Analysis Final Supplements

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SECTION I – TRUE OR FALSE

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1. If $f:[1,\infty)\to\mathbb{R}$ is a decreasing continuous function with $\lim_{x\to\infty} f(x)=0$, then $\left(\sum_{k=1}^{n} f(k)\right)\sim\left(\int_1^n f(t)dt\right)$ when $n\to\infty$

   False. Take $f(x)=\frac{1}{x^2}$, then $\sum_{k=1}^{\infty} f(k)=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$, but $\lim_{n\to\infty}\int_1^n \frac{1}{t^2}dt=\lim_{n\to\infty}\left[-\frac{1}{t}\right]_1^n=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$, but $\frac{\pi^2}{6}\not\sim 1$.

2. Let $(a_n)_{n\geq1}$ and $(b_n)_{n\geq1}$ be two sequences of positive real numbers. If $a_n\sim b_n$ when $n\to\infty$, then $e^{a_n}\sim e^{b_n}$ when $n\to\infty$

   False. Take $(a_n=n)_{n\geq1}$, $(b_n=n+1)_{n\geq1}$, then $a_n\sim b_n$ but $e^{a_n}\not\sim e^{b_n}$.

3. Let $(a_n)_{n\geq1}$ and $(b_n)_{n\geq1}$ be two sequences of positive real numbers. If $a_n\sim b_n$ when $n\to\infty$, then $\log(a_n)\sim\log(b_n)$ when $n\to\infty$

   False. Take $(a_n=1+\frac{1}{n})_{n\geq1}$, $(b_n=1-\frac{1}{n})_{n\geq1}$, then $a_n\sim b_n$.

   However, $\lim_{n\to\infty}\frac{\log(a_n)}{\log(b_n)}=\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\left(-\frac{1}{n^2}\right)\right)\div\left(\frac{n}{n-1}\left(\frac{1}{n^2}\right)\right)\right)=\lim_{n\to\infty}\frac{1-n}{1+n}=\lim_{n\to\infty}\frac{-1+\frac{1}{n}}{1+\frac{1}{n}}=-1\Rightarrow a_n\not\sim b_n$

4. Let $(u_{m,n})_{m,n\geq1}$ be a double sequence in $\mathbb{R}$. If both iterated limits $\lim_{m\to\infty}\left(\lim_{n\to\infty}u_{m,n}\right)$ and $\lim_{n\to\infty}\left(\lim_{m\to\infty}u_{m,n}\right)$ exist, they are equal.

   False. Let $(u_{m,n}=\mathbb{1}_{m\geq n})_{m,n\geq1}$, then: $\forall n\geq1$, $\lim_{n\to\infty}u_{m,n}=0\Rightarrow\lim_{m\to\infty}\left(\lim_{n\to\infty}u_{m,n}\right)=0$; $\forall m\geq1$, $\lim_{m\to\infty}u_{m,n}=1\Rightarrow\lim_{n\to\infty}\left(\lim_{m\to\infty}u_{m,n}\right)=1$

5. Let $f,g:[1,\infty)\to\mathbb{R}$ be two non-negative continuous functions. If $f(x)\sim g(x)$ when $x\to\infty$, then $\int_1^x f(t)dt\sim\int_1^x g(t)dt$ when $x\to\infty$

   False. Take $f(x)=\frac{1}{x^2}$, $g(x)=\frac{1}{x^2}+\frac{4}{x^3}$. Then, $f(x)\sim g(x)$ as $x\to\infty$. $\int_1^x f(t)dt=\int_1^x\frac{1}{t^2}dt=1-\frac{1}{x}\xrightarrow{x\to\infty}1$, $\int_1^x g(t)dt=\left[-\frac{1}{t}-\frac{2}{t^2}\right]_1^x\xrightarrow{x\to\infty}3$

6. Let $f,g:[1,\infty)\to\mathbb{R}$ be two continuous functions. If $f(x)\sim g(x)$ when $x\to\infty$, then $\int_1^\infty f(t)dt$ and $\int_1^\infty g(t)dt$ have the same behavior.

   ⌐ we need non-negative.

   False. $f(x)=\frac{\sin x}{\sqrt{x}}$, $g(x)=\frac{\sin x}{\sqrt{x}}+\frac{1}{x}$. Notice, $\frac{1}{\sqrt{x}}\xrightarrow{x\to\infty}0$ is decreasing and $\left|\int_1^x \sin t\,dt\right|\leq\left|\int_0^{2\pi}t\,dt\right|=2\pi<\infty\Rightarrow\int_1^\infty f(t)dt$ converges.

   However, $\int_1^\infty\frac{1}{t}dt$ diverges, so $\int_1^\infty g(t)dt$ diverges.

