

3-4-25 (WEEK 3)

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THEOREM (Direct result just by using linearity)

Let α be a step function (w.r.t. the partition $P=(x_k)_{0 \leq k \leq n}$)

Let $f: [a, b] \rightarrow \mathbb{R}$ be s.t.

(i) $\forall 0 \leq k \leq n-1$, at least one of f and α is right cont. at x_k

(ii) $\forall 1 \leq k \leq n$, at least one of f and α is left cont. at x_k .

Then, $f \in R(\alpha)$ and $\int_a^b f d\alpha = \sum_{k=1}^n f(x_k) [\alpha(x_k) - \alpha(x_{k-1})]$

COROLLARY

Let $a_1, \dots, a_n \in \mathbb{R}$, define a left-cont. function $f: [0, 1] \rightarrow \mathbb{R}$

$$x \mapsto \begin{cases} 0, & x=0 \\ a_k, & x \in (t_{k-1}, t_k], 1 \leq k \leq n \end{cases}$$

Then, $\sum_{k=1}^n a_k = \sum_{k=1}^n f(t_k) = \int_0^1 f(x) dL(x)$

COROLLARY (Euler's Summation Formula) \leftarrow Just need to understand how to prove, no need to memorize

Let $f: [a, b] \rightarrow \mathbb{R}$ be C^1 . Then,

(1) $\sum_{n=a}^b f(n) = \int_a^b f(x) dx + \int_a^b f'(x) \{x\} dx + f(a)\{a\} - f(b)\{b\}$, where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of x

(2) $\sum_{n=a}^b f(n) = \int_a^b f(x) dx + \int_a^b f'(x) (\{x\} - \frac{1}{2}) dx + \frac{1}{2} [f(a) + f(b)]$

Proof

We know f and f' are conti., so they belong to $R(x; a, b)$

First, we know that $f \in R(\lfloor x \rfloor; a, b)$ so $\int_a^b f(x) d\lfloor x \rfloor$ is well-defined.

So, we also have $f \in R(\{x\}; a, b)$. By integration by parts, we find $\int_a^b f(x) d\{x\} + \int_a^b \{x\} df(x) = f(b)\{b\} - f(a)\{a\}$, sub in the previous corollary, we get (1).

To show (2), we may directly apply integration by parts to f and $\{x\}$.

The other method is carried out by taking the formula for (1).

Let $a < b \in \mathbb{Z}$, we have $\sum_{n=a}^b f(n) = f(a) + \int_a^b f(x) dx + \int_a^b f'(x) \{x\} dx$ and $\int_a^b \{x\} df(x) = \frac{1}{2} [f(b) - f(a)]$ \square

\leftarrow Since $\{a\} = \{b\} = 0 \forall a, b \in \mathbb{Z}$, so " $f(a)\{a\} - f(b)\{b\}$ " = 0

COROLLARY (Abel's Summation)

Let $(a_n)_{n \geq 1}$ be real numbers. Define $A: \mathbb{R}_n \rightarrow \mathbb{R}$

$$x \mapsto \sum_{k=1}^{\lfloor x \rfloor} a_k$$

\leftarrow If $f \in C^1$, $df(t) = f'(t) dt$

For $x \geq 1$, and a conti. function $f: (1, x) \rightarrow \mathbb{R}$, we have $\sum_{k=1}^{\lfloor x \rfloor} a_k f(k) = - \int_1^x A(t) df(t) + A(x)f(x)$

DARBOUX SUMMATIONS

UPPER AND LOWER DARBOUX SUMMATIONS AND INTEGRALS

DEFINITION

Let $P \in \mathcal{P}([a, b])$ and define for $1 \leq k \leq n$, $M_k = M_k(f) := \sup \{f(x) \mid x \in [x_{k-1}, x_k]\}$ and $m_k = m_k(f) := \inf \{f(x) \mid x \in [x_{k-1}, x_k]\}$

We define the upper and lower Darboux sums as follows: $U_P(f, \alpha) = \sum_{k=1}^n M_k(f) \Delta \alpha_k$, $L_P(f, \alpha) = \sum_{k=1}^n m_k(f) \Delta \alpha_k$

Note: No tagged points are needed for these defs. Also, when $\alpha(x) = x$, we call them upper/lower Riemann sums.

LEMMA (By def)

Suppose that α is increasing / non-decreasing on $[a, b]$. Then, for any $f: [a, b] \rightarrow \mathbb{R}$ and any partition $P \in \mathcal{P}([a, b])$ and tagged points t , we have $m_k(f) \leq f(t_k) \leq M_k(f)$, $1 \leq k \leq n$ and $L_{P, t}(f, \alpha) \leq S_{P, t}(f, \alpha) \leq U_P(f, \alpha)$

PROPOSITION

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Suppose that α is increasing.

(a) $\forall P \leq P'$, we have $U_{P'}(f, \alpha) \leq U_P(f, \alpha)$ and $L_P(f, \alpha) \leq L_{P'}(f, \alpha)$

(b) $\forall P, P'$, we have $L_P(f, \alpha) \leq U_{P'}(f, \alpha)$

Proof

(a) It is enough to prove the inequalities hold when P' contains one more subdivision point than P . Let $P = (x_k)_{k=0}^n$ and $c \in (x_{i-1}, x_i)$ when $1 \leq i \leq n$.

We have: $U_P(f, \alpha) = \sum_{k=1}^n M_k(f) \Delta \alpha_k$, and $U_{P'}(f, \alpha) = \sum_{k=1, k \neq i}^n M_k(f) \Delta \alpha_k + M'(\alpha(c) - \alpha(x_{i-1})) + M''(\alpha(x_i) - \alpha(c))$, where $M' := \sup \{f(x) \mid x_{i-1} \leq x \leq c\}$ and $M'' := \sup \{f(x) \mid c \leq x \leq x_i\}$.

