Analysis Midterm Supplements

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2D CONCEPTUAL QUESTIONS

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1) (True or false) Let $f:[a,b)\to\mathbb{R}$ be a monotonic function. Then, for any $c\in(a,b)$, both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist.

True.

Proof

WLOG, say f is monotonely increasing, then $\forall x>c$, $f(x)\geq f(c)$.

Define $L=\inf\{f(x)\mid x>c\}$, then by def, $\forall\varepsilon>0$, $\exists x'>c$, s.t. $f(x')<L+\varepsilon$. However, $f(x)\geq L$ $\forall x>c$

Now, $\forall\varepsilon>0$, $\exists\delta:=x'-c>0$, s.t. $0<x-c<\delta=x'-c \Rightarrow c<x<x' \Rightarrow f(c)\leq L\leq f(x)<f(x')<L+\varepsilon \Rightarrow 0\leq f(x)-L<\varepsilon$ $\therefore \lim_{x\to c^+} f(x)=L$ □

Similarly, define $L=\sup\{f(x)\mid x<c\}$, then by def, $\forall\varepsilon>0$, $\exists x'<c$, s.t. $L-\varepsilon<f(x')$. However, $f(x)\leq L$ $\forall x<c$.

Now, $\forall\varepsilon>0$, $\exists\delta:=c-x'>0$, s.t. $0<c-x<\delta=c-x' \Rightarrow x'<x<c \Rightarrow L-\varepsilon<f(x')<f(x)\leq L \Rightarrow 0\leq L-f(x)<\varepsilon$ $\therefore \lim_{x\to c^-} f(x)=L$ □

Proof Sketch

WLOG, $f\uparrow$.

"$x\to c^+$": $L:=\inf\{f(x)\mid x>c\} \Rightarrow \forall\varepsilon>0, \exists x'>c$, s.t. $f(x')<L+\varepsilon$, but $L\leq f(x)$ $\forall x>c$. $\therefore \forall\varepsilon>0, \exists\delta:=x'-c>0$, s.t. $0<x-c<\delta\Rightarrow\ldots\Rightarrow 0\leq f(x)-L<\varepsilon$

"$x\to c^-$": $L:=\sup\{f(x)\mid x<c\}$, similar □

2) Let $f:[a,b]\to\mathbb{R}$. Consider the statements:

(i) f is continuous

(ii) f is of bounded variation

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent.

(D) Neither (i) implies (ii) nor (ii) implies (i)

D ☆ Cont. and BV sound related but they aren't!!

"(i) $\not\Rightarrow$ (ii)" : $f(x)=x\sin\frac{1}{x}$ for $x>0$

"(ii) $\not\Rightarrow$ (i)" : $f(x)=\mathbb{1}_{x\geq 0}$ for $x\in[-1,1]$ (Key counterexample)

3) Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}\sin\frac{1}{x}; & x>0\\ 0; & x=0\end{cases}$

Select the correct answer.

(A) f is both of bounded variation and Riemann-integrable on $[0,1]$.

(B) f is Riemann-integrable but not of bounded variation on $[0,1]$.

(C) f is of bounded variation but not Riemann-integrable on $[0,1]$.

(D) f is neither of bounded variation nor Riemann-integrable on $[0,1]$

B

Rmb, Riemann-integrable checks for continuity not BV.

Riemann-integrable: Cont. except for a finite number of points

Not BV:

Proof

Consider the partition $P=\{0,\frac{2}{(2n-1)\pi},\frac{2}{(2n-3)\pi},\ldots,\frac{2}{\pi},1\}$, $n\geq 1$, then $V_P(f)=\sum_{k=1}^{n+1}|\Delta f_k|=1+(2(n-1))+(1-\sin 1)=2n-\sin 1$, which is not bounded. □

4) Let $BV([a,b],\mathbb{R})$ be the space of functions of bounded variation defined on $[a,b]$. Is $\|f\|=|f(a)|+V_f([a,b])$ for all $f\in BV([a,b],\mathbb{R})$ a norm? Select all correct statements:

(A) No, it does not satisfy positive definite property.

(B) No, it does not satisfy homogenous property

(C) No, it does not satisfy triangle inequality

(D) Yes, it is a norm

D (Trivial, I can technically write the proof here but it's very straightforward)

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5) Compute $\int_0^4 x^2 d\alpha$.

Here, $\alpha: [0,4] \to \mathbb{R}$ is given by $\alpha = \begin{cases} 0, & 0 \le x < 1 \\ 2, & x = 1 \\ 1, & 1 < x \le 2 \\ 3, & 2 < x \le 4 \end{cases}$

(A) 5

(B) 7

(C) 9

(D) 11

C

Recall for step function integrators, $\int_a^b f d\alpha = f(c)[\alpha(c^+) - \alpha(c^-)]$

$\therefore \int_0^4 x^2 d\alpha = f(1)(1-0) + f(2)(3-1) = 1^2(1) + 2^2(2) = 9$

6) (True or false) Let $f: [a,b] \to \mathbb{R}$ be a continuous function and $\alpha: [a,b] \to \mathbb{R}$ be a function of bounded variation. Then, $\int_a^b f d\alpha \le \int_a^b |f| d\alpha$

False.

Take $\alpha := -x$, $f := -1$, $[a,b] := [0,1]$, $\int_a^b f d\alpha = \int_0^1 -1 d(-x) = 1$, $\int_a^b |f| d\alpha = \int_0^1 (1) d(-x) = -1$.

7) Let $f: [a,b] \to \mathbb{R}$ be a bounded function, $\alpha: [a,b] \to \mathbb{R}$ be a nondecreasing function, and $L \in \mathbb{R}$.

Consider the statements:

(i) $L_P(f, \alpha) \le L \le U_P(f, \alpha)$ for all partition P of $[a,b]$

(ii) $f \in \mathcal{R}(\alpha; a, b)$ and $\int_a^b f d\alpha = L$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

B

(ii) $\Rightarrow$ (i) by def. (i) $\not\Rightarrow$ (ii) because: consider $[a,b] = [0,1]$, $f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{Q}^c \end{cases}$, $\int_0^1 f dx$ does not exist.

8) Let $\alpha: [a,b] \to \mathbb{R}$ be of bounded variation and V be its variation function. Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Consider the statements:

(i) $f \in \mathcal{R}(\alpha; a, b)$

(ii) $f \in \mathcal{R}(V; a, b)$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

C

Proof sketch

"$\alpha \Rightarrow V$": If α is const, then $V \equiv 0$, so OK. Assume $\alpha(a) < \alpha(b)$, so $V(b) > 0$.

