfrak{m}_1\cdots\mathfrak{m}_n, M_1,\dots,M_n$ are Artinian

$0 \to \mathfrak{m}_1 \to R \to M_1 \to 0$

$\Leftrightarrow$ $0 = \mathfrak{m}_1\cdots\mathfrak{m}_n, M_1,\dots M_n$ are Noetherian $\Leftrightarrow \cdots \Leftrightarrow$ $\mathfrak{m}_1, M_1$ are Noetherian $\Leftrightarrow$ $\mathfrak{m}_0 = R$ is Noetherian

## REMARK

$R$ is Artinian $\Leftrightarrow$ $R$ is Noetherian and $\operatorname{Max} R = \operatorname{Spec} R$ $\forall R$

Proof

"$\Rightarrow$": Proposition 2 + Proposition 4 + $\operatorname{Max} R = \operatorname{Spec} R$ in Artinian rings ✓

"$\Leftarrow$": $\langle 0\rangle = \bigcap_{i=1}^{k} \mathfrak{q}_i$ ← primary decomposition (来週の水曜日はこのノートを書くよ!)

Here, $\sqrt{\mathfrak{q}_i} = \mathfrak{m}_i$

Since $\mathfrak{m}_i$ is finitely generated, $\exists\, \mathfrak{m}_i$, s.t. $\mathfrak{m}_i^{n_i} \subseteq \mathfrak{q}_i$. Hence, $\mathfrak{m}_1^{n_1}\cap\cdots\cap\mathfrak{m}_k^{n_k} = \mathfrak{m}_1^{n_1}\cdots\mathfrak{m}_k^{n_k} \subseteq \mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_k = 0$

$\because$ Noetherian

$\therefore$ $\mathfrak{m}_1^{n_1}\cdots\mathfrak{m}_k^{n_k} = 0$, so $R$ is Artinian $\square$