4-17-25 (WEEK 9)

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DOUBLE SEQUENCES, DOUBLE SERIES

DOUBLE SEQUENCES AND DOUBLE LIMITS

Let $(W, \|\cdot\|)$ be a Banach space. Let $(u_{m,n})_{m,n\geq 1}$ be a double sequence with values in W (intuition: sequence on a 2D grid)

DEFINITION

Let $\ell \in W$. We say $(u_{m,n})_{m,n\geq 1}$ converges to ℓ if for every $\varepsilon>0$, $\exists N>0$, s.t. $\|u_{m,n}-\ell\|<\varepsilon$ $\forall m,n\geq N$, denoted by $\lim_{m,n\to+\infty} u_{m,n}=\ell$, which is called the limit or double limit of $(u_{m,n})_{m,n\geq 1}$.

REMARK

This is simply another way to formulate Cauchy's criterion. The sequence $(u_m)_{m\geq 1}$ satisfies Cauchy's criterion if $\lim_{m,n\to+\infty}\|u_m-u_n\|=0$

EXAMPLE

Consider $(u_{m,n})_{m,n\geq 1}$ to be defined by $u_{m,n}=\mathbb{1}_{m<n}$ $\forall m,n\geq 1$

We have $\lim_{n\to\infty}\lim_{m\to\infty} u_{m,n}=1$, $\lim_{m\to\infty}\lim_{n\to\infty} u_{m,n}=0 \Rightarrow$ The order of limits matter (iterated limits)

THEOREM

Suppose that,

(i) $\lim_{m,n\to+\infty} u_{m,n}=\ell\in W$

(ii) $\forall m\geq 1$, $\lim_{n\to\infty} u_n$ exists

Then, $\lim_{m\to\infty}\lim_{n\to\infty} u_{m,n}=\ell$

<u>Proof</u>

Let $\varepsilon>0$. By (i), we can find $N>0$, s.t. $\|u_{m,n}-\ell\|\leq\varepsilon$ $\forall m,n\geq N$.

By (ii), define $\ell_m := \lim_{n\to\infty} u_{m,n}$

$\forall m\geq 1$, $\exists N'=N'(m)$, s.t. $\|\ell_m-u_{m,n}\|\leq\varepsilon$ $\forall n\geq N'(m)$.

$\forall m\geq N$, we have $\|\ell_m-\ell\|\leq\|\ell_m-u_{m,n}\|+\|u_{m,n}-\ell\|\leq 2\varepsilon$ $\forall n\geq \max(N,N'(m))$. $\square$

THEOREM

Let $(u_{m,n})_{m,n\geq 1}$ be a sequence with terms in a Banach space. Then, $(1)\Leftrightarrow(2)$

(1) $\forall n\geq 1$, $\sum_{m\geq 1} u_{m,n}$ is absolutely convergent, and the series $\sum_{n\geq 1}\left(\sum_{m\geq 1}\|u_{m,n}\|\right)$ converges

(2) $\forall m\geq 1$, $\sum_{n\geq 1} u_{m,n}$ is absolutely convergent, and the series $\sum_{m\geq 1}\left(\sum_{n\geq 1}\|u_{m,n}\|\right)$ converges

And when (1) or (2) holds, we have $\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} u_{m,n}\right)=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} u_{m,n}\right)$

REMARK

If $u_{m,n}\geq 0$ $\forall m,n\geq 1$, we can write $\sum_{m\geq 1}\sum_{n\geq 1} u_{m,n}=\sum_{n\geq 1}\sum_{m\geq 1} u_{m,n}$ without checking any condition. They are either both $+\infty$ or finite, and the thm above guarantees equality.

PROOF OF THEOREM

By symmetry, it suffices to prove $(1)\Rightarrow(2)$.

Suppose (1) holds. $\forall n\geq 1$, define $A_n := \sum_{m\geq 1}\|u_{m,n}\|<+\infty$

- Fix $m\geq 1$. $\sum_{n\geq 1}^{N}\|u_{m,n}\|\leq\sum_{n=1}^{N} A_n\leq\sum_{n=1}^{\infty} A_n<+\infty$. Hence, $\sum_{n\geq 1} u_{m,n}$ is absolutely convergent
- Let $M\geq 1$. By linearity on finitely many converging series, $\sum_{m=1}^{M}\sum_{n=1}^{\infty}\|u_{m,n}\|=\sum_{n=1}^{\infty}\sum_{m=1}^{M}\|u_{m,n}\|\leq\sum_{n=1}^{\infty}A_n<+\infty$. Hence, $\sum_{m\geq 1}\sum_{n\geq 1}\|u_{m,n}\|$ converges
- Check the LHS of the equality is well-defined when (2) holds:

  When (2) holds, $\forall m$, $\sum_{n\geq 1} u_{m,n}$ converges absolutely $\Rightarrow \sum_{n\geq 1} u_{m,n}$ converges. Moreover, $\sum_{m\geq 1}\sum_{n\geq 1}\|u_{m,n}\|<+\infty$ by (2) $\Rightarrow \sum_{m\geq 1}\|\sum_{n\geq 1} u_{m,n}\|$ converges

  $\therefore$ $\sum_{m\geq 1}\sum_{n\geq 1} u_{m,n}$ converges

Similarly, (1) holds $\Rightarrow$ RHS of equality is well-defined. $\therefore$ Both terms are well-defined

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Now, we show equality holds.
Let $S_n = \sum_{p=1}^{n}\sum_{q=1}^{n} u_{p,q}$ for $n \geq 1$. Notice, $S_n = \sum_{q=1}^{n}\sum_{p=1}^{n} u_{p,q}$.

