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EXAMPLE

Let $(W, \|\cdot\|)$ be a Banach space, $\mathcal{L}_c(W)$ be equipped with the operator norm $|||\cdot||| \Rightarrow$ normed algebra, fix $u \in \mathcal{L}_c(W)$

Consider $$\mathcal{E}_u : \mathbb{R} \longrightarrow \mathcal{L}_c(W)$$
$$t \longmapsto \sum_{n\geq 0} \frac{t^n}{n!} u^n$$

We want to study the regularities and properties of $\mathcal{E}_u$

- Fix $t \in \mathbb{R}$, check that $\mathcal{E}_u(t)$ is well-defined, $\sum_{n\geq 0} \frac{|t|^n}{n!} |||u^n||| \leq \sum_{n\geq 0} \frac{|t|^n}{n!} |||u|||^n = \exp(|||t||| \cdot |||u|||) < +\infty$
  $\therefore \sum_{n\geq 0} \frac{t^n}{n!} u^n$ converges absolutely, so it converges.
- To check the continuity of $\mathcal{E}_u$ at some $t_0 \in \mathbb{R}$, we may check that the series of function converges uniformly on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$
  ↳ More specifically, we check for uniform convergence on $[t_0 - \varepsilon, t_0 + \varepsilon]$ for a fixed $\varepsilon > 0$.
  Fix $M > 0$. We already know $t \mapsto \sum \frac{t^n}{n!} u^n$ converges pointwise on $[-M, M]$. It remains to show that the remainder function converges uniformly to 0.
  ↳ For $t \in [-M, M]$, and $n \in \mathbb{N}$, we have $R_n(t) = \sum_{k\geq n+1} \frac{t^k}{k!} u^k \Rightarrow |||R_n(t)||| \leq \sum_{k\geq n+1} \frac{|t|^k}{k!} |||u|||^k \leq \sum_{k\geq n+1} \frac{M^k}{k!} |||u|||^k$. As $\sum \frac{M^n}{n!} |||u|||^n$ is a convergent series, RHS $\longrightarrow 0$ as it is the remainder of a convergent series
  ↳ Additionally, this upper bound does NOT depend on $t \in [-M, M]$, so the series converges uniformly
  $\therefore \mathcal{E}_u$ is continuous on $[-M, M]$ $\forall M > 0$, i.e. $\mathcal{E}_u$ is continuous on $\mathbb{R}$
- For $n \in \mathbb{N}_0$, define $u_n(t) = \frac{t^n}{n!} u^n$ which is $\mathcal{C}^\infty$ $\forall n$ ($\because$ polynomial function)
  For $n \in \mathbb{N}_0$ and $t \in \mathbb{R}$, we have $u_n'(t) = \frac{n t^{n-1}}{n!} u^n = \begin{cases} u \cdot u_{n-1}, & n \geq 1 \\ 0, & n = 0 \end{cases}$
- Fix $M > 0$. The series $\sum_{n\geq 1} u \cdot u_{n-1} = \sum_{n\geq 0} u_n'$ converges uniformly (We also know $\sum u_n$ converges pointwise), so $\mathcal{E}_u$ is $\mathcal{C}^1$ on $[-M, M]$
- We know $\mathcal{E}_u'(t) = u \mathcal{E}_u(t)$ $\forall t \in [-M, M]$. Therefore, $\mathcal{E}_u$ is $\mathcal{C}^1$ on $\mathbb{R}$, $\mathcal{E}_u'(t) = u \cdot \mathcal{E}_u(t)$ $\forall t \in \mathbb{R}$
- Let $k \in \mathbb{N}$. If $\mathcal{E}_u$ is $\mathcal{C}^k$ for $k \geq 1$, then so is $\mathcal{E}_u' = u \cdot \mathcal{E}_u \Rightarrow \mathcal{E}_u \in \mathcal{C}^{k+1}$ $\therefore$ By induction on k, $\mathcal{E}_u$ is of class $\mathcal{C}^\infty$

POWER SERIES

We state the theorems and properties in $\mathbb{R}$ or $\mathbb{C}$, but they hold in general under normed algebra with minimal modifications.

