5-13-25 (WEEK 13)

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INTEGRATION

Let $I \subseteq \mathbb{R}$ be an interval, s.t. $J \neq \emptyset$.
Let $(f_n)_{n\geq 1}$ be a sequence of functions from I to a Banach space $(W, \|\cdot\|)$.

PROPOSITION

Suppose that on each segment $J \subseteq I$, all the f_n's are continuous and $f_n \to f$ uniformly.
Let $a \in I$, define $\varphi(x) = \int_a^x f(t)\,dt$, $\forall n \geq 1$, $\varphi_n(x) = \int_a^x f_n(t)\,dt$
Then, $\varphi_n \longrightarrow \varphi$ uniformly on every segment $J \subseteq I$.

REMARK

We may interchange the order of "$\lim_{n\to\infty}$" and "$\int_a^x$", $\underbrace{\lim_{n\to\infty} \int_a^x f_n(t)\,dt}_{\lim_{n\to\infty} \varphi_n(x)} = \underbrace{\int_a^x \lim_{n\to\infty} f_n(t)\,dt}_{\int_a^x f(t)dt = \varphi(x)}$ $\forall x \in I$

PROOF OF PROPOSITION

Let $J = [c,d] \subseteq I$ be a segment of I

For $x \in J$, we have $\|\varphi_n(x) - \varphi(x)\| = \|\int_a^x (f_n(t) - f(t))\,dt\| \leq \int_a^c \|f_n - f\|_{\infty,[c,d]}\,dt \leq \underbrace{|x-a|\,\|f_n - f\|_{\infty,[c,d]}}_{\text{(unif bounded)}} \xrightarrow{n\to\infty} 0$

$\therefore$ This convergence is uniform when $x \in [c,d]$

EXAMPLE

Let $(f_n)_{n\geq 1}$ be a sequence of continuous functions from $[0,1]$ to $\mathbb{R}$.
Suppose $f_n \longrightarrow f$ uniformly on $[0,1]$. We want to show that $\int_0^1 f_n^2 \longrightarrow \int_0^1 f^2$

For example, we may try to prove $f_n^2 \longrightarrow f^2$ uniformly on $[0,1]$

- For $x \in [0,1]$, we have $|f_n(x)^2 - f(x)^2| \leq |f_n(x) - f(x)|\,|f_n(x) + f(x)| \leq 2M \|f_n - f\|_\infty$, where M is given below.
- $(f_n)_{n\geq 1}$ is a convergent sequence in $(\mathcal{B}([0,1], \mathbb{R}), \|\cdot\|_\infty)$ — Only valid because $[0,1]$ not $[0,1)$
  This means that $(f_n)_{n\geq 1}$ is bounded, so $\|f_n\|_\infty \leq M$ for some $M > 0$ uniformly in n
- This gives us $|f_n(x)^2 - f(x)^2| \leq 2M \|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$. This implies $f_n^2 \longrightarrow f^2$ uniformly on $[0,1]$
- More generally speaking, for any integer $p > 1$, we have $\int_0^1 (f_n)^p \longrightarrow \int_0^1 f^p$

EXAMPLE

For $n \in \mathbb{N}$, define $f: [0,1] \longrightarrow \mathbb{R}$, $x \longmapsto x^n$, $f_n \longrightarrow \mathbb{1}_{\{1\}}$ pointwise

We have seen that $f_n \longrightarrow \mathbb{1}_{\{1\}}$ is NOT uniform because all the f_n's are cont. at 1 but $\mathbb{1}_{\{1\}}$ is NOT cont. at 1
However, for $n \in \mathbb{N}$, $\int_0^1 f_n(t)\,dt = \frac{1}{n+1} \xrightarrow{n\to\infty} 0$
$\int_0^1 \mathbb{1}_{\{1\}}\,dt = 0$
$\therefore$ The integrals converge uniformly

CAUTION

Convergence of integrals is much weaker than uniform convergence even if unif $\Rightarrow$ int conv.

COROLLARY

Let $\sum u_n$ be a series of continuous functions from $[a,b]$ to $(W, \|\cdot\|)$.
If $\sum u_n$ converges uniformly, then $\forall x \in [a,b]$, $\int_a^x \left(\sum_{n\geq 1} u_n(t)\right) dt = \sum_{n\geq 1} \left(\int_a^x u_n(t)\,dt\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_a^x u_k(t)\,dt\right)$, where the limit on the right side is uniform in $x \in [a,b]$.
$\therefore$ We can say that we can "integrate term by term"

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THEOREM

Let $\alpha \in BV([a,b])$. Let $(f_n)_{n\geq 1}$ be a sequence of bounded functions from $[a,b]$ to $\mathbb{R}$. Suppose $f_n \in \mathcal{R}(\alpha; a, b)$ $\forall n \geq 1$.
Suppose $f_n \longrightarrow f$ uniformly. Define $g(x) = \int_a^x f(t)\, d\alpha(t)$ and $g_n(x) = \int_a^x f_n(t)\, d\alpha(t)$ $\forall n \geq 1$

