# 5-20-25 (WEEK 14)

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## PROPOSITION

Let $f(z) := \sum_{n\ge 0} a_n z^n$, $g(z) := \sum_{n\ge 0} b_n z^n$ with radii of convergence $R_f, R_g$. For $z \in \mathbb{C}$ s.t. $|z| < \min(R_f, R_g)$, $f(z)g(z) = \sum_{n\ge 0} c_n z^n = \sum_{n\ge 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$

Proof

For $z \in \mathbb{C}$ with $|z| < \min(R_f, R_g)$, then $\sum a_n z^n$ and $\sum b_n z^n$ both converge absolutely. Then, the conclusion follows.

# REGULARITY

Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence $R > 0$.

Known: $f$ is well-defined on $D(0,R)$

## THEOREM

$f$ is continuous in the disk of convergence $D(0,R)$

Proof

Let $z \in D(0,R)$, take $r \in (0, R-|z|)$. Then, $\bar{D}(z,r) \subseteq \bar{D}(0, |z|+r) \subseteq \bar{D}(0,R)$

We know that $\sum_{n=0}^{N} a_n z^n$ is continuous $\forall N \in \mathbb{N}_0$, and the convergence $z \mapsto \sum a_n z^n$ is normal on $\bar{D}(0, |z|+r)$.

$\therefore$ $f$ is continuous on $\bar{D}(0, |z|+r)$, so also at $z$. $\square$

## THEOREM (ABEL'S THEOREM)

Suppose $\sum a_n R^n$ converges. Then,

$$\begin{aligned} [0,R] &\longrightarrow \mathbb{C} \\ x &\longmapsto \sum_{n\ge 0} a_n x^n \end{aligned}$$

is continuous

In other words, $\lim_{x \to R^-, x \in \mathbb{R}} \sum a_n x^n = \sum a_n R^n$

Note: The continuity is only "radial", not across the entire disk

Proof

By rescaling, we may assume $R = 1$

Let $R_n := \sum_{k \ge n+1} a_k R^k = \sum_{k\ge n+1} a_k$ be the $n^{\text{th}}$ remainder of the convergent series $\sum a_n$

- We already know $f$ converges pointwise on $[0,R)$ $([0,R) \subseteq \bar{D}[0,R))$
- It remains to show the uniform convergence to 0 of the remainder of the series of function

For $m > n$, and $x \in [0,1)$, we have $\sum_{k=n+1}^{m} a_k x^k = \sum_{k=n+1}^{m} (R_{k-1} - R_k) x^k = \sum_{k=n}^{m-1} R_k x^{k+1} - \sum_{k=n+1}^{m} R_k x^k = R_n x^{n+1} - \underbrace{R_m x^m}_{\xrightarrow{m\to\infty} 0} + \sum_{k=n+1}^{m-1} R_k (x^{k+1} - x^k)$

$|\cdot| \le M(x^{k+1} - x^k)$

$\sum |\cdot| \le M x^{k+1}$ indep of $m$

$\therefore$ abs conv as $m \to \infty$

This means $\sum_{k=n+1}^{\infty} a_k x^k = R_n x^{n+1} + \sum_{k\ge n+1} R_k (x^{k+1} - x^k)$ $\forall n \in \mathbb{N}$, $x \in [0,1)$

Now check that it converges uniformly to 0 on $[0,1)$

Let $\varepsilon > 0$. Take $N \in \mathbb{N}$, s.t. $|R_n| \le \varepsilon$ $\forall n \ge N$.

For $n \ge N$, $x \in [0,1)$, we have $\left|\sum_{k\ge n+1} a_k x^k\right| \le |R_n x^{n+1}| + \sum_{k\ge n+1} |R_k| (x^k - x^{k+1}) \le \varepsilon + \varepsilon x^{k+1} \le 2\varepsilon$ $\forall x \in [0,1)$, $n \ge N$.

$\therefore$ $\sum a_n z^n$ converges uniformly on $[0,R]$. $\square$

## THEOREM (TAUBER'S THEOREM)

Suppose $\lim_{x \to R^-, x \in \mathbb{R}} f(x) = \ell$ exists and $n a_n \xrightarrow{n\to\infty} 0$. Then, $\sum_{n\ge 0} a_n R^n = \ell$ (where $f(x) = \sum a_n x^n$)

Proof

WLOG, take $R = 1$. Let $S_n = \sum_{k=0}^{n} a_k$ $\forall n \in \mathbb{N}_0$.

