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3-13-25 (WEEK 4)

INTEGRALS DEPENDING ON A PARAMETER

(Sorry, today's notes would have terrible handwriting cuz lots of rain got into my stylus/iPad and it's so heavy to hold the pen so it hurts to even write, let alone neat qwq 本当にごめんね qwq orz 手がとても痛いTT)

QUESTION

Given $f: [0,1] \times [0,1] \longrightarrow \mathbb{R}$

$(x,y) \longmapsto f(x,y)$

- Integrate f w.r.t. x, how do we define the regularity of f and its integral?
- Interchange the order of integration?

PROPOSITION

Let $Q = [a,b] \times [c,d] \subseteq \mathbb{R}^2$.

Let $f: Q \to \mathbb{R}$ be continuous and $\alpha \in BV([a,b])$

Define $F: [c,d] \longrightarrow \mathbb{R}$

$y \longmapsto \int_a^b f(x,y)\,d\alpha(x)$

Then, F is continuous, that is $F(y_0) = \lim_{y \to y_0} F(y)$ $\forall y_0 \in [c,d]$.

Proof

Since Q is compact, f is uniformly continuous. Suppose α is non-increasing

Let $\varepsilon > 0$, and take $\delta > 0$, s.t. $\forall (x,y), (x',y') \in Q$, $\|(x,y) - (x',y')\|_1 \leq \delta \Rightarrow |f(x,y) - f(x',y')| \leq \varepsilon$

Fix $y_0 \in [c,d]$. For $y \in [c,d]$, s.t. $|y - y_0| \leq \delta$, we have $|F(y) - F(y_0)| = |\int_a^b (f(x,y) - f(x,y_0))\,d\alpha(x)| \leq \int_a^b |f(x,y) - f(x,y_0)|\,d\alpha(x) \overset{\text{requires uniform continuity}}{\leq} \int_a^b \varepsilon\,d\alpha(x) = \varepsilon[\alpha(b) - \alpha(a)]$ □

COUNTEREXAMPLE FOR NONCONTINUOUS f

Let $f(x,y) = \begin{cases} \frac{4xy^2}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ (Behind the scenes: My hand is now screaming in agony from the pain of writing o_O)

It's not hard to see that $\forall x \in [0,1]$, $f(x,\cdot): y \mapsto f(x,y)$ is continuous. Similarly, $f(\cdot, y)$ is continuous $\forall y \in [0,1]$. However, f is not continuous on Q because $\lim_{\varepsilon \to 0^+} f(\varepsilon, \varepsilon) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} = +\infty$

Notice, $\forall y \in (0,1]$, we have $F(y) = \int_0^1 f(x,y)\,dx = \left[\frac{x^2 - y^2}{x^2 + y^2}\right]_{x=0}^{1} = \frac{1-y^2}{1+y^2} + 1 = \frac{2}{1+y^2}$

Since $f(x,0) = 0$ $\forall x \in [0,1]$, we have $F(0) = \int_0^1 0\,dx = 0$. However, $\lim_{y \to 0^+} F(y) = 2 \neq 0$ $\therefore F$ is not continuous at 0.

COROLLARY

Let $f: Q \to \mathbb{R}$ be continuous and $g \in R(x; a, b)$.

Define $F: [c,d] \longrightarrow \mathbb{R}$

$y \longmapsto \int_a^b g(x) f(x,y)\,dx$

Then, F is continuous.

Proof

For $x \in [a,b]$, write $G(x) = \int_0^x g(t)\,dt$, and $G \in BV([a,b])$

Then, for $y \in [c,d]$, we have $F(y) = \int_a^b f(x,y)\,dG(x)$. The result follows from the proposition. □

PROPOSITION

Let $\alpha \in BV([a,b])$ and $f: Q = [a,b] \times [c,d] \longrightarrow \mathbb{R}$ be continuous.

Suppose that $\frac{\partial f}{\partial y}$ is continuous on Q. Then, for $y \in [c,d]$, $F'(y)$ exists and $\frac{d}{dy} \int_a^b f(x,y)\,d\alpha(x) = F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,d\alpha(x)$

Proof

Fix $y_0 \in [c,d]$. For $y \in [c,d] \setminus \{y_0\}$, write $\frac{F(y) - F(y_0)}{y - y_0} = \int_a^b \frac{f(x,y) - f(x,y_0)}{y - y_0}\,d\alpha(x)$, which MVT implies equals $\int_a^b \frac{\partial f}{\partial y}(x,y')\,d\alpha(x)$ for some $y' = y'(x,y)$ in between y and y_0.

When $y \to y_0$, we also have $y' \to y_0$. Also, the continuity of $\frac{\partial f}{\partial y}$ implies that $\lim_{y \to y_0} \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(x,y_0)$

Therefore, $F'(y_0) := \lim_{y \to y_0} \frac{F(y) - F(y_0)}{y - y_0} = \lim_{y \to y_0} \int_a^b \frac{\partial f}{\partial y}(x,y')\,d\alpha(x) \underset{\text{proposition above}}{=} \int_a^b \lim_{y \to y_0} \frac{\partial f}{\partial y}(x,y')\,d\alpha(x) = \int_a^b \frac{\partial f}{\partial y}(x,y_0)\,d\alpha(x)$ □

THEOREM (FUBINI'S THEOREM)

