

Analysis II (Part 1)

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RIEMANN-STIELTJES INTEGRALS

FUNCTIONS OF BOUNDED VARIATION

DEFINITION

Let $I \subseteq \mathbb{R}$ be an interval, $f: I \rightarrow \mathbb{R}$ be a function.

- (1) f is non-increasing/decreasing if $f(x) \geq f(y) \quad \forall x \leq y, x, y \in I$
- (2) f is non-decreasing/increasing if $f(x) \leq f(y) \quad \forall x \leq y, x, y \in I$
- (3) f is monotonic if (1) or (2) holds

DEFINITION

Let $f: I \rightarrow \mathbb{R}$ be monotonic.

For $x \in I$, define:

- ↪ The left limit at x to be $f(x-) = \lim_{y \downarrow x, y > x} f(y)$ if $(x-\varepsilon, x) \cap I \neq \emptyset$ for $\varepsilon > 0$
- ↪ The right limit at x to be $f(x+) = \lim_{y \uparrow x, y < x} f(y)$ if $(x, x+\varepsilon) \cap I \neq \emptyset$ for $\varepsilon > 0$

E.g. we can't just pick a point at the boundary

PROPOSITION

Let $f: [a, b] \rightarrow \mathbb{R}$ be a monotonic function. Then, the set of its discontinuities D is a countable set.

Proof ↪ monotonic \Rightarrow left/right limits are well-defined

Define $D = \{x \in I \mid f(x-) \neq f(x+)\}$

By symmetry, wLOG, assume f is increasing, then $f(x-) \leq f(x+) \quad \forall x \in D$

↑ key! This is since $x \in D$, i.e. it is discontinuous

As \mathbb{Q} is dense in \mathbb{R} , we know $\exists q_x \in \mathbb{Q} \cap (f(x-), f(x+))$

∴ This gives us a map $\{x \mapsto q_x\} \subseteq D \rightarrow \mathbb{Q}$, which is injective because $\forall x_1, x_2 \in D, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

∴ D can be injected in the countable set \mathbb{Q}

∴ D is countable. \square

DEFINITION (PARTITIONS)

Let $a < b$ and $[a, b] \subseteq \mathbb{R}$ be a segment.

A partition or a subdivision of $[a, b]$ is a finite sequence $P = (x_k)_{0 \leq k \leq n}$ satisfying $a = x_0 < x_1 < \dots < x_n = b$, where n is the length of P . We denote $\text{Supp}(P) := \{x_k \mid 0 \leq k \leq n\}$ as the support of P .

For a finite subset $A \subseteq [a, b]$ with $a, b \in A$, we may find a partition P of $[a, b]$ s.t. $\text{Supp}(P) = A$. This is called the partition corresponding to A .

We say $[x_{k-1}, x_k]$ is the k^{th} subinterval of P , $\Delta x_k := x_k - x_{k-1}$, $1 \leq k \leq n$. Then, we say the mesh size of P is $\|P\| := \max_{1 \leq k \leq n} \Delta x_k$

This is not a norm!

let P, P' be partitions. If $\text{Supp}(P) \subseteq \text{Supp}(P')$, then we say P' is finer than P , and we say $P \leq P'$. This also implies $\|P\| \leq \|P'\|$

Let P_1, P_2 be partitions. Define their joint partition or smallest common refinement to be $P := P_1 \vee P_2$, which is the partition P with support $= \text{Supp}(P_1) \cup \text{Supp}(P_2)$

We denote $P([a, b])$ as the collection of all possible partitions of $[a, b]$

REMARK

For any $P = (x_k)_{0 \leq k \leq n} \in P([a, b])$, we have $b-a = \sum_{k=1}^n \Delta x_k$

DEFINITION (BOUNDED VARIATIONS)

Let $f: [a, b] \rightarrow \mathbb{R}$ be a function, $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a, b])$, define $\Delta f_k := f(x_k) - f(x_{k-1})$ for $1 \leq k \leq n$.

Define $V_P(f) := \sum_{k=1}^n |\Delta f_k|$ and $V_f = V_f([a, b]) = \sup_{P \in \mathcal{P}([a, b])} V_P(f) \in [0, \infty]$ to be the total variation of f . We say that f is of bounded variation if $V_f < \infty$.

We write $\mathcal{BV}([a, b]) = \mathcal{BV}([a, b], \mathbb{R})$ for the collection of such functions defined on $[a, b]$.

EXAMPLE

Consider the function $f: [0, 1] \rightarrow \mathbb{R}$, $f(x) = \begin{cases} x \cos(\frac{\pi}{x}), & x \in (0, 2\pi] \\ 0, & x = 0 \end{cases}$

For $n \geq 1$, consider the partition P with support $\{0, \frac{1}{2n}, \frac{1}{2n-1}, \dots, \frac{1}{2}, 1\}$, i.e. $x_0 = 0, x_k = \frac{1}{2n+k-1} \forall 1 \leq k \leq 2n$.

By completeness, we find $V_P(f) = \sum_{k=1}^{2n} |\Delta f_k| = \left| \frac{(-1)^{k-1}}{2n} - 0 \right| + \sum_{k=2}^{2n} \left| \frac{(-1)^{k-1}}{2n+k-1} - \frac{(-1)^{k-2}}{2n+k-2} \right| = \frac{1}{2n} + \sum_{k=2}^{2n} \left(\frac{1}{2n+k-1} + \frac{1}{2n+k-2} \right) = 1 + \sum_{k=2}^{2n-1} \frac{2}{2n+k} + \frac{1}{2n}$, which is not bounded for $n \geq 1$. So f is not of bounded variation.

PROPOSITION

Let $f \in \mathcal{BV}([a, b], \mathbb{R})$, then

(1) For any partitions $P \subseteq P'$, we have $V_P(f) \leq V_{P'}(f)$

(2) $\forall \varepsilon > 0$, \exists partition $P_\varepsilon \in \mathcal{P}([a, b])$ s.t. \forall finer partition $P \supseteq P_\varepsilon$, we have $V_P(f) \leq V_f \leq V_{P_\varepsilon}(f) + \varepsilon$

Proof

(1) By induction, we only need to prove this is true whenever $|Supp(P')| = |Supp(P)| + 1$

Let $P, P' \in \mathcal{P}([a, b])$ be partitions with support s.t. $Supp(P') = Supp(P) \cup \{c\}$, $x_{k-1} < c < x_k$ for some $1 \leq k \leq n$.

$$\begin{aligned} V_{P'}(f) &= \sum_{k=1, k \neq k'}^n |f(x_k) - f(x_{k-1})| + |f(c) - f(x_{k-1})| + |f(x_k) - f(c)| \\ &\geq \sum_{k=1, k \neq k'}^n |f(x_k) - f(x_{k-1})| + |f(x_k) - f(x_{k-1})| \\ &= \sum_{k=1}^n |\Delta f_k| = V_P(f) \quad \checkmark \end{aligned}$$

\therefore By induction, the statement holds. \square

(2) Let $\varepsilon > 0$, by the characterization of supremum, we can find $P_\varepsilon \in \mathcal{P}([a, b])$ s.t. $V_f \leq V_{P_\varepsilon}(f) + \varepsilon$ " $\forall \varepsilon > 0, \exists P' \in \mathcal{P}([a, b])$ s.t. $V_f \leq V_{P'}(f) + \varepsilon$ "

$\therefore \forall$ finer partitions $P \supseteq P_\varepsilon$, from (1), $V_f \leq V_{P_\varepsilon}(f) + \varepsilon \leq V_P(f) + \varepsilon \quad \square$

directly follows from $V_f = \sup_{P \in \mathcal{P}([a, b])} V_P(f)$.

PROPOSITION

If $f: [a, b] \rightarrow \mathbb{R}$ is monotonic, then $f \in \mathcal{BV}([a, b])$ and $V_f = |f(b) - f(a)|$

Proof

WLOG, assume that f is increasing.

Then, $\forall P \in \mathcal{P}([a, b])$, $V_P(f) = \sum_{k=1}^n |\Delta f_k| = \sum_{k=1}^n \Delta f_k = f(b) - f(a)$, which is independent of P

$\therefore f \in \mathcal{BV}([a, b])$ and $V_f = |f(b) - f(a)| \quad \square$

PROPOSITION

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and differentiable on (a, b) with bounded derivative, then $f \in \mathcal{BV}([a, b])$

Proof

Let $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a, b])$ be a partition, then $V_P(f) = \sum_{k=1}^n |\Delta f_k| = \sum_{k=1}^n |f'(t_k)| \Delta x_k \stackrel{MVT}{\leq} \sup_{t \in [a, b]} |f'(t)| \sum_{k=1}^n \Delta x_k = \sup_{t \in [a, b]} |f'(t)| (b-a) \quad \square$

USEFUL PROPERTIES

- Monotonic \Rightarrow bounded variation
- Continuous + bounded derivative \Rightarrow bounded variation
- $C^1 \Rightarrow$ bounded variation

PROPERTIES

PROPOSITION

$$\text{BV}([a, b], \mathbb{R}) \subseteq \mathcal{B}([a, b], \mathbb{R})$$

ProofLet $f \in \text{BV}([a, b], \mathbb{R})$, $M := V_f([a, b]) < +\infty$ Fix $x \in (a, b)$ and consider partition $P = (a, x, b)$, then we have $V_P(f) \leq V_f([a, b]) = M$ Write $V_P(f) = |f(x) - f(a)| + |f(b) - f(x)| \stackrel{\text{△}}{\geq} |f(b) - f(a)| \geq |f(b)| - |f(a)|$ \therefore This implies $|f(x)| \leq |f(a)| + V_P(f) \leq |f(a)| + M$, i.e. $|f|$ is bounded by $\max\{|f(a)| + M, |f(b)|\}$. \square Remark: " \geq " does not hold! This feels like foreshadowing for $\int 1 \cdot 1 dx$ may not converge even if $\int dx$ converges

PROPOSITION

Let $f, g \in \text{BV}([a, b], \mathbb{R})$, then $f+g, fg, cf$ ($c \in \mathbb{R}$) are all of bounded variationProofSince it is the same idea for the other two, $f+g, cf$ proofs are not shown.Let $P = (x_k)_{k=0}^{n-1} \in \mathcal{P}([a, b])$, then we have:

$$\begin{aligned} V_P(f+g) &= \sum_{k=0}^{n-1} |f(x_k) + g(x_k) - f(x_{k-1}) - g(x_{k-1})| \\ &\leq \sum_{k=0}^{n-1} |f(x_k) - f(x_{k-1})| + \sum_{k=0}^{n-1} |g(x_k) - g(x_{k-1})| \\ &= V_P(f) + V_P(g) \\ &\leq V_f([a, b]) + V_g([a, b]) \quad (\text{const}) \end{aligned}$$

 $\therefore f+g \in \text{BV}([a, b]) \checkmark$ Additionally, by taking sup over $P \in \mathcal{P}([a, b])$, we obtain $V_{f+g} \leq V_f + V_g$ For the multiplication fg , we also fix $P \in \mathcal{P}([a, b])$. \checkmark key technique

$$\begin{aligned} V_P(fg) &= \sum_{k=0}^{n-1} |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1}) - f(x_k)g(x_{k-1}) + f(x_k)g(x_{k-1})| \\ &\leq \sum_{k=0}^{n-1} |f(x_k)||g(x_k) - g(x_{k-1})| + |g(x_k)||f(x_k) - f(x_{k-1})| \\ &\leq \sup(f) \sum_{k=0}^{n-1} |\Delta g_k| + \sup(g) \sum_{k=0}^{n-1} |\Delta f_k| \\ &\stackrel{\infty}{\leq} V_f V_g \leq V_g \quad \stackrel{\infty}{\leq} V_f V_g \leq V_f \end{aligned}$$

 \therefore It doesn't depend on the partition to be bounded $\therefore fg \in \text{BV}([a, b]) \checkmark$ Again, if we take sup over $P \in \mathcal{P}([a, b])$, we obtain $V_{fg} \leq \sup(f)V_g + \sup(g)V_f \quad \square$

PROPOSITION

Let $f \in \text{BV}([a, b])$ with $|f| \geq m > 0$ for some $m \in \mathbb{R}$, then $g = \frac{1}{f} \in \text{BV}([a, b])$ ProofLet $P \in \mathcal{P}([a, b])$. Write $P = (x_k)_{k=0}^{n-1}$.

$$V_P(g) = \sum_{k=0}^{n-1} \left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right| = \sum_{k=0}^{n-1} \left| \frac{\Delta f_k}{f(x_k)f(x_{k-1})} \right| \leq \frac{1}{m^2} \sum_{k=0}^{n-1} |\Delta f_k| = \frac{V_f(f)}{m^2} \leq \frac{V_f(f)}{m^2} \quad (\text{const})$$

Need to turn $V_P(f)$ into V_f to create an upper bound independent of the input

PROPOSITION

Let $f \in BV([a, b])$ and $c \in (a, b)$. Then, $f \in BV([a, c])$, $f \in BV([c, b])$, and $V_f([a, b]) = V_f([a, c]) + V_f([c, b])$

✓ strong enough for equality to hold! Shun / 舒海 (@shun4midx)

Proof

Let $P_1 \in \mathcal{P}([a, c])$ and $P_2 \in \mathcal{P}([c, b])$.

Define $P := P_1 \cup P_2$ to be the partition with support $\text{Supp}(P_1) \cup \text{Supp}(P_2)$. Then, we have $V_P(f) = V_{P_1}(f) + V_{P_2}(f)$ ✓ equality holds since P_1 and P_2 don't have overlapping intervals

$$\therefore V_P(f) = V_{P_1}(f) + V_{P_2}(f) \leq V_f([a, b]) \leq M < \infty$$

✓ (First prove they are indeed bounded)

Now, for " $V_f([a, b]) = V_f([a, c]) + V_f([c, b])$ ",

✗ since partitions usually don't contain c

" \leq ": Take the sup over $P_1 \in \mathcal{P}([a, c])$, and sup over $P_2 \in \mathcal{P}([c, b])$, $V_{P_1}(f) + V_{P_2}(f) \leq V_f([a, b]) \Rightarrow V_f([a, c]) + V_f([c, b]) \leq V_f([a, b])$ ✓

" \geq ": Fix $P \in \mathcal{P}([a, b])$.

