5-1-25 (WEEK 11)

Shun/羊羽海 (@shun4midx)

REMARK

How do we define $\int_a^b f$, for $f: I \longrightarrow W$? How do we obtain Δ-ineq?

(a) If W is a fin dim vector space

↳ Write $f = \sum_{i=1}^{n} f_i e_i$, define $\int_a^b f := \sum_{i=1}^{n} (\int f_i) e_i$

↳ If $\|\cdot\| = N_1, N_2, N_\infty$, it follows from direct computation

(b) If f is infinite dimensional

↳ Step function: $f: I \longrightarrow W$, $x = (x_k)_{0 \le k \le n} \in P([a,b])$, s.t. $f|_{(x_{k-1}, x_k)} \equiv f_k \in W$ is const $\forall 1 \le k \le n$. Define $\int_a^b f := \sum_{k=1}^{n} f_k \Delta x_k$, and we have

$$\left\|\int_a^b f\right\| = \left\|\sum_{k=1}^{n} f_k \Delta x_k\right\| \le \sum_{k=1}^{n} \|f_k\| \Delta x_k = \int_a^b \|f\|$$

↳ General cont. function:

(i) Find a sequence $(f_n)_{n \ge 1}$ of step functions, s.t. $\|f_n - f\|_\infty := \sup_{t \in [a,b]} |f_n(t) - f(t)| \xrightarrow{n \to \infty} 0$ $(\forall \varepsilon > 0, \exists M \ge 1, \text{ s.t. } |x-y| \le \frac{1}{M} \Rightarrow \|f(x) - f(y)\| < \frac{\varepsilon}{2})$

(ii) Check that $(\int_a^b f_n)_{n \ge 1}$ is Cauchy in W, so converges, then $\int_a^b := \lim_{n \to \infty} \int_a^b f_n$ (Define $f_n := g_t$ $\forall n \ge 1$, $\|f_n - f_m\| \le \frac{2}{N}$ if $n, m \ge N$)

(iii) Check that the limit is well-def

EXAMPLE (CONTINUED)

To get a more precise asymptotic formula for $F(x)$ when $x \to \infty$, we may start with IBP.

Write $\sqrt{2\pi} F(x) = \int_x^\infty \frac{-t e^{-\frac{t^2}{2}}}{-t} dt = \frac{e^{-\frac{x^2}{2}}}{x} - \int_x^\infty \frac{e^{-\frac{t^2}{2}}}{t^2} dt = \frac{e^{-\frac{x^2}{2}}}{x} - o(F(x))$ as $x \to \infty$

We deduce when $x \to \infty$, $F(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} (1 + o(1))$ when $x \to \infty$

$\therefore$ By induction, $\forall n \ge 0$, when $n \to \infty$, after IBP, we get $F(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{1}{x} + \sum_{k=1}^{n} (-1)^k \frac{(2k-1)!!}{x^{2k+1}}\right)(1 + o(1))$ when $x \to \infty$

PROPOSITION (COMPARISON FOR NON-INTEGRABLE FUNCTIONS)

Let $f: [a,b) \longrightarrow W$, $g: [a,b) \longrightarrow \mathbb{R}_+$ be a nonintegrable function, both are p.c.

1) If $f \underset{b}{=} O(g)$, then $\int_a^x f \underset{x \to b}{=} O(\int_a^x g)$

2) If $f \underset{b}{=} o(g)$, then $\int_a^x f \underset{x \to b}{=} o(\int_a^x g)$

3) If $W = \mathbb{R}$ and $f \underset{b}{\sim} g$, then f is nonintegrable on $[a,b)$ and $\int_a^x f \underset{x \to b}{\sim} \int_a^x g$

<u>Proof</u>

Note that $\int_a^x g \xrightarrow[x \to b^-]{} +\infty$ because g is nonintegrable

As (2) and (3) are similar, we prove (1) here only

1) Let $M > 0$ and $\delta > 0$, s.t. $\|f(x)\| \le M|g(x)|$ where $x \in (b-\delta, b)$

For $x \in (b-\delta, b)$, $\|\int_a^x f\| = \|\int_a^{b-\delta} f + \int_{b-\delta}^x f\| \le \int_a^{b-\delta} \|f\| + \int_{b-\delta}^x \|f\| \le \int_a^{b-\delta} \|f\| + M \int_a^x g$

Since $\int_a^x g \xrightarrow[x \to b]{} +\infty$, thus $\exists \delta' \in (0, \delta)$, s.t. $\forall x \in [b-\delta', b)$, $\int_a^{b-\delta} \|f\| \le M \int_a^x g$

$\therefore \forall x \in [b-\delta', b)$, $\|\int_a^x f\| \le 2M \int_a^x g$. □

EXAMPLE

$\int_2^x \frac{dt}{\ln t} \xrightarrow{x \to \infty} ?$

- $t \mapsto \frac{1}{\ln t}$ is nonneg and nonintegrable on $[2, +\infty)$

  By IBP, $\int_2^x \frac{dt}{\ln t} = \left(\frac{t}{\ln t}\right)\Big|_2^x + \int_2^x \frac{dt}{(\ln t)^2} = \frac{x}{\ln x} - \frac{2}{\ln 2} + o\left(\int_2^x \frac{dt}{\ln t}\right)$ when $t \to \infty$

  $\therefore \int_2^x \frac{dt}{(\ln t)^2} \sim \frac{x}{\ln x}$ when $t \to \infty$

IMPROPER INTEGRAL

Let $f: [a,b) \to \mathbb{R}$ be p.c.

