3-11-25 (WEEK 4)

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RIEMANN-STIELTJES INTEGRABILITY

THEOREM

Let $f, \alpha: [a,b] \longrightarrow \mathbb{R}$ be bounded. Then, $f \in \mathcal{R}(\alpha)$ if (1) or (2)

1) f is continuous and $\alpha \in BV$
2) α is continuous and $f \in BV$

COROLLARY

For $\alpha(x) = x$ and a continuous or a bounded variation f, then f is integrable

PROOF OF THEOREM

Integration by parts gives us $f \in \mathcal{R}(\alpha) \Leftrightarrow \alpha \in \mathcal{R}(f)$, therefore it is enough to show that when (1) holds with increasing α, $f \in \mathcal{R}(\alpha)$.

Let $\varepsilon > 0$. Since f is continuous on $[a,b]$ and $[a,b]$ is compact, we know that f is uniformly continuous

Take $\delta > 0$, s.t. $x, y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Let us take a partition $P \in \mathcal{P}([a,b])$ s.t. $\|P_\varepsilon\| < \delta$, then:

$U_P(f,\alpha) - L_P(f,\alpha) = \sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta\alpha_k \leq \varepsilon \sum_{k=1}^{n} \Delta\alpha_k = \varepsilon[\alpha(b) - \alpha(a)]$, thus Riemann's condition holds. □

THEOREM

Let $f, \alpha: [a,b] \to \mathbb{R}$ be bounded, α be nondecreasing. If (1) or (2) holds, then $f \notin \mathcal{R}(\alpha)$

1) $\exists c \in [a,b]$, s.t. f and α are not right-continuous at c
2) $\exists c \in [a,b]$, s.t. f and α are not left-continuous at c

Proof

By symmetry, we only need to check that $(1) \Rightarrow f \notin \mathcal{R}(\alpha)$

Suppose that $c \in [a,b]$, s.t. f and α are not right continuous at c

Let $\varepsilon > 0$, $\delta > 0$, s.t. $\exists x \in (c, c+\delta]$, $|f(x) - f(c)| > \varepsilon$ and $\exists y \in (c, c+\delta]$, s.t. $|\alpha(y) - \alpha(c)| > \varepsilon$

Let $P \in \mathcal{P}([a,b])$, s.t. $x_i = c$, $x_{i+1} = y$ for some $1 \leq i \leq n-1$.

Then, $U_P(f,\alpha) - L_P(f,\alpha) = \sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta\alpha_k \geq \varepsilon [M_{i+1}(f) - m_{i+1}(f)] \geq \varepsilon^2$ □

MEAN VALUE THEOREMS

FIRST MEAN VALUE THEOREM

Let $\alpha: [a,b] \to \mathbb{R}$ be a nondecreasing function and $f \in \mathcal{R}(\alpha; a, b)$. Let $M := \sup\{f(x) \mid x \in [a,b]\}$ and $m := \inf\{f(x) \mid x \in [a,b]\}$. Then, $\exists c \in [m, M]$, such that $\int_a^b f(x)\,d\alpha(x) = c\int_a^b d\alpha(x) = c[\alpha(b) - \alpha(a)]$. In particular, if f is continuous, then $c = f(x_0)$ for some $x_0 \in [a,b]$.

Proof

If α is a constant function, then of course $\int_a^b f\,d\alpha = 0$ and $\int_a^b d\alpha = 0 \Rightarrow$ The statement holds trivially

Suppose $\alpha(a) < \alpha(b)$. Then, for any given partition $P \in \mathcal{P}([a,b])$, we have $L_P(f,\alpha) \leq \underline{I}(f,\alpha) = \bar{I}(f,\alpha) = \int_a^b f\,d\alpha \leq U_P(f,\alpha)$