Define P' to be the partition with support $\text{Supp}(P) \cup \{c\}$

From the previous notes, we know that $V_P(f) \leq V_{P'}(f)$

Define $P_1 \in \mathcal{P}([a, c])$ with support $\text{Supp}(P') \cap [a, c]$, $P_2 \in \mathcal{P}([c, b])$ with support $\text{Supp}(P') \cap [c, b]$

Then, $V_P(f) \leq V_{P'}(f) = V_{P_1}(f) + V_{P_2}(f) \leq V_f([a, c]) + V_f([c, b])$

We conclude by taking sup over $P \in \mathcal{P}([a, b])$ ✓

DEFINITION

Let $f \in BV([a, b])$. Define the variation function to be $V: [a, b] \rightarrow \mathbb{R}$ ✓ because it is quite pointless to use this value for a .

$$x \mapsto \begin{cases} 0 & x=a \\ V_f([a, b]) & x \neq a \end{cases}$$

LEMMA

Let $f \in BV([a, b])$ and V be its variation function. Then, both V and $V-f$ are increasing

Proof

For V : We know for $x > a$, $V(x) \geq V(a) = 0$

Let $x, y \in [a, b]$ with $x < y$. Then, $V(y) - V(x) = V_f([a, y]) - V_f([a, x]) = V_f([x, y]) \geq 0$ ✓ by def of V_f

For $V-f$: Let $D := V-f$, $x, y \in [a, b]$, $x < y$. ✓ $\because f(y) - f(x) \leq V_f([x, y])$

Then, $D(y) - D(x) = (V(y) - V(x)) - (f(y) - f(x)) = V_f([x, y]) - (f(y) - f(x)) \geq 0$

THEOREM (Makes checking for bounded variation way easier)

Let $f: [a, b] \rightarrow \mathbb{R}$, then (a) \Leftrightarrow (b)

(a) $f \in BV([a, b])$

✓ Notice this isn't a unique decomposition

(b) } non-decreasing functions g_1 and g_2 , s.t. $f = g_1 - g_2$

Proof

(b) \Rightarrow (a): As monotonic functions are of bounded variation, thus their difference is also of bounded variation ✓

(a) \Rightarrow (b): Use the variation function V , then we know from before, V and $V-f$ are non-decreasing $\Rightarrow f = V - (V-f)$ suffices ✓

PROPOSITION

Let $f \in BV([a, b])$ and $x \in [a, b]$. Then, f is continuous at $x \Leftrightarrow V$ is continuous at x

Proof

From above, V is increasing

It suffices to prove that " $\forall x \in [a, b]$, $f(x+) = f(x) \Leftrightarrow V(x+) - V(x) = 0$ " ✓ "monotonic \Rightarrow limit exists"

We know the right limits $f(x+)$, $V(x+)$ are well-defined since V is increasing and $f = V - (V-f)$, V , $V-f$ are increasing

• Suppose that $V(x+) = V(x)$, i.e. V is continuous at x from the right

We note that for $y > x$, $0 \leq |f(y) - f(x)| \leq |V_f([x, y])| = |V(y) - V(x)|$.

Take the limit $y \rightarrow x+$, we find $|f(x+) - f(x)| \leq |V(x+) - V(x)| = 0$, hence $f(x+) = f(x)$

- Suppose that $f(x+\delta) = f(x)$, we need to show that $V(x+\delta) = V(x)$ definition of limit
 Let $\epsilon > 0$. By the right continuity of f at x , we may find $\delta > 0$, s.t. $y \in [x, x+\delta] \Rightarrow |f(y) - f(x)| < \epsilon$

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By the characterization of the total variation, we can take $P_\epsilon \in P([a, b])$ s.t. $\forall P \geq P_\epsilon$, we have $V_P(f) \leq V_f([a, b]) \leq V_P(f) + \epsilon$ Quite a reoccurring inequality
 $|f(x_i) - f(x_0)| = |f(x_i) - f(x)| < \epsilon$
 let $P \geq P_\epsilon$ s.t. $x_i \in [x, x+\delta]$. Then, $V_f([x, b]) \leq V_P(f) + \epsilon = |\Delta f| + \sum_{k=2}^n |\Delta f_k| + \epsilon \leq V_f([x, b]) + 2\epsilon$
 $\therefore V_f([x, b]) - V_f([x, b]) \leq 2\epsilon$

$$\text{Now, LHS} = V_f([x, x_1]) = V_f([a, x]) - V_f([a, x_1]) = V(x) - V(x_1) \leq 2\epsilon \quad \square$$

REMARK

For the theorem above, we can actually add this condition:

Let $f: [a, b] \rightarrow \mathbb{R}$, f is continuous, then (a) \Leftrightarrow (b)

(a) $f \in BV([a, b])$

(b) \exists two non-decreasing continuous functions g_1, g_2 , s.t. $f = g_1 - g_2$

Proof

It suffices to show (a) \Rightarrow (b).

Recall we proved it before using $f = V - (V - f)$.

Now, we know from the above proposition, V and f share the same continuities, thus so does $V - f$ \square

RIEMANN-STIELTJES INTEGRALS

Let $[a, b] \subseteq \mathbb{R}$ be a segment

Let $f, g, \alpha, \beta: [a, b] \rightarrow \mathbb{R}$ be bounded functions

DEFINITION

Let $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a, b])$. For every $1 \leq k \leq n$, take $t_k \in [x_{k-1}, x_k]$ and write $t = (t_k)_{0 \leq k \leq n}$. We call (P, t) a **tagged partition**, t contains tagged points of P .

Define the **Riemann-Stieltjes sum** of f w.r.t. α for (P, t) , $S_{P,t}(f, \alpha) = \sum_{k=1}^n f(t_k) \Delta \alpha_k = \sum_{k=1}^n f(t_k)(\alpha(x_k) - \alpha(x_{k-1}))$

Consider the following condition:

(RS) : $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists P \in \mathcal{P}([a, b])$ s.t. $\forall P \subseteq P$, tagged points t of P , we have $|S_{P,t}(f, \alpha) - L| < \varepsilon$

If (RS) is satisfied, we say that f is **Riemann-Stieltjes integrable** and write this unique L to be its integral, denoted $\int_a^b f d\alpha = \int_a^b f(x) d\alpha(x)$. We write $R(\alpha; a, b) = R(\alpha)$ for the set of functions f satisfying (RS).

REMARK

(1) f is called **integrand**, α is called **integrator**

(2) When $\alpha(x) = x$, we recover the notation from Riemann-integrability. We write $R(x; a, b) = R(x)$ for the set of Riemann-integrable functions

(3) We may also have x_{k-1}, x_k instead of x_{k-1}, t_k for $1 \leq k \leq n$. This allows us to use the same notation when $a = b$.

(4) Let V be a finite-dimensional vector space over \mathbb{R} , i.e. $V = \mathbb{C}$ or \mathbb{R}^d . Fix a basis (e_1, \dots, e_n) of V , we may write $f = \sum_{i=1}^n f_i e_i$, where $f: [a, b] \rightarrow \mathbb{R}$ is a real valued function. If $\int_a^b f d\alpha$ is well-defined $\forall i$, we may set $\int_a^b f d\alpha = \sum_{i=1}^n (\int_a^b f_i d\alpha) e_i$:

EXAMPLES

(1) If $\alpha: [a, b] \rightarrow \mathbb{R}$ is a **constant function**, for any bounded function $f: [a, b] \rightarrow \mathbb{R}$, $S_{P,t}(f, \alpha) = 0$ for all tagged partitions $P \in \mathcal{P}([a, b])$. This means that (RS) holds and $\int_a^b f d\alpha = 0$

(2) When $\alpha(x) = x$, all **continuous functions** are Riemann-integrable, i.e. $C([a, b]) \subseteq R(x; a, b)$

(3) Let $f, \alpha: [-1, 1] \rightarrow \mathbb{R}$ to be $f = \alpha = \mathbf{1}_{x \geq 0}$. Consider a partition $P \in \mathcal{P}([-1, 1])$ with $x_k = 0$ for some k . For any tagged points t of P , we have $S_{P,t}(f, \alpha) = f(t_k) \Delta \alpha_k = f(t_k) = \begin{cases} 1, & t_k = x_k = 0 \\ 0, & t_k \neq x_k \end{cases}$. This implies that (RS) does not hold (**KEY CONSTRUCTION EXAMPLE**)

LEMMA

Consider the following condition,

(RS') : $\exists L \in \mathbb{R}$, s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall P \in \mathcal{P}([a, b])$ with $\max_{1 \leq k \leq n} |x_k - x_{k-1}| = \|P\| < \delta$, any tagged points t , we have $|S_{P,t}(f, \alpha) - L| < \varepsilon$. We have (RS') \Rightarrow (RS)

REMARK It is true for $\alpha = x$ though

In general, (RS) \nRightarrow (RS'). Consider $f, \alpha: [-1, 1] \rightarrow \mathbb{R}$, $f, \alpha = \mathbf{1}_{x \geq 0}$

- (RS) holds
- Let $\delta \in (0, 1)$ and $P \in \mathcal{P}([0, 1])$, s.t. $\|P\| < \delta$, there exists k s.t. $x_{k-1} = x_k = \frac{\delta}{2}$. Then, $S_{P,t}(f, \alpha) = f(t_k) \Delta \alpha_k = f(t_k) = \begin{cases} 1, & t_k \in (0, \frac{\delta}{2}) \\ 0, & t_k \in [\frac{\delta}{2}, 1] \end{cases}$ \therefore (RS') does not hold.

PROPOSITION (Linearity in integrand)

Let $\alpha: [a, b] \rightarrow \mathbb{R}$ be bounded, $f, g \in R(\alpha)$. Then, $\forall c \in \mathbb{R}$, $f+cg \in R(\alpha)$ and $\int_a^b (f+cg) d\alpha = \int_a^b f d\alpha + c \int_a^b g d\alpha$

As a consequence, $R(\alpha)$ is an \mathbb{R} -vector space, the integral operator $I: R(\alpha) \rightarrow \mathbb{R}$ is a **linear form**, i.e. $I \in \mathcal{L}(R(\alpha), \mathbb{R})$

$$f \mapsto \int_a^b f d\alpha$$

Proof

Fix $c \in \mathbb{R}$ and let $h = f+cg$. Since $f \in R(\alpha)$, we may find $P' \in \mathcal{P}([a, b])$, s.t. $|S_{P',t}(f, \alpha) - \int_a^b f d\alpha| < \varepsilon$ $\forall P \supseteq P'$ and tagged points of P .

Similarly, take $P_\varepsilon'' \in P([a, b])$, s.t. $|S_{P_\varepsilon''} + f(g, \alpha) - \int_a^b g d\alpha| < \varepsilon$

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Take $P_\varepsilon = P_\varepsilon' \vee P_\varepsilon''$, then for $P_2 P_\varepsilon$ and any tagged points t of P , we have $|S_{P,t}(f, \alpha) - \int_a^b f d\alpha| < \varepsilon$ and $|S_{P,t}(g, \alpha) - \int_a^b g d\alpha| < \varepsilon$

Moreover, $S_{P,t}(h, \alpha) = \sum_{k=1}^n h(t_k) \Delta \alpha_k = \sum_{k=1}^n [f(t_k) + cg(t_k)] \Delta \alpha_k = S_{P,t}(f, \alpha) + c S_{P,t}(g, \alpha)$

$$\therefore |S_{P,t}(h, \alpha) - \int_a^b f d\alpha - c \int_a^b g d\alpha| \leq |S_{P,t}(f, \alpha) - \int_a^b f d\alpha| + |c||S_{P,t}(g, \alpha) - \int_a^b g d\alpha| \leq (1+|c|)\varepsilon$$

This means that $h=f+cg$ satisfies (RS), and we have $\int_a^b h d\alpha = \int_a^b f d\alpha + c \int_a^b g d\alpha \quad \square$

PROPOSITION (Linearity in integrator)

Let $\alpha, \beta: [a, b] \rightarrow \mathbb{R}$ be bounded, $f \in R(\alpha) \cap R(\beta)$. Then, $\forall c \in \mathbb{R}$, we have $f \in R(\alpha+c\beta)$ and $\int_a^b f d(\alpha+c\beta) = \int_a^b f d\alpha + c \int_a^b f d\beta$

(Proof is very similar to above)

DEFINITION

For a, b , any bounded function $\alpha: [a, b] \rightarrow \mathbb{R}$, $f \in R(\alpha; a, b)$, we define $\int_b^a f d\alpha = - \int_a^b f d\alpha = - \int_a^b f(x) d\alpha(x)$. We also write $R(\alpha; a, b) = R(\alpha; b, a)$. When $a=b$, $\int_a^a f d\alpha = 0$ for any bounded function f defined on $a=b$, so $R(a; a, a) \equiv \mathbb{R}$

PROPOSITION

Let $I \subseteq \mathbb{R}$ be a segment, $a, b, c \in I$. Let $\alpha: [a, b] \rightarrow \mathbb{R}$ be bounded, $f \in R(\alpha; a, b) \cap R(b, c)$. Then, we have $f \in R(\alpha; a, c)$ and $\int_a^c f d\alpha = \int_a^b f d\alpha + \int_b^c f d\alpha$ (if $a=b$ or $b=c$, then it's trivially true)

Proof

By symmetry + notation from above, WLOG, assume $a < b < c$.