DEFINITIONS AND RADIUS OF CONVERGENCE

In $(\mathbb{C}, |\cdot|)$, balls are called disks

For example, $D(a,r) = B(a,r) = \{y \in \mathbb{C} \mid |y-a| < r\} \Rightarrow$ open disk
$\bar{D}(a,r) = \bar{B}(a,r) = \{y \in \mathbb{C} \mid |y-a| \leq r\} \Rightarrow$ closed disk

DEFINITION

Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers and $c \in \mathbb{C}$.

- $\sum_{n\geq 0} a_n (z-c)^n$ is called a power series centered at $c \in \mathbb{C}$ with variable $z \in \mathbb{C}$.
- If $a_n \in \mathbb{R}$ $\forall n$ and $c \in \mathbb{R}$, then $\sum_{n\geq 0} a_n (x-c)^n$ is called a real power series centered at $c \in \mathbb{C}$ with variable $x \in \mathbb{R}$.

WLOG, we automatically assume $c = 0$ (by translation).

PROPOSITION (ABEL'S LEMMA)

Let $\sum a_n z^n$ be a power series. Let $z_0 \in \mathbb{C}$, s.t. $(a_n z_0^n)_{n\geq 0}$ is bounded. Then,

1) For $z \in \mathbb{C}$ with $|z| < |z_0|$, the series $\sum a_n z^n$ converges absolutely
2) For $r \in (0, |z_0|)$, the series of functions $\sum a_n z^n$ converges normally in the closed disk $\bar{D}(0, r)$. (In the entire disk, we have unif conv!!)

Proof

Let $M > 0$, s.t. $|a_n||z_0|^n \leq M$. Let $z \in \mathbb{C}$ with $|z| < |z_0|$. Then, $\forall n \geq 0$, $|a_n z^n| = \underbrace{|a_n z_0^n|}_{\leq M} \underbrace{\left|\frac{z}{z_0}\right|^n}_{<1}$

Hence,

(1) $\sum |a_n z^n|$ converges because it can be upper bounded by a geometric series $\sum M \left|\frac{z}{z_0}\right|^n$ which converges. $\Rightarrow \sum a_n z^n$ converges absolutely
(2) Fix $r \in (0, |z_0|)$. For $z \in \bar{D}(0, r)$, $|a_n z^n| \leq M \left(\frac{r}{|z_0|}\right)^n$, with the upper bound independent on $z \in \bar{D}(0, r)$. Hence, it is normal convergence.

DEFINITION

Let $\sum a_n z^n$ be a power series. Define $R = R(\sum a_n z^n) := \sup\{r \geq 0 \mid (|a_n| r^n)_{n\geq 0} \text{ is bounded}\}$. This is called the radius of convergence of the power series $\sum a_n z^n$

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REMARK

If we add phases ($a_n e^{i\theta_n}$) to the sequence $(a_n)_{n\geq 0}$, the power series $\sum a_n z^n$ has its radius of convergence remain unchanged

PROPOSITION

Let $\sum a_n z^n$ be a power series and R be its radius of convergence. Then,

(1) For $z\in\mathbb{C}$, with $|z|<R$, the series $\sum a_n z^n$ converges absolutely

(2) For $z\in\mathbb{C}$, with $|z|>R$, the series $\sum a_n z^n$ diverges

(3) For $r\in[0,R)$, the series $\sum a_n z^n$ converges normally on the closed disk $\bar{D}(0,r)$

Proof

(1) Let $z\in\mathbb{C}$ and $|z|<R$. Write $r=\frac{|z|+R}{2}$. Then, $|z|<r<R$.
By definition of R, we know that $(|a_n|r^n)_{n\geq 0}$ is bounded. $\therefore \sum a_n z^n$ converges absolutely

(3) Can be shown in the same way as (1)

(2) $|z|>R$ means that $(|a_n||z|^n)_{n\geq 0}$ is unbounded, so $a_n z^n \not\to 0 \Rightarrow \sum a_n z^n$ does not converge □

REMARK

1) If $R=\infty$, the power series is well-defined on $\mathbb{C}$. Such a function is called an entire function

2) When $R<\infty$, and $z\in\partial D(0,R)$, the behavior of $\sum a_n z^n$ can have any behavior

PROPOSITION (D'ALEMBERT'S CRITERION, RATIO TEST)

If $\ell := \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \in [0,+\infty]$ exists, then $R=\frac{1}{\ell}$

Proof

Let $z\in\mathbb{C}$. We want to check when $\sum a_n z^n$ converges.