Moreover, $L_P(f,\alpha) = \sum_{k=1}^{n} m_k(f)\Delta\alpha_k \geq \sum_{k=1}^{n} m\Delta\alpha_k = m\int_a^b d\alpha$ and $U_P(f,\alpha) = \sum_{k=1}^{n} M_k(f)\Delta\alpha_k \leq \sum_{k=1}^{n} M\Delta\alpha_k = M\int_a^b f\,d\alpha$

Therefore, $m\int_a^b d\alpha \leq \int_a^b f\,d\alpha \leq M\int_a^b d\alpha$

$$\Rightarrow c = \frac{\int_a^b f\,d\alpha}{\int_a^b d\alpha} \in [m, M] \quad □$$

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COROLLARY

Some assumptions give us $\left|\int_a^b f\,d\alpha\right| \le M\int_a^b d\alpha = M[\alpha(b)-\alpha(a)]$, where $M=\sup\{|f(x)| \mid x\in[a,b]\}$

(This result could also be derived from the triangle inequality)

SECOND MEAN VALUE THEOREM

Let α be continuous and f be non-decreasing. Then, $\int_a^b \alpha\, df = f(a)\int_a^{x_0} d\alpha + f(b)\int_{x_0}^b d\alpha$ for some $x_0\in[a,b]$.

<u>Proof</u>

The integration by parts gives us $\int_a^b f\,d\alpha = f(b)\alpha(b)-f(a)\alpha(a)-\int_a^b \alpha\,df$

Now, applying 1st MVT, we obtain $\int_a^b \alpha\,df = \alpha(x_0)(f(b)-f(a))$ for some $x_0\in[a,b]$.

Hence, putting these statements together, we get $\int_a^b \alpha\,df = f(a)\int_a^{x_0} d\alpha + f(b)\int_{x_0}^b d\alpha$ □

FUNDAMENTAL THEOREMS OF CALCULUS

DEFINITION

Let $I\subseteq\mathbb{R}$ be an interval, $f,F: I\to\mathbb{R}$ be functions. If $F'(x)=f(x)\ \forall x\in \operatorname{int}(I)$, we say F is a primitive or antiderivative of f

FIRST FUNDAMENTAL THEOREM OF CALCULUS

Let $\alpha\in BV([a,b])$ and $f\in R(\alpha;a,b)$. Define $F(x)=\int_a^x d\alpha\ \forall x\in[a,b]$. Then, we have:

(a) $F\in BV([a,b])$

(b) If α is continuous at some $c\in[a,b]$, then F is also continuous at c.

(c) If α is non-decreasing, then the derivative $F'(x)$ exists at $x\in(a,b)$ whenever $\alpha'(x)$ exists and f is continuous at x; additionally, for such x, we have $F'(x)=f(x)\alpha'(x)$

<u>Proof</u>

It is sufficient for us to prove the theorem for non-decreasing α.

Note that for $x<y$, $x,y\in[a,b]$, we have $|F(y)-F(x)|\le M_{x,y}|\alpha(y)-\alpha(x)|$, where $M_{x,y}=\sup\{|f(t)| \mid t\in[x,y]\}\le\sup\{|f(t)| \mid t\in[a,b]\}=:M$

(a) Given a partition $P\in\mathcal{P}([a,b])$, we write $V_P(F)=\underline{\sum_{k=1}^n|\Delta F_k|\le\sum_{k=1}^n M|\Delta\alpha_k|}=M[\alpha(b)-\alpha(a)]<\infty$ $\therefore$ We have $F\in BV([a,b])$ ✓

(b) Let $c\in[a,b]$, s.t. α is continuous at c. Let $\varepsilon>0$, take $\delta>0$, s.t. $\forall x\in[a,b]$, $|x-c|\le\delta\Rightarrow|\alpha(x)-\alpha(c)|\le\varepsilon$

Then, we can see $\forall x\in[a,b]$, $|x-c|\le\delta\Rightarrow|F(x)-F(c)|\le M\varepsilon$ ($\because \sum|\Delta F_k|\le M\sum|\Delta\alpha_k|$)