Then, (1) $f \in \mathcal{R}(\alpha; a, b)$, so g is well-defined
(2) $g_n \longrightarrow g$ uniformly

Proof

By decomposition thm, WLOG, we may assume α to be nondecreasing. The case $\alpha(a) = \alpha(b)$ is trivial, all the integrals are zero, so nothing to prove. Hence, let us assume $\alpha(a) < \alpha(b)$

Recall Riemann's condition: $\forall \varepsilon > 0, \exists P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $U_P(f,\alpha) - L_P(f,\alpha) \leq \varepsilon$ $\forall P \supseteq P_\varepsilon \Leftrightarrow f \in \mathcal{R}(\alpha; a, b)$

(1) Let us check Riemann's condition.
Given $\varepsilon > 0$. We may find $N > 0$, s.t. $\|f - f_n\|_\infty \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)}$ $\forall n \geq N$
Then, $\forall P \in \mathcal{P}([a,b])$, $|U_P(f - f_N, \alpha)| = \left|\sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} (f(x) - f_N(x)) \Delta\alpha_i\right| \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)} \left|\sum_{i=1}^{n} \Delta\alpha_i\right| = \varepsilon$. Similarly, $|L_P(f - f_N, \alpha)| < \varepsilon$.

Since $f_N \in \mathcal{R}(\alpha; a, b)$, we may find $P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $U_P(f_N, \alpha) - L_P(f_N, \alpha) \leq \varepsilon$ $\forall P \supseteq P_\varepsilon$.

$\therefore U_P(f,\alpha) - L_P(f,\alpha) \leq U_P(f - f_N, \alpha) - L_P(f - f_N, \alpha) + U_P(f_N, \alpha) - L_P(f_N, \alpha) \leq 3\varepsilon$ $\forall P \supseteq P_\varepsilon$. ✓

(2) For $n \geq 1$ and $x \in [a,b]$, $|g_n(x) - g(x)| = \left|\int_a^x (f_n(t) - f(t))\, d\alpha(t)\right| \leq \|f_n - f\|_\infty (\alpha(x) - \alpha(a)) \leq \|f_n - f\|_\infty (\alpha(b) - \alpha(a)) \xrightarrow{n \to \infty} 0$ indep of x □

COROLLARY

Let $\alpha \in BV([a,b])$. Let $\sum u_n$ be a series of bounded functions from $[a,b]$ to $\mathbb{R}$ s.t. $u_n \in \mathcal{R}(\alpha; a, b)$ $\forall n \geq 1$.
Suppose $\sum u_n$ converges uniformly. Then,
(1) $\sum u_n \in \mathcal{R}(\alpha; a, b)$
(2) $\forall x \in [a,b]$, $\int_a^x \sum_{n=1}^{\infty} u_n(t)\, d\alpha(t) = \sum_{n=1}^{\infty} \int_a^x u_n(t)\, d\alpha(t)$ and the convergence is uniform.

DERIVATIVES

Let $I \subseteq \mathbb{R}$ be an interval s.t. $\mathring{I} \neq \emptyset$, and $f_n : I \longrightarrow W$ $\forall n \geq 1$

THEOREM

Suppose (i) $\forall n \geq 1$, $f_n : I \to W$ is of class $\mathcal{C}^1$
(ii) The sequence $(f_n)_{n\geq 1}$ converges pointwise to $f \in \mathcal{F}(I, W)$
(iii) The sequence $(f_n')_{n\geq 1}$ converges uniformly to $g \in \mathcal{F}(I, W)$ on every segment I

Then, the following properties hold.
(1) The function f is of class $\mathcal{C}^1$ and $f' = g$
(2) The sequence $(f_n)_{n\geq 1}$ converges uniformly on every segment of I

Proof

Let $a \in I$, by (ii), we know $f_n(a) \xrightarrow{n \to \infty} f(a)$
(1) For $x \in I$, we have $\underbrace{\int_a^x g_n(t)\, dt}_{= f_n(x) - f_n(a)} \xrightarrow{n \to \infty} \int_a^x g(t)\, dt$ from before

$\therefore \lim_{n \to \infty} f_n(x) = f(a) + \int_a^x g(t)\, dt$ $\forall x \in I$ $\Rightarrow$ By FTC, $f'(x) = g(x)$ $\forall x \in I$
As g is continuous on I, we denote this f is $\mathcal{C}^1$.

