Analysis II (Part 2)

Shun/翔海(@shun4midx)

↳ His own notes, feel free to use them for personal use.

4-10-25 (WEEK 8) Shun/翔海(@shun4midx)

TESTS OF CONVERGENCE

THEOREM (D'ALEMBERT'S CRITERION/RATIO TEST)

Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$. Suppose that $a_n>0$ $\forall n\geq N$ for some $N\geq 1$. Suppose $\ell = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \in [0,+\infty]$ is well defined, then:

(1) If $\ell<1$, then Σa_n is convergent

(2) If $\ell>1$, then Σa_n is divergent

(3) If $\ell=1$, and $\frac{a_{n+1}}{a_n}\geq 1$ $\forall n\geq N'$ for some $N'>0$, then Σa_n is divergent

REMARK

In (3), if $\frac{a_{n+1}}{a_n}$ is not always ≥ 1 for large enough n, then they may exhibit different behaviors, eg $\overset{\text{div}}{\Sigma \frac{1}{n}}$ vs $\overset{\text{conv}}{\Sigma \frac{1}{n^2}}$

PROOF OF THEOREM

(1) Suppose $\ell<1$. Set $\ell' := \frac{\ell+1}{2}<1$. By def, $\exists N>0$, s.t. $\frac{a_{n+1}}{a_n}\leq \ell'$, $a_n>0$ for some $N>0$.
By recurrence, we have $a_n\leq (\ell')^{n-N}a_N$ $\forall n\geq N$, thus $\sum_{k=1}^{n}|a_k| = \sum_{k=1}^{N-1}|a_k| + \sum_{k=N}^{n}|a_k| \leq \underbrace{\sum_{k=1}^{N-1}|a_k|}_{\text{const}} + \underbrace{\sum_{k=N}^{n}(\ell')^{k-N}|a_N|}_{\text{conv since geometric series}}$
$\because$ RHS conv $\therefore \sum_{k=1}^{n}|a_k|$ conv too $\therefore \Sigma a_n$ converges. □

(2) Suppose $\ell>1$. Set $\ell' := \frac{\ell+1}{2}>1$, so we can find $N'>0$, s.t. $a_n>0$, $\frac{a_{n+1}}{a_n}\geq \ell'$ $\forall n\geq N$ $\therefore a_n\geq (\ell')^{n-N}a_N$ $\forall n\geq N$, so $a_n\to+\infty$ $\therefore \Sigma a_n$ diverges

(3) Suppose $\ell=1$ and $N>0$, s.t. $a_n>0$, $\frac{a_{n+1}}{a_n}\geq 1$ $\forall n\geq N$. We find $a_n\geq a_N>0$ $\forall n\geq N$. $\therefore a_n\not\to 0$, so Σa_n diverges □

EXAMPLE

For $z\in\mathbb{C}^*$, consider the series $\Sigma \frac{z^n}{n!}$

↳ If $z=0$, of course it converges

↳ If $z\neq 0$, consider the norm, $\frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|}{n+1} \xrightarrow{n\to\infty} 0<1$ $\therefore$ By d'Alembert's criterion, $\Sigma\frac{z^n}{n!}$ converges absolutely

COROLLARY

Let Σu_n be a series with general terms in a Banach space $(W, \|\cdot\|)$. Let $r:=\liminf_{n\to\infty}\frac{\|u_{n+1}\|}{\|u_n\|}$ and $R:=\limsup_{n\to\infty}\frac{\|u_{n+1}\|}{\|u_n\|}$

(1) If $R<1$, then Σu_n converges absolutely

(2) If $r>1$, then Σu_n diverges

(3) If $r\leq 1\leq R$, then we cannot conclude

Proof

The proof is very similar to d'Alembert's criterion, so we prove (1) only as an example.

(1) Suppose $R>1$, let $\ell' := \frac{R+1}{2}$. By the characterization of limsup, $\exists N>0$, s.t. $\frac{\|a_{n+1}\|}{\|a_n\|}\leq \ell' = \frac{R+1}{2}<1$ $\forall n\geq N$.
$\therefore \|a_n\|<(\ell)^{n-N}\|a_N\|$ $\forall n\geq N \Rightarrow \Sigma a_n$ converges absolutely

THEOREM (CAUCHY'S CRITERION/ROOT TEST)

Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$. Suppose that $a_n>0$ $\forall n\geq N$ for some $N>0$. Suppose $\lambda := \lim_{n\to\infty}(a_n)^{\frac{1}{n}} \in [0,+\infty]$ is well-defined, then:

(1) If $\lambda<1$, then Σu_n converges

(2) If $\lambda>1$, then Σu_n diverges

(3) If $\lambda=1$, and $(u_n)^{\frac{1}{n}}>1$ for large enough n, then Σu_n diverges

Proof

(1) Suppose $\lambda<1$. Set $\lambda' := \frac{\lambda+1}{2}<1$. We find $N>0$, s.t. $a_n>0$, $(a_n)^{\frac{1}{n}}\leq\lambda'$ $\forall n\geq N$. $\therefore a_n\leq(\lambda')$ $\forall n\geq N$.
$\therefore \sum_{k=1}^{n}|a_k| = \sum_{k=1}^{N-1}|a_k| + \sum_{k=N}^{n}|a_k| \leq \text{cst} + \underbrace{\sum_{k=N}^{n}|a_k|}_{\text{conv}}$ $\therefore \Sigma a_n$ converges absolutely

The proof to (2) and (3) is very similar. □

COROLLARY

Let $(u_n)_{n\geq 1}$ be a sequence in $\mathbb{C}$. Let $\lambda := \limsup_{n\to\infty}|u_n|^{\frac{1}{n}}$ (It is also applicable to Banach spaces)

1) If $\lambda<1$, then Σu_n converges

2) If $\lambda<1$, then Σu_n diverges

3) If $\lambda=1$, no conclusion

Shun/翔海(@shun4midx)

REMARK

The root test is stronger. Given a sequence $(a_n)_{n\geq 1}$, we have seen: $\liminf\limits_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \underbrace{\liminf\limits_{n\to\infty} \sqrt[n]{a_n} \leq \limsup\limits_{n\to\infty} \sqrt[n]{a_n}}_{\text{Root test}} \leq \underbrace{\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n}}_{\text{Ratio test}}$

If we have $\limsup\limits_{n\to\infty} (a_n)^{\frac{1}{n}} < 1 < \limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$, then the root test applies but ratio test does not apply

Here is an example of such an $(a_n)_{n\geq 1}$

Let $a_n = \frac{(1+(-1)^n)2^{n+1}+1}{4^n} = \begin{cases} \frac{1}{4^n}, & n \text{ odd} \\ 4(\frac{1}{2})^n + \frac{1}{4^n}, & n \text{ even} \end{cases}$

Here, $\frac{a_{2n+2}}{a_{2n+1}} = \frac{4(\frac{1}{2})^{2n+2} + \frac{1}{4^{2n+2}}}{\frac{1}{4^{2n+1}}} \to +\infty$, so $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty$

However, we know $|a_n| \leq \frac{2(2^{n+1})+1}{4^n} \Rightarrow 0 \leq \limsup\limits_{n\to\infty} (a_n)^{\frac{1}{n}} \leq \frac{1}{2} < 1$!

CONDITIONALLY CONVERGENT SERIES

ALTERNATING SERIES

DEFINITION

Let $\sum u_n$ be a series with terms in $\mathbb{R}$.

We say it is an alternating series if $(-1)^n u_n$ has constant sign.

We may also write $\sum u_n = \sum (-1)^n a_n$, where $(a_n)_{n\geq 1}$ is a sequence with constant sign. We may assume $a_n \geq 0$ $\forall n \geq 0$, by a global sign flip.

THEOREM

Let $(a_n)_{n\geq 1}$ be a real sequence. Suppose that a_n is nondecreasing with limit 0. Then, $\sum (-1)^n a_n$ converges and its remainder satisfies, $\forall n \geq 1$, $|R_n| \leq a_{n+1}$, $R_n = \sum_{k=n+1}^{\infty} (-1)^k a_k$

Proof

Consider the odd and even partial sums separately:

- $S_{2n+2} - S_{2n} = a_{2n+2} - a_{2n+1} \leq 0$ $\forall n \geq 1$
- $S_{2n+1} - S_{2n-1} = -a_{2n+1} + a_{2n} \geq 0$ $\forall n \geq 1$

$\therefore$ We have $(S_{2n-1})_{n\geq 1} \nearrow$ and $(S_{2n})_{n\geq 1} \searrow$

$\because$ $S_{2n} - S_{2n-1} = a_{2n} \to 0$

$\therefore$ $(S_{2n})_{n\geq 1}, (S_{2n-1})_{n\geq 1}$ are adjacent sequences, hence they both converge to the same limit S.

Also, $\forall n \geq 1$, we have $S_{2n-1} \leq S_{2n+1} \leq S \leq S_{2n}$, so $\forall n \geq 1$, $|R_{2n}| = |S - S_{2n}| \leq S_{2n} - S_{2n+1} = a_{2n+1}$ and $|R_{2n-1}| = |S - S_{2n-1}| \leq S_{2n} - S_{2n-1} = a_{2n}$

EXAMPLE

Let's study the series $\sum \frac{(-1)^{n+1}}{n}$, now that we proved its convergence.

Recall $H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)$ as $n \to \infty$. Denote $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$.

$\therefore$ $\forall n \geq 1$, $H_{2n} - S_{2n} = \sum_{k=1}^{n} \frac{2}{2k} = H_n$

In other words, $S_{2n} = H_{2n} - H_n = (\ln(2n) + \gamma + o(1)) - (\ln n + \gamma + o(1)) = \ln 2 + o(1)$ when $n \to \infty$.

$\therefore$ $\sum \frac{(-1)^{n+1}}{n}$ converges to $\ln 2$, additionally, $|S_n - \overset{S}{\ln 2}| = |R_n| \leq \frac{1}{n+1}$ (We can theoretically use this sequence to compute $\ln 2$ but its rate of convergence is too slow and hence it is not practical.)

DIRICHLET'S TEST

Consider a series $\sum u_n$ in a Banach space. Assume $\forall n \geq 1$, $u_n = a_n b_n$. We write $S_n = \sum_{k=1}^{n} b_k$ for $n \geq 1$ and $S_0 = 0$.

ABEL'S TRANSFORM

For every $n \geq 0$, we have $\sum_{k=1}^{n} u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n$ (IBP but Σ version)

Proof

$\forall n \geq 0$, we have $\sum_{k=1}^{n} u_k = \sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k (S_k - S_{k-1}) = \sum_{k=1}^{n} a_k S_k - \sum_{k=0}^{n-1} a_{k+1} S_k = \sum_{k=0}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n$

Shun/翔海(@shun4midx)

THEOREM (DIRICHLET TEST)

Let $\sum u_n$ be a series with general terms in a Banach space $(W, \|\cdot\|)$. Suppose that its general term u_n writes $u_n = a_n b_n$ with $a_n \in \mathbb{R}$ and $b_n \in W$ for all $n \geq 1$, and satisfies:

(i) the sequence $(a_n)_{n\geq 1}$ is nonnegative, nonincreasing, and tends to 0

(ii) the series $\sum b_n$ is bounded

Then, $\sum u_n$ converges.

Proof

Let us apply Abel transform eq to $\sum u_n$. $\therefore \forall n \geq 0$, $\sum_{k=1}^{n} u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n$, where S_n is the n-th partial sum of series $\sum b_n$.

Let $M > 0$, s.t. $|S_n| = |\sum_{k=1}^{n} b_k| \leq M$ $\forall n \geq 1$. Then, $|a_n S_n| \leq |a_n| M \xrightarrow{n\to\infty} 0$, so the series $\sum u_n$ and $\sum (a_n - a_{n+1}) S_n$ share the same behavior.
Moreover, $\forall k \geq 0$, $|(a_k - a_{k+1}) S_k| \leq (a_k - a_{k+1}) M$ since $(a_k)_{k\geq 1}$ is nonincreasing.

$\therefore \forall n \geq 0$, $\sum_{k=1}^{n} |(a_k - a_{k+1}) S_k| \leq \sum_{k=1}^{n} (a_k - a_{k+1}) M = (a_1 - a_{n+1}) M \leq a_1 M$. Hence, $\sum (a_n - a_{n+1}) S_n$ is absolutely convergent and hence convergent. □

EXAMPLES (Applications of Dirichlet Test)

(1) Let $(a_n)_{n\geq 0}$ be a nonincreasing sequence tending to 0. We know the alternating series $\sum (-1)^n a_n$ is convergent since the partial sum $|(-1)^1 + (-1)^2 + \dots + (-1)^n| \leq 1$ is bounded (see "bounded" suffices, no need for "converge")

(2) Let $(a_n)_{n\geq 0}$ be a nonincreasing sequence tending to 0. Let $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Consider the series $\sum a_n e^{in\theta}$.
$\forall n \geq 0$, we have $|1 + e^{i\theta} + \dots + e^{in\theta}| = \left|\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right| = \left|\sin\left(\frac{(n+1)\theta}{2}\right) \div \sin\left(\frac{\theta}{2}\right)\right| \leq \frac{1}{|\sin(\frac{\theta}{2})|}$ $\therefore$ The series converges if $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$

REARRANGEMENT OF SERIES

Let $\sum u_n$ be a series with terms in $(W, \|\cdot\|)$

DEFINITION

we don't say "permutation" cuz it can be uncountable

We say $\sum v_n$ is a **rearrangement** of $\sum u_n$ if $\exists$ a bijection $\varphi: \mathbb{N} \to \mathbb{N}$, s.t. $v_n = u_{\varphi(n)}$ $\forall n \geq 1$

THEOREM

If $\sum u_n$ converges absolutely to s, then any rearrangement of $\sum u_n$ is also absolutely convergent with the same limit

Proof

Let $\varphi: \mathbb{N} \to \mathbb{N}$ be a bijection and $\sum v_n$ with $v_n = u_{\varphi(n)}$, $n \geq 1$, be a rearrangement of $\sum u_n$.
$\forall n \geq 1$, $\sum_{k=1}^{n} \|v_k\| = \sum_{k=1}^{n} \|u_{\varphi(k)}\| \leq \sum_{n=1}^{\infty} \|u_n\| < +\infty$ $\therefore \sum_{k=1}^{\infty} \|v_k\|$ converges $\Rightarrow \sum v_k$ absolutely converges

Now, for "same limit",

partial sums

Let $\varepsilon > 0$, take $N \geq 1$, s.t. $\sum_{n=N+1}^{\infty} \|u_n\| < \varepsilon$. For $n \geq 0$, $S_n = \sum_{k=1}^{n} u_k$, $T_n = \sum_{k=1}^{n} v_k$

From the ε we chose, we know that $\|S_N - S\| = \|\sum_{n\geq N+1} u_n\| < \varepsilon$

lmao shun4midx mention

characterization of idx mapping between u, v.

Note that $\{1, \dots, N\}$ is a finite set, so it has an upper bound $\varphi(M)$, that is $\{1, \dots, N\} \subseteq \{\varphi(1), \dots, \varphi(M)\}$

For $n \geq M$, $\|T_n - S_N\| = \|\sum_{k=1}^{n} v_k - \sum_{k=1}^{N} u_k\| = \|\sum_{k=1}^{n} u_{\varphi(k)} - \sum_{k=1}^{N} u_k\| = \|\sum_{k\in A} u_k\| \leq \sum_{k\in A} \|u_k\| \leq \sum_{k\geq N+1} \|u_k\| < \varepsilon$, where $A = \{\varphi(1), \dots, \varphi(n)\} \setminus \{1, \dots, N\}$

$\therefore$ For $n \geq M$, $\|T_n - S\| \leq \|T_n - S_N\| + \|S_N - S\| < 2\varepsilon$ □

WHAT HAPPENS IF $\sum u_n$ DOES NOT CONVERGE ABSOLUTELY?

