Shun/翔海 (@shun4midx)

THEOREM (ABEL'S RULE)

Let $f:[a,b)\to\mathbb{R}$ be C^1 and $g:[a,b)\to\mathbb{R}$ be C^0.
Suppose (i) $f(x)\searrow 0$ when $x\to b^-$
(ii) $\exists M>0$, s.t. $\left|\int_a^x g(t)dt\right|\leq M$ for $x\in[a,b)$
Then, $\int_a^b f(t)g(t)dt$ converges

Proof

We want to check that Cauchy's criterion is satisfied.
Let $\varepsilon>0$, by (i), we may find $A\in[a,b)$, s.t. $|f(t)|\leq\varepsilon$ $\forall t\in[A,b)$
Let G be a primitive of g, $G(x)=\int_a^x g(t)dt$ $\forall x\in[A,b)$
$\Rightarrow$ It follows from (ii) that $|G(x)|\leq M$ for some fixed M uniformly in $x\in[a,b)$

Let $x,y\in[A,b)$ with $x\leq y$. We have $\int_x^y f(t)g(t)dt=[f(t)G(t)]_x^y-\int_x^y f'(t)G(t)dt=f(y)G(y)-f(x)G(x)-\int_x^y f'(t)G(t)$
Notice,
- $|f(y)G(y)|\leq\varepsilon M$
- $|f(x)G(x)|\leq\varepsilon M$
- $\left|\int_x^y f'(t)G(t)dt\right|\leq\int_x^y |f'(t)||G(t)|dt\leq M\int_x^y|f'(t)|dt=-M\int_x^y f'(t)dt=M(f(x)-f(y))\leq\varepsilon M$

$\therefore$ In conclusion, $\left|\int_x^y f(t)g(t)dt\right|\leq 3\varepsilon M$ for $x,y\in[A,b)$ with $x\leq y$. □

EXAMPLE

Fix $\alpha>0$. Then, the following integrals converge. $\int_1^\infty \frac{\sin t}{t^\alpha}dt$, $\int_1^\infty\frac{\cos t}{t^\alpha}dt$ (by Abel's rule). Also, this means $\int_1^\infty \frac{e^{it}}{t^\alpha}dt$ conv too.

EXAMPLE

Let $f:[1,+\infty)\longrightarrow\mathbb{C}$, $x\longmapsto\frac{e^{ix}}{\sqrt{x}}$, $g:[1,+\infty)\longrightarrow\mathbb{C}$, $x\longmapsto\frac{e^{ix}}{\sqrt{x}}+\frac{1}{x}$

- $f(x)\sim g(x)$ when $x\to+\infty$
- $\int_1^\infty f$ conv

$\therefore \int_1^\infty g$ does not conv (since otherwise $\int_1^\infty(g(t)-f(t))dt=\int_1^\infty\frac{1}{t}dt$ conv ⟶✗)

LAPLACE'S METHOD

THEOREM

Let $-\infty\leq a<b\leq+\infty$ and $g,h:(a,b)\to\mathbb{R}$ be C^1.
Suppose (i) $x\longmapsto g(x)e^{h(x)}$ is integrable on (a,b) (kinda like abs conv, not just conv)
(ii) $\exists c\in(a,b)$, s.t. (a) h is increasing on (a,c) and decreasing on (c,b) with $h''(c)<0$
(b) $g(c)\neq 0$
Then, when $\lambda\to+\infty$, we have $\int_a^b g(x)e^{\lambda h(x)}\sim\sqrt{\frac{2\pi}{-\lambda h''(c)}}\,g(c)e^{\lambda h(c)}$

