Mock Midterm

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MOCK MIDTERM (NOT DONE PART 2 YET)

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PART I: LECTURE MATERIALS (50%)

Exercise 1

Justify whether each of the following statements is true or false. If it is true, please prove it briefly; otherwise, find a counterexample.

(a) If $f:[a,b]\to\mathbb{R}$ is a monotonic function, then it is of bounded variation.

True. As f is monotonic, $V_P(f)=\sum_{k=1}^{n}|\Delta f_k|=|f(b)-f(a)|\ \forall P$ $\therefore V_f([a,b])=|f(b)-f(a)|$. □

(b) If $\alpha:[a,b]\to\mathbb{R}$ is a step function and $f:[a,b]\to\mathbb{R}$ is bounded, then $f\in R(\alpha;a,b)$.

False. $f(x)=\alpha(x)$ would mean f,α share the same discontinuities, hence $f\notin R(\alpha;a,b)$. □ $\left(f=\alpha=\mathbb{1}_{x\geq 0},\ [a,b]=[-1,1],\text{ take partition } x_k=0 \quad S_{P,t}(f,\alpha)=f(t_k)\Delta\alpha_k=f(t_k)=\begin{cases}1, & t_k=x_k=0\\ 0, & t_k<x_k\end{cases}\right)$

(c) If $f\in R(x;a,b)$, then there is a partition P of $[a,b]$ such that $U_P(f,x)=L_P(f,x)$

False. Take $f(x)=\begin{cases}1, & x\neq 0\\ 0, & x=0\end{cases}$, defined for $x\in[-1,1]$. $\forall P$, $U_P(f,x)>0$, $L_P(f,x)=0$, but $f\in R(x;a,b)$ since it is only discontinuous at a point.

(d) If $f:[a,b]\to\mathbb{R}$ is continuous, then there exists $c\in[a,b]$ such that $f(c)=\frac{1}{b-a}\int_a^b f(t)\,dt$

True. Define $F(x):=\int_a^x f(t)\,dt$, then $F\in\mathcal{C}^1$. $\therefore$ By MVT, $\frac{F(b)-F(a)}{b-a}=F'(c)$ for some $c\in[a,b]$ (Assuming $a\neq b$).
$LHS=\frac{F(b)-F(a)}{b-a}=\frac{1}{b-a}\int_a^b f(t)\,dt$, $RHS=f(c)$, by FTC. $\therefore f(c)=\frac{1}{b-a}\int_a^b f(t)\,dt$. □

(e) Let $a=(a_n)_{n\geq 1}$ and $b=(b_n)_{n\geq 1}$ be two real sequences such that $\lim_{n\to\infty}(a_n-b_n)=0$. Then, a and b have the same behavior, that is both are either convergent or divergent.

True. Assume they have different behavior, WLOG assume a diverges and b converges. By def, $\lim_{n\to\infty}(a_n-b_n)=0\Rightarrow\forall\varepsilon>0,\exists N>0$, s.t. $\forall n\geq N$, $|a_n-b_n|<\varepsilon\Rightarrow b_n-\varepsilon<a_n<b_n+\varepsilon$. Say b converges to L, by def, $\exists N'>0$, s.t. $\forall n\geq N'$, $|b_n-L|<\varepsilon\Rightarrow L-\varepsilon<b_n<L+\varepsilon$
$\therefore\forall n\geq\max\{N,N'\}$, $L-2\varepsilon<a_n<L+2\varepsilon\Rightarrow|a_n-L|<2\varepsilon$ $\therefore$ By def, a converges $\divideontimes$ $\therefore$ a and b have the same behavior. □

Exercise 2

Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Consider the following statements:

(i) f is of bounded variation

(ii) The set of discontinuities of f is countable

(iii) f is Riemann integrable, i.e. $f\in R(x;a,b)$

Justify whether each of the following statements is true or false. If it is true, please prove it briefly; otherwise, find a counterexample.

(a) (i)$\Rightarrow$(ii)

True. $f=V-(V-f)$, where $V, V-f$ are monotonely increasing, and are also bounded
$\forall$ monotonely increasing functions h, all its discontinuities are jump discontinuities.
WLOG, say h is nondecreasing, then $\forall c\in D=\{x\in[a,b]\mid h(x^-)\neq h(x^+)\}$, $h(c^+)-h(c^-)>0$.
Take the $\inf_{c\in D} h(c^+)-h(c^-)=:k$. As h is bounded, $\sup_{x\in[a,b]}h(x)-\inf_{x\in[a,b]}h(x)\leq M<\infty$
$\therefore h$ can have at most $\frac{M}{\varepsilon}<\infty$ discontinuities, i.e. so do $V, V-f$
$\therefore f=V-(V-f)$ has a finite, i.e. countable, number of discontinuities. □

(b) (ii)$\Rightarrow$(i)

False. Take $[a,b]=[0,1]$, $f(x)=\begin{cases}x\sin(\frac{1}{x}), & x>0\\ 0, & x=0\end{cases}$, then f is conti., but not of bounded variation.

