4-10-25 (WEEK 8)

Shun/翔海 (@shun4midx)

TESTS OF CONVERGENCE

THEOREM (D'ALEMBERT'S CRITERION/RATIO TEST)

Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$. Suppose that $a_n > 0$ $\forall n \geq N$ for some $N \geq 1$. Suppose $\ell = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \in [0, +\infty]$ is well defined, then:

(1) If $\ell < 1$, then $\sum a_n$ is convergent

(2) If $\ell > 1$, then $\sum a_n$ is divergent

(3) If $\ell = 1$, and $\frac{a_{n+1}}{a_n} \geq 1$ $\forall n \geq N'$ for some $N' > 0$, then $\sum a_n$ is divergent

REMARK

In (3), if $\frac{a_{n+1}}{a_n}$ is not always ≥ 1 for large enough n, then they may exhibit different behaviors, eg $\overset{\text{div}}{\sum \frac{1}{n}}$ vs $\overset{\text{conv}}{\sum \frac{1}{n^2}}$

PROOF OF THEOREM

(1) Suppose $\ell < 1$. Set $\ell' := \frac{\ell+1}{2} < 1$. By def, $\exists N > 0$, s.t. $\frac{a_{n+1}}{a_n} \leq \ell'$, $a_n > 0$ for some $N > 0$.
By recurrence, we have $a_n \leq (\ell')^{n-N} a_N$ $\forall n \geq N$, thus $\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{n} |a_k| \leq \underbrace{\sum_{k=1}^{N-1} |a_k|}_{\text{const}} + \underbrace{\sum_{k=N}^{n} (\ell')^{k-N} |a_N|}_{\text{conv since geometric series}}$
$\because$ RHS conv $\therefore \sum_{k=1}^{n} |a_k|$ conv too $\therefore \sum a_n$ converges. □

(2) Suppose $\ell > 1$. Set $\ell' := \frac{\ell+1}{2} > 1$, so we can find $N' > 0$, s.t. $a_n > 0$, $\frac{a_{n+1}}{a_n} \geq \ell'$ $\forall n \geq N$ $\therefore a_n \geq (\ell')^{n-N} a_N$ $\forall n \geq N$, so $a_n \to +\infty$ $\therefore \sum a_n$ diverges

(3) Suppose $\ell = 1$ and $N > 0$, s.t. $a_n > 0$, $\frac{a_{n+1}}{a_n} \geq 1$ $\forall n \geq N$. We find $a_n \geq a_N > 0$ $\forall n \geq N$. $\therefore a_n \not\to 0$, so $\sum a_n$ diverges □

EXAMPLE

For $z \in \mathbb{C}^*$, consider the series $\sum \frac{z^n}{n!}$

↳ If $z = 0$, of course it converges

↳ If $z \neq 0$, consider the norm, $\frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|}{n+1} \xrightarrow{n\to\infty} 0 < 1$ $\therefore$ By d'Alembert's criterion, $\sum \frac{z^n}{n!}$ converges absolutely

COROLLARY

Let $\sum u_n$ be a series with general terms in a Banach space $(W, \|\cdot\|)$. Let $r = \liminf_{n\to\infty} \frac{\|u_{n+1}\|}{\|u_n\|}$ and $R = \limsup_{n\to\infty} \frac{\|u_{n+1}\|}{\|u_n\|}$

(1) If $R < 1$, then $\sum u_n$ converges absolutely

(2) If $r > 1$, then $\sum u_n$ diverges

(3) If $r \leq 1 \leq R$, then we cannot conclude

Proof

The proof is very similar to d'Alembert's criterion, so we prove (1) only as an example.

(1) Suppose $R > 1$, let $\ell' = \frac{R+1}{2}$. By the characterization of limsup, $\exists N > 0$, s.t. $\frac{\|a_{n+1}\|}{\|a_n\|} \leq \ell' = \frac{R+1}{2} < 1$ $\forall n \geq N$.
$\therefore \|a_n\| < (\ell)^{n-N} \|a_N\|$ $\forall n \geq N \Rightarrow \sum a_n$ converges absolutely

THEOREM (CAUCHY'S CRITERION/ROOT TEST)

Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$. Suppose that $a_n > 0$ $\forall n \geq N$ for some $N > 0$. Suppose $\lambda := \lim_{n\to\infty} (a_n)^{\frac{1}{n}} \in [0, +\infty]$ is well-defined, then:

(1) If $\lambda < 1$, then $\sum u_n$ converges

(2) If $\lambda > 1$, then $\sum u_n$ diverges

(3) If $\lambda = 1$, and $(u_n)^{\frac{1}{n}} > 1$ for large enough n, then $\sum u_n$ diverges

Proof

(1) Suppose $\lambda < 1$. Set $\lambda' := \frac{\lambda+1}{2} < 1$. We find $N > 0$, s.t. $a_n > 0$, $(a_n)^{\frac{1}{n}} \leq \lambda'$ $\forall n \geq N$. $\therefore a_n \leq (\lambda')^n$ $\forall n \geq N$.
$\therefore \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{n} |a_k| \leq \text{cst} + \underbrace{\sum_{k=N}^{n} |a_k|}_{\text{conv}}$ $\therefore \sum a_n$ converges absolutely

The proof to (2) and (3) is very similar. □

COROLLARY

Let $(u_n)_{n\geq 1}$ be a sequence in $\mathbb{C}$. Let $\lambda := \limsup_{n\to\infty} |u_n|^{\frac{1}{n}}$ (It is also applicable to Banach spaces)

