4-1-25 (WEEK 7)

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PARTIAL SUMS AND REMAINDERS

(Sorry for the bad handwriting... once again, rain got into my stylus and I forgot to zip my bag in heavy rain, so...)

THEOREM

Let $\sum_n u_n$ and $\sum_n v_n$ be two series with nonnegative terms

Suppose that $u_n \sim v_n$. Then,

1) If $\sum u_n$ converges, then $\sum v_n$ converges, and $\sum_{k=n+1}^{\infty} u_k \sim \sum_{k=n+1}^{\infty} v_k$ (remainders, start from n+1 to ∞)
2) If $\sum u_n$ diverges, then $\sum v_n$ diverges, and $\sum_{k=n+1} u_k \sim \sum_{k=n+1} v_k$ (partial sums, start from 1 to n)

Proof

We have seen that $\sum u_n$ and $\sum v_n$ have the same behavior

1) Let $\varepsilon>0$, take $N>0$, s.t. $(1-\varepsilon)u_n \leq v_n \leq (1+\varepsilon)u_n \ \forall n \geq N$

   $\forall M \geq n \geq N$, we have $\sum_{k=n+1}^{M} v_k \leq \sum_{k=n+1}^{M} (1+\varepsilon)u_k \leq (1+\varepsilon)\sum_{k=n+1}^{\infty} u_k \Rightarrow \sum_{k=n+1}^{\infty} v_k \leq (1+\varepsilon)\sum_{k=n+1}^{\infty} u_k$

   Similarly, we also have $\sum_{k=n+1}^{\infty} v_k \geq (1-\varepsilon)\sum_{k=n+1}^{\infty} u_k$, which gets us $\sum_{k=n+1}^{\infty} v_k \sim \sum_{k=n+1}^{\infty} u_k$

2) Let $\varepsilon>0$, take $N>0$, s.t. $(1-\varepsilon)u_n \leq v_n \leq (1+\varepsilon)u_n \ \forall n \geq N$.

   Write $\sum_{k=1}^{n} v_k = \sum_{k=1}^{N-1} v_k + \sum_{k=N}^{n} v_k$ for $n \geq N$.

   We want to show "$(1-2\varepsilon)\sum_{k=1}^{n} u_k \overset{(b)}{\leq} \sum_{k=1}^{n} v_k \overset{(a)}{\leq} (1+2\varepsilon)\sum_{k=1}^{n} u_k$" for large enough n.

   "(a)": $\sum_{k=1}^{n} v_k \leq \sum_{k=1}^{N-1} v_k + (1+\varepsilon)\sum_{k=N}^{n} v_k$

   $\because \dfrac{\overbrace{\sum_{k=1}^{N} v_k}^{\text{(const)}}}{\underbrace{\sum_{k=1}^{n} u_k}_{\text{(div series)}}} \xrightarrow[n\to\infty]{} 0$, thus $\exists N' \geq N$, s.t. $\sum_{k=1}^{N} v_k \leq \varepsilon \sum_{k=1}^{n} u_k$

   Therefore, $\sum_{k=1}^{n} v_k \leq \varepsilon \sum_{k=1}^{n} u_k + (1+\varepsilon)\sum_{k=1}^{n} u_k = (1+2\varepsilon)\sum_{k=1}^{n} u_k$ for $n \geq N'$

   "(b)": For $n \geq N$, we have $\sum_{k=1}^{n} v_k \geq \sum_{k=1}^{N-1} v_k + (1-\varepsilon)\sum_{k=N}^{n} u_k$

   Now, take large $N' \geq N$, s.t. $\varepsilon \sum_{k=N}^{n} u_k \geq (1-2\varepsilon)\sum_{k=1}^{N-1} u_k - \sum_{k=1}^{N-1} v_k \ \forall n \geq N$ (Main idea: It diverges, so abuse the fact it's very large)

   Then, we find $\sum_{k=1}^{n} v_k \geq (1-2\varepsilon)\sum_{k=1}^{n} u_k$ □

EXAMPLE (Algorithmic approach of analyzing remainders)

Let's study the asymptotic behavior of the harmonic series

Define $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \ \forall n \geq 1$

1) Note that $\ln(1+x) \sim \infty$ when $x \to \infty$. We have $\frac{1}{n} \sim \ln(1+\frac{1}{n}) = \ln(n+1) - \ln(n)$

   The series $\sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)]$ diverges, so $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

   Moreover, we know that $H_n \sim \sum_{k=1}^{n} (\ln(k+1) - \ln(k)) = \ln(n+1) \sim \ln(n)$

   $\therefore \frac{H_n}{\ln n} \to 1$ as $n \to \infty$, i.e. $H_n = \ln n + o(\ln n)$

2) Let's understand the term $o(\ln n)$.

