20 Analysis Questions / Misconceptions

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1) (True or false) Let $f:[a,b]\to\mathbb{R}$ be a monotonic function. Then, for any $c\in(a,b)$, both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist.

True.

Proof

WLOG, say f is monotonely increasing, then $\forall x>c$, $f(x)\geq f(c)$.

Define $L=\inf\{f(x)\mid x>c\}$, then by def, $\forall \varepsilon>0, \exists x'>c$, s.t. $f(x')<L+\varepsilon$. However, $f(x)\geq L\ \forall x>c$

Now, $\forall\varepsilon>0, \exists\delta:=x'-c>0$, s.t. $0<x-c<\delta=x'-c \Rightarrow c<x<x' \Rightarrow f(c)\leq L\leq f(x)<f(x')<L+\varepsilon \Rightarrow 0\leq f(x)-L<\varepsilon$ $\therefore \lim_{x\to c^+} f(x)=L$ □

Similarly, define $L=\sup\{f(x)\mid x<c\}$, then by def, $\forall\varepsilon>0, \exists x'<c$, s.t. $L-\varepsilon<f(x')$. However, $f(x)\leq L\ \forall x<c$.

Now, $\forall\varepsilon>0, \exists\delta:=c-x'>0$, s.t. $0<c-x<\delta=c-x' \Rightarrow x'<x<c \Rightarrow L-\varepsilon<f(x')<f(x)\leq L \Rightarrow 0\leq L-f(x)<\varepsilon$ $\therefore \lim_{x\to c^-} f(x)=L$ □

Proof Sketch

WLOG, $f\uparrow$.

"$x\to c^+$": $L:=\inf\{f(x)\mid x>c\} \Rightarrow \forall\varepsilon>0, \exists x'>c$, s.t. $f(x')<L+\varepsilon$, but $L\leq f(x)\ \forall x>c$. $\therefore \forall\varepsilon>0, \exists\delta:=x'-c>0$, s.t. $0<x-c<\delta \Rightarrow \ldots \Rightarrow 0\leq f(x)-L<\varepsilon$

"$x\to c^-$": $L:=\sup\{f(x)\mid x<c\}$, similar □

2) Let $f:[a,b]\to\mathbb{R}$. Consider the statements:

(i) f is continuous

(ii) f is of bounded variation

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent.

(D) Neither (i) implies (ii) nor (ii) implies (i)

D ☆ Cont. and BV sound related but they aren't!!

"(i) $\not\Rightarrow$ (ii)" : $f(x)=x\sin\frac{1}{x}$ for $x>0$

"(ii) $\not\Rightarrow$ (i)" : $f(x)=\mathbb{1}_{x\geq 0}$ for $x\in[-1,1]$ (Key counterexample)

3) Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}\sin\frac{1}{x}, & x>0\\ 0, & x=0\end{cases}$

Select the correct answer.

(A) f is both of bounded variation and Riemann-integrable on $[0,1]$.

(B) f is Riemann-integrable but not of bounded variation on $[0,1]$.

(C) f is of bounded variation but not Riemann-integrable on $[0,1]$.

(D) f is neither of bounded variation nor Riemann-integrable on $[0,1]$

B

Rmb, Riemann-integrable checks for continuity not BV.

Riemann-integrable: Cont. except for a finite number of points

Not BV:

Proof

Consider the partition $P=\{0,\frac{2}{(2n-1)\pi},\frac{2}{(2n-3)\pi},\ldots,\frac{2}{\pi},1\}$, $n\geq 1$, then $V_P(f)=\sum_{k=1}^{n+1}|\Delta f_k|=1+(2(n-1))+(1-\sin 1)=2n-\sin 1$, which is not bounded. □

4) Let $BV([a,b],\mathbb{R})$ be the space of functions of bounded variation defined on $[a,b]$. Is $\|f\|=|f(a)|+V_f([a,b])$ for all $f\in BV([a,b],\mathbb{R})$ a norm? Select all correct statements:

(A) No, it does not satisfy positive definite property.

(B) No, it does not satisfy homogenous property

(C) No, it does not satisfy triangle inequality

(D) Yes, it is a norm

D (Trivial, I can technically write the proof here but it's very straightforward)

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5) Compute $\int_0^4 x^2 d\alpha$.

