4-29-25 (WEEK 11)

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EXAMPLE

Fix $\lambda>0$ and consider $\underset{=[0,+\infty)}{\mathbb{R}_{\geq 0}} \longrightarrow \mathbb{R}$, $x \longmapsto e^{-\lambda x}$ which is nonnegative.

For every integer $n\geq 1$, take $J_n=[0,\infty]$, and $\int_J f(x)\,dx = \int_0^n e^{-\lambda x}\,dx = \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^n = \frac{1-e^{-\lambda n}}{\lambda} \xrightarrow{n\to\infty} \frac{1}{\lambda}$. $\therefore \int_J f\,dx = \frac{1}{\lambda}$

EXAMPLE

The function $\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, $x \longmapsto |\sin x|$ is nonnegative and not integrable

For $k\in\mathbb{N}$, $\int_{k\pi}^{(k+1)\pi} |\sin x|\,dx = \int_0^\pi \sin x\,dx = 2 \Rightarrow \int_0^{n\pi} |\sin x|\,dx = 2n \xrightarrow{n\to\infty} \infty$

$\therefore$ The integral isn't well defined.

EXAMPLE (RIEMANN INTEGRALS)

1) $t\longmapsto t^{-\alpha}$ is integrable on $[a,+\infty)$ iff $\alpha>1$ for some $a>0$.
2) $t\longmapsto t^{-\alpha}$ is integrable on $[0,a)$ iff $\alpha<1$ for some $a>0$.
3) For $a<b$, $t\longmapsto (b-t)^{-\alpha}$ is integrable on $[a,b)$ iff $\alpha<1$
4) For $a<b$, $t\longmapsto (t-a)^{-\alpha}$ is integrable on $(a,b]$ iff $\alpha<1$

EXAMPLE (BERTRAND'S INTEGRALS)

Fix $\alpha,\beta\in\mathbb{R}$, consider $t\longmapsto t^{-\alpha}|\ln t|^{-\beta}$

1) For $a>1$, if $\alpha>1$, it is integrable on $[a,+\infty)$ or $\alpha=1$ and $\beta>1$
2) For $a\in(0,1)$, if $\alpha<1$, it is integrable on $(0,a]$ or $\alpha=1$ and $\beta>1$

DEFINITION

Let $I\subseteq\mathbb{R}$ be an interval and $(W,\|\cdot\|)$ is a Banach space. A function $f:I\longrightarrow W$ is called integrable if $\|f\|$ is integrable

For a sequence $(J_n)_{n\geq 0}$ satisfying for $n\geq 1$, $\underbrace{J_n\subseteq J_{n+1}\subseteq\ldots\subseteq I, \text{ and } \bigcup_{n\geq 1} J_n = I}_{\text{"interval\_def"}}$, define $\int_I f = \lim_{n\to\infty}\int_{J_n} f\in W$

Denote $L^1(I,W) := \{f: I\longrightarrow W \mid \int_I \|f\| < +\infty\}$ (L^1 norm stuff...)

PROPOSITION

The definition of $\int_I f := \lim_{n\to\infty}\int_{J_n} f$ does not depend on the choice of $(J_n)_{n\geq 0}$ if it satisfies interval_def

REMARK

- For $[a,b)$, $-\infty<a<b<\infty$, we may consider $(J_n=[a,b-\frac{1}{n}])_{n\geq 1}$
- For $[a,+\infty)$, $-\infty<a<+\infty$, we may consider $(J_n=[a,n])_{n\geq 1}$

PROOF OF PROPOSITION

We do so by considering two steps:

(1) Given $(J_n)_{n\geq 1}$ satisfying interval_def, check that $\lim_{n\to\infty}\int_{J_n} f$ is well-defined
(2) Given $(J_n)_{n\geq 1}$ and $(K_n)_{n\geq 1}$ satisfying interval_def, check that their integrals are equal

Step 1

Let $(J_n=[a_n,b_n])_{n\geq 1}$ satisfying interval_def, $u_n := \int_{J_n} f$, $U_n := \int_{J_n}\|f\|$

Let $\varepsilon>0$ and take $N\geq 1$ s.t. $|U_p-U_q|<\varepsilon$ for all $p,q\geq N$.

