Analysis II (Part 1)

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RIEMANN-STIELTJES INTEGRALS

FUNCTIONS OF BOUNDED VARIATION

DEFINITION

Let $I \subseteq \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$ be a function.

(1) f is non-increasing/decreasing if $f(x) \geq f(y)$ $\forall x \leq y$, $x, y \in I$

(2) f is non-decreasing/increasing if $f(x) \leq f(y)$ $\forall x \leq y$, $x, y \in I$

(3) f is monotonic if (1) or (2) holds

DEFINITION

Let $f: I \to \mathbb{R}$ be monotonic

For $x \in I$, define:

E.g. we can't just pick a point at the boundary

↳ The left limit at x to be $f(x-) = \lim_{y<x, y\to x} f(y)$ if $(x-\varepsilon, x) \cap I \neq \emptyset$ for $\varepsilon > 0$

↳ The right limit at x to be $f(x+) = \lim_{y>x, y\to x} f(y)$ if $(x, x+\varepsilon) \cap I \neq \emptyset$ for $\varepsilon > 0$

PROPOSITION

Let $f: [a,b] \to \mathbb{R}$ be a monotonic function. Then, the set of its discontinuities D is a countable set.

Proof

monotonic ⇒ left/right limits are well-defined

Define $D = \{x \in I \mid f(x-) \neq f(x+)\}$

By symmetry, WLOG, assume f is increasing, then $f(x-) < f(x+)$ $\forall x \in D$

Key! This is since $x \in D$, i.e. it is discontinuous

As $\mathbb{Q}$ is dense in $\mathbb{R}$, we know $\exists q_x \in \mathbb{Q} \cap (f(x-), f(x+))$

∴ This gives us a map $\begin{cases} D \to \mathbb{Q} \\ x \mapsto q_x \end{cases}$, which is injective because $\forall x_1, x_2 \in D$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

∵ D can be injected in the countable set $\mathbb{Q}$

∴ D is countable. □

DEFINITION (PARTITIONS)

Let $a<b$ and $[a,b] \in \mathbb{R}$ be a segment.

A partition or a subdivision of $[a,b]$ is a finite sequence $P = (x_k)_{0\leq k\leq n}$ satisfying $a = x_0 < x_1 < \ldots < x_n = b$, where n is the length of P

We denote $\mathrm{Supp}(P) := \{x_k \mid 0 \leq k \leq n\}$ as the support of P

For a finite subset $A \subseteq [a,b]$ with $a, b \in A$, we may find a partition P of $[a,b]$ s.t. $\mathrm{Supp}(P) = A$. This is called the partition corresponding to A.

This is not a norm!

We say $[x_{k-1}, x_k]$ is the k^{th} subinterval of P, $\Delta x_k := x_k - x_{k-1}$, $1 \leq k \leq n$. Then, we say the mesh size of P is $\|P\| := \max_{1\leq k\leq n} \Delta x_k$

Let P, P' be partitions. If $\mathrm{Supp}(P) \subseteq \mathrm{Supp}(P')$, then we say P' is finer than P, and we say $P \subseteq P'$. This also implies $\|P\| \leq \|P'\|$

Let P_1, P_2 be partitions. Define their joint partition or smallest common refinement to be $P := P_1 \vee P_2$, which is the partition P with support $= \mathrm{Supp}(P_1) \cup \mathrm{Supp}(P_2)$

We denote $\mathcal{P}([a,b])$ as the collection of all possible partitions of $[a,b]$

REMARK

For any $P = (x_k)_{0\leq k\leq n} \in \mathcal{P}([a,b])$, we have $b - a = \sum_{k=1}^{n} \Delta x_k$

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DEFINITION (BOUNDED VARIATIONS)

Let $f:[a,b]\to\mathbb{R}$ be a function, $P=(x_k)_{0\le k\le n}\in\mathcal{P}([a,b])$, define $\Delta f_k:=f(x_k)-f(x_{k-1})$ for $1\le k\le n$.

Define $V_P(f):=\sum_{k=1}^{n}|\Delta f_k|$ and $V_f=V_f([a,b])=\sup_{P\in\mathcal{P}([a,b])}V_P(f)\in[0,\infty]$ to be the **total variation** of f. We say that f is of **bounded variation** if $V_P<+\infty$.

We write $\mathcal{BV}([a,b])=\mathcal{BV}([a,b],\mathbb{R})$ for the collection of such functions defined on $[a,b]$.

EXAMPLE

Consider the function $f:[0,1]\to\mathbb{R}$, $f(x)=\begin{cases}x\cos(\frac{\pi}{x}), & x\in(0,2\pi]\\ 0, & x=0\end{cases}$

For $n\ge1$, consider the partition $\mathcal{P}$ with support $\{0,\frac{1}{2n},\frac{1}{2n-1},\dots,\frac{1}{2},1\}$, i.e. $x_0=0$, $x_k=\frac{1}{2n+1-k}$ $\forall 1\le k\le 2n$

By completeness, we find $V_P(f)=\sum_{k=1}^{2n}|\Delta f_k|=\left|\frac{(-1)^{2n}}{2n}-0\right|+\sum_{k=2}^{2n}\left|\frac{(-1)^{k-1}}{2n+1-k}-\frac{(-1)^k}{2n+2-k}\right|=\frac{1}{2n}+\sum_{k=2}^{2n}\left(\frac{1}{2n+1-k}+\frac{1}{2n+2-k}\right)=1+\sum_{k=2}^{2n-1}\frac{2}{k}+\frac{1}{2n}$, which is **not bounded** for $n\ge1$. So f is **not of bounded variation**

PROPOSITION

Let $f\in\mathcal{BV}([a,b],\mathbb{R})$, then

(1) For any partitions $P\subseteq P'$, we have $V_P(f)\le V_{P'}(f)$

(2) $\forall\varepsilon>0$, $\exists$ partition $P_\varepsilon\in\mathcal{P}([a,b])$ s.t. $\forall$ finer partition $P\supseteq P_\varepsilon$, we have $V_P(f)\le V_f\le V_P(f)+\varepsilon$

<u>Proof</u>

(1) By induction, we only need to prove this is true whenever $|Supp(P')|=|Supp(P)|+1$

Let $P,P'\in\mathcal{P}([a,b])$ be partitions with support s.t. $Supp(P')=Supp(P)\cup\{c\}$, $x_{k-1}<c<x_k$ for some $1\le k\le n$.