- Since $f \in R(\alpha; a, b)$, we may take $P_\varepsilon^{(a,b)} \in P([a, b])$, s.t. $|S_{P_\varepsilon^{(a,b)}} + f(\alpha) - \int_a^b f d\alpha| < \varepsilon$ for any $P^{(a,b)} \supseteq P_\varepsilon^{(a,b)}$ and tagged points $t^{(a,b)}$
- Similarly, take $P_\varepsilon^{(b,c)} \in P([b, c])$, s.t. $|S_{P_\varepsilon^{(b,c)}} + f(\alpha) - \int_b^c f d\alpha| < \varepsilon$ for any $P^{(b,c)} \supseteq P_\varepsilon^{(b,c)}$ and tagged points $t^{(b,c)}$

Then, define $P_\varepsilon = P_\varepsilon^{(a,b)} \vee P_\varepsilon^{(b,c)}$ and take $P \supseteq P_\varepsilon$ and tagged points t , let $P^{(a,b)} = P \cap [a, b]$, $P^{(b,c)} = P \cap [b, c]$, $t^{(a,b)} = t \cap [a, b]$, $t^{(b,c)} = t \cap [b, c]$. Then, $S_{P,t}(f, \alpha) = S_{P_\varepsilon^{(a,b)}, t}(f, \alpha) + S_{P_\varepsilon^{(b,c)}, t}(f, \alpha)$. So by triangle inequality, $|S_{P,t}(f, \alpha) - \int_a^c f d\alpha| < 2\varepsilon \quad \square$

PROPOSITION (Integration by Parts)

Let $f \in R(\alpha; a, b)$. Then, $\alpha \in R(f; a, b)$, and we have $\int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a)$

(First prove $\int \alpha df$ is Riemann-Stieltjes integrable, then check that $|\int_a^b \alpha df - f(b)\alpha(b) + f(a)\alpha(a) + \int_a^b f d\alpha|$ is small)

PROPOSITION (Integration by Parts)

Let $f \in R(\alpha; a, b)$. Then, $\alpha \in R(f; a, b)$, and we have $\int_a^b f d\alpha + \int_a^b \alpha d f = f(b)\alpha(b) - f(a)\alpha(a)$

Proof

Let $\epsilon > 0$, take $P \in P([a, b])$, s.t. $\forall P \supseteq P_\epsilon$ and tagged points, $|S_{P_\epsilon}(f, \alpha) - \int_a^b f d\alpha| < \epsilon$ (★)

Consider $P \supseteq P_\epsilon$, tagged points t of P , we write $S_{P_\epsilon}(f, \alpha) = \sum_{k=1}^n \alpha(t_k) \Delta t_k = \sum_{k=1}^n \alpha(t_k) [f(x_k) - f(x_{k-1})]$

Note that $f(b)\alpha(b) - f(a)\alpha(a) = \sum_{k=1}^n [\alpha(t_k)f(x_k) - f(x_{k-1})\alpha(x_{k-1})]$, i.e. $S_{P_\epsilon}(f, \alpha) - [f(b)\alpha(b) - f(a)\alpha(a)] = \sum_{k=1}^n [\alpha(t_k)[f(x_k) - f(x_{k-1})] - (f(x_k)\alpha(x_k) - f(x_{k-1})\alpha(x_{k-1}))]$

$$\therefore S_{P_\epsilon}(f, \alpha) - [f(b)\alpha(b) - f(a)\alpha(a)] = \sum_{k=1}^n f(x_k)[\alpha(t_k) - \alpha(x_k)] + \sum_{k=1}^n f(x_{k-1})[\alpha(x_{k-1}) - \alpha(t_k)]$$

Now, by taking a new partition, $Q = (x_0, t_1, x_1, t_2, x_2, t_3, \dots)$, $S = (x_0, x_1, x_2, x_3, x_4, \dots)$, then $S_{P_\epsilon}(f, \alpha) - [f(b)\alpha(b) - f(a)\alpha(a)] = -S_{Q, S}(f, \alpha)$

The partition Q is finer than P so also finer than P_ϵ .

From (★), we find that $| -S_{Q, S}(f, \alpha) + \int_a^b f d\alpha | < \epsilon \Leftrightarrow | S_{P_\epsilon}(f, \alpha) - [f(b)\alpha(b) - f(a)\alpha(a)] | < \epsilon$

∴ This shows that $\alpha \in R(f)$ and $\int_a^b \alpha d f = f(b)\alpha(b) - f(a)\alpha(a)$ □

PROPOSITION (Change of Variables)

↗ necessary condition for any change of variables of RS integrals

Let c, d and $g: [c, d] \rightarrow \mathbb{R}$ is a continuous injective monotonic function.

Define $a = g(c)$, $b = g(d)$. Given $f \in R(\alpha; a, b) = R(\alpha; b, n)$, define $h(x) = f(g(x))$, $\beta(x) = \alpha(g(x))$, $\forall x \in [c, d]$.

Then, $h \in R(\beta; c, d)$ and $\int_c^d f(t) d\alpha(t) = \int_a^b f d\alpha = \int_c^d h d\beta = \int_c^d f(g(x)) d\alpha(g(x))$

Proof

WLOG, suppose that g is a strictly increasing function. In particular, g is bijective.

For any tagged partition (P, t) of $[a, b]$, define its "image" tagged position (P', t') under g^{-1} as below, $P' = (y_k)_{0 \leq k \leq n}$, $t' = (t'_k)_{0 \leq k \leq n}$, with $y_k = g^{-1}(x_k)$, $t'_k = g^{-1}(t_k)$

$$\text{Then, } S_{P', t'}(h, \beta) = \sum_{k=1}^n h(t'_k) [\beta(y_k) - \beta(y_{k-1})] = \sum_{k=1}^n f(g(t'_k)) [\alpha(g(y_k)) - \alpha(g(y_{k-1}))] = \sum_{k=1}^n f(t_k) [\alpha(x_k) - \alpha(x_{k-1})] = S_{P, t}(f, \alpha)$$

∴ (h, β) satisfies (RS) $\Leftrightarrow (f, \alpha)$ satisfies (RS). Moreover, they have the same limit, i.e. $\int_a^b f d\alpha = \int_c^d h d\beta$. □

PROPOSITION

Let $f \in R(\alpha; a, b)$. Suppose that α is C^1 . Then, $f \alpha' \in R(x; a, b)$ and we have the identity $\int_a^b f(x) d\alpha(x) = \int_a^b f(x) \alpha'(x) dx$

Proof

Let $\epsilon > 0$ and $P_\epsilon^{(1)} \in P([a, b])$, s.t. $|S_{P_\epsilon^{(1)}}(f, \alpha) - \int_a^b f d\alpha| < \epsilon$ $\forall P \supseteq P_\epsilon^{(1)}$ and tagged points t .

Let $g(x) = f(x)\alpha'(x)$ for $x \in [a, b]$.

↗ $\exists \delta_0$, s.t. $\Delta x_k = \delta_k \alpha'_k$

For a fixed tagged partition (P, t) with $P \supseteq P_\epsilon^{(1)}$, we have $S_{P, t}(g, x) = \sum_{k=1}^n g(t_k) \Delta x_k$ and $S_{P, t}(f, \alpha) = \sum_{k=1}^n f(t_k) [\alpha(x_k) - \alpha(x_{k-1})]$

By taking their difference, we have $S_{P, t}(f, \alpha) - S_{P, t}(g, x) = \sum_{k=1}^n f(t_k) [\alpha'(S_k) - \alpha'(t_k)] \Delta x_k$

Since α' is continuous on $[a, b]$, it is also uniformly continuous

Let us take $\delta > 0$, s.t. $\forall s, t \in [a, b]$, $|s - t| < \delta \Rightarrow |\alpha'(s) - \alpha'(t)| < \epsilon$

Thus, by taking $P_\epsilon := P_\epsilon^{(1)} \vee P_\epsilon^{(2)}$ with $P_\epsilon^{(2)} \in P([a, b])$ with $\|P_\epsilon^{(2)}\| \leq \frac{\delta}{M}$,

Then, $\forall P \supseteq P_\epsilon$, we also have $\|P\| \leq \frac{\delta}{M}$, and $|S_{P, t}(f, \alpha) - S_{P, t}(g, x)| \leq \sum_{k=1}^n M \epsilon \cdot \Delta x_k = M(b-a) = \epsilon$, with $M = \sup_{x \in [a, b]} |f(x)| < \infty$ □

COROLLARY

If we take $f \equiv 1$, we find $\alpha(b) - \alpha(a) = \int_a^b d\alpha(x) = \int_a^b \alpha'(x) dx$, which is the second fundamental theorem of calculus

STEP FUNCTION INTEGRATORS

DEFINITION

Given a function $\alpha: [a, b] \rightarrow \mathbb{R}$, it is called a step function if there is a partition $P = P([a, b])$ s.t. $f|_{(x_{k-1}, x_k)}$ is constant for $1 \leq k \leq n$.

We define the jump at x_k to be $\alpha_k := \alpha(x_k+) - \alpha(x_k-)$, with $\alpha_0 := \alpha(x_0+) - \alpha(x_0-)$ and $\alpha_n := \alpha(x_n+) - \alpha(x_n-)$

LEMMA

Let $c \in [a, b]$, and $\alpha: [a, b] \rightarrow \mathbb{R}$ be defined by $\alpha(x) = \begin{cases} \alpha(a), & a \leq x < c \\ \alpha(b), & c \leq x \leq b \end{cases}$

Let $f: [a, b] \rightarrow \mathbb{R}$, s.t. (1) f or α is continuous from the left at c
 (2) f or α is continuous from the right at c

Then, $f \in R(\alpha; a, b)$ and we have $\int_a^b f d\alpha = f(c)[\alpha(c+) - \alpha(c-)]$

Proof

Let $P \in P([a, b])$, s.t. $x_k = c$ for some $1 \leq k \leq n-1$

Consider a choice of tagged points t at P , we have $S_{P,t}(f, \alpha) = f(t_k)[\alpha(x_k) - \alpha(x_{k-1})] + f(t_{k+1})[\alpha(x_{k+1}) - \alpha(x_k)]$

$$\begin{aligned} \Delta &= \Delta(P, t) = S_{P,t}(f, \alpha) - f(c)[\alpha(c+) - \alpha(c-)] \\ &= [f(t_k) - f(c)][\alpha(c) - \alpha(c-)] + [f(t_{k+1}) - f(c)][\alpha(c+) - \alpha(c)] \end{aligned}$$

$\Downarrow \Delta_1$ $\Downarrow \Delta_2$

Let us bound $|\Delta_1|$ from above.

- If α is left continuous at c , i.e. $\alpha(c-) = \alpha(c)$, then $\Delta_1 = 0$
- If f is left continuous at c . Let $\varepsilon > 0$. Take $\delta > 0$, s.t. $x \in (c-\delta, c) \Rightarrow |f(x) - f(c)| < \varepsilon$
 Then, if $|t_k - c| < \delta$, then $t_k \in (c-\delta, c)$ and we find $|\Delta_1| < \varepsilon |\alpha(c) - \alpha(c-)|$

The similar applies to Δ_2 .

In conclusion, for $\varepsilon > 0$, we can find $\delta > 0$, s.t. for $P \in P([a, b])$ satisfying $\|P\| < \delta$ and $c \in \text{Supp}(P)$, then $|\Delta| \leq \varepsilon |\alpha(c) - \alpha(c-)| + \varepsilon |\alpha(c+) - \alpha(c)|$ \square

3-4-25 (WEEK 3)

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THEOREM (Direct result just by using linearity)

Let α be a step function (w.r.t. the partition $P = (x_k)_{0 \leq k \leq n}$)

Let $f: [a, b] \rightarrow \mathbb{R}$ be s.t.

- (i) $\forall 0 \leq k \leq n-1$, at least one of f and α is right conti at x_k
- (ii) $\forall 1 \leq k \leq n$, at least one of f and α is left conti at x_k .

Then, $f \in R(\alpha)$ and $\int_a^b f d\alpha = \sum_{k=1}^n f(x_k) [\alpha(x_k+) - \alpha(x_k-)]$

COROLLARY

Let $a_1, \dots, a_n \in \mathbb{R}$, define a left-conti function $f: [0, 1] \rightarrow \mathbb{R}$

$$x \mapsto \begin{cases} 0, & x=0 \\ a_k, & x \in (k-1, k], \quad 1 \leq k \leq n \end{cases}$$

Then, $\sum_{k=1}^n a_k = \sum_{k=1}^n f(k) = \int_0^1 f(x) dx$

COROLLARY (Euler's Summation Formula) ← Just need to understand how to prove, no need to memorize

Let $f: [a, b] \rightarrow \mathbb{R}$ be C^1 . Then,

- (1) $\sum_{a \leq n \leq b} f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \{x\} dx + f(a)\{a\} - f(b)\{b\}$, where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of x
- (2) $\sum_{n=a}^b f(n) = \int_a^b f(x) dx + \int_a^b f'(x) (\{x\} - \frac{1}{2}) dx + \frac{1}{2}[f(a) + f(b)]$

Proof

We know f and f' are conti, so they belong to $R(x; a, b)$

First, we know that $f \in R([x]; a, b)$ so $\int_a^b f(x) dx$ is well-defined.

So, we also have $f \in R(\{x\}) = R(x - \lfloor x \rfloor)$. By integration by parts, we find $\int_a^b f(x) d(x - \lfloor x \rfloor) + \int_a^b \{x\} df(x) = f(b)\{b\} - f(a)\{a\}$, sub in the previous corollary, we get (1).

To show (2), we may directly apply integration by parts to f and $\{x\}$.

The other method is carried out by taking the formula for (1).

Let $a, b \in \mathbb{Z}$, we have $\sum_{n=a}^b f(n) = f(a) + \int_a^b f(x) dx + \int_a^b f'(x) \{x\} dx$ and $\int_a^b f'(x) dx = \frac{1}{2} \int_a^b df(x) = \frac{1}{2}[f(b) - f(a)] \quad \square$

Since $\{a\} = \{b\} = 0 \quad \forall a, b \in \mathbb{Z}$, so " $f(a)\{a\} - f(b)\{b\} = 0$ "

COROLLARY (Abel's Summation)

Let $(a_n)_{n \geq 1}$ be real numbers. Define $A: \mathbb{R}_n \rightarrow \mathbb{R}$

$$x \mapsto \sum_{k=1}^{\lfloor x \rfloor} a_k \quad \text{If } f \in C^1, df(t) = f'(t) dt$$

For $x \geq 1$, and a conti function $f: (1, x) \rightarrow \mathbb{R}$, we have $\sum_{k=1}^{\lfloor x \rfloor} a_k f(k) = - \int_1^x A(t) df(t) + A(x) f(x)$

DARBOUX SUMMATIONS

UPPER AND LOWER DARBOUX SUMMATIONS AND INTEGRALS

DEFINITION

Let $P \in P([a, b])$ and define for $\{k \leq n\}$, $M_k = M_k(f) := \sup \{f(x) \mid x \in [x_{k-1}, x_k]\}$ and $m_k = m_k(f) := \inf \{f(x) \mid x \in [x_{k-1}, x_k]\}$

We define the upper and lower Darboux sums as follows: $U_P(f, \alpha) = \sum_{k=1}^n M_k(f) \Delta x_k$, $L_P(f, \alpha) = \sum_{k=1}^n m_k(f) \Delta x_k$

Note: No tagged points are needed for these defns. Also, when $\alpha(x) = x$, we call them upper/lower Riemann sums.

LEMMA (By def)

Suppose that α is increasing/non-decreasing on $[a, b]$. Then, for any $f: [a, b] \rightarrow \mathbb{R}$ and any partition $P \in P([a, b])$ and tagged points t , we have $m_k(f) \leq f(t_k) \leq M_k(f)$, $1 \leq k \leq n$ and $L_{P,t}(f, \alpha) \leq S_{P,t}(f, \alpha) \leq U_{P,t}(f, \alpha)$

PROPOSITION

Suppose that α is increasing.

(a) $\forall P \in \mathcal{P}$, we have $U_P(f, \alpha) \leq U_P(f, \alpha)$ and $L_P(f, \alpha) \leq L_{P'}(f, \alpha)$

(b) $\forall P, P'$, we have $L_P(f, \alpha) \leq U_{P'}(f, \alpha)$

Proof

(a) It is enough to prove the inequalities hold when P' contains one more subdivision point than P . Let $P = (x_k)$ such that $c \in (x_{i-1}, x_i)$ where $1 \leq i \leq n$.

