3-4-25 (WEEK 3)

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THEOREM (Direct result just by using linearity)

Let α be a step function (w.r.t. the partition $P=(x_k)_{0\le k\le n}$)

Let $f:[a,b]\longrightarrow\mathbb{R}$ be s.t.

(i) $\forall 0\le k\le n-1$, at least one of f and α is right conti at x_k

(ii) $\forall 1\le k\le n$, at least one of f and α is left conti at x_k.

Then, $f\in R(\alpha)$ and $\int_a^b f\,d\alpha=\sum_{k=1}^n f(x_k)[\alpha(x_k+)-\alpha(x_k-)]$

COROLLARY

Let $a_1,\dots,a_n\in\mathbb{R}$, define a left-conti function $f:[0,1]\longrightarrow\mathbb{R}$

$x\longmapsto\begin{cases}0, & x=0\\ a_k, & x\in(k-1,k],\ 1\le k\le n\end{cases}$

Then, $\sum_{k=1}^n a_k=\sum_{k=1}^n f(k)=\int_0^n f(x)\,d\lfloor x\rfloor$

COROLLARY (Euler's Summation Formula) ← Just need to understand how to prove, no need to memorize

Let $f:[a,b]\to\mathbb{R}$ be C^1. Then,

(1) $\sum_{a<n\le b,\,n\in\mathbb{Z}} f(n)=\int_a^b f(x)\,dx+\int_a^b f'(x)\{x\}\,dx+f(a)\{a\}-f(b)\{b\}$, where $\{x\}:=x-\lfloor x\rfloor$ denotes the fractional part of x

(2) $\sum_{n=a}^b f(n)=\int_a^b f(x)\,dx+\int_a^b f'(x)(\{x\}-\frac{1}{2})\,dx+\frac{1}{2}[f(a)+f(b)]$

Proof

We know f and f' are conti, so they belong to $R(x;a,b)$

First, we know that $f\in R(\lfloor x\rfloor;a,b)$ so $\int_a^b f(x)\,d\lfloor x\rfloor$ is well-defined.

So, we also have $f\in R(\{x\})=R(x-\lfloor x\rfloor)$. By integration by parts, we find $\int_a^b f(x)\,d(\underbrace{x-\lfloor x\rfloor}_{\{x\}})+\int_a^b\{x\}\,df(x)=f(b)\{b\}-f(a)\{a\}$, sub in the previous corollary, we get (1).

To show (2), we may directly apply integration by parts to f and $\{x\}$.

The other method is carried out by taking the formula for (1).

Let $a<b\in\mathbb{Z}$, we have $\sum_{n=a}^b f(n)=f(a)+\int_a^b f(x)\,dx+\int_a^b f'(x)\{x\}\,dx$ and $\int_a^b\frac{1}{2}f'(x)\,dx=\frac{1}{2}\int_a^b df(x)=\frac{1}{2}[f(b)-f(a)]$ □

Since $\{a\}=\{b\}=0\ \forall a,b\in\mathbb{Z}$, so "$f(a)\{a\}-f(b)\{b\}$" $=0$

COROLLARY (Abel's Summation)

Let $(a_n)_{n\ge1}$ be real numbers. Define $A:\mathbb{R}_n\longrightarrow\mathbb{R}$

$x\longmapsto\sum_{k=1}^{\lfloor x\rfloor}a_k$

If $f\in C^1$, $df(t)=f'(t)\,dt$

For $x\ge1$, and a conti function $f:[1,x]\longrightarrow\mathbb{R}$, we have $\sum_{k=1}^{\lfloor x\rfloor}a_k f(k)=-\int_1^x A(t)\,df(t)+A(x)f(x)$

DARBOUX SUMMATIONS

UPPER AND LOWER DARBOUX SUMMATIONS AND INTEGRALS

DEFINITION

Let $P\in\mathcal{P}([a,b])$ and define for $1\le k\le n$, $M_k=M_k(f):=\sup\{f(x)\mid x\in[x_{k-1},x_k]\}$ and $m_k=m_k(f):=\inf\{f(x)\mid x\in[x_{k-1},x_k]\}$

We define the upper and lower Darboux sums as follows: $U_P(f,\alpha)=\sum_{k=1}^n M_k(f)\Delta\alpha_k$, $L_P(f,\alpha)=\sum_{k=1}^n m_k(f)\Delta\alpha_k$

Note: No tagged points are needed for these defs. Also, when $\alpha(x)=x$, we call them upper/lower Riemann sums.