Then, $V_{P'}(f)=\sum_{k=1,k\ne i}^{n}|f(x_k)-f(x_{k-1})|+|f(c)-f(x_{i-1})|+|f(x_i)-f(c)|$

$\ge\sum_{k=1,k\ne i}^{n}|f(x_k)-f(x_{k-1})|+|f(x_i)-f(x_{i-1})|$

$=\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|=V_P(f)$ ✓

$\therefore$ By induction, the statement holds. □

(2) Let $\varepsilon>0$, by the characterization of supremum, we can find $P_\varepsilon\in\mathcal{P}([a,b])$ s.t. $V_f\le V_{P_\varepsilon}(f)+\varepsilon$ directly follows from $V_f=\sup_{P\in\mathcal{P}([a,b])}V_P(f)$. "$\forall\varepsilon>0$, $\exists P'\in\mathcal{P}([a,b])$ s.t. $V_f\le V_{P'}(f)+\varepsilon$"

$\therefore$ $\forall$ finer partitions $P\supseteq P_\varepsilon$, from (1), $V_f\le V_{P_\varepsilon}(f)+\varepsilon\le V_P(f)+\varepsilon$ □

PROPOSITION

If $f:[a,b]\to\mathbb{R}$ is monotonic, then $f\in\mathcal{BV}([a,b])$ and $V_f=|f(b)-f(a)|$

<u>Proof</u>

WLOG, assume that f is increasing.

Then, $\forall P\in\mathcal{P}([a,b])$, $V_P(f)=\sum_{k=1}^{n}|\Delta f_k|=\sum_{k=1}^{n}\Delta f_k=f(b)-f(a)$, which is independent of P

$\therefore$ $f\in\mathcal{BV}([a,b])$ and $V_f=|f(b)-f(a)|$ □

PROPOSITION

If $f:[a,b]\to\mathbb{R}$ is continuous and differentiable on (a,b) with bounded derivative, then $f\in\mathcal{BV}([a,b])$

<u>Proof</u>

Let $P=(x_k)_{0\le k\le n}\in\mathcal{P}([a,b])$ be a partition, then $V_P(f)=\sum_{k=1}^{n}|\Delta f_k|\overset{\text{MVT}}{=}\sum_{k=1}^{n}|f'(t_k)|\Delta x_k\le\sup_{t\in[a,b]}|f'(t)|\sum_{k=1}^{n}\Delta x_k=\sup_{t\in[a,b]}|f'(t)|(b-a)$ □

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USEFUL PROPERTIES

- Monotonic $\Rightarrow$ bounded variation
- Continuous + bounded derivative $\Rightarrow$ bounded variation
- $\mathcal{C}^1 \Rightarrow$ bounded variation

PROPERTIES

PROPOSITION

$BV([a,b],\mathbb{R}) \subseteq \mathcal{B}([a,b],\mathbb{R})$

Proof

Let $f \in BV([a,b],\mathbb{R})$, $M := V_f([a,b]) < +\infty$

Fix $x \in (a,b)$ and consider partition $P = (a, x, b)$, then we have $V_P(f) \leq V_f([a,b]) = M$

Write $V_P(f) = |f(x) - f(a)| + |f(b) - f(x)| \overset{\triangle \text{ ineq}}{\geq} |f(x) - f(a)| \geq |f(x)| - |f(a)|$

$\therefore$ This implies $|f(x)| \leq |f(a)| + V_P(f) \leq |f(a)| + M$, i.e. $|f|$ is bounded by $\max\{|f(a)|+M, |f(b)|\}$. □

Remark: "$\supseteq$" does not hold! This feels like foreshadowing for $\int |\cdot| dx$ may not converge even if $\int dx$ converges

PROPOSITION

Let $f, g \in BV([a,b],\mathbb{R})$, then $f \pm g$, fg, cf $(c \in \mathbb{R})$ are all of bounded variation

Proof

Since it is the same idea for the other two, $f-g$, cf proofs are not shown.

Let $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a,b])$, then we have:

$$
\begin{aligned}
V_P(f+g) &= \sum_{k=1}^{n} |f(x_k) + g(x_k) - f(x_{k-1}) - g(x_{k-1})| \\
&\leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| \\
&= V_P(f) + V_P(g) \\
&\leq V_f([a,b]) + V_g([a,b]) \quad \text{(const)}
\end{aligned}
$$

$\therefore f+g \in BV([a,b])$ ✓

Additionally, by taking sup over $P \in \mathcal{P}([a,b])$, we obtain $V_{f+g} \leq V_f + V_g$

For the multiplication fg, we also fix $P \in \mathcal{P}([a,b])$.

$$
\begin{aligned}
V_P(fg) &= \sum_{k=1}^{n} |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1}) \underbrace{- f(x_k)g(x_{k-1}) + f(x_k)g(x_{k-1})}_{\text{key technique}}| \\
&\leq \sum_{k=1}^{n} |f(x_k)||g(x_k) - g(x_{k-1})| + |g(x_k)||f(x_k) - f(x_{k-1})| \\
&\leq \underbrace{\sup(f)}_{\infty} \underbrace{\sum_{k=1}^{n} |\Delta g_k|}_{V_P(g) \leq V_g} + \underbrace{\sup(g)}_{\infty} \underbrace{\sum_{k=1}^{n} |\Delta f_k|}_{V_P(f) \leq V_f}
\end{aligned}
$$

$\because$ It doesn't depend on the partition to be bounded

$\therefore fg \in BV([a,b])$ ✓

Again, if we take sup over $P \in \mathcal{P}([a,b])$, we obtain $V_{fg} \leq \sup(f) V_g + \sup(g) V_f$ □

PROPOSITION

Let $f \in BV([a,b])$ with $\underbrace{|f| \geq m > 0}_{\text{No zero denominator!}}$ for some $m \in \mathbb{R}$, then $g = \frac{1}{f} \in BV([a,b])$

Proof

Let $P \in \mathcal{P}([a,b])$. Write $P = (x_k)_{0 \leq k \leq n}$.

$$V_P(g) = \sum_{k=1}^{n} \left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right| = \sum_{k=1}^{n} \left| \frac{\Delta f_k}{f(x_k) f(x_{k-1})} \right| \leq \frac{1}{m^2} \sum_{k=1}^{n} |\Delta f_k| = \frac{V_P(f)}{m^2} \leq \frac{V_f}{m^2} \text{ (const)}$$

Need to turn $V_P(f)$ into V_f to create an upper bound independent of the input

PROPOSITION

strong enough for equality to hold! Shun/羊羽海(@shun4midx)

Let $f\in BV([a,b])$ and $c\in(a,b)$. Then, $f\in BV([a,c])$, $f\in BV([c,b])$, and $V_f([a,b]) = V_f([a,c]) + V_f([c,b])$

Proof

Let $P_1 \in \mathcal{P}([a,c])$ and $P_2 \in \mathcal{P}([c,b])$.

