THEOREM (Proof for last section of notes)

For any increasing integrator α, TFAE:

(1) $f \in R(\alpha; a, b)$

(2) f satisfies Riemann's condition

(3) $\underline{I}(f,\alpha) = \overline{I}(f,\alpha)$

Proof

"(1)$\Rightarrow$(2)": Suppose that f satisfies (RS). The proof is trivial if $\alpha(a)=\alpha(b)$, so we may assume $\alpha(a)<\alpha(b)$.

Let $\varepsilon>0$, then take $P_\varepsilon \in \mathcal{P}([a,b])$ s.t. $\forall$ tagged partition (P,t), (P,t'), with $P \supseteq P_\varepsilon$, we have $|S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$, $|S_{P,t'}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$

By triangle inequality, $|\sum_{k=1}^n [f(t_k) - f(t_k')]\Delta\alpha_k| < 2\varepsilon$

For each $1 \le k \le n$, we take $t_k, t_k' \in [x_{k-1}, x_k]$, s.t. $M_k(f) \le f(t_k) + \frac{\varepsilon}{\alpha(b)-\alpha(a)}$ and $m_k(f) \ge f(t_k) - \frac{\varepsilon}{\alpha(b)-\alpha(a)}$

This means, $M_k(f) - m_k(f) \le f(t_k) - f(t_k') + \frac{2\varepsilon}{\alpha(b)-\alpha(a)}$.

Therefore, $U_P(f,\alpha) - L_P(f,\alpha) = \sum_{k=1}^n [M_k(f) - m_k(f)]\Delta\alpha_k \le \sum_{k=1}^n [f(t_k) - f(t_k')]\Delta\alpha_k + \frac{2\varepsilon}{\alpha(b)-\alpha(a)} \overbrace{\sum_{k=1}^n \Delta\alpha_k}^{\alpha(b)-\alpha(a)} \le 4\varepsilon$, so (2) holds.

"(2)$\Rightarrow$(3)": Suppose that (2) holds. Let $\varepsilon>0$ and take $P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $0 \le U_P(f,\alpha) - L_P(f,\alpha) \le \varepsilon$ for all $P \supseteq P_\varepsilon$.

For all $P \supseteq P_\varepsilon$, $\overline{I}(f,\alpha) \le \underline{U_P(f,\alpha) \le L_P(f,\alpha) + \varepsilon} \le \underline{I}(f,\alpha) + \varepsilon$

As this is true $\forall \varepsilon$, thus $\overline{I}(f,\alpha) \le \underline{I}(f,\alpha)$, which implies $\overline{I}(f,\alpha) = \underline{I}(f,\alpha)$ ($\because$ "$\ge$" is trivially true)

"(3)$\Rightarrow$(1)": Suppose that (3) holds. Let $L = \underline{I}(f,\alpha) = \overline{I}(f,\alpha)$.

Let $\varepsilon>0$, take $P_\varepsilon^{(1)} \in \mathcal{P}([a,b])$ s.t. $L_P(f,\alpha) + \varepsilon \ge \underline{I}(f,\alpha) = L$ $\forall P \supseteq P_\varepsilon^{(1)}$

Take $P_\varepsilon^{(2)} \in \mathcal{P}([a,b])$, s.t. $U_P(f,\alpha) - \varepsilon \le \overline{I}(f,\alpha) = L$ $\forall P \supseteq P_\varepsilon^{(2)}$

Take $P_\varepsilon := P_\varepsilon^{(1)} \vee P_\varepsilon^{(2)}$, then $\forall P \supseteq P_\varepsilon$, tagged points t, we have:

$\hookrightarrow$ $S_{P,t}(f,\alpha) \le U_P(f,\alpha) \le L + \varepsilon$

$\hookrightarrow$ $S_{P,t}(f,\alpha) \ge L_P(f,\alpha) \ge L - \varepsilon$

$\therefore$ In other words, $|S_{P,t}(f,\alpha) - L| \le \varepsilon$. Namely, $f \in R(\alpha; a, b)$ and $\int_a^b f\,d\alpha = L$ $\square$

PROPOSITION (APPLICATION)

Suppose that α is nondecreasing on $[a,b]$. If $f \in R(\alpha; a, b)$, then $f^2 \in R(\alpha; a, b)$

Proof

Let $f \in R(\alpha; a, b)$ and $P \in \mathcal{P}([a,b])$ be a partition.

key step

For $1 \le k \le n$, we have $M_k(f^2) - m_k(f^2) = \underline{M_k(|f|)^2 - m_k(|f|)^2} = (M_k(|f|) + m_k(|f|))(M_k(|f|) - m_k(|f|)) \le 2M[M_k(|f|) - m_k(|f|)]$, where we have $M = \sup\{|f(x)| \mid x \in [a,b]\} < \infty$