By definition, F is continuous at c. ✓

(c) Let $x\in(a,b)$, s.t. $\alpha'(x)$ exists and f is continuous at x. By definition and MVT, $F'(x)=\lim_{y\to x}\frac{F(y)-F(x)}{y-x}=\lim_{y\to x}\frac{c[\alpha(y)-\alpha(x)]}{y-x}$

Since $c=c(x,y)\to f(x)$ when $y\to x$, by the continuity of f at x and $\alpha'(x)$ exists, we deduce $F'(x)=f(x)\alpha'(x)$ □

COROLLARY (Freshman Calculus I: Riemann Integrals)

Let $\alpha(x)=x$ and $f\in R(x;a,b)$ be a Riemann-integrable function.

Define $F(x)=\int_a^x f(t)\,dt$ and $G(x)=\int_a^x g(t)\,dt$

Then, the following properties hold:

(a) F and G are continuous functions of bounded variation on $[a,b]$.

(b) If f is continuous at $x\in(a,b)$, then $F'(x)=f(x)$

(c) For $f\in R(G;a,b)$, $g\in R(F;a,b)$ and $fg\in R(x;a,b)$, we have $\int_a^b f(x)g(x)\,dx=\int_a^b f(x)\,dG(x)=\int_a^b g(x)\,dF(x)$

SECOND FUNDAMENTAL THEOREM OF CALCULUS

Let $f\in R(x;a,b)$ and $F:[a,b]\to\mathbb{R}$, s.t. F is continuous and F' is well-defined on (a,b). Suppose that $F'(x)=f(x)$ for every $x\in(a,b)$. Then, we have $\int_a^b F'(x)\,dx=\int_a^b f(x)\,dx=F(b)-F(a)$

<u>Proof</u>

Let $\varepsilon>0$. Since $f\in R(x;a,b)$, we may find $P_\varepsilon\in\mathcal{P}([a,b])$, s.t. $\left|S_{P,t}(f,x)-\int_a^b f\,dx\right|<\varepsilon$ for any partition $P\supseteq P_\varepsilon$ and tagged points t.

For $P\supseteq P_\varepsilon$, and tagged points t, we have $S_{P,t}(f,x)=\sum_{k=1}^n f(t_k)\Delta x_k=\sum_{k=1}^n F'(t_k)\Delta x_k$

Why we need a special partition.
It is valid too, since $t_k\in(x_{k-1},x_k)$

Let us make a special choice of t, for a given partition $P\supseteq P_\varepsilon$, $\Delta F_k=\underline{F(x_k)-F(x_{k-1})=F'(t_k)\Delta x_k}=f(t_k)\Delta x_k$

Therefore, $S_{P,t}(f,x)=\sum_{k=1}^n f(t_k)\Delta x_k=\sum_{k=1}^n\Delta F_k=F(b)-F(a)$, this means $\int_a^b f(x)\,dx=F(b)-F(a)$

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COROLLARY

Let $f \in \mathcal{R}(\alpha; a, b)$, $\alpha: [a,b] \to \mathbb{R}$ be continuous s.t. $\alpha' \in \mathcal{R}(x; a, b)$. Then, $\int_a^b f \, d\alpha = \int_a^b f \alpha' \, dx$

Proof

Just take $g = \alpha'$. $\square$

PROPOSITION (Change of Variables) Very important to check!

Let $g: [c,d] \to \mathbb{R}$ be $\mathcal{C}^1$. Let $f: g([c,d]) \to \mathbb{R}$ be continuous, and define $F(x) = \int_{g(c)}^{x} f(s) \, ds$ $\forall x \in g([c,d])$

Then, we have $\forall x \in [c,d]$, $\int_c^x f \circ g(t) g'(t) = F \circ g(x) = \int_{g(c)}^{g(x)} f(s) \, ds$