For $x \in [0,1)$, we have $S_n - f(x) = \sum_{k=0}^{n} a_k - \sum_{k\ge 0} a_k x^k = \underbrace{\sum_{k=0}^{n} a_k (1-x^k)}_{(1)} - \underbrace{\sum_{k\ge n+1} a_k x^k}_{(2)}$

As $1 - x^k = (1-x)(1+x+\dots+x^{k-1})$, $|1-x^k| \le |1-x| k$, $k \in \mathbb{N}_0$, $x \in [0,1)$,

$\therefore$ $|(1)| \le \sum_{k=0}^{n} (1-x) \cdot k|a_k| = (1-x) \sum_{k=0}^{n} |k a_k|$

(Cesàro sum)

We know $|n a_n| \xrightarrow{n\to\infty} 0 \Rightarrow \frac{1}{n}\sum_{k=0}^{n-1} |k a_k| \xrightarrow{n\to\infty} 0$. If we take $x_n = 1 - \frac{1}{n}$, then $|(1)| \le \frac{1}{n} \sum_{k=0}^{n} |k a_k| \xrightarrow{n\to\infty} 0$

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- Let $\varepsilon>0$. Take $N$ s.t. $|a_n n|\leq\varepsilon\ \forall n\geq N$. For $n\geq N$, we have for $x\in[0,1)$, $\sum_{k\geq n+1}|a_k|x^k\leq\sum_{k\geq n+1}\frac{\varepsilon}{k}x^k\leq\frac{\varepsilon}{n}\sum_{k\geq n+1}x^k\leq\frac{\varepsilon}{n}\frac{x^{n+1}}{1-x}\leq\frac{\varepsilon}{n(1-x)}$

  If we take $x_n=1-\frac{1}{n}$, then $|(2)|\leq\varepsilon$

$\therefore$ In conclusion, let $\varepsilon>0$, take $N$, s.t. $|a_n n|\leq\varepsilon\ \forall n\geq N$. For $n\geq N$, take $x_n=1-\frac{1}{n}$. Then, $|S_n-f(x_n)|\leq\frac{1}{n}\sum_{k=0}^{n}|ka_k|+\varepsilon\Rightarrow\limsup_{n\to\infty}|S_n-f(x_n)|\leq\varepsilon$

This holds $\forall\varepsilon>0$, so we deduce that $\lim|S_n-f(x_n)|=0$

By the assumption, $f(x_n)\xrightarrow{n\to\infty}\ell\Rightarrow\lim_{n\to\infty}S_n=\lim_{n\to\infty}(S_n-f(x_n))+\lim_{n\to\infty}f(x_n)=\ell$ □

## COROLLARY

Let $\sum a_n$, $\sum b_n$ be convergent series. Define $c_n=\sum_{k=0}^{n}a_kb_{n-k}$. Suppose $\sum c_n$ is convergent. Then, $\sum_{n\geq0}c_n=(\sum_{n\geq0}a_n)(\sum_{n\geq0}b_n)$

**Proof**

Sketch: Abel's thm + equality in the disk: $(\sum a_nz^n)(\sum b_nz^n)=\sum c_nz^n\ \forall z\in(-1,1)$, take lim as $z\to1^-$. □

## DEFINITION

Let $A\subseteq\mathbb{C}$ and $f:A\longrightarrow\mathbb{C}$. For $z\in A$, we say $f$ is $\mathbb{C}$-differentiable (or simply differentiable) at $z_0\in\mathring{A}$ if the following limit exists,

$\frac{df}{dz}(z_0)=\frac{d}{dz}f(z_0)=f'(z_0):=\lim_{z\to z_0,z\in\mathbb{C}}\frac{f(z)-f(z_0)}{z-z_0}\in\mathbb{C}$, which is also called the $\mathbb{C}$-derivative of $f$ at $z_0$.

## REMARK

This notation is much stronger than the differential defined in the first term. Additionally, scaling + rotation $\not\Leftarrow$ linear + continuous

Note that $\frac{d}{dz}(z^n)=nz^{n-1}$ for all $n\in\mathbb{N}_0$.

Let $f(z)=\sum a_nz^n$ be a power series with $R>0$.