$V_P(f) \le V_P \le v_P(f) + \varepsilon$

Let $\varepsilon > 0$, $P_\varepsilon^{(1)} \in \mathcal{P}([a,b])$, s.t. $\forall P \supseteq P_\varepsilon^{(1)}$, tagged pts t, $\left|\sum_{k=1}^n [f(t_k) - f(t_k')] \Delta\alpha_k\right| \le \varepsilon$.

Let $M = \sup|f|$, $P_\varepsilon^{(2)} \in \mathcal{P}([a,b])$, s.t. $\forall P \supseteq P_\varepsilon^{(2)}$, $V(b) = V_\alpha([a,b]) \le V_P(\alpha) + \frac{\varepsilon}{M}$

V nondecreasing $\Rightarrow$ check Riemann condition: $\sum_{k=1}^n [M_k(f) - m_k(f)] \Delta V_k = \underbrace{\sum_{k=1}^n [M_k(f) - m_k(f)] |\Delta\alpha_k|}_{(*)} + \underbrace{\sum_{k=1}^n [M_k(f) - m_k(f)][\Delta V_k - |\Delta\alpha_k|]}_{(**)}$

$\hookrightarrow$ $(**) \le 2M \sum_{k=1}^n (\Delta V_k - |\Delta\alpha_k|) = 2M\left(V(b) - \sum_{k=1}^n |\Delta\alpha_k|\right) \le 2\varepsilon$ $(P_\varepsilon^{(2)})$

$\hookrightarrow$ $K^+ := \{1 \le k \le n \mid \Delta\alpha_k \ge 0\}$, $K^- := \{1 \le k \le n \mid \Delta\alpha_k < 0\}$, $\varepsilon' = \frac{\varepsilon}{V(b)}$

$\Rightarrow$ $k \in K^+$: Choose $t_k, t_k' \in [x_{k-1}, x_k]$, s.t. $f(t_k) - f(t_k') \ge M_k(f) - m_k(f) - \varepsilon'$; $k \in K^-$: $f(t_k') - f(t_k) \ge M_k(f) - m_k(f) - \varepsilon'$

$\Rightarrow$ $(*) = \sum_{k \in K^+} [M_k(f) - m_k(f)] \Delta\alpha_k + \sum_{k \in K^-} [M_k(f) - m_k(f)][-\Delta\alpha_k] \le \sum_{k=1}^n [f(t_k) - f(t_k')] \Delta\alpha_k + \varepsilon' \sum_{k=1}^n |\Delta\alpha_k| \le 2\varepsilon$ $\square$

"$V \Rightarrow \alpha$": $\forall i$, $|\alpha(x_i) - \alpha(x_{i-1})| \le V(x_i) - V(x_{i-1})$, so $\sum_{k=1}^n [M_k(f) - m_k(f)] |\alpha(x_k) - \alpha(x_{k-1})| \le \sum_{k=1}^n [M_k(f) - m_k(f)] |V(x_k) - V(x_{k-1})|$ ✓

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9) Let $\alpha:[a,b]\to\mathbb{R}$ be a nondecreasing function and $f:[a,b]\to\mathbb{R}$ be a bounded function. Consider the statements:

(i) $f\in\mathcal{R}(\alpha;a,b)$

(ii) $f\in\mathcal{R}(x;a,b)$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

D

(had to check answer)

(i) $\not\Rightarrow$ (ii), because if $\alpha=0$, $f\in\mathcal{R}(\alpha;a,b)$ $\forall f$, even $f\notin\mathcal{R}(x;a,b)$. (ii) $\not\Rightarrow$ (i), because $f=\alpha=$ step discontinuity is not integrable ($\because$ they have the same discontinuities)

10) Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $\alpha:[a,b]\to\mathbb{R}$ be a function. Consider the statements:

(i) $f\in\mathcal{R}(\alpha;a,b)$

(ii) α is of bounded variation

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

B

(i) $\not\Rightarrow$ (ii), say we take $f=0$, $\alpha\notin BV([a,b])$.

(ii) $\Rightarrow$ (i):

Proof sketch

Let $\varepsilon>0$. $[a,b]$ cpt $\Rightarrow$ f unif cont. $\Rightarrow$ Take $\delta>0$, s.t. $\forall x,y\in[a,b]$, $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$

Take partition $P\supseteq P_\varepsilon$, $\|P_\varepsilon\|<\delta$, then $U_P(f,\alpha)-L_P(f,\alpha)=\sum_{k=1}^{n}[M_k(f)-m_k(f)]\Delta\alpha_k\le\varepsilon\sum_{k=1}^{n}\Delta\alpha_k=\varepsilon[\alpha(b)-\alpha(a)]\Rightarrow$ Riemann cond $\square$

11) Let $f:[a,b]\to\mathbb{R}$ be of bounded variation. Consider the statements:

(i) $f\in\mathcal{R}(f;a,b)$

(ii) f is continuous

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i)

(B) (ii) implies (i) but (i) does not imply (ii)

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i)

C

(ii) $\Rightarrow$ (i) is a direct result of what I just proved, f is BV, f is cont. $\Rightarrow f\in\mathcal{R}(f;a,b)$

(i) $\Rightarrow$ (ii):

Proof sketch

Consider "f not cont. $\Rightarrow f\notin\mathcal{R}(f;a,b)$"

WLOG f not right cont. at c, $\exists\,\varepsilon>0$, $\delta>0$, s.t. $\exists\, x\in(c,c+\delta)$, $|f(x)-f(c)|>\varepsilon$

Let $P\in\mathcal{P}([a,b])$, $x_i=c$, $x_{i+1}=y$, $1\le i\le n-1$. Then, $U_P(f,\alpha)-L_P(f,\alpha)=\sum_{k=1}^{n}[M_k(f)-m_k(f)]\Delta f_k\geq\varepsilon[M_{i+1}(f)-m_{i+1}(f)]\geq\varepsilon^2$ $\square$

12) (True or false) Let $f:[a,b]\to\mathbb{R}$ be a Riemann-integrable function. Then, $F(x)=\int_a^x f(t)\,dt$ is a Lipschitz function on $[a,b]$.

