

PARTIAL SUMS AND REMAINDERS

(Sorry for the bad handwriting... once again, rain got into my stylus and I forgot to zip my bag in heavy rain, so...)

THEOREM

Let $\sum u_n$ and $\sum v_n$ be two series with nonnegative terms

Suppose that $u_n \sim v_n$. Then,

- 1) If $\sum u_n$ converges, then $\sum v_n$ converges, and $\sum_{k=n+1}^{\infty} u_k \sim \sum_{k=n+1}^{\infty} v_k$ (remainders, start from $n+1$ to ∞)
- 2) If $\sum u_n$ diverges, then $\sum v_n$ diverges, and $\sum_{k=1}^n u_k \sim \sum_{k=1}^n v_k$ (partial sums, start from 1 to n)

Proof

We have seen that $\sum u_n$ and $\sum v_n$ have the same behavior

- 1) Let $\varepsilon > 0$, take $N > 0$, s.t. $(1-\varepsilon)u_n \leq v_n \leq (1+\varepsilon)u_n \quad \forall n \geq N$
 $\forall M \geq n \geq N$, we have $\sum_{k=n+1}^M v_k \leq \sum_{k=n+1}^M (1+\varepsilon)u_k \leq (1+\varepsilon) \sum_{k=n+1}^M u_k \Rightarrow \sum_{k=n+1}^{\infty} v_k \leq (1+\varepsilon) \sum_{k=n+1}^{\infty} u_k$
 Similarly, we also have $\sum_{k=n+1}^{\infty} v_k \geq (1-\varepsilon) \sum_{k=n+1}^{\infty} u_k$, which gets us $\sum_{k=n+1}^{\infty} v_k \sim \sum_{k=n+1}^{\infty} u_k$

- 2) Let $\varepsilon > 0$, take $N > 0$, s.t. $(1-\varepsilon)u_n \leq v_n \leq (1+\varepsilon)u_n \quad \forall n \geq N$.

Write $\sum_{k=1}^n v_k = \sum_{k=1}^N v_k + \sum_{k=N+1}^n v_k$ for $n \geq N$.

We want to show " $(1-2\varepsilon) \sum_{k=1}^n u_k \leq \sum_{k=1}^n v_k \leq (1+2\varepsilon) \sum_{k=1}^n u_k$ " for large enough n .

"(a)": $\sum_{k=1}^n v_k \leq \sum_{k=1}^N v_k + (1+\varepsilon) \sum_{k=N+1}^n u_k$

$$\therefore \frac{\sum_{k=1}^n v_k}{\sum_{k=1}^n u_k} \xrightarrow{n \rightarrow \infty} 0, \text{ thus } \exists N' \geq N, \text{ s.t. } \sum_{k=1}^n v_k \leq \varepsilon \sum_{k=1}^n u_k$$

(div series)

Therefore, $\sum_{k=1}^n v_k \leq \varepsilon \sum_{k=1}^n u_k + (1+\varepsilon) \sum_{k=1}^n u_k = (1+2\varepsilon) \sum_{k=1}^n u_k$ for $n \geq N'$

"(b)": For $n \geq N$, we have $\sum_{k=1}^n v_k \geq \sum_{k=1}^N v_k + (1-\varepsilon) \sum_{k=N+1}^n u_k$

Now, take large $N' \geq N$, s.t. $\varepsilon \sum_{k=1}^n u_k \geq (1-2\varepsilon) \sum_{k=1}^n u_k - \sum_{k=1}^N v_k \quad \forall n \geq N$

Then, we find $\sum_{k=1}^n v_k \geq (1-2\varepsilon) \sum_{k=1}^n u_k$ \square

(Main idea: It diverges, so abuse the fact it's very large)

EXAMPLE (Algorithmic approach of analyzing remainders)

Let's study the asymptotic behavior of the harmonic series

Define $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \quad \forall n \geq 1$

- 1) Note that $\ln(1+x) \sim x$ when $x \rightarrow 0$. We have $\frac{1}{n} \sim \ln(1+\frac{1}{n}) = \ln(n+1) - \ln(n)$
 The series $\sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)]$ diverges, so $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

Moreover, we know that $H_n \sim \sum_{k=1}^n (\ln(k+1) - \ln(k)) = \ln(n+1) \sim \ln(n)$

$\therefore \frac{H_n}{\ln n} \rightarrow 1$ as $n \rightarrow \infty$, i.e. $H_n = \ln n + o(\ln n)$

- 2) Let's understand the term $o(\ln n)$.

