3-28-25 (WEEK 6)

Shun/翔海(@shun4midx)

NAKAYAMA'S LEMMA AND ARTIN-REES LEMMA

Here, R is commutative and $M \in {}_R\mathfrak{m}$.

DEFINITION

The Jacobson radical of R is $J_R := \bigcap_{m \in \mathrm{Max}R} m$ (nilradical was intersection of prime ideals)

PROPERTIES

(1) $I \subsetneq R \Rightarrow \langle I, J_R \rangle \subsetneq R$: $I \subsetneq R \Rightarrow \exists m \in \mathrm{Max}R$ s.t. $I \subseteq m$, $J_R \subseteq m \Rightarrow \langle I, J_R \rangle \subseteq m \subsetneq R$

(2) $n_R \subseteq J_R$

(3) $x \in J_R \Leftrightarrow 1 - rx \in R^\times$ $\forall r \in R$: "$\Rightarrow$": $1-rx \notin R^\times \Rightarrow \langle 1-rx \rangle \subseteq m \Rightarrow 1-\underbrace{rx}_{\in m} \in m \Rightarrow 1 \in m$ $\divideontimes$ $\therefore 1-rx \in R^\times$ ✓

"$\Leftarrow$": If $x \notin m$ for a $m \in \mathrm{Max}R$, then $\langle x \rangle + m = R$, say $1 = rx + \underbrace{z}_{\in m} \Rightarrow m \ni z = 1 - rx \in R^\times \Rightarrow m = R$ $\divideontimes$ $\therefore x \in J_R$.

NAKAYAMA LEMMA

If M is finitely generated and $I \subseteq J_R$, s.t. $IM = M$, then $M = 0$

Proof

Assume that $M \neq 0$. Let n be the smallest integer, s.t. M is generated by n elements, say $x_1, \dots, x_n$

$\because IM = M \ni x_n$

$\therefore x_n = a_1 x_1 + \dots + a_{n-1}x_{n-1} + a_n x_n$ with $a_i \in I$.

$\Rightarrow \underbrace{(1-a_n)}_{\in R^\times \because a_n \in I \subseteq J_R} x_n = a_1 x_1 + \dots + a_{n-1} x_{n-1}$ $\therefore M = \langle x_1, \dots, x_{n-1} \rangle$ $\divideontimes$ $\square$

COROLLARY 1

For a finitely generated M, $N \subseteq M$, $I \subseteq J_R$, then $I + MN \Rightarrow M = N$

Proof

M is finitely generated $\Rightarrow M/N$ is finitely generated

We know $I(M/N) = \frac{IM+N}{N} = \frac{M}{N} \Rightarrow$ By Nakayama lemma, $M/N = 0 \Rightarrow M = N$ $\square$

COROLLARY 2

For a local (R, m), finitely generated M, if $M/mM = \langle \bar{x}_1, \dots, \bar{x}_n \rangle_{R/m}$ with $\dim_{R/m} M/mM = n$, then $M = \langle x_1, \dots, x_n \rangle_R$

(only has 1 max)

Proof

Let $N = \langle x_1, \dots, x_n \rangle_R$. Then, $\frac{N+mM}{mM} = \langle \bar{x}_1, \dots, \bar{x}_n \rangle = M/mM \Rightarrow N + mM = M \Rightarrow N = M$ $\square$

COROLLARY 3

For a local (R, m), finitely generated M, N, $f: M \to N$ in ${}_R\mathfrak{m}$, $\bar{f}: M/mM \to N/mN$ is a linear transformation

Proof

(1) "$\bar{f}$ is onto $\Rightarrow f$ is onto": $\mathrm{Im}(\bar{f}) = \frac{f(M)+mN}{mN} = N/mN \Rightarrow f(M) + mN = N \Rightarrow f(M) = N$

(2) Assume M, N are free, then $\bar{f}$ is 1-1 $\Rightarrow f$ is 1-1:

$M = \langle x_1, \dots, x_\ell \rangle \cong R^\ell$ (free $\Rightarrow$ no relation)

$\Updownarrow$ free basis

$M/mM = \langle \bar{x}_1, \dots, \bar{x}_\ell \rangle \cong (R/m)^\ell$

basis

Similarly, $N = \langle y_1, \dots, y_k \rangle \cong R^k \Leftrightarrow N/mN = \langle \bar{y}_1, \dots, \bar{y}_k \rangle$