True (This is a direct result of MVT: $\forall x,y\in[a,b]$, $|F(y)-F(x)|=|\int_x^y f(t)\,dt|=c|y-x|$, $c\in(x,y)\Rightarrow|F(y)-F(x)|\le K|y-x|$ for some K $\forall y,x$)

Proof sketch (of MVT)

$L_P(f,\alpha)\le\int_a^b f\,d\alpha\le U_P(f,\alpha)$, $m[\alpha(b)-\alpha(a)]\le L_P(f,\alpha)$, $M[\alpha(b)-\alpha(a)]\le U_P(f,\alpha)\Rightarrow m\int_a^b d\alpha\le\int_a^b f\,d\alpha\le M\int_a^b d\alpha$

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13) Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Consider the statements:
(i) The set of discontinuities of f is a measure zero set.
(ii) There is a continuous function $g:[a,b]\to\mathbb{R}$ such that $\{x\in[a,b] \mid f(x)\neq g(x)\}$ is a measure zero set.
Select the correct answer.
(A) (i) implies (ii) but (ii) does not imply (i)
(B) (ii) implies (i) but (i) does not imply (ii)
(C) (i) and (ii) are equivalent
(D) Neither (i) implies (ii) nor (ii) implies (i)

D (Had to check answer for counterexample)

(i) $\not\Rightarrow$ (ii): $f=\mathbb{1}_{x\geq\frac{1}{2}}$ on $[0,1]$
(ii) $\not\Rightarrow$ (i): $f=\mathbb{1}_{x\in\mathbb{Q}}$ on $[0,1]$

14) Select the correct primitives of $\frac{1}{\sqrt{x^2+1}}, \frac{1}{\sqrt{x^2-1}}, \frac{1}{\sqrt{1-x^2}}$
(A) $\sin^{-1}x$, $\sinh^{-1}x$, $\cosh^{-1}x$
(B) $\sin^{-1}x$, $\cosh^{-1}x$, $\sinh^{-1}x$
(C) $\sinh^{-1}x$, $\cosh^{-1}x$, $\sin^{-1}x$
(D) $\sinh^{-1}x$, $\sin^{-1}x$, $\cosh^{-1}x$
(E) $\cosh^{-1}x$, $\sin^{-1}x$, $\sinh^{-1}x$
(F) $\cosh^{-1}x$, $\sinh^{-1}x$, $\sin^{-1}x$

C

15) Let $(a_n)_{n\geq1}$ be a real sequence. Consider the statements:
(i) $(a_n)_{n\geq1}$ converges to 0
(ii) $\sum_{n=1}^{\infty} a_n$ converges
Select the correct answer.
(A) (i) implies (ii) but (ii) does not imply (i)
(B) (ii) implies (i) but (i) does not imply (ii)
(C) (i) and (ii) are equivalent
(D) Neither (i) implies (ii) nor (ii) implies (i)

B

(i) $\not\Rightarrow$ (ii): $(a_n=\frac{1}{n})_{n\geq1}$

(ii) $\Rightarrow$ (i)

Proof sketch

Denote $S_n:=\sum_{i=1}^{n} a_i$. $\because (S_n)_{n\geq1}$ converges $\Rightarrow (S_n)_{n\geq1}$ is Cauchy $\Rightarrow \forall\varepsilon>0, \exists N>0$, s.t. $\forall m>n>N$, $|S_m-S_n|=|\sum_{i=n+1}^{m} a_i|<\varepsilon \Rightarrow \forall n>N, |a_{n+1}|<\varepsilon$ □

16) Let $(a_n)_{n\geq1}$ be a real sequence. Consider the statements:
(i) $(a_n)_{n\geq1}$ converges
(ii) $a_n=O(1)$ as $n\to\infty$
Select the correct answer.
(A) (i) implies (ii) but (ii) does not imply (i)
(B) (ii) implies (i) but (i) does not imply (ii)
(C) (i) and (ii) are equivalent
(D) Neither (i) implies (ii) nor (ii) implies (i)

A

(ii) $\not\Rightarrow$ (i): $a_n=1,2,1,2,1,\dots$

(i) $\Rightarrow$ (ii):

Proof sketch

for $n\geq N$.

Set $c_n=a_n$ $\forall n\geq1$, as all convergent sequences are bounded, then for $b_n=1$ $\forall n$, $a_n=c_nb_n$, c_n is bounded. $\therefore a_n=O(1)$ □

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17) Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two real sequences. Consider the statements:

(i) Both $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n\to\infty$.

(ii) $a_n \sim b_n$ as $n \to \infty$

Select the correct answer.

(A) (i) implies (ii) but (ii) does not imply (i).

(B) (ii) implies (i) but (i) does not imply (ii).

(C) (i) and (ii) are equivalent

(D) Neither (i) implies (ii) nor (ii) implies (i).

B

(i) $\not\Rightarrow$ (ii): $(a_n = 1)_{n\geq 1}$ and $(b_n = 2)_{n\geq 1}$

(ii) $\Rightarrow$ (i):

Proof sketch

$a_n \sim b_n \Rightarrow \exists (c_n)_{n\geq 1} \xrightarrow{n\to\infty} 1$, $N>0$, s.t. $a_n = c_n b_n\ \forall n \geq N \Rightarrow a_n = O(b_n)$. Take $(c'_n = \frac{1}{c_n})_{n\geq 1}$ for $n \geq N$, as $c_n \neq 0$, this is defined $\Rightarrow b_n = O(a_n)$. □

18) Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two nonnegative real sequences such that $a_n = O(b_n)$ as $n\to\infty$. Select all correct statements:

(A) If $(a_n)_{n\geq 1}$ converges, then $(b_n)_{n\geq 1}$ converges

(B) If $(b_n)_{n\geq 1}$ converges, then $(a_n)_{n\geq 1}$ converges

(C) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges

(D) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

D

Why not A? $(a_n = 0)_{n\geq 1}$, $(b_n = n)_{n\geq 1}$

Why not B? $(b_n = n)_{n\geq 1}$, $(a_n = \begin{cases} 1, & \text{odd } n \\ 2, & \text{even } n \end{cases})_{n\geq 1}$

Why not C? $(a_n = 0)_{n\geq 1}$, $(b_n = \frac{1}{n})_{n\geq 1}$

Why D:

Proof sketch

$a_n = O(b_n) \Rightarrow \exists N > 0$, s.t. $\forall n \geq N$, $a_n = c_n b_n$ for some bounded $(c_n)_{n\geq 1}$, i.e. $c_n \leq M < \infty$. As nonneg, $0 \leq \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} c_n b_n \leq M \sum_{n=1}^{\infty} b_n$ □

19) Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two nonnegative real sequences such that $a_n \sim b_n$ as $n\to\infty$. Select all correct statements:

(A) If $(a_n)_{n\geq 1}$ converges, then $(b_n)_{n\geq 1}$ converges

(B) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges

(C) If both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then $\sum_{k=1}^{n} a_k \sim \sum_{k=1}^{n} b_k$ as $n\to\infty$

(D) If both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then $\sum_{k=n}^{\infty} a_k \sim \sum_{k=n}^{\infty} b_k$ as $n \to \infty$

A, B, D

Why not C? (Had to check answer) $a_1 = 1$, $b_1 = 2$, $a_n = b_n = \frac{1}{n^2}$, $n \geq 2$ (☆Remember properly, when remainders ~ or partial sums ~)

Proof sketch

A: Let $\varepsilon > 0$, take $N > 0$, s.t. $(1-\varepsilon)a_n \leq b_n \leq (1+\varepsilon)a_n\ \forall n \geq N$ ... OK ✓

B: Same as last question

D: Let $\varepsilon > 0$, take $N > 0$, s.t. $(1-\varepsilon)u_n \leq v_n \leq (1+\varepsilon)v_n$. $\therefore \forall M \geq n \geq N$, $\sum_{k=n+1}^{M} v_k \leq \sum_{k=n+1}^{M} (1+\varepsilon)u_k \leq (1+\varepsilon)\sum_{k=n+1}^{\infty} u_k \Rightarrow \sum_{k=n+1}^{\infty} v_k \leq (1+\varepsilon)\sum_{k=n+1}^{\infty} u_k$. Also, $\sum_{k=n+1}^{\infty} v_k \geq (1-\varepsilon)\sum_{k=n+1}^{\infty} u_k$

20) Select the correct asymptotic notations (as $n\to\infty$) of the series $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} \frac{1}{k}$, $\sum_{k=1}^{n} \frac{1}{2^k}$

(A) $O(1)$, $O(n^2)$, $O(\log n)$

(B) $O(1)$, $O(\log n)$, $O(n^2)$

(C) $O(n^2)$, $O(1)$, $O(\log n)$

(D) $O(n^2)$, $O(\log n)$, $O(1)$

(E) $O(\log n)$, $O(1)$, $O(n^2)$

(F) $O(\log n)$, $O(n^2)$, $O(1)$

D, trivial

ボクはバカだよ ᵔᴥᵔ

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MOCK MIDTERM (NOT DONE PART 2 YET)

PART I: LECTURE MATERIALS (50%)

Exercise 1

Justify whether each of the following statements is true or false. If it is true, please prove it briefly; otherwise, find a counterexample.

(a) If $f:[a,b]\to\mathbb{R}$ is a monotonic function, then it is of bounded variation.

True. As f is monotonic, $V_P(f)=\sum_{k=1}^{n}|\Delta f_k|=|f(b)-f(a)|\ \forall P$ $\therefore V_f([a,b])=|f(b)-f(a)|$. □

(b) If $\alpha:[a,b]\to\mathbb{R}$ is a step function and $f:[a,b]\to\mathbb{R}$ is bounded, then $f\in R(\alpha;a,b)$.

False. $f(x)=\alpha(x)$ would mean f,α share the same discontinuities, hence $f\notin R(\alpha;a,b)$. □ $\left(f=\alpha=\mathbb{1}_{x\geq 0},\ [a,b]=[-1,1],\text{ take partition } x_k=0,\ S_{P,t}(f,\alpha)=f(t_k)\Delta\alpha_k=f(t_k)=\begin{cases}1, & t_k=x_k=0\\ 0, & t_k<x_k\end{cases}\right)$

(c) If $f\in R(x;a,b)$, then there is a partition P of $[a,b]$ such that $U_P(f,x)=L_P(f,x)$

False. Take $f(x)=\begin{cases}1, & x\neq 0\\ 0, & x=0\end{cases}$, defined for $x\in[-1,1]$. $\forall P$, $U_P(f,x)>0$, $L_P(f,x)=0$, but $f\in R(x;a,b)$ since it is only discontinuous at a point.

(d) If $f:[a,b]\to\mathbb{R}$ is continuous, then there exists $c\in[a,b]$ such that $f(c)=\frac{1}{b-a}\int_a^b f(t)\,dt$

True. Define $F(x):=\int_a^x f(t)\,dt$, then $F\in\mathcal{C}^1$. $\therefore$ By MVT, $\frac{F(b)-F(a)}{b-a}=F'(c)$ for some $c\in[a,b]$ (Assuming $a\neq b$).
$LHS=\frac{F(b)-F(a)}{b-a}=\frac{1}{b-a}\int_a^b f(t)\,dt$, $RHS=f(c)$, by FTC. $\therefore f(c)=\frac{1}{b-a}\int_a^b f(t)\,dt$. □

(e) Let $a=(a_n)_{n\geq 1}$ and $b=(b_n)_{n\geq 1}$ be two real sequences such that $\lim_{n\to\infty}(a_n-b_n)=0$. Then, a and b have the same behavior, that is both are either convergent or divergent.

True. Assume they have different behavior, WLOG assume a diverges and b converges. By def, $\lim_{n\to\infty}(a_n-b_n)=0\Rightarrow\forall\varepsilon>0,\exists N>0$, s.t. $\forall n\geq N$, $|a_n-b_n|<\varepsilon\Rightarrow b_n-\varepsilon<a_n<b_n+\varepsilon$. Say b converges to L, by def, $\exists N'>0$, s.t. $\forall n\geq N'$, $|b_n-L|<\varepsilon\Rightarrow L-\varepsilon<b_n<L+\varepsilon$
$\therefore\forall n\geq\max\{N,N'\}$, $L-2\varepsilon<a_n<L+2\varepsilon\Rightarrow|a_n-L|<2\varepsilon$ $\therefore$ By def, a converges $\divideontimes$ $\therefore$ a and b have the same behavior. □

Exercise 2

Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Consider the following statements:

(i) f is of bounded variation

(ii) The set of discontinuities of f is countable

(iii) f is Riemann integrable, i.e. $f\in R(x;a,b)$

Justify whether each of the following statements is true or false. If it is true, please prove it briefly; otherwise, find a counterexample.

