EXAMPLE

Let $f_n : [0,1] \longrightarrow \mathbb{R}$ for $n \geq 1$
$x \longmapsto x^n(1-x)$

- For $x \in [0,1)$, $|f_n(x)| = x^n(1-x) \leq x^n \xrightarrow{n\to\infty} 0$
- For $x=1$, $f_n(x) = 0$ $\forall n$
- Thus, $f_n \xrightarrow{n\to\infty} 0$ pointwise
- Check that this convergence is uniform
  For $n \geq 1$, $f_n'(x) = nx^{n-1}(1-x) - x^n = nx^{n-1}(1 - \frac{n+1}{n}x)$

	0		$\frac{n}{n+1}$		1
f_n'		$+$		$-$	
f_n	0	$\nearrow$		$\searrow$	0

$\Rightarrow |f_n(x)| \leq f_n(\frac{n}{n+1}) = \frac{1}{n+1}(\frac{n}{n+1})^n$ $\forall x \in [0,1]$
$\leq \frac{1}{n+1} \xrightarrow{n\to\infty} 0$ (indep of x)
$\therefore f_n \to 0$ uniformly □

REMARK

If $f_n \to f$ pointwise but not uniformly, how do we prove this?

- $f_n \not\to f$ uniformly: $\exists \varepsilon > 0$, s.t. $\forall N > 0$, $\exists x \in [0,1], n \geq N$, s.t. $|f_N(x) - f(x)| \geq \varepsilon$
  ↳ This means $\exists (n_k)_{k\geq1}$ and $(x_k)_{k\geq1}$, s.t. $|f_{n_k}(x_k) - f(x_k)| \geq c$ for some $c > 0$

EXAMPLE

Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ $\forall n \in \mathbb{N}$
$x \longmapsto \frac{x+\sqrt{n}}{x+n}$

- $\forall$ fixed $x \in \mathbb{R}$, $f_n(x) \xrightarrow{x\to\infty} 0$, so $f_n \to 0$ pointwise
- For $x = n$, we have $f_n(n) = \frac{n+\sqrt{n}}{2n} \xrightarrow{n\to\infty} \frac{1}{2}$

$\therefore f_n(n) \geq \frac{1}{4}$ for $n > N$ for some $N > 1$
$\therefore$ The convergence is not uniform

THEOREM (DINI'S THEOREM)

Let (K,d) be a compact metric space. Let $(f_n)_{n\geq1}$ be a sequence of continuous functions $K \to \mathbb{R}$
Suppose (i) The sequence (f_n) is increasing, i.e. $f_n(x) \leq f_{n+1}(x)$ $\forall x \in K$
(ii) $f_n \longrightarrow f$ pointwise, f is continuous
Then, $f_n \longrightarrow f$ uniformly

Proof

When we say "(K,d) is compact", we want to use the Borel-Lebesgue property (finite subcovering)
Let $g_n := f_n - f \geq 0$ $\forall n \in \mathbb{N}$. Fix $\varepsilon > 0$. Consider $\forall n > 0$, $E_n = \{x \in K \mid g_n(x) < \varepsilon\} = g_n^{-1}((-\infty, \varepsilon))$, which is open in (K,d), as it is the preimage of an open set under a conti. function.

Note, $\forall x \in K$, $g_n(x) \searrow 0$, so $\exists N > 0$, s.t. $x \in E_n$ $\forall n \geq N$
$\therefore (E_n)_{n\geq1}$ is an increasing sequence of open sets, and $\bigcup_{n\geq1}^{\infty} E_n = K$
$\therefore (E_n)_{n\geq1}$ is an open covering of the compact space (K,d), so $\exists N > 0$, s.t. $K = \bigcup_{n\geq1}^{N} E_n = E_N$. This means $0 \leq g_N(x) < \varepsilon$ $\forall x \in K$
$\therefore 0 \leq g_n(x) < \varepsilon$ $\forall n \geq N$ □

ALTERNATE VERSION

Let $I = [a,b]$ be a segment and a sequence $(f_n)_{n\geq1}$ from $I \longrightarrow \mathbb{R}$
Suppose (i) $f_n \nearrow$ on I $\forall n \geq 1$
(ii) $f_n \longrightarrow f$ pointwise, f is continuous
Then, $f_n \to f$ uniformly.

