2-27-25 (WEEK 2)

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PROPOSITION (Integration by Parts)

Let $f\in\mathcal{R}(\alpha;a,b)$. Then, $\alpha\in\mathcal{R}(f;a,b)$, and we have $\int_a^b f\,d\alpha+\int_a^b \alpha\,df=f(b)\alpha(b)-f(a)\alpha(a)$

<u>Proof</u>

Let $\varepsilon>0$, take $P_\varepsilon\in\mathcal{P}([a,b])$, s.t. $\forall P\supseteq P_\varepsilon$ and tagged points, $|S_{P,t}(\alpha,f)-\int_a^b f\,d\alpha|<\varepsilon$ $(*)$

Consider $P\supseteq P_\varepsilon$, tagged points t of P, we write $S_{P,t}(\alpha,f)=\sum_{k=1}^n \alpha(t_k)\Delta f_k=\sum_{k=1}^n \alpha(t_k)[f(x_k)-f(x_{k-1})]$

Note that $f(b)\alpha(b)-f(a)\alpha(a)=\sum_{k=1}^n[f(x_k)\alpha(x_k)-f(x_{k-1})\alpha(x_{k-1})]$, i.e. $S_{P,t}(f,\alpha)-[f(b)\alpha(b)-f(a)\alpha(a)]=\sum_{k=1}^n[\alpha(t_k)[f(x_k)-f(x_{k-1})]-[f(x_k)\alpha(x_k)-f(x_{k-1})\alpha(x_{k-1})]]$

$\therefore S_{P,t}(\alpha,f)-[f(b)\alpha(b)-f(a)\alpha(a)]=\sum_{k=1}^n f(x_k)[\alpha(t_k)-\alpha(x_k)]+\sum_{k=1}^n f(x_{k-1})[\alpha(x_{k-1})-\alpha(t_k)]$

Now, by taking a new partition, $Q=(x_0,t_1,x_1,t_2,x_2,t_3,\dots)$, $S=(x_0,x_1,x_1,x_2,x_2,x_3,x_3,\dots)$, then $S_{P,t}(\alpha,f)-[f(b)\alpha(b)-f(a)\alpha(a)]=-S_{Q,S}(f,\alpha)$

The partition Q is finer than P so also finer than P_ε.

From $(*)$, we find that $|-S_{Q,S}(f,\alpha)+\int_a^b f\,d\alpha|<\varepsilon \Leftrightarrow |S_{P,t}(\alpha,f)-[f(b)\alpha(b)-f(a)\alpha(a)]|<\varepsilon$

$\therefore$ This shows that $\alpha\in\mathcal{R}(f)$ and $\int_a^b\alpha\,df=f(b)\alpha(b)-f(a)\alpha(b)$ $\square$

PROPOSITION (Change of Variables)

necessary condition for any change of variables of RS integrals

Let $c\le d$ and $g:[c,d]\to\mathbb{R}$ is a continuous injective monotonic function.

Define $a=g(c)$, $b=g(d)$. Given $f\in\mathcal{R}(\alpha;a,b)=\mathcal{R}(\alpha;b,a)$, define $h(x)=f\circ g(x)$, $\beta(x)=\alpha\circ g(x)$, $\forall x\in[c,d]$.

Then, $h\in\mathcal{R}(\beta;c,d)$ and $\int_{g(c)}^{g(d)} f(t)\,d\alpha(t)=\int_a^b f\,d\alpha=\int_c^d h\,d\beta=\int_c^d f(g(x))\,d\alpha(g(x))$

<u>Proof</u>

WLOG, suppose that g is a strictly increasing function. In particular, g is bijective.

For any tagged partition (P,t) of $[a,b]$, define its "image" tagged position (P',t') under g^{-1} as below, $P'=(y_k)_{0\le k\le n}$, $t'=(t_k')_{0\le k\le n}$, with $y_k=g^{-1}(x_k)$, $t_k'=g^{-1}(t_k)$

Then, $S_{P',t'}(h,\beta)=\sum_{k=1}^n h(t_k')[\beta(y_k)-\beta(y_{k-1})]=\sum_{k=1}^n f(g(t_k'))[\alpha(g(y_k))-\alpha(g(y_{k-1}))]=\sum_{k=1}^n f(t_k)[\alpha(x_k)-\alpha(x_{k-1})]=S_{P,t}(h,\beta)$

$\therefore (h,\beta)$ satisfies (RS) $\Leftrightarrow (f,\alpha)$ satisfies (RS). Moreover, they have the same limit, i.e. $\int_a^b f\,d\alpha=\int_c^d h\,d\beta$. $\square$

PROPOSITION

Let $f\in\mathcal{R}(\alpha;a,b)$. Suppose that α is C^1. Then, $f\alpha'\in\mathcal{R}(x;a,b)$ and we have the identity $\int_a^b f(x)\,d\alpha(x)=\int_a^b f(x)\alpha'(x)\,dx$

<u>Proof</u>

Let $\varepsilon>0$ and $P_\varepsilon^{(1)}\in\mathcal{P}([a,b])$, s.t. $|S_{P,t}(f,\alpha)-\int_a^b f\,d\alpha|<\varepsilon$ $\forall P\supseteq P_\varepsilon^{(1)}$ and tagged points t.