EXAMPLE

Recall: We know that $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = \ln 2$ but is not absolutely convergent

A possible rearrangement: $(1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + \dots + (\frac{1}{2k-1} - \frac{1}{2k-2}) - \frac{1}{4k} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots = \frac{1}{2} \ln 2$

RIEMANN SERIES THEOREM

Shun/翔海(@shun4midx)

Let $\sum u_n$ be a real-valued series that is conditionally convergent

For $-\infty \leq x \leq y \leq \infty$, then $\exists$ a bijection $\varphi: \mathbb{N} \to \mathbb{N}$, s.t. the rearrangement $\sum u_{\varphi(n)}$ satisfies $\liminf\limits_{n\to\infty} \sum_{k=1}^{n} u_{\varphi(k)} = x$ and $\limsup\limits_{n\to\infty} \sum_{k=1}^{n} u_{\varphi(k)} = y$

(Proof sketch: Let $\sum a_n$ have the sum of all +ve terms, $\sum b_n$ have the sum of all -ve terms. Both diverge, so we can find distinct values to match)

REMARK

If we take $x=y$, then the arrangement satisfies $\sum u_{\varphi(n)} = x = y$ (i.e. it conv)

CAUCHY SERIES

DEFINITION

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $(A, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. Consider a binary operator $A \times A \to A$ denoted by $\cdot$

(1) We say that $(A, \cdot)$ is an algebra if:

- ↳ Right distributivity: $\forall x,y,z \in A$, $(x+y)\cdot z = x\cdot z + y \cdot z$
- ↳ Left distributivity: $\forall x,y,z \in A$, $z\cdot(x+y) = z\cdot x + z \cdot y$
- ↳ Scalar multiplication: $\forall x,y,z \in A$, $a,b \in \mathbb{K}$, $(ax)\cdot(by) = (ab)(x\cdot y)$

(2) We say $(A, \|\cdot\|)$ is a normed algebra if $(A,\cdot)$ is an algebra and $\forall x,y \in A$, $\|xy\| \leq \|x\| \|y\|$

EXAMPLE

(1) $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$ are normed algebras

(2) For $n \geq 1$, $M_{n\times n}(\mathbb{K})$ equipped with matrix norm $|||\cdot|||$ is a normed algebra, where $|||A||| = \sup\limits_{x\in\mathbb{K}^n, \|x\|=1} \|Ax\|$

(3) Let U be a normed vector space. $\mathcal{L}(U) = \{\text{linear maps}: U \to U\}$ with the operator norm $|||\cdot|||$ is a normed algebra

THEOREM (CAUCHY PRODUCT)

Let $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$ be absolutely convergent series with general terms in a normed algebra $(A, \cdot, \|\cdot\|)$. We define its Cauchy series $\sum_{n\geq 0} c_n$ to be $\forall n \geq 0$, $c_n = \sum_{k=0}^{n} a_n b_{n-k}$. Then, $\sum_{n\geq 0} c_n$ converges absolutely to $(\sum_{n\geq 0} a_n)(\sum_{n\geq 0} b_n)$

Proof

Denote $A := \sum_{p=0}^{\infty} \|a_p\|$ and $B := \sum_{q=0}^{\infty} \|b_q\|$

Let $n \geq 0$, then $\sum_{k=0}^{n} \|c_k\| \leq \sum_{k=0}^{n} \Big(\sum_{\substack{p+q=k \\ p,q\geq 0}} \|a_p\| \|b_q\| \Big) \leq \sum_{0\leq p,q \leq n} \|a_p\| \|b_q\| = \Big(\sum_{p=0}^{n} \|a_p\|\Big)\Big(\sum_{q=0}^{n} \|b_q\|\Big) \leq AB$ $\therefore \sum c_n$ converges absolutely (rearrangement ok)

Now, define $\forall n \geq 0$, $\Delta_n = \sum_{k=0}^{2n} c_k - \Big(\sum_{p=0}^{n} a_p\Big)\Big(\sum_{q=0}^{n} b_q\Big)$

Thus, $\forall n \geq 0$, $\Delta_n = \sum_{\substack{p+q\leq 2n \\ p,q\geq 0}} a_p b_q - \sum_{0 \leq p,q \leq n} a_p b_q = \sum_{p\geq n+1, q\geq 0, p+q\leq 2n} a_p b_q + \sum_{q \geq n+1, p\geq 0, p+q \leq 2n} a_p b_q$

$\therefore$ By Δ ineq, $\forall n \geq 0$, $\|\Delta_n\| \leq \sum_{p\geq n+1, q\geq 0, p+q\leq 2n} \|a_p\| \|b_q\| + \sum_{q\geq n+1, p\geq 0, p+q\leq 2n} \|a_p\| \|b_q\|$

$\leq \sum_{p\geq n+1, q\geq 0} \|a_p\| \|b_q\| + \sum_{q\geq n+1, p\geq 0} \|a_p\| \|b_q\| = B \sum_{p=n+1}^{\infty} \|a_p\| + A \sum_{q=n+1}^{\infty} \|b_q\| \xrightarrow{n\to\infty} 0$ □

EXAMPLE (Applications)

1) On $\mathbb{R}$, define the exponential function $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$

For $x,y \in \mathbb{R}$, $e^x \cdot e^y = \Big(\sum_{n=1}^{\infty} \frac{x^n}{n!}\Big)\Big(\sum_{n=1}^{\infty} \frac{y^n}{n!}\Big) = \sum_{n\geq 0} c_n$

given commutativity

Here, we know $c_n = \sum_{\substack{p+q=k \\ p,q\geq 0}} \frac{x^p}{p!} \frac{y^q}{q!} = \sum \binom{p+q}{p} \frac{1}{(p+q)!} x^p x^q = \frac{1}{n!}(x+y)^n \Rightarrow e^x e^y = e^{x+y}$

Notably, Cauchy product makes us able to extend this concept to other normed algebras.

4-17-25 (WEEK 9) Shun/翔海 (@shun4midx)

DOUBLE SEQUENCES, DOUBLE SERIES

DOUBLE SEQUENCES AND DOUBLE LIMITS

Let $(W, \|\cdot\|)$ be a Banach space. Let $(u_{m,n})_{m,n\geq 1}$ be a double sequence with values in W (intuition: sequence on a 2D grid)

DEFINITION
Let $\ell \in W$. We say $(u_{m,n})_{m,n\geq 1}$ converges to ℓ if for every $\varepsilon>0$, $\exists N>0$, s.t. $\|u_{m,n}-\ell\|<\varepsilon$ $\forall m,n\geq N$, denoted by $\lim\limits_{m,n\to+\infty} u_{m,n}=\ell$, which is called the limit or double limit of $(u_{m,n})_{m,n\geq 1}$.

REMARK
This is simply another way to formulate Cauchy's criterion. The sequence $(u_m)_{m\geq 1}$ satisfies Cauchy's criterion if $\lim\limits_{m,n\to+\infty}\|u_m-u_n\|=0$

EXAMPLE
Consider $(u_{m,n})_{m,n\geq 1}$ to be defined by $u_{m,n}=\mathbb{1}_{m\geq n}$ $\forall m,n\geq 1$

We have $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} u_{m,n}=1$, $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} u_{m,n}=0 \Rightarrow$ The order of limits matter (iterated limits)

THEOREM
Suppose that,

(i) $\lim\limits_{m,n\to+\infty} u_{m,n}=\ell\in W$

(ii) $\forall m\geq 1$, $\lim\limits_{n\to\infty} u_n$ exists

Then, $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} u_{m,n}=\ell$

Proof

Let $\varepsilon>0$. By (i), we can find $N>0$, s.t. $\|u_{m,n}-\ell\|\leq\varepsilon$ $\forall m,n\geq N$.

By (ii), define $\ell_m := \lim\limits_{n\to\infty} u_{m,n}$

$\forall m\geq 1$, $\exists N'=N'(m)$, s.t. $\|\ell_m-u_{m,n}\|\leq\varepsilon$ $\forall n\geq N'(m)$.

$\forall m\geq N$, we have $\|\ell_m-\ell\|\leq\|\ell_m-u_{m,n}\|+\|u_{m,n}-\ell\|\leq 2\varepsilon$ $\forall n\geq\max(N,N'(m))$. $\square$

THEOREM
Let $(u_{m,n})_{m,n\geq 1}$ be a sequence with terms in a Banach space. Then, (1) $\Leftrightarrow$ (2)

(1) $\forall n\geq 1$, $\sum\limits_{m\geq 1} u_{m,n}$ is absolutely convergent, and the series $\sum\limits_{n\geq 1}\left(\sum\limits_{m\geq 1}\|u_{m,n}\|\right)$ converges

(2) $\forall m\geq 1$, $\sum\limits_{n\geq 1} u_{m,n}$ is absolutely convergent, and the series $\sum\limits_{m\geq 1}\left(\sum\limits_{n\geq 1}\|u_{m,n}\|\right)$ converges

And when (1) or (2) holds, we have $\sum\limits_{m=1}^{\infty}\left(\sum\limits_{n=1}^{\infty} u_{m,n}\right)=\sum\limits_{n=1}^{\infty}\left(\sum\limits_{m=1}^{\infty} u_{m,n}\right)$

REMARK
If $u_{m,n}\geq 0$ $\forall m,n\geq 1$, we can write $\sum\limits_{m\geq 1}\sum\limits_{n\geq 1} u_{m,n}=\sum\limits_{n\geq 1}\sum\limits_{m\geq 1} u_{m,n}$ without checking any condition. They are either both $+\infty$ or finite, and the thm above guarantees equality.

PROOF OF THEOREM
By symmetry, it suffices to prove (1) $\Rightarrow$ (2).

Suppose (1) holds. $\forall n\geq 1$, define $A_n := \sum\limits_{m\geq 1}\|u_{m,n}\|<+\infty$

- Fix $m\geq 1$. $\sum\limits_{n\geq 1}^{N}\|u_{m,n}\|\leq\sum\limits_{n\geq 1}^{N} A_n\leq\sum\limits_{n\geq 1}^{\infty} A_n<+\infty$. Hence, $\sum\limits_{n\geq 1} u_{m,n}$ is absolutely convergent
- Let $M\geq 1$. By linearity on finitely many converging series, $\sum\limits_{m\geq 1}^{M}\sum\limits_{n\geq 1}^{\infty}\|u_{m,n}\|=\sum\limits_{n\geq 1}^{\infty}\sum\limits_{m\geq 1}^{M}\|u_{m,n}\|\leq\sum\limits_{n\geq 1}^{\infty} A_n<+\infty$. Hence, $\sum\limits_{m\geq 1}\sum\limits_{n\geq 1}\|u_{m,n}\|$ converges
- Check the LHS of the equality is well-defined when (2) holds:

  When (2) holds, $\forall m$, $\sum\limits_{n\geq 1} u_{m,n}$ converges absolutely $\Rightarrow\sum\limits_{n\geq 1} u_{m,n}$ converges. Moreover, $\sum\limits_{m\geq 1}\sum\limits_{n\geq 1}\|u_{m,n}\|<+\infty$ by (2) $\Rightarrow\sum\limits_{m\geq 1}\|\sum\limits_{n\geq 1} u_{m,n}\|$ converges

  $\therefore$ $\sum\limits_{m\geq 1}\sum\limits_{n\geq 1} u_{m,n}$ converges

Similarly, (1) holds $\Rightarrow$ RHS of equality is well-defined. $\therefore$ Both terms are well-defined

Now, we show equality holds.
Let $S_n = \sum_{p=1}^{n}\sum_{q=1}^{n} u_{p,q}$ for $n \geq 1$. Notice, $S_n = \sum_{q=1}^{n}\sum_{p=1}^{n} u_{p,q}$.

Goal: Show that $S_n \xrightarrow{n\to\infty}$ LHS or RHS of equality, then conclude by symmetry.

We introduce the following quantities: $\forall m, q \geq 1$, $a_{m,q} = \sum_{p=1}^{m} u_{p,q}$, $a_q = \sum_{p\geq 1} u_{p,q}$
Let $\varepsilon > 0$. Let $Q > 0$, s.t. $\sum_{q\geq Q} A_q \leq \varepsilon$

For $n \geq Q$, we have:
$\underbrace{\sum_{q=1}^{\infty} a_q - S_n}_{\text{(RHS)}} = \sum_{q=1}^{\infty} a_q - \sum_{q=1}^{n} a_q = \sum_{q=1}^{Q}(a_q - a_{n,q}) + \sum_{q=Q+1}^{n}(a_q - a_{n,q}) + \sum_{q=n+1}^{\infty} a_q$

By Δ-ineq, we get: $\left\|\sum_{q=1}^{\infty} a_q - S_n\right\| \leq \left\|\sum_{q=1}^{Q}(a_q - a_{n,q})\right\| + \sum_{q=Q+1}^{\infty} A_q \leq \left\|\sum_{q=1}^{Q}(a_q - a_{n,q})\right\| + \varepsilon$
We cannot take the limit, but we can take limsup: $\limsup_{n\to\infty} \left\|\sum_{q=1}^{\infty} a_q - S_n\right\| \leq \varepsilon$
As the choice of $\varepsilon > 0$ is arbitrary, hence: $\limsup_{n\to\infty}\left\|\sum_{q=1}^{\infty} a_q - S_n\right\| = 0 \Rightarrow \lim_{n\to\infty}\left\|\sum_{q=1}^{\infty} a_q - S_n\right\| = 0$, that is, $S_n \xrightarrow{n\to\infty} \sum_{q=1}^{\infty} a_q$. □
(Limsup not only helps proving limit exists, but also proves it equals 0 at the same time)

INFINITE PRODUCT

CONVERGENCE AND DIVERGENCE

DEFINITION

Let $(u_n)_{n\geq 1}$ be a sequence in $K = \mathbb{R}$ or $\mathbb{C}$
Define $P_0 = 1$, $\underbrace{P_n}_{n\text{th partial product}} = \prod_{k=1}^{n} \underbrace{u_k}_{k\text{th factor}}$, $n \geq 1$, called partial products

DEFINITION

Define $Z := \{n \geq 1 \mid u_n = 0\}$

1) If $|Z| = \infty$, then we say $\prod u_n$ diverges to 0
2) If $Z = \emptyset$, then:
   (a) If $P_n \to P \neq 0$, we say the infinite product $\prod u_n$ converges to P, denoted as $\prod u_n = P$
   (b) If $P_n \xrightarrow{n\to\infty} 0$, we say the infinite product diverges to 0
   (c) Otherwise, we say it diverges
3) If $|Z| < +\infty$, then let $N > 0$, s.t. $u_n \neq 0$ $\forall n \geq N$
   Define $v_n = u_{N+n-1}$ $\forall n \geq 1$, $P_0' = 1$, $P_n' = \prod_{k=1}^{n} v_k = \prod_{k=N}^{N+n-1} u_k$ (omit all zero terms)
   (a) If $\prod v_n$ converges to $P' \neq 0$, then we say $\prod u_n$ converges to $u_1 \cdots u_{N-1} P'$ (which is a "convergence" to 0, w/o the 0 it still converges)
   (b) If $\prod v_n$ diverges to 0, then we say $\prod u_n$ diverges to 0
   (c) Otherwise, we say it diverges

REMARK

Removing/adding finitely many zeroes in $(u_n)_{n\geq 1}$ does not change the behavior (convergence or divergence) of the product.

PROPOSITION (CAUCHY'S CRITERION)

The infinite product $\prod u_n$ converges $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n \geq N, k \geq 1$, we have $|u_{n+1} \cdots u_{n+k} - 1| < \varepsilon$

minus 1 cuz to conv, the elements must approach 1

$\frac{|P_{n+k} - P_n|}{|u_1 \cdots u_n|} = \left|\frac{P_{n+k} - P_n}{P_n}\right|$

Proof

WLOG, let's assume $u_n \neq 0$ $\forall n \geq 1$.

- Suppose $\prod u_n = P \neq 0$. Then, $P_n \xrightarrow{n\to\infty} P$, so $(P_n)_{n\geq 1}$ is bounded
  Let $M > 0$, s.t. $|P_n| \geq M$ $\forall n \geq 1$
  Let $\varepsilon > 0$, take $N > 0$, s.t. $|P_{n+k} - P_n| < \varepsilon$ $\forall n \geq N, k > 0$.
  $\therefore \left|\frac{P_n - P_{n+k}}{P_n}\right| < \frac{\varepsilon}{|P_n|} \leq \frac{\varepsilon}{M}$

Shun/羊羽海(@shun4midx)

- Suppose that the Cauchy's criterion is satisfied

  Let $\varepsilon = \frac{1}{2}$. Take $N > 0$, s.t. $|Q_n - 1| < \varepsilon = \frac{1}{2}$ where $Q_n = \prod_{k=N+1}^{n} u_k$ $\forall n \geq N$. $\therefore \frac{1}{2} < |Q_n| < \frac{3}{2}$

  Now, let $\varepsilon > 0$, $N' \geq N$, then we have the following $\forall n \geq N'$

  Thus, $\left|\frac{Q_{n+k}}{Q_n} - 1\right| = \left|\frac{Q_{n+k} - Q_n}{Q_n}\right| < \varepsilon \Rightarrow |Q_{n+k} - Q_n| < \frac{3}{2}\varepsilon$

  $\therefore$ $(Q_n)_{n\geq 1}$ converges $\square$

Shun/翔海 (@shun4midx)

THEOREM

Let $(a_n)_{n\geq 1}$ be a sequence with strictly positive terms. Then, $\prod(1+a_n)$ converges $\Leftrightarrow$ $\sum a_n$ converges

Proof

Observe: $\prod(1+a_n)$ converges $\Leftrightarrow$ $\sum \ln(1+a_n)$ converges

"$\Rightarrow$": Suppose $\prod(1+a_n)$ converges, then $\sum \ln(1+a_n)$ converges.

$\therefore \ln(1+a_n) \xrightarrow{n\to\infty} 0$, i.e. $a_n \xrightarrow{n\to\infty} 0$

$\therefore$ We have $\ln(1+a_n) \sim a_n$ as $n\to\infty$

By comparison thm, $\sum \ln(1+a_n)$ conv $\Rightarrow$ $\sum a_n$ converges

"$\Leftarrow$": Suppose $\sum a_n$ conv, so $a_n \xrightarrow{n\to\infty} 0$

$\therefore \ln(1+a_n) \sim a_n$

By comparison thm, $\sum a_n$ conv $\Rightarrow$ $\sum \ln(1+a_n)$ conv $\therefore \prod(1+a_n)$ conv $\square$

REMARK

The positivity assumption is important!

Consider $a_n = \frac{(-1)^n}{\sqrt{n}}$ for $n\geq 1$

↳ $\sum a_n$ conv since alternating series

↳ For $n\geq 1$, $(1+a_{2n})(1+a_{2n+1}) = \left(1+\frac{1}{\sqrt{2n}}\right)\left(1-\frac{1}{\sqrt{2n+1}}\right) = 1+\frac{1}{\sqrt{2n}}-\frac{1}{\sqrt{2n+1}}-\frac{1}{\sqrt{2n(2n+1)}} = 1+\frac{1}{\sqrt{2n}}-\frac{1}{\sqrt{2n}}\left(\frac{1}{\sqrt{1+\frac{1}{2n}}}\right)-\frac{1}{\sqrt{2n(2n+1)}} = 1-\frac{1}{2n}+o(n^{-\frac{3}{2}})$

However, $\prod(1-\frac{1}{2n})$ div because $\sum\frac{1}{2n}$ div $\Rightarrow \prod(1+a_{2n})(1+a_{2n+1})$ div $\Rightarrow \prod(1+a_n)$ div

REMARK

If $(a_n)_{n\geq 1}$ is a sequence with $a_n\in(-1,0)$, we still have the same proof for "$\sum a_n$ conv $\Leftrightarrow \prod(1+a_n)$ conv"

DEFINITION

Let $(a_n)_{n\geq 1}$ be a nonzero complex-valued sequence (or over a normed algebra). We say that $\prod(1+a_n)$ conv absolutely if $\prod(1+|a_n|)$ conv

PROPOSITION

If $\prod(1+a_n)$ conv abs, then it must also conv

Proof

For $n\geq 1$ and $k\geq 0$, by $\triangle$ ineq, we have $\left|\prod_{i=n}^{n+k}(1+a_i)-1\right| \leq \prod_{j=1}^{k}(1+|a_{n+j}|)-1$ (simple expansion)

Therefore, if $\prod(1+|a_n|)$ satisfies the Cauchy's condition, so does $\prod(1+a_n)$ $\square$

APPLICATION TO THE RIEMANN ζ FUNCTION

Let $(p_n)_{n\geq 1}$ be the sequence of ordered primes, i.e. $p_1=2$, $p_2=3$, $p_3=5$, ...

