2-20-25 (WEEK 1)

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USEFUL PROPERTIES

- Monotonic $\Rightarrow$ bounded variation
- Continuous + bounded derivative $\Rightarrow$ bounded variation
- $\mathcal{C}^1 \Rightarrow$ bounded variation

PROPERTIES

PROPOSITION

$BV([a,b],\mathbb{R}) \subseteq \mathcal{B}([a,b],\mathbb{R})$

Proof

Let $f \in BV([a,b],\mathbb{R})$, $M := V_f([a,b]) < +\infty$

Fix $x \in (a,b)$ and consider partition $P = (a, x, b)$, then we have $V_P(f) \leq V_f([a,b]) = M$

Write $V_P(f) = |f(x) - f(a)| + |f(b) - f(x)| \overset{\triangle \text{ ineq}}{\geq} |f(x) - f(a)| \geq |f(x)| - |f(a)|$

$\therefore$ This implies $|f(x)| \leq |f(a)| + V_P(f) \leq |f(a)| + M$, i.e. $|f|$ is bounded by $\max\{|f(a)| + M, |f(b)|\}$. $\square$

Remark: "$\supseteq$" does not hold! This feels like foreshadowing for $\int |\cdot| dx$ may not converge even if $\int dx$ converges

PROPOSITION

Let $f, g \in BV([a,b],\mathbb{R})$, then $f \pm g, fg, cf$ $(c \in \mathbb{R})$ are all of bounded variation

Proof

Since it is the same idea for the other two, $f-g$, cf proofs are not shown.

Let $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a,b])$, then we have:

$$\begin{aligned}
V_P(f+g) &= \sum_{k=1}^{n} |f(x_k) + g(x_k) - f(x_{k-1}) - g(x_{k-1})| \\
&\leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| \\
&= V_P(f) + V_P(g) \\
&\leq V_f([a,b]) + V_g([a,b]) \quad \text{(const)}
\end{aligned}$$

$\therefore f+g \in BV([a,b])$ ✓

Additionally, by taking sup over $P \in \mathcal{P}([a,b])$, we obtain $V_{f+g} \leq V_f + V_g$

For the multiplication fg, we also fix $P \in \mathcal{P}([a,b])$.

key technique

$$\begin{aligned}
V_P(fg) &= \sum_{k=1}^{n} |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1}) \underline{- f(x_k)g(x_{k-1}) + f(x_k)g(x_{k-1})}| \\
&\leq \sum_{k=1}^{n} |f(x_k)||g(x_k) - g(x_{k-1})| + |g(x_{k-1})||f(x_k) - f(x_{k-1})| \\
&\leq \underbrace{\sup(f)}_{<\infty} \underbrace{\sum_{k=1}^{n} |\Delta g_k|}_{V_P(g) \leq V_g} + \underbrace{\sup(g)}_{<\infty} \underbrace{\sum_{k=1}^{n} |\Delta f_k|}_{V_P(f) \leq V_f}
\end{aligned}$$

$\because$ It doesn't depend on the partition to be bounded

$\therefore fg \in BV([a,b])$ ✓

Again, if we take sup over $P \in \mathcal{P}([a,b])$, we obtain $V_{fg} \leq \sup(f) V_g + \sup(g) V_f$ $\square$

PROPOSITION

Let $f \in BV([a,b])$ with $\underline{|f| \geq m > 0}$ for some $m \in \mathbb{R}$, then $g = \frac{1}{f} \in BV([a,b])$

No zero denominator!

Proof

Let $P \in \mathcal{P}([a,b])$. Write $P := (x_k)_{0 \leq k \leq n}$.

Need to turn $V_P(f)$ into V_f to create an upper bound independent of the input

$$V_P(g) = \sum_{k=1}^{n} \left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right| = \sum_{k=1}^{n} \left| \frac{\Delta f_k}{f(x_k)f(x_{k-1})} \right| \leq \frac{1}{m^2} \sum_{k=1}^{n} |\Delta f_k| = \frac{V_P(f)}{m^2} \leq \frac{V_f}{m^2} \quad \text{(const)}$$

PROPOSITION strong enough for equality to hold! Shun/羊羽海 (@shun4midx)

Let $f\in BV([a,b])$ and $c\in(a,b)$. Then, $f\in BV([a,c])$, $f\in BV([c,b])$, and $V_f([a,b]) = V_f([a,c])+V_f([c,b])$

Proof

Let $P_1\in\mathcal{P}([a,c])$ and $P_2\in\mathcal{P}([c,b])$.

Define $P:=P_1\vee P_2$ to be the partition with support $Supp(P_1)\cup Supp(P_2)$. Then, we have $V_P(f) = V_{P_1}(f)+V_{P_2}(f)$ equality holds since P_1 and P_2 don't have overlapping intervals

$\because V_P(f)=V_{P_1}(f)+V_{P_2}(f)\leq V_f([a,b])\leq M<\infty$

$\therefore f\in BV([a,c])\cap BV([c,b])$ ✓ (First prove they are indeed bounded)

Now, for "$V_f([a,b])=V_f([a,c])+V_f([c,b])$", # since partitions usually don't contain c

"$\leq$": Take the sup over $P_1\in\mathcal{P}([a,c])$, and sup over $P_2\in\mathcal{P}([c,b])$, $V_{P_1}(f)+V_{P_2}(f)\leq V_f([a,b]) \Rightarrow V_f([a,c])+V_f([c,b])\leq V_f([a,b])$ ✓

"$\geq$": Fix $P\in\mathcal{P}([a,b])$.