Define $P := P_1 \vee P_2$ to be the partition with support $Supp(P_1)\cup Supp(P_2)$. Then, we have $V_P(f) = V_{P_1}(f) + V_{P_2}(f)$ — equality holds since P_1 and P_2 don't have overlapping intervals

$\because V_P(f) = V_{P_1}(f) + V_{P_2}(f) \leq V_f([a,b]) \leq M < \infty$

$\therefore f\in BV([a,c])\cap BV([c,b])$ ✓ (First prove they are indeed bounded)

Now, for "$V_f([a,b]) = V_f([a,c]) + V_f([c,b])$",

"$\leq$": Take the sup over $P_1\in\mathcal{P}([a,c])$, and sup over $P_2\in\mathcal{P}([c,b])$, $V_{P_1}(f) + V_{P_2}(f) \leq V_f([a,b]) \Rightarrow V_f([a,c]) + V_f([c,b]) \leq V_f([a,b])$ ✓ (since partitions usually don't contain c)

"$\geq$": Fix $P\in\mathcal{P}([a,b])$.

Define P' to be the partition with support $Supp(P)\cup\{c\}$

From the previous notes, we know that $V_P(f) \leq V_{P'}(f)$

Define $P_1 \in \mathcal{P}([a,c])$ with support $Supp(P')\cap[a,c]$, $P_2\in\mathcal{P}([c,b])$ with support $Supp(P')\cap[c,b]$

Then, $V_P(f) \leq V_{P'}(f) = V_{P_1}(f) + V_{P_2}(f) \leq V_f([a,c]) + V_f([c,b])$

We conclude by taking sup (key step) over $P\in\mathcal{P}([a,b])$ □

DEFINITION

Let $f\in BV([a,b])$. Define the **variation function** to be
$$V: [a,b] \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} 0, & x=a \\ V_f([a,b]), & x\in(a,b] \end{cases}$$
because it is quite pointless to use this value for a.

LEMMA

Let $f\in BV([a,b])$ and V be its variation function. Then, both V and $V-f$ are increasing

Proof

For V: We know for $x>a$, $V(x)\geq V(a)=0$

Let $x,y\in(a,b]$ with $x<y$. Then, $V(y)-V(x) = V_f([a,y]) - V_f([a,x]) = V_f([x,y]) \geq 0$ (by def of V_f)

For $V-f$: Let $D := V-f$, $x,y\in(a,b]$, $x<y$.

Then, $D(y)-D(x) = [V(y)-V(x)] - [f(y)-f(x)] = V_f([x,y]) - [f(y)-f(x)] \geq 0$ ($\because f(y)-f(x)\leq V_f([x,y])$)

THEOREM (Makes checking for bounded variation way easier)

Let $f:[a,b]\to\mathbb{R}$, then (a) $\Leftrightarrow$ (b)

(a) $f\in BV([a,b])$

(b) $\exists$ non-decreasing functions g_1 and g_2, s.t. $f = g_1 - g_2$ (Notice this isn't a unique decomposition)

Proof

(b)$\Rightarrow$(a): As monotonic functions are of bounded variation, thus their difference is also of bounded variation ✓

(a)$\Rightarrow$(b): Use the variation function V, then we know from before, V and $V-f$ are non-decreasing $\Rightarrow f = V-(V-f)$ suffices ✓

PROPOSITION

Let $f\in BV([a,b])$ and $x\in[a,b]$. Then, f is continuous at $x \Leftrightarrow V$ is continuous at x

Proof

It suffices to prove that "$\forall x\in[a,b]$, $f(x+) = f(x) \Leftrightarrow V(x+) - V(x)$" (From above, V is increas.)

We know the right limits $f(x+)$, $V(x+)$ are well-defined since V is increasing ("monotonic $\Rightarrow$ limit exists") and $f = V-(V-f)$, V, $V-f$ are increasing

- Suppose that $V(x+) = V(x)$, i.e. V is continuous at x from the right

  We note that for $y>x$, $0 \leq |f(y)-f(x)| \leq |V_f([x,y])| = |V(y)-V(x)|$.

  Take the limit $y\to x+$, we find $|f(x+)-f(x)| \leq |V(x+)-V(x)| = 0$, hence $f(x+) = f(x)$

- Suppose that $f(x+) = f(x)$, we need to show that $V(x+) = V(x)$

  definition of limit

  Let $\varepsilon > 0$. By the right continuity of f at x, we may find $\delta > 0$, s.t. $y \in [x, x+\delta) \Rightarrow |f(y) - f(x)| < \varepsilon$

  Quite a reoccuring inequality

  By the characterization of the total variation, we can take $P_\varepsilon \in \mathcal{P}([a,b])$ s.t. $\forall P \supseteq P_\varepsilon$, we have $V_P(f) \leq V_f([a,b]) \leq V_P(f) + \varepsilon$

  $|f(x_1) - f(x_0)| = |f(x_1) - f(x)| < \varepsilon$

  Let $P \supseteq P_\varepsilon$ s.t. $x_1 \in [x, x+\delta)$. Then, $V_f([x,b]) \leq V_P(f) + \varepsilon = |\Delta f_1| + \sum_{k=2}^{n} |\Delta f_k| + \varepsilon \leq V_f([x_1, b]) + 2\varepsilon$

  $\therefore V_f([x,b]) - V_f([x_1,b]) \leq 2\varepsilon$

  Now, LHS $= V_f([x, x_1]) = V_f([a,x]) - V_f([a,x_1]) = V(x) - V(x_1) \leq 2\varepsilon$ □

REMARK

For the theorem above, we can actually add this condition:

Let $f: [a,b] \to \mathbb{R}$, **f is continuous**, then (a) $\Leftrightarrow$ (b)

(a) $f \in BV([a,b])$

(b) $\exists$ two non-dereasing **continuous** functions g_1, g_2, s.t. $f = g_1 - g_2$

<u>Proof</u>

It suffices to show (a) $\Rightarrow$ (b).

Recall we proved it before using $f = V - (V - f)$.

Now, we know from the above proposition, V and f share the same continuities, thus so does $V - f$ □

2-25-25 (WEEK 2) Shun/羊羽海 (@shun4midx)

RIEMANN-STIELTJES INTEGRALS

Let $[a,b] \subseteq \mathbb{R}$ be a segment

Let $f, g, \alpha, \beta : [a,b] \longrightarrow \mathbb{R}$ be bounded functions

DEFINITION

Let $P = (x_k)_{0 \le k \le n} \in \mathcal{P}([a,b])$. For every $1 \le k \le n$, take $t_k \in [x_{k-1}, x_k]$ and write $t = (t_k)_{0 \le k \le n}$. We call (P, t) a tagged partition, t contains tagged points of P.