THEOREM (EULER'S PRODUCT)

For $s>1$, we have $\zeta(s) := \sum_{n\geq 1}\frac{1}{n^s} = \prod_{k=1}^{\infty}\frac{1}{1-p_k^{-s}}$. Moreover, the infinite product conv abs

Proof

For $n\geq 1$, let $P_n = \prod_{k=1}^{n}\frac{1}{1-p_k^{-s}}$

Goal: Show that $P_n \xrightarrow{n\to\infty} \zeta(s) \neq 0$

For any $k\geq 1$, we have $\frac{1}{1-\underbrace{p_k^{-s}}_{\in(0,1)}} = \underbrace{\sum_{m\geq 0}(p_k^{-s})^m}_{\text{Geom series}} = \sum_{m\geq 0}p_k^{-sm} = 1+\sum_{m\geq 1}p_k^{-sm} =: 1+a_k$

Note that $a_k>0$ $\forall k\geq 1$. $\sum_{k=1}^{N}a_k = \sum_{k=1}^{N}\sum_{m_k\geq 1}p_k^{-sm_k} \leq \zeta(s)$

$\therefore \sum a_k$ conv ($\because$ it abs conv), so $\prod(1+a_n)$ conv

Hence, $P_n = \prod_{k=1}^{n}\frac{1}{1-p_k^{-s}} = \prod_{k=1}^{n}\sum_{m\geq 0}\frac{1}{p^{ms}} = \sum_{m_1,\dots,m_n\geq 0}\frac{1}{(p_1^{m_1}\cdots p_n^{m_n})^s}$. Define $A_n = \{N\in\mathbb{N} \mid N$ has all prime factors along $p_1,\dots,p_n\}$. $\therefore P_n = \sum_{N\in A_n}\frac{1}{N^s} \Rightarrow |P_n-\zeta(s)| \leq \sum_{N\geq p_{n+1}}\frac{1}{N^s}$ $\square$

REMARK

We may show that $\sum_{k\geq 1}\frac{1}{p_k}$ div using the Riemann zeta function.

COMPLEMENT ON RIEMANN INTEGRALS

Shun / 翔海 (@shun4midx)

DEFINITION

↳ A function $f:[a,b]\to\mathbb{R}$ is said to be **piecewise continuous** on $[a,b]$ if $\exists$ a partition $P=(x_k)_{0\le k\le n}\in\mathcal{P}([a,b])$, s.t. $f|_{(x_{i-1},x_i)}$ is conti and can be extended continuously to $[x_{i-1},x_i]$, $\forall i$

↳ Let $I\subseteq\mathbb{R}$ be a subset. A function $f:I\to\mathbb{R}$ is said to be **piecewise continuous** if $f|_I$ is p.c. $\forall$ segments $J\subseteq I$

↳ Let $PC(I,\mathbb{R})$ be the set of piecewise continuous functions on I

↳ For any normed vector space $(W,\|\cdot\|)$, we define $PC(I,V)$ similarly

EXAMPLES OF PIECEWISE CONTINUOUS FUNCTIONS

1) $\begin{cases} x\longmapsto \frac{1}{x} \\ \mathbb{R}^*:=\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}\end{cases}$

2) $x\longmapsto \ln x$

   $(0,+\infty)\mapsto\mathbb{R}$

3) $x\longmapsto \sin(\frac{1}{x})$

   $\mathbb{R}^*\longrightarrow\mathbb{R}$

PROPOSITION

Let $I=[a,b]$ be a segment of $\mathbb{R}$. If $f:I\longrightarrow\mathbb{R}$ is a piecewise continuous function, then it is **bounded** and **Riemann-integrable** on I.

Proof

Let $P=(x_k)_{0\le k\le n}\in\mathcal{P}([a,b])$ be a partition satisfying the definition. Then, on $[x_{i-1},x_i]$, we may define a conti $g_i:[x_{i-1},x_i]\to\mathbb{R}$, s.t. $g_i|_{(x_{i-1},x_i)}\equiv f|_{(x_{i-1},x_i)}$

Therefore, $g_i\in R(x;x_{i-1},x_i)$ and $f\in R(x;x_{i-1},x_i)$

To conclude, we use the cyclic relation to observe $f\in R(x;a,b)$ □

INTEGRABILITY ON AN INTERVAL

Let I be an interval, $PC_+(I):=PC_+(I,\mathbb{R}_+)$

DEFINITION

Let $f\in PC_+(I)$. We say that f is **integrable** on I, if $\exists M>0$, s.t. $\int_J f\le M$ for any segment $J\subseteq I$, and we write $\int_I f=\sup_{J\subseteq I}\int_J f$

$\lfloor J$ is a segment

REMARK

If I is an interval, $a=\inf I$, $b=\sup I$, we may write $\int_I f=\int_a^b I$ (this is the generalization of notation)

PROPOSITION

Let $f\in PC_+(I)$ be an integrable function. Then, for any sequence of segments $(J_n=[a_n,b_n])_{n\ge1}$ with $\forall n\ge1$, $J_n\subseteq J_{n+1}\subseteq\dots\subseteq I$ and $\bigcup_{n\ge1}J_n=I$, we have $\int_I f=\sup_{n\ge1}\int_{J_n}f=\lim_{n\to\infty}\int_{J_n}f=\lim_{n\to\infty}\int_{a_n}^{b_n}f(x)dx$ (Focus is $\lim_{n\to\infty}\int_{J_n}f=\int_I f$)

Proof

Given such a sequence $(J_n)_{n\ge1}$, we want to show $\int_{J_n}f\xrightarrow{n\to\infty}\int_I f$

↳ $\forall n\ge1$, $J_n\subseteq I$ is a subsegment $\Rightarrow\int_{J_n}f\le\int_I f\Rightarrow\limsup_{n\to\infty}\int_{J_n}f\le\int_I f$

↳ Given $\varepsilon>0$, by the characterization of sup, we may find a subsegment $J\subseteq I$, s.t. $\int_J f+\varepsilon\ge\int_I f$

Since $\bigcup_{n\ge1}J_n=I$ and $(J_n)_{n\ge1}$ is increasing for inclusion, we may find $N>0$, s.t. $J_n\supseteq J$ $\forall n\ge N$.

$\Rightarrow\int_I f\le\int_{J_n}f+\varepsilon$ $\forall n\ge N$

$\Rightarrow\int_I f\le\liminf_{n\to\infty}\int_{J_n}f+\varepsilon$

This relation holds $\forall\varepsilon>0$, so we have $\int_I f\le\liminf_{n\to\infty}\int_{J_n}f$

$\therefore\lim_{n\to\infty}\int_{J_n}f=\int_I f$ □

 Shun/羊羽海 (@shun4midx)

EXAMPLE

Fix $\lambda>0$ and consider $\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, $x \longmapsto e^{-\lambda x}$, which is nonnegative. ($\mathbb{R}_{\geq 0} = [0,+\infty)$)

For every integer $n\geq 1$, take $J_n=[0,\infty]$, and $\int_J f(x)\,dx = \int_0^n e^{-\lambda x}\,dx = \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^n = \frac{1-e^{-\lambda n}}{\lambda} \xrightarrow{n\to\infty} \frac{1}{\lambda}$. $\therefore \int_J f\,dx = \frac{1}{\lambda}$

EXAMPLE

The function $\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, $x \longmapsto |\sin x|$ is nonnegative and not integrable

For $k\in\mathbb{N}$, $\int_{k\pi}^{(k+1)\pi} |\sin x|\,dx = \int_0^\pi \sin x\,dx = 2 \Rightarrow \int_0^{n\pi} |\sin x|\,dx = 2n \xrightarrow{n\to\infty} \infty$

$\therefore$ The integral isn't well defined.

EXAMPLE (RIEMANN INTEGRALS)

1) $t\longmapsto t^{-\alpha}$ is integrable on $[a,+\infty)$ iff $\alpha>1$ for some $a>0$.
2) $t\longmapsto t^{-\alpha}$ is integrable on $[0,a)$ iff $\alpha<1$ for some $a>0$.
3) For $a<b$, $t\longmapsto (b-t)^{-\alpha}$ is integrable on $[a,b)$ iff $\alpha<1$
4) For $a<b$, $t\longmapsto (t-a)^{-\alpha}$ is integrable on $(a,b]$ iff $\alpha<1$

EXAMPLE (BERTRAND'S INTEGRALS)

Fix $\alpha,\beta\in\mathbb{R}$, consider $t\longmapsto t^{-\alpha}|\ln t|^{-\beta}$

1) For $a>1$, if $\alpha>1$, it is integrable on $[a,+\infty)$ or $\alpha=1$ and $\beta>1$
2) For $a\in(0,1)$, if $\alpha<1$, it is integrable on $(0,a]$ or $\alpha=1$ and $\beta>1$

DEFINITION

Let $I\subseteq\mathbb{R}$ be an interval and $(W,\|\cdot\|)$ is a Banach space. A function $f: I\longrightarrow W$ is called **integrable** if $\|f\|$ is integrable

For a sequence $(J_n)_{n\geq 0}$ satisfying for $n\geq 1$, $\underbrace{J_n \subseteq J_{n+1} \subseteq \ldots \subseteq I, \text{ and } \bigcup_{n\geq 1} J_n = I}_{\text{"interval\_def"}}$, define $\int_I f = \lim_{n\to\infty} \int_{J_n} f \in W$

Denote $L^1(I,W) := \{f: I\longrightarrow W \mid \int_I \|f\| < +\infty\}$ (L^1 norm stuff...)

PROPOSITION

The definition of $\int_I f := \lim_{n\to\infty}\int_{J_n} f$ **does not depend** on the choice of $(J_n)_{n\geq 0}$ if it satisfies interval_def

REMARK

- For $[a,b)$, $-\infty<a<b<\infty$, we may consider $(J_n=[a,b-\frac{1}{n}])_{n\geq 1}$
- For $[a,+\infty)$, $-\infty<a<+\infty$, we may consider $(J_n=[a,n])_{n\geq 1}$

PROOF OF PROPOSITION

We do so by considering two steps:

(1) Given $(J_n)_{n\geq 1}$ satisfying interval_def, check that $\lim_{n\to\infty}\int_{J_n} f$ is well-defined
(2) Given $(J_n)_{n\geq 1}$ and $(K_n)_{n\geq 1}$ satisfying interval_def, check that their integrals are equal

Step 1

Let $(J_n=[a_n,b_n])_{n\geq 1}$ satisfying interval_def, $u_n := \int_{J_n} f$, $U_n := \int_{J_n}\|f\|$

Let $\varepsilon>0$ and take $N\geq 1$ s.t. $|U_p-U_q|<\varepsilon$ for all $p,q\geq N$.

For $p,q\geq N$, $p\geq q$, $u_p-u_q = \int_{J_p} f - \int_{J_q} f = \int_{[a_p,a_q]} f - \int_{[b_p,b_q]} f$

$\therefore \|u_p-u_q\| \leq \int_{[a_p,a_q]}\|f\| + \int_{[b_q,b_p]}\|f\| = U_p-U_q < \varepsilon$

This means that $(u_n)_{n\geq 1}$ is a Cauchy sequence in W, so it converges. ✓

Step 2

Let $(J_n)_{n\geq1}$ and $(K_n)_{n\geq1}$ satisfy interval_def.

From (1), we know that $u_n = \int_{J_n} f \xrightarrow{n\to\infty} u$ and $v_n = \int_{J_n} f \xrightarrow{n\to\infty} v$

For $n\geq1$, let $L_n := J_n \cup K_n$. Due to interval_def, we may find $N\geq1$ s.t. $J_n \cap K_n \neq \emptyset$ $\forall n\geq N$. Thus, $(L_{N+n})_{n\geq1}$ satisfies interval_def.
Now, we want to show that $w=u$, then by symmetry we also have $w=v$, so $u=v$.

- $\forall n\geq1$, $J_n \subseteq L_n$
- $\forall n\geq1$, $U_n := \int_{J_n} \|f\|$, $W_n := \int_{L_n} \|f\|$

since we found W_n, U_n are nonneg $\Rightarrow$ their integrals/limits are well-def, hence $W_n - U_n \to 0$

$\therefore \|w_n - u_n\| = \|\int_{L_n\setminus J_n} f\| \leq \underline{\int_{L_n\setminus J_n} \|f\| = W_n - U_n} \xrightarrow{n\to\infty} 0$
$\therefore \lim_{n\to\infty}(w_n - u_n) = 0 \Rightarrow w=u$. □

PROPERTIES

Properties that can be preserved when we take a limit still hold for integrable functions, such as

- $$\begin{aligned} L^1(I,W) &\longrightarrow W \\ f &\longmapsto \int_I f \end{aligned}$$ is a linear map
- $\int_I f + \int_J f = \int_{I\cup J} f$ if $I\cap J = \emptyset$
- $\triangle$ ineq, $\|\int_I f\| \leq \int_I \|f\|$, IBP, change of variables

PROPOSITION

Let $f\in \mathcal{PC}(I,W)$ be a piecewise continuous function on I. TFAE
(1) f is integrable on $[a,b)$
(2) (Partial integral) $x\mapsto \int_a^x \|f(t)\|dt$ is bounded on $[a,b)$
(3) (Partial integral) $x\mapsto \int_a^x \|f(t)\|dt$ has a limit when $x\to b^-$
(4) (Remainder integral) The limit of $x\mapsto \int_x^b \|f(t)\|\,dt$ when $x\to b^-$ is 0
(5) (Cauchy's criterion) $\forall \varepsilon>0$, $\exists A\in I$, s.t. $\forall x,y\in[A,b)$, $x<y$, $\int_x^y \|f(t)\|dt < \varepsilon$

PROPOSITION

Let $f\in\mathcal{PC}(I,W)$ and $c\in \mathring{I}$. Define $I_- := I\cap(-\infty,c]$ and $I_+ := [c,+\infty)$, then TFAE
(1) f is integrable on I
(2) f is integrable on I_- and I_+
And in this case, we have $\int_I f = \int_{I_-} f + \int_{I_+} f$

PROPOSITION

Let $f\in\mathcal{PC}(I,W)$, $\varphi\in\mathcal{PC}_+(I)$.
(1) If $\|f\|\leq\varphi$ on I and φ is integrable, then f is integrable and $\|\int_I f\| \leq \int_I \varphi$
(2) If $f\in\mathcal{PC}_+(I)$ and is nonintegrable with $f\leq\varphi$, then φ is nonintegrable

Proof
(1) For any subsegment $J\subseteq I$, we have $\int_a^x \|f(t)\|\,dt \leq \int_a^x \varphi(t)dt \leq \int_a^b \varphi(t)dt = \int_I \varphi$
(2) By contradiction □

EXAMPLE

Check that $f: t\longmapsto \frac{1}{\sqrt{t(1-t)}}$ is integrable on $(0,1)$.
Let $c=\frac{1}{2}$, $I_- = (0,\frac{1}{2}]$, $I_+ = [\frac{1}{2},1)$

- For $t\in I_-$, $f(t)\leq\frac{2}{\sqrt{t}}$, which is integrable on $(0,\frac{1}{2}]$, and so is f
- For $t\in I_+$, $f(t)\leq \frac{2}{\sqrt{1-t}}$, which is integrable on $[\frac{1}{2},1)$, so is f

COMPARISON OF INTEGRALS

Shun/翔海 (@shun4midx)

DEFINITION

Let $f:[a,b)\longrightarrow W$, $g:[a,b)\longrightarrow \mathbb{R}$ be piecewise continuous

i.e. $\forall a\in(b-\delta,b)$

We say $f \underset{b}{=} O(g)$ or $f(x)=O(g(x))$ when $x\to b^-$ if $\exists M>0, \delta>0$, s.t. $\forall x\in [a,b)\cap B(b,\delta)$, we have $\|f(x)\|\leq M\|g(x)\|$

We say $f \underset{b}{=} o(g)$ or $f(x)=o(g(x))$ when $x\to b^-$ if $\forall \varepsilon>0, \exists \delta>0$, s.t. $\forall x\in(b-\delta,b)$, $\|f(x)\|\leq \varepsilon\|g(x)\|$

We say $f \underset{b}{\sim} g$ or $f(x)\sim b(x)$ when $x\to b^-$ if $f-g \underset{b}{=} o(g)$

PROPOSITION

Crucial!!

Let $f:[a,b)\longrightarrow W$ be p.c. and $g:[a,b)\longrightarrow \mathbb{R}_+$ be integrable

(1) If $f \underset{b}{=} O(g)$, then f is integrable on $[a,b)$ and $\int_x^b f \underset{x\mapsto b}{=} O(\int_x^b g)$

(2) If $f \underset{b}{=} o(g)$, then f is integrable on $[a,b)$ and $\int_x^b f \underset{x\mapsto b}{=} o(\int_x^b g)$

(3) If $W=\mathbb{R}$ and $f \underset{b}{\sim} g$, then f is integrable on $[a,b)$ and $\int_x^b f \underset{x\mapsto b}{\sim} \int_x^b g$

<u>Proof</u>

As (2), (3) are similar, we only prove (1) here.