DEFINITION AND PROPERTIES

DEFINITION

(1) We say that the integral $\int_{[a,b)} f = \int_{[a,b)} f(t)\,dt$ **converges** if $x \mapsto \int_a^x f = \int_{[a,x]} f$ converges when $x \to b^-$

(2) Otherwise, we say that the integral $\int_{[a,b)} f$ **diverges**

REMARK

If $f: [a,b) \to \mathbb{R}$ is nonneg integrable, then this definition coincides with what we saw earlier, $\lim_{x \to b^-} \int_a^x f = \sup_{J \subseteq [a,b], J \text{ seg}} \int_J f$

PROPOSITION (CAUCHY'S CRITERION)

Shun/翔海(@shun4midx)

Let $f:[a,b)\to\mathbb{R}$ be a piecewise continuous function. TFAE
(1) The integral $\int_{[a,b)} f$ converges
(2) $\forall \varepsilon>0, \exists c\in[a,b)$ s.t. $\forall x,y\in[c,b)$ with $x<y$, we have $|\int_x^y f(t)\,dt|<\varepsilon$

PROPOSITION

Let $f:[a,b)\to\mathbb{R}$ be a piecewise conti function. Let $c\in[a,b)$, then:
(1) Both integrals $\int_{[a,b)} f$ and $\int_{[c,b)} f$ have the same behavior
(2) If they both converge, we have $\int_{[a,b)} f = \int_{[a,c)} f + \int_{[c,b)} f$

Proof

These are direct consequences of "$\forall c\in[a,b), \forall x\in[c,b), \int_a^x f = \int_a^c f + \int_c^x f$"

COROLLARY

If $\int_{[a,b)} f$ conv, then $\int_{[x,b)} f \xrightarrow{x\to b} 0$
- $\int_{[a,x)} f$: partial integral
- $\int_{[x,b)} f$: remainder integral

PROPOSITION

If $f:[a,b]\to\mathbb{R}$ is a bounded p.c. function, and $f\in R(x;a,b)$, then $\int_{[a,b)} f = \int_a^b f$ ($\therefore$ We can generalize the notation $\int_a^b$)

PROPOSITION

Let $f:[a,b)\to\mathbb{R}$ be bounded p.c. and F is a primitive of f. Then, TFAE
(1) $\int_a^b f$ converges
(2) F has a finite limit at b^-

Proof

$\int_a^x f = F(x)-F(a) \xrightarrow{x\to b^-} F(b^-)-F(a)$

The function $x\mapsto \int_x^b f$ is differentiable with derivative $-f$.

DEFINITION

Let $-\infty\le a<b\le+\infty$. Fix $c\in(a,b)$. Let $f:(a,b)\to\mathbb{R}$ be p.c. We say that the improper integral $\int_{(a,b)} f = \int_a^b f$ converges if both $\int_{(a,c]} f$ and $\int_{[c,b)} f$

REMARK

The choice of $c\in(a,b)$ is arbitrary, from the previous propositions

EXAMPLE

Let $f:(0,1)\longrightarrow\mathbb{R}$

$x\longmapsto \frac{1}{x}-\frac{1}{1-x}$

Indeed, x is not integrable.

$f(x)\underset{x\to 0^+}{\sim}\frac{1}{x}$ and $\frac{1}{x}$ is not integrable around 0^+

$f(x)\underset{x\to 1^-}{\sim}-\frac{1}{1-x}$ and $\frac{1}{1-x}$ is not integrable around 1^-

We may check the convergence of $\int_{(0,\frac{1}{2}]} f$ and $\int_{[\frac{1}{2},1)} f$

We find $\int_{\frac{1}{n}}^{\frac{1}{2}} f = 2\ln\frac{1}{2}-\ln\frac{1}{n}-\ln(1-\frac{1}{n}) \xrightarrow{n\to\infty} \infty$, so $\int_{(0,\frac{1}{2}]}$ diverges

CONDITIONAL CONVERGENCE

DEFINITION

Let $f:I\to\mathbb{R}$, I is an interval. We say that $\int_I f$ converges conditionally if
- $\int_I f$ converges
- f is not integrable on I (i.e. $\int_I |f|$ diverges)

EXAMPLE

$\int_\pi^\infty \frac{\sin x}{x}\,dx$ converges conditionally

- "Not integrable": For $k \in \mathbb{N}$, $\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{(k+1)\pi} \underbrace{\int_{k\pi}^{(k+1)\pi} |\sin x|\, dx}_{}$ and $\sum \frac{1}{k+1}$ div
  - const >0
  - Check $\int 1 \cdot 1 = \infty$
  - $\therefore \int_\pi^\infty \left|\frac{\sin x}{x}\right| dx = +\infty$
- "Converges": For $t > \pi$, $\int_\pi^t \frac{\sin x}{x}\,dx = \left[-\frac{\cos x}{x}\right]_\pi^t - \int_\pi^t \frac{\cos x}{x^2}\,dx = \underbrace{-\frac{\cos t}{t}}_{\to 0} + \underbrace{\frac{\cos \pi}{\pi}}_{\text{const}} - \int_\pi^t \frac{\cos x}{x^2}\,dx$
  - $\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$, so $x \mapsto \frac{\cos x}{x}$ is integrable on (π, ∞)

THEOREM (ABEL'S RULE) (Proof in next section of notes)

Let $f : [a,b) \to \mathbb{R}$ be $\mathcal{C}^1$, $g : [a,b) \to \mathbb{R}$ be $\mathcal{C}^0$,

(i) f is decreasing with $\lim_{x \to b} f(x) = 0$

(ii) $\exists M > 0$, s.t. $\forall x \in [a,b)$, we have $\left|\int_a^x g(t)\,dt\right| \leq M$

Then, the integral $\int_a^b f(t)\,g(t)\,dt$ is convergent