(a) (i) $\Rightarrow$ (ii)

True. $f=V-(V-f)$, where V, $V-f$ are monotonely increasing, and are also bounded
$\forall$ monotonely increasing functions h, all its discontinuities are jump discontinuities.
WLOG, say h is nondecreasing, then $\forall c\in D=\{x\in[a,b]\mid h(x^-)\neq h(x^+)\}$, $h(c^+)-h(c^-)>0$.
Take the $\inf_{c\in D} h(c^+)-h(c^-)=:k$. As h is bounded, $\sup_{x\in[a,b]}h(x)-\inf_{x\in[a,b]}h(x)\leq M<\infty$
$\therefore h$ can have at most $\frac{M}{k}<\infty$ discontinuities, i.e. so do V, $V-f$
$\therefore f=V-(V-f)$ has a finite, i.e. countable, number of discontinuities. □

(b) (ii) $\Rightarrow$ (i)

False. Take $[a,b]=[0,1]$, $f(x)=\begin{cases}x\sin(\frac{1}{x}), & x>0\\ 0, & x=0\end{cases}$, then f is conti, but not of bounded variation.

(c) (i) $\Rightarrow$ (iii)

True. Take $\varepsilon>0$, then $\forall P\supseteq P_\varepsilon$, $\|P_\varepsilon\|<\varepsilon$, $\left|\sum_{k=1}^{n}[f(x_k)-f(x_{k-1})][x_k-x_{k-1}]\right|<\varepsilon\sum_{k=1}^{n}(f(x_k)-f(x_{k-1}))=\varepsilon V_f([a,b])$ $\therefore$ By Riemann condition, $f\in R(x;a,b)$. □

(d) (iii) $\Rightarrow$ (i)

False Take $[a,b]=[0,1]$, $f(x)=\begin{cases}x\sin(\frac{1}{x}), & x>0\\ 0, & x=0\end{cases}$, then f is conti $\Rightarrow f\in R(x;a,b)$ but not BV.

(e) (ii) $\Rightarrow$ (iii)

False. Take $f(x)=\mathbb{1}_{x\in\mathbb{Q}}$, $[a,b]=[0,1]$.

(f) (iii) $\Rightarrow$ (ii)

False. Take $f(x)=\mathbb{1}_{x\in\mathcal{C}}$, $[a,b]=[0,1]$.

Exercise 3

Let $\alpha:[a,b]\to\mathbb{R}$ be a function of bounded variation and $V(x)=V_f([a,x])$ be its variation function.

(a) Show that both $V+\alpha$ and $V-\alpha$ are nondecreasing functions.

"$V-\alpha$": $\forall y>x\in[a,b]$, $(V-\alpha)(y)-(V-\alpha)(x)=\underline{(V(y)-V(x))-(\alpha(y)-\alpha(x))}=V_\alpha([x,y])-(\alpha(y)-\alpha(x))$

As $\underline{|\alpha(x)-\alpha(y)|\leq V_\alpha([x,y])}$, thus $\alpha(y)-\alpha(x)\leq V_\alpha([x,y])$, so $(V-\alpha)(y)-(V-\alpha)(x)\geq 0$, i.e. $V-\alpha$ is nondecreasing. □

"$V+\alpha$": $\forall y>x\in[a,b]$, $(V+\alpha)(y)-(V-\alpha)(x)=(V(y)-V(x))-(\alpha(x)-\alpha(y))=V_\alpha([x,y])-(\alpha(x)-\alpha(y))$

As $|\alpha(x)-\alpha(y)|\leq V_\alpha([x,y])$, thus $\alpha(x)-\alpha(y)\leq V_\alpha([x,y])$, so $(V+\alpha)(y)-(V+\alpha)(x)\geq 0$, i.e. $V+\alpha$ is nondecreasing. □

(b) Suppose further that α is $\mathcal{C}^1$. Show that $V(x)=\int_a^x|\alpha'(t)|dt$ and $\alpha(x)=\int_a^x\alpha'(t)dt+\alpha(a)$.

By FTC, $\int_a^x\alpha'(t)dt+\alpha(a)=[\alpha(t)]_a^x+\alpha(a)=\alpha(x)-\alpha(a)+\alpha(a)=\alpha(x)$. $\therefore \alpha(x)=\int_a^x\alpha'(t)dt+\alpha(a)$. □

Now, consider $V(x)$,

Seeing derivatives in integrals and absolute value ⇒ can consider MVT, not just FTC

By def, $V(x)=V_\alpha([a,x])=\underline{\sum_{i=1}^{n}|\alpha(x_i)-\alpha(x_{i-1})|}$ for some $P\in\mathcal{P}([a,x])$

As $\alpha\in\mathcal{C}^1$, by MVT, $V(x)=\underline{\sum_{i=1}^{n}|\alpha'(\xi_i)||x_i-x_{i-1}|}$ for some $\xi_i\in(x_{i-1},x_i)$ $\forall i$

As we take finer P, $\|P\|\longrightarrow 0$, then we obtain $V(x)=\int_a^x|\alpha'(t)|dt$ □

Exercise 4

Let $a=(a_n)_{n\geq1}$ and $b=(b_n)_{n\geq1}$ be two sequences in $\mathbb{R}$. Write down the definition of the three following notions:

(a) a is dominated by b, denoted by $a_n=O(b_n)$

$\exists$ a bounded sequence $c=(c_n)_{n\geq1}$, $N>0$, s.t. $\forall n\geq N$, $a_n=c_nb_n$.

(b) a is negligible compared to b, denoted by $a_n=o(b_n)$

$\exists$ a bounded sequence $c=(c_n)_{n\geq1}$ with limit 0, $N>0$, s.t. $\forall n\geq N$, $a_n=c_nb_n$

(c) a is equivalent to b, denoted by $a_n\sim b_n$

$\exists$ a bounded sequence $c=(c_n)_{n\geq1}$ with limit 1, $N>0$, s.t. $\forall n\geq N$, $a_n=c_nb_n$.

