3-25-25 (WEEK 6)

# LESBEGUE'S CRITERION

(Too tired and depressed to make my own notes for the main thm proof, sorry... I'll include all the other relevant notation and lemmas here... short notes for a reason. :( )

## PROPOSITION

Let $S=(S_n)_{n\geq 1}$ be a sequence of measure zero subsets. Then, their union $S=\bigcup_{n\geq 1} S_n$ also has measure zero

Proof

Let $\varepsilon>0$. $\forall n\geq 1$, since $S_n$ has measure zero, we may find a countable family $\{U_{n,i} \mid i\geq 1\}$ of open intervals covering $U_n$, and such that $\sum_{i\geq 1}|U_{n,i}|\leq \frac{\varepsilon}{2^n}$

Notice, $\mathcal{U}:=\{U_{n,i} \mid i,n\geq 1\}$ is still countable, since it is a countable union of countable families.

It also covers $S$, and $\sum_{n,i\geq 1}|U_{n,i}|=\sum_{n\geq 1}\sum_{i\geq 1}|U_{n,i}|=\sum_{n\geq 1}\frac{\varepsilon}{2^n}=\varepsilon$ □

## DEFINITION

Let $f:[a,b]\to\mathbb{R}$ be a bounded function. For any subset $A\subseteq[a,b]$, define the oscillation of $f$ on $A$ to be $\Omega_f(A):=\sup\{f(x)-f(y) \mid x,y\in A\}$

For $x\in[a,b]$, define the oscillation of $f$ at $x$ to be $\omega_f(x):=\lim_{h\to 0^+}\Omega_f(B(x,h)\cap[a,b])$ ← The idea is to view the point as an infinitely small ball

## REMARK

"$\Omega_f(A)$" has actually appeared before already in Darboux sums

Notice, $\forall A\subseteq B\subseteq[a,b]$, we have $\Omega_f(A)\leq\Omega_f(B)$. Then, the function $h\mapsto\Omega_f(B(x,h)\cap[a,b])$ is nondecreasing and the limit as $h\to 0^+$ is always well-defined since it is also bounded.

## PROPOSITION

$f$ is continuous at $x \iff \omega_f(x)=0$

Proof

Main idea

"$\Rightarrow$": By def, fix $\varepsilon>0$, then $\exists\delta>0$, s.t. $\forall y\in B(x,\delta)$, $|f(x)-f(y)|<\varepsilon$

$\therefore$ $\forall h\in(0,\delta)$, we have $\Omega_f(B(x,h)\cap[a,b])\leq 2\varepsilon$

As $\varepsilon>0$ can be arbitrarily small, thus $\omega_f(x)=0$

"$\Leftarrow$": Fix $\varepsilon>0$. By def, $\exists\delta>0$, s.t. $h<\delta\Rightarrow\Omega_f(B(x,h)\cap[a,b])<\varepsilon$ (expand the limit def of $\omega_f$)

$\therefore$ $\forall y\in B(x,\delta)$, $|f(x)-f(y)|<\varepsilon$, so by def, $f$ is cont. at $x$. □

## PROPOSITION

$\forall\varepsilon>0$, $J_\varepsilon:=\{x\in[a,b] \mid \omega_f(x)\geq\varepsilon\}$ is a closed set

Proof

Take opposite

Assume not. Let $\varepsilon>0$ be s.t. $J_\varepsilon$ is not closed. Let $x\in\overline{J_\varepsilon}\setminus J_\varepsilon$, i.e. $\omega_f(x)<\varepsilon$

By def of limit, $\exists\delta>0$, s.t. $h\in(0,\delta)\Rightarrow\Omega_f(B(x,h)\cap[a,b])<\varepsilon$

$\therefore$ $\forall y\in B(x,h)$, we also have $\omega_f(y)<\varepsilon$, i.e. $B(x,h)\cap J_\varepsilon=\emptyset$, which contradicts "$x\in\overline{J_\varepsilon}$". ⇸

## LEMMA

Suppose $\omega_f(x)<\varepsilon$ $\forall x\in[a,b]$. Then, $\exists\delta>0$, s.t. $\forall[c,d]\subseteq[a,b]$ with $|d-c|<\delta$, we have $\Omega_f([c,d])<\varepsilon$

Proof

$\forall x\in[a,b]$, by def of limit, $\exists\delta_x>0$, s.t. $\Omega_f(B(x,\delta_x)\cap[a,b])<\varepsilon$

closed + bounded for over $\mathbb{R}$

$\because$ $\{B(x,\frac{\delta_x}{2}) \mid x\in[a,b]\}$ is an open covering of the compact set $[a,b]$

$\therefore$ By Borel-Lebesgue property, we can extract a finite subcovering

We only need to prove "$\exists$" $\delta$ anyway

Let $x_1,\dots,x_n\in[a,b]$ be s.t. $\{B(x_i,\frac{\delta_i}{2}) \mid 1\leq i\leq n\}$ covers $[a,b]$. Take $\delta=\min\{\frac{\delta_i}{2} \mid 1\leq i\leq n\}$.

For any segment $[c,d]\subseteq[a,b]$ with $|d-c|<\delta$, $\exists 1\leq i\leq n$, s.t. $[c,d]\cap B(x_i,\frac{\delta_i}{2})\neq\emptyset$ (obv, but it is a covering, so...)

strategic choice of $\delta=\min\{\frac{\delta_i}{2}\}$

$\therefore$ $[c,d]\subseteq B(x_i,\frac{\delta_i}{2}+\delta)\cap[a,b]\subseteq B(x_i,\delta_i)\cap[a,b]$

$\therefore$ From our remark, $\Omega_f([c,d])\leq\Omega_f(B(x_i,\delta_i)\cap[a,b])<\varepsilon$. □