We have: $U_P(f, \alpha) = \sum_{k=1}^n M_k(f) \Delta \alpha_{x_k}$, and $U_{P'}(f, \alpha) = \sum_{k=1, k \neq i}^n M_k(f) \Delta \alpha_{x_k} + M'(x(c) - \alpha(x_{i-1})) + M''(\alpha(x_i) - \alpha(c))$, where $M' := \sup \{f(x) | x_{i-1} \leq x \leq c\}$ and $M'' := \sup \{f(x) | c \leq x \leq x_i\}$.

Since $M' \leq M_i(f)$, $M'' \leq M_i(f)$, we deduce $M'(x(c) - \alpha(x_{i-1})) + M''(\alpha(x_i) - \alpha(c)) \leq M_i(f)(\alpha(x_i) - \alpha(x_{i-1}))$, so $U_{P'}(f, \alpha) \leq U_P(f, \alpha)$ (L_P is similar)

(b) Let $P, P' \in \mathcal{P}([a, b])$, and $P'' := P \vee P'$. Then, from (a), $L_P(f, \alpha) \leq L_{P''}(f, \alpha) \leq U_{P''}(f, \alpha) \leq U_P(f, \alpha)$ \square

Core idea to link two unrelated partitions

DEFINITION

Suppose α is increasing. The upper/lower Stieltjes integrals of f w.r.t. α are defined by:

$$\bar{I}(f, \alpha) = \int_a^b f d\alpha := \inf \{U_P(f, \alpha) | P \in \mathcal{P}([a, b])\}$$

$$I(f, \alpha) = \int_a^b f d\alpha := \sup \{L_P(f, \alpha) | P \in \mathcal{P}([a, b])\}$$

PROPOSITION

Suppose α is increasing, then $I(f, \alpha) \leq \bar{I}(f, \alpha)$ (Proof: Trivial)

REMARK

The equality above may not hold. Say $\alpha(x) = x$, $f(x) = 1$ defined on $[0, 1]$.

Then, $U_P(f, x) = 1$ and $L_P(f, x) = 0 \quad \forall P \in \mathcal{P}([a, b]) \Rightarrow$ By def, $\bar{I}(f, x) = 1$, $I(f, x) = 0$, so $I(f, x) \neq \bar{I}(f, x)$

PROPOSITION (Linearity)

Let $a \leq c \leq b$. Let $\alpha: [a, b] \rightarrow \mathbb{R}$ be bounded and increasing

Let $f, g: [a, b] \rightarrow \mathbb{R}$ be bounded. We have:

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha, \text{ also } \int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

$$\int_a^b (f+g) d\alpha \leq \int_a^b f d\alpha + \int_a^b g d\alpha, \text{ also } \int_a^b (f+g) d\alpha \geq \int_a^b f d\alpha + \int_a^b g d\alpha$$

RIEMANN'S CONDITION**DEFINITION**

Let $\alpha: [a, b] \rightarrow \mathbb{R}$ be increasing. We say that f satisfies Riemann's condition w.r.t. α on $[a, b]$ if $\forall \varepsilon > 0$, $\exists P \in \mathcal{P}([a, b])$, s.t. $\forall P \in \mathcal{P}$, we have $U_P(f, \alpha) - L_P(f, \alpha) < \varepsilon$ (Note: Tagged points don't matter here)

REMARK

Thanks to the propositions above, it suffices to find $P \in \mathcal{P}([a, b])$ with $U_P(f, \alpha) - L_P(f, \alpha) < \varepsilon$ to satisfy Riemann's condition

THEOREM

The assumption for α to be increasing is actually not very restrictive. Recall the decomposition theorem for function of bounded variation ($\alpha \in BV \Leftrightarrow \alpha = V - (V - \alpha)$)

For any increasing integrator α , TFAE:

bounded variation ($\alpha \in BV \Leftrightarrow \alpha = V - (V - \alpha)$)

increasing

(1) $f \in R(\alpha; a, b)$

(2) f satisfies Riemann's condition

(3) $I(f, \alpha) = \bar{I}(f, \alpha)$

Proof: Next set of notes!

SOME APPLICATIONS

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PROPOSITION

Given non-decreasing α , and $f, g \in \mathcal{P}([a, b])$, suppose $f(x) \leq g(x) \quad \forall x \in [a, b]$, then we have $\int_a^b f(x) d\alpha(x) \leq \int_a^b g(x) d\alpha(x)$

Proof

For any partition $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a, b])$, we have $M_k(f) \leq M_k(g)$ and $m_k(f) \leq m_k(g)$, so we have $L_P(f, \alpha) \leq L_P(g, \alpha)$ and $U_P(f, \alpha) \leq U_P(g, \alpha)$. Therefore, the upper and lower Stieltjes integrals satisfy: $\bar{I}(f, \alpha) \leq \bar{I}(g, \alpha)$ and $\bar{I}(f, \alpha) \leq \bar{I}(g, \alpha)$. By the Thm above, thus $\int_a^b f d\alpha \leq \int_a^b g d\alpha \quad \square$

PROPOSITION

Given non-decreasing α , $f \in R(\alpha; a, b)$, then $|f| \in R(\alpha; a, b)$ and $|\int_a^b f(x) d\alpha(x)| \leq \int_a^b |f(x)| d\alpha(x)$

Proof

For any partition $P \in \mathcal{P}([a, b])$ and $1 \leq k \leq n$, we have $M_k(f) - m_k(f) = \sup \{f(x) - f(y) \mid x, y \in (x_{k-1}, x_k)\}$

We may check that $M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f)$, hence $U_P(|f|, \alpha) - L_P(|f|, \alpha) \leq U_P(f, \alpha) - L_P(f, \alpha)$, i.e. $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha \quad \square$

3-6-2S (WEEK 3)

THEOREM (Proof for last section of notes)

For any increasing integrator α , TFAE:

$$(1) f \in R(\alpha; a, b)$$

(2) f satisfies Riemann's condition

$$(3) \bar{I}(f, \alpha) = \bar{I}(f, \alpha)$$

Proof

"(1) \Rightarrow (2)": Suppose that f satisfies (RS). The proof is trivial if $\alpha(a) = \alpha(b)$, so we may assume $\alpha(a) < \alpha(b)$.

Let $\varepsilon > 0$, then take $P_\varepsilon \in \mathcal{P}([a, b])$ s.t. \forall tagged partition $(P, t), (P, t')$, with $P \supseteq P_\varepsilon$, we have $|S_{P,t}(f, \alpha) - \int_a^b f d\alpha| < \varepsilon$, $|S_{P,t'}(f, \alpha) - \int_a^b f d\alpha| < \varepsilon$

By triangle inequality, $|\sum_{k=1}^n (f(t_k) - f(t'_k)) \Delta \alpha_k| < 2\varepsilon$

For each $1 \leq k \leq n$, we take $t_k, t'_k \in [x_{k-1}, x_k]$, s.t. $M_k(f) \leq f(t_k) + \frac{\varepsilon}{\alpha(b)-\alpha(a)}$ and $m_k(f) \geq f(t'_k) - \frac{\varepsilon}{\alpha(b)-\alpha(a)}$

This means, $M_k(f) - m_k(f) \leq f(t_k) - f(t'_k) + \frac{2\varepsilon}{\alpha(b)-\alpha(a)}$.

Therefore, $U_P(f, \alpha) - L_P(f, \alpha) = \sum_{k=1}^n (M_k(f) - m_k(f)) \Delta \alpha_k \leq \sum_{k=1}^n [f(t_k) - f(t'_k)] \Delta \alpha_k + \frac{2\varepsilon}{\alpha(b)-\alpha(a)} \sum_{k=1}^n \Delta \alpha_k \stackrel{\text{if } \alpha(b) = \alpha(a)}{\leq} 4\varepsilon$, so (2) holds.

"(2) \Rightarrow (3)": Suppose that (2) holds. Let $\varepsilon > 0$ and take $P_\varepsilon \in \mathcal{P}([a, b])$, s.t. $0 \leq U_P(f, \alpha) - L_P(f, \alpha) \leq \varepsilon$ for all $P \supseteq P_\varepsilon$.

For all $P \supseteq P_\varepsilon$, $\bar{I}(f, \alpha) \leq U_P(f, \alpha) \leq L_P(f, \alpha) + \varepsilon \leq \bar{I}(f, \alpha) + \varepsilon$

As this is true $\forall \varepsilon$, thus $\bar{I}(f, \alpha) \leq \bar{I}(f, \alpha)$, which implies $\bar{I}(f, \alpha) = \bar{I}(f, \alpha)$ ($\because \geq$ is trivially true)

"(3) \Rightarrow (1)": Suppose that (3) holds. Let $L = \bar{I}(f, \alpha) = \bar{I}(f, \alpha)$.

Let $\varepsilon > 0$, take $P_\varepsilon^{(1)} \in \mathcal{P}([a, b])$ s.t. $L_P(f, \alpha) + \varepsilon \geq \bar{I}(f, \alpha) = L \quad \forall P \supseteq P_\varepsilon^{(1)}$

Take $P_\varepsilon^{(2)} \in \mathcal{P}([a, b])$, s.t. $U_P(f, \alpha) - \varepsilon \leq \bar{I}(f, \alpha) = L \quad \forall P \supseteq P_\varepsilon^{(2)}$

Take $P_\varepsilon := P_\varepsilon^{(1)} \cup P_\varepsilon^{(2)}$, then $\forall P \supseteq P_\varepsilon$, tagged points t , we have:

$$\hookrightarrow S_{P,t}(f, \alpha) \leq U_P(f, \alpha) \leq L + \varepsilon$$

$$\hookrightarrow S_{P,t}(f, \alpha) \geq L_P(f, \alpha) \geq L - \varepsilon$$

\therefore In other words, $|S_{P,t}(f, \alpha) - L| \leq \varepsilon$. Namely, $f \in R(\alpha; a, b)$ and $\int_a^b f d\alpha = L \quad \square$

PROPOSITION (APPLICATION)

Suppose that α is nondecreasing on $[a, b]$. If $f \in R(\alpha; a, b)$, then $f^2 \in R(\alpha'; a, b)$

Proof

Let $f \in R(\alpha; a, b)$ and $P \in \mathcal{P}([a, b])$ be a partition. \checkmark key step

For $1 \leq k \leq n$, we have $M_k(f^2) - m_k(f^2) = M_k(|f|)^2 - m_k(|f|)^2 = (M_k(|f|) + m_k(|f|))(M_k(|f|) - m_k(|f|)) \leq 2M[M_k(|f|) - m_k(|f|)]$, where we have

$$M = \sup \{|f(x)| \mid x \in [a, b]\} < \infty$$

\hookrightarrow This suffices because after one more step

$\because |f|$ satisfies Riemann's condition ($U_P - L_P < \varepsilon$)

\hookrightarrow (i.e. \sum), we get $U_P - L_P$ again

$\therefore f^2$ satisfies Riemann's condition \square

COROLLARY

Suppose that α is nondecreasing on $[a, b]$. If $f, g \in R(\alpha; a, b)$, then $f \cdot g \in R(\alpha; a, b)$

Proof

Use $f \cdot g = \frac{1}{2}((f+g)^2 - f^2 - g^2)$, since $+/-$ operations preserve integrability, thus $f \cdot g \in R(\alpha; a, b)$ too. \square

REMARK

We can use induction to further show that for nondecreasing α and $f \in R(\alpha; a, b)$, $\forall n \in \mathbb{N}_0$, then $f^n \in R(\alpha; a, b)$.

IMPORTANT

The converse of our proposition does not hold. Consider $f: [0, 1] \rightarrow \mathbb{R}$. We have $f^2 \in R(\alpha; a, b)$ but $f \notin R(\alpha; a, b)$.

$$x \mapsto \begin{cases} -1, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$$

INTEGRATORS OF BOUNDED VARIATION

GOAL

We want to use the decomposition theorem for functions of bounded variation to extend the previous results to general integrators, i.e. of bounded variation. We can mainly do so by recalling: $\alpha \in BV \Rightarrow \alpha = V - (V - \alpha)$, for a variation function V .

THEOREM

Let $\alpha \in BV([a, b])$ and V be its variation function. Then, $f \in R(\alpha; a, b) \Rightarrow f \in R(V; a, b)$

Proof

When α is a constant function, then $V=0$, and the theorem holds trivially. Now, let us assume $\alpha(a) < \alpha(b)$, so $V(b) > 0$

Let $\varepsilon > 0$. Take $P_\varepsilon^{(1)} \in P([a, b])$ s.t. $\forall P \geq P_\varepsilon$ and tagged points t_i , then $\left| \sum_{k=1}^n [f(t_k) - f(t_{k-1})] \Delta \alpha_k \right| \leq \varepsilon$.

Let $M = \sup |f|$. Take $P_\varepsilon^{(2)} \in P([a, b])$, s.t. $\forall P \geq P_\varepsilon^{(2)}$, $V(b) = V_\alpha([a, b]) \leq V_p(b) + \frac{\varepsilon}{M} = \sum_{k=1}^n |\Delta \alpha_k| + \frac{\varepsilon}{M}$ (Prop from before: $\forall \varepsilon > 0$, $\exists P_\varepsilon$, s.t. $P \geq P_\varepsilon \Rightarrow V_p(f) \leq V_p(f) + \varepsilon$)

We now factcheck that $f \in R(\alpha; a, b)$. Since V is non-decreasing, we only need to check that it satisfies Riemann's condition, i.e. we need to bound

$$\sum_{k=1}^n [M_{k+}(f) - m_{k-}(f)] \Delta V_k = \sum_{k=1}^n [M_{k+}(f) - m_{k-}(f)] |\Delta \alpha_k| + \sum_{k=1}^n [M_{k+}(f) - m_{k-}(f)] [\Delta V_k - |\Delta \alpha_k|]$$

(*) (*)

Consider (*).

Let $M = \sup |f|$, then $(*) \leq 2M \sum_{k=1}^n |\Delta V_k - |\Delta \alpha_k|| = 2M (V(b) - \sum_{k=1}^n |\Delta \alpha_k|) \leq 2\varepsilon$.

To bound (**), we distinguish the indices k w.r.t. the sign of $\Delta \alpha_k$.

