4-15-25 (WEEK 9)

Shun/羊羽海 (@shun4midx)

THEOREM (DIRICHLET TEST)

Let $\sum u_n$ be a series with general terms in a Banach space $(W, \|\cdot\|)$. Suppose that its general term u_n writes $u_n = a_n b_n$ with $a_n \in \mathbb{R}$ and $b_n \in W$ for all $n \geq 1$, and satisfies:

(i) the sequence $(a_n)_{n\geq 1}$ is nonnegative, nonincreasing, and tends to 0

(ii) the series $\sum b_n$ is bounded

Then, $\sum u_n$ converges.

Proof

Let us apply Abel transform eq to $\sum u_n$. $\therefore \forall n \geq 0$, $\sum_{k=1}^{n} u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n$, where S_n is the n-th partial sum of series $\sum b_n$.

Let $M > 0$, s.t. $|S_n| = |\sum_{k=1}^{n} b_k| \leq M$ $\forall n \geq 1$. Then, $|a_n S_n| \leq |a_n| M \xrightarrow{n \to \infty} 0$, so the series $\sum u_n$ and $\sum (a_n - a_{n+1}) S_n$ share the same behavior.

Moreover, $\forall k \geq 0$, $|(a_k - a_{k+1}) S_k| \leq (a_k - a_{k+1}) M$ since $(a_k)_{k\geq 1}$ is nonincreasing.

$\therefore \forall n \geq 0$, $\underline{\sum_{k=1}^{n} |(a_k - a_{k+1}) S_k| \leq \sum_{k=1}^{n} (a_k - a_{k+1}) M} = (a_1 - a_{n+1}) M \leq a_1 M$. Hence, $\sum (a_n - a_{n+1}) S_n$ is absolutely convergent and hence convergent. □

EXAMPLES (Applications of Dirichlet Test)

(1) Let $(a_n)_{n\geq 0}$ be a nonincreasing sequence tending to 0. We know the alternating series $\sum (-1)^n a_n$ is convergent since the partial sum $|(-1)^1 + (-1)^2 + \dots + (-1)^n| \leq 1$ is bounded (see "bounded" suffices, no need for "converge")

(2) Let $(a_n)_{n\geq 0}$ be a nonincreasing sequence tending to 0. Let $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Consider the series $\sum a_n e^{in\theta}$.

$\forall n \geq 0$, we have $|1 + e^{i\theta} + \dots + e^{in\theta}| = \left|\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right| = \left|\sin\left(\frac{(n+1)\theta}{2}\right) \div \sin\left(\frac{\theta}{2}\right)\right| \leq \frac{1}{|\sin(\frac{\theta}{2})|}$ $\therefore$ The series converges if $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$

REARRANGEMENT OF SERIES

Let $\sum u_n$ be a series with terms in $(W, \|\cdot\|)$

DEFINITION

we don't say "permutation" cuz it can be uncountable

We say $\sum v_n$ is a rearrangement of $\sum u_n$ if $\exists$ a bijection $\varphi: \mathbb{N} \to \mathbb{N}$, s.t. $v_n = u_{\varphi(n)}$ $\forall n > 1$

THEOREM

If $\sum u_n$ converges absolutely to s, then any rearrangement of $\sum u_n$ is also absolutely convergent with the same limit

Proof

Let $\varphi: \mathbb{N} \to \mathbb{N}$ be a bijection and $\sum v_n$ with $v_n = u_{\varphi(n)}$, $n \geq 1$, be a rearrangement of $\sum u_n$.