1) If $\lambda < 1$, then $\sum u_n$ converges

2) If $\lambda < 1$, then $\sum u_n$ diverges

3) If $\lambda = 1$, no conclusion

REMARK

The root test is stronger. Given a sequence $(a_n)_{n\geq 1}$, we have seen: $\liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \underbrace{\liminf\limits_{n\to\infty} \sqrt[n]{a_n} \leq \limsup\limits_{n\to\infty} \sqrt[n]{a_n}}_{\text{Root test}} \leq \underbrace{\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n}}_{\text{Ratio test}}$

If we have $\limsup\limits_{n\to\infty} (a_n)^{\frac{1}{n}} < 1 < \limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$, then the root test applies but ratio test does not apply

Here is an example of such an $(a_n)_{n\geq 1}$

Let $a_n = \frac{(1+(-1)^n)2^{n+1}+1}{4^n} = \begin{cases} \frac{1}{4^n}, & n \text{ odd} \\ 4(\frac{1}{2})^n + \frac{1}{4^n}, & n \text{ even} \end{cases}$

Here, $\frac{a_{2n+2}}{a_{2n+1}} = \frac{4(\frac{1}{2})^{2n+2} + \frac{1}{4^{2n+2}}}{\frac{1}{4^{2n+1}}} \to +\infty$, so $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty$

However, we know $|a_n| \leq \frac{2(2^{n+1})+1}{4^n} \Rightarrow 0 \leq \limsup\limits_{n\to\infty} (a_n)^{\frac{1}{n}} \leq \frac{1}{2} < 1$!

CONDITIONALLY CONVERGENT SERIES

ALTERNATING SERIES

DEFINITION

Let $\sum u_n$ be a series with terms in $\mathbb{R}$.

We say it is an alternating series if $(-1)^n u_n$ has constant sign.

We may also write $\sum u_n = \sum (-1)^n a_n$, where $(a_n)_{n\geq 1}$ is a sequence with constant sign. We may assume $a_n \geq 0$ $\forall n \geq 0$, by a global sign flip.

THEOREM

Let $(a_n)_{n\geq 1}$ be a real sequence. Suppose that a_n is nondecreasing with limit 0. Then, $\sum (-1)^n a_n$ converges and its remainder satisfies, $\forall n \geq 1$, $|R_n| \leq a_{n+1}$, $R_n = \sum\limits_{k=n+1}^{\infty} (-1)^k a_k$

Proof

Consider the odd and even partial sums separately:

- $S_{2n+2} - S_{2n} = a_{2n+2} - a_{2n+1} \leq 0$ $\forall n \geq 1$
- $S_{2n+1} - S_{2n-1} = -a_{2n+1} + a_{2n} \geq 0$ $\forall n \geq 1$

$\therefore$ We have $(S_{2n-1})_{n\geq 1} \nearrow$ and $(S_{2n})_{n\geq 1} \searrow$

$\because$ $S_{2n} - S_{2n-1} = a_{2n} \to 0$

$\therefore$ $(S_{2n})_{n\geq 1}, (S_{2n-1})_{n\geq 1}$ are adjacent sequences, hence they both converge to the same limit S.

Also, $\forall n \geq 1$, we have $S_{2n-1} \leq S_{2n+1} \leq S \leq S_{2n}$, so $\forall n \geq 1$, $|R_{2n}| = |S - S_{2n}| \leq S_{2n} - S_{2n+1} = a_{2n+1}$ and $|R_{2n-1}| = |S - S_{2n-1}| \leq S_{2n} - S_{2n-1} = a_{2n}$

EXAMPLE

Let's study the series $\sum \frac{(-1)^{n+1}}{n}$, now that we proved its convergence.

Recall $H_n = \sum\limits_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)$ as $n \to \infty$. Denote $S_n = \sum\limits_{k=1}^{n} \frac{(-1)^{k+1}}{k}$.

$\therefore$ $\forall n \geq 1$, $H_{2n} - S_{2n} = \sum\limits_{k=1}^{n} \frac{2}{2k} = H_n$

In other words, $S_{2n} = H_{2n} - H_n = (\ln(2n) + \gamma + o(1)) - (\ln n + \gamma + o(1)) = \ln 2 + o(1)$ when $n \to \infty$.

$\therefore$ $\sum \frac{(-1)^{n+1}}{n}$ converges to $\ln 2$, additionally, $|S_n - \ln 2| = |R_n| \leq \frac{1}{n+1}$ (We can theoretically use this sequence to compute $\ln 2$ but its rate of convergence is too slow and hence it is not practical.)

DIRICHLET'S TEST

Consider a series $\sum u_n$ in a Banach space. Assume $\forall n \geq 1$, $u_n = a_n b_n$. We write $S_n = \sum\limits_{k=1}^{n} b_k$ for $n \geq 1$ and $S_0 = 0$.

ABEL'S TRANSFORM

For every $n \geq 0$, we have $\sum\limits_{k=1}^{n} u_k = \sum\limits_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n$ (IBP but Σ version)

Proof

$\forall n \geq 0$, we have $\sum\limits_{k=1}^{n} u_k = \sum\limits_{k=1}^{n} a_k b_k = \sum\limits_{k=1}^{n} a_k (S_k - S_{k-1}) = \sum\limits_{k=1}^{n} a_k S_k - \sum\limits_{k=0}^{n-1} a_{k+1} S_k = \sum\limits_{k=0}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n$