Define the Riemann-Stieltjes sum of f w.r.t. α for (P,t), $S_{P,t}(f,\alpha) = \sum_{k=1}^{n} f(t_k)\Delta\alpha_k = \sum_{k=1}^{n} f(t_k)(\alpha(x_k) - \alpha(x_{k-1}))$

Consider the following condition:

(RS): $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists P_\varepsilon \in \mathcal{P}([a,b])$ s.t. $\forall P \supseteq P_\varepsilon$, tagged points t of P, we have $|S_{P,t}(f,\alpha) - L| < \varepsilon$

If (RS) is satisfied, we say that f is Riemann-Stieltjes integrable and write this unique L to be its integral, denoted $\int_a^b f\,d\alpha = \int_a^b f(x)\,d\alpha(x)$

We write $\mathcal{R}(\alpha; a, b) = \mathcal{R}(\alpha)$ for the set of functions f satisfying (RS).

REMARK

(1) f is called integrand, α is called integrator

(2) When $\alpha(x) = x$, we recover the notation from Riemann-integrability. We write $\mathcal{R}(x; a, b) = \mathcal{R}(x)$ for the set of Riemann-integrable functions

(3) We may also have $x_{k-1} \le x_k$ instead of $x_{k-1} < x_k$ for $1 \le k \le n$. This allows us to use the same notation when $a = b$.

(4) Let V be a finite-dimensional vector space over $\mathbb{R}$, i.e. $V = \mathbb{C}$ or $\mathbb{R}^d$. Fix a basis $(e_1, \dots, e_n)$ of V, we may write $f = \sum_{i=1}^{n} f_i e_i$, where $f_i : [a,b] \to \mathbb{R}$ is a real valued function. If $\int_a^b f_i\,d\alpha$ is well-defined $\forall i$, we may set $\int_a^b f\,d\alpha = \sum_{i=1}^{n} \left(\int_a^b f_i\,d\alpha\right) e_i$

EXAMPLES

(1) If $\alpha : [a,b] \to \mathbb{R}$ is a constant function, for any bounded function $f : [a,b] \to \mathbb{R}$, $S_{P,t}(f,\alpha) = 0$ for all tagged partitions $P \in \mathcal{P}([a,b])$. This means that (RS) holds and $\int_a^b f\,d\alpha = 0$

(2) When $\alpha(x) = x$, all continuous functions are Riemann-integrable, i.e. $\mathcal{C}([a,b]) \subseteq \mathcal{R}(x; a, b)$

(3) Let $f, \alpha : [-1,1] \longrightarrow \mathbb{R}$ to be $f = \alpha = \mathbb{1}_{x \ge 0}$. Consider a partition $P \in \mathcal{P}([-1,1])$ with $x_k = 0$ for some k. For any tagged points t of P, we have $S_{P,t}(f,\alpha) = f(t_k)\Delta\alpha_k = f(t_k) = \begin{cases} 1, & t_k = x_k = 0 \\ 0, & t_k < x_k \end{cases}$. This implies that (RS) does not hold (KEY CONSTRUCTION EXAMPLE)

LEMMA

Consider the following condition,

(RS'): $\exists L \in \mathbb{R}$, s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall P \in \mathcal{P}([a,b])$ with $\max_{1 \le k \le n} |x_k - x_{k-1}| = \|P\| < \delta$, any tagged points t, we have $|S_{P,t}(f,\alpha) - L| < \varepsilon$

We have (RS') $\Rightarrow$ (RS)

REMARK

It is true for $\alpha = x$ though

In general, (RS) $\not\Rightarrow$ (RS'). Consider $f, \alpha : [-1,1] \to \mathbb{R}$, $f, \alpha = \mathbb{1}_{x \ge 0}$

- (RS) holds
- Let $\delta \in (0,1)$ and $P \in \mathcal{P}([0,1])$, s.t. $\|P\| < \delta$, there exists k s.t. $x_{k-1} = x_k = \frac{\delta}{2}$.

  Then, $S_{P,t}(f,\alpha) = f(t_k)\Delta\alpha_k = f(t_k) = \begin{cases} 1, & t_k \in [0, x_k] \\ 0, & t_k \in [x_{k-1}, 0) \end{cases}$ $\therefore$ (RS') does not hold.

PROPOSITION (linearity in integrand)

Let $\alpha : [a,b] \longrightarrow \mathbb{R}$ be bounded, $f, g \in \mathcal{R}(\alpha)$. Then, $\forall c \in \mathbb{R}$, $f + cg \in \mathcal{R}(\alpha)$ and $\int_a^b (f + cg)\,d\alpha = \int_a^b f\,d\alpha + c\int_a^b g\,d\alpha$

As a consequence, $\mathcal{R}(\alpha)$ is an $\mathbb{R}$-vector space, the integral operator $I : \mathcal{R}(\alpha) \longrightarrow \mathbb{R}$, $f \longmapsto \int_a^b f\,d\alpha$ is a linear form, i.e. $I \in \mathcal{L}(\mathcal{R}(\alpha), \mathbb{R})$

Proof

Fix $c \in \mathbb{R}$ and let $h = f + cg$. Since $f \in \mathcal{R}(\alpha)$, we may find $P'_\varepsilon \in \mathcal{P}([a,b])$, s.t. $|S_{P'_\varepsilon,t}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$ $\forall P \supseteq P'_\varepsilon$ and tagged points of P.

Similarly, take $P_\varepsilon'' \in \mathcal{P}([a,b])$, s.t. $|S_{P_\varepsilon'',t}(g,\alpha) - \int_a^b g\,d\alpha| < \varepsilon$ $\forall P \supseteq P_\varepsilon''$.
Take $P_\varepsilon = P_\varepsilon' \vee P_\varepsilon''$, then for $P \supseteq P_\varepsilon$ and any tagged points t of P, we have $|S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$ and $|S_{P,t}(g,\alpha) - \int_a^b g\,d\alpha| < \varepsilon$

Moreover, $S_{P,t}(h,\alpha) = \sum_{k=1}^n h(t_k)\Delta\alpha_k = \sum_{k=1}^n [f(t_k) + cg(t_k)]\Delta\alpha_k = S_{P,t}(f,\alpha) + cS_{P,t}(g,\alpha)$
$\therefore |S_{P,t}(h,\alpha) - \int_a^b f\,d\alpha - c\int_a^b g\,d\alpha| \leq |S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha| + |c||S_{P,t}(g,\alpha) - \int_a^b g\,d\alpha| \leq (1+|c|)\varepsilon$

This means that $h = f + cg$ satisfies (RS), and we have $\int_a^b h\,d\alpha = \int_a^b f\,d\alpha + c\int_a^b g\,d\alpha$ $\square$

PROPOSITION (linearity in integrator)
Let $\alpha, \beta : [a,b] \to \mathbb{R}$ be bounded, $f \in \mathcal{R}(\alpha) \cap \mathcal{R}(\beta)$. Then, $\forall c \in \mathbb{R}$, we have $f \in \mathcal{R}(\alpha + c\beta)$ and $\int_a^b f\,d(\alpha + c\beta) = \int_a^b f\,d\alpha + c\int_a^b f\,d\beta$
(Proof is very similar to above)