7. Let $f:[1,\infty)\to\mathbb{R}$ be a non-negative continuous function. If the limit $\lim_{n\to\infty}\left(\int_1^n f(t)dt\right)$ exists, then f is integrable on $[1,\infty)$

   True. For any sequence of $J_n=[1,n]$, we have $\bigcup_{n\geq1}J_n=I$. Then, $\int_1^\infty f(x)dx=\lim_{n\to\infty}\int_1^n f(t)dt$, so f is integrable on $[1,\infty)$. □

8. Let $f:[1,\infty)\to\mathbb{R}$ be a continuous function. If the limit $\lim_{n\to\infty}\left(\int_1^n f(t)dt\right)$ exists, then the improper integral $\int_1^\infty f(t)dt$ is convergent.

   False. Take $f(x)=\sin(2\pi x)$. Then, $\lim_{n\to\infty}\left(\int_1^n\sin(2\pi t)dt\right)=\frac{1}{2\pi}\lim_{n\to\infty}\left[\cos(2\pi)-\cos(2\pi n)\right]=0$. However, $\int_1^\infty f(t)dt$ is convergent.

9. Let $f:[1,\infty)\to\mathbb{R}$ be a continuous function. If there exists $M>0$ such that $\left|\int_1^x f(t)dt\right|<M$ for all $x\in[1,\infty)$, then $\int_1^\infty f(t)dt$ exists.

   False. Take $f(x)=\sin(2\pi x)$. Then, $\left|\int_1^x f(t)dt\right|=\left|\int_1^x\sin(2\pi t)dt\right|<\left|\int_0^{2\pi}t\,dt\right|=2\pi$ $\forall x\in[1,\infty)$. However, $\int_1^\infty f(t)dt$ does not exist.

10. Let $(f_n)_{n\geq1}$ be a sequence of uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$. If $(f_n)_{n\geq1}$ converges uniformly to a function f, then f is also uniformly continuous.

    True. By def, $\forall\varepsilon>0$, $\exists N>0$, s.t. $\forall n\geq N$, $x\in\mathbb{R}$, $|f_n(x)-f(x)|\leq\varepsilon$

    Also, by def, $\forall x,y\in\mathbb{R}$, $\forall\varepsilon>0$, $\exists\delta_n>0$, s.t. $|x-y|<\delta_n\Rightarrow|f_n(x)-f_n(y)|<\varepsilon$, for every n.

    $\therefore$ $\forall\varepsilon'=3\varepsilon>0$, $\forall x,y\in\mathbb{R}$, $\exists\delta_N$, s.t. $|f(x)-f(y)|\leq|f_n(x)-f(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)|<3\varepsilon=\varepsilon'\Rightarrow$ by def, f is unif conti. □

11. Let $(f_n)_{n\geq1}$ be a sequence of functions from $[0,1]$ to $\mathbb{R}$. Suppose that for any $n\in\mathbb{N}$, there exists $M_n>0$ such that $|f_n(x)|<M_n$ for all $x\in[0,1]$. If $\sum_{n\geq1}M_n$ converges, then $\sum_{n\geq1}f_n$ converges uniformly on $[0,1]$.

    True. As $\sum_{n\geq1}M_n$ converges, its remainder must converge to 0, so $\forall\varepsilon>0$, $\exists N>0$, s.t. $\forall n\geq N$, $\left|\sum_{k=n}^{\infty}M_k\right|<\varepsilon$. As $M_n>0$ $\forall n$, thus $\sum_{k=n}^{\infty}M_n<\varepsilon$

    $\therefore$ $\forall\varepsilon>0$, then $\forall m,n\geq N$, $\left|\sum_{k=1}^{n}f_k(x)-\sum_{k=1}^{m}f_k(x)\right|=\left|\sum_{k=m+1}^{n}f_k(x)\right|\leq\sum_{k=m+1}^{n}|f_k(x)|<\sum_{k=m+1}^{n}M_k<\sum_{k=m+1}^{\infty}M_n<\varepsilon$.

    $\therefore$ By def, $\sum_{n\geq1}f_n$ converges uniformly on $[0,1]$. □

    (Correct, but here is a simpler approach: $\sum_{n\geq1}M_n$ converges $\Rightarrow$ $\sum_{n\geq1}f_n$ converges normally $\Rightarrow$ $\sum_{n\geq1}f_n$ converges uniformly)

12. Let $(f_n)_{n\geq1}$ be a sequence of non-negative continuous functions from $[0,1]$ to $\mathbb{R}$. If there exists $M>0$ such that $\left\|\sum_{k=1}^{n}f_k\right\|<M$ for all $n\in\mathbb{N}$, then $\sum_{n\geq1}f_n$ converges uniformly

    False. Consider $f_n(x)=x^{n-1}-x^n$ for $n\geq1$, then $\left\|\sum_{k=1}^{n}x^{k-1}-x^k\right\|<1$ $\forall n\geq1$. However $\sum_{k=1}^{n}f_k(x)=1-x^n\xrightarrow{n\to\infty}f(x)=\mathbb{1}_{\{0,1\}}$, which is not uniform

    (Dini's Thm requires continuity to hold true.)

