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LOCALIZATION

Recall: "units" = elements that exist multiplicative inverse

Today, R is assumed to be commutative

Let S be a multiplicatively closed set with $1 \in S$, $0 \notin S$

DEFINITION

Suppose that there is a ring B and a ring homo $f: R \to B$ s.t.

(1) $f(x)$ is a unit of B $\forall x \in S$

(2) If $g: R \to A$ is another ring homo s.t. $g(x)$ is a unit of A $\forall x \in S$, then $\exists!$ ring homo $h: B \to A$ s.t.
$$\begin{tikzcd} B \arrow[r, "h"] & A \\ R \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$
(Universal Property)

Such B, if it exists, is unique up to isom and is called the localization of R w.r.t. S, denoted by R_S

THEOREM

R_S exists

Proof

Set $R_S = R \times S / \sim$ where $(a,s) \sim (b,t) \Leftrightarrow \exists u \in S$ s.t. $(at-bs)u=0$

Step 1: "$\sim$" is an equivalence relation

- $(a,s) \sim (a,s)$ since $(as-as)1=0$
- "$(a,s) \sim (b,t) \Rightarrow (b,t) \sim (a,s)$" since $(at-bs)u=0 \Rightarrow (bs-at)u=0$
- $(a,s) \sim (b,t)$, $(b,t) \sim (c,u) \Rightarrow (a,s) \sim (c,u)$ since $(at-bs)v=0$, $(bu-ct)w=0 \Rightarrow (at-bs)vuw=0$, $(bu-ct)wvs=0 \Rightarrow (au-cs)\underbrace{tvw}_{\in S}=0$

Define $\frac{a}{t} := [(a,t)]$

Step 2: R_S has a ring structure: $\frac{a}{s}+\frac{b}{t}=\frac{at+bs}{st}$, $\frac{a}{s}\cdot\frac{b}{t}=\frac{ab}{st}$

- Well-defined:

  $\frac{a}{s}=\frac{a'}{s'} \Rightarrow (as'-a's)v=0 \Rightarrow (as'-a's)vwtt'=0$

  $\frac{b}{t}=\frac{b'}{t'} \Rightarrow (bt'-b't)w=0 \Rightarrow (bt'-b't)wvss'=0$

  $+)\ [(at+bs)s't'-(a't'+b's')st]vw=0$

  $\therefore as'v=a'sv$, $bt'w=b'tw \Rightarrow (abs't'-a'b'st)vw=0$

Actually, $(R_S, +, \cdot)$ forms a ring

Step 3: $f: R \to R_S$, $a \mapsto \frac{a}{1}$ satisfies the universal property

(1) $\forall x \in S$, $f(x)=\frac{x}{1}$ $\because \frac{1}{x}\cdot\frac{x}{1}=\frac{x}{x}=\frac{1}{1}$ $\therefore \frac{x}{1}$ is a unit in R_S

(2) Let $g: R \to A$ with $g(x)$ being a unit of A $\forall x \in S$

If $\exists$ a ring homo $R_S \to A$ with $g=hf$, then $h(\frac{a}{s})=h(\frac{a}{1}\cdot\frac{1}{s})=h(\frac{a}{1})h(\frac{1}{s})=h(\frac{a}{1})(h(\frac{s}{1}))^{-1}=hf(a)(hf(s))^{-1}=g(a)g(s)^{-1}=\frac{g(a)}{g(s)}$

So, we define $h(\frac{a}{s})=g(a)g(s)^{-1}$

It is well-defined as follows: $\frac{a}{s}=\frac{b}{t} \Rightarrow (at-bs)u=0 \Rightarrow (g(a)g(t)-g(b)g(s))g(u)=0 \Rightarrow g(a)g(s)^{-1}=g(b)g(t)^{-1}$

PROPERTIES

- If S contains no zero divisor, then $f: R \to R_S$ is injective

  $x \mapsto \frac{x}{1}=\frac{0}{s}$, i.e. $\exists u \in S$, s.t. $\underbrace{us}_{\in S}x=0 \Rightarrow us$ is not a zero divisor, so $x=0$

- If R is an integral domain (prime ideal) and $S=R\setminus\{0\}$, then R_S is called the quotient field of R which is the smallest field containing R:

  If for a field F, say $g: R \hookrightarrow F$, with $g(a) \neq 0$ being a unit, by universal property, $\exists! h: R_S \to F$ which is injective since $h(\frac{a}{s})=g(a)g(s)^{-1}=0 \Rightarrow g(a)=0 \Rightarrow a=0 \Rightarrow \frac{a}{s}=\frac{0}{1}$

  Pick $0 \neq f \notin R$ (non-nilpotent), consider $S=\{1, f, f^2, \ldots\} \Rightarrow R_S = R_f$

- $S = R/P$ for some $P \in \operatorname{Spec} R$, $R_S = R_P$

REMARK

$S \subseteq T$ with $1 \in S$, $T \not\ni 0$, S, T are mcs (multiplicatively closed sets)
When will $R_S \cong R_T$?
Ans: When $T \subseteq \bar{S} = R \setminus \bigcup_{P \cap S = \emptyset} P$ ($\frac{t}{1} \cdot \frac{a}{s} = \frac{1}{1} \Rightarrow \exists u \in S$, s.t. $\underbrace{tau}_{\in S} = \underbrace{s}_{\in S}$, $t \in \bar{S} \Rightarrow \langle t \rangle \cap S \neq \emptyset \Rightarrow s = ta$)