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Let $\alpha \in BV([a,b])$, $\beta \in BV([a,b])$, and $f: Q \to \mathbb{R}$ be continuous.
Fix $(x,y) \in Q$, define $F(y) := \int_a^b f(x,y)\,d\alpha(x)$, $G(x) := \int_c^d f(x,y)\,d\beta(y)$.
Then, $F \in R(\beta)$ and $G \in R(\alpha)$ and $\int_c^d F(y)\,d\beta(y) = \int_a^b G(x)\,d\alpha(x) \Leftrightarrow \int_a^b \int_c^d f(x,y)\,d\beta(y)\,d\alpha(x) = \int_c^d \int_a^b f(x,y)\,d\alpha(x)\,d\beta(y)$

COUNTEREXAMPLE FOR NONCONTINUOUS f

Let $f(x,y) = \begin{cases} 1, & x \in \mathbb{Q} \\ 2y, & x \notin \mathbb{Q} \end{cases}$

- We know $\overline{\int_0^1} f(x,y)\,dy = \underline{\int_0^1} f(x,y)\,dy = \begin{cases} \int_0^1 1\,dy = 1, & x \in \mathbb{Q} \\ \int_0^1 2y\,dy = 1, & x \notin \mathbb{Q} \end{cases}$. Thus, $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \int_0^1 1\,dx = 1$
- Meanwhile, $\overline{\int_0^1} f(x,y)\,dx = \max(1, 2y)$ and $\underline{\int_0^1} f(x,y)\,dx = \min(1, 2y) \Rightarrow$ Other than $y = \frac{1}{2}$, $\int_0^1 f(x,y)\,dx$ is not well-defined.

PROOF OF FUBINI'S THEOREM (Behind the scenes: My hand is cramping so hard I'm using my elbow to control my pen movement)

WLOG, suppose α and β are nondecreasing. Since Q is compact, thus f is uniformly continuous
Let $\varepsilon > 0$, take $\delta > 0$, s.t. $\forall (x,y), (x',y') \in Q$, $\|(x,y) - (x',y')\|_\infty \leq \delta \Rightarrow |f(x,y) - f(x',y')| \leq \varepsilon$

Let $P_x = (x_k)_{0 \leq k \leq m} \in \mathcal{P}([a,b])$, $P_y = (y_\ell)_{0 \leq \ell \leq n} \in \mathcal{P}([c,d])$, s.t. $\|P_x\|, \|P_y\| < \delta$. Now, let us rewrite one of our integrals.

$\int_a^b \int_c^d f(x,y)\,d\beta(y)\,d\alpha(x) = \sum_{k=1}^m \sum_{\ell=1}^n \int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} f(x,y)\,d\beta(y)\,d\alpha(x) \overset{MVT}{=} \sum_{k=1}^m \sum_{\ell=1}^n \int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} f(x, y_\ell')\,d\beta(y)\,d\alpha(x)$ for some $y_\ell' \in [y_{\ell-1}, y_\ell]$.

By MVT, $= \sum_{k=1}^m \sum_{\ell=1}^n \int_{x_{k-1}}^{x_k} f(x, y_\ell')(\beta(y_\ell) - \beta(y_{\ell-1}))\,d\alpha(x)$

By MVT, $= \sum_{k=1}^m \sum_{\ell=1}^n f(x_k', y_\ell')[\alpha(x_k) - \alpha(x_{k-1})](\beta(y_\ell) - \beta(y_{\ell-1}))$, for some $(x_k', y_\ell') \in [x_{k-1}, x_k] \times [y_{\ell-1}, y_\ell]$

Similarly, we have $\int_c^d \int_a^b f(x,y)\,d\alpha(x)\,d\beta(y) = \sum_{\ell=1}^n \sum_{k=1}^m f(x_k'', y_\ell'')[\alpha(x_k) - \alpha(x_{k-1})][\beta(y_\ell) - \beta(y_{\ell-1})]$

Taking their difference, we find: $\left|\int_a^b \int_c^d f(x,y)\,d\beta(y)\,d\alpha(x) - \int_c^d \int_a^b f(x,y)\,d\alpha(x)\,d\beta(y)\right| \leq \sum_{k=1}^m \sum_{\ell=1}^n \underbrace{|f(x_k', y_\ell') - f(x_k'', y_\ell'')|}_{\leq \varepsilon \text{ as assumed}} \Delta\alpha_k \Delta\beta_\ell = \varepsilon[\alpha(b) - \alpha(a)][\beta(d) - \beta(c)]$

RIEMANN INTEGRALS

DEFINITION

Let $S \subseteq \mathbb{R}$ be a subset. We may say that S has **measure zero** if for every $\varepsilon > 0$, $\exists$ a countable family $\{U_i := (a_i, b_i) \mid i \in I\}$ of open intervals such that:
(i) $S \subseteq \bigcup_{i \in I} (a_i, b_i)$ ("S can be covered by these open intervals")
(ii) The sum of lengths satisfy $\sum_{i \in I} |U_i| = \sum_{i \in I} (b_i - a_i) \leq \varepsilon$
where $|U_i| = b_i - a_i$ denotes the length of the open interval U_i for $i \in I$.

EXAMPLE

1) If $S \subseteq \mathbb{R}$ is a finite set, then S has measure zero
2) If $S = \{s_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ is a countable set, then S has measure zero
   For $\varepsilon > 0$, take $U_i := (s_i - \frac{\varepsilon}{2^{i+1}}, s_i + \frac{\varepsilon}{2^{i+1}})$. Then, $\sum_{i=1}^\infty |U_i| = \sum_{i=1}^\infty \frac{\varepsilon}{2^i} = \varepsilon$

THEOREM (LEBESGUE'S CRITERION) (Sneak peak to next main section of notes)

Let $f: [a,b] \to \mathbb{R}$ be bounded, and D be the set of its discontinuities. Then, $f \in R(x; a, b) \Leftrightarrow D$ has measure zero
(Sorry no proof today qwq my hand is hurting too much)