$\forall n \geq 1$, $\sum_{k=1}^{n} \|v_k\| = \sum_{k=1}^{n} \|u_{\varphi(k)}\| \leq \sum_{n=1}^{\infty} \|u_n\| < +\infty$ $\therefore \sum_{k=1}^{n} \|v_k\|$ converges $\Rightarrow \sum v_k$ absolutely converges

Now, for "same limit",

partial sums

Let $\varepsilon > 0$, take $N \geq 1$, s.t. $\sum_{n=N+1}^{\infty} \|u_n\| < \varepsilon$. For $n \geq 0$, $\underline{S_n = \sum_{k=1}^{n} u_k,\ T_n = \sum_{i=1}^{n} v_i}$

From the ε we chose, we know that $\|S_N - S\| = \|\sum_{n \geq N+1} u_n\| < \varepsilon$

lmao shun4midx mention

characterization of idx mapping between u,v.

Note that $\{1, \dots, N\}$ is a finite set, so it has an upper bound $\varphi(M)$, that is $\{1, \dots, N\} \subseteq \{\varphi(1), \dots, \varphi(M)\}$

For $n \geq M$, $\|T_n - S_N\| = \|\sum_{k=1}^{n} v_k - \sum_{k=1}^{N} u_k\| = \|\sum_{k=1}^{n} u_{\varphi(k)} - \sum_{k=1}^{N} u_k\| = \|\sum_{k \in A} u_k\| \leq \sum_{k \in A} \|u_k\| \leq \sum_{k \geq N+1} \|u_k\| < \varepsilon$, where $A = \{\varphi(1), \dots, \varphi(N)\} \setminus \{1, \dots, N\}$

$\therefore$ For $n \geq M$, $\|T_n - S\| \leq \|T_n - S_N\| + \|S_N - S\| < 2\varepsilon$ □

WHAT HAPPENS IF $\sum u_n$ DOES NOT CONVERGE ABSOLUTELY?

EXAMPLE

Recall: We know that $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = \ln 2$ but is not absolutely convergent

A possible rearrangement: $(1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + \dots + (\frac{1}{2k-1} - \frac{1}{2k-2}) - \frac{1}{4k} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots = \frac{1}{2}\ln 2$

Shun/翔海 (@shun4midx)

RIEMANN SERIES THEOREM

Let $\sum u_n$ be a real-valued series that is conditionally convergent

For $-\infty \leq x \leq y \leq \infty$, then $\exists$ a bijection $\varphi: \mathbb{N} \to \mathbb{N}$, s.t. the rearrangement $\sum u_{\varphi(n)}$ satisfies $\liminf\limits_{n\to\infty} \sum\limits_{k=1}^{n} u_{\varphi(k)} = x$ and $\limsup\limits_{n\to\infty} \sum\limits_{k=1}^{n} u_{\varphi(k)} = y$

(Proof sketch: Let $\sum a_n$ have the sum of all +ve terms, $\sum b_n$ have the sum of all -ve terms. Both diverge, so we can find distinct values to match)

REMARK

If we take $x=y$, then the arrangement satisfies $\sum u_{\varphi(n)} = x = y$ (i.e. it conv)

CAUCHY SERIES

DEFINITION

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $(A, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. Consider a binary operator $A \times A \to A$ denoted by $\cdot$

(1) We say that $(A, \cdot)$ is an algebra if:

- ↳ Right distributivity: $\forall x,y,z \in A$, $(x+y)\cdot z = x\cdot z + y \cdot z$
- ↳ Left distributivity: $\forall x,y,z \in A$, $z\cdot(x+y) = z\cdot x + z\cdot y$
- ↳ Scalar multiplication: $\forall x,y,z \in A$, $a,b \in \mathbb{K}$, $(ax)\cdot(by) = (ab)(x\cdot y)$

(2) We say $(A, \|\cdot\|)$ is a normed algebra if $(A,\cdot)$ is an algebra and $\forall x,y \in A$, $\|xy\| \leq \|x\|\|y\|$

EXAMPLE

(1) $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$ are normed algebras

(2) For $n \geq 1$, $M_{n\times n}(\mathbb{K})$ equipped with matrix norm $|||\cdot|||$ is a normed algebra, where $|||A||| = \sup\limits_{x\in\mathbb{K}^n, \|x\|=1} \|Ax\|$