    Dini's Thm: (K,d): cpt, $(f_n)_{n\geq1}$: cont: $K\to\mathbb{R}$. Suppose (i) $(f_n)_{n\geq1}$ incr, i.e. $f_n(x)\leq f_{n+1}(x)$ $\Rightarrow$ $(f_n)_{n\geq1}$ conv unif to f

    (ii) $(f_n)_{n\geq1}$ conv ptwise to cont: $f:K\to\mathbb{R}$

13. If a sequence of continuous functions $f_n:[0,1]\to\mathbb{R}$ converges uniformly to f on $[0,1]$, then the indefinite integral $\int_0^x f_n(t)dt$ converges to $\int_0^x f(t)dt$ uniformly on $[0,1]$

    True. By def, $\forall\varepsilon>0$, $\exists N>0$, s.t. $\forall n\geq N$, $|f_n(x)-f(x)|<\varepsilon\Rightarrow\left|\int_0^x f_n(t)dt-\int_0^x f(t)dt\right|\leq\int_0^x|f_n(t)-f(t)|dt\leq\int_0^1\varepsilon\,dt=\varepsilon$, N indep of x

    $\therefore$ By def, $\int_0^x f_n(t)dt\xrightarrow{n\to\infty}\int_0^x f(t)dt$ unif on $[0,1]$

14. If a sequence of continuous functions $f_n:[0,\infty)\to\mathbb{R}$ converges uniformly to f on $[0,\infty)$, then the indefinite integral $\int_0^x f_n(t)dt$ converges to $\int_0^x f(t)dt$ uniformly on $[0,\infty)$

    False. Consider $f_n(x)=\frac{1}{n}$ $\forall n\geq1$, $x\in[0,\infty)$, we know $f_n\xrightarrow{n\to\infty}0$ uniformly, but $\int_0^x f_n(t)dt=\frac{x}{n}\xrightarrow{n\to\infty}0$ is not uniform.

15. Let $f(z)=\sum_{n\geq0}a_nz^n$ be a power series with $a_n\in\mathbb{C}$. Then, the radius of convergence of $f(z)$ is $\left(\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}\right)^{-1}$. Here, we view $\frac{1}{0}:=+\infty$ and $\frac{1}{+\infty}=0$.

    True. Notice, the behavior of $\lim_{n\to\infty}\left(|a_nz^n|^{\frac{1}{n}}\right)=|z|\left(\frac{1}{R_f}\right)\begin{cases}>1, & |z|>R_f\\<1, & |z|<R_f\end{cases}$. By the root test, $\sum_{n\geq0}a_nz^n$ converges for $|z|<R_f$ but diverges for $|z|>R_f$. Hence, by def, $R_f=\left(\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}\right)^{-1}$ □

16. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be $a_n\in\mathbb{C}$. If the radius of convergence of $f(z)$ is R, then there must exist some $z_0\in\mathbb{C}$ with $|z_0|=R$ such that $f(z_0)$ is convergent.
False. Take $a_n=1\ \forall n$, then $R(\sum_{n\geq 0}z^n)=1$, however, $\forall |z|=1$, $\sum_{n=0}^{\infty}z^n$ never converges (Correct example even if I thought it wasn't)

17. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with $a_n\in\mathbb{C}$. If there exists $z_0\in\mathbb{C}$ such that $f(z_0)$ converges, then $f(z)$ converges absolutely for all $z\in\mathbb{C}$ with $|z|\leq|z_0|$
False. Let $a_n=\frac{1}{n}\ \forall n$. $\forall z\in\mathbb{C}$, s.t. $|z|=1$ but $z\neq 1$, we know $\sum_{n=0}^{\infty}\frac{z^n}{n}$ converges. However, if $z=1$, $\sum_{n=0}^{\infty}\frac{1}{n}$ diverges.

