3-27-25 (WEEK 6)

SEQUENCES AND SERIES

We take sequences in metric spaces

We take series in normed vector spaces in order to take summations

↳ Sometimes, we also need completeness, i.e. a Banach space

BASIC NOTATIONS

REMINDERS ($\mathbb{R}$-valued sequences)

DEFINITION

Let $(a_n)_{n\geq 1}$ be a real-valued sequence

We say $(a_n)_{n\geq 1}$ **converges** to $l \in \mathbb{R}$, i.e. $x_n \xrightarrow[n\to\infty]{} l$ if $\forall \varepsilon > 0, \exists N \geq 1$, s.t. $|x_n - l| \leq \varepsilon$ $\forall n \geq N$

CAUCHY'S CONDITION

In a **complete** vector space, to check for convergence, it is enough to check: $\forall \varepsilon > 0, \exists N \geq 1$, s.t. $\forall m, n \geq N$, $|a_m - a_n| < \varepsilon$ (no need to know the limit l to compute)

PROPOSITION

(1) If $(a_n)_{n\geq 1}$ is nondecreasing and bounded above by some $M < \infty$, then $(a_n)_{n\geq 1}$ converges to a limit $l \leq M$

(2) If $(a_n)_{n\geq 1}$ is nonincreasing and bounded below by some $M > -\infty$, then $(a_n)_{n\geq 1}$ converges to a limit $l \geq M$

DEFINITION

Given two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of real numbers, we say they are **adjacent** if one is increasing and the other is decreasing with $a_n - b_n \xrightarrow[n\to\infty]{} 0$

PROPOSITION

If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are adjacent, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

DEFINITION

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two real sequences. Here are some asymptotic notations (CSを勉強するボクは見って、笑った)

1) We say that a is **dominated** by b, denoted by $a_n = O(b_n)$, if $\exists$ bounded sequence $c := (c_n)_{n\geq 1}$ and $N \in \mathbb{N}$, s.t. $a_n = c_n b_n$ $\forall n \geq N$
2) We say that a is **negligible** compared to b, i.e. $a_n = o(b_n)$, if $\exists$ sequence $\varepsilon = (\varepsilon_n)_{n\geq 1}$ that **converges to 0** and $N \in \mathbb{N}$, s.t. $a_n = \varepsilon_n b_n$ $\forall n \in \mathbb{N}$
3) We say that a is **equivalent** to b, i.e. $a_n \sim b_n$ if $\exists$ sequence $c = (c_n)_{n\geq 1}$ that **converges to 1** and $n \in \mathbb{N}$, s.t. $a_n = c_n b_n$ $\forall n \in \mathbb{N}$

Remark: $\sim$ is an equivalence relation in $\mathbb{R}^{\mathbb{N}}$, but

EXAMPLES

1) Define $a_n = \frac{1}{n}$, $b_n = \frac{1}{n} + \frac{1}{n^2}$ for $n \geq 1$, then $a_n = O(b_n)$ and $a_n \sim b_n$
2) Let $(a_n)_{n\geq 1} = (0, 1, 1, \ldots)$ and $(b_n)_{n\geq 1} = (1, 1, \ldots)$. Then, $a_n = O(b_n)$ and $a_n \sim b_n$
3) Let $a_n = n^2$, $b_n = 2^n$ for $n \geq 1$

DEFINITIONS

Let $(u_n)_{n\geq 0}$ be a sequence in a normed vector space $(W, \|\cdot\|)$

- Define $S_0 := 0$, $S_n := u_1 + \ldots + u_n$ for $n \geq 1$
- The **series** with general term u_n is the sequence $(S_n)_{n\geq 1}$, denoted as $\sum_{n\geq 1} u_n$ — same notation but different meaning
- For $n \geq 0$, S_n is called the n^{th} partial sum of $\sum u_n$
- We say that the series $\sum u_n$ converges if the sequence $(S_n)_{n\geq 0}$ converges in $(W, \|\cdot\|)$. In this case, we write $\sum_{n\geq 1} u_n$ for the limit
- In the case that $\sum_{n\geq 1} u_n$ converges, we define its **n^{th} remainder** by $R_n = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{n} u_k = \sum_{k=n+1}^{\infty} u_k$