By assumption, take $M>0, \delta>0$, s.t. $\forall x\in(b-\delta,b)$, $\|f(x)\|\leq Mg(t)$

$\therefore$ $\forall x\in(b-\delta,b)$, we have $\|\int_x^b f\|\leq \int_x^b\|f\|\leq M\int_x^b g$ $\square$

EXAMPLE (THE GAMMA FUNCTION)

Define the gamma function $\Gamma: (0,+\infty)\longrightarrow \mathbb{R}$

$$x \longmapsto \int_0^{+\infty} t^{x-1}e^{-t}\,dt$$

- $t\mapsto t^{x-1}e^{-t}$ is nonnegative
- Around 0^+, $t^{x-1}e^{-t}\sim t^{x-1}$ and $t\mapsto t^{x-1}$ is nonnegative around 0 and integrable, so $t^{x-1}e^{-t}$ is too
- Around 0^-, $t^{x-1}e^{-t}=O(\frac{1}{t^2})$ and $t\mapsto\frac{1}{t^2}$ is integrable around $+\infty$, so $t^{x-1}e^{-t}$ is too
- E.g.: $\Gamma(1)=1$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$
- Hence, $\Gamma(x+1)=\int_0^{+\infty}t^xe^{-t}\,dt=-(t^xe^{-t})\Big|_0^{+\infty}+\int_0^{\infty}xt^{x-1}e^{-t}\,dt=x\Gamma(x)$ $\forall x>0$, which is the factorial

EXAMPLE

Goal: Find the asymptotic behavior of arccos around $x=1^-$

Note that $\int_x^1\frac{1}{\sqrt{1-t^2}}\,dt=\arccos x$, $x\in[0,1]$.

As $\frac{1}{\sqrt{1-t^2}}=\frac{1}{\sqrt{(1-t)(1+t)}}\underset{t\to 1}{\sim}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-t}}$, thus $\arccos x\sim\sqrt{2(1-x)}$ (integrate both sides)

EXAMPLE

We know that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}}\,dx=1$

We want to find the asymptotic behavior of $F(x)=\int_x^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt$ when $x\longrightarrow+\infty$

1) Note that $e^{-\frac{t^2}{2}}=o(te^{-\frac{t^2}{2}})\Rightarrow F(x)=o(\int_x^{+\infty}\frac{1}{\sqrt{2\pi}}te^{-\frac{t^2}{2}}\,dt)=o([-\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}]\Big|_x^{\infty})=o(e^{-\frac{x^2}{2}})$

5-1-25 (WEEK 11) Shun/翔海(@shun4midx)

REMARK

How do we define $\int_a^b f$, for $f: I \longrightarrow W$? How do we obtain Δ-ineq?

(a) If W is a fin dim vector space

↳ Write $f = \sum_{i=1}^{n} f_i e_i$, define $\int_a^b f := \sum_{i=1}^{n} (\int f_i) e_i$

↳ If $\|\cdot\| = N_1, N_2, N_\infty$, it follows from direct computation

(b) If f is infinite dimensional

↳ Step function: $f: I \longrightarrow W$, $x = (x_k)_{0 \le k \le n} \in \mathcal{P}([a,b])$, s.t. $f|_{(x_{k-1}, x_k)} \equiv f_k \in W$ is const $\forall 1 \le k \le n$. Define $\int_a^b f := \sum_{k=1}^{n} f_k \Delta x_k$, and we have
$\|\int_a^b f\| = \|\sum_{k=1}^{n} f_k \Delta x_k\| \le \sum_{k=1}^{n} \|f_k\| \Delta x_k = \int_a^b \|f\|$

↳ General cont. function:

(i) Find a sequence $(f_n)_{n \ge 1}$ of step functions, s.t. $\|f_n - f\|_\infty := \sup_{t \in [a,b]} |f_n(t) - f(t)| \xrightarrow{n \to \infty} 0$ ($\forall \varepsilon > 0, \exists M \ge 1$, s.t. $|x-y| \le \frac{1}{M} \Rightarrow \|f(x) - f(y)\| < \frac{\varepsilon}{2}$)

(ii) Check that $(\int_a^b f_n)_{n \ge 1}$ is Cauchy in W, so converges, then $\int_a^b := \lim_{n \to \infty} \int_a^b f_n$ (Define $f_n := g_{\frac{1}{n}}$ $\forall n \ge 1$, $\|f_n - f_m\| \le \frac{2}{N}$ if $n, m \ge N$)

(iii) Check that the limit is well-def

EXAMPLE (CONTINUED)

To get a more precise asymptotic formula for $F(x)$ when $x \to \infty$, we may start with IBP.

Write $\sqrt{2\pi} F(x) = \int_x^\infty \frac{-te^{-\frac{t^2}{2}}}{-t} dt = \frac{e^{-\frac{x^2}{2}}}{x} - \int_x^\infty \frac{e^{-\frac{t^2}{2}}}{t^2} dt = \frac{e^{-\frac{x^2}{2}}}{x} - o(F(x))$ as $x \to \infty$

We deduce when $x \to \infty$, $F(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} (1 + o(1))$ when $x \to \infty$

$\therefore$ By induction, $\forall n \ge 0$, when $n \to \infty$, after IBP, we get $F(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{1}{x} + \sum_{k=1}^{n} (-1)^k \frac{(2k-1)!!}{x^{2k+1}}\right)(1 + o(1))$ when $x \to \infty$

PROPOSITION (COMPARISON FOR NON-INTEGRABLE FUNCTIONS)

Let $f: [a,b) \longrightarrow W$, $g: [a,b) \longrightarrow \mathbb{R}_+$ be a nonintegrable function, both are p.c.

1) If $f \underset{b}{=} O(g)$, then $\int_a^x f \underset{x \to b}{=} O(\int_a^x g)$

2) If $f \underset{b}{=} o(g)$, then $\int_a^x f \underset{x \to b}{=} o(\int_a^x g)$

3) If $W = \mathbb{R}$ and $f \underset{b}{\sim} g$, then f is nonintegrable on $[a,b)$ and $\int_a^x f \underset{x \to b}{\sim} \int_a^x g$

Proof

Note that $\int_a^x g \xrightarrow{x \to b} +\infty$ because g is nonintegrable

As (2) and (3) are similar, we prove (1) here only

1) Let $M > 0$ and $\delta > 0$, s.t. $\|f(x)\| \le M|g(x)|$ where $x \in [b-\delta, b)$

For $x \in [b-\delta, b)$, $\|\int_a^x f\| = \|\int_a^{b-\delta} f + \int_{b-\delta}^x f\| \le \int_a^{b-\delta} \|f\| + \int_{b-\delta}^x \|f\| \le \int_a^{b-\delta} \|f\| + M \int_a^x g$

Since $\int_a^x g \xrightarrow{x \to b} +\infty$, thus $\exists \delta' \in (0, \delta]$, s.t. $\forall x \in [b-\delta', b)$, $\int_a^{b-\delta} \|f\| \le M \int_a^x g$

$\therefore \forall x \in [b-\delta', b)$, $\|\int_a^x f\| \le 2M \int_a^x g$. $\square$

EXAMPLE

$\int_2^x \frac{dt}{\ln t} \xrightarrow{x \to \infty} ?$

- $t \mapsto \frac{1}{\ln t}$ is nonneg and nonintegrable on $[2, +\infty)$

By IBP, $\int_2^x \frac{dt}{\ln t} = \left(\frac{t}{\ln t}\right)\Big|_2^x + \int_2^x \frac{dt}{(\ln t)^2} = \frac{x}{\ln x} - \frac{2}{\ln 2} + o\left(\int_2^x \frac{dt}{\ln t}\right)$ when $t \to \infty$

$\therefore \int_2^x \frac{dt}{(\ln t)} \sim \frac{x}{\ln x}$ when $t \to \infty$

IMPROPER INTEGRAL

Let $f: [a,b) \to \mathbb{R}$ be p.c.

DEFINITION AND PROPERTIES

DEFINITION

(1) We say that the integral $\int_{[a,b)} f = \int_{[a,b)} f(t)\,dt$ converges if $x \mapsto \int_a^x f = \int_{[a,x]} f$ converges when $x \to b^-$

(2) Otherwise, we say that the integral $\int_{[a,b)} f$ diverges

REMARK

If $f: [a,b) \to \mathbb{R}$ is nonneg integrable, then this definition coincides with what we saw earlier, $\lim_{x \to b} \int_a^x f = \sup_{J \subseteq [a,b], J \text{ seg}} \int_J f$

PROPOSITION (CAUCHY'S CRITERION)

Let $f:[a,b)\to\mathbb{R}$ be a piecewise continuous function. TFAE

(1) The integral $\int_{[a,b)} f$ converges

(2) $\forall \varepsilon>0, \exists c\in[a,b)$ s.t. $\forall x,y\in[c,b)$ with $x<y$, we have $\left|\int_x^y f(t)\,dt\right|<\varepsilon$

PROPOSITION

Let $f:[a,b)\to\mathbb{R}$ be a piecewise conti function. Let $c\in[a,b)$, then:

(1) Both integrals $\int_{[a,b)} f$ and $\int_{[c,b)} f$ have the same behavior

(2) If they both converge, we have $\int_{[a,b)} f=\int_{[a,c)} f+\int_{[c,b)} f$

Proof

These are direct consequences of "$\forall c\in[a,b), \forall x\in[c,b), \int_a^x f=\int_a^c f+\int_c^x f$"

COROLLARY

If $\int_{[a,b)} f$ conv, then $\int_{[x,b)} f \xrightarrow{x\to b} 0$

- $\int_{[a,x)} f$: partial integral
- $\int_{[x,b)} f$: remainder integral

PROPOSITION

If $f:[a,b]\to\mathbb{R}$ is a bounded p.c. function, and $f\in R(x;a,b)$, then $\int_{[a,b)} f=\int_a^b f$ ($\therefore$ We can generalize the notation $\int_a^b$)

PROPOSITION

Let $f:[a,b)\to\mathbb{R}$ be bounded p.c. and F is a primitive of f. Then, TFAE

(1) $\int_a^b f$ converges

(2) F has a finite limit at b^-

Proof

$\int_a^x f=F(x)-F(a)\xrightarrow{x\to b^-} F(b^-)-F(a)$

The function $x\mapsto\int_x^b f$ is differentiable with derivative $-f$.

DEFINITION

Let $-\infty\leq a<b\leq+\infty$. Fix $c\in(a,b)$. Let $f:(a,b)\to\mathbb{R}$ be p.c. We say that the improper integral $\int_{(a,b)} f=\int_a^b f$ converges if both $\int_{(a,c]} f$ and $\int_{[c,b)} f$

REMARK

The choice of $c\in(a,b)$ is arbitrary, from the previous propositions

EXAMPLE

Let $f:(0,1)\longrightarrow\mathbb{R}$

$x\longmapsto\frac{1}{x}-\frac{1}{1-x}$

Indeed, x is not integrable.

$f(x)\underset{x\to0^+}{\sim}\frac{1}{x}$ and $\frac{1}{x}$ is not integrable around 0^+

$f(x)\underset{x\to1^-}{\sim}-\frac{1}{1-x}$ and $\frac{1}{1-x}$ is not integrable around 1^-

We may check the convergence of $\int_{(0,\frac{1}{2}]} f$ and $\int_{[\frac{1}{2},1)} f$

We find $\int_{\frac{1}{n}}^{\frac{1}{2}} f=2\ln\frac{1}{2}-\ln\frac{1}{n}-\ln(1-\frac{1}{n})\xrightarrow{n\to\infty}\infty$, so $\int_{(0,\frac{1}{2}]}$ diverges

CONDITIONAL CONVERGENCE

DEFINITION

Let $f:I\to\mathbb{R}$, I is an interval. We say that $\int_I f$ converges conditionally if

- $\int_I f$ converges
- f is not integrable on I (i.e. $\int_I|f|$ diverges)

EXAMPLE

$\int_\pi^\infty \frac{\sin x}{x}\,dx$ converges conditionally

- "Not integrable": For $k\in\mathbb{N}$, $\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{(k+1)\pi}\underbrace{\int_{k\pi}^{(k+1)\pi} |\sin x|\,dx}_{}$ ← const >0 and $\sum \frac{1}{k+1}$ div
  - Check $\int 1\cdot 1=\infty$
  - $\therefore \int_\pi^\infty \left|\frac{\sin x}{x}\right| dx = +\infty$
- "Converges": For $t>\pi$, $\int_\pi^t \frac{\sin x}{x}\,dx = \left[-\frac{\cos x}{x}\right]_\pi^t - \int_\pi^t \frac{\cos x}{x^2}\,dx = \underbrace{-\frac{\cos t}{t}}_{\to 0} + \underbrace{\frac{\cos \pi}{\pi}}_{\text{const}} - \int_\pi^t \frac{\cos x}{x^2}\,dx$
  - $\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$, so $x\mapsto \frac{\cos x}{x}$ is integrable on $[\pi,\infty)$

THEOREM (ABEL'S RULE) (Proof in next section of notes)

Let $f:[a,b)\to\mathbb{R}$ be C^1, $g:[a,b)\to\mathbb{R}$ be C^0,

(i) f is decreasing with $\lim_{x\to b} f(x)=0$

(ii) $\exists M>0$, s.t. $\forall x\in[a,b)$, we have $\left|\int_a^x g(t)\,dt\right|\leq M$

Then, the integral $\int_a^b f(t)g(t)\,dt$ is convergent

5-6-25 (WEEK 12) Shun/翔海 (@shun4midx)

THEOREM (ABEL'S RULE)

Let $f:[a,b)\to\mathbb{R}$ be C^1 and $g:[a,b)\to\mathbb{R}$ be C^0.

Suppose (i) $f(x)\searrow 0$ when $x\to b^-$

(ii) $\exists M>0$, s.t. $\left|\int_a^x g(t)dt\right|\leq M$ for $x\in[a,b)$

Then, $\int_a^b f(t)g(t)dt$ converges

Proof

We want to check that Cauchy's criterion is satisfied.

Let $\varepsilon>0$, by (i), we may find $A\in[a,b)$, s.t. $|f(t)|\leq\varepsilon$ $\forall t\in[A,b)$

Let G be a primitive of g, $G(x)=\int_a^x g(t)dt$ $\forall x\in[A,b)$

$\Rightarrow$ It follows from (ii) that $|G(x)|\leq M$ for some fixed M uniformly in $x\in[a,b)$

Let $x,y\in[A,b)$ with $x\leq y$. We have $\int_x^y f(t)g(t)dt=[f(t)G(t)]_x^y-\int_x^y f'(t)G(t)dt=f(y)G(y)-f(x)G(x)-\int_x^y f'(t)G(t)$

Notice,
- $|f(y)G(y)|\leq\varepsilon M$
- $|f(x)G(x)|\leq\varepsilon M$
- $\left|\int_x^y f'(t)G(t)dt\right|\leq\int_x^y|f'(t)||G(t)|dt\leq M\int_x^y|f'(t)|dt=-M\int_x^y f'(t)dt=M(f(x)-f(y))\leq\varepsilon M$

$\therefore$ In conclusion, $\left|\int_x^y f(t)g(t)dt\right|\leq 3\varepsilon M$ for $x,y\in[A,b)$ with $x\leq y$. $\square$

EXAMPLE

Fix $\alpha>0$. Then, the following integrals converge. $\int_1^\infty\frac{\sin t}{t^\alpha}dt$, $\int_1^\infty\frac{\cos t}{t^\alpha}dt$ (by Abel's rule). Also, this means $\int_1^\infty\frac{e^{it}}{t^\alpha}dt$ conv too.

EXAMPLE

Let $f:[1,+\infty)\longrightarrow\mathbb{C}$, $x\longmapsto\frac{e^{ix}}{\sqrt{x}}$, $g:[1,+\infty)\longrightarrow\mathbb{C}$, $x\longmapsto\frac{e^{ix}}{\sqrt{x}}+\frac{1}{x}$

- $f(x)\sim g(x)$ when $x\to+\infty$
- $\int_1^\infty f$ conv

$\therefore\int_1^\infty g$ does not conv (since otherwise $\int_1^\infty(g(t)-f(t))dt=\int_1^\infty\frac{1}{t}dt$ conv $\rightarrow\!\!\times\!\!\leftarrow$)

LAPLACE'S METHOD

THEOREM

Let $-\infty\leq a<b\leq+\infty$ and $g,h:(a,b)\to\mathbb{R}$ be C^1.