DEFINITION
For $a < b$, any bounded function $\alpha : [a,b] \to \mathbb{R}$, $f \in \mathcal{R}(\alpha; a, b)$, we define $\int_b^a f\,d\alpha = -\int_a^b f\,d\alpha = -\int_a^b f(x)\,d\alpha(x)$. We also write $\mathcal{R}(\alpha; a, b) = \mathcal{R}(\alpha; b, a)$
When $a = b$, $\int_a^b f\,d\alpha = 0$ for any bounded function f defined on $a = b$, so $\mathcal{R}(\alpha; a, a) \cong \mathbb{R}$

PROPOSITION
Let $I \subseteq \mathbb{R}$ be a segment, $a, b, c \in I$. Let $\alpha : [a,b] \to \mathbb{R}$ be bounded, $f \in \mathcal{R}(\alpha; a, b) \cap (\mathcal{R}; b, c)$. Then, we have $f \in \mathcal{R}(\alpha; a, c)$ and
$\int_a^c f\,d\alpha = \int_a^b f\,d\alpha + \int_b^c d\alpha$ (if $a = b$ or $b = c$, then it's trivially true)

<u>Proof</u>
By symmetry + notation from above, WLOG, assume $a < b < c$.
- Since $f \in \mathcal{R}(\alpha; a, b)$, we may take $P_\varepsilon^{(a,b)} \in \mathcal{P}([a,b])$, s.t. $|S_{P_\varepsilon^{(a,b)},t}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$ for any $P^{(a,b)} \supseteq P_\varepsilon^{(a,b)}$ and tagged points $t^{(a,b)}$
- Similarly, take $P_\varepsilon^{(b,c)} \in \mathcal{P}([b,c])$, s.t. $|S_{P_\varepsilon^{(b,c)},t}(f,\alpha) - \int_b^c f\,d\alpha| < \varepsilon$ for any $P^{(b,c)} \supseteq P_\varepsilon^{(b,c)}$ and tagged points $t^{(b,c)}$

Then, define $P_\varepsilon = P_\varepsilon^{(a,b)} \vee P_\varepsilon^{(b,c)}$ and take $P \supseteq P_\varepsilon$ and tagged points t, let $P^{(a,b)} = P \cap [a,b]$, $P^{(b,c)} = P \cap [b,c]$, $t^{(a,b)} = t \cap [a,b]$, $t^{(b,c)} = t \cap [b,c]$
Then, $S_{P,t}(f,\alpha) = S_{P^{(a,b)},t}(f,\alpha) + S_{P^{(b,c)},t}(f,\alpha)$, so by triangle inequality, $|S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha - \int_b^c f\,d\alpha| < 2\varepsilon$ $\square$

PROPOSITION (Integration by Parts)
Let $f \in \mathcal{R}(\alpha; a, b)$. Then, $\alpha \in \mathcal{R}(f; a, b)$, and we have $\int_a^b f\,d\alpha + \int_a^b \alpha\,df = f(b)\alpha(b) - f(a)\alpha(a)$
(First prove $\int \alpha\,df$ is Riemann-Stieltjes integrable, then check that $|\int_a^b \alpha\,df - f(b)\alpha(b) + f(a)\alpha(a) + \int_a^b f\,d\alpha|$ is small)

2-27-25 (WEEK 2)

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PROPOSITION (Integration by Parts)

Let $f \in \mathcal{R}(\alpha; a, b)$. Then, $\alpha \in \mathcal{R}(f; a, b)$, and we have $\int_a^b f\,d\alpha + \int_a^b \alpha\,df = f(b)\alpha(b) - f(a)\alpha(a)$

Proof

Let $\varepsilon > 0$, take $P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $\forall P \supseteq P_\varepsilon$ and tagged points, $|S_{P,t}(\alpha, f) - \int_a^b f\,d\alpha| < \varepsilon$ $(*)$

Consider $P \supseteq P_\varepsilon$, tagged points t of P, we write $S_{P,t}(\alpha, f) = \sum_{k=1}^n \alpha(t_k)\Delta f_k = \sum_{k=1}^n \alpha(t_k)[f(x_k) - f(x_{k-1})]$

Note that $f(b)\alpha(b) - f(a)\alpha(a) = \sum_{k=1}^n [f(x_k)\alpha(x_k) - f(x_{k-1})\alpha(x_{k-1})]$, i.e. $S_{P,t}(f,\alpha) - [f(b)\alpha(b) - f(a)\alpha(a)] = \sum_{k=1}^n [\alpha(t_k)[f(x_k) - f(x_{k-1})] - [f(x_k)\alpha(x_k) - f(x_{k-1})\alpha(x_{k-1})]]$

$\therefore S_{P,t}(\alpha, f) - [f(b)\alpha(b) - f(a)\alpha(a)] = \sum_{k=1}^n f(x_k)[\alpha(t_k) - \alpha(x_k)] + \sum_{k=1}^n f(x_{k-1})[\alpha(x_{k-1}) - \alpha(t_k)]$

Now, by taking a new partition, $Q = (x_0, t_1, x_1, t_2, x_2, t_3, \dots)$, $S = (x_0, x_1, x_1, x_2, x_2, x_3, x_3, \dots)$, then $S_{P,t}(\alpha, f) - [f(b)\alpha(b) - f(a)\alpha(a)] = -S_{Q,S}(f, \alpha)$

The partition Q is finer than P so also finer than P_ε.

From $(*)$, we find that $|-S_{Q,S}(f,\alpha) + \int_a^b f\,d\alpha| < \varepsilon \Leftrightarrow |S_{P,t}(\alpha, f) - [f(b)\alpha(b) - f(a)\alpha(a)]| < \varepsilon$

$\therefore$ This shows that $\alpha \in \mathcal{R}(f)$ and $\int_a^b \alpha\,df = f(b)\alpha(b) - f(a)\alpha(b)$ □

PROPOSITION (Change of Variables)

necessary condition for any change of variables of RS integrals

Let $c \leq d$ and $g: [c,d] \to \mathbb{R}$ is a continuous injective monotonic function.

Define $a := g(c)$, $b := g(d)$. Given $f \in \mathcal{R}(\alpha; a, b) = \mathcal{R}(\alpha; b, a)$, define $h(x) := f \circ g(x)$, $\beta(x) = \alpha \circ g(x)$, $\forall x \in [c,d]$.

Then, $h \in \mathcal{R}(\beta; c, d)$ and $\int_{g(c)}^{g(d)} f(t)\,d\alpha(t) = \int_a^b f\,d\alpha = \int_c^d h\,d\beta = \int_c^d f(g(x))\,d\alpha(g(x))$

Proof

WLOG, suppose that g is a strictly increasing function. In particular, g is bijective.

