2-25-25 (WEEK 2) Shun/羊羽海(@shun4midx)

RIEMANN-STIELTJES INTEGRALS

Let $[a,b] \subseteq \mathbb{R}$ be a segment

Let $f, g, \alpha, \beta : [a,b] \longrightarrow \mathbb{R}$ be bounded functions

DEFINITION

Let $P = (x_k)_{0 \le k \le n} \in \mathcal{P}([a,b])$. For every $1 \le k \le n$, take $t_k \in [x_{k-1}, x_k]$ and write $t = (t_k)_{0 \le k \le n}$. We call (P, t) a tagged partition, t contains tagged points of P.

Define the Riemann-Stieltjes sum of f w.r.t. α for (P,t), $S_{P,t}(f,\alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k = \sum_{k=1}^{n} f(t_k)(\alpha(x_k) - \alpha(x_{k-1}))$

Consider the following condition:

(RS): $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists P_\varepsilon \in \mathcal{P}([a,b])$ s.t. $\forall P \supseteq P_\varepsilon$, tagged points t of P, we have $|S_{P,t}(f,\alpha) - L| < \varepsilon$

If (RS) is satisfied, we say that f is Riemann-Stieltjes integrable and write this unique L to be its integral, denoted $\int_a^b f\,d\alpha = \int_a^b f(x)\,d\alpha(x)$

We write $\mathcal{R}(\alpha; a, b) = \mathcal{R}(\alpha)$ for the set of functions f satisfying (RS).

REMARK

(1) f is called integrand, α is called integrator

(2) When $\alpha(x) = x$, we recover the notation from Riemann-integrability. We write $\mathcal{R}(x; a, b) = \mathcal{R}(x)$ for the set of Riemann-integrable functions

(3) We may also have $x_{k-1} \le x_k$ instead of $x_{k-1} < x_k$ for $1 \le k \le n$. This allows us to use the same notation when $a = b$.

(4) Let V be a finite-dimensional vector space over $\mathbb{R}$, i.e. $V = \mathbb{C}$ or $\mathbb{R}^d$. Fix a basis $(e_1, \dots, e_n)$ of V, we may write $f = \sum_{i=1}^{n} f_i e_i$, where $f_i : [a,b] \to \mathbb{R}$ is a real valued function. If $\int_a^b f_i\,d\alpha$ is well-defined $\forall i$, we may set $\int_a^b f\,d\alpha = \sum_{i=1}^{n} \left(\int_a^b f_i\,d\alpha\right) e_i$

EXAMPLES

(1) If $\alpha : [a,b] \to \mathbb{R}$ is a constant function, for any bounded function $f : [a,b] \to \mathbb{R}$, $S_{P,t}(f,\alpha) = 0$ for all tagged partitions $P \in \mathcal{P}([a,b])$. This means that (RS) holds and $\int_a^b f\,d\alpha = 0$

(2) When $\alpha(x) = x$, all continuous functions are Riemann-integrable, i.e. $\mathcal{C}([a,b]) \subseteq \mathcal{R}(x; a, b)$

(3) Let $f, \alpha : [-1,1] \longrightarrow \mathbb{R}$ to be $f = \alpha = \mathbb{1}_{x \ge 0}$. Consider a partition $P \in \mathcal{P}([-1,1])$ with $x_k = 0$ for some k. For any tagged points t of P, we have $S_{P,t}(f,\alpha) = f(t_k)\Delta\alpha_k = f(t_k) = \begin{cases} 1, & t_k = x_k = 0 \\ 0, & t_k < x_k \end{cases}$. This implies that (RS) does not hold (KEY CONSTRUCTION EXAMPLE)

LEMMA

Consider the following condition,

(RS'): $\exists L \in \mathbb{R}$, s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall P \in \mathcal{P}([a,b])$ with $\max_{1 \le k \le n} |x_k - x_{k-1}| = \|P\| < \delta$, any tagged points t, we have $|S_{P,t}(f,\alpha) - L| < \varepsilon$

We have $(RS') \Rightarrow (RS)$

REMARK

It is true for $\alpha = x$ though

In general, $(RS) \not\Rightarrow (RS')$. Consider $f, \alpha : [-1,1] \to \mathbb{R}$, $f, \alpha = \mathbb{1}_{x \ge 0}$

- (RS) holds
- Let $\delta \in (0,1)$ and $P \in \mathcal{P}([0,1])$, s.t. $\|P\| < \delta$, there exists k s.t. $x_{k-1} = x_k = \frac{\delta}{2}$.

  Then, $S_{P,t}(f,\alpha) = f(t_k)\Delta\alpha_k = f(t_k) = \begin{cases} 1, & t_k \in [0, x_k] \\ 0, & t_k \in [x_{k-1}, 0) \end{cases}$ $\therefore$ (RS') does not hold.

PROPOSITION (linearity in integrand)

Let $\alpha : [a,b] \longrightarrow \mathbb{R}$ be bounded, $f, g \in \mathcal{R}(\alpha)$. Then, $\forall c \in \mathbb{R}$, $f + cg \in \mathcal{R}(\alpha)$ and $\int_a^b (f + cg)\,d\alpha = \int_a^b f\,d\alpha + c\int_a^b g\,d\alpha$

As a consequence, $\mathcal{R}(\alpha)$ is an $\mathbb{R}$-vector space, the integral operator $I : \mathcal{R}(\alpha) \longrightarrow \mathbb{R}$, $f \longmapsto \int_a^b f\,d\alpha$ is a linear form, i.e. $I \in \mathcal{L}(\mathcal{R}(\alpha), \mathbb{R})$

Proof

Fix $c \in \mathbb{R}$ and let $h = f + cg$. Since $f \in \mathcal{R}(\alpha)$, we may find $P'_\varepsilon \in \mathcal{P}([a,b])$, s.t. $|S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$ $\forall P \supseteq P'_\varepsilon$ and tagged points of P.