Suppose (i) $x\longmapsto g(x)e^{h(x)}$ is integrable on (a,b) ← kinda like abs conv, not just conv

(ii) $\exists c\in(a,b)$, s.t. (a) h is increasing on (a,c) and decreasing on (c,b) with $h''(c)<0$

(b) $g(c)\neq 0$

Then, when $\lambda\to+\infty$, we have $\int_a^b g(x)e^{\lambda h(x)}\sim\sqrt{\frac{2\pi}{-\lambda h''(c)}}\,g(c)e^{\lambda h(c)}$

PROOF SKETCH

By Taylor expansion, $h(x)\approx h(c)-\frac{1}{2}(-h''(c))(x-c)^2$

Then, $\int_a^b g(x)e^{\lambda h(x)}dx\approx\int_{c-\varepsilon}^{c+\varepsilon}e^{\lambda(h(c)+\frac{1}{2}h''(c)(x-c)^2)}dx=g(c)e^{\lambda h(c)}\int_{c-\varepsilon}^{c+\varepsilon}e^{\frac{1}{2}h''(c)(x-c)^2}dx$

Let $y=\sqrt{\lambda}(x-c)$, then $=g(c)e^{\lambda h(c)}\frac{1}{\sqrt{\lambda}}\int_{-\varepsilon\sqrt{\lambda}}^{\varepsilon\sqrt{\lambda}}e^{-\frac{1}{2}h''(c)y^2}dy\approx g(c)e^{\lambda h(c)}\frac{1}{\sqrt{\lambda}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}h''(c)y^2}dy=\sqrt{\frac{2\pi}{-\lambda h''(c)}}\,g(c)e^{\lambda h(c)}$ (Used in steepest descent in ML)

Remark: These two "$\approx$" are not rigorous and require explanation, hence why this is a "proof sketch" rather than a "proof"

APPLICATION (STIRLING'S FORMULA)

Recall for $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$, $x>0$, $\Gamma(n+1)=n!$ $\forall n\in\mathbb{N}$

$\Gamma(n+1)=\int_0^\infty t^n e^{-t}dt=\int_0^\infty e^{n\ln t-t}dt$

Let $t=ns$, $=\int_0^\infty ne^{n\ln(ns)-ns}ds=n^{n+1}\int_0^\infty e^{n(\ln s-s)}ds$

Define $h:(0,+\infty)\longrightarrow\mathbb{R}$

$s\longmapsto\ln s-s$

We have $h'(s) = \frac{1}{s} - 1$, $h''(s) = -\frac{1}{s^2}$

	0	1	∞
$h'(s)$	$+$		$-$
h	$\nearrow$		$\searrow$

h is $\nearrow$ on $(0,1)$, $\searrow$ on $(1,+\infty)$, and $h''(1) = -1 < 0$

$\therefore$ By Laplace's method, we find $\int_0^\infty e^{n(\ln s - s)}\,ds \underset{n\to+\infty}{\sim} \sqrt{\frac{2\pi}{n}}\, e^{-n} \Rightarrow n! = \Gamma(n+1) \underset{n\to+\infty}{\sim} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

SEQUENCES AND SERIES OF FUNCTIONS

NOTATION

$F(A,M) := \{f: A \to M \text{ functions}\}$

$B(A,M) := \{f: A \to M \text{ bounded functions}\}$

NOTIONS OF CONVERGENCE

SEQUENCES OF FUNCTIONS

DEFINITION

Let $(f_n)_{n\geq 1}$ be a sequence of functions from A to M, that is, they are elements of $F(A,M)$

- Let $f \in F(A,M)$, we say $(f_n)_{n\geq 1}$ converges pointwise to f if $\forall x \in A$, $\exists f_n(x) \xrightarrow{n\to\infty} f(x)$ in (M,d)
- We say $(f_n)_{n\geq 1}$ converges pointwise if $\exists f \in F(A,M)$, s.t. $(f_n)_{n\geq 1}$ converges pointwise to f
- Let $B \subseteq A$ be a subset. We say $(f_n)_{n\geq 1}$ converges pointwise on B if $((f_n)|_B)_{n\geq 1}$ converges pointwise

EXAMPLE

For $n \geq 1$, let $f_n: [0,1] \longrightarrow \mathbb{R}$, $x \longmapsto x^n$

The sequence of functions $(f_n)_{n\geq 1}$ converges pointwise to $\mathbb{1}_{\{1\}}$ on $[0,1]$

REMARK

(1) If $f_n \xrightarrow{n\to\infty} f$ pointwise, then f is unique (but depends on d)

(2) If $(M,d) = (W, \|\cdot\|)$ is a finite-dimensional normed vector space, and $f_n \longrightarrow f$ pointwise, the limit does not depend on the norm

(3) Some properties are preserved by pointwise convergence: linearity, product, inequality, monotonicity, etc

(4) Analytic properties (continuity, differentiability, integrability, etc) may NOT be preserved

DEFINITION

Let $(f_n)_{n\geq 1}$ be a sequence of functions from A to M

- Let $f \in F(A,M)$, we say $(f_n)_{n\geq 1}$ converges uniformly to f if $\forall \varepsilon > 0$, $\exists N(\varepsilon) > 0$, s.t. $\forall n \geq N$, $x \in A$, $d(f_n(x), f(x)) < \varepsilon$
- We say $(f_n)_{n\geq 1}$ converges uniformly if $\exists f \in F(A,M)$, s.t. $(f_n)_{n\geq 1}$ converges uniformly to f
- We say $(f_n)_{n\geq 1}$ converges uniformly on $B \subseteq A$ if $((f_n)|_B)_{n\geq 1}$ converges uniformly

REMARK

Let's write the pointwise convergence using quantifiers.

We say $f_n \to f$ pointwise if $\forall \varepsilon > 0$, $\exists N(x,\varepsilon) > 0$, s.t. $\forall n \geq N$, $d(f_n(x), f(x)) \leq \varepsilon$

This means that uniform convergence $\Rightarrow$ pointwise convergence. In particular, the uniform limit is unique.

PROPOSITION (CAUCHY'S CRITERION)

Suppose (M,d) is complete. Let $(f_n)_{n\geq 1}$ be a sequence of functions from A to M. Then, $f_n \xrightarrow{n\to\infty} f$ uniformly if $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall m, n \geq N$, $\forall x \in A$, $d(f_n(x), f_m(x)) \leq \varepsilon$

COROLLARY

Shun/翔海(@shun4midx)

If $(f_n)_{n\geq 1}$ converges uniformly to f, it converges pointwise to f.

REMARK

To show that $f_n \to f$ uniformly, we start by proving the pointwise convergence, then check that this convergence is uniform

DEFINITION

- Let us equip $B(A,M)$ with the following distance: $\forall f,g \in B(A,M)$, $d_\infty(f,g) = d_{\infty,A}(f,g) := \sup_{x\in A} d(f(x),g(x))$
  If $f_n \in B(A,M)$ and $f_n \to f$ uniformly $\Leftrightarrow d_\infty(f_n,f) \xrightarrow{n\to\infty} 0$
- Let $(W,\|\cdot\|)$ be a normed vector space. Let us equip $B(A,W)$ with the following norm, $\forall f \in B(A,W)$, $\|f\|_\infty = \|f\|_{\infty,A} = \sup_{x\in A} \|f(x)\|$, called the norm of uniform convergence. $f_n \to f$ uniformly $\Leftrightarrow \|f_n - f\| \xrightarrow{n\to\infty} 0$

PROPOSITION

Let $(W,\|\cdot\|)$ be a Banach space. Then, $B(A,W)$ is also a Banach space

<u>Proof</u>

Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $B(A,W)$. We want to check that $(f_n)_{n\geq 1}$ converges in $(B(A,W), \|\cdot\|_\infty)$

- Let $x \in A$, note that $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in W, so it converges to a limit we call $f(x)$
- Check that $f \in B(A,W)$. First, a Cauchy sequence is bounded, so $\|f_n\|_\infty < M$ for some $M>0$ uniformly in n

For $x \in A$, $f(x) = \lim_{n\to\infty} f_n(x)$, so $\|f(x)\| = \lim_{x\to\infty} \|f_n(x)\| < M$. This means that $f \in B(A,W)$

- Check that $f_n \to f$ in $(B(A,W), \|\cdot\|_\infty)$. Let $\varepsilon > 0$, take $N>0$, s.t. $\|f_n - f_m\|_\infty \leq \varepsilon$ $\forall m,n \geq N$

$\forall x \in A$, we have $\|f_n(x) - f(x)\| = \lim_{m\to\infty} \|f_n(x) - f_m(x)\| \leq \varepsilon$ $\forall n \geq N$. This means, $\|f_n - f\|_\infty \leq \varepsilon$ $\forall n \geq N$

5-8-25 (WEEK 12)

Shun/翔海 (@shun4midx)

EXAMPLE

Let $f_n: [0,1] \longrightarrow \mathbb{R}$ for $n \geq 1$
$x \longmapsto x^n(1-x)$

- For $x \in [0,1)$, $|f_n(x)| = x^n(1-x) \leq x^n \xrightarrow{n\to\infty} 0$
- For $x=1$, $f_n(x)=0$ $\forall n$
- Thus, $f_n \xrightarrow{n\to\infty} 0$ pointwise
- Check that this convergence is uniform

  For $n>1$, $f_n'(x) = nx^{n-1}(1-x) - x^n = nx^{n-1}(1-\frac{n+1}{n}x)$

	0		$\frac{n}{n+1}$		1
f_n'		$+$		$-$	
f_n	0	$\nearrow$		$\searrow$	0

$\Rightarrow |f_n(x)| \leq f_n(\frac{n}{n+1}) = \frac{1}{n+1}(\frac{n}{n+1})^n$ $\forall x \in [0,1]$
$\leq \frac{1}{n+1} \xrightarrow{n\to\infty} 0$ (indep of x)

$\therefore f_n \rightarrow 0$ uniformly $\square$

REMARK

If $f_n \rightarrow f$ pointwise but not uniformly, how do we prove this?

- $f_n \not\rightarrow f$ uniformly: $\exists \varepsilon>0$, s.t. $\forall N>0$, $\exists x \in [0,1], n \geq N$, s.t. $|f_N(x) - f(x)| \geq \varepsilon$
  - ↳ This means $\exists (n_k)_{k\geq 1}$ and $(x_k)_{k \geq 1}$, s.t. $|f_{n_k}(x_k) - f(x_k)| \geq c$ for some $c>0$

EXAMPLE

Let $f_n: \mathbb{R} \longrightarrow \mathbb{R}$ $\forall n \in \mathbb{N}$
$x \longmapsto \frac{x+\sqrt{n}}{x+n}$

- $\forall$ fixed $x \in \mathbb{R}$, $f_n(x) \xrightarrow{x\to\infty} 0$, so $f_n \rightarrow 0$ pointwise
- For $x=n$, we have $f_n(n) = \frac{n+\sqrt{n}}{2n} \xrightarrow{n\to\infty} \frac{1}{2}$

$\therefore f_n(n) \geq \frac{1}{4}$ for $n>N$ for some $N>1$

$\therefore$ The convergence is not uniform

THEOREM (DINI'S THEOREM)

Let (K,d) be a compact metric space. Let $(f_n)_{n\geq 1}$ be a sequence of continuous functions $K \rightarrow \mathbb{R}$

Suppose (i) The sequence (f_n) is increasing, i.e. $f_n(x) \leq f_{n+1}(x)$ $\forall x \in K$
(ii) $f_n \longrightarrow f$ pointwise, f is continuous

Then, $f_n \longrightarrow f$ uniformly

Proof

When we say "(K,d) is compact", we want to use the Borel-Lebesgue property (finite subcovering)

Let $g_n := f_n - f \geq 0$ $\forall n \in \mathbb{N}$. Fix $\varepsilon>0$. Consider $\forall n>0$, $E_n = \{x \in K \mid g_n(x) < \varepsilon\} = g_n^{-1}((-\infty, \varepsilon))$, which is open in (K,d), as it is the preimage of an open set under a conti. function.

Note, $\forall x \in K$, $g_n(x) \searrow 0$, so $\exists N>0$, s.t. $x \in E_n$ $\forall n \geq N$

$\therefore (E_n)_{n\geq 1}$ is an increasing sequence of open sets, and $\bigcup_{n\geq 1}^{\infty} E_n = K$

$\therefore (E_n)_{n\geq 1}$ is an open covering of the compact space (K,d), so $\exists N>0$, s.t. $K = \bigcup_{n\geq 1}^{N} E_n = E_N$. This means $0 \leq g_N(x) < \varepsilon$ $\forall x \in K$

$\therefore 0 \leq g_n(x) < \varepsilon$ $\forall n \geq N$ $\square$

ALTERNATE VERSION

Let $I=[a,b]$ be a segment and a sequence $(f_n)_{n\geq 1}$ from $I \longrightarrow \mathbb{R}$

Suppose (i) $f_n \nearrow$ on I $\forall n \geq 1$
(ii) $f_n \longrightarrow f$ pointwise, f is continuous

Then, $f_n \rightarrow f$ uniformly.

SERIES OF FUNCTIONS

Shun/翔海 (@shun4midx)

Let $(u_n)_{n\geq 1}$ be a sequence of functions from A to W < normed vector space / Banach space — if we need "Cauchy ⇒ conv"

DEFINITION

We say that $\sum u_n$ converges pointwise if $\forall x\in A$, the series $\sum u_n(x)$ converges. We write $\sum_{n\geq 1} u_n : A \longrightarrow W$, $x \longmapsto \sum_{n\geq 1} u_n(x)$

- $S_n(x) := \sum_{k=1}^{n} u_k(x)$ is called the k^{th} partial sum
- If $\sum u_n$ converges pointwise, then $R_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$ is the n^{th} remainder
- We say $\sum u_n$ converges uniformly if $(S_n)_{n\geq 0}$ converges uniformly

PROPOSITION

The series of functions $\sum u_n$ converges uniformly iff

(i) $\sum u_n$ converges pointwise

(ii) The sequence of remainders $(R_n)_{n\geq 0}$ converge uniformly to 0

EXAMPLE

Consider $\sum \frac{(-1)^n}{n} x^n$ for $x\in[0,1]$

- For $x\in[0,1]$, $\left(\frac{x^n}{n}\right) \searrow 0$, so the alternating series converges
  $\Rightarrow$ The series of functions $\sum \frac{(-1)^n}{n} x^n$ converges pointwise
- We may estimate the remainder of an alternating series. For $x\in[0,1]$, $|R_n(x)| \leq \left|\frac{(-1)^{n+1}x^{n+1}}{n+1}\right| \leq \frac{1}{n+1} \xrightarrow{n\to\infty} 0$
  $\therefore R_n \to 0$ uniformly

REMARK

$(f_n)_{n\geq 1}$ converges uniformly $\Leftrightarrow \sum (f_{n+1}-f_n)$ converges uniformly

PROPOSITION (CAUCHY'S CRITERION)

$\sum u_n$ converges uniformly $\Leftrightarrow \forall \varepsilon>0, \exists N>0$, s.t. $\forall n\geq N, k\geq 1$, $\underbrace{\|u_{n+1}+\ldots+u_{n+k}\|_\infty}_{S_{n+k}-S_n} < \varepsilon$

DEFINITION

Let $u_n \in \mathcal{B}(A,W) =: \mathcal{E}$ for $n\geq 1$. We say the series of functions $\sum u_n$ converges normally (function version of conv abs!) if $\sum \|u_n\|$ converges

(If $\mathcal{E}$ is a Banach space, $u_n\in\mathcal{E}$, $\sum\|u_n\|_\infty$ converges means that $\sum u_n$ converges absolutely in $(\mathcal{E}, \|\cdot\|_\infty)$)

PROPOSITION

Let $\sum u_n$ be a series of bounded functions from A to W that converges normally on A. Then, we have:

(1) $\forall a\in A$, $\sum u_n(a)$ converges absolutely

(2) The series of functions $\sum u_n$ converges uniformly

Proof

(1) Let $a\in A$, $n\in\mathbb{N}$. Then, $\sum_{k=1}^{n} |u_k(a)| \leq \sum_{k=1}^{n} \|u_k\|_\infty \leq \sum_{k=1}^{\infty} \|u_k\|_\infty \leq +\infty$, so $\sum u_k(a)$ converges absolutely

(2) $\sum_{n\geq 1} \|u_n\|_\infty$ converges so it satisfies Cauchy's property

$\forall n\geq 1, k\geq 1$, $\|u_n(x)+\ldots+u_{n+k}(x)\|_\infty \leq \|u_n(x)\|_\infty + \ldots + \|u_{n+k}(x)\|_\infty \leq \|u_n\|_\infty + \ldots + \|u_{n+k}\|_\infty$

$\therefore$ We have the Cauchy's condition for $\sum u_n(x)$ $\forall x\in A$

$\therefore$ $\sum u_n$ converges uniformly □

EXAMPLE

Let $(f_n)_{n\geq 1}$ be a sequence of functions from $[0,1]$ to $\mathbb{R}$. $f_1 \equiv 1$, $\forall n\geq 1$, $\forall x\in[0,1]$, $f_{n+1}(x) = 1 + \frac{1}{2}\int_0^x f_n(t)\,dt$

- Show that $(f_n)_{n\geq 1}$ converges uniformly. To achieve this, let us check that $\sum (f_{n+1}-f_n)$ converges normally
- For $n\geq 1$, $x\in[0,1]$, we have $|f_{n+2}(x)-f_{n+1}(x)| = \frac{1}{2}\left|\int_0^x (f_{n+1}(t)-f_n(t))\,dt\right| \leq \frac{1}{2}\|f_{n+1}-f_n\|_\infty$
  $\Rightarrow \|f_{n+2}-f_{n+1}\|_\infty \leq \|f_{n+1}-f_n\|_\infty$ $\forall n\geq 1 \Rightarrow \|f_{n+1}-f_n\|_\infty \leq \left(\frac{1}{2}\right)^{n-1}\|f_2-f_1\|_\infty \Rightarrow$ ($\because$ it is a GS) $\sum\|f_{n+1}-f_n\|_\infty$ converges

COUNTEREXAMPLE

Back to $\sum \frac{(-1)^n}{n} x^n$

- Known: Uniform convergence on $[0,1]$
- Let $u_n(x) = \frac{(-1)^n}{n} x^n$, $x \in [0,1]$. $\|u_n\|_\infty = \frac{1}{n}$, $\sum \frac{1}{n} = \infty$, so $\sum u_n$ does not converge normally!
- However, for $a \in [0,1)$, $\|u_n|_{[0,a]}\|_\infty = \frac{a^n}{n}$. In fact, $\sum \frac{a^n}{n}$ converges, so $\sum u_n$ converges normally on $[0,a]$

PROPERTIES OF THE UNIFORM LIMIT

Let (X, d_X) and (M, d_M) be metric spaces

Let $(f_n)_{n\geq 1}$ be a sequence of functions in $\mathcal{B}(X,M)$.

CONTINUITY

PROPOSITION

For $a \in X$, suppose f_n is continuous at a for $n \geq 1$. If $f_n \to f$ uniformly, then f is continuous at a

(Btw, the contrapositive provides us with another way to prove nonuniform convergence: find a discontinuous point of f)

Proof

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, take $N > 0$, s.t. $\forall n \geq N$, $x \in X$, $d_M(f_n(x), f(x)) \leq \varepsilon$ (*)

We can use the continuity of f_N at a.

