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THEOREM

Let $(a_n)_{n\geq 1}$ be a sequence with strictly positive terms. Then, $\prod(1+a_n)$ converges $\Leftrightarrow$ $\sum a_n$ converges

Proof

Observe: $\prod(1+a_n)$ converges $\Leftrightarrow$ $\sum \ln(1+a_n)$ converges

"$\Rightarrow$": Suppose $\prod(1+a_n)$ converges, then $\sum \ln(1+a_n)$ converges.

$\therefore \ln(1+a_n) \xrightarrow{n\to\infty} 0$, i.e. $a_n \xrightarrow{n\to\infty} 0$

$\therefore$ We have $\ln(1+a_n) \sim a_n$ as $n\to\infty$

By comparison thm, $\sum \ln(1+a_n)$ conv $\Rightarrow$ $\sum a_n$ converges

"$\Leftarrow$": Suppose $\sum a_n$ conv, so $a_n \xrightarrow{n\to\infty} 0$

$\therefore \ln(1+a_n) \sim a_n$

By comparison thm, $\sum a_n$ conv $\Rightarrow$ $\sum \ln(1+a_n)$ conv $\therefore \prod(1+a_n)$ conv $\square$

REMARK

The positivity assumption is important!

Consider $a_n = \frac{(-1)^n}{\sqrt{n}}$ for $n\geq 1$

↳ $\sum a_n$ conv since alternating series

↳ For $n\geq 1$, $(1+a_{2n})(1+a_{2n+1}) = \left(1+\frac{1}{\sqrt{2n}}\right)\left(1-\frac{1}{\sqrt{2n+1}}\right) = 1+\frac{1}{\sqrt{2n}}-\frac{1}{\sqrt{2n+1}}-\frac{1}{\sqrt{2n(2n+1)}} = 1+\frac{1}{\sqrt{2n}}-\frac{1}{\sqrt{2n}}\left(\frac{1}{\sqrt{1+\frac{1}{2n}}}\right)-\frac{1}{\sqrt{2n(2n+1)}} = 1-\frac{1}{2n}+o(n^{-\frac{3}{2}})$

However, $\prod(1-\frac{1}{2n})$ div because $\sum\frac{1}{2n}$ div $\Rightarrow \prod(1+a_{2n})(1+a_{2n+1})$ div $\Rightarrow \prod(1+a_n)$ div

REMARK

If $(a_n)_{n\geq 1}$ is a sequence with $a_n \in (-1,0)$, we still have the same proof for "$\sum a_n$ conv $\Leftrightarrow \prod(1+a_n)$ conv"

DEFINITION

Let $(a_n)_{n\geq 1}$ be a nonzero complex-valued sequence (or over a normed algebra). We say that $\prod(1+a_n)$ conv absolutely if $\prod(1+|a_n|)$ conv

PROPOSITION

If $\prod(1+a_n)$ conv abs, then it must also conv

Proof

For $n\geq 1$ and $k\geq 0$, by Δ ineq, we have $\left|\prod_{i=n}^{n+k}(1+a_i)-1\right| \leq \prod_{j=1}^{k}(1+|a_{n+j}|)-1$ (simple expansion)

Therefore, if $\prod(1+|a_n|)$ satisfies the Cauchy's condition, so does $\prod(1+a_n)$ $\square$

APPLICATION TO THE RIEMANN ζ FUNCTION

Let $(p_n)_{n\geq 1}$ be the sequence of ordered primes, i.e. $p_1=2$, $p_2=3$, $p_3=5, \ldots$

THEOREM (EULER'S PRODUCT)

For $s>1$, we have $\zeta(s) := \sum_{n\geq 1}\frac{1}{n^s} = \prod_{k=1}^{\infty}\frac{1}{1-p_k^{-s}}$. Moreover, the infinite product conv abs

Proof

For $n\geq 1$, let $P_n = \prod_{k=1}^{n}\frac{1}{1-p_k^{-s}}$

Goal: Show that $P_n \xrightarrow{n\to\infty} \zeta(s) \neq 0$

For any $k\geq 1$, we have $\underbrace{\frac{1}{1-p_k^{-s}}}_{\in(0,1)} = \underbrace{\sum_{m\geq 0}(p_k^{-s})^m}_{\text{Geom series}} = \sum_{m\geq 0}p_k^{-sm} = 1+\sum_{m\geq 1}p_k^{-sm} =: 1+a_k$

Note that $a_k>0$ $\forall k\geq 1$. $\sum_{k=1}^{N}a_k = \sum_{k=1}^{N}\sum_{m_k\geq 1}p_k^{-sm_k} \leq \zeta(s)$

$\therefore \sum a_k$ conv (: it abs conv), so $\prod(1+a_n)$ conv

Hence, $P_n = \prod_{k=1}^{n}\frac{1}{1-p_k^{-s}} = \prod_{k=1}^{n}\sum_{m\geq 0}\frac{1}{p^{ms}} = \sum_{m_1,\ldots,m_n\geq 0}\frac{1}{(p_1^{m_1}\cdots p_n^{m_n})^s}$. Define $A_n = \{N\in\mathbb{N} \mid N$ has all prime factors along $p_1,\ldots,p_n\}$. $\therefore P_n = \sum_{N\in A_n}\frac{1}{N^s} \Rightarrow |P_n-\zeta(s)| \leq \sum_{N\geq p_{n+1}}\frac{1}{N^s}$ $\square$

REMARK

We may show that $\sum_{k\geq 1}\frac{1}{p_k}$ div using the Riemann zeta function.