Consider the following four sequences: (A) $\log\log n$ (B) c^n, $c>1$ (C) n^s, $s>0$ (D) $(\log n)^k$, $k>0$

Match (a^1-a^4) with (A-D) to make up correct statements: $a^{i+1}=o(a^i)$ as $n\to\infty$ for $i=1,2,3$

$a^1=B$, $a^2=C$, $a^3=D$, $a^4=A$

PART II: PROBLEMS (50%)

Problem 5

Let $\alpha:[a,b]\to\mathbb{R}$ be a strictly increasing and continuous function. Given a bounded function $g:[a,b]\to\mathbb{R}$, we say that g is α-differentiable at $x\in(a,b)$ if the limit $D_\alpha g(x)=\lim_{y\to x}\frac{g(y)-g(x)}{\alpha(y)-\alpha(x)}$ exists. The one-sided α-derivatives, $D_\alpha^+g(x)$ $(x\in[a,b))$ and $D_\alpha^-g(x)$ $(x\in(a,b])$, are defined similarly.

(1) Prove that α is α-differentiable at every $x\in[a,b]$ and $D_\alpha\alpha(x)=1$

$\forall x\in[a,b]$, by def, $\lim_{y\to x}\frac{\alpha(y)-\alpha(x)}{\alpha(y)-\alpha(x)}=\lim_{y\to x}1=1$ $\therefore\alpha$ is α-diffable $\forall x\in[a,b]$ and $D_\alpha\alpha(x)=1$. □

(2) Prove that if g is α-differentiable at $x\in[a,b]$, then g is continuous at x

By def, g is α-diffable ⇒ $\forall\varepsilon>0,\exists\delta>0$, s.t. $|y-x|<\delta\Rightarrow\left|\frac{g(y)-g(x)}{\alpha(y)-\alpha(x)}-D_\alpha g(x)\right|<\varepsilon$, as the derivative exists. (Forgot to say unif cont.)

$\because$ α is cont: $\therefore\exists\delta'>0$, s.t. $|y-x|<\delta'\Rightarrow|\alpha(y)-\alpha(x)|<\varepsilon$

For $|y-x|<\min\{\delta,\delta'\}$, $|g(y)-g(x)-D_\alpha g(x)[\alpha(y)-\alpha(x)]|<\varepsilon^2\Rightarrow||g(y)-g(x)|-D_\alpha g(x)\,\varepsilon|<\varepsilon^2\Rightarrow|g(y)-g(x)|<\varepsilon'$, where $\varepsilon'=\min\{\varepsilon^2+D_\alpha g(x)\varepsilon,\ |D_\alpha g(x)\varepsilon-\varepsilon^2|\}$

$\therefore$ g is cont. at x. □

(3) Prove the following fundamental theorem of calculus: Given $f\in\mathcal{R}(\alpha;a,b)$, define the function $F:[a,b]\to\mathbb{R}$ by $F(x)=\int_a^x f(t)\,d\alpha(t)$. Then, F is continuous. Further, if f is continuous at $x\in[a,b]$, then F is α-differentiable at x and $D_\alpha F(x)=f(x)$.

As $f\in\mathcal{R}(\alpha;a,b)$, thus f is bounded on $[a,b]$ ($\because\alpha$ is strictly increasing). Say $|f(x)|\leq M<\infty$ $\forall x\in[a,b]$. Then, $\forall x,y\in[a,b]$,

$|F(y)-F(x)|=|\int_x^y f(t)\,d\alpha(t)|\leq|\int_x^y M\,d\alpha(t)|=M[\alpha(y)-\alpha(x)]$. As α is unif cont., $\forall\varepsilon>0,\exists\delta>0$, s.t. $|F(y)-F(x)|<M\varepsilon$. $\therefore F$ is cont. □

Notice, by MVT, $\exists c\in(x,y)$, s.t. $\int_x^y f(t)\,d\alpha(t)=f(c)[\alpha(y)-\alpha(x)]$, where $F(y)-F(x)=\int_x^y f(t)dt$, if $\alpha(y)\neq\alpha(x)$

As f is cont., $[a,b]$ is cpt, so f is unif cont.:

$\forall\varepsilon>0,\exists\delta>0$, s.t. $\forall x,y\in[a,b]$, $|y-x|<\delta\Rightarrow|f(y)-f(x)|<\varepsilon$, so $|f(c)-f(x)|=\left|\frac{F(y)-F(x)}{\alpha(y)-\alpha(x)}-f(x)\right|<\varepsilon$

$\therefore$ By def, $\lim_{y\to x}\frac{F(y)-F(x)}{\alpha(y)-\alpha(x)}=f(x)$, i.e. $D_\alpha F(x)=f(x)$. □

Problem 6

Suppose $f:[a,b]\to\mathbb{R}$ is a $\mathcal{C}^2$ function and there exists a unique point $c\in(a,b)$ such that $f(c)=\max_{x\in[a,b]} f(x)$ and $f''(c)<0$. In this problem, you are freely allowed to use the fact that $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.

(1) Show that $f(x)=f(c)+\frac{1}{2}f''(c)(x-c)^2+o((x-c)^2)$ as $x\to c$.

As $f(c)=\max_{x\in[a,b]}f(x)$, thus $f'(c)=0$.

$\therefore$ Applying Taylor's expansion, $f(x)=f(c)+f'(c)(x-c)+\frac{1}{2}f''(c)(x-c)^2+o((x-c)^2)=f(c)+\frac{1}{2}f''(c)(x-c)^2+o((x-c)^2)$. $\square$

(2) For any $\varepsilon>0$, show that there exists $\delta>0$ such that $f(c)+\frac{1}{2}(f''(c)+\varepsilon)(x-c)^2\geq f(x)\geq f(c)+\frac{1}{2}(f''(c)-\varepsilon)(x-c)^2$ for all $x\in[c-\delta,c+\delta]$

By def, for $g(x):=f(x)-f(c)-\frac{1}{2}f''(c)(x-c)^2$, $g(x)=o((x-c)^2)\Rightarrow\forall\varepsilon>0,\exists\delta>0$, s.t. $|x-c|\leq\delta\Rightarrow\left|\frac{g(x)}{(x-c)^2}\right|\leq\varepsilon$, i.e. $|g(x)|\leq\varepsilon(x-c)^2$

$\therefore\forall\varepsilon>0,\exists\delta>0$, s.t. $\forall x\in[c-\delta,c+\delta]$, $f(c)+\frac{1}{2}(f''(c)+\varepsilon)(x-c)^2\geq f(x)\geq f(c)+\frac{1}{2}(f''(c)-\varepsilon)(x-c)^2$. $\square$

(3) By considering the estimate $\int_a^b e^{nf(x)}dx\geq\int_{c-\delta}^{c+\delta}e^{nf(x)}dx$, deduce that $\liminf_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{e^{nf(c)}\sqrt{2\pi/(-nf''(c))}}\geq 1$