We can find $\eta > 0$, s.t. $d_X(x,a) \leq \eta \Rightarrow d_M(f_N(x), f_N(a)) \leq \varepsilon$ (**)

Then, for $x \in B_X(a,\eta)$, we have: $d_M(f(x), f(a)) \leq \underbrace{d_M(f(x), f_N(x))}_{(*)} + \underbrace{d_M(f_N(x), f_N(a))}_{(**)} + \underbrace{d_M(f_N(a), f(a))}_{(*)} \leq 3\varepsilon$ □

COROLLARY

Suppose f_n is continuous on X for all $n \geq 1$. If $f_n \to f$ uniformly, then f is continuous on X.

COROLLARY

Let $\sum u_n$ be a series of continuous functions on X. If $\sum u_n$ converges uniformly, then $\sum u_n$ is continuous

EXAMPLE

Let $u_n: \mathbb{R}^+ \longrightarrow \mathbb{R}$ for $n \geq 0$

$x \longmapsto \frac{x^n}{n!}$

We want to discuss the series of functions $u(x) = \sum_{n\geq 0} u_n(x)$

- For $x \geq 0$, $\sum_{n\geq 0} \frac{x^n}{n!}$ conv by the ratio test. $\therefore u(x)$ is well-defined
- $x \longmapsto u_n(x)$ is continuous $\forall n \geq 0$
- Question: Is u a continuous function?
- Uniform convergence on $\mathbb{R}^+$? Fix $n \in \mathbb{N}$, $|R_n(x)| = |\sum_{n\geq 0} u_n(x) - \sum_{n=0}^{N} u_n(x)| = |\sum_{n\geq N+1}^{\infty} u_n(x)| \geq u_{N+1}(x) = \frac{x^{N+1}}{(N+1)!} \xrightarrow{x\to\infty} \infty$ $\therefore R_n \not\to 0$ uniformly
- Given $M > 0$, let us show that $\sum u_n$ converges uniformly on $[0,M]$. Then, the continuity of u on $[0,M]$ follows.
  This is true $\forall M > 0$, so we deduce the continuity of u on the whole $\mathbb{R}^+$
- For $x \in [0,M]$, $|R_n(x)| = \sum_{k\geq n+1} u_k(x) \leq \sum_{k\geq n+1} u_k(M) = \sum_{k\geq n+1} \frac{M^k}{k!} \xrightarrow{n\to\infty} 0$ indep of $x \in [0,M]$ ($\because$ They are remainders of the convergent $\sum_{n\geq 0} \frac{M^n}{n!}$)

$\Rightarrow \sum u_n$ converges uniformly on $[0,M]$

5-13-25 (WEEK 13)

Shun/翔海 (@shun4midx)

INTEGRATION

Let $I \subseteq \mathbb{R}$ be an interval, s.t. $J \neq \emptyset$.
Let $(f_n)_{n\geq 1}$ be a sequence of functions from I to a Banach space $(W, \|\cdot\|)$.

PROPOSITION

Suppose that on each segment $J \subseteq I$, all the f_n's are continuous and $f_n \to f$ uniformly.
Let $a \in I$, define $\varphi(x) = \int_a^x f(t)\,dt$, $\forall n \geq 1$, $\varphi_n(x) = \int_a^x f_n(t)\,dt$
Then, $\varphi_n \longrightarrow \varphi$ uniformly on every segment $J \subseteq I$.

REMARK

We may interchange the order of "$\lim\limits_{n\to\infty}$" and "$\int_a^x$", $\overbrace{\lim\limits_{n\to\infty}\int_a^x f_n(t)\,dt}^{\lim\limits_{n\to\infty}\varphi_n(x)} = \overbrace{\int_a^x \lim\limits_{n\to\infty} f_n(t)\,dt}^{\int_a^x f(t)dt = \varphi(x)}$ $\forall x \in I$

PROOF OF PROPOSITION

Let $J = [c,d] \subseteq I$ be a segment of I
For $x \in J$, we have $\|\varphi_n(x) - \varphi(x)\| = \|\int_a^x (f_n(t) - f(t))\,dt\| \leq \int_a^c \|f_n - f\|_{\infty,[c,d]}\,dt \leq \overbrace{|x-a|\,\|f_n - f\|_{\infty,[c,d]}}^{\text{(unif bounded)}} \xrightarrow{n\to\infty} 0$
$\therefore$ This convergence is uniform when $x \in [c,d]$

EXAMPLE

Let $(f_n)_{n\geq 1}$ be a sequence of continuous functions from $[0,1]$ to $\mathbb{R}$.
Suppose $f_n \longrightarrow f$ uniformly on $[0,1]$. We want to show that $\int_0^1 f_n^2 \longrightarrow \int_0^1 f^2$

For example, we may try to prove $f_n^2 \longrightarrow f^2$ uniformly on $[0,1]$

- For $x \in [0,1]$, we have $|f_n(x)^2 - f(x)^2| \leq |f_n(x) - f(x)|\,|f_n(x) + f(x)| \leq 2M\|f_n - f\|_\infty$, where M is given below.
- $(f_n)_{n\geq 1}$ is a convergent sequence in $(\mathcal{B}([0,1],\mathbb{R}), \|\cdot\|_\infty)$ — Only valid because $[0,1]$ not $[0,1)$
  This means that $(f_n)_{n\geq 1}$ is bounded, so $\|f_n\|_\infty \leq M$ for some $M > 0$ uniformly in n
- This gives us $|f_n(x)^2 - f(x)^2| \leq 2M\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$. This implies $f_n^2 \longrightarrow f^2$ uniformly on $[0,1]$
- More generally speaking, for any integer $p > 1$, we have $\int_0^1 (f_n)^p \longrightarrow \int_0^1 f^p$

EXAMPLE

For $n \in \mathbb{N}$, define $f: [0,1] \longrightarrow \mathbb{R}$, $x \longmapsto x^n$, $f_n \longrightarrow \mathbb{1}_{\{1\}}$ pointwise

We have seen that $f_n \longrightarrow \mathbb{1}_{\{1\}}$ is NOT uniform because all the f_n's are cont. at 1 but $\mathbb{1}_{\{1\}}$ is NOT cont. at 1
However, for $n \in \mathbb{N}$, $\int_0^1 f_n(t)\,dt = \frac{1}{n+1} \xrightarrow{n\to\infty} 0$
$\int_0^1 \mathbb{1}_{\{1\}}\,dt = 0$
$\therefore$ The integrals converge uniformly

CAUTION

Convergence of integrals is much weaker than uniform convergence even if unif $\Rightarrow$ int conv.

COROLLARY

Let $\sum u_n$ be a series of continuous functions from $[a,b]$ to $(W, \|\cdot\|)$.
If $\sum u_n$ converges uniformly, then $\forall x \in [a,b]$, $\int_a^x \left(\sum_{n\geq 1} u_n(t)\right) dt = \sum_{n\geq 1}\left(\int_a^x u_n(t)\,dt\right) = \lim\limits_{n\to\infty} \sum_{k=1}^{n} \left(\int_a^x u_k(t)\,dt\right)$, where the limit on the right side is uniform in $x \in [a,b]$.
$\therefore$ We can say that we can "integrate term by term"

Shun/翔海(@shun4midx)

THEOREM

Let $\alpha \in BV([a,b])$. Let $(f_n)_{n\geq 1}$ be a sequence of bounded functions from $[a,b]$ to $\mathbb{R}$. Suppose $f_n \in \mathcal{R}(\alpha;a,b)$ $\forall n\geq 1$.
Suppose $f_n \longrightarrow f$ uniformly. Define $g(x) = \int_a^x f(t)\,d\alpha(t)$ and $g_n(x) = \int_a^x f_n(t)\,d\alpha(t)$ $\forall n\geq 1$

Then, (1) $f \in \mathcal{R}(\alpha;a,b)$, so g is well-defined
(2) $g_n \to g$ uniformly

Proof

By decomposition thm, WLOG, we may assume α to be nondecreasing. The case $\alpha(a)=\alpha(b)$ is trivial, all the integrals are zero, so nothing to prove. Hence, let us assume $\alpha(a)<\alpha(b)$

Recall Riemann's condition: $\forall \varepsilon>0, \exists P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $U_P(f,\alpha) - L_P(f,\alpha) \leq \varepsilon$ $\forall P \supseteq P_\varepsilon \Leftrightarrow f \in \mathcal{R}(\alpha;a,b)$

(1) Let us check Riemann's condition.

Given $\varepsilon>0$. We may find $N>0$, s.t. $\|f-f_n\|_\infty \leq \frac{\varepsilon}{\alpha(b)-\alpha(a)}$ $\forall n\geq N$

Then, $\forall P \in \mathcal{P}([a,b])$, $|U_P(f-f_N,\alpha)| = \left|\sum_{i=1}^n \sup_{x\in[x_{i-1},x_i]} (f(x)-f_N(x))\Delta\alpha_i\right| \leq \frac{\varepsilon}{\alpha(b)-\alpha(a)}\left|\sum_{i=1}^n \Delta\alpha_i\right| = \varepsilon$. Similarly, $|L_P(f-f_N,\alpha)| < \varepsilon$.

Since $f_N \in \mathcal{R}(\alpha;a,b)$, we may find $P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $U_P(f_N,\alpha) - L_P(f_N,\alpha) \leq \varepsilon$ $\forall P \supseteq P_\varepsilon$.

$\therefore U_P(f,\alpha) - L_P(f,\alpha) \leq U_P(f-f_N,\alpha) - L_P(f-f_N,\alpha) + U_P(f_N,\alpha) - L_P(f_N,\alpha) \leq 3\varepsilon$ $\forall P \supseteq P_\varepsilon$. ✓

(2) For $n\geq 1$ and $x\in[a,b]$, $|g_n(x)-g(x)| = \left|\int_a^x (f_n(t)-f(t))\,d\alpha(t)\right| \leq \|f_n-f\|_\infty(\alpha(x)-\alpha(a)) \leq \|f_n-f\|_\infty(\alpha(b)-\alpha(a)) \xrightarrow{n\to\infty} 0$ indep of x □

COROLLARY

Let $\alpha \in BV([a,b])$. Let $\sum u_n$ be a series of bounded functions from $[a,b]$ to $\mathbb{R}$ s.t. $u_n \in \mathcal{R}(\alpha;a,b)$ $\forall n\geq 1$.
Suppose $\sum u_n$ converges uniformly. Then,
(1) $\sum u_n \in \mathcal{R}(\alpha;a,b)$
(2) $\forall x\in[a,b]$, $\int_a^x \sum_{n=1}^\infty u_n(t)\,d\alpha(t) = \sum_{n=1}^\infty \int_a^x u_n(t)\,d\alpha(t)$ and the convergence is uniform.

DERIVATIVES

Let $I \subseteq \mathbb{R}$ be an interval s.t. $\mathring{I} \neq \emptyset$, and $f_n: I \longrightarrow W$ $\forall n\geq 1$

THEOREM

Suppose (i) $\forall n\geq 1$, $f_n: I \to W$ is of class $\mathcal{C}^1$
(ii) The sequence $(f_n)_{n\geq 1}$ converges pointwise to $f \in \mathcal{F}(I,W)$
(iii) The sequence $(f_n')_{n\geq 1}$ converges uniformly to $g \in \mathcal{F}(I,W)$ on every segment I

Then, the following properties hold.
(1) The function f is of class $\mathcal{C}^1$ and $f'=g$
(2) The sequence $(f_n)_{n\geq 1}$ converges uniformly on every segment of I

Proof

Let $a\in I$, by (ii), we know $f_n(a) \xrightarrow{n\to\infty} f(a)$

(1) For $x\in I$, we have $\underbrace{\int_a^x g_n(t)\,dt}_{``f_n(x)-f_n(a)"} \xrightarrow{n\to\infty} \int_a^x g(t)\,dt$ from before

$\therefore \lim_{n\to\infty} f_n(x) = f(a) + \int_a^x g(t)\,dt$ $\forall x\in I$ $\Rightarrow$ By FTC, $f'(x) = g(x)$ $\forall x\in I$

As g is continuous on I, we denote this f is $\mathcal{C}^1$.

(2) For $x\in I$, we have $f_n(x)-f(x) = \left(\int_a^x g_n(t)\,dt - \int_a^x g(t)\,dt\right) + (f_n(a)-f(a)) = \int_a^x (g_n(t)-g(t))\,dt + (f_n(a)-f(a))$

$\therefore \|f_n(x)-f(x)\| \leq \underbrace{|a-x|\,\|g_n-g\|_{\infty,[a,x]}}_{x\in J \text{ segment}} + \underbrace{\|f_n(a)-f(a)\|}_{\xrightarrow{n\to\infty} 0 \text{ indep of } x}$

$\therefore$ There is convergence on segments. □

Shun/羊羽海 (@shun4midx)

COROLLARY

Let $p\geq 1$ be an integer, $f_n: I \to W$ be of class $\mathcal{C}^p$ $\forall n\geq 1$

Suppose (i) $(f_n^{(k)})_{n\geq 1}$ converges pointwise for $0\leq k\leq p-1$

(ii) $(f_n^{(p)})_{n\geq 1}$ converges uniformly on every segment

Then $f := \lim_{n\to\infty} f_n \in \mathcal{C}^p$ and for $0\leq k\leq p$, we have $f^{(k)}(x) = \lim_{n\to\infty} f_n^{(k)}(x)$ $\forall x\in \mathring{I}$

COROLLARY

Let $(u_n)_{n\geq 1}$ be a sequence of $\mathcal{C}^1$ functions $I \to W$.

Suppose (i) $\sum u_n$ converges pointwise

(ii) $\sum u_n'$ converges uniformly on every segment

Then, $\sum_{n\geq 1} u_n \in \mathcal{C}^1$ and $\left(\sum_{n\geq 1} u_n\right)' = \sum_{n\geq 1} u_n'$

We often say this as "We can differentiate the series term by term"

EXAMPLE

We want to check that ζ is of class $\mathcal{C}^1$.

Recall: $\zeta: (1,+\infty) \longrightarrow \mathbb{R}$

$s \longmapsto \sum_{n\geq 1} \frac{1}{n^s}$

For $n\geq 1$, let $u_n: (1,+\infty) \longrightarrow \mathbb{R}$ which is of class $\mathcal{C}^1$.

$s \longmapsto \frac{1}{n^s} = \exp(-s\ln n)$

- $\sum u_n(s)$ is well-defined $\forall s>1$, so $\sum u_n$ converges pointwise
- Let $a,b\in(1,+\infty)$, s.t. $1<a<b<+\infty$. Take $J=[a,b]$.

  For $n\geq 1$, we have $u_n'(s) = (-\ln n)\left(\frac{1}{n^s}\right)$ $\forall s>1$.

  Take $c\in(1,a)$. We see that $\|(u_n')|_J\|_\infty = \frac{\ln n}{n^a} = O\left(\frac{1}{n^c}\right)$. Since $\sum \frac{1}{n^s}$ converges, we deduce that $\sum (u_n')|_J$ converges normally, so uniformly.

  Apply corollary to conclude that $\zeta \in \mathcal{C}^1$ on $(1,+\infty)$ and $\zeta'(s) = \sum_{n\geq 1} -\frac{\ln n}{n^s}$ $\forall s>1$

COROLLARY

Let $p\geq 1$ be an integer, $(u_n)_{n\geq 1}$ be a sequence of $\mathcal{C}^p$ from $I \to W$

Suppose (i) For $0\leq k\leq p-1$, $\sum u_n^{(k)}$ converges pointwise

(ii) $\sum u_n^{(p)}$ converges uniformly on every segment

Then, $\sum_{n\geq 1} u_n \in \mathcal{C}^p$ and for $0\leq k\leq p$, we have $\left(\sum_{n\geq 1} u_n\right)^{(p)} = \sum_{n\geq 1} u_n^{(p)}$

Remark: This implies ζ is $\mathcal{C}^\infty$ and $\zeta^{(k)}(s) = \sum_{n\geq 1} \frac{(-\ln n)^k}{n^s}$ $\forall k\in\mathbb{N}$

EXAMPLE

Let $(W, \|\cdot\|)$ be a Banach space. Consider $\mathcal{L}_c(W)$ equipped over $|||\cdot|||$, which is still a Banach space. Moreover, it is also a normed algebra

Given $u\in\mathcal{L}_c(W)$, define $E_u: \mathbb{R} \longrightarrow \mathcal{L}_c(W)$. Define $u_n: \mathbb{R} \longrightarrow \mathcal{L}_c(W)$

$t \longmapsto \sum_{n\geq 0} \frac{t^n}{n!} u^n$ $\qquad$ $t \longmapsto \frac{t^n}{n!} u^n$

- $\forall t\in\mathbb{R}$, $\|u_n(t)\| \leq \frac{|t|^n |||u^n|||}{n!} \leq \frac{|t|^n |||u|||^n}{n!}$

  Since $\sum_{n\geq 0} \frac{|t|^n |||u|||^n}{n!} = \exp(|t|\,|||u|||) < +\infty$, thus $\sum u_n(t)$ conv abs and hence converges

Shun/翔海 (@shun4midx)

EXAMPLE

Let $(W, \|\cdot\|)$ be a Banach space, $\mathcal{L}_c(W)$ be equipped with the operator norm $|||\cdot||| \Rightarrow$ normed algebra, fix $u \in \mathcal{L}_c(W)$

Consider
$$\begin{aligned} \mathcal{E}_u : \mathbb{R} &\longrightarrow \mathcal{L}_c(W) \\ t &\longmapsto \sum_{n\geq 0} \frac{t^n}{n!} u^n \end{aligned}$$