   Let $A_n := H_n - \ln n$ for $n \geq 1$

   $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots$

   For $n \geq 2$, $A_n - A_{n-1} = H_n - H_{n-1} - \ln n + \ln(n-1) = \frac{1}{n} + \ln(1-\frac{1}{n}) = \frac{1}{n} + (-\frac{1}{n} - \frac{1}{2n^2} + o(\frac{1}{n^2})) = -\frac{1}{2n^2} + o(\frac{1}{n^2})$

   $\Rightarrow A_n - A_{n-1} \sim -\frac{1}{2n^2}$

   By the thm above, hence $\sum (A_n - A_{n-1})$ converges, i.e. $(A_n)_{n\geq 1}$ converges □

   Now, let $\gamma := \lim_{n\to\infty} A_n$, and call it Euler's constant

   Then, we get $A_n = H_n - \ln n = \gamma + o(1) \Rightarrow H_n = \ln n + \gamma + o(1)$

3) Compare the partial sums for $\sum_{k=n+1}^{\infty} (A_k - A_{k+1}) = \gamma - A_n$ and $\sum_{k=n+1}^{\infty} \frac{1}{k^2} \sim \frac{1}{n}$ by Riemann series.

   We get: $\gamma - A_n = \gamma - H_n + \ln n \sim -\frac{1}{2n} \Rightarrow \gamma - H_n + \ln n = -\frac{1}{2n} + o(\frac{1}{n}) \Rightarrow H_n = \ln n + \gamma + \frac{1}{2n} + o(\frac{1}{n})$

4) Let $D_n = H_n - \ln n - \gamma - \frac{1}{2n}$ for $n \geq 1$.

   $D_n - D_{n-1} = H_n - H_{n-1} - \ln n + \ln(n-1) - \frac{1}{2n} + \frac{1}{2(n-1)}$

   $= \frac{1}{n} + \ln(1-\frac{1}{n}) + \frac{1}{2n} - \frac{1}{2n}\left(\frac{1}{1-\frac{1}{n}}\right)$ $\frac{1}{1-x} = 1 + x + x^2 + \dots$

   $= \frac{1}{n} - (\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + o(\frac{1}{n^3})) - \frac{1}{2n} + \frac{1}{2n}(1 + \frac{1}{n} + \frac{1}{n^2} + o(\frac{1}{n^2})) = \frac{1}{6n^3} + o(\frac{1}{n^3})$

   Again, by Riemann series, $\sum \frac{1}{n^3}$ converges $\Rightarrow \sum (D_n - D_{n-1})$ converges, moreover, $\sum_{k=n+1}^{\infty} (D_k - D_{k-1}) \sim \sum_{k=n+1}^{\infty} \frac{1}{k^3} \sim \frac{1}{2n^2} \Rightarrow D_n \sim \frac{1}{2n^2} \Rightarrow H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + o(\frac{1}{n^2})$

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5) We have the following expression: $H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k\geq 1}^{N} \frac{B_{2k}}{2k n^{2k}} + o\left(\frac{1}{n^{2N}}\right)$ when $n \to \infty$, where $(B_{2k})_{k\geq 1}$ are Bernoulli numbers

REMARK

Using this similar approach, we can derive the infamous Stirling's formula ~~(i.e. every CS student's nightmare when learning about asymptotic notation :clown:)~~

COMPARISON BETWEEN SERIES AND INTEGRALS

PROPOSITION

Let $f: [1, +\infty) \to \mathbb{R}_+$ be a nondecreasing function with $\lim_{x\to\infty} f(x) = 0$. $\forall n \geq 1$, define $S_n = \sum_{k=1}^{n} f(k)$, $I_n = \int_1^n f(t)\,dt$, $D_n = S_n - I_n$

Then, the following properties hold:

(1) For $n \geq 1$, we have $0 \leq f(n+1) \leq D_{n+1} \leq D_n \leq f(1)$

(2) The sequence $(D_n)_{n\geq 1}$ converges, and denote $D := \lim_{n\to\infty} D_n$

(3) The series $\sum f(n)$ and the integral $\int_1^\infty f(t)\,dt := \lim_{x\to\infty} \int_1^x f(t)\,dt$ have the same behavior (both converge or both diverge)

(4) $\forall n \geq 1$, we have $0 \leq D_n - D \leq f(n)$

Proof

1) For $k \geq 1$, we have $f(k+1) \leq \int_k^{k+1} f(t)\,dt \leq f(k)$

Then, $I_{n+1} = \sum_{k=1}^{n} \int_k^{k+1} f(t)\,dt \leq \sum_{k=1}^{n} f(k) = S_n \Rightarrow f(n+1) = S_{n+1} - S_n \leq S_{n+1} - I_{n+1} = D_{n+1}$

$D_{n+1} - D_n = (S_{n+1} - S_n) - (I_{n+1} - I_n) = f(n+1) - \int_n^{n+1} f(t)\,dt \leq 0$ $\therefore (D_n)_{n\geq 1}$ is decreasing

$\therefore$ $D_{n+1} \leq D_n \leq \ldots \leq D_1 = S_1 - I_1 = f(1)$ ✓

2) We know $(D_n)_{n\geq 1}$ is decreasing and bounded from below by 0, hence it converges

3) We know $D = \lim_{n\to\infty} D_n = \lim_{n\to\infty} (S_n - I_n)$ converges

If $\lim_{n\to\infty} S_n$ exists, then $\lim_{n\to\infty} I_n = D - \lim_{n\to\infty} S_n$ exists. Same argument for "$\lim_{n\to\infty} I_n$ exists".