Consider $\int_{c-\delta}^{c+\delta}e^{nf(x)}dx$, by (2),

$$\int_{c-\delta}^{c+\delta}e^{nf(c)}e^{\frac{n}{2}(f''(c)+\varepsilon)(x-c)^2}dx\geq\int_{c-\delta}^{c+\delta}e^{nf(x)}dx\geq\int_{c-\delta}^{c+\delta}e^{nf(c)}e^{\frac{n}{2}(f''(c)-\varepsilon)(x-c)^2}dx$$

$$\Rightarrow e^{nf(c)}\int_{-\delta}^{\delta}e^{\frac{n}{2}(f''(c)+\varepsilon)x^2}dx\geq\int_{c-\delta}^{c+\delta}e^{nf(x)}dx\geq e^{nf(c)}\int_{-\delta}^{\delta}e^{\frac{n}{2}(f''(c)-\varepsilon)x^2}dx$$

$$1\leq\liminf_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{e^{nf(c)}\int_{-\delta}^{\delta}e^{\frac{n}{2}(f''(c)-\varepsilon)x^2}dx}\overset{\varepsilon\to0\text{ as }n\to\infty}{=}\liminf_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{2e^{nf(c)}\int_0^{\delta}e^{-(\sqrt{-\frac{n}{2}f''(c)}\,x)^2}dx}=\liminf_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{2e^{nf(c)}\left(\frac{1}{\sqrt{-\frac{n}{2}f''(c)}}\right)\int_0^{\infty}e^{-x^2}dx}$$

$\delta\sqrt{-\frac{n}{2}f''(c)}\to\infty$ as $n\to\infty$

$$=\liminf_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{2\sqrt{2}e^{nf(c)}\left(\frac{1}{\sqrt{-nf''(c)}}\right)\left(\frac{\sqrt{\pi}}{2}\right)}=\liminf_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{e^{nf(c)}\sqrt{\frac{2\pi}{n(-f''(c))}}}\ \square$$

(4) Pick $\eta>0$ such that for an $|x-c|\geq\delta$, $f(x)\leq f(c)-\eta$. By considering the decomposition $\int_a^b e^{nf(x)}dx=\int_a^{c-\delta}e^{nf(x)}dx+\int_{c-\delta}^{c+\delta}e^{nf(x)}dx+\int_{c+\delta}^{b}e^{nf(x)}dx$, deduce that $\limsup_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{e^{nf(c)}\sqrt{2\pi/(-nf''(c))}}\leq 1$

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(5) Conclude that $\int_a^b e^{nf(x)}dx\sim e^{nf(c)}\sqrt{\frac{2\pi}{n(-f''(c))}}$ as $n\to\infty$

From (3) and (4), we get:

$$1\leq\liminf_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{e^{nf(c)}\sqrt{\frac{2\pi}{n(-f''(c))}}}\leq\limsup_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{e^{nf(c)}\sqrt{\frac{2\pi}{n(-f''(c))}}}\leq1\Rightarrow\lim_{n\to\infty}\frac{\int_a^b e^{nf(x)}dx}{e^{nf(c)}\sqrt{\frac{2\pi}{n(-f''(c))}}}=1$$

Let $A_n=\int_a^b e^{nf(x)}dx$, $B_n=e^{nf(c)}\sqrt{\frac{2\pi}{n(-f''(c))}}$, then set $C_n=\frac{A_n}{B_n}$, we get $\lim_{n\to\infty}C_n=1$ $\therefore$ By def, $A_n\sim B_n$, i.e. $\int_a^b e^{nf(x)}dx\sim e^{nf(c)}\sqrt{\frac{2\pi}{n(-f''(c))}}$ as $n\to\infty$. $\square$

(6) Let $f(x)=-\sin^2x$ be defined on $[-\frac{\pi}{2},\frac{\pi}{2}]$ Prove that $\int_0^{\frac{\pi}{2}}e^{-n\sin^2x}dx\sim\sqrt{\frac{\pi}{4n}}$ as $n\to\infty$.

From (5), notice $f''(x)=\frac{d}{dx}(-2\sin x\cos x)=\frac{d}{dx}(-\sin(2x))=-2\cos(2x)$

Set $a=-\frac{\pi}{2}$, $b=\frac{\pi}{2}$, $c=0$, we get: $f(c)=0$, $f''(c)=-2$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-n\sin^2x}dx\sim e^{n(0)}\sqrt{\frac{2\pi}{n(-(-2))}}=\sqrt{\frac{\pi}{n}}\text{ as }n\to\infty$$

$\because$ LHS $=2\int_0^{\frac{\pi}{2}}e^{-n\sin^2x}dx$

$\therefore\int_0^{\frac{\pi}{2}}e^{-n\sin^2x}dx\sim\frac{1}{2}\sqrt{\frac{\pi}{n}}=\sqrt{\frac{\pi}{4n}}$ as $n\to\infty$. $\square$

Problem 7

Let $n\in\mathbb{N}$ and $f:[1,\infty)\to\mathbb{R}$ be a $\mathcal{C}^\infty$ function. Recall the Euler's summation formula: $\sum_{k=1}^{n}f(k)=\int_1^n f(x)dx+\int_1^n f'(x)(\{x\}-\frac{1}{2})dx+\frac{1}{2}(f(n)+f(1))$.

Write $\varphi_1:[1,\infty)\to\mathbb{R}$ to be the function $\{x\}-\frac{1}{2}$.

(1) Show that there is a constant $c\in\mathbb{R}$ such that $\varphi_2:[1,\infty)\to\mathbb{R}$ defined by $\varphi_2(x):=2\int_1^x\varphi_1(t)dt+c$ satisfies the property $\int_k^{k+1}\varphi_2(x)dx=0$ for all $k\in\mathbb{N}$. Moreover, determine the value of such constant c.

$$\int_k^{k+1}\varphi_2(x)dx=\int_k^{k+1}\left(2\int_1^x(\{t\}-\tfrac{1}{2})dt+c\right)dx=c+2\int_k^{k+1}\int_1^x\{t\}-\tfrac{1}{2}dt\,dx=c+2\int_k^{k+1}\left[\lfloor x\rfloor\int_0^1 t-\tfrac{1}{2}dt+\int_0^{\{x\}}t-\tfrac{1}{2}dt\right]dx$$

$$=c+\int_k^{k+1}\{x\}^2-\{x\}dx=c+\int_0^1x^2-x\,dx=c-\tfrac{1}{6}$$

$\therefore\forall k\in\mathbb{N}$, $c=\frac{1}{6}\Rightarrow\int_k^{k+1}\varphi_2(x)dx=0$ $\square$

In the succeeding questions, we fix this constant c satisfying the property (P).