For $p,q\geq N$, $p\geq q$, $u_p-u_q = \int_{J_p} f - \int_{J_q} f = \int_{[a_p,a_q]} f - \int_{[b_p,b_q]} f$

$\therefore \|u_p-u_q\| \leq \int_{[a_p,a_q]}\|f\| + \int_{[b_q,b_p]}\|f\| = U_p-U_q<\varepsilon$

This means that $(u_n)_{n\geq 1}$ is a Cauchy sequence in W, so it converges. ✓

Step 2

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Let $(J_n)_{n\geq1}$ and $(K_n)_{n\geq1}$ satisfy interval_def.

From (1), we know that $u_n = \int_{J_n} f \xrightarrow{n\to\infty} u$ and $v_n = \int_{J_n} f \xrightarrow{n\to\infty} v$

For $n\geq1$, let $L_n := J_n \cup K_n$. Due to interval_def, we may find $N\geq1$ s.t. $J_n \cap K_n \neq \emptyset$ $\forall n\geq N$. Thus, $(L_{N+n})_{n\geq1}$ satisfies interval_def.
Now, we want to show that $w=u$, then by symmetry we also have $w=v$, so $u=v$.

- $\forall n\geq1$, $J_n \subseteq L_n$
- $\forall n\geq1$, $U_n := \int_{J_n} \|f\|$, $W_n := \int_{L_n} \|f\|$

since we found W_n, U_n are nonneg $\Rightarrow$ their integrals/limits are well-def, hence $W_n - U_n \longrightarrow 0$

$\therefore \|w_n - u_n\| = \|\int_{L_n \setminus J_n} f\| \leq \underline{\int_{L_n\setminus J_n} \|f\| = W_n - U_n} \xrightarrow{n\to\infty} 0$
$\therefore \lim_{n\to\infty}(w_n - u_n) = 0 \Rightarrow w=u.$ □

PROPERTIES

Properties that can be preserved when we take a limit still hold for integrable functions, such as

- $\begin{aligned} L^1(I,W) &\longrightarrow W \\ f &\longmapsto \int_I f \end{aligned}$ is a linear map
- $\int_I f + \int_J f = \int_{I\cup J} f$ if $I\cap J = \emptyset$
- $\triangle$ ineq, $\|\int_I f\| \leq \int_I \|f\|$, IBP, change of variables

PROPOSITION

Let $f\in \mathcal{PC}(I,W)$ be a piecewise continuous function on I. TFAE
(1) f is integrable on $[a,b)$
(2) (Partial integral) $x\mapsto \int_a^x \|f(t)\|dt$ is bounded on $[a,b)$
(3) (Partial integral) $x\mapsto \int_a^x \|f(t)\|dt$ has a limit when $x\to b^-$
(4) (Remainder integral) The limit of $x\mapsto \int_x^b \|f(t)\| dt$ when $x\to b^-$ is 0
(5) (Cauchy's criterion) $\forall \varepsilon>0$, $\exists A\in I$, s.t. $\forall x,y\in[A,b)$, $x<y$, $\int_x^y \|f(t)\|dt < \varepsilon$

PROPOSITION

Let $f\in\mathcal{PC}(I,W)$ and $c\in \mathring{I}$. Define $I_- := I\cap(-\infty,c]$ and $I_+ := [c,+\infty)$, then TFAE
(1) f is integrable on I
(2) f is integrable on I_- and I_+
And in this case, we have $\int_I f = \int_{I_-} f + \int_{I_+} f$

PROPOSITION

Let $f\in\mathcal{PC}(I,W)$, $\varphi\in\mathcal{PC}_+(I)$.
(1) If $\|f\|<\varphi$ on I and φ is integrable, then f is integrable and $\|\int_I f\| \leq \int_I \varphi$
(2) If $f\in\mathcal{PC}_+(I)$ and is nonintegrable with $f\leq\varphi$, then φ is nonintegrable

Proof
(1) For any subsegment $J\subseteq I$, we have $\int_a^x \|f(t)\|dt \leq \int_a^x \varphi(t)dt \leq \int_a^b \varphi(t)dt = \int_I \varphi$
(2) By contradiction □

EXAMPLE

Check that $f: t\longmapsto \frac{1}{\sqrt{t(1-t)}}$ is integrable on $(0,1)$.
Let $c=\frac{1}{2}$, $I_- = (0,\frac{1}{2}]$, $I_+ = [\frac{1}{2},1)$

- For $t\in I_-$, $f(t)\leq \frac{2}{\sqrt{t}}$, which is integrable on $(0,\frac{1}{2}]$, and so is f
- For $t\in I_+$, $f(t)\leq \frac{2}{\sqrt{1-t}}$, which is integrable on $[\frac{1}{2},1)$, so is f