(c) (i)$\Rightarrow$(iii)

True. Take $\varepsilon>0$, then $\forall P\supseteq P_\varepsilon$, $\|P_\varepsilon\|<\varepsilon$, $\left|\sum_{k=1}^{n}[f(x_k)-f(x_{k-1})][x_k-x_{k-1}]\right|<\varepsilon\sum_{k=1}^{n}(f(x_k)-f(x_{k-1}))=\varepsilon V_f([a,b])$ $\therefore$ By Riemann condition, $f\in R(x;a,b)$. □

(d) (iii)$\Rightarrow$(i)

False Take $[a,b]=[0,1]$, $f(x)=\begin{cases}x\sin(\frac{1}{x}), & x>0\\ 0, & x=0\end{cases}$, then f is conti. $\Rightarrow f\in R(x;a,b)$ but not BV.

(e) (ii)$\Rightarrow$(iii)

False. Take $f(x)=\mathbb{1}_{x\in\mathbb{Q}}$, $[a,b]=[0,1]$.

(f) (iii)$\Rightarrow$(ii)

False. Take $f(x)=\mathbb{1}_{x\in\mathcal{C}}$, $[a,b]=[0,1]$.

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Exercise 3

Let $\alpha:[a,b]\to\mathbb{R}$ be a function of bounded variation and $V(x)=V_f([a,x])$ be its variation function.

(a) Show that both $V+\alpha$ and $V-\alpha$ are nondecreasing functions.

"$V-\alpha$": $\forall y>x\in[a,b]$, $(V-\alpha)(y)-(V-\alpha)(x)=(V(y)-V(x))-(\alpha(y)-\alpha(x))=V_\alpha([x,y])-(\alpha(y)-\alpha(x))$

As $|\alpha(x)-\alpha(y)|\leq V_\alpha([x,y])$, thus $\alpha(y)-\alpha(x)\leq V_\alpha([x,y])$, so $(V-\alpha)(y)-(V-\alpha)(x)\geq 0$, i.e. $V-\alpha$ is nondecreasing. $\square$

"$V+\alpha$": $\forall y>x\in[a,b]$, $(V+\alpha)(y)-(V-\alpha)(x)=(V(y)-V(x))-(\alpha(x)-\alpha(y))=V_\alpha([x,y])-(\alpha(x)-\alpha(y))$

As $|\alpha(x)-\alpha(y)|\leq V_\alpha([x,y])$, thus $\alpha(x)-\alpha(y)\leq V_\alpha([x,y])$, so $(V+\alpha)(y)-(V+\alpha)(x)\geq 0$, i.e. $V+\alpha$ is nondecreasing. $\square$

(b) Suppose further that α is $\mathcal{C}^1$. Show that $V(x)=\int_a^x|\alpha'(t)|\,dt$ and $\alpha(x)=\int_a^x\alpha'(t)\,dt+\alpha(a)$.

By FTC, $\int_a^x\alpha'(t)\,dt+\alpha(a)=[\alpha(t)]_a^x+\alpha(a)=\alpha(x)-\alpha(a)+\alpha(a)=\alpha(x)$. $\therefore \alpha(x)=\int_a^x\alpha'(t)\,dt+\alpha(a)$. $\square$

Now, consider $V(x)$,

By def, $V(x)=V_\alpha([a,x])=\sum_{i=1}^{n}|\alpha(x_i)-\alpha(x_{i-1})|$ for some $P\in\mathcal{P}([a,x])$

Seeing derivatives in integrals and absolute value ⇒ can consider MVT, not just FTC

As $\alpha\in\mathcal{C}^1$, by MVT, $V(x)=\sum_{i=1}^{n}|\alpha'(\xi_i)||x_i-x_{i-1}|$ for some $\xi_i\in(x_{i-1},x_i)$ $\forall i$

As we take finer P, $\|P\|\longrightarrow 0$, then we obtain $V(x)=\int_a^x|\alpha'(t)|\,dt$ $\square$

Exercise 4

Let $a=(a_n)_{n\geq1}$ and $b=(b_n)_{n\geq1}$ be two sequences in $\mathbb{R}$. Write down the definition of the three following notions:

(a) a is dominated by b, denoted by $a_n=O(b_n)$

$\exists$ a bounded sequence $c=(c_n)_{n\geq1}$, $N>0$, s.t. $\forall n\geq N$, $a_n=c_nb_n$.

(b) a is negligible compared to b, denoted by $a_n=o(b_n)$

$\exists$ a bounded sequence $c=(c_n)_{n\geq1}$ with limit 0, $N>0$, s.t. $\forall n\geq N$, $a_n=c_nb_n$

(c) a is equivalent to b, denoted by $a_n\sim b_n$

$\exists$ a bounded sequence $c=(c_n)_{n\geq1}$ with limit 1, $N>0$, s.t. $\forall n\geq N$, $a_n=c_nb_n$.

Consider the following four sequences: (A) $\log\log n$ (B) c^n, $c>1$ (C) n^s, $s>0$ (D) $(\log n)^k$, $k>0$

Match (a^1-a^4) with $(A-D)$ to make up correct statements: $a^{i+1}=o(a^i)$ as $n\to\infty$ for $i=1,2,3$

$a^1=B$, $a^2=C$, $a^3=D$, $a^4=A$