REMARK

$(S_n)_{n\geq 0}$ converges $\Leftrightarrow \sum(S_{n+1}-S_n)$ converges, since $\underbrace{\sum_{n=0}^{N-1}(S_{n+1}-S_n)=S_N-S_0}_{\text{Telescoping series}}=S_N$

PROPOSITION

(1) If the series $\sum u_n$ converges, then $(S_n)_{n\geq 1}$ is a Cauchy sequence
(2) If $(W,\|\cdot\|)$ is a Banach space, then the series $\sum u_n$ converges iff $(S_n)_{n\geq 1}$ is Cauchy

Proof

CVG $\Rightarrow$ Cauchy in general metric space
Cauchy $\Rightarrow$ CVG in complete space

COROLLARY (CAUCHY'S CRITERION)

Suppose that $(W,\|\cdot\|)$ is a Banach space. The series $\sum u_n$ converges iff $\forall \varepsilon>0$, $\exists N\geq 1$, s.t. $\forall n\geq N$, $\forall k\geq 1$, $\|u_{n+1}+\dots+u_{n+k}\|<\varepsilon$ Use norms

Proof

For $n\geq 1$, $k\geq 1$, $S_{n+k}-S_n=u_{n+1}+\dots+u_{n+k}$. Then, by the above proposition, QED.

COROLLARY

If $\sum_{n\geq 1}u_n$ is a convergent series, then $\lim_{n\to\infty}u_n=0$

Proof

It is a satisfaction of the fact that $(S_n)_{n\geq 0}$ is a Cauchy sequence. □

REMARK

The converse does not hold, $\sum\frac{1}{n}=\infty$

DEFINITION

Suppose that $(W,\|\cdot\|)$ is a Banach space, and let $\sum u_n$ be a series with general terms in W

- If the series $\sum\|u_n\|$ converges, we say that the series $\sum u_n$ converges absolutely (w/o norm)
- If the series $\sum u_n$ converges but not absolutely, then we say $\sum u_n$ converges conditionally

EXAMPLE

$\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}=\ln 2$ is convergent but not absolutely convergent

THEOREM

For a Banach space $(W,\|\cdot\|)$, if $\sum u_n$ converges absolutely, then $\sum u_n$ converges

Proof

$\forall n,k\geq 1$, we have $\|u_{n+1}+\dots+u_{n+k}\|\leq\|u_{n+1}\|+\dots+\|u_{n+k}\|$
$\therefore$ Cauchy's condition for $\sum\|u_n\|\Rightarrow$ Cauchy's condition for $\sum u_n$ (Shows how useful Cauchy's criterion is)

APPLICATIONS

Useful in metric spaces like vector spaces of matrices or function spaces, we only need to examine numbers due to the norm.

SERIES WITH NONNEGATIVE TERMS

COMPARISON BETWEEN SERIES

PROPOSITION

Let $\sum u_n$ be a series with nonnegative terms, then $\sum_{n\geq 1}u_n$ converges $\Leftrightarrow$ $\underbrace{(S_n)_{n\geq 0}}_{\text{sequence of partial sums}}$ is bounded from above

PROPOSITION (COMPARISON TEST)

We consider two nonnegative series $\sum u_n$ and $\sum v_n$ satisfying $\forall n \geq 1,\ 0 \leq u_n \leq v_n$

(1) If $\sum v_n$ converges, then $\sum u_n$ converges

(2) If $\sum u_n$ diverges, then $\sum v_n$ diverges

Proof

Let $(S_n)_{n\geq 0}$ be the partial sums of $\sum u_n$ and $(T_n)_{n\geq 0}$ be the partial sums of $\sum v_n$. Then, $\forall n \geq 0$, $S_n \leq T_n$. Conclude by prop above. $\square$

THEOREM

Let $\sum u_n$ and $\sum v_n$ be series with nonnegative terms

(1) If $v_n = O(u_n)$, and $\sum u_n$ converges, then $\sum v_n$ converges

(2) If $u_n \sim v_n$, then $\sum u_n$ and $\sum v_n$ either both converge or both diverge

Proof

(2) is a direct consequence of (1), $\because v(n) = O(u_n)$ and $u_n = O(v_n) \Leftrightarrow u_n \sim v_n$

$\therefore$ It suffices to prove (1)

Suppose $v_n = O(u_n)$

Let $M > 0$ and $N \geq 1$, s.t. $v_n \leq M u_n$ $\forall n \geq N$.