PROOF SKETCH

By Taylor expansion, $h(x)\approx h(c)-\frac{1}{2}(-h''(c))(x-c)^2$
Then, $\int_a^b g(x)e^{\lambda h(x)}dx\approx\int_{c-\varepsilon}^{c+\varepsilon}e^{\lambda(h(c)+\frac{1}{2}h''(c)(x-c)^2)}dx=g(c)e^{\lambda h(c)}\int_{c-\varepsilon}^{c+\varepsilon}e^{\frac{1}{2}h''(c)(x-c)^2}dx$
Let $y=\sqrt{\lambda}(x-c)$, then $=g(c)e^{\lambda h(c)}\frac{1}{\sqrt{\lambda}}\int_{-\varepsilon\sqrt{\lambda}}^{\varepsilon\sqrt{\lambda}}e^{-\frac{1}{2}h''(c)y^2}dy\approx g(c)e^{\lambda h(c)}\frac{1}{\sqrt{\lambda}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}h''(c)y^2}dy=\sqrt{\frac{2\pi}{-\lambda h''(c)}}\,g(c)e^{\lambda h(c)}$ (Used in steepest descent in ML)
Remark: These two "≈" are not rigorous and require explanation, hence why this is a "proof sketch" rather than a "proof"

APPLICATION (STIRLING'S FORMULA)

Recall for $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$, $x>0$, $\Gamma(n+1)=n!$ $\forall n\in\mathbb{N}$
$\Gamma(n+1)=\int_0^\infty t^n e^{-t}dt=\int_0^\infty e^{n\ln t-t}dt$
Let $t=ns$, $=\int_0^\infty ne^{n\ln(ns)-ns}ds=n^{n+1}\int_0^\infty e^{n(\ln s-s)}ds$

Define $h:(0,+\infty)\longrightarrow\mathbb{R}$, $s\longmapsto\ln s-s$

We have $h'(s)=\frac{1}{s}-1$, $h''(s)=-\frac{1}{s^2}$

	0	1	∞
$h'(s)$	+		−
h	↗		↘

h is $\nearrow$ on $(0,1)$, $\searrow$ on $(1,+\infty)$, and $h''(1)=-1<0$

$\therefore$ By Laplace's method, we find $\int_0^\infty e^{n(\ln s-s)}ds \underset{n\to+\infty}{\sim} \sqrt{\frac{2\pi}{n}}e^{-n} \Rightarrow n!=\Gamma(n+1) \underset{n\to+\infty}{\sim} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

SEQUENCES AND SERIES OF FUNCTIONS

NOTATION

$\mathcal{F}(A,M):=\{f:A\to M \text{ functions}\}$

$\mathcal{B}(A,M):=\{f:A\to M \text{ bounded functions}\}$

NOTIONS OF CONVERGENCE

SEQUENCES OF FUNCTIONS

DEFINITION

Let $(f_n)_{n\geq1}$ be a sequence of functions from A to M, that is, they are elements of $\mathcal{F}(A,M)$

- Let $f\in\mathcal{F}(A,M)$, we say $(f_n)_{n\geq1}$ converges pointwise to f if $\forall x\in A$, $\exists f_n(x)\xrightarrow{n\to\infty} f(x)$ in (M,d)
- We say $(f_n)_{n\geq1}$ converges pointwise if $\exists f\in\mathcal{F}(A,M)$, s.t. $(f_n)_{n\geq1}$ converges pointwise to f
- Let $B\subseteq A$ be a subset. We say $(f_n)_{n\geq1}$ converges pointwise on B if $((f_n)|_B)_{n\geq1}$ converges pointwise

EXAMPLE

For $n\geq1$, let $f_n:[0,1]\longrightarrow\mathbb{R}$, $x\longmapsto x^n$

The sequence of functions $(f_n)_{n\geq1}$ converges pointwise to $\mathbb{1}_{\{1\}}$ on $[0,1]$

REMARK

(1) If $f_n\xrightarrow{n\to\infty} f$ pointwise, then f is unique (but depends on d)

(2) If $(M,d)=(W,\|\cdot\|)$ is a finite-dimensional normed vector space, and $f_n\longrightarrow f$ pointwise, the limit does not depend on the norm

(3) Some properties are preserved by pointwise convergence: linearity, product, inequality, monoticity, etc

(4) Analytic properties (continuity, differentiability, integrability, etc) may NOT be preserved