For any tagged partition (P, t) of $[a,b]$, define its "image" tagged position (P', t') under g^{-1} as below, $P' = (y_k)_{0 \leq k \leq n}$, $t' = (t'_k)_{0 \leq k \leq n}$, with $y_k = g^{-1}(x_k)$, $t'_k = g^{-1}(t_k)$

Then, $S_{P',t'}(h, \beta) = \sum_{k=1}^n h(t'_k)[\beta(y_k) - \beta(y_{k-1})] = \sum_{k=1}^n f(g(t'_k))[\alpha(g(y_k)) - \alpha(g(y_{k-1}))] = \sum_{k=1}^n f(t_k)[\alpha(x_k) - \alpha(x_{k-1})] = S_{P,t}(h, \beta)$

$\therefore (h, \beta)$ satisfies (RS) $\Leftrightarrow (f, \alpha)$ satisfies (RS). Moreover, they have the same limit, i.e. $\int_a^b f\,d\alpha = \int_c^d h\,d\beta$. □

PROPOSITION

Let $f \in \mathcal{R}(\alpha; a, b)$. Suppose that α is C^1. Then, $f\alpha' \in \mathcal{R}(x; a, b)$ and we have the identity $\int_a^b f(x)\,d\alpha(x) = \int_a^b f(x)\alpha'(x)\,dx$

Proof

Let $\varepsilon > 0$ and $P_\varepsilon^{(1)} \in \mathcal{P}([a,b])$, s.t. $|S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$ $\forall P \supseteq P_\varepsilon^{(1)}$ and tagged points t.

Let $g(x) := f(x)\alpha'(x)$ for $x \in [a,b]$.

$\exists s_k$, s.t. $\Delta\alpha_k = s_k \Delta x_k$

For a fixed tagged partition (P, t) with $P \supseteq P_\varepsilon'$, we have $S_{P,t}(g, x) = \sum_{k=1}^n g(t_k)\Delta x_k$ and $S_{P,t}(f,\alpha) = \sum_{k=1}^n f(t_k)[\alpha(x_k) - \alpha(x_{k-1})]$

By taking their difference, we have $S_{P,t}(f,\alpha) - S_{P,t}(g,x) = \sum_{k=1}^n f(t_k)[\alpha'(s_k) - \alpha'(t_k)]\Delta x_k$

Since α' is continuous on $[a,b]$, it is also uniformly continuous

Let us take $\delta > 0$, s.t. $\forall s, t \in [a,b]$, $|s - t| < \delta \Rightarrow |\alpha'(s) - \alpha'(t)| < \varepsilon$

Thus, by taking $P_\varepsilon := P_\varepsilon^{(1)} \vee P_\varepsilon^{(2)}$ with $P_\varepsilon^{(2)} \in \mathcal{P}([a,b])$ with $\|P_\varepsilon^{(2)}\| \leq \frac{\delta}{2}$,

Then, $\forall P \supseteq P_\varepsilon$, we also have $\|P\| \leq \frac{\delta}{2}$, and $|S_{P,t}(f,\alpha) - S_{P,t}(g,x)| \leq \sum_{k=1}^n M\varepsilon \cdot \Delta x = M(b-a) = \varepsilon$, with $M = \sup_{x \in [a,b]} |f(x)| < \infty$ □

COROLLARY

If we take $f \equiv 1$, we find $\alpha(b) - \alpha(a) = \int_a^b d\alpha(x) = \int_a^b \alpha'(x)\,dx$, which is the second fundamental theorem of calculus

STEP FUNCTION INTEGRATORS

DEFINITION

Given a function $\alpha: [a,b] \to \mathbb{R}$, it is called a step function if there is a partition $P = \mathcal{P}([a,b])$ s.t. $f|_{(x_{k-1}, x_k)}$ is constant for $1 \leq k \leq n$.

We define the jump at x_k to be $\alpha_k := \alpha(x_k+) - \alpha(x_k-)$, with $\alpha_0 := \alpha(x_0+) - \alpha(x_0)$ and $\alpha_n := \alpha(x_n) - \alpha(x_n-)$

LEMMA

Let $c\in[a,b]$, and $\alpha:[a,b]\to\mathbb{R}$ be defined by $\alpha(x)=\begin{cases}\alpha(a), & a\le x<c\\ \alpha(b), & c<x\le b\end{cases}$

Let $f:[a,b]\to\mathbb{R}$, s.t. (1) f or α is continuous from the left at c

(2) f or α is continuous from the right at c

Then, $f\in R(\alpha;a,b)$ and we have $\int_a^b f\,d\alpha=f(c)[\alpha(c+)-\alpha(c-)]$

<u>Proof</u>

Let $P\in\mathcal{P}([a,b])$, s.t. $x_k=c$ for some $1\le k\le n-1$

Consider a choice of tagged points t at P, we have $S_{P,t}(f,\alpha)=f(t_k)[\overbrace{\alpha(x_k)}^{\alpha(c)}-\overbrace{\alpha(x_{k-1})}^{\alpha(c-)}]+f(t_{k+1})[\overbrace{\alpha(x_{k+1})}^{\alpha(c+)}-\overbrace{\alpha(x_k)}^{\alpha(c)}]$

$$\Rightarrow \Delta=\Delta(P,t)=S_{P,t}(f,\alpha)-f(c)[\alpha(c+)-\alpha(c-)]$$
$$=\underbrace{[f(t_k)-f(c)][\alpha(c)-\alpha(c-)]}_{\overset{!!}{\Delta_1}}+\underbrace{[f(t_{k-1})-f(c)][\alpha(c+)-\alpha(c)]}_{\overset{!!}{\Delta_2}}$$

Let us bound $|\Delta_1|$ from above.

- If α is left continuous at c, i.e. $\alpha(c-)=\alpha(c)$, then $\Delta_1=0$
- If f is left continuous at c. Let $\varepsilon>0$. Take $\delta_1>0$, s.t. $x\in(c-\delta_1,c)\Rightarrow|f(x)-f(c)|<\varepsilon$

  Then, if $\|P\|<\delta_1$, then $t_k\in(c-\delta_1,c)$ and we find $|\Delta_1|<\varepsilon|\alpha(c)-\alpha(c-)|$

The similar applies to Δ_2.

In conclusion, for $\varepsilon>0$, we can find $\delta>0$, s.t. for $P\in\mathcal{P}([a,b])$ satisfying $\|P\|<\delta$ and $c\in\mathrm{Supp}(P)$, then $|\Delta|\le\varepsilon|\alpha(c)-\alpha(c-)|+\varepsilon|\alpha(c+)-\alpha(c)|$ $\square$

3-4-25 (WEEK 3)

THEOREM (Direct result just by using linearity)

Let α be a step function (w.r.t. the partition $P=(x_k)_{0\le k\le n}$)

Let $f:[a,b]\longrightarrow\mathbb{R}$ be s.t.