CONSTRUCTION FOR MODULES

For an R-module M, $1 \in S \not\ni 0$ mcs in R, define $M_S := \{(m,s) \mid m \in M, s \in S\}/\sim$, $(m,s) \sim (n,t) \Leftrightarrow \exists u \in S$, s.t. $u(tm - sn) = 0$
$\Rightarrow$ "$\sim$" is an equivalence relation, $\frac{m}{s} := [(m,s)]$ in M_S
Notice: M_S is an R_S-module

PROPOSITION

If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact (in ${}_R\mathcal{M}$), then $0 \to M_S' \xrightarrow{f_S} M_S \xrightarrow{g_S} M_S'' \to 0$ is exact

$$\frac{a}{s} \longmapsto \frac{f(a)}{s}$$
$$\frac{b}{s} \longmapsto \frac{g(b)}{s}$$

Proof

- f_S is 1-1: $f_S(\frac{a}{t}) = f_S(\frac{a'}{t'}) \Rightarrow \frac{f(a)}{t} = \frac{f(a')}{t'} \Rightarrow \exists u \in S$, s.t. $u(t'f(a) - tf(a')) = 0 \Rightarrow f(ut'a - uta') = 0 \overset{f \text{ is 1-1}}{\Rightarrow} ut'a - uta' = 0 \Rightarrow \frac{a}{t} = \frac{a'}{t'}$ ✓
- g_S is onto: $\forall \frac{b}{t} \in M_S''$, let $g(a) = b \Rightarrow g_S(\frac{a}{t}) = \frac{b}{t}$ ✓
- $\operatorname{Im} f_S \subseteq \operatorname{Ker} g_S$: $g_S f_S(\frac{a}{t}) = \frac{gf(a)}{t} = \frac{0}{t} = 0$ ✓
- $\operatorname{Ker} g_S \subseteq \operatorname{Im} f_S$: Let $\frac{a}{t} \in \operatorname{Ker} g_S$, i.e. $g_S(\frac{a}{t}) = \frac{g(a)}{t} = \frac{0}{1} \Rightarrow \exists u \in S$ s.t. $ug(a) = 0 \Rightarrow \underbrace{g(ua) = 0}_{\operatorname{Ker} g \subseteq \operatorname{Im} f} \Rightarrow f(b) = a$ $\therefore \frac{a}{t} = \frac{ua}{ut} = \frac{f(b)}{ut} = f_S(\frac{b}{ut})$

FACT

$R_S \otimes_R M \cong M_S$

Proof

$f: R_S \times M \to M_S$ is bilinear $\Rightarrow \exists$ R-module homo $\bar{f}: R_S \otimes_R M \longrightarrow M_S$
$(\frac{a}{t}, m) \mapsto \frac{am}{t}$ $\qquad \frac{a}{t} \otimes m \longmapsto \frac{am}{t}$

- $\bar{f}$ is onto: $\forall \frac{m}{t} \in M_S$, $\bar{f}(\frac{1}{t} \otimes m) = \frac{m}{t}$ ✓
- $\bar{f}$ is 1-1: Let $\sum_i (\frac{a_i}{t_i}) \otimes m_i \in R_S \otimes M$
  Set $t = \prod_i t_i$, $\bar{t}_i = \prod_{j \neq i} t_j$, then $\sum_i (\frac{a_i}{t_i}) \otimes m_i = \sum_i (\frac{a_i \bar{t}_i}{t}) \otimes m_i = \frac{1}{t} \otimes \sum_i \bar{t}_i a_i m_i$

  If $\frac{1}{t} \otimes m \in \operatorname{Ker} f$, i.e. $f(\frac{1}{t} \otimes m) = \frac{m}{t} = 0$ in M_S, i.e. $\exists u \in S$, s.t. $um = 0$.
  Then, $\frac{1}{t} \otimes m = \frac{u}{tu} \otimes m = \frac{1}{tu} \otimes \underbrace{um}_{0} = 0$ ✓

THEOREM

R_S is a flat R-module

Proof

Given $0 \to M' \to M \to M'' \to 0$ in ${}_R\mathcal{M}$, by prop, $0 \to M_S' \to M_S \to M_S'' \to 0$ is exact □

$$\begin{array}{ccc} M_S' & M_S & M_S'' \\ \cong & \cong & \cong \\ R_S \otimes_R M' & R_S \otimes_R M & R_S \otimes_R M'' \end{array}$$

REMARK

Given $M \in {}_R\mathcal{M}$, TFAE:
(1) $M = 0$
(2) $M_P = 0$ $\forall P \in \operatorname{Spec} R$
(3) $M_Q = 0$ $\forall Q \in \operatorname{Max} R$

Proof

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) is straightforward.
Consider proving (3) $\Rightarrow$ (1),
Assume $\exists\, 0 \neq z \in M$

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Define $\mathrm{ann}(z) := \{r \in R \mid rz = \overset{\in_R M}{0}\} \subseteq R \Rightarrow \exists Q \in \mathrm{Max}\,R$ s.t. $\mathrm{ann}(z) \subseteq Q$

By assumption, $\frac{z}{1} = \frac{0}{1}$ in $M_Q \Rightarrow \exists u \notin Q$, s.t. $uz = 0 \Rightarrow u \in \mathrm{ann}(z) \subseteq Q \;\; \rightarrow\!\leftarrow$