DEFINITION

Let $(f_n)_{n\geq1}$ be a sequence of functions from A to M

- Let $f\in\mathcal{F}(A,M)$, we say $(f_n)_{n\geq1}$ converges uniformly to f if $\forall\varepsilon>0$, $\exists N(\varepsilon)>0$, s.t. $\forall n\geq N$, $x\in A$, $d(f_n(x),f(x))<\varepsilon$
- We say $(f_n)_{n\geq1}$ converges uniformly if $\exists f\in\mathcal{F}(A,M)$, s.t. $(f_n)_{n\geq1}$ converges uniformly to f
- We say $(f_n)_{n\geq1}$ converges uniformly on $B\subseteq A$ if $((f_n)|_B)_{n\geq1}$ converges uniformly

REMARK

Let's write the pointwise convergence using quantifiers.

We say $f_n\to f$ pointwise if $\forall\varepsilon>0$, $\exists N(x,\varepsilon)>0$, s.t. $\forall n\geq N$, $d(f_n(x),f(x))\leq\varepsilon$

This means that uniform convergence $\Rightarrow$ pointwise convergence. In particular, the uniform limit is unique.

PROPOSITION (CAUCHY'S CRITERION)

Suppose (M,d) is complete. Let $(f_n)_{n\geq1}$ be a sequence of functions from A to M. Then, $f_n\xrightarrow{n\to\infty} f$ uniformly if $\forall\varepsilon>0,\exists N>0$, s.t. $\forall m,n\geq N$, $\forall x\in A$, $d(f_n(x),f_m(x))\leq\varepsilon$

COROLLARY

Shun / 翔海 (@shun4midx)

If $(f_n)_{n\geq 1}$ converges uniformly to f, it converges pointwise to f.

REMARK

To show that $f_n \to f$ uniformly, we start by proving the pointwise convergence, then check that this convergence is uniform

DEFINITION

- Let us equip $B(A,M)$ with the following distance: $\forall f,g \in B(A,M)$, $d_\infty(f,g) = d_{\infty,A}(f,g) := \sup_{x\in A} d(f(x), g(x))$
  If $f_n \in B(A,M)$ and $f_n \to f$ uniformly $\Leftrightarrow d_\infty(f_n, f) \xrightarrow{n\to\infty} 0$
- Let $(W, \|\cdot\|)$ be a normed vector space. Let us equip $B(A,W)$ with the following norm, $\forall f \in B(A,W)$, $\|f\|_\infty = \|f\|_{\infty,A} = \sup_{x\in A} \|f(x)\|$, called the norm of uniform convergence. $f_n \to f$ uniformly $\Leftrightarrow \|f_n - f\| \xrightarrow{n\to\infty} 0$

PROPOSITION

Let $(W, \|\cdot\|)$ be a Banach space. Then, $B(A,W)$ is also a Banach space

Proof

Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $B(A,W)$. We want to check that $(f_n)_{n\geq 1}$ converges in $(B(A,W), \|\cdot\|_\infty)$

- Let $x \in A$, note that $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in W, so it converges to a limit we call $f(x)$
- Check that $f \in B(A,W)$. First, a Cauchy sequence is bounded, so $\|f_n\|_\infty < M$ for some $M > 0$ uniformly in n

For $x \in A$, $f(x) = \lim_{n\to\infty} f_n(x)$, so $\|f(x)\| = \lim_{x\to\infty} \|f_n(x)\| < M$. This means that $f \in B(A,W)$

- Check that $f_n \to f$ in $(B(A,W), \|\cdot\|_\infty)$. Let $\varepsilon > 0$, take $N > 0$, s.t. $\|f_n - f_m\|_\infty \leq \varepsilon$ $\forall m,n \geq N$

$\forall x \in A$, we have $\|f_n(x) - f(x)\| = \lim_{m\to\infty} \|f_n(x) - f_m(x)\| \leq \varepsilon$ $\forall n \geq N$. This means, $\|f_n - f\|_\infty \leq \varepsilon$ $\forall n \geq N$