Let $A_n = H_n - \ln n$ for $n \geq 1$

For $n \geq 2$, $A_n - A_{n-1} = H_n - H_{n-1} - \ln n + \ln(n-1) = \frac{1}{n} + \ln(1-\frac{1}{n}) = \frac{1}{n} + (-\frac{1}{n} - \frac{1}{2n^2} + o(\frac{1}{n^2})) = -\frac{1}{2n^2} + o(\frac{1}{n^2})$

$\Rightarrow A_n - A_{n-1} \sim -\frac{1}{2n^2}$

By the thm above, hence $\sum (A_n - A_{n-1})$ converges, i.e. $(A_n)_{n \geq 1}$ converges \square

Now, let $\gamma := \lim_{n \rightarrow \infty} A_n$, and call it Euler's constant

Then, we get $A_n = H_n - \ln n = \gamma + o(1) \Rightarrow H_n = \ln n + \gamma + o(1)$

- 3) Compare the partial sums for $\sum_{k=n+1}^{\infty} (A_k - A_{k+1}) = \gamma - A_n$ and $\sum_{k=n+1}^{\infty} \frac{1}{k^2} \sim \frac{1}{n}$ by Riemann series.

We get: $\gamma - A_n = \gamma - H_n + \ln n \sim -\frac{1}{2n} \Rightarrow \gamma - H_n + \ln n = -\frac{1}{2n} + o(\frac{1}{n}) \Rightarrow H_n = \ln n + \gamma + \frac{1}{2n} + o(\frac{1}{n})$

- 4) Let $D_n = H_n - \ln n - \gamma - \frac{1}{2n}$ for $n \geq 1$.

$D_n - D_{n-1} = H_n - H_{n-1} - \ln n + \ln(n-1) - \frac{1}{2n} + \frac{1}{2(n-1)}$

$= \frac{1}{n} + \ln(1-\frac{1}{n}) + \frac{1}{2n} - \frac{1}{2(n-1)} \quad \frac{1}{1-x} = 1+x+x^2+\dots$

$= \frac{1}{n} - (\frac{1}{n} + \frac{1}{2n^2} + o(\frac{1}{n^2})) - \frac{1}{2n} + \frac{1}{2n} (1 + \frac{1}{n} + \frac{1}{n^2} + o(\frac{1}{n^2})) = \frac{1}{6n^3} + o(\frac{1}{n^3})$

Again, by Riemann series, $\sum \frac{1}{n^3}$ converges $\Rightarrow \sum (D_n - D_{n-1})$ converges, moreover, $\sum_{k=n+1}^{\infty} (D_k - D_{k-1}) \sim \sum_{k=n+1}^{\infty} \frac{1}{k^3} \sim \frac{1}{2n^2} \Rightarrow D_n \sim \frac{1}{2n^2} \Rightarrow H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + o(\frac{1}{n^2})$

5) We have the following expression: $H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k n^{2k}} + o\left(\frac{1}{n^{2m}}\right)$ when $n \rightarrow \infty$, where $(B_{2k})_{k \geq 1}$ are Bernoulli numbers

REMARK

Using this similar approach, we can derive the infamous Stirling's formula (i.e. every CS student's nightmare when learning about asymptotic notation is about)

COMPARISON BETWEEN SERIES AND INTEGRALS

PROPOSITION

Let $f: [1, +\infty) \rightarrow \mathbb{R}_+$ be a nondecreasing function with $\lim_{x \rightarrow \infty} f(x) = 0$. $\forall n \geq 1$, define $S_n = \sum_{k=1}^n f(k)$, $I_n = \int_1^n f(t) dt$, $D_n = S_n - I_n$. Then, the following properties hold:

- (1) For $n \geq 1$, we have $0 \leq f(n+1) \leq D_{n+1} \leq D_n \leq f(1)$
- (2) The sequence $(D_n)_{n \geq 1}$ converges, and denote $D := \lim_{n \rightarrow \infty} D_n$
- (3) The series $\sum f(n)$ and the integral $\int_1^{\infty} f(t) dt := \lim_{x \rightarrow \infty} \int_1^x f(t) dt$ have the same behavior (both converge or both diverge)
- (4) $\forall n \geq 1$, we have $0 \leq D_n - D \leq f(n)$

Proof

- 1) For $k \geq 1$, we have $f(k+1) \leq \int_k^{k+1} f(t) dt \leq f(k)$
Then, $I_{n+1} = \sum_{k=1}^{n+1} \int_k^{k+1} f(t) dt \leq \sum_{k=1}^{n+1} f(k) = S_{n+1} \Rightarrow f(n+1) = S_{n+1} - S_n \leq S_{n+1} - I_{n+1} = D_{n+1}$
 $D_{n+1} - D_n = (S_{n+1} - S_n) - (I_{n+1} - I_n) = f(n+1) - \int_n^{n+1} f(t) dt \leq 0 \quad \therefore (D_n)_{n \geq 1} \text{ is decreasing}$
 $\therefore D_{n+1} \leq D_n \leq \dots \leq D_1 = S_1 - I_1 = f(1) \quad \checkmark$
- 2) We know $(D_n)_{n \geq 1}$ is decreasing and bounded from below by 0, hence it converges
- 3) We know $D = \lim_{n \rightarrow \infty} D_n = \lim_{n \rightarrow \infty} (S_n - I_n)$ converges
If $\lim_{n \rightarrow \infty} S_n$ exists, then $\lim_{n \rightarrow \infty} I_n = D - \lim_{n \rightarrow \infty} S_n$ exists. Same argument for " $\lim_{n \rightarrow \infty} I_n$ exists".
- 4) We know $D_n - D = \sum_{k=n+1}^{\infty} (D_k - D_{k+1}) \geq 0$. We have $D_k - D_{k+1} = \int_k^{k+1} f(t) dt - f(k+1) \leq f(k) - f(k+1) \quad \therefore \sum_{k=n+1}^{\infty} (D_k - D_{k+1}) \leq \sum_{k=n+1}^{\infty} (f(k) - f(k+1)) \Rightarrow D_n - D \leq f(n) \quad \square$