We want to study the regularities and properties of $\mathcal{E}_u$

- Fix $t \in \mathbb{R}$, check that $\mathcal{E}_u(t)$ is well-defined, $\sum_{n\geq 0} \frac{|t|^n}{n!} |||u^n||| \leq \sum_{n\geq 0} \frac{|t|^n}{n!} |||u|||^n = \exp(|||t||| \cdot |||u|||) < +\infty$
  $\therefore \sum_{n\geq 0} \frac{t^n}{n!} u^n$ converges absolutely, so it converges.
- To check the continuity of $\mathcal{E}_u$ at some $t_0 \in \mathbb{R}$, we may check that the series of function converges uniformly on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$
  $\hookrightarrow$ More specifically, we check for uniform convergence on $[t_0 - \varepsilon, t_0 + \varepsilon]$ for a fixed $\varepsilon > 0$.
  Fix $M > 0$. We already know $t \mapsto \sum \frac{t^n}{n!} u^n$ converges pointwise on $[-M, M]$. It remains to show that the remainder function converges uniformly to 0.
  $\hookrightarrow$ For $t \in [-M, M]$, and $n \in \mathbb{N}$, we have $R_n(t) = \sum_{k\geq n+1} \frac{t^k}{k!} u^k \Rightarrow |||R_n(t)||| \leq \sum_{k\geq n+1} \frac{|t|^k}{k!} |||u|||^k \leq \sum_{k\geq n+1} \frac{M^k}{k!} |||u|||^k$. As $\sum \frac{M^n}{n!} |||u|||^n$ is a convergent series, RHS $\longrightarrow 0$ as it is the remainder of a convergent series
  $\hookrightarrow$ Additionally, this upper bound does NOT depend on $t \in [-M, M]$, so the series converges uniformly
  $\therefore \mathcal{E}_u$ is continuous on $[-M, M]$ $\forall M > 0$, i.e. $\mathcal{E}_u$ is continuous on $\mathbb{R}$
- For $n \in \mathbb{N}_0$, define $u_n(t) = \frac{t^n}{n!} u^n$ which is $\mathcal{C}^\infty$ $\forall n$ ($\because$ polynomial function)
  For $n \in \mathbb{N}_0$ and $t \in \mathbb{R}$, we have $u_n'(t) = \frac{n t^{n-1}}{n!} u^n = \begin{cases} u \cdot u_{n-1}, & n \geq 1 \\ 0, & n = 0 \end{cases}$
- Fix $M > 0$. The series $\sum_{n\geq 1} u \cdot u_{n-1} = \sum_{n\geq 1} u_n'$ converges uniformly (We also know $\sum u_n$ converges pointwise), so $\mathcal{E}_u$ is $\mathcal{C}^1$ on $[-M, M]$
- We know $\mathcal{E}_u'(t) = u \mathcal{E}_u(t)$ $\forall t \in [-M, M]$. Therefore, $\mathcal{E}_u$ is $\mathcal{C}^1$ on $\mathbb{R}$, $\mathcal{E}_u'(t) = u \cdot \mathcal{E}_u(t)$ $\forall t \in \mathbb{R}$
- Let $k \in \mathbb{N}$. If $\mathcal{E}_u$ is $\mathcal{C}^k$ for $k \geq 1$, then so is $\mathcal{E}_u' = u \cdot \mathcal{E}_u \Rightarrow \mathcal{E}_u \in \mathcal{C}^{k+1}$ $\therefore$ By induction on k, $\mathcal{E}_u$ is of class $\mathcal{C}^\infty$

POWER SERIES

We state the theorems and properties in $\mathbb{R}$ or $\mathbb{C}$, but they hold in general under normed algebra with minimal modifications.

DEFINITIONS AND RADIUS OF CONVERGENCE

In $(\mathbb{C}, |\cdot|)$, balls are called disks

For example, $D(a,r) = B(a,r) = \{y \in \mathbb{C} \mid |y - a| < r\} \Rightarrow$ open disk
$\bar{D}(a,r) = \bar{B}(a,r) = \{y \in \mathbb{C} \mid |y - a| \leq r\} \Rightarrow$ closed disk

DEFINITION

Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers and $c \in \mathbb{C}$.

- $\sum_{n\geq 0} a_n (z - c)^n$ is called a power series centered at $c \in \mathbb{C}$ with variable $z \in \mathbb{C}$.
- If $a_n \in \mathbb{R}$ $\forall n$ and $c \in \mathbb{R}$, then $\sum_{n\geq 0} a_n (x - c)^n$ is called a real power series centered at $c \in \mathbb{C}$ with variable $x \in \mathbb{R}$.

WLOG, we automatically assume $c = 0$ (by translation).

PROPOSITION (ABEL'S LEMMA)

Let $\sum a_n z^n$ be a power series. Let $z_0 \in \mathbb{C}$, s.t. $(a_n z_0^n)_{n\geq 0}$ is bounded. Then,

1) For $z \in \mathbb{C}$ with $|z| < |z_0|$, the series $\sum a_n z^n$ converges absolutely
2) For $r \in (0, |z_0|)$, the series of functions $\sum a_n z^n$ converges normally in the closed disk $\bar{D}(0, r)$. (In the entire disk, we have unif conv!!)

Proof

Let $M > 0$, s.t. $|a_n| |z_0|^n \leq M$. Let $z \in \mathbb{C}$ with $|z| < |z_0|$. Then, $\forall n \geq 0$, $|a_n z^n| = \overbrace{|a_n z_0^n|}^{\leq M} \overbrace{\left|\frac{z}{z_0}\right|^n}^{<1}$

Hence,

(1) $\sum |a_n z^n|$ converges because it can be upper bounded by a geometric series $\sum M \left|\frac{z}{z_0}\right|^n$ which converges. $\Rightarrow \sum a_n z^n$ converges absolutely
(2) Fix $r \in (0, |z_0|)$. For $z \in \bar{D}(0, r)$, $|a_n z^n| \leq M \left(\frac{r}{|z_0|}\right)^n$, with the upper bound independent on $z \in \bar{D}(0, r)$. Hence, it is normal convergence.

DEFINITION

Let $\sum a_n z^n$ be a power series. Define $R = R(\sum a_n z^n) := \sup\{r \geq 0 \mid (|a_n| r^n)_{n\geq 0}$ is bounded$\}$. This is called the radius of convergence of the power series $\sum a_n z^n$

REMARK

If we add phases ($a_n e^{i\theta_n}$) to the sequence $(a_n)_{n\geq 0}$, the power series $\sum a_n z^n$ has its radius of convergence remain unchanged

PROPOSITION

Let $\sum a_n z^n$ be a power series and R be its radius of convergence. Then,

(1) For $z \in \mathbb{C}$, with $|z|<R$, the series $\sum a_n z^n$ converges absolutely

(2) For $z \in \mathbb{C}$, with $|z|>R$, the series $\sum a_n z^n$ diverges

(3) For $r \in [0,R)$, the series $\sum a_n z^n$ converges normally on the closed disk $\bar{D}(0,r)$

Proof

(1) Let $z \in \mathbb{C}$ and $|z|<R$. Write $r = \frac{|z|+R}{2}$. Then, $|z|<r<R$.
By definition of R, we know that $(|a_n| r^n)_{n\geq 0}$ is bounded. $\therefore \sum a_n z^n$ converges absolutely

(3) Can be shown in the same way as (1)

(2) $|z|>R$ means that $(|a_n||z|^n)_{n\geq 0}$ is unbounded, so $a_n z^n \not\to 0 \Rightarrow \sum a_n z^n$ does not converge $\square$

REMARK

1) If $R=\infty$, the power series is well-defined on $\mathbb{C}$. Such a function is called an entire function

2) When $R<\infty$, and $z \in \partial D(0,R)$, the behavior of $\sum a_n z^n$ can have any behavior

PROPOSITION (D'ALEMBERT'S CRITERION, RATIO TEST)

If $\ell := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \in [0,+\infty]$ exists, then $R = \frac{1}{\ell}$

Proof

Let $z \in \mathbb{C}$. We want to check when $\sum a_n z^n$ converges.

We have: $\frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = \left|\frac{a_{n+1}}{a_n}\right| |z| \longrightarrow \ell |z| \begin{cases} <1, & |z|<\frac{1}{\ell} \\ =1, & |z|=\frac{1}{\ell} \\ >1, & |z|>\frac{1}{\ell} \end{cases}$

This means, by def, $R=\frac{1}{\ell}$. $\square$

PROPOSITION (CAUCHY'S CRITERION, ROOT TEST)

Let $\lambda := \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Then, $R = \frac{1}{\lambda}$.

Proof

Similar to above.

EXAMPLES ON THE BOUNDARY OF R

1) Consider the power series $\sum z^n$. Obviously, $R(\sum z^n) = 1$.
This means for $r \in [0,1)$, $\sum z^n$ converges normally on $\bar{D}[0,r]$.
For $z \in \partial D(0,1)$, $|z^n|=1 \Rightarrow \sum z^n$ diverges

2) Consider the power series $\sum \frac{z^n}{n^2}$ with $R(\sum \frac{z^n}{n^2}) = 1$
Here, $\left\|\frac{z^n}{n^2}\right\|_{\infty, D(0,1)} = \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges. Hence, $\sum \frac{z^n}{n^2}$ converges normally on $\bar{D}(0,1)$.

3) Consider $\sum \frac{z^n}{n}$, $R(\sum \frac{z^n}{n}) = 1$
- For $z=1$, $\sum \frac{1}{n}$ diverges
- For $z=e^{i\theta}$, $\theta \in \mathbb{R} \setminus (2\pi\mathbb{Z})$, $\sum \frac{e^{i\theta}}{n}$ converges because $\frac{1}{n} \downarrow 0$ and $\left|\sum_{k=0}^{n} e^{ik\theta}\right| = \left|\frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}\right| = \left|\sin \frac{n+1}{2}\theta\right| \div \left|\sin\frac{\theta}{2}\right| \leq \frac{1}{|\sin\frac{\theta}{2}|}$, which is bounded.

OPERATIONS ON POWER SERIES

PROPOSITION

Let $f(z) = \sum a_n z^n$ and $g(z) = \sum a_n z^n$ be power series with radius of convergence R_f and R_g respectively.

Let $R = R(\sum (a_n+b_n) z^n)$, then $R \geq \min(R_f, R_g)$

Moreover, if $R_f \neq R_g$, then $R = \min(R_f, R_g)$. $\forall z \in \mathbb{C}$ with $|z| < \min(R_f, R_g)$, we also have $\sum_{n\geq 0} (a_n+b_n) z^n = \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} b_n z^n$

Proof

Let $z \in \mathbb{C}$ with $|z| < \min(R_f, R_g)$. We know $(a_n z^n)_{n\geq 0}$ and $(b_n z^n)_{n\geq 0}$ are bounded, so $((a_n+b_n) z^n)_{n\geq 0}$ is bounded $\Rightarrow |z| \leq R$

$\therefore$ By taking $|z| \longmapsto \min(R_f, R_g)$ from below, we find that $\min(R_f, R_g) \leq R$ $\square$

Now, consider when $R_f \neq R_g$, by symmetry, suppose $R_f < R_g$

Let $z \in \mathbb{C}$, s.t. $R_f < z < R_g$. We know $(a_n z^n)_{n\geq 0}$ is bounded and $(b_n z^n)_{n\geq 0}$ is unbounded $\Rightarrow ((a_n + b_n) z^n)_{n\geq 0}$ is unbounded

This means that $|z| \geq R$. By taking $|z| \to R_f^+$ ← (limit, not "positive"), we find $R_f \geq R$.

We have already shown that $R \geq R_f$ $\therefore$ $R = R_f$ $\square$

When $|z| < \min(R_f, R_g)$, $\sum a_n z^n$ and $\sum b_n z^n$ converge $\therefore$ $\sum (a_n + b_n) z^n$ converges to $\sum a_n z^n + \sum b_n z^n$ $\square$ (limit proof of $\sum^{N}$, $N \to \infty$)

DEFINITION

Let $\sum a_n z^n$ and $\sum b_n z^n$ be power series. Their Cauchy product $\sum c_n z^n$ is given by $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Shun/翔海 (@shun4midx)

PROPOSITION

Let $f(z) := \sum_{n\geq 0} a_n z^n$, $g(z) := \sum_{n\geq 0} b_n z^n$ with radii of convergence R_f, R_g. For $z \in \mathbb{C}$ s.t. $|z| < \min(R_f, R_g)$, $f(z)g(z) = \sum_{n\geq 0} c_n z^n = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$

<u>Proof</u>

For $z \in \mathbb{C}$ with $|z| < \min(R_f, R_g)$, then $\sum a_n z^n$ and $\sum b_n z^n$ both converge absolutely. Then, the conclusion follows.

REGULARITY

Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence $R > 0$.

Known: f is well-defined on $D(0, R)$

THEOREM

f is continuous in the disk of convergence $D(0, R)$

<u>Proof</u>

Let $z \in D(0, R)$, take $r \in (0, R - |z|)$. Then, $\bar{D}(z, r) \subseteq \bar{D}(0, |z| + r) \subseteq \bar{D}(0, R)$

We know that $\sum_{n=0}^{N} a_n z^n$ is continuous $\forall N \in \mathbb{N}_0$, and the convergence $z \mapsto \sum a_n z^n$ is normal on $\bar{D}(0, |z| + r)$.

$\therefore$ f is continuous on $\bar{D}(0, |z| + r)$, so also at z. □

THEOREM (ABEL'S THEOREM)

Suppose $\sum a_n R^n$ converges. Then,

$$\begin{aligned} [0, R] &\longrightarrow \mathbb{C} \\ x &\longmapsto \sum_{n\geq 0} a_n x^n \end{aligned}$$

is continuous

In other words, $\lim_{x \to R^-, x \in \mathbb{R}} \sum a_n x^n = \sum a_n R^n$

Note: The continuity is only "radial", not across the entire disk

<u>Proof</u>

By rescaling, we may assume $R = 1$

Let $R_n := \sum_{k\geq n+1} a_k R^k = \sum_{k\geq n+1} a_k$ be the n^{th} remainder of the convergent series $\sum a_n$

- We already know f converges pointwise on $[0, R]$ $([0, R] \subseteq \bar{D}[0, R])$
- It remains to show the uniform convergence to 0 of the remainder of the series of function

For $m > n$, and $x \in [0, 1]$, we have $\sum_{k=n+1}^{m} a_k x^k = \sum_{k=n+1}^{m} (R_{k-1} - R_k) x^k = \sum_{k=n}^{m-1} R_k x^{k+1} - \sum_{k=n+1}^{m} R_k x^k = R_n x^{n+1} - \underbrace{R_m x^m}_{\xrightarrow{m\to\infty} 0} + \underbrace{\sum_{k=n+1}^{m-1} R_k (x^{k+1} - x^k)}$

$|\cdot| \leq M(x^{k+1} - x^k)$

$\sum |\cdot| \leq M x^{k+1}$ indep of m

$\therefore$ abs conv as $m \to \infty$

This means $\sum_{k=n+1}^{\infty} a_k x^k = R_n x^{n+1} + \sum_{k\geq n+1} R_k (x^{k+1} - x^k)$ $\forall n \in \mathbb{N}$, $x \in [0, 1]$

Now check that it converges uniformly to 0 on $[0, 1]$

Let $\varepsilon > 0$. Take $N \in \mathbb{N}$, s.t. $|R_n| \leq \varepsilon$ $\forall n \geq N$.

For $n \geq N$, $x \in [0, 1)$, we have $\left|\sum_{k\geq n+1} a_k x^k\right| \leq |R_n x^{n+1}| + \sum_{k\geq n+1} |R_k| (x^k - x^{k+1}) \leq \varepsilon + \varepsilon x^{k+1} \leq 2\varepsilon$ $\forall x \in [0, 1]$, $n \geq N$.

$\therefore$ $\sum a_n z^n$ converges uniformly on $[0, R]$. □

THEOREM (TAUBER'S THEOREM)

Suppose $\lim_{x \to R^-, x \in \mathbb{R}} f(x) = \ell$ exists and $n a_n \xrightarrow{n\to\infty} 0$. Then, $\sum_{n\geq 0} a_n R^n = \ell$ (where $f(x) = \sum a_n x^n$)

<u>Proof</u>

WLOG, take $R = 1$. Let $S_n = \sum_{k=0}^{n} a_k$ $\forall n \in \mathbb{N}_0$.

For $x \in [0, 1)$, we have $S_n - f(x) = \sum_{k=0}^{n} a_k - \sum_{k\geq 0} a_k x^k = \underbrace{\sum_{k=0}^{n} a_k (1 - x^k)}_{(1)} - \underbrace{\sum_{k\geq n+1} a_k x^k}_{(2)}$

As $1 - x^k = (1 - x)(1 + x + \ldots + x^{k-1})$, $|1 - x^k| \leq |1 - x| k$, $k \in \mathbb{N}_0$, $x \in [0, 1)$,

$\therefore$ $|(1)| \leq \sum_{k=0}^{n} (1 - x) \cdot k|a_k| = (1 - x) \sum_{k=0}^{n} |k a_k|$

(Cesàro sum)

We know $|n a_n| \xrightarrow{n\to\infty} 0 \Rightarrow \frac{1}{n} \sum_{k=0}^{n-1} |k a_k| \xrightarrow{n\to\infty} 0$. If we take $x_n = 1 - \frac{1}{n}$, then $|(1)| \leq \frac{1}{n} \sum_{k=0}^{n} |k a_k| \xrightarrow{n\to\infty} 0$

- Let $\varepsilon>0$. Take N s.t. $|a_n n| \le \varepsilon$ $\forall n \ge N$. For $n \ge N$, we have for $x \in [0,1)$, $\sum_{k \ge n+1} |a_k| x^k \le \sum_{k \ge n+1} \frac{\varepsilon}{k} x^k \le \frac{\varepsilon}{n} \sum_{k \ge n+1} x^k \le \frac{\varepsilon}{n} \frac{x^{n+1}}{1-x} \le \frac{\varepsilon}{n(1-x)}$

  If we take $x_n = 1-\frac{1}{n}$, then $|(2)| \le \varepsilon$

$\therefore$ In conclusion, let $\varepsilon>0$, take N, s.t. $|a_n n| \le \varepsilon$ $\forall n \ge N$. For $n \ge N$, take $x_n = 1-\frac{1}{n}$. Then, $|S_n - f(x_n)| \le \frac{1}{n}\sum_{k=0}^{n} |k a_k| + \varepsilon \Rightarrow \limsup_{n\to\infty} |S_n - f(x_n)| \le \varepsilon$

This holds $\forall \varepsilon>0$, so we deduce that $\lim |S_n - f(x_n)| = 0$

By the assumption, $f(x_n) \xrightarrow{n\to\infty} \ell \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}(S_n - f(x_n)) + \lim_{n\to\infty} f(x_n) = \ell$ $\square$

COROLLARY

Let $\sum a_n$, $\sum b_n$ be convergent series. Define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Suppose $\sum c_n$ is convergent. Then, $\sum_{n\ge0} c_n = \left(\sum_{n\ge0} a_n\right)\left(\sum_{n\ge0} b_n\right)$

Proof

Sketch: Abel's thm + equality in the disk: $(\sum a_n z^n)(\sum b_n z^n) = \sum c_n z^n$ $\forall z \in (-1,1)$, take lim as $z \to 1^-$. $\square$

DEFINITION

Let $A \subseteq \mathbb{C}$ and $f: A \longrightarrow \mathbb{C}$. For $z \in A$, we say f is $\mathbb{C}$-differentiable (or simply differentiable) at $z_0 \in \mathring{A}$ if the following limit exists,

$\frac{df}{dz}(z_0) = \frac{d}{dz}f(z_0) = f'(z_0) := \lim_{z\to z_0, z\in\mathbb{C}} \frac{f(z)-f(z_0)}{z-z_0} \in \mathbb{C}$, which is also called the $\mathbb{C}$-derivative of f at z_0.