18. Let $f:[-1,1]\to\mathbb{R}$ be a C^∞ function and $T_n(x)=\frac{f^{(k)}(0)}{k!}$ be its nth Taylor polynomial. If $(T_n)_{n\geq 0}$ converges uniformly to a function on $[-1,1]$, then $(T_n)_{n\geq 0}$ converges to f on $[-1,1]$.
False. For $f(x)=\begin{cases}e^{-\frac{1}{x}}, & x>0\\ 0, & x\leq 0\end{cases}$, it is C^∞, but $T_n(x)\equiv 0\ \forall n\geq 0$, so T_n converges uniformly to $0\neq f$.

SECTION 2 — MULTIPLE CHOICE

1. Let $(a_n)_{n\geq 1}$ be a sequence of non-negative real numbers. Consider the statements:
(i) $\lim_{n\to\infty}a_n=0$
(ii) The series $\sum_{n\geq 1}(-1)^na_n$ is convergent
Select the correct answer.
(A) (i) implies (ii) but (ii) does not imply (i)
(B) (ii) implies (i) but (i) does not imply (ii)
(C) (i) and (ii) are equivalent
(D) Neither (i) implies (ii) nor (ii) implies (i)

B

Proof
(i) $\not\Rightarrow$ (ii): $\frac{(-1)^n}{\sqrt{n+(-1)^n}}=\frac{(-1)^n}{\sqrt{n}}-\frac{1}{n}+o(\frac{1}{n})$, so $\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n+(-1)^n}}$ diverges, but $\lim_{n\to\infty}\frac{1}{\sqrt{n+(-1)^n}}=0$ (Need "decreasing to 0" for this to hold)
(ii) $\Rightarrow$ (i): By Cauchy's condition at $k=1$, $\lim_{n\to\infty}(-1)^na_n=0$. $\therefore \lim_{n\to\infty}a_n=0$ $\square$
$\sum u_n$ converges $\Leftrightarrow \forall\varepsilon>0, \exists N>0$, s.t. $\forall n\geq N, \forall k\geq 1$, $\|u_{n+1}+\dots+u_{n+k}\|<\varepsilon$ (Derived from Cauchy seq on $(\sum_{n=1}^{m}u_n)_{m\geq 1}$)

2. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers and $(c_n)_{n\geq 0}$ be their Cauchy product.
Consider the statements:
(i) Both $\sum_{n\geq 0}a_n$ and $\sum_{n\geq 0}b_n$ are convergent series
(ii) The series of Cauchy product $\sum_{n\geq 0}c_n$ is convergent.
Select the correct answer.
(A) (i) implies (ii) but (ii) does not imply (i)
(B) (ii) implies (i) but (i) does not imply (ii)
(C) (i) and (ii) are equivalent
(D) Neither (i) implies (ii) nor (ii) implies (i)

D

(ii) $\not\Rightarrow$ (i): The Cauchy product of these two divergent series $(-1-2-2-\dots)(-1+2-2+2-\dots)$ is convergent $(=1+0+\dots=1)$
(i) $\not\Rightarrow$ (ii): The Cauchy product of $\left(\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n+1}}\right)^2$. $\forall n$, $c_n=\sum_{k=0}^{n}\frac{(-1)^k}{\sqrt{k+1}}\left(\frac{(-1)^{n-k}}{\sqrt{n-k+1}}\right)=(-1)^n\sum_{k=0}^{n}\frac{1}{\sqrt{k+1}\sqrt{n-k+1}}$, $\left|\sum_{k=0}^{n}\frac{1}{\sqrt{k+1}\sqrt{n-k+1}}\right|\geq\sum_{k=0}^{n}\frac{2}{n+2}=\frac{2n+2}{n+2}\xrightarrow{n\to\infty}2$
$\therefore$ The Cauchy product diverges.

3. Let $(u_{m,n})_{m,n\geq 1}$ be a double sequence of real numbers. Consider the statements:

(i) The limit $\lim_{n\to\infty} u_{m,n}$ exists for all $m\geq 1$ and the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} u_{m,n})$ exists

(ii) The double limit $\lim_{m,n\to\infty} u_{m,n}$ exists.

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

D

(i) $\not\Rightarrow$ (ii): Let $u_{m,n} = \mathbb{1}_{m>n}$, then $\lim_{n\to\infty}\lim_{m\to\infty} u_{m,n} = 1$, $\lim_{m\to\infty}\lim_{n\to\infty} u_{m,n} = 0 \Rightarrow \lim_{m,n\to\infty} u_{m,n}$ DNE

(ii) $\not\Rightarrow$ (i): Let $u_{m,n} = \begin{cases} 1, & m-n=1 \\ -1, & m-n=-1 \\ 0, & \text{otherwise} \end{cases}$, and $S_{m,n} = \lim_{i\to\infty}\lim_{j\to\infty}\sum_{i=0}^{m}\sum_{j=0}^{n} u_{m,n}$. Of course, $\lim_{m,n\to\infty} S_{m,n} = 0$, however $\lim_{m\to\infty}\lim_{n\to\infty} S_{m,n}$ is undefined.