COMPARISON OF INTEGRALS

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DEFINITION

Let $f:[a,b)\longrightarrow W$, $g:[a,b)\longrightarrow \mathbb{R}$ be piecewise continuous

i.e. $\forall a\in(b-\delta,b)$

We say $f \underset{b}{=} O(g)$ or $f(x)=O(g(x))$ when $x\to b^-$ if $\exists M>0, \delta>0$, s.t. $\forall x\in [a,b)\cap B(b,\delta)$, we have $\|f(x)\|\leq M\|g(x)\|$

We say $f \underset{b}{=} o(g)$ or $f(x)=o(g(x))$ when $x\to b^-$ if $\forall \varepsilon>0, \exists \delta>0$, s.t. $\forall x\in(b-\delta,b)$, $\|f(x)\|\leq \varepsilon\|g(x)\|$

We say $f\underset{b}{\sim} g$ or $f(x)\sim b(x)$ when $x\to b^-$ if $f-g \underset{b}{=} o(g)$

PROPOSITION

Let $f:[a,b)\longrightarrow W$ be p.c. and $g:[a,b)\longrightarrow \mathbb{R}_+$ be integrable

Crucial!!

(1) If $f\underset{b}{=}O(g)$, then f is integrable on $[a,b)$ and $\int_x^b f \underset{x\to b}{=} O(\int_x^b g)$

(2) If $f\underset{b}{=}o(g)$, then f is integrable on $[a,b)$ and $\int_x^b f \underset{x\to b}{=} o(\int_x^b g)$

(3) If $W=\mathbb{R}$ and $f\underset{b}{\sim} g$, then f is integrable on $[a,b)$ and $\int_x^b f \underset{x\to b}{\sim} \int_x^b g$

Proof

As (2), (3) are similar, we only prove (1) here.

By assumption, take $M>0$, $\delta>0$, s.t. $\forall x\in(b-\delta,b)$, $\|f(x)\|\leq Mg(t)$

$\therefore \forall x\in(b-\delta,b)$, we have $\|\int_x^b f\|\leq \int_x^b\|f\|\leq M\int_x^b g$ $\square$

EXAMPLE (THE GAMMA FUNCTION)

Define the gamma function $\Gamma: (0,+\infty)\longrightarrow\mathbb{R}$

$$x \longmapsto \int_0^{+\infty} t^{x-1}e^{-t}\,dt$$

- $t\mapsto t^{x-1}e^{-t}$ is nonnegative
- Around 0^+, $t^{x-1}e^{-t}\sim t^{x-1}$ and $t\mapsto t^{x-1}$ is nonnegative around 0 and integrable, so $t^{x-1}e^{-t}$ is too
- Around 0^-, $t^{x-1}e^{-t}=O(\frac{1}{t^2})$ and $t\mapsto\frac{1}{t^2}$ is integrable around $+\infty$, so $t^{x-1}e^{-t}$ is too
- E.g.: $\Gamma(1)=1$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$
- Hence, $\Gamma(x+1)=\int_0^{+\infty}t^xe^{-t}\,dt=-(t^xe^{-t})\Big|_0^{+\infty}+\int_0^{\infty}xt^{x-1}e^{-t}\,dt=x\Gamma(x)$ $\forall x>0$, which is the factorial

EXAMPLE

Goal: Find the asymptotic behavior of arccos around $x=1^-$

Note that $\int_x^1\frac{1}{\sqrt{1-t^2}}\,dt=\arccos x$, $x\in(0,1]$.

As $\frac{1}{\sqrt{1-t^2}}=\frac{1}{\sqrt{(1-t)(1+t)}}\underset{t\to 1}{\sim}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-t}}$, thus $\arccos x\sim\sqrt{2(1-x)}$ (integrate both sides)

EXAMPLE

We know that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}}\,dx=1$

We want to find the asymptotic behavior of $F(x)=\int_x^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt$ when $x\longrightarrow+\infty$

1) Note that $e^{-\frac{t^2}{2}}=o(te^{-\frac{t^2}{2}})\Rightarrow F(x)=o(\int_x^{+\infty}\frac{1}{\sqrt{2\pi}}te^{-\frac{t^2}{2}}\,dt)=o([-\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}]\Big|_x^{\infty})=o(e^{-\frac{x^2}{2}})$