2-18-24 (WEEK 1) Shun/翔海 (@shun4midx)

RIEMANN-STIELTJES INTEGRALS

FUNCTIONS OF BOUNDED VARIATION

DEFINITION

Let $I \subseteq \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$ be a function.

(1) f is non-increasing/decreasing if $f(x) > f(y)\ \forall x \leq y,\ x, y \in I$

(2) f is non-decreasing/increasing if $f(x) < f(y)\ \forall x \leq y,\ x, y \in I$

(3) f is monotonic if (1) or (2) holds

DEFINITION

Let $f: I \to \mathbb{R}$ be monotonic

For $x \in I$, define:

↳ The left limit at x to be $f(x-) = \lim_{y<x,\, y\to x} f(y)$ if $(x-\varepsilon, x) \cap I \neq \emptyset$ for $\varepsilon > 0$ ← E.g. we can't just pick a point at the boundary

↳ The right limit at x to be $f(x+) := \lim_{y>x,\, y\to x} f(y)$ if $(x, x+\varepsilon) \cap I \neq \emptyset$ for $\varepsilon > 0$

PROPOSITION

Let $f: [a,b] \to \mathbb{R}$ be a monotonic function. Then, the set of its discontinuities D is a countable set.

Proof

Define $D = \{x \in I \mid f(x-) \neq f(x+)\}$ ← monotonic ⇒ left/right limits are well-defined

By symmetry, WLOG, assume f is increasing, then $f(x-) < f(x+)\ \forall x \in D$ ← Key! This is since $x \in D$, i.e. it is discontinuous

As $\mathbb{Q}$ is dense in $\mathbb{R}$, we know $\exists q_x \in \mathbb{Q} \cap (f(x-), f(x+))$

∴ This gives us a map $\begin{cases} D \to \mathbb{Q} \\ x \mapsto q_x \end{cases}$, which is injective because $\forall x_1, x_2 \in D$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

∵ D can be injected in the countable set $\mathbb{Q}$

∴ D is countable. □

DEFINITION (PARTITIONS)

Let $a<b$ and $[a,b] \in \mathbb{R}$ be a segment.

A partition or a subdivision of $[a,b]$ is a finite sequence $P = (x_k)_{0\leq k\leq n}$ satisfying $a = x_0 < x_1 < \dots < x_n = b$, where n is the length of P

We denote $\mathrm{Supp}(P) := \{x_k \mid 0 \leq k \leq n\}$ as the support of P

For a finite subset $A \subseteq [a,b]$ with $a, b \in A$, we may find a partition P of $[a,b]$ s.t. $\mathrm{Supp}(P) = A$. This is called the partition corresponding to A.

We say $[x_{k-1}, x_k]$ is the k^{th} subinterval of P, $\Delta x_k := x_k - x_{k-1}$, $1 \leq k \leq n$. Then, we say the mesh size of P is $\|P\| := \max_{1\leq k\leq n} \Delta x_k$ ← This is not a norm!

Let P, P' be partitions. If $\mathrm{Supp}(P) \subseteq \mathrm{Supp}(P')$, then we say P' is finer than P, and we say $P \subseteq P'$. This also implies $\|P\| \leq \|P'\|$

Let P_1, P_2 be partitions. Define their joint partition or smallest common refinement to be $P := P_1 \vee P_2$, which is the partition P with support $= \mathrm{Supp}(P_1) \cup \mathrm{Supp}(P_2)$

We denote $\mathcal{P}([a,b])$ as the collection of all possible partitions of $[a,b]$

REMARK

For any $P = (x_k)_{0\leq k\leq n} \in \mathcal{P}([a,b])$, we have $b - a = \sum_{k=1}^{n} \Delta x_k$

DEFINITION (BOUNDED VARIATIONS)

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Let $f:[a,b]\to\mathbb{R}$ be a function, $P=(x_k)_{0\le k\le n}\in\mathcal{P}([a,b])$, define $\Delta f_k := f(x_k)-f(x_{k-1})$ for $1\le k\le n$.
Define $V_P(f) := \sum_{k=1}^{n}|\Delta f_k|$ and $V_f = V_f([a,b]) = \sup_{P\in\mathcal{P}([a,b])} V_P(f)\in[0,\infty]$ to be the total variation of f. We say that f is of bounded variation if $V_f<+\infty$.

We write $\mathcal{BV}([a,b]) = \mathcal{BV}([a,b],\mathbb{R})$ for the collection of such functions defined on $[a,b]$.