4. Let $f_n:\mathbb{R}\to\mathbb{R}$ be a sequence of functions. Consider the statements:

(i) $\sum_{n\geq 1} f_n$ converges uniformly

(ii) f_n converges uniformly to the zero function when $n\to\infty$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

A

(ii) $\not\Rightarrow$ (i): $f_n(x) = \frac{1}{n}$ $\forall n\geq 1$

(i) $\Rightarrow$ (ii): By Cauchy's condition, $\sum_{n\geq 1} f_n$ conv uniformly $\Leftrightarrow \forall \varepsilon>0, \exists N>0$, s.t. $\forall n\geq N, \forall k\geq 1, \|f_{n+1}+\dots+f_{n+k}\|_\infty < \varepsilon$, N indep of x

$\therefore \forall\varepsilon>0, \exists N>0$, s.t. $\forall n\geq N$, $\|f_{n+1}\|_\infty<\varepsilon$, N indep of $x \Rightarrow$ By def, f_n conv uniformly to 0 when $n\to\infty$

5. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. Consider the statements:

(i) $\sum_{n\geq 0} a_n$ converges

(ii) $f(x) = \sum_{n\geq 0} a_n x^n$ is well-defined on $[0,1)$ and $\lim_{x\to 1^-} f(x)$ exists.

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

A

(ii) $\not\Rightarrow$ (i): For $x\in[0,1)$, $\sum_{n\geq 0}(-1)^n x^n = \frac{1}{1+x}$ is well-defined and $\lim_{x\to 1^-} f(x) = \frac{1}{2}$, but $\sum_{n\geq 0}(-1)^n$ diverges

(i) $\Rightarrow$ (ii): $\sum_{n\geq 0} a_n$ converges $\Rightarrow \lim_{n\to\infty}\sqrt[n]{a_n}\leq 1 \Rightarrow \lim_{n\to\infty} x\sqrt[n]{a_n} < 1 \Rightarrow f(x) = \sum_{n\geq 1} a_n x^n$ is well-defined on $[0,1)$. ~~Also, $\lim_{x\to 1^-} f(x) = \sum_{n\geq 1} a_n$.~~ $\square$

Abel's theorem: Let $S_n = \sum_{k\geq n+1} a_k$

$\forall m>n$, we know $\sum_{k=n+1}^{m} a_k x^k = \sum_{k=n+1}^{m}(R_{k-1}-R_k)x^k = R_n x^{n+1} - R_m x^m + \sum_{k=n+1}^{m-1} R_k(x^{k+1}-x^k)$

$\therefore \sum_{k=n+1}^{\infty} a_k x^k = R_n x^{n+1} + \sum_{k\geq n+1} R_k(x^{k+1}-x^k)$

Let $\varepsilon>0$, take $N\in\mathbb{N}$, s.t. $|R_n|\leq\varepsilon$ $\forall n\geq N$, N indep of x

$\hookrightarrow \left|\sum_{k\geq n+1} a_k x^k\right| \leq |R_n x^{n+1}| + \sum_{k\geq n+1}|R_n|(x^{k+1}-x^k) \leq \varepsilon + \varepsilon x^{k+1} \leq 2\varepsilon$

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6. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. Suppose that the power series $f(x)=\sum_{n\geq 0} a_n x^n$ and $F(x)=\sum_{n\geq 0}\frac{a_n}{n+1}x^{n+1}$ have radius of convergence $R=1$. Consider the statements:
   (i) $f(x)$ is well-defined on $[0,1)$
   (ii) $F(x)$ is well-defined on $[0,1)$
   Select the correct answer.
   (A) (i) implies (ii) but (ii) does not imply (i)
   (B) (ii) implies (i) but (i) does not imply (ii)
   (C) (i) and (ii) are equivalent
   (D) Neither (i) implies (ii) nor (ii) implies (i)
   A (remember the direction properly, it's not integration, just a Dirichlet's test)
   (ii) $\not\Rightarrow$ (i): $a_n = n+1$
   (i) $\Rightarrow$ (ii): We know $\sum_{n\geq 0} a_n x^n$ is bounded, and $\frac{x}{n+1} \xrightarrow{n\to\infty} 0$ (decreases). $\therefore$ By Dirichlet's test, $\sum_{n\geq 0}\frac{a_n}{n+1}x^{n+1}$ converges.