REMARK

This notation is much stronger than the differential defined in the first term. Additionally, scaling + rotation $\Rightarrow$ linear + continuous

Note that $\frac{d}{dz}(z^n) = nz^{n-1}$ for all $n \in \mathbb{N}_0$.

Let $f(z) = \sum a_n z^n$ be a power series with $R>0$.

THEOREM

Let $f \in \mathcal{C}^1$. Then,

- $R(\sum n a_n z^{n-1}) = R(\sum a_n z^n)$
- $\forall z \in D(0,R)$, $f'(z) = \sum_{n\ge1} n a_n z^{n-1} = \sum_{n\ge0} (n+1) a_{n+1} z^n$

Proof

Let $R' := R(\sum n a_n z^{n-1})$.

- "$R' \le R$": Let $z \in D(0,R')$. We know $(n a_n z^{n-1})_{n\ge0}$ is bounded $\Rightarrow (a_n z^n)_{n\ge0}$ is bounded $\Rightarrow |z| \le R \Rightarrow R' \le R$ by taking z to the boundary
- "$R \le R'$": Let $z \in D(0,R)$. We know $(a_n z^n)_{n\ge0}$ is bounded

  Take $z' \in D(0,|z|)$, $n a_n (z')^{n-1} = \underbrace{a_n z^{n-1}}_{\text{bounded}} \underbrace{\left(\frac{z'}{z}\right)^{n-1} \cdot n}_{\to 0 \text{ if } |\frac{z'}{z}|<1} \Rightarrow (n a_n |z'|^n)_{n\ge0}$ is bounded

  $\therefore |z'| \le R' \Rightarrow |z| \le R' \Rightarrow R \le R'$

$\therefore$ In conclusion, $R = R'$ $\square$

Note, $\sum n a_n z^{n-1}$ converges normally on $\bar{D}(0,r)$ for $r<R$. $\sum a_n z^n$ converges normally on $\bar{D}(0,r)$ for $r<R$.

As every $z \mapsto a_n z^n$ is $\mathcal{C}^1$, so $\sum a_n z^n \in \mathcal{C}^1$ on $\bar{D}(0,R)$ and can be differentiated term by term $\square$

(By induction, $f \in \mathcal{C}^\infty$)

COROLLARY

f is $\mathcal{C}^\infty$ on $D(0,R)$.

$\forall z \in D(0,R)$, $f^{(p)}(z) = \sum_{n\ge p} n(n-1)\cdots(n-p+1) a_n z^{n-p} = \sum_{n\ge p} \binom{n}{p} p!\, a_n z^{n-p}$

In particular, this gives $\forall p \in \mathbb{N}_0$, $a_p = \frac{f^{(p)}(0)}{p!}$ and $\forall z \in D(0,R)$, $f(z) = \sum_{p\ge0} \frac{f^{(p)}(0)}{p!} z^p$

EXAMPLE

$\forall z \in D(0,1)$, $\frac{1}{1-z} = \sum_{n\ge0} z^n \overset{\text{derivative}}{\Rightarrow} \frac{1}{(1-z)^2} = \sum_{n\ge0} n z^{n-1} = \sum_{n\ge0} (n+1) z^n$ $\forall z \in (0,1)$. By induction, $\frac{p!}{(1-z)^{p+1}} = \sum_{n\ge0} \binom{p+1}{p} z^n$

Also, $\frac{1+z}{(1-z)^3} = -\frac{1-z}{(1-z)^3} + \frac{2z}{(1-z)^3} = \sum_{n\ge1} n^2 z^{n-1} \Rightarrow \sum_{n\ge1} \frac{n^2}{2^n} = 6$ $\leftarrow z=\frac{1}{2}$ (Multiply the above by z and diff again)

Shun/翔海 (@shun4midx)

COROLLARY

Define $F: D(0,R) \longrightarrow \mathbb{C}$

$z \longmapsto \sum_{n\geq 0} \frac{a_n}{n+1} z^{n+1}$

Then, F has the same radius of convergence and $F' = f$ on $D(0,R)$

APPLICATION

$\forall x \in (-1,1)$, $\frac{1}{1-x} = \sum_{n\geq 0} x^n \Rightarrow -\ln(1-x) = \sum_{n\geq 0} \frac{x^{n+1}}{n+1}$

By Abel's thm, $\sum \frac{(-1)^n}{n+1}$ converges.

Hence, $\lim_{x\to -1} \sum_{n\geq 0} \frac{x^{n+1}}{n+1} = \sum \frac{(-1)^{n+1}}{n+1} \Rightarrow \lim_{x\to -1} \ln(1-x) = \sum \frac{(-1)^n}{n+1} \Rightarrow \ln 2 = \sum \frac{(-1)^n}{n+1}$

COEFFICIENTS OF POWER SERIES

COROLLARY (UNIQUENESS OF POWER SERIES)

Let $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ with $R_f := R(\sum a_n z^n) > 0$, $R_g := R(\sum b_n z^n) > 0$

Suppose that there exists $r \in [0, \min(R_f, R_g)]$ s.t. $f \equiv g$ on $(-r, r)$. Then, $a_n \equiv b_n$ $\forall n > 0$

Proof

Let $r > 0$ s.t. $f \equiv g$ on $(-r, r)$.

Since f and g are both C^∞ and they are equal, we deduce that $f^{(k)} \equiv g^{(k)}$ on $(-r, r)$, $\forall k \in \mathbb{N}_0$.

This implies $f^{(k)}(0) = g^{(k)}(0)$ $\forall k > 0$. Hence, $a_k = b_k$ $\forall k > 0$. □

EXAMPLE

Let $f(z) = \sum_{n\geq 0} a_n z^n$ with $R > 0$. Suppose that f is an even function, i.e. $f(-z) = f(z)$ $\forall z \in (-R, R)$. Then, $(-1)^n a_n = a_n$ $\forall n \geq 0$. In particular, $a_n = 0$ $\forall$ odd n.

THEOREM (CAUCHY'S FORMULA)

Let $f(z) = \sum_{n\geq 0} a_n z^n$ with $R > 0$. Then, for $r \in (0, R)$ and $n \in \mathbb{N}_0$, we have $r^n a_n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta$

Proof

We write $\int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta = \int_0^{2\pi} \sum_{k\geq 0} a_k r^k e^{i(k-n)\theta} d\theta \overset{*}{=} \sum_{k\geq 0} a_k r^k \int_0^{2\pi} e^{i(k-n)\theta} d\theta = a_n r^n 2\pi$

Now, let us check that we can indeed interchange Σ and $\int$.

The series of function is given by
$$\begin{aligned} [0, 2\pi] &\longrightarrow \mathbb{C} \\ \theta &\longmapsto \sum_{k\geq 0} a_k r^k e^{i(k-n)\theta} \end{aligned}$$

For every $k \geq 0$, $\| \theta \longmapsto a_k r^k e^{i(k-n)\theta} \|_{\infty, [0, 2\pi]} = |a_k| r^k$

Since $r < R$, we know $\sum |a_k| r^k$ converges, so the series of functions converges normally and uniformly □

EXPANSION IN POWER SERIES

DEFINITION

Let $A \subseteq \mathbb{C}$ be an open set and $f: A \longrightarrow \mathbb{C}$.

- Let $R > 0$, if $0 \in A$ and there exists a power series $\sum a_n z^n$ such that $\forall z \in D(0, R)$, $f(z) = \sum_{n\geq 0} a_n z^n$, then we say that f can be expanded into a power series around 0 or expanded into a power series on $D(0, R)$

  In particular, we know that $R(\sum a_n z^n) \geq R$ and f is C^∞ on $D(0, R)$.
- Let $z_0 \in A$. We say that f can be expanded into a power series around z_0 if $z \mapsto f(z + z_0)$ can be expanded into a power series around 0 ($\Leftrightarrow f(z) = \sum a_n (z - z_0)^n$)

PROPOSITION

Let $A \subseteq \mathbb{C}$ be open and $0 \in A$. Then, (1) $\Leftrightarrow$ (2)

(1) f can be represented as a power series around 0

(2) There exists r, s.t. the remainder $(R_n)_{n\geq 0}$ converges pointwise to 0 on $D(0, r)$, where $R_n(z) = f(z) - \sum_{k=0}^{n} a_k z^k = \sum_{k\geq n+1} a_k z^k \xrightarrow[\text{ptwise}]{n\to\infty} 0$ and $a_k = \frac{f^{(k)}(0)}{k!}$

When (2) holds, we have $R(\sum a_k z^k) > r$ and $f = \sum a_k z^k$ on $D(0, r)$.

Proof

"$\Rightarrow$": By def ✓

"$\Leftarrow$": Let r, s.t. (2) holds. For $z \in D(0, r)$, $\sum a_k z^k$ converges, so $(a_k z^k)_{k\geq 0}$ is bounded, and $|z| \leq R(\sum a_k z^k) \Rightarrow r \leq R(\sum a_k z^k)$ □

REMARK

How to check $R_n(z) \xrightarrow{n\to\infty} 0$?

(1) The remainder R_n can be estimated using

↳ Taylor-integral formula: $R_n(z) = z^{n+1} \int_0^1 \frac{(1-t)^n}{n!} f^{(n+1)}(t) dt$

↳ Taylor-Lagrange: $R_n(z) = \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(\theta z)$, $\theta \in [0, 1]$

(2) It is NOT enough to check that $R(\sum \frac{f^{(k)}(0)}{k!} z^k) > 0$

Shun/翔海(@shun4midx)

EXAMPLE

Consider $f: \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longmapsto \begin{cases} e^{-\frac{1}{x}}, & x>0 \\ 0, & x \le 0 \end{cases}$$

We are going to check that $f^{(k)}(0)=0 \ \forall k>0$, $R(\sum a_n z^n)=\infty$

But clearly, $f(x) \neq 0$ for $x>0$, so we cannot expand into a power series around 0.

- $\forall x>0$, $f'(x)=\frac{1}{x^2}e^{-\frac{1}{x}}=\frac{1}{x^2}f(x)$, $f''(x)=(-\frac{2}{x^3}-\frac{1}{x^2})f(x)$

  We have $f'(x), f''(x) \xrightarrow{x\to\infty} 0$

  In general, $f^{(k)}(x)=P_k(\frac{1}{x})e^{-\frac{1}{x}}$ for some polynomial P, $\deg P \le 2k$.

  $\Rightarrow f^{(k)}(x) \xrightarrow{x\to\infty} 0$

  $\therefore$ The function f is C^∞ on $\mathbb{R}$ with $f^{(k)}(x)=0 \ \forall k$

- Note that $f(x)=o(x^n)$, Taylor expansion tells us $f(x)=0+\dots+0+\underline{f(x)}$ ← as $o(x^n)$

EXAMPLE

1) $z \mapsto e^z$ can be expanded around 0

   For $z\in\mathbb{C}$, $R_n(z)=e^z-\sum_{k=0}^{n}\frac{z^k}{k!}$

   We know that $R_n(z)=\frac{z^{n+1}}{(n+1)!}f^{(n+1)}(\theta z)$ for some $\theta=\theta(z)\in(0,1)$

   $\therefore$ For $z\in\mathbb{C}$, $|R_n(z)|=\frac{|z|^{n+1}}{(n+1)!}e^{\theta+\mathrm{Re}(z)} \xrightarrow{n\to\infty} 0$

2) $z \mapsto \frac{1}{1-z}$ is defined on $\mathbb{C}\setminus\{1\}$.

   The expansion around 0: $\forall z\in D(0,1)$, $\frac{1}{1-z}=\sum_{n\ge0} z^n$

   For $z\in D(0,1)$, $|R_n(z)| \le \sum_{k\ge n+1}|z|^k=\frac{|z|^{n+1}}{1-|z|} \xrightarrow{n\to\infty} 0$

3) For a polynomial $P\in\mathbb{R}[x]$, $\forall z\in\mathbb{C}$, $P(z)=\sum_{n\ge0}\frac{P^{(n)}(0)}{n!}z^n$ is a finite series

PROPOSITION

If f can be written as a power series in $D(0,R)$, $R>0$, then for any $z_0\in D(0,R)$, f can also be written as a power series around z_0

<u>Proof</u>

Let $R>0$ and $(a_n)_{n\ge0}$ s.t. $f(z)=\sum a_n z^n$ for $z\in D(0,R)$. Fix $z_0\in D(0,R)$, take $r\in(0,R-|z|)$. We want to write f as a power series in $D(z,r)$, i.e. in the form $\sum b_n(z-z_0)^n$

For $z\in D(z,r)$, write $\sum_{n\ge0}a_n z^n=\sum_{n\ge0}a_n((z-z_0)+z_0)^n \overset{\text{binomial expansion}}{=} \sum_{n\ge0}a_n\sum_{k=0}^{n}\binom{n}{k}z^{n-k}(z-z_0)^k=\sum_{n\ge0}\sum_{k\ge0}\mathbb{1}_{n\ge k}\,a_n\binom{n}{k}z^{n-k}(z-z_0)^k \overset{?}{=} \sum_{k\ge0}\Big(\underbrace{\sum_{n\ge0}\mathbb{1}_{n\ge k}\,a_n\binom{n}{k}z_0^{n-k}}_{b_n}\Big)(z-z_0)^k$

- For $n\ge0$, $\sum_{k\ge0}\mathbb{1}_{n\ge k}a_n=\sum_{k=0}^{n}a_n$ is a finite sum, so it converges absolutely
- $\sum_{n\ge0}\sum_{k\ge0}|a_n|\,\mathbb{1}_{n\ge k}\binom{n}{k}|(z-z_0)^k|\,|z|^{n-k}=\sum_{n\ge0}|a_n|\underbrace{(|z-z_0|+|z_0|)}_{<R}{}^n<+\infty$, because we are inside the disk of convergence. □

APPLICATIONS TO ODE

- Know that the solution can be expanded into a power series

  ↳ Write $f(z)=\sum a_n z^n \longrightarrow$ plug into the ODE $\Rightarrow$ get relations between coefficients
- Don't know that the solution can be written as a power series, we can assume that there is such a solution. Then, apply the previous step and check $R>0$.

EXAMPLE

Consider $f: \mathbb{R} \longrightarrow \mathbb{R}$, power sum around 0?

$$x \longmapsto e^{x^2}\int_0^x e^{-t^2}\,dt$$

- $x \mapsto e^{x^2}$ can be written as a power series on $\mathbb{R}$ (centered anywhere)
- $t \mapsto e^{-t^2}$ the same, we can integrate on $\mathbb{R}$
- Cauchy product has a radius of convergence $=\infty$

  $\Big\}\Rightarrow$ can be expanded around any $x\in\mathbb{R}$
- $f'(x)=2xe^{x^2}\int e^{-t^2}dt+1=2xf(x)+1$, $f(0)=0$ (ODE)

Shun/翔海(@shun4midx)

Suppose $f(x) = \sum_{n\geq 0} a_n x^n$ on $\mathbb{R}$.

$$\left.\begin{array}{l} f'(x) = \sum_{n\geq 1} n a_n x^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1} x^n \\ 1+2xf(x) = 1+\sum_{n\geq 1} 2a_{n-1} x^n \end{array}\right\} \overset{\text{uniqueness of coef}}{\Rightarrow} \begin{cases} (n+1)a_{n+1} = 2a_{n-1} & \forall n \geq 1 \\ a_1 = 1 \end{cases}$$

← recursive relation ✓

, with initial condition $a_0 = 0$ from $f(0) = 0$

(We may check that this power series has $R > 0$)

EXAMPLE

Let $\alpha \in \mathbb{C}$. Consider
$$\begin{aligned} f : (-1,1) &\longrightarrow \mathbb{C} \\ x &\longmapsto (1+x)^\alpha = e^{\alpha \ln(1+x)} \end{aligned}$$

$\forall x \in (-1,1)$, $f'(x) = \frac{\alpha}{1+x} f(x)$, $f(0) = 1$

Suppose $f(x) = \sum_{n\geq 0} a_n x^n$ $\forall x \in (-1,1) \Rightarrow f'(x) = \sum_{n\geq 0} (n+1) a_{n+1} x^n \Rightarrow (1+x) f'(x) = \sum_{n\geq 0} ((n+1)a_{n+1} + n a_n) x^n = \sum_{n\geq 0} \alpha a_n x^n$

$\therefore$ $(n+1)a_{n+1} = (\alpha - n) a_n$ $\forall n \geq 0$.

$\therefore$ $a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} =: \binom{\alpha}{n} \Rightarrow f(x) = \sum_{n\geq 0} \binom{\alpha}{n} x^n$, check $R > 0$.