Analysis II Definitions

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Definitions

2-18-25 (Week 1): Riemann-Stieltjes Integrals (Functions of Bounded Variation)

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a function.

(1) f is **non-increasing/decreasing** if $f(x) \geq / > f(y)\ \forall x \leq y,\ x, y \in I$

(2) f is **non-decreasing/increasing** if $f(x) \leq / < f(y)\ \forall x \leq y,\ x, y \in I$

(3) f is **monotonic** if (1) or (2) holds

Definition 1.2. Let $f : I \to \mathbb{R}$ be monotonic. For $x \in I$, define:

- The **left limit** at x to be $f(x-) = \lim_{y<x,\ y\to x} f(y)$ if $(x - \varepsilon, x) \cap I \neq \emptyset$ for $\varepsilon > 0$ *(e.g. we cannot just pick a point at the boundary)*
- The **right limit** at x to be $f(x+) = \lim_{y>x,\ y\to x} f(y)$ if $(x, x + \varepsilon) \cap I \neq \emptyset$ for $\varepsilon > 0$

Definition 1.3. Let $a < b$ and $[a, b] \in \mathbb{R}$ be a segment.

- A **partition** or a **subdivision** of $[a, b]$ is a finite sequence $P = (x_k)_{0\leq k\leq n}$ s.t. $a = x_0 < x_1 < \cdots < x_n = b$, where n is the **length** of P. We denote $\text{Supp}(P) := \{x_k \mid 0 \leq k \leq n\}$ as the **support** of P.
- For a finite subset $A \subseteq [a, b]$ with $a, b \in A$, we may find a partition P of $[a, b]$ s.t. $\text{Supp}(P) = A$. This is called the **partition corresponding to** A.
- We say $[x_{k-1}, x_k]$ is the k^{th} **subinterval** of P, $\Delta x_k := x_k - x_{k-1}$, $1 \leq k \leq n$. Then, we say the **mesh size** of P is $||P|| := \max_{1\leq k\leq n} \Delta x_k$
- Let P, P' be partitions. If $\text{Supp}(P) \subseteq \text{Supp}(P')$, then we say P' is **finer** than P, and we say $P \subseteq P'$. This also implies $||P|| \leq ||P'||$.
- Let P_1, P_2 be partitions. Define their **joint partition** or **smallest comon refinement** to be $P := P_1 \vee P_2$, which is the partition P with support = $\text{Supp}(P_1) \cup \text{Supp}(P_2)$.
- We denote $\mathcal{P}([a, b])$ as the collection of **all** possible partitions of $[a, b]$.

Definition 1.4. Let $f : [a, b] \to \mathbb{R}$ be a function, $P = (x_k)_{0\leq k\leq n} \in \mathcal{P}([a, b])$, define $\Delta f_k := f(x_k) - f(x_{k-1})$ for $1 \leq k \leq n$. Define $V_P(f) := \sum_{k=1}^{n} |\Delta f_k|$ and $V_f = V_f([a, b]) = \sup_{P\in\mathcal{P}([a,b])} V_P(f) \in [0, \infty]$ to be the **total variation** of f. We say that f is of **bounded variation** if $V_P < +\infty$. We write $\mathcal{BV}([a, b]) = \mathcal{BV}([a, b], \mathbb{R})$ for the collection of such functions defined on $[a, b]$.

2-20-25 (Week 1): Properties of Bounded Variation

Definition 2.1. Let $f \in \mathcal{BV}$. Define its **variation function** to be

$$V : [a, b] \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} 0 & \text{if } x = a \\ V_f([a, b]) & \text{if } x \in (a, b] \end{cases}$$

2-25-25 (Week 2): Riemann-Stieltjes Integrals

Definition 3.1. Let $P = (x_k)_{0\le k\le n} \in \mathcal{P}([a,b])$. For every $1 \le k \le n$, take $t_k \in [x_{k-1}, x_k]$ and write $t = (t_k)_{0\le k\le n}$. We call (P,t) a **tagged partition**, where t contains **tagged points** of P.

Then, the **R-S sum** of f w.r.t. α for (P,t), is $\underline{S_{P,t}(f,\alpha) = \sum_{k=1}^n f(t_k)\Delta\alpha_k = \sum_{k=1}^n f(t_k)[\alpha(x_k) - \alpha(x_{k-1})]}$. *(Notice that t is used for f and x is used for α).*

Definition 3.2. The **(RS) condition** is when $\exists L \in \mathbb{R}$, s.t. $\forall \varepsilon > 0$, $\exists \mathcal{P}_\varepsilon \in \mathcal{P}([a,b])$, s.t. $\forall P \supseteq \mathcal{P}_\varepsilon$, **tagged points** t of P, we have $\underline{|S_{P,t}(f,\alpha) - L| < \varepsilon}$. If **(RS)** holds, we say f is **R-S integrable**, and define the unique L to be its **integral**, $\underline{\int_a^b f d\alpha = \int_a^b f(x) d\alpha(x)}$.

Definition 3.3. We write $\underline{R(\alpha; a, b) = R(\alpha)}$ for the set of **functions** f satisfying **(RS)**.