Since $\bar{f}$ is 1-1, $\dim \mathrm{Im}\bar{f} = \ell$, say $\langle \bar{w}_1, \dots, \bar{w}_\ell \rangle = \mathrm{Im}\bar{f}$ "Assume not"

Let $v_i \in M$, s.t. $f(v_i) = w_i$ with $\underbrace{\langle \bar{v}_1, \dots, \bar{v}_\ell \rangle_{(R/m)}}_{= M/mM} \cong \mathrm{Im}\bar{f}$ $\dashrightarrow \langle \bar{w}_1, \dots, \bar{w}_\ell, \bar{w}_{\ell+1}, \dots, \bar{w}_k \rangle \Rightarrow N/mN = \langle w_1, \dots, w_\ell, w_{\ell+1}, \dots, w_k \rangle$

$\langle \bar{w}_1, \dots, \bar{w}_\ell \rangle_{R/m} \Rightarrow \mathrm{Im}f = \langle w_1, \dots, w_\ell \rangle_R$

$\therefore M = \langle v_1, \dots, v_\ell \rangle$

Now, for $x \in \ker f$, say $x = \sum_{i=0}^{\ell} a_i v_i$, $f(x) = \sum_{i=0}^{\ell} a_i w_i \Rightarrow a_i = 0$ $\forall i \Rightarrow x = 0 \Rightarrow f$ is 1-1 $\square$ $\therefore$ (1)+(2) $\Rightarrow M, N$: finite free, $\bar{f}$ isom $\Rightarrow f$ isom $\Rightarrow M \cong N$

DEFINITION

- A filtration of M is $M=M_0\supseteq M_1\supseteq M_2\supseteq\dots$
- Let $I\subseteq R$. $\{M_i\}_{i=0,1,\dots}$ is an I-filtration if $IM_n\subseteq M_{n+1}\ \forall n$ (e.g. $M_i=I^iM$)
- I-filtration $\{M_i\}_{i=0,1,\dots}$ is stable if $IM_n=M_{n+1}\ \forall n>>0$.
- $R=\bigoplus_{i\geq 0}^{\infty}R_i$ is a graded ring R if $R_iR_j\subseteq R_{i+j}$
- $M=\bigoplus_{i=0}^{\infty}M_i$ is a graded module over a graded ring if $R_iM_j\subseteq M_{i+j}$

THEOREM

Let R be a graded ring. Then, Noeth $R\Leftrightarrow$ Noeth R_0 and $R=R_0[a_1,\dots,a_m]$, $a_i\in R$

Proof

"$\Leftarrow$": $R=R[a_1,\dots,a_m]\cong R_0[x_1,\dots,x_m]/\mathrm{Ker}\,\varphi$: Noeth

$$\begin{array}{ccc} & & 0\\ & & \uparrow\\ R_0[x_1,\dots,x_m] & \xrightarrow{\varphi} & R[a_1,\dots,a_m]\\ x_i & \mapsto & a_i\end{array}$$

"$\Rightarrow$": Let $R^+=\bigoplus_{i\geq 1}^{\infty}R_i\subseteq R$ (ideal $\because R^+R\subseteq R$) and $R_0\cong R/R^+$

$R^+=\langle z_1,\dots,z_\ell\rangle_R$ and $z_i=\underbrace{z_{i,i_1}}_{\in R_{i_1}}+\underbrace{z_{i,i_2}}_{\in R_{i_2}}+\dots+\underbrace{z_{i,i_{\ell_i}}}_{\in R_{i\ell_i}}$

$\Rightarrow\langle z_{i,i_j}\mid i=1,\dots,\ell;\ j=1,\dots,\ell_i\rangle_R$

$\Rightarrow\langle a_1,\dots,a_m\rangle_R$, $a_i\in R_{d_i>0}$ $\forall i=1,\dots,n$

Claim: $R_k\subseteq R_0[a_1,\dots,a_m]\ \forall k\geq 0$ ($\Rightarrow R=R_0[a_1,\dots,a_m]$)

Proof

By induction on k, $k=0$: OK

For $k>0$, let $a\in R_k\subseteq R^+$, $a=\sum_{i=1}^{m}\underbrace{r_i}_{\in R_{k-d_i},\,<k}\underbrace{a_i}_{\in R_{d_i}}$ $\therefore r_i\in$ RHS $\square$