Let $K^+ := \{k \leq n \mid \Delta \alpha_k > 0\}$ and $K^- := \{k \leq n \mid \Delta \alpha_k < 0\}$, $\varepsilon' = \frac{\varepsilon}{\sqrt{b-a}}$ (really important tactic to deal with absolute values)

For $k \in K^+$, choose $t_k, t_{k'} \in [x_{k-1}, x_k]$, s.t. $f(t_k) - f(t_{k'}) \geq M_{k+}(f) - m_{k-}(f) - \varepsilon'$

For $k \in K^-$, choose $t_k, t_{k'} \in [x_{k-1}, x_k]$, s.t. $f(t_{k'}) - f(t_k) \geq M_{k+}(f) - m_{k-}(f) - \varepsilon'$

Therefore, we find $(***) = \sum_{k \in K^+} [M_{k+}(f) - m_{k-}(f)] |\Delta \alpha_k| = \sum_{k \in K^+} [M_{k+}(f) - m_{k-}(f)] \Delta \alpha_k + \sum_{k \in K^-} [M_{k+}(f) - m_{k-}(f)] (-\Delta \alpha_k)$
 $\leq \sum_{k=1}^n (f(t_k) - f(t_{k'})) \Delta \alpha_k + \varepsilon' \sum_{k=1}^n |\Delta \alpha_k| \leq \varepsilon + \varepsilon' V(b) = 2\varepsilon \quad \square$

↑ sub from above ↑

COROLLARY (Formal restating of our goal)

Let $\alpha \in BV([a, b])$, bounded $f: [a, b] \rightarrow \mathbb{R}$, then (1) \Leftrightarrow (2).

(1) $f \in R(\alpha; a, b)$

(2) \exists nondecreasing α_1, α_2 , s.t. $f \in R(\alpha_1) \cap R(\alpha_2)$ and $\alpha = \alpha_1 - \alpha_2$

Proof

"(2) \Rightarrow (1)": By linearity

"(1) \Rightarrow (2)": Write $\alpha = V - (V - \alpha)$ and then use the theorem above. \square

PROPOSITION

Let $\alpha \in BV([a, b])$ and $f \in R(\alpha; a, b)$. Then, for any $[c, d] \subseteq [a, b]$, we have $f \in R(\alpha; c, d)$

Proof If only BV , we can use $\alpha = V - (V - \alpha) \Rightarrow f \in R(V; a, b) \cap R(V - \alpha; a, b) \Rightarrow f \in R(\alpha; a, b)$, as we have the prop above

Suppose α is non-decreasing on $[a, b]$. If we can check that $f \in R(\alpha; a, x)$ for all $x \in [a, b]$, then we get $f \in R(\alpha; a, c) \cap R(\alpha; a, d) = R(\alpha; c, d)$.

For $x \in [a, b]$ and a partition $P \in P([a, b])$, define $\Delta p(x) := U_p(f|_{[a, x]}, \alpha|_{[a, x]}) - L_p(f|_{[a, x]}, \alpha|_{[a, x]})$

Fix $x \in [a, b]$, let $\varepsilon > 0$, then take $P_\varepsilon \in P([a, b])$, s.t. $\Delta p(b) \leq \varepsilon \quad \forall P \geq P_\varepsilon$.

WLOG, we may assume $x \in \text{Supp}(P_\varepsilon)$ (" $\forall P \geq P_\varepsilon$ " means any points chosen still make the inequality true).

Now, let $P_\varepsilon' := P_\varepsilon \cap [a, x]$. Now, $\forall P \geq P_\varepsilon'$, define $P := P' \cup P_\varepsilon \setminus P_\varepsilon'$. Then, we have $\Delta p'(x) \leq \Delta p(b) \leq \varepsilon$. \square

THEOREM

Let $\alpha \in BV([a, b])$, $f, g \in R(\alpha; a, b)$.

Define for all $x \in [a, b]$, $F(x) = \int_a^x f(t) d\alpha(t)$, $G(x) = \int_a^x g(t) d\alpha(t)$. Then, $f \in R(G; a, b)$, $g \in R(F; a, b)$ and $fg \in R(\alpha; a, b)$, where:

$$\int_a^b f(x) g(x) d\alpha(x) = \int_a^b f(x) d(G(x)) = \int_a^b g(x) d(F(x))$$

Proof

Similar to the above, it is sufficient to prove this is the case for non-increasing α .

Suppose α is non-decreasing, we already know that $fg \in R(\alpha)$. By symmetry, we only need to check $f \in R(G)$ and $\int_a^b fg d\alpha = \int_a^b f dG$.

Let $P \in P([a, b])$ and fix tagged points t , then $S_{P,t}(f, G) = \sum_{k=1}^n f(t_k) \Delta G_k = \sum_{k=1}^n f(t_k) \int_{x_{k-1}}^{x_k} g(t) d\alpha(t)$ and $\int_a^b f g d\alpha = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(t) g(t) d\alpha(t)$

Now, $|S_{P,t}(f, G) - \int_a^b f g d\alpha| = \left| \sum_{k=1}^n \int_{x_{k-1}}^{x_k} [f(t_k) - f(x)] g(x) d\alpha(x) \right|$

$$\leq M \sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f(t_k) - f(x)| d\alpha(x), \text{ where } |g(x)| \leq M \quad \forall x \in [a, b], \text{ since it is bounded.}$$

$$\leq M \sum_{k=1}^n \int_{x_{k-1}}^{x_k} [M_u(f) - m_u(f)] d\alpha(x)$$

$$= M[U_p(f, \alpha) - L_p(f, \alpha)]$$

\therefore By def, f satisfies Riemann's condition w.r.t. $\alpha \Rightarrow |S_{P,t}(f, G) - \int_a^b f g d\alpha| \leq M\varepsilon' \square$

RIEMANN-STIELTJES INTEGRABILITY

THEOREM

Let $f, \alpha: [a, b] \rightarrow \mathbb{R}$ be bounded. Then, $f \in R(\alpha)$ if (1) or (2)

- 1) f is continuous and $\alpha \in BV$
- 2) α is continuous and $f \in BV$

COROLLARY

For $\alpha(x)=x$ and α continuous or a bounded variation f , then f is integrable

PROOF OF THEOREM

Integration by parts gives us $f \in R(\alpha) \Leftrightarrow \alpha \in R(f)$, therefore it is enough to show that when (1) holds with increasing α , $f \in R(\alpha)$.

Let $\varepsilon > 0$. Since f is continuous on $[a, b]$ and $[a, b]$ is compact, we know that f is uniformly continuous.

Take $\delta > 0$, s.t. $x, y \in [a, b]$, $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$

Let us take a partition $P \in P([a, b])$ s.t. $\|P\| < \delta$, then:

$$U_p(f, \alpha) - L_p(f, \alpha) = \sum_{k=1}^n [M_k(f) - m_k(f)] \Delta \alpha_k \leq \varepsilon \sum_{k=1}^n \Delta \alpha_k = \varepsilon [\alpha(b) - \alpha(a)], \text{ thus Riemann's condition holds. } \square$$

THEOREM

Let $f, \alpha: [a, b] \rightarrow \mathbb{R}$ be bounded, α be nondecreasing. If (1) or (2) holds, then $f \notin R(\alpha)$

- 1) $\exists c \in [a, b]$, s.t. f and α are not right-continuous at c
- 2) $\exists c \in [a, b]$, s.t. f and α are not left-continuous at c

Proof

By symmetry, we only need to check that (1) $\Rightarrow f \notin R(\alpha)$

Suppose that $c \in [a, b]$, s.t. f and α are not right continuous at c

Let $\varepsilon > 0$, $\delta > 0$, s.t. $\exists x \in (c, c+\delta)$, $|f(x) - f(c)| > \varepsilon$ and $\exists y \in (c, c+\delta)$, s.t. $|\alpha(y) - \alpha(c)| > \varepsilon$

Let $P \in P([a, b])$, s.t. $x_i = c$, $x_{i+1} = y$ for some $1 \leq i \leq n-1$.

$$\text{Then, } U_p(f, \alpha) - L_p(f, \alpha) = \sum_{k=1}^n [M_k(f) - m_k(f)] \Delta \alpha_k \geq \varepsilon [M_{i+1}(f) - m_{i+1}(f)] \geq \varepsilon^2 \quad \square$$

MEAN VALUE THEOREMS

FIRST MEAN VALUE THEOREM

Let $\alpha: [a, b] \rightarrow \mathbb{R}$ be a nondecreasing function and $f \in R(\alpha; a, b)$. Let $M := \sup \{f(x) \mid x \in [a, b]\}$ and $m := \inf \{f(x) \mid x \in [a, b]\}$. Then, $\exists c \in [m, M]$, such that $\int_a^b f(x) d\alpha(x) = c \int_a^b d\alpha(x) = c[\alpha(b) - \alpha(a)]$. In particular, if f is continuous, then $c = f(x_0)$ for some $x_0 \in [a, b]$.

Proof

If α is a constant function, then of course $\int_a^b f d\alpha = 0$ and $\int_a^b d\alpha = 0 \Rightarrow$ The statement holds trivially

Suppose $\alpha(a) < \alpha(b)$. Then, for any given partition $P \in P([a, b])$, we have $L_p(f, \alpha) \leq I(f, \alpha) = \bar{I}(f, \alpha) = \int_a^b f d\alpha \leq U_p(f, \alpha)$
 Moreover, $L_p(f, \alpha) = \sum_{k=1}^n m_k(f) \Delta \alpha_k \geq \sum_{k=1}^n m \Delta \alpha_k = m \int_a^b d\alpha$ and $U_p(f, \alpha) = \sum_{k=1}^n M_k(f) \Delta \alpha_k \leq \sum_{k=1}^n M \Delta \alpha_k = M \int_a^b d\alpha$

Therefore, $m \int_a^b d\alpha \leq \int_a^b f d\alpha \leq M \int_a^b d\alpha$

$$\Rightarrow c = \frac{\int_a^b f d\alpha}{\int_a^b d\alpha} \in [m, M] \quad \square$$

COROLLARY

Some assumptions give us $|\int_a^b f(x) dx| \leq M \int_a^b d\alpha = M [\alpha(b) - \alpha(a)]$, where $M = \sup \{f(x) | x \in [a, b]\}$
 (This result could also be derived from the triangle inequality)

SECOND MEAN VALUE THEOREM

Let α be continuous and f be non-decreasing. Then, $\int_a^b \alpha df = f(a) \int_a^{x_0} d\alpha + f(b) \int_{x_0}^b d\alpha$ for some $x_0 \in [a, b]$.

Proof

The integration by parts gives us $\int_a^b f(x) d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df$

Now, applying 1st MVT, we obtain $\int_a^b \alpha df = \alpha(x_0)(f(b) - f(a))$ for some $x_0 \in [a, b]$.

Hence, putting these statements together, we get $\int_a^b f(x) d\alpha = f(a) \int_a^{x_0} d\alpha + f(b) \int_{x_0}^b d\alpha \quad \square$

FUNDAMENTAL THEOREMS OF CALCULUS**DEFINITION**

Let $I \subseteq \mathbb{R}$ be an interval, $f, F: I \rightarrow \mathbb{R}$ be functions. If $F'(x) = f(x) \forall x \in \text{int}(I)$, we say F is a primitive or antiderivative of f .

FIRST FUNDAMENTAL THEOREM OF CALCULUS

Let $\alpha \in BV([a, b])$ and $f \in R(\alpha; a, b)$. Define $F(x) = \int_a^x f(t) d\alpha$ $\forall x \in [a, b]$. Then, we have:

(a) $F \in BV([a, b])$

(b) If α is continuous at some $c \in [a, b]$, then F is also continuous at c .

(c) If α is non-decreasing, then the derivative $F'(x)$ exists at $x \in (a, b)$ whenever $\alpha'(x)$ exists and f is continuous at x ; additionally, for such x , we have $F'(x) = f(x)\alpha'(x)$

Proof

It is sufficient for us to prove the theorem for non-decreasing α .

Note that for $x < y$, $x, y \in [a, b]$, we have $|F(y) - F(x)| \leq M_{x,y} |\alpha(y) - \alpha(x)|$, where $M_{x,y} = \sup \{f(t) | t \in [x, y]\} \leq \sup \{f(t) | t \in [a, b]\} =: M$

(a) Given a partition $P \in \mathcal{P}([a, b])$, we write $V_P(F) = \sum_{k=1}^n |\Delta F_k| \leq \sum_{k=1}^n M |\Delta \alpha_k| = M (\alpha(b) - \alpha(a)) < \infty$ \therefore We have $F \in BV([a, b])$ \checkmark

(b) Let $c \in [a, b]$, s.t. α is continuous at c . Let $\varepsilon > 0$, take $\delta > 0$, s.t. $\forall x \in [a, b]$, $|x - c| \leq \delta \Rightarrow |\alpha(x) - \alpha(c)| \leq \varepsilon$

Then, we can see $\forall x \in [a, b]$, $|x - c| \leq \delta \Rightarrow |F(x) - F(c)| \leq M \varepsilon$ ($\because \varepsilon \leq M \varepsilon \leq M \leq |\Delta \alpha_k|$)

By definition, F is continuous at c . \checkmark

(c) Let $x \in (a, b)$, s.t. $\alpha'(x)$ exists and f is continuous at x . By definition and MVT, $F'(x) = \lim_{y \rightarrow x} \frac{F(y) - F(x)}{y - x} = \lim_{y \rightarrow x} \frac{c[\alpha(y) - \alpha(x)]}{y - x}$

Since $c = c(x, y) \rightarrow f(x)$ when $y \rightarrow x$, by the continuity of f at x and $\alpha'(x)$ exists, we deduce $F'(x) = f(x)\alpha'(x)$ \square

COROLLARY (Freshman Calculus I : Riemann Integrals)

Let $\alpha(x) = x$ and $f \in R(x; a, b)$ be a Riemann-integrable function.

Define $F(x) = \int_a^x f(t) dt$ and $G(x) = \int_a^x g(t) dt$

Then, the following properties hold:

(a) F and G are continuous functions of bounded variation on $[a, b]$.

(b) If f is continuous at $x \in (a, b)$, then $F'(x) = f(x)$

(c) For $f \in R(G; a, b)$, $g \in R(F; a, b)$ and $f_g \in R(x; a, b)$, we have $\int_a^b f(x)g(x) dx = \int_a^b f(x)dG(x) = \int_a^b g(x)dF(x)$

SECOND FUNDAMENTAL THEOREM OF CALCULUS

Let $f \in R(x; a, b)$ and $F: [a, b] \rightarrow \mathbb{R}$, s.t. F is continuous and F' is well-defined on (a, b) . Suppose that $F'(x) = f(x)$ for every $x \in (a, b)$. Then, we have $\int_a^b F'(x) dx = \int_a^b f(x) dx = F(b) - F(a)$

Proof

Let $\varepsilon > 0$. Since $f \in R(x; a, b)$, we may find $P \in \mathcal{P}([a, b])$, s.t. $|S_{P,\varepsilon}(f, x) - \int_a^b f(x) dx| < \varepsilon$ for any partition $P \supset P_\varepsilon$ and tagged points t .