Let $g(x)=f(x)\alpha'(x)$ for $x\in[a,b]$.

$\exists \zeta_k$, s.t. $\Delta\alpha_k=\zeta_k\Delta x_k$

For a fixed tagged partition (P,t) with $P\supseteq P_\varepsilon'$, we have $S_{P,t}(g,x)=\sum_{k=1}^n g(t_k)\Delta x_k$ and $S_{P,t}(f,\alpha)=\sum_{k=1}^n f(t_k)[\alpha(x_k)-\alpha(x_{k-1})]$

By taking their difference, we have $S_{P,t}(f,\alpha)-S_{P,t}(g,x)=\sum_{k=1}^n f(t_k)[\alpha'(\zeta_k)-\alpha'(t_k)]\Delta x_k$

Since α' is continuous on $[a,b]$, it is also uniformly continuous

Let us take $\delta>0$, s.t. $\forall s,t\in[a,b]$, $|s-t|<\delta\Rightarrow|\alpha'(s)-\alpha'(t)|<\varepsilon$

Thus, by taking $P_\varepsilon:=P_\varepsilon^{(1)}\vee P_\varepsilon^{(2)}$ with $P_\varepsilon^{(2)}\in\mathcal{P}([a,b])$ with $\|P_\varepsilon^{(2)}\|\le\frac{\delta}{2}$,

Then, $\forall P\supseteq P_\varepsilon$, we also have $\|P\|\le\frac{\delta}{2}$, and $|S_{P,t}(f,\alpha)-S_{P,t}(g,x)|\le\sum_{k=1}^n M\varepsilon\cdot\Delta x=M(b-a)\varepsilon$, with $M=\sup_{x\in[a,b]}|f(x)|<\infty$ $\square$

COROLLARY

If we take $f\equiv1$, we find $\alpha(b)-\alpha(a)=\int_a^b d\alpha(x)=\int_a^b\alpha'(x)\,dx$, which is the second fundamental theorem of calculus

STEP FUNCTION INTEGRATORS

DEFINITION

Given a function $\alpha:[a,b]\to\mathbb{R}$, it is called a step function if there is a partition $P=\mathcal{P}([a,b])$ s.t. $f|_{(x_{k-1},x_k)}$ is constant for $1\le k\le n$.

We define the jump at x_k to be $\alpha_k:=\alpha(x_k+)-\alpha(x_k-)$, with $\alpha_0:=\alpha(x_0+)-\alpha(x_0)$ and $\alpha_n:=\alpha(x_n)-\alpha(x_n-)$

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LEMMA

Let $c\in[a,b]$, and $\alpha:[a,b]\to\mathbb{R}$ be defined by $\alpha(x)=\begin{cases}\alpha(a), & a\le x<c\\ \alpha(b), & c<x\le b\end{cases}$

Let $f:[a,b]\to\mathbb{R}$, s.t. (1) f or α is continuous from the left at c

(2) f or α is continuous from the right at c

Then, $f\in R(\alpha;a,b)$ and we have $\int_a^b f\,d\alpha=f(c)[\alpha(c+)-\alpha(c-)]$

Proof

Let $P\in\mathcal{P}([a,b])$, s.t. $x_k=c$ for some $1\le k\le n-1$

Consider a choice of tagged points t at P, we have $S_{P,t}(f,\alpha)=f(t_k)[\underbrace{\alpha(x_k)}_{\alpha(c)}-\underbrace{\alpha(x_{k-1})}_{\alpha(c-)}]+f(t_{k+1})(\underbrace{\alpha(x_{k+1})}_{\alpha(c+)}-\underbrace{\alpha(x_k)}_{\alpha(c)})$

$\Rightarrow \Delta=\Delta(P,t)=S_{P,t}(f,\alpha)-f(c)[\alpha(c+)-\alpha(c-)]$

$=\underbrace{[f(t_k)-f(c)][\alpha(c)-\alpha(c-)]}_{\Delta_1}+\underbrace{[f(t_{k-1})-f(c)][\alpha(c+)-\alpha(c)]}_{\Delta_2}$

Let us bound $|\Delta_1|$ from above.

- If α is left continuous at c, i.e. $\alpha(c-)=\alpha(c)$, then $\Delta_1=0$
- If f is left continuous at c. Let $\varepsilon>0$. Take $\delta_1>0$, s.t. $x\in(c-\delta_1,c)\Rightarrow|f(x)-f(c)|<\varepsilon$

  Then, if $\|P\|<\delta_1$, then $t_k\in(c-\delta_1,c)$ and we find $|\Delta_1|<\varepsilon|\alpha(c)-\alpha(c-)|$

The similar applies to Δ_2.

In conclusion, for $\varepsilon>0$, we can find $\delta>0$, s.t. for $P\in\mathcal{P}([a,b])$ satisfying $\|P\|<\delta$ and $c\in\operatorname{Supp}(P)$, then $|\Delta|\le\varepsilon|\alpha(c)-\alpha(c-)|+\varepsilon|\alpha(c+)-\alpha(c)|$ $\square$