$\because$ $|f|$ satisfies Riemann's condition

$\therefore$ f^2 satisfies Riemann's condition $\square$

COROLLARY

Suppose that α is nondecreasing on $[a,b]$. If $f, g \in R(\alpha; a, b)$, then $fg \in R(\alpha; a, b)$

Proof

Use $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$, since $+/-$ operations preserve integrability, thus $fg \in R(\alpha; a, b)$ too. $\square$

REMARK

We can use induction to further show that for nondecreasing α and $f \in R(\alpha; a, b)$, $\forall n \in \mathbb{N}_0$, then $f^n \in R(\alpha; a, b)$.

IMPORTANT

The converse of our proposition does not hold. Consider $f: [0,1] \to \mathbb{R}$, $x \mapsto \begin{cases} -1, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$. We have $f^2 \in R(\alpha; a, b)$ but $f \notin R(\alpha; a, b)$.

INTEGRATORS OF BOUNDED VARIATION

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GOAL

We want to use the decomposition theorem for functions of bounded variation to extend the previous results to general integrators, i.e. of bounded variation. We can mainly do so by recalling: $\alpha \in BV \Rightarrow \alpha = V-(V-\alpha)$, for a variation function V.

THEOREM

Let $\alpha \in BV([a,b])$ and V be its variation function. Then, $f \in R(\alpha; a,b) \Rightarrow f \in R(V; a,b)$

Proof

When α is a constant function, then $V \equiv 0$, and the theorem holds trivially. Now, let us assume $\alpha(a) < \alpha(b)$, so $V(b) > 0$

Let $\varepsilon > 0$. Take $P_\varepsilon^{(1)} \in \mathcal{P}([a,b])$ s.t. $\forall P \supseteq P_\varepsilon$ and tagged points t, then $\left|\sum_{k=1}^n [f(t_k) - f(t_k')]\Delta\alpha_k\right| \le \varepsilon$.

Let $M = \sup|f|$. Take $P_\varepsilon^{(2)} \in \mathcal{P}([a,b])$, s.t. $\forall P \supseteq P_\varepsilon^{(2)}$, $V(b) = V_\alpha([a,b]) \le V_P(\alpha) + \frac{\varepsilon}{M} = \sum_{k=1}^n |\Delta\alpha_k| + \frac{\varepsilon}{M}$ (Prop from before: $\forall \varepsilon>0, \exists P_\varepsilon$, s.t. $P \supseteq P_\varepsilon \Rightarrow V_P(f) \le V_f \le V_P(f)+\varepsilon$)

We now factcheck that $f \in R(\alpha; a,b)$. Since V is non-decreasing, we only need to check that it satisfies Riemann's condition, i.e. we need to bound

$$\sum_{k=1}^n [M_k(f) - m_k(f)]\Delta V_k = \underbrace{\sum_{k=1}^n [M_k(f)-m_k(f)]|\Delta\alpha_k|}_{(**)} + \underbrace{\sum_{k=1}^n [M_k(f)-m_k(f)][\Delta V_k - |\Delta\alpha_k|]}_{(*)}$$

Consider $(*)$.

Let $M = \sup|f|$, then $(*) \le 2M\sum_{k=1}^n (\Delta V_k - |\Delta\alpha_k|) = 2M\left(V(b) - \sum_{k=1}^n |\Delta\alpha_k|\right) \le 2\varepsilon$.

To bound $(**)$, we distinguish the indices k w.r.t. the sign of $\Delta\alpha_k$.

Let $K^+ := \{1 \le k \le n \mid \Delta\alpha_k \ge 0\}$ and $K^- := \{1 \le k \le n \mid \Delta\alpha_k < 0\}$, $\varepsilon' = \frac{\varepsilon}{V(b)}$ (really important tactic to deal with absolute values)

For $k \in K^+$, choose $t_k, t_k' \in [x_{k-1}, x_k]$, s.t. $f(t_k) - f(t_k') \ge M_k(f) - m_k(f) - \varepsilon'$

For $k \in K^-$, choose $t_k, t_k' \in [x_{k-1}, x_k]$, s.t. $f(t_k') - f(t_k) \ge M_k(f) - m_k(f) - \varepsilon'$

sub from above

Therefore, we find $(**) = \sum_{k=1}^n [M_k(f) - m_k(f)]|\Delta\alpha_k| = \sum_{k\in K^+} [M_k(f) - m_k(f)]\Delta\alpha_k + \sum_{k \in K^-} [M_k(f) - m_k(f)](-\Delta\alpha_k)$

$\le \sum_{k=1}^n [f(t_k) - f(t_k')]\Delta\alpha_k + \varepsilon'\sum_{k=1}^n |\Delta\alpha_k| \le \varepsilon + \varepsilon' V(b) = 2\varepsilon$ □

COROLLARY (Formal restating of our goal)

Let $\alpha \in BV([a,b])$, bounded $f: [a,b] \to \mathbb{R}$, then $(1) \Leftrightarrow (2)$.