For $P \supset P_\varepsilon$, and tagged points t , we have $S_{P,\varepsilon}(f, x) = \sum_{k=1}^n f(t_k) \Delta x_k = \sum_{k=1}^n F'(t_k) \Delta x_k$

Let us make a special choice of t , for a given partition $P \supset P_\varepsilon$, $\Delta F_k = F(x_k) - F(x_{k-1}) = F'(t_k) \Delta x_k = f(t_k) \Delta x_k$ Why we need a special partition. It is valid too, since $t \in (x_{k-1}, x_k)$
 Therefore, $S_{P,\varepsilon}(f, x) = \sum_{k=1}^n f(t_k) \Delta x_k = \sum_{k=1}^n \Delta F_k = F(b) - F(a)$, this means $\int_a^b f(x) dx = F(b) - F(a)$

COROLLARY

Let $f \in L^1(x; a, b)$, $\alpha: [a, b] \rightarrow \mathbb{R}$ be continuous s.t. $\alpha' \in L^1(x; a, b)$. Then, $\int_a^b f d\alpha = \int_a^b f \alpha' dx$

Proof

Just take $g = \alpha'$. \square

PROPOSITION (Change of Variables) Very important to check!

Let $g: [c, d] \rightarrow \mathbb{R}$ be C^1 . Let $f: g([c, d]) \rightarrow \mathbb{R}$ be continuous, and define $F(x) = \int_{g(c)}^x f(s) ds \quad \forall x \in g([c, d])$

Then, we have $\forall x \in [c, d]$, $\int_c^x f \circ g(t) g'(t) dt = F(g(x)) = \int_{g(c)}^{g(x)} f(s) ds$

INTEGRALS DEPENDING ON A PARAMETER

QUESTION

Given $f: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$

$$(x, y) \longmapsto f(x, y)$$

- Integrate f w.r.t. x , how do we define the regularity of f and its integral?
- Interchange the order of integration?

Sorry, today's notes would have terrible handwriting cuz lots of rain got into my stylus/iPad and it's so heavy to hold the pen so it hurts to even write, let alone neat writing. 木ヨリ2つ打たなきよる子がとても痛い TT

PROPOSITION

Let $\Omega = [a, b] \times [c, d] \subseteq \mathbb{R}^2$.

Let $f: \Omega \rightarrow \mathbb{R}$ be continuous and $\alpha \in BV([a, b])$

Define $F: [c, d] \rightarrow \mathbb{R}$

$$y \longmapsto \int_a^b f(x, y) d\alpha(x)$$

Then, F is continuous, that is $F(y_0) = \lim_{y \rightarrow y_0} F(y) \quad \forall y \in [c, d]$.

Proof

Since Ω is compact, f is uniformly continuous. Suppose α is non-increasing

Let $\varepsilon > 0$, and take $\delta > 0$, s.t. $\forall (x, y), (x', y') \in \Omega, |(x, y) - (x', y')| \leq \delta \Rightarrow |f(x, y) - f(x', y')| \leq \varepsilon$

requires uniform continuity

Fix $y_0 \in [c, d]$. For $y \in [c, d]$, s.t. $|y - y_0| \leq \delta$, we have $|F(y) - F(y_0)| = |\int_a^b (f(x, y) - f(x, y_0)) d\alpha(x)| \leq \int_a^b |f(x, y) - f(x, y_0)| d\alpha(x) \leq \int_a^b \varepsilon d\alpha(x) = \varepsilon [\alpha(b) - \alpha(a)] \square$

COUNTEREXAMPLE FOR NONCONTINUOUS f

Let $f(x, y) = \begin{cases} \frac{4xy^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ (Behind the scenes: My hand is now screaming in agony from the pain of writing 0-0)

It's not hard to see that $\forall x \in [0, 1], f(x, \cdot): y \mapsto f(x, y)$ is continuous. Similarly, $f(\cdot, y)$ is continuous $\forall y \in [0, 1]$. However, f is not continuous on Ω because $\lim_{\varepsilon \rightarrow 0^+} f(0, \varepsilon) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} = +\infty$

Notice, $\forall x \in [0, 1]$, we have $F(y) = \int_0^1 f(x, y) dx = \left[\frac{x^2-y^2}{x^2+y^2} \right]_{x=0}^1 = \frac{1-y^2}{1+y^2} + 1 = \frac{2}{1+y^2}$

Since $f(x, 0) = 0 \quad \forall x \in [0, 1]$, we have $F(0) = \int_0^1 0 dx = 0$. However, $\lim_{y \rightarrow 0^+} F(y) = 2 \neq 0 \therefore F$ is not continuous at 0.

COROLLARY

Let $f: \Omega \rightarrow \mathbb{R}$ be continuous and $g \in R(x; a, b)$.

Define $F: [c, d] \rightarrow \mathbb{R}$

$$y \longmapsto \int_a^b g(x) f(x, y) dx$$

Then, F is continuous.

Proof

For $x \in [a, b]$, write $G(x) = \int_0^x g(t) dt$, and $G \in BV([a, b])$

Then, for $y \in [c, d]$, we have $F(y) = \int_a^b f(x, y) dG(x)$. The result follows from the proposition. \square

PROPOSITION

Let $\alpha \in BV([a, b])$ and $f: \Omega = [a, b] \times [c, d] \rightarrow \mathbb{R}$ be continuous.

Suppose that $\frac{\partial f}{\partial y}$ is continuous on Ω . Then, for $y \in [c, d]$, $F'(y)$ exists and $\frac{d}{dy} \int_a^b f(x, y) d\alpha(x) = F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y) d\alpha(x)$

Proof

Fix $y_0 \in [c, d]$. For $y \in [c, d] \setminus \{y_0\}$, write $\frac{F(y) - F(y_0)}{y - y_0} = \int_a^b \frac{f(x, y) - f(x, y_0)}{y - y_0} d\alpha(x)$, which MVT implies equals $\int_a^b \frac{\partial f}{\partial y}(x, y') d\alpha(x)$ for some $y' = y'(x, y)$ in between y and y_0 .

When $y \rightarrow y_0$, we also have $y' \rightarrow y_0$. Also, the continuity of $\frac{\partial f}{\partial y}$ implies that $\lim_{y \rightarrow y_0} \frac{\partial f}{\partial y}(x, y) = \frac{\partial f}{\partial y}(x, y_0)$

Therefore, $F'(y_0) = \lim_{y \rightarrow y_0} \frac{F(y) - F(y_0)}{y - y_0} = \lim_{y \rightarrow y_0} \int_a^b \frac{\partial f}{\partial y}(x, y') d\alpha(x) = \int_a^b \lim_{y \rightarrow y_0} \frac{\partial f}{\partial y}(x, y') d\alpha(x) = \int_a^b \frac{\partial f}{\partial y}(x, y_0) d\alpha(x) \square$

proposition above

THEOREM (FUBINI'S THEOREM)

Let $\alpha \in BV([a, b])$, $\beta \in BV([c, d])$, and $f: Q \rightarrow \mathbb{R}$ be continuous.

Fix $(x, y) \in Q$, define $F(y) = \int_a^b f(x, y) d\alpha(x)$, $G(x) = \int_c^d f(x, y) d\beta(y)$.

Then, $F \in R(\beta)$ and $G \in R(\alpha)$ and $\int_c^d F(y) d\beta(y) = \int_a^b G(x) d\alpha(x) \Leftrightarrow \int_a^b \int_c^d f(x, y) d\beta(y) d\alpha(x) = \int_c^d \int_a^b f(x, y) d\alpha(x) d\beta(y)$.

COUNTEREXAMPLE FOR NONCONTINUOUS f

Let $f(x, y) = \begin{cases} 1, & x \in Q \\ 2y, & x \notin Q \end{cases}$

• We know $\int_0^1 f(x, y) dy = \int_0^1 1 dy = \int_0^1 2y dy = 1, x \in Q$. Thus, $\int_0^1 \int_0^1 f(x, y) dy dx = \int_0^1 1 dx = 1$

• Meanwhile, $\int_0^1 f(x, y) dx = \max(1, 2y)$ and $\int_0^1 f(x, y) dx = \min(1, 2y) \Rightarrow$ Other than $y = \frac{1}{2}$, $\int_0^1 f(x, y) dx$ is not well-defined.

PROOF OF FUBINI'S THEOREM (Behind the scenes: My hand is cramping so hard I'm using my elbow to control my pen movement)

WLOG, suppose α and β are nondecreasing. Since Q is compact, thus f is uniformly continuous.

Let $\epsilon > 0$, take $\delta > 0$, s.t. $\forall (x, y), (x', y') \in Q, |(x, y) - (x', y')|_\infty \leq \delta \Rightarrow |f(x, y) - f(x', y')| \leq \epsilon$

Let $P_x = (x_k)_{0 \leq k \leq m} \in P([a, b])$, $P_y = (y_k)_{0 \leq k \leq n} \in P([c, d])$, s.t. $\|P_x\|, \|P_y\| \leq \delta$. Now, let us rewrite one of our integrals.

$$\int_a^b \int_c^d f(x, y) d\beta(y) d\alpha(x) = \sum_{k=1}^m \sum_{j=1}^{n+1} \int_{x_{k-1}}^{x_k} \int_{y_{j-1}}^{y_j} f(x, y) d\beta(y) d\alpha(x) \stackrel{\text{MVT}}{=} \sum_{k=1}^m \sum_{j=1}^{n+1} \int_{x_{k-1}}^{x_k} f(x, y_k) d\beta(y) d\alpha(x) \text{ for some } y_k' \in [y_{j-1}, y_j].$$

$$\text{By MVT, } = \sum_{k=1}^m \sum_{j=1}^{n+1} \int_{x_{k-1}}^{x_k} f(x, y_k) (\beta(y_k) - \beta(y_{k-1})) d\alpha(x)$$

$$\text{By MVT, } = \sum_{k=1}^m \sum_{j=1}^{n+1} f(x_k, y_k) [\alpha(x_k) - \alpha(x_{k-1})] (\beta(y_k) - \beta(y_{k-1})), \text{ for some } (x_k', y_k') \in [x_{k-1}, x_k] \times [y_{k-1}, y_k].$$

$$\text{Similarly, we have } \int_c^d \int_a^b f(x, y) d\alpha(x) d\beta(y) = \sum_{k=1}^m \sum_{j=1}^{n+1} f(x_k', y_k') [\alpha(x_k) - \alpha(x_{k-1})] [\beta(y_k) - \beta(y_{k-1})]$$

$$\text{Taking their difference, we find: } |\int_a^b \int_c^d f(x, y) d\beta(y) d\alpha(x) - \int_c^d \int_a^b f(x, y) d\alpha(x) d\beta(y)| \leq \sum_{k=1}^m \sum_{j=1}^{n+1} |f(x_k', y_k') - f(x_k'', y_k'')| \xrightarrow{\epsilon \text{ as assumed}} \Delta \alpha \cdot \Delta \beta \cdot \epsilon = \epsilon [\alpha(b) - \alpha(a)][\beta(d) - \beta(c)]$$

RIEMANN INTEGRALS**DEFINITION**

Let $S \subseteq \mathbb{R}$ be a subset. We may say that S has measure zero if for every $\epsilon > 0$, \exists a countable family $\{U_i = (a_i, b_i) | i \in I\}$ of open intervals such that:

(i) $S \subseteq \bigcup_{i \in I} (a_i, b_i)$ ("S can be covered by these open intervals")

(ii) The sum of lengths satisfy $\sum_{i \in I} |U_i| = \sum_{i \in I} (b_i - a_i) \leq \epsilon$

where $|U_i| = b_i - a_i$ denotes the length of the open interval U_i for $i \in I$.

EXAMPLE

1) If $S \subseteq \mathbb{R}$ is a finite set, then S has measure zero

2) If $S = \{s_n | n \in \mathbb{N}\} \subseteq \mathbb{R}$ is a countable set, then S has measure zero

For $\epsilon > 0$, take $U_i = (s_i - \frac{\epsilon}{2^{i+1}}, s_i + \frac{\epsilon}{2^{i+1}})$. Then, $\sum_{i \in I} |U_i| = \sum_{i \in I} \frac{\epsilon}{2^{i+1}} = \epsilon$

THEOREM (LEBESGUE'S CRITERION) (Sneak peak to next main section of notes)

Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded, and D be the set of its discontinuities. Then, $f \in R(x; a, b) \Leftrightarrow D$ has measure zero

(Sorry no proof today qwq my hand is hurting too much)

3-25-25 (WEEK 6)

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LESBEGUE'S CRITERION

(Too tired and depressed to make my own notes for the main thm proof, sorry... I'll include all the other relevant notation and lemmas here... short notes for a reason. :)

PROPOSITION

Let $S = \{S_n\}_{n \geq 1}$ be a sequence of measure zero subsets. Then, their union $S = \bigcup_{n \geq 1} S_n$ also has measure zero.

Proof

Let $\epsilon > 0$. $\forall n \geq 1$, since S_n has measure zero, we may find a countable family $\{U_{n,i} | i \geq 1\}$ of open intervals covering S_n , and such that $\sum_{i \geq 1} |U_{n,i}| \leq \frac{\epsilon}{2^n}$.

Notice, $\mathcal{U} := \{U_{n,i} | i, n \geq 1\}$ is still countable, since it is a countable union of countable families.

It also covers S , and $\sum_{n \geq 1} \sum_{i \geq 1} |U_{n,i}| = \sum_{n \geq 1} \frac{\epsilon}{2^n} = \epsilon \quad \square$

DEFINITION

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function. For any subset $A \subseteq [a, b]$, define the oscillation of f on A to be $\Omega_f(A) := \sup \{f(x) - f(y) | x, y \in A\}$.
For $x \in [a, b]$, define the oscillation of f at x to be $\omega_f(x) := \lim_{h \rightarrow 0} \Omega_f(B(x, h) \cap [a, b])$. ← The idea is to view this point as an infinitely small ball

REMARK

" $\Omega_f(A)$ " has actually appeared before already in Darboux sums

Notice, $\forall A \subseteq B \subseteq [a, b]$, we have $\Omega_f(A) \leq \Omega_f(B)$. Then, the function $h \mapsto \Omega_f(B(x, h) \cap [a, b])$ is nondecreasing and the limit as $h \rightarrow 0^+$ is always well-defined since it is also bounded.