SERIES OF FUNCTIONS

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Let $(u_n)_{n\geq 1}$ be a sequence of functions from A to W < normed vector space / Banach space — if we need "Cauchy ⇒ conv"

DEFINITION

We say that $\sum u_n$ converges pointwise if $\forall x\in A$, the series $\sum u_n(x)$ converges. We write $\sum_{n\geq 1} u_n : A \longrightarrow W$, $x \longmapsto \sum_{n\geq 1} u_n(x)$

- $S_n(x) = \sum_{k=1}^{n} u_k(x)$ is called the k^{th} partial sum
- If $\sum u_n$ converges pointwise, then $R_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$ is the n^{th} remainder
- We say $\sum u_n$ converges uniformly if $(S_n)_{n\geq 0}$ converges uniformly

PROPOSITION

The series of functions $\sum u_n$ converges uniformly iff

(i) $\sum u_n$ converges pointwise

(ii) The sequence of remainders $(R_n)_{n\geq 0}$ converge uniformly to 0

EXAMPLE

Consider $\sum \frac{(-1)^n}{n} x^n$ for $x\in[0,1)$

- For $x\in[0,1)$, $\left(\frac{x^n}{n}\right) \searrow 0$, so the alternating series converges
  $\Rightarrow$ The series of functions $\sum \frac{(-1)^n}{n} x^n$ converges pointwise
- We may estimate the remainder of an alternating series. For $x\in[0,1)$, $|R_n(x)| \leq \left|\frac{(-1)^{n+1}x^{n+1}}{n+1}\right| \leq \frac{1}{n+1} \xrightarrow{n\to\infty} 0$
  $\therefore R_n \to 0$ uniformly

REMARK

$(f_n)_{n\geq 1}$ converges uniformly $\Leftrightarrow \sum (f_{n+1} - f_n)$ converges uniformly

PROPOSITION (CAUCHY'S CRITERION)

$\sum u_n$ converges uniformly $\Leftrightarrow \forall \varepsilon>0, \exists N>0$, s.t. $\forall n\geq N, k\geq 1$, $\underbrace{\|u_{n+1}+\dots+u_{n+k}\|_\infty}_{S_{n+k}-S_n} < \varepsilon$

DEFINITION

Let $u_n \in \mathcal{B}(A,W) =: \mathcal{E}$ for $n\geq 1$. We say the series of functions $\sum u_n$ converges normally (function version of conv abs!) if $\sum \|u_n\|$ converges

(If $\mathcal{E}$ is a Banach space, $u_n\in\mathcal{E}$, $\sum\|u_n\|_\infty$ converges means that $\sum u_n$ converges absolutely in $(\mathcal{E}, \|\cdot\|_\infty)$)

PROPOSITION

Let $\sum u_n$ be a series of bounded functions from A to W that converges normally on A. Then, we have:

(1) $\forall a\in A$, $\sum u_n(a)$ converges absolutely

(2) The series of functions $\sum u_n$ converges uniformly

<u>Proof</u>

(1) Let $a\in A$, $n\in\mathbb{N}$. Then, $\sum_{k=1}^{n} |u_k(a)| \leq \sum_{k=1}^{n} \|u_k\|_\infty \leq \sum_{k=1}^{\infty} \|u_k\|_\infty < +\infty$, so $\sum u_k(a)$ converges absolutely

(2) $\sum_{n\geq 1} \|u_n\|_\infty$ converges so it satisfies Cauchy's property

$\forall n\geq 1, k\geq 1$, $\|u_n(x)+\dots+u_{n+k}(x)\|_\infty \leq \|u_n(x)\|_\infty + \dots + \|u_{n+k}(x)\|_\infty \leq \|u_n\|_\infty + \dots + \|u_{n+k}\|_\infty$

$\therefore$ We have the Cauchy's condition for $\sum u_n(x)$ $\forall x\in A$

$\therefore \sum u_n$ converges uniformly □

EXAMPLE

Let $(f_n)_{n\geq 1}$ be a sequence of functions from $[0,1)$ to $\mathbb{R}$. $f_1 \equiv 1$, $\forall n\geq 1$, $\forall x\in[0,1)$, $f_{n+1}(x) = 1 + \frac{1}{2}\int_0^x f_n(t)\,dt$