Here, $\alpha: [0,4] \to \mathbb{R}$ is given by $\alpha = \begin{cases} 0, & 0 \leq x < 1 \\ 2, & x=1 \\ 1, & 1 < x \leq 2 \\ 3, & 2 < x \leq 4 \end{cases}$

(A) 5
(B) 7
(C) 9
(D) 11

C

Recall for step function integrators, $\int_a^b f d\alpha = f(c)[\alpha(c^+) - \alpha(c^-)]$

$\therefore \int_0^4 x^2 d\alpha = f(1)(1-0) + f(2)(3-1) = 1^2(1) + 2^2(2) = 9$

6) (True or false) Let $f: [a,b] \to \mathbb{R}$ be a continuous function and $\alpha: [a,b] \to \mathbb{R}$ be a function of bounded variation. Then, $\int_a^b f d\alpha \leq \int_a^b |f| d\alpha$

False.

Take $\alpha := -x$, $f := -1$, $[a,b] := [0,1]$, $\int_a^b f d\alpha = \int_0^1 -1 d(-x) = 1$, $\int_a^b |f| d\alpha = \int_0^1 (1) d(-x) = -1$.

7) Let $f: [a,b] \to \mathbb{R}$ be a bounded function, $\alpha: [a,b] \to \mathbb{R}$ be a nondecreasing function, and $L \in \mathbb{R}$.

Consider the statements:

(i) $L_P(f,\alpha) \leq L \leq U_P(f,\alpha)$ for all partition P of $[a,b]$

(ii) $f \in R(\alpha; a,b)$ and $\int_a^b f d\alpha = L$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)
(B) (ii) implies (i) but (i) does not imply (ii)
(C) (i) and (ii) are equivalent
(D) Neither (i) implies (ii) nor (ii) implies (i)

B

(ii) $\Rightarrow$ (i) by def. (i) $\not\Rightarrow$ (ii) because: consider $[a,b] = [0,1]$, $f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{Q}^c \end{cases}$, $\int_0^1 f dx$ does not exist.

8) Let $\alpha: [a,b] \to \mathbb{R}$ be of bounded variation and V be its variation function. Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Consider the statements:

(i) $f \in R(\alpha; a,b)$

(ii) $f \in R(V; a,b)$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)
(B) (ii) implies (i) but (i) does not imply (ii)
(C) (i) and (ii) are equivalent
(D) Neither (i) implies (ii) nor (ii) implies (i)

C

Proof sketch

"$\alpha \Rightarrow V$": If α is const, then $V \equiv 0$, so OK. Assume $\alpha(a) < \alpha(b)$, so $V(b) > 0$.

$V_P(f) \leq V_P \leq V_P(f) + \varepsilon$

Let $\varepsilon > 0$, $P_\varepsilon^{(1)} \in \mathcal{P}([a,b])$, s.t. $\forall P \supseteq P_\varepsilon^{(1)}$, tagged pts t, $\left|\sum_{k=1}^n [f(t_k) - f(t_k')] \Delta\alpha_k\right| \leq \varepsilon$.

Let $M = \sup |f|$, $P_\varepsilon^{(2)} \in \mathcal{P}([a,b])$, s.t. $\forall P \supseteq P_\varepsilon^{(2)}$, $V(b) = V_\alpha([a,b]) \leq V_P(\alpha) + \frac{\varepsilon}{M}$

V nondecreasing $\Rightarrow$ check Riemann condition: $\sum_{k=1}^n [M_k(f) - m_k(f)] \Delta V_k = \underbrace{\sum_{k=1}^n [M_k(f) - m_k(f)] |\Delta\alpha_k|}_{(*)} + \underbrace{\sum_{k=1}^n [M_k(f) - m_k(f)][\Delta V_k - |\Delta\alpha_k|]}_{(**)}$

$\hookrightarrow$ $(**) \leq 2M \sum_{k=1}^n (\Delta V_k - |\Delta\alpha_k|) = 2M \left(V(b) - \sum_{k=1}^n |\Delta\alpha_k|\right) \leq 2\varepsilon$ $(P_\varepsilon^{(2)})$