Shun/羊羽海(@shun4midx)

Similarly, take $P_\varepsilon'' \in \mathcal{P}([a,b])$, s.t. $|S_{P_\varepsilon'',t}(g,\alpha) - \int_a^b g\,d\alpha| < \varepsilon \ \forall P \supseteq P_\varepsilon''$.
Take $P_\varepsilon = P_\varepsilon' \vee P_\varepsilon''$, then for $P \supseteq P_\varepsilon$ and any tagged points t of P, we have $|S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$ and $|S_{P,t}(g,\alpha) - \int_a^b g\,d\alpha| < \varepsilon$

Moreover, $S_{P,t}(h,\alpha) = \sum_{k=1}^n h(t_k)\Delta\alpha_k = \sum_{k=1}^n [f(t_k) + cg(t_k)]\Delta\alpha_k = S_{P,t}(f,\alpha) + cS_{P,t}(g,\alpha)$
$\therefore |S_{P,t}(h,\alpha) - \int_a^b f\,d\alpha - c\int_a^b g\,d\alpha| \le |S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha| + |c||S_{P,t}(g,\alpha) - \int_a^b g\,d\alpha| \le (1+|c|)\varepsilon$

This means that $h = f + cg$ satisfies (RS), and we have $\int_a^b h\,d\alpha = \int_a^b f\,d\alpha + c\int_a^b g\,d\alpha$ □

PROPOSITION (linearity in integrator)
Let $\alpha, \beta : [a,b] \to \mathbb{R}$ be bounded, $f \in \mathcal{R}(\alpha) \cap \mathcal{R}(\beta)$. Then, $\forall c \in \mathbb{R}$, we have $f \in \mathcal{R}(\alpha + c\beta)$ and $\int_a^b f\,d(\alpha + c\beta) = \int_a^b f\,d\alpha + c\int_a^b f\,d\beta$
(Proof is very similar to above)

DEFINITION
For $a < b$, any bounded function $\alpha : [a,b] \to \mathbb{R}$, $f \in \mathcal{R}(\alpha; a, b)$, we define $\int_b^a f\,d\alpha := -\int_a^b f\,d\alpha := -\int_a^b f(x)\,d\alpha(x)$. We also write $\mathcal{R}(\alpha; a, b) = \mathcal{R}(\alpha; b, a)$
When $a = b$, $\int_a^b f\,d\alpha = 0$ for any bounded function f defined on $a = b$, so $\mathcal{R}(\alpha; a, a) \cong \mathbb{R}$

PROPOSITION
Let $I \subseteq \mathbb{R}$ be a segment, $a, b, c \in I$. Let $\alpha : [a,b] \to \mathbb{R}$ be bounded, $f \in \mathcal{R}(\alpha; a, b) \cap \mathcal{R}(; b, c)$. Then, we have $f \in \mathcal{R}(\alpha; a, c)$ and $\int_a^c f\,d\alpha = \int_a^b f\,d\alpha + \int_b^c d\alpha$ (if $a = b$ or $b = c$, then it's trivially true)

<u>Proof</u>

By symmetry + notation from above, WLOG, assume $a < b < c$.
- Since $f \in \mathcal{R}(\alpha; a, b)$, we may take $P_\varepsilon^{(a,b)} \in \mathcal{P}([a,b])$, s.t. $|S_{P_\varepsilon^{(a,b)},t}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$ for any $P^{(a,b)} \supseteq P_\varepsilon^{(a,b)}$ and tagged points $t^{(a,b)}$
- Similarly, take $P_\varepsilon^{(b,c)} \in \mathcal{P}([b,c])$, s.t. $|S_{P_\varepsilon^{(b,c)},t}(f,\alpha) - \int_b^c f\,d\alpha| < \varepsilon$ for any $P^{(b,c)} \supseteq P_\varepsilon^{(b,c)}$ and tagged points $t^{(b,c)}$

Then, define $P_\varepsilon := P_\varepsilon^{(a,b)} \vee P_\varepsilon^{(b,c)}$ and take $P \supseteq P_\varepsilon$ and tagged points t, let $P^{(a,b)} = P \cap [a,b]$, $P^{(b,c)} = P \cap [b,c]$, $t^{(a,b)} = t \cap [a,b]$, $t^{(b,c)} = t \cap [b,c]$
Then, $S_{P,t}(f,\alpha) = S_{P^{(a,b)},t}(f,\alpha) + S_{P^{(b,c)},t}(f,\alpha)$, so by triangle inequality, $|S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha - \int_b^c f\,d\alpha| < 2\varepsilon$ □

PROPOSITION (Integration by Parts)
Let $f \in \mathcal{R}(\alpha; a, b)$. Then, $\alpha \in \mathcal{R}(f; a, b)$, and we have $\int_a^b f\,d\alpha + \int_a^b \alpha\,df = f(b)\alpha(b) - f(a)\alpha(a)$
(First prove $\int \alpha\,df$ is Riemann-Stieltjes integrable, then check that $|\int_a^b \alpha\,df - f(b)\alpha(b) + f(a)\alpha(a) + \int_a^b f\,d\alpha|$ is small)