PROPOSITION

f is continuous at $x \iff \omega_f(x) = 0$

Proof

" \Rightarrow ": By def, fix $\epsilon > 0$, then $\exists \delta > 0$, s.t. $\forall y \in B(x, \delta)$, $|f(x) - f(y)| < \epsilon$

∴ $\forall h \in (0, \delta)$, we have $\Omega_f(B(x, h) \cap [a, b]) \leq 2\epsilon$
As $\epsilon > 0$ can be arbitrarily small, thus $\omega_f(x) = 0$

" \Leftarrow ": Fix $\epsilon > 0$. By def, $\exists \delta > 0$, s.t. $h < \delta \Rightarrow \Omega_f(B(x, h) \cap [a, b]) < \epsilon$ (expand the limit def of ω_f)

∴ $\forall y \in B(x, \delta)$, $|f(x) - f(y)| < \epsilon$, so by def, f is cont. at x . \square

PROPOSITION

$\forall \epsilon > 0$, $J_\epsilon := \{x \in [a, b] | \omega_f(x) \geq \epsilon\}$ is a closed set

Proof

Assume not. Let $\epsilon > 0$ be s.t. J_ϵ is not closed. Let $x \in \overline{J_\epsilon} \setminus J_\epsilon$, i.e., $\omega_f(x) < \epsilon$

By def of limit, $\exists \delta > 0$, s.t. $h < \delta \Rightarrow \Omega_f(B(x, h) \cap [a, b]) < \epsilon$

∴ $\forall y \in B(x, h)$, we also have $\omega_f(y) < \epsilon$, i.e., $B(x, h) \cap J_\epsilon = \emptyset$, which contradicts " $x \in \overline{J_\epsilon}$ ". ←

LEMMA

Suppose $\omega_f(x) < \epsilon \ \forall x \in [a, b]$. Then, $\exists \delta > 0$, s.t. $\forall [c, d] \subseteq [a, b]$ with $|d - c| < \delta$, we have $\Omega_f([c, d]) < \epsilon$

Proof

$\forall x \in [a, b]$, by def of limit, $\exists \delta_x > 0$, s.t. $\Omega_f(B(x, \delta_x) \cap [a, b]) < \epsilon$

closed + bounded for over \mathbb{R}

∴ $\{B(x, \frac{\delta_x}{2}) | x \in [a, b]\}$ is an open covering of the compact set $[a, b]$

∴ By Borel-Lebesgue property, we can extract a finite subcovering

Let $x_1, \dots, x_n \in [a, b]$ be s.t. $\{B(x_i, \frac{\delta_i}{2}) | 1 \leq i \leq n\}$ covers $[a, b]$. Take $\delta = \min \{\frac{\delta_i}{2} | 1 \leq i \leq n\}$.
We only need to prove " $\exists \delta$ " anyway

For any segment $[c, d] \subseteq [a, b]$ with $|d - c| < \delta$, $\exists 1 \leq i \leq n$, s.t. $[c, d] \cap B(x_i, \frac{\delta_i}{2}) \neq \emptyset$ (obv, but it is a covering, so...)

∴ $[c, d] \subseteq B(x_i, \frac{\delta_i}{2} + \delta) \cap [a, b] \subseteq B(x_i, \delta_i) \cap [a, b]$ (strategic choice of $\delta = \min \{\frac{\delta_i}{2}\}$)

∴ From our remark, $\Omega_f([c, d]) \leq \Omega_f(B(x_i, \delta_i) \cap [a, b]) < \epsilon$. \square

SEQUENCES AND SERIES

We take sequences in metric spaces

We take series in normed vector spaces in order to take summations

↳ Sometimes, we also need completeness, i.e. a Banach space

BASIC NOTATIONS

REMINDERS (R-valued sequences)

DEFINITION

Let $(a_n)_{n \geq 1}$ be a real-valued sequence

We say $(a_n)_{n \geq 1}$ converges to $l \in \mathbb{R}$, i.e. $a_n \xrightarrow{n \rightarrow \infty} l$ if $\forall \epsilon > 0, \exists N \geq 1$, s.t. $|a_n - l| \leq \epsilon \quad \forall n \geq N$

CAUCHY'S CONDITION

In a complete vector space, to check for convergence, it is enough to check: $\forall \epsilon > 0, \exists N \geq 1$, s.t. $\forall m, n \geq N$, $|a_m - a_n| \leq \epsilon$ (no need to know the limit l to compute)

PROPOSITION

(1) If $(a_n)_{n \geq 1}$ is nondecreasing and bounded above by some $M < \infty$, then $(a_n)_{n \geq 1}$ converges to a limit $l \leq M$

(2) If $(a_n)_{n \geq 1}$ is nonincreasing and bounded below by some $M > -\infty$, then $(a_n)_{n \geq 1}$ converges to a limit $l \geq M$

DEFINITION

Given two sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ of real numbers, we say they are adjacent if one is increasing and the other is decreasing with $a_n - b_n \xrightarrow{n \rightarrow \infty} 0$

PROPOSITION

If $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ are adjacent, then $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$

DEFINITION

Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be two real sequences. Here are some asymptotic notations (CSを勉強するが今は見て、笑った)

1) We say that a is dominated by b , denoted by $a_n = O(b_n)$, if \exists bounded sequence $c = (c_n)_{n \geq 1}$ and $N \in \mathbb{N}$, s.t. $a_n = c_n b_n \quad \forall n \geq N$

2) We say that a is negligible compared to b , i.e. $a_n = o(b_n)$, if \exists sequence $\varepsilon = (\varepsilon_n)_{n \geq 1}$ that converges to 0 and $N \in \mathbb{N}$, s.t. $a_n = \varepsilon_n b_n \quad \forall n \geq N$

3) We say that a is equivalent to b , i.e. $a_n \sim b_n$ if \exists sequence $c = (c_n)_{n \geq 1}$ that converges to 1 and $N \in \mathbb{N}$, s.t. $a_n = c_n b_n \quad \forall n \geq N$

Remark: \sim is an equivalence relation in $\mathbb{R}^{\mathbb{N}}$, but

EXAMPLES

1) Define $a_n = \frac{1}{n}$, $b_n = \frac{1}{n} + \frac{1}{n^2}$ for $n \geq 1$, then $a_n = O(b_n)$ and $a_n \sim b_n$

2) Let $(a_n)_{n \geq 1} = (0, 1, 1, \dots)$ and $(b_n)_{n \geq 1} = (1, 1, \dots)$. Then, $a_n = O(b_n)$ and $a_n \sim b_n$

3) Let $a_n = n^2$, $b_n = 2^n$ for $n \geq 1$

DEFINITIONS

Let $(u_n)_{n \geq 0}$ be a sequence in a normed vector space $(W, \| \cdot \|)$

- Define $s_0 := 0$, $s_n := u_1 + \dots + u_n$ for $n \geq 1$
- The series with general term u_n is the sequence $(s_n)_{n \geq 1}$, denoted as $\sum_{n=1}^{\infty} u_n$ — same notation but different meaning
- For $n \geq 0$, s_n is called the n^{th} partial sum of $\sum u_n$
- We say that the series $\sum u_n$ converges if the sequence $(s_n)_{n \geq 0}$ converges in $(W, \| \cdot \|)$. In this case, we write $\sum_{n=1}^{\infty} u_n$ for the limit
- In the case that $\sum_{n=1}^{\infty} u_n$ converges, we define its n^{th} remainder by $R_n = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^n u_k = \sum_{k=n+1}^{\infty} u_k$

REMARK

$(S_n)_{n \geq 0}$ converges $\Leftrightarrow \sum(S_{n+1} - S_n)$ converges, since $\sum_{n=0}^{N-1} (S_{n+1} - S_n) = S_N - S_0 = S_N$

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PROPOSITION

- (1) If the series $\sum u_n$ converges, then $(S_n)_{n \geq 1}$ is a Cauchy sequence
- (2) If $(W, \|\cdot\|)$ is a Banach space, then the series $\sum u_n$ converges iff $(S_n)_{n \geq 1}$ is Cauchy

Proof

CVG \Rightarrow Cauchy in general metric space

Cauchy \Rightarrow CVG in complete space

COROLLARY (CAUCHY'S CRITERION)

Suppose that $(W, \|\cdot\|)$ is a Banach space. The series $\sum u_n$ converges iff $\forall \varepsilon > 0, \exists N \geq 1$, s.t. $\forall n \geq N, \forall k \geq 1, \|u_{n+1} + \dots + u_{n+k}\| < \varepsilon$

Proof

For $n \geq 1, k \geq 1$, $S_{n+k} - S_n = u_{n+1} + \dots + u_{n+k}$. Then, by the above proposition, QED.

Use norms

COROLLARY

If $\sum u_n$ is a convergent series, then $\lim_{n \rightarrow \infty} u_n = 0$

Proof

It is a satisfaction of the fact that $(S_n)_{n \geq 0}$ is a Cauchy sequence. \square

REMARK

The converse does not hold, $\sum \frac{1}{n} = \infty$

DEFINITION

Suppose that $(W, \|\cdot\|)$ is a Banach space, and let $\sum u_n$ be a series with general terms in W

- If the series $\sum \|u_n\|$ converges, we say that the series $\sum u_n$ converges absolutely (w/o norm)
- If the series $\sum u_n$ converges but not absolutely, then we say $\sum u_n$ converges conditionally

EXAMPLE

$\sum \frac{(-1)^{n+1}}{n} = \ln 2$ is convergent but not absolutely convergent

THEOREM

For a Banach space $(W, \|\cdot\|)$, if $\sum u_n$ converges absolutely, then $\sum u_n$ converges

Proof

$\forall n, k \geq 1$, we have $\|u_{n+1} + \dots + u_{n+k}\| \leq \|u_{n+1}\| + \dots + \|u_{n+k}\|$

\therefore Cauchy's condition for $\sum \|u_n\| \Rightarrow$ Cauchy's condition for $\sum u_n$ (Shows how useful Cauchy's criterion is.)

APPLICATIONS

Useful in metric spaces like vector spaces of matrices or function spaces, we only need to examine numbers due to the norm.

SERIES WITH NONNEGATIVE TERMS

COMPARISON BETWEEN SERIES

PROPOSITION

Let $\sum u_n$ be a series with nonnegative terms, then $\sum u_n$ converges $\Leftrightarrow (S_n)_{n \geq 0}$ is bounded from above

Sequence of partial sums

PROPOSITION (COMPARISON TEST)

We consider two nonnegative series $\sum u_n$ and $\sum v_n$ satisfying $\forall n \geq 1, 0 \leq u_n \leq v_n$.

(1) If $\sum v_n$ converges, then $\sum u_n$ converges.

(2) If $\sum u_n$ diverges, then $\sum v_n$ diverges.

Proof

Let $(S_n)_{n \geq 0}$ be the partial sums of $\sum u_n$ and $(T_n)_{n \geq 0}$ be the partial sums of $\sum v_n$. Then, $\forall n \geq 0, S_n \leq T_n$. Conclude by prop above. \square

THEOREM

Let $\sum u_n$ and $\sum v_n$ be series with nonnegative terms.

(1) If $v_n = O(u_n)$, and $\sum u_n$ converges, then $\sum v_n$ converges.

(2) If $u_n \sim v_n$, then $\sum u_n$ and $\sum v_n$ either both converge or both diverge.

Proof

(2) is a direct consequence of (1), $\because v_n = O(u_n)$ and $u_n = O(v_n) \Leftrightarrow u_n \sim v_n$.

\therefore It suffices to prove (1)

Suppose $v_n = O(u_n)$.

Let $M > 0$ and $N \geq 1$, s.t. $u_n \leq M v_n \quad \forall n \geq N$.

Then, $\forall n \geq N, \sum_{k=1}^n v_k = \sum_{k=1}^N v_k + \sum_{k=N+1}^n v_k \leq \sum_{k=1}^N v_k + M \sum_{k=N+1}^n u_k$

Since $\sum u_n$ converges and $(\sum_{k=N+1}^n u_k)_{n \geq N}$ is bounded from above $\therefore \sum v_n$ converges \square

REMARK

Define $u_n = \frac{(-1)^n}{n}$ and $v_n = n$, $n \geq 1$. It is clear that $u_n = O(v_n)$, but $\sum u_n$ converges and $\sum v_n$ diverges. \therefore "non-negative" is a really important same with $u_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}$ and $v_n = \frac{(-1)^n}{\sqrt{n}}$, with $u_n \sim v_n$.

EXAMPLE

Let's study the behavior of $\sum \frac{1}{n^2}$.

For $k \geq 2$, $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \leq \frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$

$\therefore \sum_{n=k}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{k-1} - \frac{1}{n} \xrightarrow{n \rightarrow \infty} \frac{1}{k-1} \quad \forall k \geq 2$

$\therefore \sum \frac{1}{n^2}$ converges

Moreover, $\sum_{k=2}^{\infty} \frac{1}{k^2} \geq 1 + \sum_{k=2}^{\infty} \frac{1}{k^2} \geq 1 + 1 = 2$

Consider $R_n := \sum_{k=n+1}^{\infty} \frac{1}{k^2}$. We know that $\sum_{k=n+1}^{\infty} \left(\frac{1}{k} - \frac{1}{k-1} \right) = \frac{1}{n+1} \leq R_n \leq \sum_{k=n+1}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{n}$ } We can try and consider the denominator
 $\therefore R_n \sim \frac{1}{n}$ as $n \rightarrow +\infty$

PROPOSITION (RIEMANN SERIES)

Let $\alpha \in \mathbb{R}$. The Riemann series $\sum \frac{1}{n^\alpha}$. We note $\sum \frac{1}{n^\alpha}$ converges $\Leftrightarrow \alpha > 1$

Proof

For $\alpha > \beta, n \geq 1$, define $\frac{1}{\alpha^n} \geq \frac{1}{\beta^n}$

• $\alpha = 1$: $\sum \frac{1}{n}$ is divergent, so $\forall \alpha < 1$, $\sum \frac{1}{n^\alpha}$ is divergent.

To check divergence, see $\frac{1}{k} = \int_k^{k+1} \frac{dx}{x} \geq \int_k^{k+1} \frac{dx}{k+1} = \ln(k+1) - \ln(k) \quad \forall k \geq 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{n=1}^{\infty} (\ln(n+1) - \ln(n)) = \ln(n+1) \xrightarrow{n \rightarrow \infty} +\infty$

$\therefore \sum \frac{1}{n}$ diverges

• $\alpha > 1$. For $k \geq 2$, we have $\frac{1}{k^\alpha} \leq \int_{k-1}^k \frac{dx}{x^\alpha} = \frac{1}{\alpha} \left(\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}} \right)$

Moreover, $\sum_{k=2}^{\infty} \frac{1}{k^\alpha} \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}} \right]$ is convergent, so $\sum_{k=2}^{\infty} \frac{1}{k^\alpha}$ is convergent. More specifically, $\sum_{k=2}^{\infty} \frac{1}{k^\alpha} \leq \sum_{k=2}^{\infty} \frac{1}{k^\alpha} \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}} \right] = \frac{1}{\alpha-1}$

REMARK (Studying the remainder) - integral trick

Let us study the remainder $\sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} = \frac{1}{n^\alpha} + \sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} \leq \frac{1}{n^\alpha} + \frac{1}{n-1} \frac{1}{n^{\alpha-1}}$

Similarly, $\forall n \geq 2, \frac{1}{k^\alpha} \geq \int_k^{k+1} \frac{dx}{x^\alpha} = \frac{1}{\alpha} \left[\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{k^{\alpha-1}} \right] \Rightarrow \sum_{k=n+1}^{\infty} \frac{1}{k^\alpha} \geq \frac{1}{\alpha-1} \frac{1}{n^{\alpha-1}}$

$\therefore R_n \sim \frac{1}{\alpha-1} \frac{1}{n^{\alpha-1}}$

The integral trick again from the previous part

PARTIAL SUMS AND REMAINDERS

(Sorry for the bad handwriting... once again, rain got into my stylus and I forgot to zip my bag in heavy rain, so...)