ARTIN-REES LEMMA

GENERAL FORM

For Noeth R, $I\subseteq R$, M is a finitely generated R-module, $\{M_i\}$ is a stable I-filtration

If $N\subseteq M$ and $N_n=N\cap M_n$, then $\{N_i\}$ is also a stable I-filtration

Proof

For a Noeth M, finitely generated M: V_i, $M=\langle v_1,\dots,v_m\rangle_R\Rightarrow 0\to\mathrm{Ker}\,\varphi\to \underset{\text{Noeth}}{R^{\oplus m}}\to\underset{\text{Noeth}}{M}\to 0$

- Define $S=S_I(R)=\bigoplus_{n\geq 0}^{\infty}I^nt^n\subseteq R[t]\subseteq\bigoplus_{n\geq 0}^{\infty}Rt^n$

  $\because$ R is Noeth $\Rightarrow I=\langle a_1,\dots,a_m\rangle$ and $S=R[a_1t,\dots,a_mt]$ $\therefore S$ is Noeth

  Define $\tilde{M}=\bigoplus_{n\geq 0}^{\infty}M_nt^n$ which is a graded S-module $(I^\ell t^\ell M_jt^j\subseteq M_{j+\ell}t^{\ell+j})$

  $\cup$

  $U_m=M_0+M_1t+\dots+M_mt^m$: A finitely generated R-module $=\langle \star_1,\dots,\star_s\rangle_R$

  $L_m=\langle U_m\rangle_S=U_m\oplus IM_mt^{m+1}\oplus I^2M_mt^{m+2}\oplus\dots$

  Also, $L_m\subseteq L_{m+1}$ and $\bigcup_{n\geq 0}^{\infty}L_m=\tilde{M}$

  $\therefore L_m$ is a finitely generated S-module $=\langle \star_1,\dots,\star_s\rangle_S$

Observe, with how S is Noeth, $\tilde{M}$ is finitely generated over $S\Leftrightarrow\tilde{M}$ is Noeth $\Leftrightarrow\tilde{M}=L_{n_0}$ for some $n_0\Leftrightarrow M_{n_0+m}=I^mM_{n_0}\ \forall m\geq 0\Leftrightarrow\{M_i\}$ is I-stable

Now, $I(N\cap M_n)\subseteq IN\cap IM_n\subseteq N\cap M_n=N_{n+1}\Rightarrow\{N_i\}$ is an I-filtration

Similarly, $\tilde{N}=\bigoplus_{n\geq 0}^{\infty}N_nt^n$ is an S-submodule of $\tilde{M}\Rightarrow\tilde{N}$ is a finitely generated S-module $\square$

COROLLARY

For a Noeth R, finitely generated R-module M, $I\subseteq R$, $N\subseteq M$, then $I^{n_0+m}M\cap N=I^m(I^{n_0}M\cap N)\ \forall m\geq 0$

Proof

Let $M_n=I^nM$, then $N^n=I^nM\cap N$. By thm, $\{N_n\}$ is I-stable, i.e. $\exists n_0$, s.t. $I^mN_{n_0}=N_{n_0+m}$ $\square$

KRULL THEOREM

For a Noeth R, $I \subseteq J_R$, finitely generated R-module M, then $\bigcap_{n=0}^{\infty} I^n M = \{0\}$

Proof

Noeth ⇒ finitely generated

Let $N := \bigcap_{n=0}^{\infty} I^n M \subseteq M$ and $N \cap I^n M = N$

By Artin-Rees lemma, $\exists n_0 \in \mathbb{N}$, s.t. $I^m (N \cap I^n M) = I^{m+n_0} M \cap N$ $\forall m \geq 0$. If $m=1$, we get $IN = N$. By Nakayama lemma, $N = 0$. $\square$

COROLLARY + REMARK

For a Noeth local (R, m), we get $\bigcap_{n=0}^{\infty} m^n = \{0\}$

Then, $\forall x \in R$, $\exists k$, s.t. $x \in m^k$ but $x \notin m^{k+1}$ ⇒ We can define "order" with $o(x) = k$

By defining "distance" as $2^{-o(x)}$, we can do completion like in analysis.

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