Example 3.1. **KEY CONSTRUCTION EXAMPLE.** *(What if f and α share the* ***same discontinuities****)* Let $f, \alpha : [-1,1] \to \mathbb{R}$ to be $\boxed{f = \alpha = \mathbb{1}_{x\ge 0}}$. Consider a partition $P \in \mathcal{P}([-1,1])$ with $\underline{x_k = 0}$ for some k. $\forall$ tagged points t of P, $\underline{S_{P,t}(f,\alpha) = f(t_k)\Delta\alpha_k = f(t_k) = \boxed{\mathbb{1}_{t_k = x_k = 0}}}$. Hence, $\boxed{\textbf{(RS) does not hold}}$

Definition 3.4. The **(RS') condition** is when $\exists L \in \mathbb{R}$, s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall P \in \mathcal{P}([a,b])$ with $\underline{\max_{1\le k\le n} |x_k - x_{k-1}| = ||P|| < \delta}$, any tagged points t, we have $\underline{|S_{P,t}(f,\alpha) - L| < \varepsilon}$. By def, **(RS')** $\Rightarrow$ **(RS)**.

Example 3.2. Let $f = \mathbb{1}_{x>0}$, $\alpha = \mathbb{1}_{x\ge 0}$, $\delta \in (0,1)$ and $P \in \mathcal{P}([0,1])$, s.t. $\underline{||P|| < \delta}$, then $\exists\ k$, s.t. $\underline{x_{k-1} = -\frac{\delta}{2},\ x_k = \frac{\delta}{2}}$. Then, $S_{P,t}(f,\alpha) = f(t_k)[\alpha(x_k) - \alpha(x_{k-1})] = f(t_k) = \boxed{\mathbb{1}_{t_k > 0}}$, which **depends on tagged points**. Here we have $\boxed{\textbf{(RS) but not (RS')}}$

Definition 3.5. For $a < b$, any bounded $\alpha : [a,b] \to \mathbb{R}$, $f \in R(\alpha; a, b)$, we define $\underline{\int_b^a f d\alpha = -\int_a^b f d\alpha}$. We also write $\underline{R(\alpha; a, b) = R(\alpha; b, a)}$. *(This is for our convenience in future theorems)*

2-27-25 (Week 2): Step Function Integrators

Definition 4.1. Given $\alpha : [a,b] \to \mathbb{R}$, it is a **step function** if $\exists P \in \mathcal{P}([a,b])$, s.t. $\underline{f|_{[x_{k-1}, x_k]}}$ is **constant** for $1 \le k \le n$. We define the **jump** at x_k to be $\underline{\alpha_k := \alpha(x_k^+) - \alpha(x_k^-)}$, with $\alpha_0 := \alpha(x_0^+) - \alpha(x_0)$ and $\alpha_n := \alpha(x_n) - \alpha(x_n^-)$.

3-4-25 (Week 3): Darboux Summations and Riemann's Condition

Definition 5.1. Let $P \in \mathcal{P}([a,b])$ and define for $1 \le k \le n$, $M_k = M_k(f) := \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}$ and $m_k = m_k(f) := \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}$. We define the **upper and lower Darboux sums** as $\underline{U_P(f,\alpha) = \sum_{k=1}^n M_k(f)\Delta\alpha_k}$ and $\underline{L_P(f,\alpha) = \sum_{k=1}^n m_k(f)\Delta\alpha_k}$ *(Note, no tagged points are needed for these defs. Also, when $\alpha(x) = x$, these are the upper and lower Riemann sums)*

Definition 5.2. Suppose α is **nondecreasing**, then the **upper/lower Stieltjes integrals** of f w.r.t. α are $\underline{\overline{I}(f,\alpha) = \overline{\int_a^b} f d\alpha := \inf\{U_P(f,\alpha) \mid P \in \mathcal{P}([a,b])\}}$ and $\underline{\underline{I}(f,\alpha) = \underline{\int_a^b} f d\alpha := \inf\{U_P(f,\alpha) \mid P \in \mathcal{P}([a,b])\}}$.

Definition 5.3. Let $\alpha : [a,b] \to \mathbb{R}$ be **nondecreasing**. We say f satisfies **Riemann's condition** w.r.t. α on $[a,b]$ if $\forall \varepsilon > 0$, $exists\ P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $\forall\ P \supset P_\varepsilon$, we have $\underline{0 \le U_P(f,\alpha) - L_P(f,\alpha) < \varepsilon}$ *(Again, tagged points don't matter here)*

3-6-25 (Week 3): Riemann's Condition

Example 6.1. IMPORTANT. The **converse** of $f \in R(\alpha; a, b) \Rightarrow f^2 \in R(\alpha; a, b)$ **does not hold**. Consider over $x \in [0, 1]$, define $\boxed{f(x) = 2 \cdot \mathbb{1}_{x \notin \mathbb{Q}} - 1}$. We have $\underline{f^2 \in R(\alpha; a, b)}$ but $\underline{f \notin R(\alpha; a, b)}$.