$\hookrightarrow$ $K^+ := \{1 \leq k \leq n \mid \Delta\alpha_k \geq 0\}$, $K^- := \{1 \leq k \leq n \mid \Delta\alpha_k < 0\}$, $\varepsilon' = \frac{\varepsilon}{V(b)}$

$\Rightarrow$ $k \in K^+$: Choose $t_k, t_k' \in [x_{k-1}, x_k]$, s.t. $f(t_k) - f(t_k') \geq M_k(f) - m_k(f) - \varepsilon'$; $k \in K^-$: $f(t_k') - f(t_k) \geq M_k(f) - m_k(f) - \varepsilon'$

$\Rightarrow$ $(*) = \sum_{k \in K^+} [M_k(f) - m_k(f)] \Delta\alpha_k + \sum_{k \in K^-} [M_k(f) - m_k(f)][-\Delta\alpha_k] \leq \sum_{k=1}^n [f(t_k) - f(t_k')] \Delta\alpha_k + \varepsilon' \sum_{k=1}^n |\Delta\alpha_k| \leq 2\varepsilon$ □

"$V \Rightarrow \alpha$": $\forall i$, $|\alpha(x_i) - \alpha(x_{i-1})| \leq V(x_i) - V(x_{i-1})$, so $\sum_{k=1}^n [M_k(f) - m_k(f)] |\alpha(x_k) - \alpha(x_{k-1})| \leq \sum_{k=1}^n [M_k(f) - m_k(f)] |V(x_k) - V(x_{k-1})|$ ✓

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9) Let $\alpha:[a,b]\to\mathbb{R}$ be a nondecreasing function and $f:[a,b]\to\mathbb{R}$ be a bounded function. Consider the statements:

(i) $f\in\mathcal{R}(\alpha;a,b)$

(ii) $f\in\mathcal{R}(x;a,b)$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

D

(had to check answer)

(i) $\not\Rightarrow$ (ii), because if $\alpha=0$, $f\in\mathcal{R}(\alpha;a,b)$ $\forall f$, even $f\notin\mathcal{R}(x;a,b)$. (ii) $\not\Rightarrow$ (i), because $f=\alpha=$ step discontinuity is not integrable ($\because$ they have the same discontinuities)

10) Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $\alpha:[a,b]\to\mathbb{R}$ be a function. Consider the statements:

(i) $f\in\mathcal{R}(\alpha;a,b)$

(ii) α is of bounded variation

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

B

(i) $\not\Rightarrow$ (ii), say we take $f=0$, $\alpha\notin BV([a,b])$.

(ii) $\Rightarrow$ (i):

Proof sketch

Let $\varepsilon>0$. $[a,b]$ cpt $\Rightarrow$ f unif cont. $\Rightarrow$ Take $\delta>0$, s.t. $\forall x,y\in[a,b]$, $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$

Take partition $P\supseteq P_\varepsilon$, $\|P_\varepsilon\|<\delta$, then $U_P(f,\alpha)-L_P(f,\alpha)=\sum_{k=1}^{n}[M_k(f)-m_k(f)]\Delta\alpha_k\le\varepsilon\sum_{k=1}^{n}\Delta\alpha_k=\varepsilon[\alpha(b)-\alpha(a)]\Rightarrow$ Riemann cond $\square$

11) Let $f:[a,b]\to\mathbb{R}$ be of bounded variation. Consider the statements:

(i) $f\in\mathcal{R}(f;a,b)$

(ii) f is continuous

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

C

(ii) $\Rightarrow$ (i) is a direct result of what I just proved, f is BV, f is cont. $\Rightarrow f\in\mathcal{R}(f;a,b)$

(i) $\Rightarrow$ (ii):

Proof sketch

Consider "f not cont. $\Rightarrow f\notin\mathcal{R}(f;a,b)$"

WLOG f not right cont. at c, $\exists\,\varepsilon>0$, $\delta>0$, s.t. $\exists\,x\in(c,c+\delta)$, $|f(x)-f(c)|>\varepsilon$

Let $P\in\mathcal{P}([a,b])$, $x_i=c$, $x_{i+1}=y$, $1\le i\le n-1$. Then, $U_P(f,\alpha)-L_P(f,\alpha)=\sum_{k=1}^{n}[M_k(f)-m_k(f)]\Delta f_k\ge\varepsilon[M_{i+1}(f)-m_{i+1}(f)]\ge\varepsilon^2$ $\square$

12) [True or false] Let $f:[a,b]\to\mathbb{R}$ be a Riemann-integrable function. Then, $F(x)=\int_a^x f(t)\,dt$ is a Lipschitz function on $[a,b]$.

