5-22-25 (WEEK 14) Shun/翔海 (@shun4midx)

COEFFICIENTS OF POWER SERIES

COROLLARY (UNIQUENESS OF POWER SERIES)

Let $f(z)=\sum_{n\geq 0} a_n z^n$ and $g(z)=\sum_{n\geq 0} b_n z^n$ with $R_f := R(\sum a_n z^n) > 0$, $R_g := R(\sum b_n z^n) > 0$

Suppose that there exists $r\in (0, \min(R_f, R_g)]$ s.t. $f\equiv g$ on $(-r,r)$. Then, $a_n \equiv b_n$ $\forall n\geq 0$

Proof

Let $r>0$ s.t. $f\equiv g$ on $(-r,r)$.

Since f and g are both C^∞ and they are equal, we deduce that $f^{(k)} \equiv g^{(k)}$ on $(-r,r)$, $\forall k\in\mathbb{N}_0$.

This implies $f^{(k)}(0) = g^{(k)}(0)$ $\forall k\geq 0$. Hence, $a_k = b_k$ $\forall k\geq 0$. □

EXAMPLE

Let $f(z)=\sum_{n\geq 0} a_n z^n$ with $R>0$. Suppose that f is an even function, i.e. $f(-z)=f(z)$ $\forall z\in(-R,R)$. Then, $(-1)^n a_n = a_n$ $\forall n\geq 0$. In particular, $a_n = 0$ $\forall$ odd n.

THEOREM (CAUCHY'S FORMULA)

Let $f(z)=\sum_{n\geq 0} a_n z^n$ with $R>0$. Then, for $r\in(0,R)$ and $n\in\mathbb{N}_0$, we have $r^n a_n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{-in\theta}\,d\theta$

Proof

We write $\int_0^{2\pi} f(re^{i\theta})e^{-in\theta}\,d\theta = \int_0^{2\pi}\sum_{k\geq 0} a_k r^k e^{i(k-n)\theta}\,d\theta \overset{?}{=} \sum_{k\geq 0} a_k r^k \int_0^{2\pi} e^{i(k-n)\theta}\,d\theta = a_n r^n 2\pi$

Now, let us check that we can indeed interchange Σ and $\int$.

The series of function is given by
$$\begin{aligned} [0,2\pi] &\longrightarrow \mathbb{C} \\ \theta &\longmapsto \sum_{k\geq 0} a_k r^k e^{i(k-n)\theta} \end{aligned}$$

For every $k\geq 0$, $\|\theta \longmapsto a_k r^k e^{i(k-n)\theta}\|_{\infty,[0,2\pi]} = |a_k| r^k$

Since $r<R$, we know $\sum |a_k| r^k$ converges, so the series of functions converges normally and uniformly □

EXPANSION IN POWER SERIES

DEFINITION

Let $A\subseteq\mathbb{C}$ be an open set and $f: A\longrightarrow\mathbb{C}$.

- Let $R>0$, if $0\in A$ and there exists a power series $\sum a_n z^n$ such that $\forall z\in D(0,R)$, $f(z)=\sum_{n\geq 0} a_n z^n$, then we say that f can be expanded into a power series around 0 or expanded into a power series on $D(0,R)$

  In particular, we know that $R(\sum a_n z^n)\geq R$ and f is C^∞ on $D(0,R)$.
- Let $z_0\in A$. We say that f can be expanded into a power series around z_0 if $z\mapsto f(z+z_0)$ can be expanded into a power series around 0 ($\Rightarrow f(z)=\sum a_n (z-z_0)^n$)

PROPOSITION

Let $A\subseteq\mathbb{C}$ be open and $0\in A$. Then, (1) $\Leftrightarrow$ (2)

(1) f can be represented as a power series around 0

(2) There exists r, s.t. the remainder $(R_n)_{n\geq 0}$ converges pointwise to 0 on $D(0,r)$, where $R_n(z) = f(z) - \sum_{k=0}^{n} a_k z^k = \sum_{k\geq n+1} a_k z^k \xrightarrow[\text{ptwise}]{n\to\infty} 0$ and $a_k = \frac{f^{(k)}(0)}{k!}$

When (2) holds, we have $R(\sum a_k z^k) > r$ and $f = \sum a_k z^k$ on $D(0,r)$.

