Analysis II Definitions

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Definitions

2-18-25 (Week 1): Riemann-Stieltjes Integrals (Functions of Bounded Variation)

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a function.

(1) f is **non-increasing/decreasing** if $f(x) \geq / > f(y)\ \forall x \leq y,\ x, y \in I$

(2) f is **non-decreasing/increasing** if $f(x) \leq / < f(y)\ \forall x \leq y,\ x, y \in I$

(3) f is **monotonic** if (1) or (2) holds

Definition 1.2. Let $f : I \to \mathbb{R}$ be monotonic. For $x \in I$, define:

- The **left limit** at x to be $\underline{f(x-) = \lim_{y<x,\ y\to x} f(y)}$ if $(x - \varepsilon, x) \cap I \neq \emptyset$ for $\varepsilon > 0$ *(e.g. we cannot just pick a point at the boundary)*
- The **right limit** at x to be $\underline{f(x+) = \lim_{y>x,\ y\to x} f(y)}$ if $(x, x + \varepsilon) \cap I \neq \emptyset$ for $\varepsilon > 0$

Definition 1.3. Let $a < b$ and $[a, b] \in \mathbb{R}$ be a segment.

- A **partition** or a **subdivision** of $[a, b]$ is a finite sequence $P = (x_k)_{0\leq k\leq n}$ s.t. $a = x_0 < x_1 < \cdots < x_n = b$, where n is the **length** of P. We denote $\underline{\text{Supp}(P) := \{x_k \mid 0 \leq k \leq n\}}$ as the **support** of P.
- For a finite subset $A \subseteq [a, b]$ with $a, b \in A$, we may find a partition P of $[a, b]$ s.t. $\text{Supp}(P) = A$. This is called the **partition corresponding to** A.
- We say $[x_{k-1}, x_k]$ is the k^{th} **subinterval** of P, $\underline{\Delta x_k := x_k - x_{k-1}}$, $1 \leq k \leq n$. Then, we say the **mesh size** of P is $\underline{||P|| := \max_{1\leq k\leq n} \Delta x_k}$
- Let P, P' be partitions. If $\text{Supp}(P) \subseteq \text{Supp}(P')$, then we say P' is **finer** than P, and we say $\underline{P \subseteq P'}$. This also implies $\underline{||P|| \leq ||P'||}$.
- Let P_1, P_2 be partitions. Define their **joint partition** or **smallest comon refinement** to be $\underline{P := P_1 \vee P_2}$, which is the partition P with support = $\underline{\text{Supp}(P_1) \cup \text{Supp}(P_2)}$.
- We denote $\underline{\mathcal{P}([a, b])}$ as the collection of **all** possible partitions of $[a, b]$.

Definition 1.4. Let $f : [a, b] \to \mathbb{R}$ be a function, $P = (x_k)_{0\leq k\leq n} \in \mathcal{P}([a, b])$, define $\underline{\Delta f_k := f(x_k) - f(x_{k-1})}$ for $1 \leq k \leq n$. Define $\underline{V_P(f) := \sum_{k=1}^{n} |\Delta f_k|}$ and $\underline{V_f = V_f([a, b]) = \sup_{P\in\mathcal{P}([a,b])} V_P(f)} \in [0, \infty]$ to be the **total variation** of f. We say that f is of **bounded variation** if $V_P < +\infty$. We write $\underline{\mathcal{BV}([a, b]) = \mathcal{BV}([a, b], \mathbb{R})}$ for the collection of such functions defined on $[a, b]$.