True (This is a direct result of MVT: $\forall x,y\in[a,b]$, $|F(y)-F(x)|=|\int_x^y f(t)\,dt|=c|y-x|$, $c\in(x,y)\Rightarrow|F(y)-F(x)|\le K|y-x|$ for some K $\forall y,x$)

Proof sketch (of MVT)

$L_P(f,\alpha)\le\int_a^b f\,d\alpha\le U_P(f,\alpha)$, $m[\alpha(b)-\alpha(a)]\le L_P(f,\alpha)$, $M[\alpha(b)-\alpha(a)]\le U_P(f,\alpha)\Rightarrow m\int_a^b d\alpha\le\int_a^b f\,d\alpha\le M\int_a^b d\alpha$

13) Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Consider the statements: Shun/翔海(@shun4midx)

(i) The set of discontinuities of f is a measure zero set.

(ii) There is a continuous function $g:[a,b]\to\mathbb{R}$ such that $\{x\in[a,b]\,/\,f(x)\neq g(x)\}$ is a measure zero set.

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

D (Had to check answer for counterexample)

(i) $\not\Rightarrow$ (ii): $f=\mathbb{1}_{x\geq\frac{1}{2}}$ on $[0,1]$

(ii) $\not\Rightarrow$ (i): $f=\mathbb{1}_{x\in\mathbb{Q}}$ on $[0,1]$

14) Select the correct primitives of $\frac{1}{\sqrt{x^2+1}}, \frac{1}{\sqrt{x^2-1}}, \frac{1}{\sqrt{1-x^2}}$

(A) $\sin^{-1}x$, $\sinh^{-1}x$, $\cosh^{-1}x$

(B) $\sin^{-1}x$, $\cosh^{-1}x$, $\sinh^{-1}x$

(C) $\sinh^{-1}x$, $\cosh^{-1}x$, $\sin^{-1}x$

(D) $\sinh^{-1}x$, $\sin^{-1}x$, $\cosh^{-1}x$

(E) $\cosh^{-1}x$, $\sin^{-1}x$, $\sinh^{-1}x$

(F) $\cosh^{-1}x$, $\sinh^{-1}x$, $\sin^{-1}x$

C

15) Let $(a_n)_{n\geq1}$ be a real sequence. Consider the statements:

(i) $(a_n)_{n\geq1}$ converges to 0

(ii) $\sum_{n=1}^{\infty}a_n$ converges

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

B

(i) $\not\Rightarrow$ (ii): $(a_n=\frac{1}{n})_{n\geq1}$

(ii) $\Rightarrow$ (i)

Proof sketch

Denote $S_n:=\sum_{i=1}^{n}a_i$. $\therefore (S_n)_{n\geq1}$ converges $\Rightarrow (S_n)_{n\geq1}$ is Cauchy $\Rightarrow \forall\varepsilon>0, \exists N>0$, s.t. $\forall m>n>N$, $|S_m-S_n|=|\sum_{i=n+1}^{m}a_i|<\varepsilon \Rightarrow \forall n>N, |a_{n+1}|<\varepsilon$ □

16) Let $(a_n)_{n\geq1}$ be a real sequence. Consider the statements:

(i) $(a_n)_{n\geq1}$ converges

(ii) $a_n=O(1)$ as $n\to\infty$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

A

(ii) $\not\Rightarrow$ (i): $a_n=1,2,1,2,1,\dots$

(i) $\Rightarrow$ (ii):

Proof sketch

Set $c_n=a_n\ \forall n\geq1$, as all convergent sequences are bounded, then for $b_n=1\ \forall n$, $a_n=c_nb_n$, c_n is bounded for $n\geq N$. $\therefore a_n=O(1)$ □

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17) Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two real sequences. Consider the statements:

(i) Both $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n\to\infty$.

(ii) $a_n \sim b_n$ as $n\to\infty$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i).

(B) (ii) implies (i) but (i) does not imply (ii).

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i).