We have: $\frac{|a_{n+1}z^{n+1}|}{|a_n z^n|} = \left|\frac{a_{n+1}}{a_n}\right| |z| \longrightarrow \ell|z| \begin{cases} <1, & |z|<\frac{1}{\ell} \\ =1, & |z|=\frac{1}{\ell} \\ >1, & |z|>\frac{1}{\ell} \end{cases}$

This means, by def, $R=\frac{1}{\ell}$. □

PROPOSITION (CAUCHY'S CRITERION, ROOT TEST)

Let $\lambda := \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Then, $R=\frac{1}{\lambda}$.

Proof

Similar to above.

EXAMPLES ON THE BOUNDARY OF R

1) Consider the power series $\sum z^n$. Obviously, $R(\sum z^n)=1$.
This means for $r\in(0,1)$, $\sum z^n$ converges normally on $\bar{D}(0,r)$.
For $z\in\partial D(0,1)$, $|z^n|=1 \Rightarrow \sum z^n$ diverges

2) Consider the power series $\sum \frac{z^n}{n^2}$ with $R(\sum \frac{z^n}{n^2})=1$
Here, $\|\frac{z^n}{n^2}\|_{\infty, \bar{D}(0,1)} = \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges. Hence, $\sum \frac{z^n}{n^2}$ converges normally on $\bar{D}(0,1)$.

3) Consider $\sum \frac{z^n}{n}$, $R(\sum \frac{z^n}{n})=1$
- For $z=1$, $\sum \frac{1}{n}$ diverges
- For $z=e^{i\theta}$, $\theta\in\mathbb{R}\setminus(2\pi\mathbb{Z})$, $\sum \frac{e^{i\theta}}{n}$ converges because $\frac{1}{n}\searrow 0$ and $\left|\sum_{k=0}^{n} e^{ik\theta}\right| = \left|\frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}\right| = \left|\sin\frac{n+1}{2}\theta\right| \div \left|\sin\frac{\theta}{2}\right| \leq \frac{1}{|\sin\frac{\theta}{2}|}$, which is bounded.

OPERATIONS ON POWER SERIES

PROPOSITION

Let $f(z) := \sum a_n z^n$ and $g(z) := \sum a_n z^n$ be power series with radius of convergence R_f and R_g respectively.

Let $R := R(\sum(a_n+b_n)z^n)$, then $R \geq \min(R_f, R_g)$

Moreover, if $R_f \neq R_g$, then $R=\min(R_f,R_g)$. $\forall z\in\mathbb{C}$ with $|z|<\min(R_f,R_g)$, we also have $\sum_{n\geq 0}(a_n+b_n)z^n = \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} b_n z^n$

Proof

Let $z\in\mathbb{C}$ with $|z|<\min(R_f,R_g)$. We know $(a_n z^n)_{n\geq 0}$ and $(b_n z^n)_{n\geq 0}$ are bounded, so $((a_n+b_n)z^n)_{n\geq 0}$ is bounded $\Rightarrow |z|\leq R$

$\therefore$ By taking $|z| \longmapsto \min(R_f,R_g)$ from below, we find that $\min(R_f,R_g)\leq R$ □

Now, consider when $R_f \neq R_g$, by symmetry, suppose $R_f < R_g$

Let $z \in \mathbb{C}$, s.t. $R_f < z < R_g$. We know $(a_n z^n)_{n \geq 0}$ is bounded and $(b_n z^n)_{n \geq 0}$ is unbounded $\Rightarrow ((a_n + b_n) z^n)_{n \geq 0}$ is unbounded

This means that $|z| \geq R$. By taking $|z| \to R_f^+$, ← (limit, not "positive") we find $R_f \geq R$.

We have already shown that $R \geq R_f$ $\therefore R = R_f$ $\square$

When $|z| < \min(R_f, R_g)$, $\sum a_n z^n$ and $\sum b_n z^n$ converge $\therefore \sum (a_n + b_n) z^n$ converges to $\sum a_n z^n + \sum b_n z^n$ $\square$ (limit proof of $\sum^N$, $N \to \infty$)

DEFINITION

Let $\sum a_n z^n$ and $\sum b_n z^n$ be power series. Their Cauchy product $\sum c_n z^n$ is given by $c_n = \sum_{k=0}^{n} a_k b_{n-k}$