Proof

"$\Rightarrow$": By def ✓

"$\Leftarrow$": Let r, s.t. (2) holds. For $z\in D(0,r)$, $\sum a_k z^k$ converges, so $(a_k z^k)_{k\geq 0}$ is bounded, and $|z|\leq R(\sum a_k z^k) \Rightarrow r\leq R(\sum a_k z^k)$ □

REMARK

How to check $R_n(z) \xrightarrow{n\to\infty} 0$?

(1) The remainder R_n can be estimated using

↳ Taylor-integral formula: $R_n(z) = z^{n+1}\int_0^1 \frac{(1-t)^n}{n!} f^{(n+1)}(t)\,dt$

↳ Taylor-Lagrange: $R_n(z) = \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(\theta z)$, $\theta\in[0,1]$

(2) It is NOT enough to check that $R(\sum \frac{f^{(k)}(0)}{k!} z^k) > 0$

Shun/翔海(@shun4midx)

EXAMPLE

Consider $f: \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longmapsto \begin{cases} e^{-\frac{1}{x}}, & x>0 \\ 0, & x \leq 0 \end{cases}$$

We are going to check that $f^{(k)}(0)=0 \ \forall k \geq 0$, $R(\sum a_n z^n)=\infty$

But clearly, $f(x) \neq 0$ for $x>0$, so we cannot expand into a power series around 0.

- $\forall x>0$, $f'(x)=\frac{1}{x^2}e^{-\frac{1}{x}}=\frac{1}{x^2}f(x)$, $f''(x)=(-\frac{2}{x^3}-\frac{1}{x^2})f(x)$

  We have $f'(x), f''(x) \xrightarrow{x\to\infty} 0$

  In general, $f^{(k)}(x)=P_k(\frac{1}{x})e^{-\frac{1}{x}}$ for some polynomial P, $\deg P \leq 2k$.

  $\Rightarrow f^{(k)}(x) \xrightarrow{x\to\infty} 0$

  $\therefore$ The function f is C^∞ on $\mathbb{R}$ with $f^{(k)}(x)=0 \ \forall k$

- Note that $f(x)=o(x^n)$, Taylor expansion tells us $f(x)=0+\dots+0+\underbrace{f(x)}_{\text{as } o(x^n)}$

EXAMPLE

1) $z \mapsto e^z$ can be expanded around 0

   For $z \in \mathbb{C}$, $R_n(z)=e^z-\sum_{k=0}^{n}\frac{z^k}{k!}$

   We know that $R_n(z)=\frac{z^{n+1}}{(n+1)!}f^{(n+1)}(\theta z)$ for some $\theta=\theta(z)\in(0,1)$

   $\therefore$ For $z\in\mathbb{C}$, $|R_n(z)|=\frac{|z|^{n+1}}{(n+1)!}e^{\theta+\mathrm{Re}(z)} \xrightarrow{n\to\infty} 0$

2) $z \mapsto \frac{1}{1-z}$ is defined on $\mathbb{C}\setminus\{1\}$.

   The expansion around 0: $\forall z\in D(0,1)$, $\frac{1}{1-z}=\sum_{n\geq 0} z^n$

   For $z\in D(0,1)$, $|R_n(z)| \leq \sum_{k\geq n+1}|z|^k=\frac{|z|^{n+1}}{1-|z|} \xrightarrow{n\to\infty} 0$

3) For a polynomial $P\in\mathbb{R}[x]$, $\forall z\in\mathbb{C}$, $P(z)=\sum_{n\geq 0}\frac{P^{(n)}(0)}{n!}z^n$ is a finite series

PROPOSITION

If f can be written as a power series in $D(0,R)$, $R>0$, then for any $z_0\in D(0,R)$, f can also be written as a power series around z_0

Proof

Let $R>0$ and $(a_n)_{n\geq 0}$ s.t. $f(z)=\sum a_n z^n$ for $z\in D(0,R)$. Fix $z_0\in D(0,R)$, take $r\in(0,R-|z|)$. We want to write f as a power series in $D(z_0,r)$, i.e. in the form $\sum b_n(z-z_0)^n$