(3) Let U be a normed vector space. $\mathcal{L}(U) = \{\text{linear maps}: U \to U\}$ with the operator norm $|||\cdot|||$ is a normed algebra

THEOREM (CAUCHY PRODUCT)

Let $\sum\limits_{n\geq 0} a_n$ and $\sum\limits_{n\geq 0} b_n$ be absolutely convergent series with general terms in a normed algebra $(A, \cdot, \|\cdot\|)$. We define its Cauchy series $\sum\limits_{n\geq 0} c_n$ to be $\forall n \geq 0$, $c_n = \sum\limits_{k=0}^{n} a_n b_{n-k}$. Then, $\sum\limits_{n\geq 0} c_n$ converges absolutely to $\left(\sum\limits_{n\geq 0} a_n\right)\left(\sum\limits_{n\geq 0} b_n\right)$

Proof

Denote $A := \sum\limits_{p=0}^{\infty} \|a_p\|$ and $B := \sum\limits_{q=0}^{\infty} \|b_q\|$

Let $n \geq 0$, then $\sum\limits_{k=0}^{n} \|c_k\| \leq \sum\limits_{k=0}^{n} \left(\sum\limits_{\substack{p+q=k \\ p,q\geq 0}} \|a_p\| \|b_q\| \right) \leq \sum\limits_{0\leq p,q\leq n} \|a_p\| \|b_q\| = \left(\sum\limits_{p=0}^{n} \|a_p\|\right)\left(\sum\limits_{q=0}^{n} \|b_q\|\right) \leq AB$ $\therefore \sum c_n$ converges absolutely (rearrangement ok)

Now, define $\forall n \geq 0$, $\Delta_n = \sum\limits_{k=0}^{2n} c_k - \left(\sum\limits_{p=0}^{n} a_p\right)\left(\sum\limits_{q=0}^{n} b_q\right)$

Thus, $\forall n \geq 0$, $\Delta_n = \sum\limits_{\substack{p+q\leq 2n \\ p,q\geq 0}} a_p b_q - \sum\limits_{0\leq p,q\leq n} a_p b_q = \sum\limits_{p\geq n+1, q\geq 0, p+q\leq 2n} a_p b_q + \sum\limits_{q\geq n+1, p\geq 0, p+q\leq 2n} a_p b_q$

$\therefore$ By Δ ineq, $\forall n \geq 0$, $\|\Delta_n\| \leq \sum\limits_{p\geq n+1, q\geq 0, p+q\leq 2n} \|a_p\| \|b_q\| + \sum\limits_{q\geq n+1, p\geq 0, p+q\leq 2n} \|a_p\| \|b_q\|$

$\leq \sum\limits_{p\geq n+1, q\geq 0} \|a_p\| \|b_q\| + \sum\limits_{q\geq n+1, p\geq 0} \|a_p\| \|b_q\| = \underline{B \sum\limits_{p=n+1}^{\infty} \|a_p\| + A \sum\limits_{q=n+1}^{\infty} \|b_q\|} \xrightarrow{n\to\infty} 0$ □

EXAMPLE (Applications)

1) On $\mathbb{R}$, define the exponential function $e^x = \sum\limits_{n=1}^{\infty} \frac{x^n}{n!}$

For $x, y \in \mathbb{R}$, $e^x \cdot e^y = \left(\sum\limits_{n=1}^{\infty} \frac{x^n}{n!}\right)\left(\sum\limits_{n=1}^{\infty} \frac{y^n}{n!}\right) = \sum\limits_{n\geq 0} c_n$

Here, we know $c_n = \sum\limits_{\substack{p+q=k \\ p,q\geq 0}} \frac{x^p}{p!} \frac{y^q}{q!} = \sum \binom{p+q}{p} \frac{1}{(p+q)!} x^p x^q \overset{\text{given commutativity}}{=} \frac{1}{n!}(x+y)^n \Rightarrow e^x e^y = e^{x+y}$

Notably, Cauchy product makes us able to extend this concept to other normed algebras.