EXAMPLE

Consider the function $f:[0,1]\to\mathbb{R}$, $f(x)=\begin{cases} x\cos\left(\frac{\pi}{x}\right), & x\in(0,1] \\ 0, & x=0\end{cases}$

For $n\ge1$, consider the partition $\mathcal{P}$ with support $\{0,\frac{1}{2n},\frac{1}{2n-1},\dots,\frac{1}{2},1\}$, i.e. $x_0=0$, $x_k=\frac{1}{2n+1-k}$ $\forall 1\le k\le 2n$

By completeness, we find $V_P(f)=\sum_{k=1}^{2n}|\Delta f_k| = \left|\frac{(-1)^{2n}}{2n}-0\right| + \sum_{k=2}^{2n}\left|\frac{(-1)^{k-1}}{2n+1-k}-\frac{(-1)^k}{2n+2-k}\right| = \frac{1}{2n}+\sum_{k=2}^{2n}\left(\frac{1}{2n+1-k}+\frac{1}{2n+2-k}\right) = 1+\sum_{k=2}^{2n-1}\frac{2}{k}+\frac{1}{2n}$, which is not bounded for $n\ge1$. So f is not of bounded variation

PROPOSITION

Let $f\in\mathcal{BV}([a,b],\mathbb{R})$, then

(1) For any partitions $P\subseteq P'$, we have $V_P(f)\le V_{P'}(f)$

(2) $\forall\varepsilon>0$, $\exists$ partition $P_\varepsilon\in\mathcal{P}([a,b])$ s.t. $\forall$ finer partition $P\supseteq P_\varepsilon$, we have $V_P(f)\le V_f\le V_P(f)+\varepsilon$

<u>Proof</u>

(1) By induction, we only need to prove this is true whenever $|\mathrm{Supp}(P')|=|\mathrm{Supp}(P)|+1$

Let $P,P'\in\mathcal{P}([a,b])$ be partitions with support s.t. $\mathrm{Supp}(P')=\mathrm{Supp}(P)\cup\{c\}$, $x_{k-1}<c<x_k$ for some $1\le k\le n$.

Then,
$$\begin{aligned} V_{P'}(f) &= \sum_{k=1,k\ne i}^{n}|f(x_k)-f(x_{k-1})| + |f(c)-f(x_{i-1})| + |f(x_i)-f(c)| \\ &\ge \sum_{k=1,k\ne i}^{n}|f(x_k)-f(x_{k-1})| + |f(x_i)-f(x_{i-1})| \\ &= \sum_{k=1}^{n}|f(x_k)-f(x_{k-1})| = V_P(f) \checkmark \end{aligned}$$

$\therefore$ By induction, the statement holds. $\square$

(2) Let $\varepsilon>0$, by the characterization of supremum, we can find $P_\varepsilon\in\mathcal{P}([a,b])$ s.t. $V_f\le V_{P_\varepsilon}(f)+\varepsilon$ directly follows from $V_f=\sup_{P\in\mathcal{P}([a,b])}V_P(f)$. "$\forall\varepsilon>0$, $\exists P'\in\mathcal{P}([a,b])$ s.t. $V_f\le V_{P'}(f)+\varepsilon$"

$\therefore$ $\forall$ finer partitions $P\supseteq P_\varepsilon$, from (1), $V_f\le V_{P_\varepsilon}(f)+\varepsilon\le V_P(f)+\varepsilon$ $\square$

PROPOSITION

If $f:[a,b]\to\mathbb{R}$ is monotonic, then $f\in\mathcal{BV}([a,b])$ and $V_f=|f(b)-f(a)|$

<u>Proof</u>

WLOG, assume that f is increasing.

Then, $\forall P\in\mathcal{P}([a,b])$, $V_P(f)=\sum_{k=1}^{n}|\Delta f_k| = \sum_{k=1}^{n}\Delta f_k = f(b)-f(a)$, which is independent of P

$\therefore$ $f\in\mathcal{BV}([a,b])$ and $V_f=|f(b)-f(a)|$ $\square$

PROPOSITION

If $f:[a,b]\to\mathbb{R}$ is continuous and differentiable on (a,b) with bounded derivative, then $f\in\mathcal{BV}([a,b])$

<u>Proof</u>

Let $P=(x_k)_{0\le k\le n}\in\mathcal{P}([a,b])$ be a partition, then $V_P(f)=\sum_{k=1}^{n}|\Delta f_k| \overset{\text{MVT}}{=} \sum_{k=1}^{n}|f'(t_k)|\Delta x_k \le \sup_{t\in(a,b)}|f'(t)|\sum_{k=1}^{n}\Delta x_k = \sup_{t\in(a,b)}|f'(t)|(b-a)$ $\square$