3-11-25 (Week 4): Fundamental Theorems of Calculus

Definition 7.1. Let $I \subseteq \mathbb{R}$ be an interval, $f,\ F : I \to \mathbb{R}$ be functions. If $\underline{F'(x) = f(x)}\ \forall x \in \text{int}(I)$, we say F is a **primitive** or **antiderivative** of f.

3-13-25 (Week 4): Integrals Depending on a Parameter and Riemann Integrals

Definition 8.1. Let $S \subseteq \mathbb{R}$ be a subset. We say S has **measure zero** if $\forall \varepsilon > 0,\ \exists$ a **countable** family $\{U_i = (a_i, b_i) \mid i \in I\}$ of open intervals s.t.:

- $\underline{S \subseteq \cup_{i \in I}(a_i, b_i)}$ *("S can be covered by these open intervals")*
- The sum of lengths satisfy $\underline{\sum_{i \in I} |U_i| = \sum_{i \in I}(b_i - a_i) \leq \varepsilon}$

where $\underline{|U_i| = b_i - a_i}$ denotes the length of the open interval U_i for $i \in I$.

3-25-25 (Week 6): Lesbegue's Criterion

Definition 9.1. Let $f : [a, b] \to \mathbb{R}$ be a **bounded** function. For any subset $A \subseteq [a, b]$, define the **oscillation** of f on A to be $\boxed{\Omega_f(A) := \sup\{f(x) - f(y) \mid x, y \in A}$.

For $x \in [a, b]$, define the **oscillation** of f at x to be $\boxed{\omega_f(x) := \lim_{h \to 0^+} \Omega_f(B(x, h) \cap [a, b])}$. *(The idea is to view the point as an infinitely small ball. Also, $\Omega_f(A)$ has actually appeared before in **Darboux sums**)*

3-27-25 (Week 6): Sequences and Series

Definition 10.1. Let $(a_n)_{n \geq 1}$ be a **real-valued sequence**. We say it **converges** to $l \in \mathbb{R}$ if $\forall \varepsilon > 0,\ \exists N \geq 1$, s.t. $\underline{|x_0 - l| \leq \varepsilon}\ \forall n \geq N$.

Definition 10.2. In a **complete** vector space, to check for convergence, we just need **Cauchy's Condition**: $\forall \varepsilon > 0,\ \exists N > 0$, s.t. $\forall m, n \geq N$, $\underline{|a_m - a_n| < \varepsilon}$ *(Good because we don't need to know the limit l)*

Definition 10.3. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be two real sequences. Here are some asymptotic notations.

- We say a is **dominated** by b, denoted by $\boxed{a_n = O(b_n)}$, if $\exists$ **bounded sequence** $c = (c_n)_{n \geq 1}$ and $N \in \mathbb{N}$, s.t. $\underline{a_n = c_n b_n}\ \forall n \geq N$
- We say a is **negligible** compared to b, denoted by $\boxed{a_n = o(b_n)}$, if $\exists$ sequence $\varepsilon = (\varepsilon_n)_{n \geq 1}$ that **converges to 0** and $N \in \mathbb{N}$, s.t. $\underline{a_n = \varepsilon_n b_n}\ \forall n \geq N$
- We say a is **equivalent** to b, i.e. $\boxed{a_n \sim b_n}$, if $\exists$ sequence $c = (c_n)_{n \geq 1}$ that **converges to 1** and $N \in \mathbb{N}$, s.t. $\underline{a_n = c_n b_n}\ \forall n \geq N$

Definition 10.4. Let $(u_n)_{n\geq 0}$ be a sequence in a **normed vector space** $(W, ||\cdot||)$

- Define $S_0 := 0$, $S_n = u_1 + \cdots + u_n$ for $n \geq 1$. The series with general term u_n is the sequence $(S_n)_{n\geq 1}$, denoted as $\underline{\sum_{n\geq 1} u_n}$. This is called the **n-th partial sum** of $\sum u_n$

- We say $\sum u_n$ converges if $(S_n)_{n\geq 0}$ converges in $(W, ||\cdot||)$. We denote the limit as $\sum_{n\geq 1} u_n$

- If $\sum_{n\geq 1} u_n$ **converges**, we define its **n-th remainder** by $\boxed{R_n = \Sigma_{k=1}^{\infty} u_k - \Sigma_{k=1}^{n} u_k = \Sigma_{k=n+1}^{\infty} u_k}$

Definition 10.5. Given a **Banach space** $(W, ||\cdot||)$, $\sum u_n$ **converges** iff **Cauchy's Criterion** holds, i.e. $\forall \varepsilon > 0,\ \exists N > 0$, s.t. $\forall n \geq N, \forall k \geq 1,$ $\boxed{||u_{n+1} + \cdots + u_{n+k}|| < \varepsilon}$. *(This is not the definition, this requires proof, but this is the definition of this useful criterion, so I decided to still put it here!)*

Definition 10.6. Suppose $(W, ||\cdot||)$ is a **Banach space**, and let $\sum u_n$ be a series with general terms in W

- If $\sum ||u_n||$ converges, we say the series $\sum u_n$ **converges absolutely** *(Notice, this is conv w/o norm)*

- If $\sum u_n$ converges but **not absolutely**, we say $\sum u_n$ **converges conditionally**