THEOREM

Let $\sum u_n$ and $\sum v_n$ be two series with nonnegative terms

Suppose that $u_n \sim v_n$. Then,

- 1) If $\sum u_n$ converges, then $\sum v_n$ converges, and $\sum_{k=n+1}^{\infty} u_k \sim \sum_{k=n+1}^{\infty} v_k$ (remainders, start from $n+1$ to ∞)
- 2) If $\sum u_n$ diverges, then $\sum v_n$ diverges, and $\sum_{k=1}^n u_k \sim \sum_{k=1}^n v_k$ (partial sums, start from 1 to n)

Proof

We have seen that $\sum u_n$ and $\sum v_n$ have the same behavior

- 1) Let $\epsilon > 0$, take $N > 0$, s.t. $(1-\epsilon)u_n \leq (1+\epsilon)v_n \leq (1+\epsilon)u_n \quad \forall n \geq N$

$\forall M \geq n \geq N$, we have $\sum_{k=n+1}^M v_k \leq \sum_{k=n+1}^M (1+\epsilon)u_k \leq (1+\epsilon)\sum_{k=n+1}^M u_k \Rightarrow \sum_{k=n+1}^M v_k \leq (1+\epsilon)\sum_{k=n+1}^M u_k$

Similarly, we also have $\sum_{k=n+1}^M v_k \geq (1-\epsilon)\sum_{k=n+1}^M u_k$, which gets us $\sum_{k=n+1}^M v_k \sim \sum_{k=n+1}^M u_k$

- 2) Let $\epsilon > 0$, take $N > 0$, s.t. $(1-\epsilon)u_n \leq v_n \leq (1+\epsilon)u_n \quad \forall n \geq N$.

Write $\sum_{k=1}^n v_k = \sum_{k=1}^n u_k + \sum_{k=N+1}^n v_k$ for $n \geq N$.

We want to show " $(1-2\epsilon)\sum_{k=1}^n u_k \leq \sum_{k=1}^n v_k \leq (1+2\epsilon)\sum_{k=1}^n u_k$ " for large enough n .

$$\text{"(a)": } \sum_{k=1}^n v_k \leq \sum_{k=1}^n u_k + (1+\epsilon)\sum_{k=N+1}^n v_k$$

$$\therefore \frac{\sum_{k=1}^n v_k}{\sum_{k=1}^n u_k} \xrightarrow{n \rightarrow \infty} 1, \text{ thus } \exists N' \geq N, \text{ s.t. } \sum_{k=1}^{N'} v_k \leq \epsilon \sum_{k=1}^{N'} u_k \quad (\text{div series})$$

Therefore, $\sum_{k=1}^n v_k \leq \epsilon \sum_{k=1}^n u_k + (1+\epsilon) \sum_{k=1}^n u_k = (1+2\epsilon) \sum_{k=1}^n u_k$ for $n \geq N'$

"(b)": For $n \geq N$, we have $\sum_{k=1}^n v_k \geq \sum_{k=1}^n u_k - (1-\epsilon) \sum_{k=N+1}^n u_k$

Now, take large $N' \geq N$, s.t. $\sum_{k=N+1}^{N-1} u_k \geq (1-2\epsilon) \sum_{k=N+1}^{N-1} u_k - \sum_{k=N+1}^{N-1} v_k \quad \forall n \geq N$

Then, we find $\sum_{k=1}^n v_k \geq (1-2\epsilon) \sum_{k=1}^n u_k \quad \square$

(Math idea: It diverges, so abuse the fact it's very large)

EXAMPLE (Algorithmic approach of analyzing remainders)

Let's study the asymptotic behavior of the harmonic series

Define $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \quad \forall n \geq 1$

- 1) Note that $\ln(1+x) \sim \infty$ when $x \rightarrow \infty$. We have $\frac{1}{n} \sim \ln(1+\frac{1}{n}) = \ln(n+1) - \ln(n)$

The series $\sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)]$ diverges, so $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

Moreover, we know that $H_n \sim \sum_{k=1}^n (\ln(k+1) - \ln(k)) = \ln(n+1) \sim \ln(n)$

$\therefore \frac{H_n}{\ln(n)} \rightarrow 1$ as $n \rightarrow \infty$, i.e. $H_n = \ln(n) + o(\ln(n))$

- 2) Let's understand the term $o(\ln(n))$.

Let $A_n := H_n - \ln(n)$ for $n \geq 1$

$$\ln(1-x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

For $n \geq 2$, $A_n - A_{n-1} = H_n - H_{n-1} - \ln(n) + \ln(n-1) = \frac{1}{n} + \ln(1 - \frac{1}{n}) = \frac{1}{n} + (-\frac{1}{n} - \frac{1}{2n^2} + o(\frac{1}{n^2})) = -\frac{1}{2n^2} + o(\frac{1}{n^2})$

$$\Rightarrow A_n - A_{n-1} \sim -\frac{1}{2n^2}$$

By the thm above, hence $\sum (A_n - A_{n-1})$ converges, i.e. $(A_n)_{n \geq 1}$ converges \square

Now, let $\gamma := \lim_{n \rightarrow \infty} A_n$, and call it Euler's constant

Then, we get $A_n = H_n - \ln(n) = \gamma + o(1) \Rightarrow H_n = \ln(n) + \gamma + o(1)$

- 3) Compare the partial sums for $\sum_{k=n+1}^{\infty} (A_k - A_{k-1}) = \gamma - A_n$ and $\sum_{k=n+1}^{\infty} \frac{1}{k^2} \sim \frac{1}{n}$ by Riemann series.

We get: $\gamma - A_n = \gamma - H_n + \ln(n) \sim -\frac{1}{2n} \Rightarrow \gamma - H_n + \ln(n) = -\frac{1}{2n} + o(\frac{1}{n}) \Rightarrow H_n = \ln(n) + \gamma + \frac{1}{2n} + o(\frac{1}{n})$

- 4) Let $D_n = H_n - \ln(n) - \gamma - \frac{1}{2n}$ for $n \geq 1$.

$$\begin{aligned} D_n - D_{n-1} &= H_n - H_{n-1} - \ln(n) + \ln(n-1) - \frac{1}{2n} + \frac{1}{2(n-1)} \\ &= \frac{1}{n} + \ln(1 - \frac{1}{n}) + \frac{1}{2n} - \frac{1}{2(n-1)} \left(1 - \frac{1}{n}\right) \quad \frac{1}{1-x} = 1 + x + x^2 + \dots \\ &= \frac{1}{n} - \left(\frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + o(\frac{1}{n})\right) - \frac{1}{2n} + \frac{1}{2n} \left(1 + \frac{1}{n} + \frac{1}{n^2} + o(\frac{1}{n^2})\right) = \frac{1}{6n^2} + o(\frac{1}{n^2}) \end{aligned}$$

Again, by Riemann series, $\sum \frac{1}{n^3}$ converges $\Rightarrow \sum (D_n - D_{n-1})$ converges, moreover, $\sum_{k=n+1}^{\infty} (D_k - D_{k-1}) \sim \sum_{k=n+1}^{\infty} \frac{1}{k^3} \sim \frac{1}{2n^2}$ $\Rightarrow D_n \sim \frac{1}{2n^2} \Rightarrow H_n = \ln(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + o(\frac{1}{n^2})$

5) We have the following expression: $H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^n \frac{B_{2k}}{2k n^{2k}} + O\left(\frac{1}{n^{2k}}\right)$ when $n \rightarrow \infty$, where $(B_{2k})_{k \geq 1}$ are Bernoulli numbers

REMARK

Using this similar approach, we can derive the infamous Stirling's formula (i.e. every CS student's nightmare when learning about asymptotic notation it seems.)

COMPARISON BETWEEN SERIES AND INTEGRALS

PROPOSITION

Let $f: [1, +\infty) \rightarrow \mathbb{R}_+$ be a nondecreasing function with $\lim_{x \rightarrow \infty} f(x) = 0$. $\forall n \geq 1$, define $S_n = \sum_{k=1}^n f(k)$, $I_n = \int_1^n f(t) dt$, $D_n = S_n - I_n$

Then, the following properties hold:

- (1) For $n \geq 1$, we have $0 \leq f(n+1) \leq D_{n+1} \leq D_n \leq f(1)$
- (2) The sequence $(D_n)_{n \geq 1}$ converges, and denote $D = \lim_{n \rightarrow \infty} D_n$
- (3) The series $\sum f(n)$ and the integral $\int_1^\infty f(t) dt := \lim_{n \rightarrow \infty} \int_1^n f(t) dt$ have the same behavior (both converge or both diverge)
- (4) $\forall n \geq 1$, we have $0 \leq D_n - D \leq f(n)$

Proof.

1) For $k \geq 1$, we have $f(k+1) \leq \int_k^{k+1} f(t) dt \leq f(k)$

Then, $I_{n+1} = \sum_{k=1}^n \int_k^{k+1} f(t) dt \leq \sum_{k=1}^n f(k) = S_n \Rightarrow f(n+1) = S_{n+1} - S_n \leq S_{n+1} - I_{n+1} = D_{n+1}$

$D_{n+1} - D_n = (S_{n+1} - S_n) - (I_{n+1} - I_n) = f(n+1) - \int_{n+1}^{n+2} f(t) dt \leq 0 \therefore (D_n)_{n \geq 1}$ is decreasing

$\therefore D_{n+1} \leq D_n \leq \dots \leq D_1 = S_1 - I_1 = f(1) \checkmark$

2) We know $(D_n)_{n \geq 1}$ is decreasing and bounded from below by 0, hence it converges

3) We know $D = \lim_{n \rightarrow \infty} D_n = \lim_{n \rightarrow \infty} (S_n - I_n)$ converges

If $\lim_{n \rightarrow \infty} S_n$ exists, then $\lim_{n \rightarrow \infty} I_n = D - \lim_{n \rightarrow \infty} S_n$ exists. Same argument for " $\lim_{n \rightarrow \infty} I_n$ exists".

4) We know $D_n - D = \sum_{k=n}^{\infty} (D_k - D_{k+1}) \geq 0$. We have $D_k - D_{k+1} = \int_k^{k+1} f(t) dt - f(k+1) \leq f(k) - f(k+1) \therefore \sum_{k=n}^{\infty} (D_k - D_{k+1}) \leq \sum_{k=n}^{\infty} (f(k) - f(k+1)) \Rightarrow D_n - D \leq f(n) \square$

REMARK

From (4), we find $0 \leq \sum_{k=1}^n f(k) - \int_1^n f(t) dt - D \leq f(n) \Rightarrow \sum_{k=1}^n f(k) = \int_1^n f(t) dt + D + O(f(n))$

If we take $f(x) = \frac{1}{x}$, we find $H_n = \sum_{k=1}^n \frac{1}{k} = \ln n + D + O\left(\frac{1}{n}\right)$ ($O\left(\frac{1}{n}\right) < o(1)$)

EXAMPLE (Riemann-Zeta function)

Let $s \in \mathbb{R}$, $f(x) = x^{-s}$ $\forall x \geq 1$

Consider the series $\sum_{n \geq 1} n^{-s} = \begin{cases} \text{convergent, } s > 1 \\ \text{divergent, } s \leq 1 \end{cases}$

We define the Riemann-Zeta function, i.e. $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$

Similarly, we can deduce $\sum_{k=1}^{\infty} \frac{1}{k^s} = \sum_{k=1}^{\infty} \left(\frac{1}{k^{s-1}} - 1 \right) + D(s) + O\left(\frac{1}{n^s}\right) = C(s) + \frac{1}{s-1} \frac{1}{n^{s-1}} + O\left(\frac{1}{n^s}\right)$

Since $\sum_{k=1}^{\infty} \frac{1}{k^s} \xrightarrow{n \rightarrow \infty} \zeta(s) = C(s)$, hence $\sum_{k=1}^{\infty} \frac{1}{k^s} = \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{s-1} \frac{1}{n^{s-1}} + O\left(\frac{1}{n^s}\right)$

PROPOSITION (BERTAND'S SERIES)

For $\alpha, \beta \in \mathbb{R}$, consider the series $\sum_{n \geq 2} \frac{1}{n^{\alpha} (\ln n)^{\beta}}$

1) When $\alpha > 1$, the series converges

2) When $\alpha = 1, \beta > 1$, the series converges

3) Otherwise, it diverges

Proof

1) Let $\alpha > 1$ and $\beta \in \mathbb{R}$.

Notice, $\frac{1}{n^{\alpha} (\ln n)^{\beta}} = o\left(\frac{1}{n^{\alpha}}\right)$ ($\frac{1}{n^{\alpha}} \div \frac{1}{n^{\alpha}} = \frac{1}{n^{\alpha}} \xrightarrow{n \rightarrow \infty} 0$)

Since $\sum \frac{1}{n^{\alpha}}$ converges $\therefore \sum \frac{1}{n^{\alpha} (\ln n)^{\beta}}$ also converges

2) Let $\alpha = 1, \beta > 1$, $f(x) = \frac{1}{x (\ln x)^{\beta}}, x \geq 2$.

$\int_2^n f(x) dx = \int_2^n \frac{1}{x (\ln x)^{\beta}} dx = \int_{\ln 2}^{\ln n} \frac{1}{y^{\beta}} dy$ is convergent

$\therefore \sum f(x)$ converges

3) Finally, when $\alpha = \beta = 1$, $\int_2^{\infty} \frac{1}{x \ln x} = \int_{\ln 2}^{\ln \infty} \frac{1}{y} dy$ is divergent

Now, for $\alpha = 1, \beta < 1$, we have $\frac{1}{x \ln x} \leq \frac{1}{x (\ln x)^{\beta}} \Rightarrow \sum \frac{1}{x (\ln x)^{\beta}}$ diverges

When $\alpha < 1$, thus $\frac{1}{n^{(\ln n)^{\alpha}}} = o(\frac{1}{n^{(\ln n)^{\beta}}})$, since $\sum \frac{1}{n^{(\ln n)^{\beta}}}$ diverges, thus $\sum \frac{1}{n^{(\ln n)^{\alpha}}}$ diverges too. \square