B

(i) $\not\Rightarrow$ (ii): $(a_n = 1)_{n\geq 1}$ and $(b_n = 2)_{n\geq 1}$

(ii) $\Rightarrow$ (i):

Proof sketch

$a_n \sim b_n \Rightarrow \exists\, (c_n)_{n\geq 1} \xrightarrow{n\to\infty} 1$, $N>0$, s.t. $a_n = c_n b_n$ $\forall n\geq N \Rightarrow a_n = O(b_n)$. Take $(c'_n = \frac{1}{c_n})_{n\geq 1}$ for $n\geq N$, as $c_n \neq 0$, this is defined $\Rightarrow b_n = O(a_n)$. $\square$

18) Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two nonnegative real sequences such that $a_n = O(b_n)$ as $n\to\infty$. Select all correct statements:

(A) If $(a_n)_{n\geq 1}$ converges, then $(b_n)_{n\geq 1}$ converges

(B) If $(b_n)_{n\geq 1}$ converges, then $(a_n)_{n\geq 1}$ converges

(C) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges

(D) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

D

Why not A? $(a_n = 0)_{n\geq 1}$, $(b_n = n)_{n\geq 1}$

Why not B? $(b_n = n)_{n\geq 1}$, $\left(a_n = \begin{cases} 1, & \text{odd } n \\ 2, & \text{even } n \end{cases}\right)_{n\geq 1}$

Why not C? $(a_n = 0)_{n\geq 1}$, $(b_n = \frac{1}{n})_{n\geq 1}$

Why D:

Proof sketch

$a_n = O(b_n) \Rightarrow \exists N>0$, s.t. $\forall n\geq N$, $a_n = c_n b_n$ for some bounded $(c_n)_{n\geq 1}$, i.e. $c_n \leq M < \infty$. As nonneg, $0 \leq \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} c_n b_n \leq M \sum_{n=1}^{\infty} b_n$ $\square$

19) Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two nonnegative real sequences such that $a_n \sim b_n$ as $n\to\infty$. Select all correct statements:

(A) If $(a_n)_{n\geq 1}$ converges, then $(b_n)_{n\geq 1}$ converges

(B) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges

(C) If both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then $\sum_{k=1}^{n} a_k \sim \sum_{k=1}^{n} b_k$ as $n\to\infty$

(D) If both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then $\sum_{k=n}^{\infty} a_k \sim \sum_{k=n}^{\infty} b_k$ as $n\to\infty$

A, B, D

Why not C? (Had to check answer) $a_1 = 1$, $b_1 = 2$, $a_n = b_n = \frac{1}{n^2}$, $n\geq 2$ (☆Remember properly, when remainders ~ or partial sums ~)

Proof sketch

A: Let $\varepsilon > 0$, take $N>0$, s.t. $(1-\varepsilon)a_n \leq b_n \leq (1+\varepsilon)a_n$ $\forall n\geq N$ ... OK ✓

B: Same as last question

D: Let $\varepsilon>0$, take $N>0$, s.t. $(1-\varepsilon)u_n \leq v_n \leq (1+\varepsilon)v_n$. $\therefore \forall M\geq n\geq N$, $\sum_{k=n+1}^{M} v_k \leq \sum_{k=n+1}^{M} (1+\varepsilon)u_k \leq (1+\varepsilon)\sum_{k=n+1}^{\infty} u_k \Rightarrow \sum_{k=n+1}^{\infty} v_k \leq (1+\varepsilon)\sum_{k=n+1}^{\infty} u_k$. Also, $\sum_{k=n+1}^{\infty} v_k \geq (1-\varepsilon)\sum_{k=n+1}^{\infty} u_k$

20) Select the correct asymptotic notations (as $n\to\infty$) of the series $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} \frac{1}{k}$, $\sum_{k=1}^{n} \frac{1}{2^k}$

(A) $O(1)$, $O(n^2)$, $O(\log n)$

(B) $O(1)$, $O(\log n)$, $O(n^2)$

(C) $O(n^2)$, $O(1)$, $O(\log n)$

(D) $O(n^2)$, $O(\log n)$, $O(1)$

(E) $O(\log n)$, $O(1)$, $O(n^2)$

(F) $O(\log n)$, $O(n^2)$, $O(1)$

D, trivial

ボクはバかだよ∴