Goal: Show that $S_n \xrightarrow{n\to\infty}$ LHS or RHS of equality, then conclude by symmetry.

We introduce the following quantities: $\forall m,q \geq 1$, $a_{m,q} = \sum_{p=1}^{m} u_{p,q}$, $a_q = \sum_{p\geq 1} u_{p,q}$
Let $\varepsilon > 0$. Let $Q > 0$, s.t. $\sum_{q\geq Q} A_q \leq \varepsilon$

For $n \geq Q$, we have:

$$\underbrace{\sum_{q=1}^{\infty} a_q - S_n}_{\text{(RHS)}} = \sum_{q=1}^{\infty} a_q - \sum_{q=1}^{n} a_q = \sum_{q=1}^{Q} (a_q - a_{n,q}) + \sum_{q=Q+1}^{n} (a_q - a_{n,q}) + \sum_{q=n+1}^{\infty} a_q$$

By Δ-ineq, we get: $\left\| \sum_{q=1}^{\infty} a_q - S_n \right\| \leq \left\| \sum_{q=1}^{Q} (a_q - a_{n,q}) \right\| + \sum_{q=Q+1}^{\infty} A_q \leq \left\| \sum_{q=1}^{Q} (a_q - a_{n,q}) \right\| + \varepsilon$
We cannot take the limit, but we can take limsup: $\limsup_{n\to\infty} \left\| \sum_{q=1}^{\infty} a_q - S_n \right\| \leq \varepsilon$
As the choice of $\varepsilon > 0$ is arbitrary, hence: $\limsup_{n\to\infty} \left\| \sum_{q=1}^{\infty} a_q - S_n \right\| = 0 \Rightarrow \lim_{n\to\infty} \left\| \sum_{q=1}^{\infty} a_q - S_n \right\| = 0$, that is, $S_n \xrightarrow{n\to\infty} \sum_{q=1}^{\infty} a_q$. $\square$
(Limsup not only helps proving limit exists, but also proves it equals 0 at the same time)

INFINITE PRODUCT

CONVERGENCE AND DIVERGENCE

DEFINITION

Let $(u_n)_{n\geq 1}$ be a sequence in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$
Define $P_0 = 1$, $P_n = \prod_{k=1}^{n} u_k$, $n \geq 1$, called partial products
(P_n: nth partial product; u_k: kth factor)

DEFINITION

Define $Z := \{n \geq 1 \mid u_n = 0\}$
1) If $|Z| = \infty$, then we say $\prod u_n$ diverges to 0
2) If $Z = \emptyset$, then:
   (a) If $P_n \to P \neq 0$, we say the infinite product $\prod u_n$ converges to P, denoted as $\prod u_n = P$
   (b) If $P_n \xrightarrow{n\to\infty} 0$, we say the infinite product diverges to 0
   (c) Otherwise, we say it diverges
3) If $|Z| < +\infty$, then let $N > 0$, s.t. $u_n \neq 0$ $\forall n \geq N$
   Define $v_n = u_{N+n-1}$ $\forall n \geq 1$, $P_0' = 1$, $P_n' = \prod_{k=1}^{n} v_k = \prod_{k=N}^{N+n-1} u_k$ (omit all zero terms)
   (a) If $\prod v_n$ converges to $P' \neq 0$, then we say $\prod u_n$ converges to $u_1 \cdots u_{N-1} P'$ (which is a "convergence" to 0, w/o the 0 it still converges)
   (b) If $\prod v_n$ diverges to 0, then we say $\prod u_n$ diverges to 0
   (c) Otherwise, we say it diverges

REMARK

Removing/adding finitely many zeroes in $(u_n)_{n\geq 1}$ does not change the behavior (convergence or divergence) of the product.

PROPOSITION (CAUCHY'S CRITERION)

The infinite product $\prod u_n$ converges $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n \geq N, k \geq 1$, we have $|u_{n+1} \cdots u_{n+k} - 1| < \varepsilon$
(minus 1 cuz to conv, the elements must approach 1)
($\frac{|P_{n+k} - P_n|}{|u_1 \cdots u_n|} = \left|\frac{P_{n+k} - P_n}{P_n}\right|$)

Proof

WLOG, let's assume $u_n \neq 0$ $\forall n \geq 1$.
- Suppose $\prod u_n = P \neq 0$. Then, $P_n \xrightarrow{n\to\infty} P$, so $(P_n)_{n\geq 1}$ is bounded
  Let $M > 0$, s.t. $|P_n| \geq M$ $\forall n \geq 1$
  Let $\varepsilon > 0$, take $N > 0$, s.t. $|P_{n+k} - P_n| < \varepsilon$ $\forall n \geq N$, $k > 0$.
  $\therefore \left|\frac{P_n - P_{n+k}}{P_n}\right| < \frac{\varepsilon}{|P_n|} \leq \frac{\varepsilon}{M}$

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- Suppose that the Cauchy's criterion is satisfied
  Let $\varepsilon=\frac{1}{2}$. Take $N>0$, s.t. $|Q_n-1|<\varepsilon=\frac{1}{2}$ where $Q_n=\prod_{k=N+1}^{n}u_k$ $\forall n\geq N$. $\therefore \frac{1}{2}<|Q_n|<\frac{3}{2}$

  Now, let $\varepsilon>0$, $N'\geq N$, then we have the following $\forall n\geq N'$
  Thus, $\left|\frac{Q_{n+k}}{Q_n}-1\right|=\left|\frac{Q_{n+k}-Q_n}{Q_n}\right|<\varepsilon \Rightarrow |Q_{n+k}-Q_n|<\frac{3}{2}\varepsilon$
  $\therefore$ $(Q_n)_{n\geq 1}$ converges $\square$