7. Let $f:\mathbb{R}\to\mathbb{R}$ be a C^∞ function and $T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k$ be its n-th Taylor polynomial. Consider the statements:
   (i) f is a polynomial function
   (ii) $(T_n)_{n\geq 1}$ converges uniformly to f on $\mathbb{R}$
   Select the correct answer.
   (A) (i) implies (ii) but (ii) does not imply (i)
   (B) (ii) implies (i) but (i) does not imply (ii)
   (C) (i) and (ii) are equivalent
   (D) Neither (i) implies (ii) nor (ii) implies (i)
   C
   (i) $\Rightarrow$ (ii): By def of poly, OK
   (ii) $\Rightarrow$ (i): $(T_n)_{n\geq 1}$ conv unif to $f \Rightarrow \exists N>0$, s.t. $\forall n\geq N$, $\forall x\in\mathbb{R}$, $|T_n(x)-f(x)|\leq 1$
   Claim: $\forall n\geq N$, $T_n - T_N$ is const
   Proof
   Assume not, then $T_n - T_N = a_k x^k + \dots + a_0$, $a_k\neq 0 \xrightarrow{x\to\infty} \infty$
   However, $\forall x\in\mathbb{R}$, $|T_n(x)-T_N(x)|\leq |T_n(x)-f(x)|+|f(x)-T_N(x)|\leq 2$ ※
   $\therefore T_n - T_N = c$, $c\in\mathbb{R}$.
   $\therefore f = \underbrace{T_1}_{\text{poly}} + \underbrace{\sum_{k=1}^{N-1}(T_{k+1}-T_k)}_{\text{poly}} + \underbrace{\sum_{k=N}^{\infty}(T_{k+1}-T_k)}_{\text{const}}$ is a poly

SECTION 3: MULTIPLE SELECTION

1. Let $(u_n)_{n\geq 1}$ be a sequence of real numbers with $u_n > -1$ for all $n\geq 1$. Consider the statements:
   (i) The infinite product $\prod_{n\geq 1}(1+u_n)$ is convergent.
   (ii) $\lim_{n\to\infty}(1+u_n)=1$
   (iii) The series $\sum_{n\geq 1}u_n$ is convergent
   Select ALL correct statements.
   (A) (i) $\Rightarrow$ (ii) (B) (ii) $\Rightarrow$ (i) (C) (i) $\Rightarrow$ (iii) (D) (iii) $\Rightarrow$ (i) (E) (ii) $\Rightarrow$ (iii) (F) (iii) $\Rightarrow$ (ii)
   AF
   (Recall $\ln(1+a_n)\sim a_n$ as $n\to\infty$)
   B: $u_n = -\frac{1}{n}$
   C: $u_{2n-1} = -\frac{1}{\sqrt{n}}$, $u_{2n} = \frac{1}{\sqrt{n}}+\frac{1}{n} \Rightarrow (1+u_{2m-1})(1+u_m) = 1-\frac{1}{m^{1.5}} \Rightarrow \prod_{n\geq 1}(1+u_n)$ conv.
   D: Notice, set $u_n = \frac{(-1)^n}{\sqrt{n}} \Rightarrow \sum_{n\geq 1}u_n$ conv, however, $\prod_{n\geq 1}\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$ div $\Leftarrow \left(1+\frac{1}{\sqrt{2n}}\right)\left(1-\frac{1}{\sqrt{2n+1}}\right) = 1-\frac{1}{2n}+o\left(\frac{1}{n}\right)$
   E: $u_n = -\frac{1}{n}$
   A: By Cauchy's condition, $\forall\varepsilon>0$, $\exists N>0$, s.t. $\forall n\geq N$, $\forall k\geq 1$, $|(1+u_{n+1})\cdots(1+u_{n+k})-1|<\varepsilon$, so $\overbrace{|1+u_{n+1}-1|<\varepsilon}^{k=1} \Rightarrow$ by def, $\lim_{n\to\infty} 1+u_n = 1$ □
   F: $\sum_{n\geq 1}u_n$ conv $\Rightarrow \lim_{n\to\infty}u_n = 0 \Rightarrow \lim_{n\to\infty}(1+u_n)=1$

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2. Let $f:[1,\infty)\to\mathbb{R}$ be a continuous, decreasing function. Consider the statements:
(i) f is integrable on $[1,\infty)$
(ii) $\lim_{x\to\infty} f(x)=0$
(iii) $f(x)=o(\frac{1}{x})$ when $x\to\infty$
Select ALL correct statements.
(A) (i)$\Rightarrow$(ii) (B) (ii)$\Rightarrow$(i) (C) (i)$\Rightarrow$(iii) (D) (iii)$\Rightarrow$(i) (E) (ii)$\Rightarrow$(iii) (F) (iii)$\Rightarrow$(ii)