LEMMA (By def)

Suppose that α is increasing/non-decreasing on $[a,b]$. Then, for any $f:[a,b]\longrightarrow\mathbb{R}$ and any partition $P\in\mathcal{P}([a,b])$ and tagged points t, we have $m_k(f)\le f(t_k)\le M_k(f)$, $1\le k\le n$ and $L_P(f,\alpha)\le S_{P,t}(f,\alpha)\le U_P(f,\alpha)$

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PROPOSITION

Suppose that α is increasing.

(a) $\forall P\subseteq P'$, we have $U_{P'}(f,\alpha)\leq U_P(f,\alpha)$ and $L_P(f,\alpha)\leq L_{P'}(f,\alpha)$

(b) $\forall P,P'$, we have $L_P(f,\alpha)\leq U_{P'}(f,\alpha)$

Proof

(a) It is enough to prove the inequalities hold when P' contains one more substitution point than P. Let $P=(x_k)_{0\leq k\leq n}$ and $c\in(x_{i-1},x_i)$ when $1\leq i\leq n$.

We have: $U_P(f,\alpha)=\sum_{k=1}^{n}M_k(f)\Delta\alpha_k$, and $U_{P'}(f,\alpha)=\sum_{k=1,k\neq i}^{n}M_k(f)\Delta\alpha_k+M'[\alpha(c)-\alpha(x_{i-1})]+M''[\alpha(x_i)-\alpha(c)]$, where $M':=\sup\{f(x)\mid x_{i-1}\leq x\leq c\}$ and $M'':=\sup\{f(x)\mid c\leq x\leq x_i\}$.

Since $M'\leq M_i(f)$, $M''\leq M_i(f)$, we deduce $M'[\alpha(c)-\alpha(x_{i-1})]+M''[\alpha(x_i)-\alpha(c)]\leq M_i(f)[\alpha(x_i)-\alpha(x_{i-1})]$, so $U_{P'}(f,\alpha)\leq U_P(f,\alpha)$ (L_P is similar)

(b) Let $P,P'\in\mathcal{P}([a,b])$, and $P'':=P'\vee P$. Then, from (a), $L_P(f,\alpha)\leq L_{P''}(f,\alpha)\leq U_{P''}(f,\alpha)\leq U_{P'}(f,\alpha)$ $\square$

Core idea to link two unrelated partitions

DEFINITION

Suppose α is increasing. The upper/lower Stieltjes integrals of f w.r.t. α are defined by:

↳ $\overline{I}(f,\alpha)=\overline{\int_a^b}f\,d\alpha:=\inf\{U_P(f,\alpha)\mid P\in\mathcal{P}([a,b])\}$

↳ $\underline{I}(f,\alpha)=\underline{\int_a^b}f\,d\alpha:=\sup\{L_P(f,\alpha)\mid P\in\mathcal{P}([a,b])\}$

PROPOSITION

Suppose α is increasing, then $\underline{I}(f,\alpha)\leq\overline{I}(f,\alpha)$ (Proof: Trivial)

REMARK

The equality above may not hold. Say $\alpha(x)=x$, $f(x)=\mathbb{1}_{\mathbb{Q}}$ defined on $[0,1]$.