For $z\in D(z_0,r)$, write $\sum_{n\geq 0}a_n z^n=\sum_{n\geq 0}a_n((z-z_0)+z_0)^n \overset{\text{binomial expansion}}{=} \sum_{n\geq 0}a_n\sum_{k=0}^{n}\binom{n}{k}z^{n-k}(z-z_0)^k=\sum_{n\geq 0}\sum_{k\geq 0}\mathbb{1}_{n\geq k}\,a_n\binom{n}{k}z^{n-k}(z-z_0)^k \overset{?}{=} \sum_{k\geq 0}\underbrace{\left(\sum_{n\geq 0}\mathbb{1}_{n\geq k}\,a_n\binom{n}{k}z_0^{n-k}\right)}_{b_n}(z-z_0)^k$

- For $n\geq 0$, $\sum_{k\geq 0}\mathbb{1}_{n\geq k}a_n=\sum_{k=0}^{n}a_n$ is a finite sum, so it converges absolutely
- $\sum_{n\geq 0}\sum_{k\geq 0}|a_n|\,\mathbb{1}_{n\geq k}\binom{n}{k}|(z-z_0)^k|\,|z|^{n-k}=\sum_{n\geq 0}|a_n|\underbrace{(|z-z_0|+|z_0|)}_{<R}{}^n<+\infty$, because we are inside the disk of convergence. $\square$

APPLICATIONS TO ODE

- Know that the solution can be expanded into a power series
  - ↳ Write $f(z)=\sum a_n z^n \longrightarrow$ plug into the ODE $\Rightarrow$ get relations between coefficients
- Don't know that the solution can be written as a power series, we can assume that there is such a solution. Then, apply the previous step and check $R>0$.

EXAMPLE

Consider $f: \mathbb{R} \longrightarrow \mathbb{R}$, power sum around 0?

$$x \longmapsto e^{x^2}\int_0^x e^{-t^2}dt$$

- $x \mapsto e^{x^2}$ can be written as a power series on $\mathbb{R}$ (centered anywhere)
- $t \mapsto e^{-t^2}$ the same, we can integrate on $\mathbb{R}$
- Cauchy product has a radius of convergence $=\infty$

  $\Big\}\Rightarrow$ can be expanded around any $x\in\mathbb{R}$
- $f'(x)=2xe^{x^2}\int e^{-t^2}dt+1=2xf(x)+1$, $f(0)=0$ (ODE)

Shun/翔海 (@shun4midx)

Suppose $f(x) = \sum_{n\geq 0} a_n x^n$ on $\mathbb{R}$.

$$\left.\begin{aligned} f'(x) &= \sum_{n\geq 1} n a_n x^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1} x^n \\ 1+2xf(x) &= 1 + \sum_{n\geq 1} 2a_{n-1} x^n \end{aligned}\right\} \overset{\text{uniqueness of coef}}{\Rightarrow} \begin{cases} (n+1)a_{n+1} = 2a_{n-1} \ \forall n \geq 1 \\ a_1 = 1 \end{cases}$$

← recursive relation ✓

, with initial condition $a_0 = 0$ from $f(0) = 0$

(We may check that this power series has $R > 0$)

EXAMPLE

Let $\alpha \in \mathbb{C}$. Consider

$$\begin{aligned} f : (-1,1) &\longrightarrow \mathbb{C} \\ x &\longmapsto (1+x)^\alpha = e^{\alpha \ln(1+x)} \end{aligned}$$

$\forall x \in (-1,1)$, $f'(x) = \frac{\alpha}{1+x} f(x)$, $f(0) = 1$

Suppose $f(x) = \sum_{n\geq 0} a_n x^n \ \forall x \in (-1,1) \Rightarrow f'(x) = \sum_{n\geq 0} (n+1) a_{n+1} x^n \Rightarrow (1+x) f'(x) = \sum_{n\geq 0} ((n+1)a_{n+1} + n a_n) x^n = \sum_{n\geq 0} \alpha a_n x^n$

$\therefore$ $(n+1)a_{n+1} = (\alpha - n) a_n$ $\forall n \geq 0$.

$\therefore$ $a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} =: \binom{\alpha}{n} \Rightarrow f(x) = \sum_{n\geq 0} \binom{\alpha}{n} x^n$, check $R > 0$.