(i) $\forall 0\le k\le n-1$, at least one of f and α is right conti at x_k

(ii) $\forall 1\le k\le n$, at least one of f and α is left conti at x_k.

Then, $f\in\mathcal{R}(\alpha)$ and $\int_a^b f\,d\alpha=\sum_{k=1}^n f(x_k)[\alpha(x_k^+)-\alpha(x_k^-)]$

COROLLARY

Let $a_1,\dots,a_n\in\mathbb{R}$, define a left-conti function

$$f:[0,1]\longrightarrow\mathbb{R},\quad x\longmapsto\begin{cases}0, & x=0\\ a_k, & x\in(k-1,k],\ 1\le k\le n\end{cases}$$

Then, $\sum_{k=1}^n a_k=\sum_{k=1}^n f(k)=\int_0^n f(x)\,d\lfloor x\rfloor$

COROLLARY (Euler's Summation Formula) ← Just need to understand how to prove, no need to memorize

Let $f:[a,b]\to\mathbb{R}$ be C^1. Then,

(1) $\sum_{a<n\le b,\,n\in\mathbb{Z}} f(n)=\int_a^b f(x)\,dx+\int_a^b f'(x)\{x\}\,dx+f(a)\{a\}-f(b)\{b\}$, where $\{x\}:=x-\lfloor x\rfloor$ denotes the fractional part of x

(2) $\sum_{n=a}^b f(n)=\int_a^b f(x)\,dx+\int_a^b f'(x)(\{x\}-\frac{1}{2})\,dx+\frac{1}{2}[f(a)+f(b)]$

Proof

We know f and f' are conti, so they belong to $\mathcal{R}(x;a,b)$

First, we know that $f\in\mathcal{R}(\lfloor x\rfloor;a,b)$ so $\int_a^b f(x)\,d\lfloor x\rfloor$ is well-defined.

So, we also have $f\in\mathcal{R}(\{x\})=\mathcal{R}(x-\lfloor x\rfloor)$. By integration by parts, we find $\int_a^b f(x)\,d(\underbrace{x-\lfloor x\rfloor}_{\{x\}})+\int_a^b\{x\}\,df(x)=f(b)\{b\}-f(a)\{a\}$, sub in the previous corollary, we get (1).

To show (2), we may directly apply integration by parts to f and $\{x\}$.

The other method is carried out by taking the formula for (1).

Let $a<b\in\mathbb{Z}$, we have $\underbrace{\sum_{n=a}^b f(n)=f(a)+\int_a^b f(x)\,dx+\int_a^b f'(x)\{x\}\,dx}_{\text{Since }\{a\}=\{b\}=0\ \forall a,b\in\mathbb{Z},\text{ so "}f(a)\{a\}-f(b)\{b\}\text{"}=0}$ and $\int_a^b\frac{1}{2}f'(x)\,dx=\frac{1}{2}\int_a^b df(x)=\frac{1}{2}[f(b)-f(a)]$ □

COROLLARY (Abel's Summation)

Let $(a_n)_{n\ge1}$ be real numbers. Define

$$A:\mathbb{R}_n\longrightarrow\mathbb{R},\quad x\longmapsto\sum_{k=1}^{\lfloor x\rfloor}a_k$$

For $x\ge1$, and a conti function $f:[1,x]\longrightarrow\mathbb{R}$, we have $\sum_{k=1}^{\lfloor x\rfloor}a_k f(k)=-\int_1^x A(t)\,\underbrace{df(t)}_{\text{If } f\in C^1,\ df(t)=f'(t)dt}+A(x)f(x)$

DARBOUX SUMMATIONS

UPPER AND LOWER DARBOUX SUMMATIONS AND INTEGRALS

DEFINITION

Let $P\in\mathcal{P}([a,b])$ and define for $1\le k\le n$, $M_k=M_k(f):=\sup\{f(x)\mid x\in[x_{k-1},x_k]\}$ and $m_k=m_k(f):=\inf\{f(x)\mid x\in[x_{k-1},x_k]\}$

We define the upper and lower Darboux sums as follows: $U_P(f,\alpha)=\sum_{k=1}^n M_k(f)\Delta\alpha_k$, $L_P(f,\alpha)=\sum_{k=1}^n m_k(f)\Delta\alpha_k$

Note: No tagged points are needed for these defs. Also, when $\alpha(x)=x$, we call them upper/lower Riemann sums.

LEMMA (By def)

Suppose that α is increasing/non-decreasing on $[a,b]$. Then, for any $f:[a,b]\longrightarrow\mathbb{R}$ and any partition $P\in\mathcal{P}([a,b])$ and tagged points t, we have $m_k(f)\le f(t_k)\le M_k(f)$, $1\le k\le n$ and $L_P(f,\alpha)\le S_{P,t}(f,\alpha)\le U_P(f,\alpha)$

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PROPOSITION

Suppose that α is increasing.

(a) $\forall P \subseteq P'$, we have $U_{P'}(f,\alpha) \leq U_P(f,\alpha)$ and $L_P(f,\alpha) \leq L_{P'}(f,\alpha)$

(b) $\forall P, P'$, we have $L_P(f,\alpha) \leq U_{P'}(f,\alpha)$

Proof

(a) It is enough to prove the inequalities hold when P' contains one more subdivision point than P. Let $P=(x_k)_{0\leq k\leq n}$ and $c\in(x_{i-1}, x_i)$ when $1\leq i\leq n$.

We have: $U_P(f,\alpha)=\sum_{k=1}^{n} M_k(f)\Delta\alpha_k$, and $U_{P'}(f,\alpha)=\sum_{k=1, k\neq i}^{n} M_k(f)\Delta\alpha_k + M'[\alpha(c)-\alpha(x_{i-1})]+M''[\alpha(x_i)-\alpha(c)]$, where $M' := \sup\{f(x) \mid x_{i-1}\leq x\leq c\}$ and $M'' := \sup\{f(x) \mid c\leq x\leq x_i\}$.

Since $M'\leq M_i(f)$, $M''\leq M_i(f)$, we deduce $M'[\alpha(c)-\alpha(x_{i-1})]+M''[\alpha(x_i)-\alpha(c)] \leq M_i(f)[\alpha(x_i)-\alpha(x_{i-1})]$, so $U_{P'}(f,\alpha)\leq U_P(f,\alpha)$ (L_P is similar)

(b) Let $P, P'\in\mathcal{P}([a,b])$, and $P'' := P'\vee P$. Then, from (a), $L_P(f,\alpha)\leq L_{P''}(f,\alpha)\leq U_{P''}(f,\alpha)\leq U_{P'}(f,\alpha)$ □

Core idea to link two unrelated partitions

DEFINITION

Suppose α is increasing. The upper/lower Stieltjes integrals of f w.r.t. α are defined by:

- $\overline{I}(f,\alpha) = \overline{\int_a^b} f\,d\alpha := \inf\{U_P(f,\alpha) \mid P\in\mathcal{P}([a,b])\}$
- $\underline{I}(f,\alpha) = \underline{\int_a^b} f\,d\alpha := \sup\{L_P(f,\alpha) \mid P\in\mathcal{P}([a,b])\}$

PROPOSITION

Suppose α is increasing, then $\underline{I}(f,\alpha)\leq\overline{I}(f,\alpha)$ (Proof: Trivial)

REMARK

The equality above may not hold. Say $\alpha(x)=x$, $f(x)=\mathbb{1}_{\mathbb{Q}}$ defined on $[0,1]$.