COMPLEMENT ON RIEMANN INTEGRALS

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DEFINITION

- A function $f:[a,b]\to\mathbb{R}$ is said to be **piecewise continuous** on $[a,b]$ if $\exists$ a partition $P=(x_k)_{0\le k\le n}\in\mathcal{P}([a,b])$, s.t. $f|_{(x_{i-1},x_i)}$ is conti and can be extended continuously to $[x_{i-1},x_i]$, $\forall i$
- Let $I\subseteq\mathbb{R}$ be a subset. A function $f:I\to\mathbb{R}$ is said to be **piecewise continuous** if $f|_J$ is p.c. $\forall$ segments $J\subseteq I$
- Let $PC(I,\mathbb{R})$ be the set of piecewise continuous functions on I
- For any normed vector space $(W,\|\cdot\|)$, we define $PC(I,V)$ similarly

EXAMPLES OF PIECEWISE CONTINUOUS FUNCTIONS

1) $\begin{cases} x\longmapsto \frac{1}{x} \\ \mathbb{R}^*:=\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}\end{cases}$

2) $x\longmapsto \ln x$

   $(0,+\infty)\mapsto\mathbb{R}$

3) $x\longmapsto \sin(\frac{1}{x})$

   $\mathbb{R}^*\longrightarrow\mathbb{R}$

PROPOSITION

Let $I=[a,b]$ be a segment of $\mathbb{R}$. If $f:I\longrightarrow\mathbb{R}$ is a piecewise continuous function, then it is **bounded** and **Riemann-integrable** on I.

Proof

Let $P=(x_k)_{0\le k\le n}\in\mathcal{P}([a,b])$ be a partition satisfying the definition. Then, on $[x_{i-1},x_i]$, we may define a conti $g_i:[x_{i-1},x_i]\to\mathbb{R}$, s.t. $g_i|_{(x_{i-1},x_i)}\equiv f|_{(x_{i-1},x_i)}$

Therefore, $g_i\in R(x;x_{i-1},x_i)$ and $f\in R(x;x_{i-1},x_i)$

To conclude, we use the cyclic relation to observe $f\in R(x;a,b)$ □

INTEGRABILITY ON AN INTERVAL

Let I be an interval, $PC_+(I):=PC_+(I,\mathbb{R}_+)$

DEFINITION

Let $f\in PC_+(I)$. We say that f is **integrable** on I, if $\exists M>0$, s.t. $\int_J f\le M$ for any segment $J\subseteq I$, and we write $\int_I f=\sup_{J\subseteq I}\int_J f$

$\llcorner$ J is a segment

REMARK

If I is an interval, $a=\inf I$, $b=\sup I$, we may write $\int_I f=\int_a^b I$ (this is the generalization of notation)

PROPOSITION

Let $f\in PC_+(I)$ be an integrable function. Then, for any sequence of segments $(J_n=[a_n,b_n])_{n\ge1}$ with $\forall n\ge1$, $J_n\subseteq J_{n+1}\subseteq\ldots\subseteq I$ and $\bigcup_{n\ge1}J_n=I$, we have $\int_I f=\sup_{n\ge1}\int_{J_n}f=\lim_{n\to\infty}\int_{J_n}f=\lim_{n\to\infty}\int_{a_n}^{b_n}f(x)dx$ (Focus is $\lim_{n\to\infty}\int_{J_n}f=\int_I f$)

Proof

Given such a sequence $(J_n)_{n\ge1}$, we want to show $\int_{J_n}f\xrightarrow{n\to\infty}\int_I f$

- $\forall n\ge1$, $J_n\subseteq I$ is a subsegment $\Rightarrow\int_{J_n}f\le\int_I f\Rightarrow \limsup_{n\to\infty}\int_{J_n}f\le\int_I f$
- Given $\varepsilon>0$, by the characterization of sup, we may find a subsegment $J\subseteq I$, s.t. $\int_J f+\varepsilon\ge\int_I f$

  Since $\bigcup_{n\ge1}J_n=I$ and $(J_n)_{n\ge1}$ is increasing for inclusion, we may find $N>0$, s.t. $J_n\supseteq J$ $\forall n\ge N$.

  $\Rightarrow \int_I f\le\int_{J_n}f+\varepsilon$ $\forall n\ge N$

  $\Rightarrow \int_I f\le\liminf_{n\to\infty}\int_{J_n}f+\varepsilon$

  This relation holds $\forall\varepsilon>0$, so we have $\int_I f\le\liminf_{n\to\infty}\int_{J_n}f$

$\therefore \lim_{n\to\infty}\int_{J_n}f=\int_I f$ □