4) We know $D_n - D = \sum_{k\geq n} (D_k - D_{k+1}) \geq 0$. We have $D_k - D_{k+1} = \int_k^{k+1} f(t)\,dt - f(k+1) \leq f(k) - f(k+1)$ $\therefore \sum_{k\geq n} (D_k - D_{k+1}) \leq \sum_{k\geq n} (f(k) - f(k+1)) \Rightarrow D_n - D \leq f(n)$ □

REMARK

From (4), we find $0 \leq \sum_{k=1}^{n} f(k) - \int_1^n f(t)\,dt - D \leq f(n) \Rightarrow \sum_{k=1}^{n} f(k) = \int_1^n f(t)\,dt + D + O(f(n))$

If we take $f(x) = \frac{1}{x}$, we find $H_n = \int_1^n \frac{dx}{x} + D + O\left(\frac{1}{n}\right) = \ln n + D + O\left(\frac{1}{n}\right)$ $\left(O\left(\frac{1}{n}\right) < o(1)\right)$

EXAMPLE (Riemann-Zeta function)

Let $s \in \mathbb{R}$, $f(x) = x^{-s}$ $\forall x \geq 1$

Consider the series $\sum_{n\geq s} n^{-s} = \begin{cases} \text{convergent}, & s > 1 \\ \text{divergent}, & s \leq 1 \end{cases}$

We define the Riemann-Zeta function, i.e. $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$

Similarly, we can deduce $\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{s-1}\left(\frac{1}{n^{s-1}} - 1\right) + D(s) + O\left(\frac{1}{n^s}\right) = C(s) + \frac{1}{s-1}\frac{1}{n^{s-1}} + O\left(\frac{1}{n^s}\right)$

Since $\sum_{k=1}^{n} \frac{1}{k^s} \xrightarrow[n\to\infty]{} \zeta(s) = C(s)$, hence $\sum_{k=1}^{\infty} \frac{1}{k^s} = \zeta(s) = \sum_{k=1}^{n} \frac{1}{n^s} = \frac{1}{s-1}\frac{1}{n^{s-1}} + O\left(\frac{1}{n^s}\right)$

PROPOSITION (BERTAND'S SERIES)

For $\alpha, \beta \in \mathbb{R}$, consider the series $\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$

1) When $\alpha > 1$, the series converges
2) When $\alpha = 1, \beta > 1$, the series converges
3) Otherwise, it diverges

Proof

1) Let $\alpha > 1$ and $\beta \in \mathbb{R}$.

Notice, $\frac{1}{n^\alpha (\ln n)^\beta} = o\left(\frac{1}{n^{\frac{1+\alpha}{2}}}\right)$ $\left[\frac{1}{n^\alpha (\ln n)^\beta} \div \frac{1}{n^{\frac{1+\alpha}{2}}} = \frac{1}{n^{\frac{\alpha-1}{2}} (\ln n)^\beta} \to 0\right]$

Since $\sum \frac{1}{n^{\frac{1+\alpha}{2}}}$ converges $\therefore \sum \frac{1}{n^\alpha (\ln n)^\beta}$ also converges

2) Let $\alpha = 1, \beta > 1$, $f(x) = \frac{1}{x(\ln x)^\beta}$, $x \geq 2$.

$$\int_2^n f(x)\,dx = \int_2^n \frac{1}{x(\ln x)^\beta}\,dx = \int_{\ln 2}^{\ln n} \frac{1}{y^\beta}\,d\beta \text{ is convergent}$$

$\therefore \sum f(x)$ converges

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3) Finally, when $\alpha=\beta=1$, $\int_2^n \frac{dx}{x\ln x} = \int_{\ln 2}^{\ln n} \frac{1}{y}\,dy$ is divergent
Now, for $\alpha=1$, $\beta<1$, we have $\frac{1}{x\ln x} \leq \frac{1}{x(\ln x)^\beta} \Rightarrow \sum \frac{1}{x(\ln x)^\beta}$ diverges

When $\alpha<1$, thus $\frac{1}{n^{(1+\alpha)/2}} = o\left(\frac{1}{n^\alpha(\ln n)^\beta}\right)$, since $\sum \frac{1}{n^{(1+\alpha)/2}}$ diverges, thus $\sum \frac{1}{n^\alpha(\ln n)^\beta}$ diverges too. $\square$