ACF

B: $f(x)=\frac{1}{x}$
D: $f(x)=\frac{1}{\sqrt{x}}$
E: $f(x)=\frac{1}{x}$
F: As $f(x)=o(\frac{1}{x})$ as $x\to\infty$, by def, $\lim_{x\to\infty} xf(x)=0$ $\therefore \lim_{x\to\infty} f(x)=0$
A: As f is decreasing, then $\lim_{x\to\infty} f(x)$ exists or is $-\infty$. As f is integrable, thus $\lim_{x\to\infty} f(x)=0$ □
C: As (i)$\Rightarrow$(ii), f must be nonnegative. Then, Cauchy condition gives $\varepsilon > \int_{\frac{x}{2}}^{x} f(t)\,dt \geq \frac{x}{2}f(x)$ for large enough x □

3. Consider the statements:
(i) $\int_0^1 \frac{\sin x}{x^\alpha}dx$ converges if and only if $\alpha\in I$
(ii) $\int_1^\infty \frac{\sin x}{x^\alpha}dx$ converges if and only if $\alpha\in J$
Select the correct answers to I and J.
(A) $I=(-\infty,0)$ (G) $J=(0,+\infty)$
(B) $I=(-\infty,0]$ (H) $J=[0,+\infty)$
(C) $I=(-\infty,1)$ (I) $J=(1,+\infty)$
(D) $I=(-\infty,1]$ (J) $J=[1,+\infty)$
(E) $I=(-\infty,2)$ (K) $J=(2,+\infty)$
(F) $I=(-\infty,2]$ (L) $J=[2,+\infty)$

EG

4) Let $f_n:[0,1]\to\mathbb{R}$ be the sequence of functions defined by $f_n(x)=x^n$ for all $n\geq1$. Select ALL correct statements.
(A) $(f_n)_{n\geq1}$ converges pointwise on $[0,1]$
(B) $(f_n)_{n\geq1}$ converges pointwise on $[0,1)$
(C) For any $\delta\in(0,1)$, $(f_n)_{n\geq1}$ converges pointwise on $[0,1-\delta]$
(D) $(f_n)_{n\geq1}$ converges uniformly on $[0,1]$
(E) $(f_n)_{n\geq1}$ converges uniformly on $[0,1)$
(F) For any $\delta\in(0,1)$, $(f_n)_{n\geq1}$ converges uniformly on $[0,1-\delta]$
(G) $\sum_{n\geq1} f_n$ converges uniformly on $[0,1]$
(H) $\sum_{n\geq1} f_n$ converges uniformly on $[0,1)$
(I) For any $\delta\in(0,1)$, $\sum_{n\geq1} f_n$ converges uniformly on $[0,1-\delta]$

ABCFI (No E!! There is a diff between $[0,1)$, and $\forall\delta\in(0,1)$, $[0,1-\delta]$)

5) Let $f_n:[0,1]\to\mathbb{R}$ be a sequence of continuous functions converging pointwise to a function $f:[0,1]\to\mathbb{R}$. Consider the statements:
(i) f is continuous
(ii) $(f_n)_{n\geq1}$ converges uniformly to f on $[0,1]$
(iii) $\int_0^1 f_n(t)\,dt$ converges to $\int_0^1 f(t)\,dt$ when $n\to\infty$
Select ALL correct statements.
(A) (i)$\Rightarrow$(ii) (B) (ii)$\Rightarrow$(i) (C) (i)$\Rightarrow$(iii) (D) (iii)$\Rightarrow$(i) (E) (ii)$\Rightarrow$(iii) (F) (iii)$\Rightarrow$(ii)

BE

A/C: $f_n(x)=(n-2n^2|x-\frac{1}{2n}|)\mathbb{1}_{0\leq x\leq\frac{1}{n}}$, $n\geq1$, $f=0$

D/F: $f_n(x)=x^n$, $n\geq1$, $f=\mathbb{1}_{\{x=1\}}$ — even one point not unif would still have the same integral.

B: (Sketch) By def.
F: (Sketch) $\int_0^1 \|f_n(t)-f(t)\|\,dt \leq$ ____

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6. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. Consider the statements:
   (i) The power series $\sum_{n\geq 0} a_n x^n$ converges pointwise on $\mathbb{R}$
   (ii) For any $R>0$, the power series $\sum_{n\geq 0} a_n x^n$ converges uniformly on $[-R,R]$
   (iii) The power series $\sum_{n\geq 0} a_n x^n$ converges uniformly on $\mathbb{R}$
   Select ALL correct statements.
   (A) (i)⇒(ii) (B) (ii)⇒(i) (C) (i)⇒(iii) (D) (iii)⇒(i) (E) (ii)⇒(iii) (F) (iii)⇒(ii)

   ABDF

   "Uniformly" ⇒ C/E: $a_n = \frac{1}{n!}$ $\forall n\geq 0$
   B: $\bigcup_{R>0} [-R,R] = \mathbb{R}$
   F: $[-R,R] \subseteq \mathbb{R}$
   D: $\|S_n - S\|_\infty \xrightarrow{n\to\infty} 0 \Rightarrow \|R_n\|_\infty \xrightarrow{n\to\infty} 0$ □
   A: ?? Normal convergence?