Define P' to be the partition with support $Supp(P)\cup\{c\}$

From the previous notes, we know that $V_P(f)\leq V_{P'}(f)$

Define $P_1\in\mathcal{P}([a,c])$ with support $Supp(P')\cap[a,c]$, $P_2\in\mathcal{P}([c,b])$ with support $Supp(P')\cap[c,b]$

Then, $V_P(f)\leq V_{P'}(f)=V_{P_1}(f)+V_{P_2}(f)\leq V_f([a,c])+V_f([c,b])$

We conclude by taking sup over $P\in\mathcal{P}([a,b])$ □ key step

DEFINITION

Let $f\in BV([a,b])$. Define the variation function to be $V:[a,b]\longrightarrow\mathbb{R}$, $x\longmapsto\begin{cases}0, & x=a\\ V_f([a,b]), & x\in(a,b]\end{cases}$ because it is quite pointless to use this value for a.

LEMMA

Let $f\in BV([a,b])$ and V be its variation function. Then, both V and $V-f$ are increasing

Proof

For V: We know for $x>a$, $V(x)\geq V(a)=0$

Let $x,y\in(a,b]$ with $x<y$. Then, $V(y)-V(x)=V_f([a,y])-V_f([a,x]) = V_f([x,y])\geq 0$ by def of V_f

For $V-f$: Let $D:=V-f$, $x,y\in(a,b]$, $x<y$. $\because f(y)-f(x)\leq V_f([x,y])$

Then, $D(y)-D(x)=[V(y)-V(x)]-[f(y)-f(x)]=V_f([x,y])-[f(y)-f(x)]\geq 0$

THEOREM (Makes checking for bounded variation way easier)

Let $f:[a,b]\to\mathbb{R}$, then (a)$\Leftrightarrow$(b)

(a) $f\in BV([a,b])$

(b) $\exists$ non-decreasing functions g_1 and g_2, s.t. $f=g_1-g_2$ Notice this isn't a unique decomposition

Proof

(b)$\Rightarrow$(a): As monotonic functions are of bounded variation, thus their difference is also of bounded variation ✓

(a)$\Rightarrow$(b): Use the variation function V, then we know from before, V and $V-f$ are non-decreasing $\Rightarrow f=V-(V-f)$ suffices ✓

PROPOSITION

Let $f\in BV([a,b])$ and $x\in[a,b]$. Then, f is continuous at $x\Leftrightarrow V$ is continuous at x

Proof

It suffices to prove that "$\forall x\in[a,b]$, $f(x+)=f(x)\Leftrightarrow V(x+)-V(x)$" From above, V is increas.

We know the right limits $f(x+)$, $V(x+)$ are well-defined since V is increasing and $f=V-(V-f)$, V, $V-f$ are increasing "monotonic $\Rightarrow$ limit exists"

- Suppose that $V(x+)=V(x)$, i.e. V is continuous at x from the right

  We note that for $y>x$, $0\leq|f(y)-f(x)|\leq|V_f([x,y])|=|V(y)-V(x)|$.

  Take the limit $y\to x+$, we find $|f(x+)-f(x)|\leq|V(x+)-V(x)|=0$, hence $f(x+)=f(x)$

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- Suppose that $f(x+) = f(x)$, we need to show that $V(x+) = V(x)$

  definition of limit

  Let $\varepsilon > 0$. By the right continuity of f at x, we may find $\delta > 0$, s.t. $y \in [x, x+\delta) \Rightarrow |f(y) - f(x)| < \varepsilon$

  Quite a reoccuring inequality

  By the characterization of the total variation, we can take $P_\varepsilon \in \mathcal{P}([a,b])$ s.t. $\forall P \supseteq P_\varepsilon$, we have $V_P(f) \le V_f([a,b]) \le V_P(f) + \varepsilon$

  $|f(x_1) - f(x_0)| = |f(x_1) - f(x)| < \varepsilon$

  Let $P \supseteq P_\varepsilon$ s.t. $x_1 \in [x, x+\delta)$. Then, $V_f([x,b]) \le V_P(f) + \varepsilon = |\Delta f_1| + \sum_{k=2}^{n} |\Delta f_k| + \varepsilon \le V_f([x_1, b]) + 2\varepsilon$

  $\therefore V_f([x,b]) - V_f([x_1, b]) \le 2\varepsilon$

  Now, $LHS = V_f([x, x_1]) = V_f([a,x]) - V_f([a, x_1]) = V(x) - V(x_1) \le 2\varepsilon$ □

REMARK

For the theorem above, we can actually add this condition:

Let $f: [a,b] \to \mathbb{R}$, f is continuous, then (a) $\Leftrightarrow$ (b)

(a) $f \in BV([a,b])$

(b) $\exists$ two non-decreasing continuous functions g_1, g_2, s.t. $f = g_1 - g_2$

Proof

It suffices to show (a) $\Rightarrow$ (b).

Recall we proved it before using $f = V - (V - f)$.

Now, we know from the above proposition, V and f share the same continuities, thus so does $V - f$ □