## THEOREM

Let $f\in\mathcal{C}^1$. Then,

- $R(\sum na_nz^{n-1})=R(\sum a_nz^n)$
- $\forall z\in D(0,R)$, $f'(z)=\sum_{n\geq1}na_nz^{n-1}=\sum_{n\geq0}(n+1)a_{n+1}z^n$

**Proof**

Let $R':=R(\sum na_nz^{n-1})$.

- "$R'\leq R$": Let $z\in D(0,R')$. We know $(na_nz^{n-1})_{n\geq0}$ is bounded $\Rightarrow(a_nz^n)_{n\geq0}$ is bounded $\Rightarrow|z|\leq R\Rightarrow R'\leq R$ by taking $z$ to the boundary
- "$R\leq R'$": Let $z\in D(0,R)$. We know $(a_nz^n)_{n\geq0}$ is bounded

  Take $z'\in D(0,|z|)$, $na_n(z')^{n-1}=\underbrace{a_nz^{n-1}}_{\text{bounded}}\underbrace{(\frac{z'}{z})^{n-1}\cdot n}_{\to0\text{ if }|\frac{z'}{z}|<1}\Rightarrow(na_n|z'|^n)_{n\geq0}$ is bounded

  $\therefore|z'|\leq R'\Rightarrow|z|\leq R'\Rightarrow R\leq R'$

$\therefore$ In conclusion, $R=R'$ □

Notice, $\sum na_nz^{n-1}$ converges normally on $\bar{D}(0,r)$ for $r<R$. $\sum a_nz^n$ converges normally on $\bar{D}(0,r)$ for $r<R$.

As every $z\mapsto a_nz^n$ is $\mathcal{C}^1$, so $\sum a_nz^n\in\mathcal{C}^1$ on $\bar{D}(0,R)$ and can be differentiated term by term □

(By induction, $f\in\mathcal{C}^\infty$)

## COROLLARY

$f$ is $\mathcal{C}^\infty$ on $D(0,R)$.

$\forall z\in D(0,R)$, $f^{(p)}(z)=\sum_{n\geq p}n(n-1)\cdots(n-p+1)a_nz^{n-p}=\sum_{n\geq p}\binom{n}{p}p!a_nz^{n-p}$

In particular, this gives $\forall p\in\mathbb{N}_0$, $a_p=\frac{f^{(p)}(0)}{p!}$ and $\forall z\in D(0,R)$, $f(z)=\sum_{p\geq0}\frac{f^{(p)}(0)}{p!}z^p$

## EXAMPLE

$\forall z\in D(0,1)$, $\frac{1}{1-z}=\sum_{n\geq0}z^n\xRightarrow{\text{derivative}}\frac{1}{(1-z)^2}=\sum_{n\geq0}nz^{n-1}=\sum_{n\geq0}(n+1)z^n\ \forall z\in(0,1)$. By induction, $\frac{p!}{p!(1-z)^{p+1}}=\sum_{n\geq0}\binom{p+1}{p}z^n$

Also, $\frac{1+z}{(1-z)^3}=-\frac{1-z}{(1-z)^3}+\frac{2z}{(1-z)^3}=\sum_{n\geq1}n^2z^{n-1}\Rightarrow\sum_{n\geq1}\frac{n^2}{2^n}\overset{z=\frac{1}{2}}{=}6$ (Multiply the above by z and diff again)

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## COROLLARY

Define $F: D(0,R) \longrightarrow \mathbb{C}$

$z \longmapsto \sum_{n\geq 0} \frac{a_n}{n+1} z^{n+1}$

Then, $F$ has the same radius of convergence and $F' = f$ on $D(0,R)$

## APPLICATION

$\forall x \in (-1,1)$, $\frac{1}{1-x} = \sum_{n\geq 0} x^n \Rightarrow -\ln(1-x) = \sum_{n\geq 0} \frac{x^{n+1}}{n+1}$

By Abel's thm, $\sum \frac{(-1)^n}{n+1}$ converges.

Hence, $\lim_{x\to -1} \sum_{n\geq 0} \frac{x^{n+1}}{n+1} = \sum \frac{(-1)^{n+1}}{n+1} \Rightarrow \lim_{x\to -1} \ln(1-x) = \sum \frac{(-1)^n}{n+1} \Rightarrow \ln 2 = \sum \frac{(-1)^n}{n+1}$