- Show that $(f_n)_{n\geq 1}$ converges uniformly. To achieve this, let us check that $\sum(f_{n+1}-f_n)$ converges normally
- For $n\geq 1$, $x\in[0,1)$, we have $|f_{n+2}(x) - f_{n+1}(x)| = \frac{1}{2}\left|\int_0^x (f_{n+1}(t) - f_n(t))\,dt\right| \leq \frac{1}{2}\|f_{n+1}-f_n\|_\infty$
  $\Rightarrow \|f_{n+2}-f_{n+1}\|_\infty \leq \|f_{n+1}-f_n\|_\infty$ $\forall n\geq 1 \Rightarrow \|f_{n+1}-f_n\|_\infty \leq \left(\frac{1}{2}\right)^{n-1}\|f_2-f_1\|_\infty \Rightarrow$ ($\because$ it is a GS) $\sum\|f_{n+1}-f_n\|_\infty$ converges

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COUNTEREXAMPLE

Back to $\sum \frac{(-1)^n}{n} x^n$

- Known: Uniform convergence on $[0,1]$
- Let $u_n(x) = \frac{(-1)^n}{n} x^n$, $x \in [0,1]$. $\|u_n\|_\infty = \frac{1}{n}$, $\sum \frac{1}{n} = \infty$, so $\sum u_n$ does not converge normally!
- However, for $a \in [0,1)$, $\|u_n|_{[0,a]}\|_\infty = \frac{a^n}{n}$. In fact, $\sum \frac{a^n}{n}$ converges, so $\sum u_n$ converges normally on $[0,a]$

PROPERTIES OF THE UNIFORM LIMIT

Let (X, d_X) and (M, d_M) be metric spaces

Let $(f_n)_{n\geq 1}$ be a sequence of functions in $\mathcal{B}(X,M)$.

CONTINUITY

PROPOSITION

For $a \in X$, suppose f_n is continuous at a for $n \geq 1$. If $f_n \to f$ uniformly, then f is continuous at a

(Btw, the contrapositive provides us with another way to prove nonuniform convergence: find a discontinuous point of f)

Proof

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, take $N > 0$, s.t. $\forall n \geq N$, $x \in X$, $d_M(f_n(x), f(x)) \leq \varepsilon$ $(*)$

We can use the continuity of f_N at a.

We can find $\eta > 0$, s.t. $d_X(x,a) \leq \eta \Rightarrow d_M(f_N(x), f_N(a)) \leq \varepsilon$ $(**)$

Then, for $x \in B_X(a, \eta)$, we have: $d_M(f(x), f(a)) \leq \underbrace{d_M(f(x), f_N(x))}_{(*)} + \underbrace{d_M(f_N(x), f_N(a))}_{(**)} + \underbrace{d_M(f_N(a), f(a))}_{(*)} \leq 3\varepsilon$ □

COROLLARY

Suppose f_n is continuous on X for all $n \geq 1$. If $f_n \to f$ uniformly, then f is continuous on X.

COROLLARY

Let $\sum u_n$ be a series of continuous functions on X. If $\sum u_n$ converges uniformly, then $\sum u_n$ is continuous

EXAMPLE

Let $u_n: \mathbb{R}^+ \longrightarrow \mathbb{R}$ for $n \geq 0$

$x \longmapsto \frac{x^n}{n!}$

We want to discuss the series of functions $u(x) = \sum_{n\geq 0} u_n(x)$

- For $x \geq 0$, $\sum_{n\geq 0} \frac{x^n}{n!}$ conv by the ratio test. $\therefore u(x)$ is well-defined
- $x \longmapsto u_n(x)$ is continuous $\forall n \geq 0$
- Question: Is u a continuous function?
- Uniform convergence on $\mathbb{R}^+$? Fix $n \in \mathbb{N}$, $|R_n(x)| = |\sum_{n\geq 0} u_n(x) - \sum_{n=0}^{N} u_n(x)| = |\sum_{n=N+1}^{\infty} u_n(x)| \geq u_{N+1}(x) = \frac{x^{N+1}}{(N+1)!} \xrightarrow{x \to \infty} \infty$ $\therefore R_n \not\to 0$ uniformly
- Given $M > 0$, let us show that $\sum u_n$ converges uniformly on $[0,M]$. Then, the continuity of u on $[0,M]$ follows.
  This is true $\forall M > 0$, so we deduce the continuity of u on the whole $\mathbb{R}^+$
- For $x \in [0,M]$, $|R_n(x)| = \sum_{k\geq n+1} u_k(x) \leq \sum_{k\geq n+1} u_k(M) = \sum_{k\geq n+1} \frac{M^k}{k!} \xrightarrow{n\to\infty} 0$ indep of $x \in [0,M]$ ($\because$ They are remainders of the convergent $\sum_{n\geq 0} \frac{M^n}{n!}$)
  $\Rightarrow \sum u_n$ converges uniformly on $[0,M]$