(2) For $x \in I$, we have $f_n(x) - f(x) = \left(\int_a^x g_n(t)\, dt - \int_a^x g(t)\, dt\right) + (f_n(a) - f(a)) = \int_a^x (g_n(t) - g(t))\, dt + (f_n(a) - f(a))$

$\therefore \|f_n(x) - f(x)\| \leq \underbrace{|a - x|\, \|g_n - g\|_{\infty, [a,x]}}_{x \in J \text{ segment}} + \underbrace{\|f_n(a) - f(a)\|}_{\xrightarrow{n\to\infty} 0 \text{ indep of } x}$

$\therefore$ There is convergence on segments. □

COROLLARY

Let $p\geq 1$ be an integer, $f_n: I \to W$ be of class $\mathcal{C}^p$ $\forall n\geq 1$
Suppose (i) $(f_n^{(k)})_{n\geq 1}$ converges pointwise for $0\leq k\leq p-1$
(ii) $(f_n^{(p)})_{n\geq 1}$ converges uniformly on every segment
Then $f := \lim_{n\to\infty} f_n \in \mathcal{C}^p$ and for $0\leq k\leq p$, we have $f^{(k)}(x) = \lim_{n\to\infty} f_n^{(k)}(x)$ $\forall x\in \mathring{I}$

COROLLARY

Let $(u_n)_{n\geq 1}$ be a sequence of $\mathcal{C}^1$ functions $I \to W$.
Suppose (i) $\sum u_n$ converges pointwise
(ii) $\sum u_n'$ converges uniformly on every segment
Then, $\sum_{n\geq 1} u_n \in \mathcal{C}^1$ and $\left(\sum_{n\geq 1} u_n\right)' = \sum_{n\geq 1} u_n'$
We often say this as "We can differentiate the series term by term"

EXAMPLE

We want to check that ζ is of class $\mathcal{C}^1$.
Recall: $\zeta : (1, +\infty) \longrightarrow \mathbb{R}$
$s \longmapsto \sum_{n\geq 1} \frac{1}{n^s}$
For $n\geq 1$, let $u_n : (1,+\infty) \longrightarrow \mathbb{R}$ which is of class $\mathcal{C}^1$.
$s \longmapsto \frac{1}{n^s} = \exp(-s\ln n)$

- $\sum u_n(s)$ is well-defined $\forall s>1$, so $\sum u_n$ converges pointwise
- Let $a,b\in(1,+\infty)$, s.t. $1<a<b<+\infty$. Take $J=[a,b]$.
  For $n\geq 1$, we have $u_n'(s) = (-\ln n)\left(\frac{1}{n^s}\right)$ $\forall s>1$.

  Take $c\in(1,a)$. We see that $\|(u_n)|_J\|_\infty = \frac{\ln n}{n^a} = O\left(\frac{1}{n^c}\right)$ Since $\sum \frac{1}{n^c}$ converges, we deduce that $\sum (u_n)|_J$ converges normally, so uniformly.
  Apply corollary to conclude that $\zeta \in \mathcal{C}^1$ on $(1,+\infty)$ and $\zeta'(s) = \sum_{n\geq 1} -\frac{\ln n}{n^s}$ $\forall s>1$

COROLLARY

Let $p\geq 1$ be an integer, $(u_n)_{n\geq 1}$ be a sequence of $\mathcal{C}^p$ from $I \to W$
Suppose (i) For $0\leq k\leq p-1$, $\sum u_n^{(k)}$ converges pointwise
(ii) $\sum u_n^{(p)}$ converges uniformly on every segment
Then, $\sum_{n\geq 1} u_n \in \mathcal{C}^p$ and for $0\leq k\leq p$, we have $\left(\sum_{n\geq 1} u_n\right)^{(p)} = \sum_{n\geq 1} u_n^{(p)}$
Remark: This implies ζ is $\mathcal{C}^\infty$ and $\zeta^{(k)}(s) = \sum_{n\geq 1} \frac{(-\ln n)^k}{n^s}$ $\forall k\in\mathbb{N}$

EXAMPLE

Let $(W, \|\cdot\|)$ be a Banach space. Consider $\mathcal{L}_c(W)$ equipped over $|||\cdot|||$, which is still a Banach space. Moreover, it is also a normed algebra
Given $u\in\mathcal{L}_c(W)$, define $E_u : \mathbb{R} \longrightarrow \mathcal{L}_c(W)$. Define $u_n : \mathbb{R} \longrightarrow \mathcal{L}_c(W)$
$t \longmapsto \sum_{n\geq 0} \frac{t^n}{n!} u^n$ $\qquad t \longmapsto \frac{t^n}{n!} u^n$

- $\forall t\in\mathbb{R}$, $\|u_n(t)\| \leq \frac{|t|^n |||u^n|||}{n!} \leq \frac{|t|^n |||u|||^n}{n!}$
  Since $\sum_{n\geq 0} \frac{|t|^n |||u|||^n}{n!} = \exp(|t|\,|||u|||) < +\infty$, thus $\sum u_n(t)$ conv abs and hence converges