Then, $U_P(f,x)=1$ and $L_P(f,x)=0$ $\forall P\in\mathcal{P}([a,b])\Rightarrow$ By def, $\overline{I}(f,x)=1$, $\underline{I}(f,x)=0$, so $\underline{I}(f,x)\lneq\overline{I}(f,x)$

PROPOSITION (Linearity)

Let $a\leq c\leq b$. Let $\alpha:[a,b]\to\mathbb{R}$ be bounded and increasing

Let $f,g:[a,b]\to\mathbb{R}$ be bounded. We have:

- $\overline{\int_a^b}f\,d\alpha=\overline{\int_a^c}f\,d\alpha+\overline{\int_c^b}f\,d\alpha$, also $\underline{\int_a^b}f\,d\alpha=\underline{\int_a^c}f\,d\alpha+\underline{\int_c^b}f\,d\alpha$
- $\overline{\int_a^b}(f+g)\,d\alpha\leq\overline{\int_a^b}f\,d\alpha+\overline{\int_a^b}g\,d\alpha$, also $\underline{\int_a^b}(f+g)\,d\alpha\geq\underline{\int_a^b}f\,d\alpha+\underline{\int_a^b}g\,d\alpha$

RIEMANN'S CONDITION

DEFINITION

Let $\alpha:[a,b]\to\mathbb{R}$ be increasing. We say that f satisfies Riemann's condition w.r.t. α on $[a,b]$ if $\forall\varepsilon>0$, $\exists P_\varepsilon\in\mathcal{P}([a,b])$, s.t. $\forall P\supseteq P_\varepsilon$, we have $0\leq U_P(f,\alpha)-L_P(f,\alpha)<\varepsilon$ (Note: Tagged points don't matter here)

REMARK

Thanks to the propositions above, it suffices to find $P_\varepsilon\in\mathcal{P}([a,b])$ with $U_{P_\varepsilon}(f,\alpha)-L_{P_\varepsilon}(f,\alpha)<\varepsilon$ to satisfy Riemann's condition

THEOREM

For any increasing integrator α, TFAE:

The assumption for α to be increasing is actually not very restrictive. Recall the decomposition theorem for function of bounded variation ($\alpha\in BV\Leftrightarrow\alpha=\underbrace{V}_{\text{increasing}}-\underbrace{(V-\alpha)}_{\text{increasing}}$)

(1) $f\in R(\alpha;a,b)$

(2) f satisfies Riemann's condition

(3) $\underline{I}(f,\alpha)=\overline{I}(f,\alpha)$

Proof: Next set of notes!

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SOME APPLICATIONS

PROPOSITION

Given non-decreasing α, and $f, g \in \mathcal{R}([a,b])$, suppose $f(x) \leq g(x)\ \forall x \in [a,b]$, then we have $\int_a^b f(x)\,d\alpha(x) \leq \int_a^b g(x)\,d\alpha(x)$

Proof

For any partition $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a,b])$, we have $M_k(f) \leq M_k(g)$ and $m_k(f) \leq m_k(g)$, so we have $L_P(f,\alpha) \leq L_P(g,\alpha)$ and $U_P(f,\alpha) \leq U_P(g,\alpha)$

Therefore, the upper and lower Stieltjes integrals satisfy: $\overline{I}(f,\alpha) \leq \overline{I}(g,\alpha)$ and $\underline{I}(f,\alpha) \leq \underline{I}(g,\alpha)$. By the Thm above, thus $\int_a^b f\,d\alpha \leq \int_a^b g\,d\alpha$ □

PROPOSITION

Given non-decreasing α, $f \in R(\alpha; a, b)$, then $|f| \in R(\alpha; a, b)$ and $\left|\int_a^b f(x)\,d\alpha(x)\right| \leq \int_a^b |f(x)|\,d\alpha(x)$

Proof

For any partition $P \in \mathcal{P}([a,b])$ and $1 \leq k \leq n$, we have $M_k(f) - m_k(f) = \sup\{f(x) - f(y) \mid x, y \in [x_{k-1}, x_k]\}$

We may check that $M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f)$, hence $U_P(|f|,\alpha) - L_P(|f|,\alpha) \leq U_P(f,\alpha) - L_P(f,\alpha)$, i.e. $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$ □