(1) $f \in R(\alpha; a,b)$

(2) $\exists$ nondecreasing α_1, α_2, s.t. $f \in R(\alpha_1) \cap R(\alpha_2)$ and $\alpha = \alpha_1 - \alpha_2$

Proof

"(2)⇒(1)": By linearity ✓

"(1)⇒(2)": Write $\alpha = V - (V-\alpha)$ and then use the theorem above. □

PROPOSITION

Let $\alpha \in BV([a,b])$ and $f \in R(\alpha; a,b)$. Then, for any $[c,d] \subseteq [a,b]$, we have $f \in R(\alpha; c,d)$

Proof

If only BV, we can use $\alpha = V-(V-\alpha) \Rightarrow f \in R(V;a,b) \cap R(V-\alpha; a,b) \Rightarrow f \in R(\alpha; a,b)$, as we have the prop above

Suppose α is non-decreasing on $[a,b]$. If we can check that $f \in R(\alpha; a,x)$ for all $x \in (a,b]$, then we get $f \in R(\alpha; a,c) \cap R(\alpha; a,d) = R(\alpha; c,d)$.

For $x \in (a,b]$ and a partition $P \in \mathcal{P}([a,b])$, define $\Delta_P(x) := U_P(f|_{[a,x]}, \alpha|_{[a,x]}) - L_P(f|_{[a,x]}, \alpha|_{[a,x]})$

Fix $x \in (a,b]$, let $\varepsilon > 0$, then take $P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $\Delta_P(b) \le \varepsilon$ $\forall P \supseteq P_\varepsilon$.

WLOG, we may assume $x \in \mathrm{Supp}(P_\varepsilon)$ (∵ "$\forall P \supseteq P_\varepsilon$" means any points chosen still make the inequality true).

Now, let $P_\varepsilon' := P_\varepsilon \cap [a,x]$. Now, $\forall P' \supseteq P_\varepsilon'$, define $P := P' \vee P_\varepsilon \supseteq P_\varepsilon$. Then, we have $\Delta_{P'}(x) \le \Delta_P(b) \le \varepsilon$. □

THEOREM

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Let $\alpha \in BV([a,b])$, $f, g \in R(\alpha; a, b)$.
Define for all $x \in [a,b]$, $F(x) = \int_a^x f(t)\,d\alpha(t)$, $G(x) = \int_a^x g(x)\,d\alpha(x)$. Then, $f \in R(G; a, b)$, $g \in R(F; a, b)$ and $fg \in R(\alpha; a, b)$, where:

$$\int_a^b f(x)g(x)\,d\alpha(x) = \int_a^b f(x)\,dG(x) = \int_a^b g(x)\,dF(x)$$

Proof

Similar to the above, it is sufficient to prove this is the case for non-increasing α.
Suppose α is non-decreasin, we already know that $fg \in R(\alpha)$. By symmetry, we only need to check $f \in R(G)$ and $\int_a^b fg\,dx = \int_a^b f\,dG$.

Let $P \in \mathcal{P}([a,b])$ and fix tagged points t, then $S_{P,t}(f,G) = \sum_{k=1}^n f(t_k)\Delta G_k = \sum_{k=1}^n f(t_k)\int_{x_{k-1}}^{x_k} g(t)\,d\alpha(t)$ and $\int_a^b fg\,d\alpha = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(t)g(t)\,d\alpha(t)$

Now, $\left|S_{P,t}(f,G) - \int_a^b fg\,d\alpha\right| = \left|\sum_{k=1}^n \int_{x_{k-1}}^{x_k} [f(t_k) - f(x)]\,g(x)\,d\alpha(x)\right|$

$\leq M\sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f(t_k) - f(x)|\,d\alpha(x)$, where $|g(x)| \leq M$ $\forall x \in [a,b]$, since it is bounded.

$\leq M\sum_{k=1}^n \int_{x_{k-1}}^{x_k} [M_k(f) - m_k(f)]\,d\alpha(x)$

$= M[U_P(f,\alpha) - L_P(f,\alpha)]$

$\therefore$ By def, f satisfies Riemann's condition w.r.t. $\alpha \Rightarrow \left|S_{P,t}(f,G) - \int_a^b fg\,d\alpha\right| \leq M\varepsilon'$ $\square$