IMPORTANT THMS AND CONDITIONS

CHAPTER 6

Bertrand's series? $\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$ conv iff $\alpha>1$ or ($\alpha=1$ and $\beta>1$)
Abel's transform: $\sum_{k=1}^{n} u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n$

$(a_n)_{n\geq 1} \searrow 0$ + $\sum b_n$ bounded ⇒ $\sum a_n b_n$ conv
$\sum_m a_{m,n}$ abs conv ⇒ $\sum_n \left(\sum_m \|a_{m,n}\|\right)$ conv

CHAPTER 7

Comparison: Conv ⇒ only remainder, div ⇒ only partial integral
Cauchy's condition: $\int_{[a,b)} f$ conv ⇔ $\forall \varepsilon>0, \exists c\in[a,b)$, s.t. $\forall x,y \in [c,b)$, $x<y \Rightarrow \left|\int_x^y f(t)\,dt\right| < \varepsilon$
Abel's rule: $f\in \mathcal{C}^1$, g conti ⇒ $f \searrow 0$ + $\forall x\in[a,b)$, $\left|\int_a^x g(t)\,dt\right| \leq M < \infty \Rightarrow \int_a^b f(t) g(t)\,dt$ conv

CHAPTER 8

Unif conv ⇔ $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$
Dini's thm: (K,d): cpt + $(f_n)_{n\geq 1}$: conti, (i) $f_n(x) \leq f_{n+1}(x)$ ← incr (ii) $f_n \xrightarrow{n\to\infty} f$ ptwise ⇒ $f_n \xrightarrow{n\to\infty} f$ unif
$\sum u_n$ conv normally ⇒ $\sum u_n$ conv unif + $\sum u_n(a)$ conv abs $\forall a\in A$

$f_n \xrightarrow{n\to\infty} f$ unif + f_n conti at a $\forall n\geq 1$ ⇒ f conti at a
$(f_n)_{n\geq 1}$: conti + $f_n \xrightarrow{n\to\infty} f$ unif ⇒ f: conti
$(\sum u_n)_{n\geq 1}$: conti + conv unif ⇒ limit function $\sum u_n$: conti

$f_n \longrightarrow f$ unif ⇒ $\int f_n \longrightarrow \int f$ unif
$\int \leftrightarrow \sum$ when $\sum u_n$ conv normally (also, $\lim_{n\to\infty} \sum_{k=1}^{n} \int_a^x u_k(t)\,dt$ would be unif conv)
$\forall n\geq 1$, $f_n \in \mathcal{C}^1$, $(f_n)_{n\geq 1}$ conv ptwise, (f_n') conv unif to g ⇒ $f' = g$ + $(f_n)_{n\geq 1}$ conv unif
$\sum u_n$ conv ptwise + $\sum u_n'$ conv unif ⇒ $(\sum u_n)' = \sum u_n'$

not just conv
$\forall z\in\mathbb{C}$, $|z|<|z_0| \Rightarrow \sum a_n z^n$: abs conv
$R = \sup\{r\geq 0 \mid (|a_n| r^n)_{n\geq 0}$ is bdded$\}$

$z \mapsto \sum_{n\geq 0} a_n z^n$: conti
Abel's thm: $\sum a_n R^n$: conv ⇒ $\sum_{n\geq 0} a_n x^n \xrightarrow{x\to R^-} \sum_{n\geq 0} a_n R^n$
Tauber's thm: $f(x) \xrightarrow{x\to R^-} \lambda$ + $n a_n \xrightarrow{n\to\infty} 0 \Rightarrow \sum_{n\geq 0} a_n R^n \xrightarrow{n\to\infty} \ell$
If $\sum c_n$ conv, then $\sum c_n = (\sum a_n)(\sum b_n)$
$R\left(\sum_{n\geq 1} n a_n z^{n-1}\right) = R\left(\sum_{n\geq 0} a_n z^n\right)$
$\forall r\in(0,R)$, $n\in\mathbb{N}_0$, $r^n a_n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta}) e^{-in\theta}\,d\theta$; Taylor ok if remainder $\to 0$ ptwise