(2) Show that $\varphi_3:[1,\infty)\to\mathbb{R}$ defined by $\varphi_3(x)=3\int_1^x\varphi_2(t)dt$ is bounded.

We know $\varphi_3(1)=3\int_1^1\varphi_2(t)dt=0$. As $\varphi_2\in\mathcal{C}^1$, by FTC, $\varphi_3'(x)=3\varphi_2(x)=6\int_1^x\varphi_1(t)dt+3c=3c+6\lfloor x\rfloor\int_0^1 t-\frac{1}{2}dt+6\int_0^{\{x\}}t-\frac{1}{2}dt$

$$=3c+3\{x\}^2-3\{x\}=3(\{x\}-\tfrac{1}{2})^2+(3c-\tfrac{3}{4})<\infty$$

$\therefore$ For every interval $[x_0,x_0+1]$, $\varphi_3'(x)$ starts and ends at the same value, and is symmetric, $\forall x_0\in\mathbb{N}$, which means the value of $\varphi_3(x)$ resets every interval of length 1, and its in between values are bounded too. $\therefore\varphi_3(x)$ is bounded. $\square$

(3) Derive the formula $\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{1}{6}\int_1^n f'''(x)\varphi_3(x)\,dx + \frac{c}{2}[f'(n) - f'(1)] + \frac{1}{2}[f(n) + f(1)]$

Consider $\int_1^n f'(x)(\{x\} - \frac{1}{2})\,dx = \int_1^n f'(x)\varphi_1(x)\,dx = \int_1^n f'(x)\,d(\frac{1}{2}\varphi_2(x)) = [\frac{1}{2}f'(n)\varphi_2(n) - \frac{1}{2}f'(1)\varphi_2(1)] - \frac{1}{2}\int_1^n \varphi_2(x) f''(x)\,dx$

$= \frac{c}{2}[f'(n) - f'(1)] - \frac{1}{6}\int_1^n f''(x)\,d(\varphi_3(x)) = \frac{c}{2}[f'(n) - f'(1)] \cancel{- \frac{1}{6}[f'''(n)\varphi_3(n) - f'''(1)\varphi_3(1)]} + \frac{1}{6}\int_1^n \varphi_3(x) + f'''(x)\,dx$

$\therefore$ Sub into Euler's summation formula, we get $\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{1}{6}\int_1^n f'''(x)\varphi_3(x)\,dx + \frac{c}{2}(f'(n) - f'(1)] + \frac{1}{2}[f(n) + f(1)]$. $\square$

(4) Suppose that $\lim_{x\to\infty}\int_1^x |f'''(t)|\,dt$ converges. Show that the limit $\lim_{x\to\infty}\int_1^x f'''(t)\varphi_3(t)\,dt$ exists.

Notice, $\varphi_3(t)$ is bounded, say $|\varphi_3(t)| \le M < \infty$ $\forall t$.

Then, $\lim_{x\to\infty}\left|\int_1^x f'''(t)\varphi_3(t)\,dt\right| \le \lim_{x\to\infty}\int_1^x |f'''(t)||\varphi_3(t)|\,dt \le M \lim_{x\to\infty}\int_1^x |f'''(t)|\,dt$, which converges

$\therefore$ $\lim_{x\to\infty}\int_1^x f'''(t)\varphi_3(t)\,dt$ exists. $\square$

(5) With the same assumption in (4), conclude that $\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + C + E(n)$, where $C = \frac{1}{2}f(1) - \frac{c}{2}f'(1) + \frac{1}{6}\int_1^\infty f'''(x)\varphi_3(x)\,dx$,
$E(n) = \frac{1}{2}f(n) + \frac{c}{2}f'(n) - \frac{1}{6}\int_n^\infty f'''(x)\varphi_3(x)\,dx$

By (3), $\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{1}{6}(\int_1^\infty f'''(x)\varphi_3(x)\,dx - \int_n^\infty f'''(x)\varphi_3(x)\,dx) + \frac{c}{2}[f'(n) - f'(1)] + \frac{1}{2}[f(n) + f(1)] = \int_1^n f(x)\,dx + C + E(n)$. $\square$

(6) Deduce that for $s > 1$, $\sum_{k=1}^{n} \frac{1}{k^s} = C' - \frac{1}{(s-1)n^{s-1}} + \frac{1}{2n^s} - \frac{cs}{2n^{s+1}} + O(\frac{1}{n^{s+2}})$ as $n \to \infty$ for some constant $C' \in \mathbb{R}$.

By (5), set $f(k) = \frac{1}{k^s}$, then $\int_1^n f(x)\,dx = \int_1^n x^{-s}\,dx = -\frac{1}{(s-1)n^{s-1}} + \frac{1}{s-1}$, $\frac{1}{2}f(n) = \frac{1}{2n^s}$, $\frac{c}{2}f'(n) = \frac{c(-s-1)}{2n^{s+1}} = -\frac{cs}{2n^{s+1}} - \frac{c}{2n^{s+1}}$

$C = \frac{1}{2}f(1) - \frac{c}{2}f'(1) + \frac{1}{6}\int_1^\infty f'''(x)\varphi_3(x)\,dx = \tilde{C}$, since $\int_1^\infty f'''(x)\varphi_3(x)\,dx$ exists and is constant.

$\int_n^\infty f'''(x)\varphi_3(x)\,dx = \int_n^\infty O(\frac{1}{x^{s+3}})\varphi_3(x)\,dx$, $|\int_n^\infty O(\frac{1}{x^{s+3}})\varphi_3(x)\,dx| \le M\int_n^\infty O(\frac{1}{x^{s+3}})\,dx = O(\frac{1}{x^{s+2}})$

$\therefore$ Combining everything, we get $\sum_{k=1}^{n} \frac{1}{k^s} = C' - \frac{1}{(s-1)n^{s-1}} + \frac{1}{2n^s} - \frac{cs}{2n^{s+1}} + O(\frac{1}{n^{s+2}})$ as $n \to \infty$. $\square$