Since $M' \leq M_i(f)$, $M'' \leq M_i(f)$, we deduce $M'(\alpha(c) - \alpha(x_{i-1})) + M''(\alpha(x_i) - \alpha(c)) \leq M_i(f)(\alpha(x_i) - \alpha(x_{i-1}))$, so $U_{P'}(f, \alpha) \leq U_P(f, \alpha)$ (L_P is similar).

(b) Let $P, P' \in \mathcal{P}([a, b])$, and $P'' = P' \vee P$. Then, from (a), $L_P(f, \alpha) \leq L_{P''}(f, \alpha) \leq U_{P''}(f, \alpha) \leq U_{P'}(f, \alpha) \square$

Core idea to link two unrelated partitions

DEFINITION

Suppose α is increasing. The upper/lower Stieltjes integrals of f w.r.t. α are defined by:

$\hookrightarrow \bar{I}(f, \alpha) = \bar{\int}_a^b f d\alpha := \inf \{U_P(f, \alpha) \mid P \in \mathcal{P}([a, b])\}$

$\hookrightarrow \underline{I}(f, \alpha) = \underline{\int}_a^b f d\alpha := \sup \{L_P(f, \alpha) \mid P \in \mathcal{P}([a, b])\}$

PROPOSITION

Suppose α is increasing, then $\underline{I}(f, \alpha) \leq \bar{I}(f, \alpha)$ (Proof: Trivial)

REMARK

The equality above may not hold. Say $\alpha(x) = x$, $f(x) = \mathbb{1}_Q$ defined on $[0, 1]$.

Then, $U_P(f, \alpha) = 1$ and $L_P(f, \alpha) = 0 \ \forall P \in \mathcal{P}([a, b]) \Rightarrow$ By def, $\bar{I}(f, \alpha) = 1$, $\underline{I}(f, \alpha) = 0$, so $\underline{I}(f, \alpha) \neq \bar{I}(f, \alpha)$

PROPOSITION (Linearity)

Let $a \leq c \leq b$. Let $\alpha: [a, b] \rightarrow \mathbb{R}$ be bounded and increasing

Let $f, g: [a, b] \rightarrow \mathbb{R}$ be bounded. We have:

• $\bar{\int}_a^b f d\alpha = \bar{\int}_a^c f d\alpha + \bar{\int}_c^b f d\alpha$, also $\underline{\int}_a^b f d\alpha = \underline{\int}_a^c f d\alpha + \underline{\int}_c^b f d\alpha$

• $\bar{\int}_a^b (f+g) d\alpha \leq \bar{\int}_a^b f d\alpha + \bar{\int}_a^b g d\alpha$, also $\underline{\int}_a^b (f+g) d\alpha \geq \underline{\int}_a^b f d\alpha + \underline{\int}_a^b g d\alpha$

RIEMANN'S CONDITION

DEFINITION

Let $\alpha: [a, b] \rightarrow \mathbb{R}$ be increasing. We say that f satisfies Riemann's condition w.r.t. α on $[a, b]$ if $\forall \varepsilon > 0, \exists P \in \mathcal{P}([a, b])$, s.t. $\forall P \geq P_\varepsilon$, we have $0 \leq U_P(f, \alpha) - L_P(f, \alpha) < \varepsilon$ (Note: Tagged points don't matter here)

REMARK

Thanks to the propositions above, it suffices to find $P_\varepsilon \in \mathcal{P}([a, b])$ with $U_{P_\varepsilon}(f, \alpha) - L_{P_\varepsilon}(f, \alpha) < \varepsilon$ to satisfy Riemann's condition

THEOREM The assumption for α to be increasing is actually not very restrictive. Recall the decomposition theorem for function of

For any increasing integrator α , IFAE: bounded variation $(\alpha \in BV \Leftrightarrow \alpha = V - (V - \alpha))$

(1) $f \in R(\alpha; a, b)$

(2) f satisfies Riemann's condition

(3) $\underline{I}(f, \alpha) = \bar{I}(f, \alpha)$

Proof: Next set of notes!

SOME APPLICATIONS

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PROPOSITION

Given non-decreasing α , and $f, g \in \mathcal{P}([a, b])$, suppose $f(x) \leq g(x) \forall x \in [a, b]$, then we have $\int_a^b f(x) d\alpha(x) \leq \int_a^b g(x) d\alpha(x)$

Proof

For any partition $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a, b])$, we have $M_k(f) \leq M_k(g)$ and $m_k(f) \leq m_k(g)$, so we have $L_P(f, \alpha) \leq L_P(g, \alpha)$ and $U_P(f, \alpha) \leq U_P(g, \alpha)$. Therefore, the upper and lower Stieltjes integrals satisfy: $\bar{I}(f, \alpha) \leq \bar{I}(g, \alpha)$ and $\underline{I}(f, \alpha) \leq \underline{I}(g, \alpha)$. By the Thm above, thus $\int_a^b f d\alpha \leq \int_a^b g d\alpha$ \square

PROPOSITION

Given non-decreasing α , $f \in \mathcal{R}(\alpha; a, b)$, then $|f| \in \mathcal{R}(\alpha; a, b)$ and $|\int_a^b f(x) d\alpha(x)| \leq \int_a^b |f(x)| d\alpha(x)$

Proof

For any partition $P \in \mathcal{P}([a, b])$ and $1 \leq k \leq n$, we have $M_k(f) - m_k(f) = \sup\{f(x) - f(y) \mid x, y \in [x_{k-1}, x_k]\}$

We may check that $M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f)$, hence $U_P(|f|, \alpha) - L_P(|f|, \alpha) \leq U_P(f, \alpha) - L_P(f, \alpha)$, i.e. $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$ \square