REMARK

From (4), we find $0 \leq \sum_{k=1}^n f(k) - \int_1^n f(t) dt - D \leq f(n) \Rightarrow \sum_{k=1}^n f(k) = \int_1^n f(t) dt + D + o(f(n))$
If we take $f(x) = \frac{1}{x}$, we find $H_n = \int_1^n \frac{1}{x} dx + D + o\left(\frac{1}{n}\right) = \ln n + D + o\left(\frac{1}{n}\right) \quad (o\left(\frac{1}{n}\right) < o(1))$

EXAMPLE (Riemann-Zeta function)

Let $s \in \mathbb{R}$, $f(x) = x^{-s} \quad \forall x \geq 1$

Consider the series $\sum_{n=1}^{\infty} n^{-s} = \begin{cases} \text{convergent, } s > 1 \\ \text{divergent, } s \leq 1 \end{cases}$

We define the Riemann-Zeta function, i.e. $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

Similarly, we can deduce $\sum_{k=1}^n \frac{1}{k^s} = \sum_{k=1}^n \left(\frac{1}{k^{s-1}} - 1 \right) + D(s) + o\left(\frac{1}{n^s}\right) = C(s) + \frac{1}{s-1} \frac{1}{n^{s-1}} + o\left(\frac{1}{n^s}\right)$

Since $\sum_{k=1}^{\infty} \frac{1}{k^s} \xrightarrow{n \rightarrow \infty} \zeta(s) = C(s)$, hence $\sum_{k=1}^{\infty} \frac{1}{k^s} = \zeta(s) = \sum_{k=1}^n \frac{1}{k^s} = \frac{1}{s-1} \frac{1}{n^{s-1}} + o\left(\frac{1}{n^s}\right)$

PROPOSITION (BERTAND'S SERIES)

For $\alpha, \beta \in \mathbb{R}$, consider the series $\sum_{n=2}^{\infty} \frac{1}{n^{\alpha} (\ln n)^{\beta}}$

- 1) When $\alpha > 1$, the series converges
- 2) When $\alpha = 1, \beta > 1$, the series converges
- 3) Otherwise, it diverges

Proof

- 1) Let $\alpha > 1$ and $\beta \in \mathbb{R}$.

Notice, $\frac{1}{n^{\alpha} (\ln n)^{\beta}} = o\left(\frac{1}{n^{\alpha}}\right) \quad \left(\frac{1}{n^{\alpha} (\ln n)^{\beta}} \div \frac{1}{n^{\alpha}} = \frac{1}{(\ln n)^{\beta}} \rightarrow 0 \right)$

Since $\sum \frac{1}{n^{\alpha}}$ converges $\therefore \sum \frac{1}{n^{\alpha} (\ln n)^{\beta}}$ also converges

- 2) Let $\alpha = 1, \beta > 1$, $f(x) = \frac{1}{x (\ln x)^{\beta}}, x \geq 2$.

$$\int_2^n f(x) dx = \int_2^n \frac{1}{x (\ln x)^{\beta}} dx = \int_{\ln 2}^{\ln n} \frac{1}{y^{\beta}} dy \quad \text{is convergent}$$

$\therefore \sum f(x)$ converges

3) Finally, when $\alpha = \beta = 1$, $\int_2^n \frac{dx}{x \ln x} = \int_{\ln 2}^{\ln n} \frac{1}{y} dy$ is divergent
Now, for $\alpha = 1, \beta < 1$, we have $\frac{1}{x \ln x} \leq \frac{1}{x (\ln x)^\beta} \Rightarrow \sum \frac{1}{x (\ln x)^\beta}$ diverges

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When $\alpha < 1$, thus $\frac{1}{n^{1+\alpha} (\ln n)^\beta} = o\left(\frac{1}{n^{1+\alpha} (\ln n)^\beta}\right)$, since $\sum \frac{1}{n^{1+\alpha} (\ln n)^\beta}$ diverges, thus $\sum \frac{1}{n^{1+\alpha} (\ln n)^\beta}$ diverges too. \square