Then, $\forall n \geq N$, $\sum_{k=1}^{n} v_k = \sum_{k=1}^{N-1} v_k + \sum_{k=N}^{n} v_k \leq \sum_{k=1}^{N-1} v_k + M \sum_{k=N}^{n} v_k$

Since $\sum u_n$ converges and $\left(\sum_{k=N}^{n} u_k\right)_{n \geq N}$ is bounded from above $\therefore \sum v_n$ converges $\square$

REMARK

Define $u_n = \frac{(-1)^n}{n}$ and $v_n = \frac{1}{n}$, $n \geq 1$. It is clear that $u_n = O(v_n)$, but $\sum u_n$ converges and $\sum v_n$ diverges $\therefore$ "non-negative" is a really important

Same with $u_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}$ and $v_n = \frac{(-1)^n}{\sqrt{n}}$, with $u_n \sim v_n$.

EXAMPLE

Let's study the behavior of $\sum \frac{1}{n^2}$.

For $k \geq 2$, $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \leq \frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$

$\therefore \sum_{k=n}^{N} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{n-1} - \frac{1}{N} \xrightarrow[N\to\infty]{} \frac{1}{n-1}$ $\forall n \geq 2$

$\therefore \sum \frac{1}{n^2}$ converges

Moreover, $\sum_{k\geq 1} \frac{1}{k^2} \geq 1 + \sum_{k\geq 2} \frac{1}{k^2} \geq 1 + \overset{\frac{1}{2-1}}{1} = 2$

Consider $R_n := \sum_{k\geq n+1} \frac{1}{k^2}$. We know that $\sum_{k\geq n+1} \left(\frac{1}{k} - \frac{1}{k-1}\right) = \frac{1}{n+1} \leq R_n \leq \sum_{k \geq n+1} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{n}$

$\therefore R_n \sim \frac{1}{n}$ as $n \to +\infty$

We can try and consider the denominator

PROPOSITION (RIEMANN SERIES)

Let $\alpha \in \mathbb{R}$. The Riemann series $\sum \frac{1}{n^\alpha}$. We note $\sum \frac{1}{n^\alpha}$ converges $\Leftrightarrow \alpha > 1$

Proof

For $\alpha > \beta$, $n \geq 1$, define $\frac{1}{\alpha^n} \geq \frac{1}{\beta^n}$

- $\alpha = 1$: $\sum \frac{1}{n}$ is divergent, so $\forall \alpha < 1$, $\sum \frac{1}{n^\alpha}$ is divergent

  To check divergence, see $\frac{1}{k} = \int_k^{k+1} \frac{dt}{k} \geq \int_k^{k+1} \frac{dt}{t} = \ln(k+1) - \ln(k)$ $\forall k \geq 1$

  $\therefore \forall n \geq 1$, $\sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=1}^{n} (\ln(k+1) - \ln(k)) = \ln(n+1) \xrightarrow{n\to\infty} +\infty$

  $\therefore \sum \frac{1}{n}$ diverges

- $\alpha > 1$. For $k \geq 2$, we have $\frac{1}{k^\alpha} \leq \int_{k-1}^{k} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\left(\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}}\right)$

  Moreover, $\sum_{k\geq 2} \frac{1}{1-\alpha}\left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}}\right]$ is convergent, so $\sum_{k\geq 2} \frac{1}{k^\alpha}$ is convergent. More specifically, $\sum_{k\geq 2} \frac{1}{k^\alpha} \leq \sum_{k\geq 2} \frac{1}{1-\alpha}\left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}}\right] = \frac{1}{\alpha-1}$

REMARK (Studying the remainder)

integral trick

The integral trick again from the previous part

Let us study the remainder $\sum_{k\geq n} \frac{1}{k^\alpha} = \frac{1}{n^\alpha} + \sum_{k\geq n+1} \frac{1}{k^\alpha} \leq \frac{1}{n^\alpha} + \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}$

Similarly, $\forall n \geq 2$, $\frac{1}{k^\alpha} \geq \int_k^{k+1} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\left[\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{k^\alpha}\right] \Rightarrow \sum_{k\geq n} \frac{1}{k^\alpha} \geq \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}$

$\therefore R_n \sim \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}$