Then, $U_P(f,x)=1$ and $L_P(f,x)=0$ $\forall P\in\mathcal{P}([a,b]) \Rightarrow$ By def, $\overline{I}(f,x)=1$, $\underline{I}(f,x)=0$, so $\underline{I}(f,x)\lneq\overline{I}(f,x)$

PROPOSITION (Linearity)

Let $a\leq c\leq b$. Let $\alpha:[a,b]\to\mathbb{R}$ be bounded and increasing

Let $f,g:[a,b]\to\mathbb{R}$ be bounded. We have:

- $\overline{\int_a^b} f\,d\alpha = \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha$, also $\underline{\int_a^b} f\,d\alpha = \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$
- $\overline{\int_a^b} (f+g)\,d\alpha \leq \overline{\int_a^b} f\,d\alpha + \overline{\int_a^b} g\,d\alpha$, also $\underline{\int_a^b} (f+g)\,d\alpha \geq \underline{\int_a^b} f\,d\alpha + \underline{\int_a^b} g\,d\alpha$

RIEMANN'S CONDITION

DEFINITION

Let $\alpha:[a,b]\to\mathbb{R}$ be increasing. We say that f satisfies Riemann's condition w.r.t. α on $[a,b]$ if $\forall\varepsilon>0$, $\exists P_\varepsilon\in\mathcal{P}([a,b])$, s.t. $\forall P\supseteq P_\varepsilon$, we have $0\leq U_P(f,\alpha)-L_P(f,\alpha)<\varepsilon$ (Note: Tagged points don't matter here)

REMARK

Thanks to the propositions above, it suffices to find $P_\varepsilon\in\mathcal{P}([a,b])$ with $U_{P_\varepsilon}(f,\alpha)-L_{P_\varepsilon}(f,\alpha)<\varepsilon$ to satisfy Riemann's condition

THEOREM

The assumption for α to be increasing is actually not very restrictive. Recall the decomposition theorem for function of bounded variation ($\alpha\in BV \Leftrightarrow \alpha = \underbrace{V}_{\text{increasing}} - \underbrace{(V-\alpha)}_{\text{increasing}}$)

For any increasing integrator α, TFAE:

(1) $f\in R(\alpha; a,b)$

(2) f satisfies Riemann's condition

(3) $\underline{I}(f,\alpha)=\overline{I}(f,\alpha)$

Proof: Next set of notes!

SOME APPLICATIONS

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PROPOSITION

Given non-decreasing α, and $f, g \in \mathcal{P}([a,b])$, suppose $f(x) \leq g(x)$ $\forall x \in [a,b]$, then we have $\int_a^b f(x)\,d\alpha(x) \leq \int_a^b g(x)\,d\alpha(x)$

<u>Proof</u>

For any partition $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a,b])$, we have $M_k(f) \leq M_k(g)$ and $m_k(f) \leq m_k(g)$, so we have $L_P(f,\alpha) \leq L_P(g,\alpha)$ and $U_P(f,\alpha) \leq U_P(g,\alpha)$

Therefore, the upper and lower Stieltjes integrals satisfy: $\overline{I}(f,\alpha) \leq \overline{I}(g,\alpha)$ and $\underline{I}(f,\alpha) \leq \underline{I}(g,\alpha)$. By the Thm above, thus $\int_a^b f\,d\alpha \leq \int_a^b g\,d\alpha$ □

PROPOSITION

Given non-decreasing α, $f \in R(\alpha; a, b)$, then $|f| \in R(\alpha; a, b)$ and $\left|\int_a^b f(x)\,d\alpha(x)\right| \leq \int_a^b |f(x)|\,d\alpha(x)$

<u>Proof</u>

For any partition $P \in \mathcal{P}([a,b])$ and $1 \leq k \leq n$, we have $M_k(f) - m_k(f) = \sup\{f(x) - f(y) \mid x, y \in [x_{k-1}, x_k]\}$

We may check that $M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f)$, hence $U_P(|f|,\alpha) - L_P(|f|,\alpha) \leq U_P(f,\alpha) - L_P(f,\alpha)$, i.e. $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$ □

3-6-25 (WEEK 3)

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THEOREM (Proof for last section of notes)

For any increasing integrator α, TFAE:

(1) $f \in R(\alpha; a, b)$

(2) f satisfies Riemann's condition

(3) $\underline{I}(f,\alpha) = \overline{I}(f,\alpha)$

Proof

"(1) $\Rightarrow$ (2)": Suppose that f satisfies (RS). The proof is trivial if $\alpha(a) = \alpha(b)$, so we may assume $\alpha(a) < \alpha(b)$.

Let $\varepsilon > 0$, then take $P_\varepsilon \in \mathcal{P}([a,b])$ s.t. $\forall$ tagged partition (P,t), (P,t'), with $P \supseteq P_\varepsilon$, we have $|S_{P,t}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$, $|S_{P,t'}(f,\alpha) - \int_a^b f\,d\alpha| < \varepsilon$

By triangle inequality, $|\sum_{k=1}^{n} [f(t_k) - f(t_k')]\Delta\alpha_k| < 2\varepsilon$

For each $1 \le k \le n$, we take $t_k, t_k' \in [x_{k-1}, x_k]$, s.t. $M_k(f) \le f(t_k) + \frac{\varepsilon}{\alpha(b)-\alpha(a)}$ and $m_k(f) \ge f(t_k) - \frac{\varepsilon}{\alpha(b)-\alpha(a)}$

This means, $M_k(f) - m_k(f) \le f(t_k) - f(t_k') + \frac{2\varepsilon}{\alpha(b)-\alpha(a)}$.

Therefore, $U_P(f,\alpha) - L_P(f,\alpha) = \sum_{k=1}^{n} [M_k(f) - m_k(f)]\Delta\alpha_k \le \sum_{k=1}^{n} [f(t_k) - f(t_k')]\Delta\alpha_k + \frac{2\varepsilon}{\alpha(b)-\alpha(a)} \overbrace{\sum_{k=1}^{n} \Delta\alpha_k}^{\alpha(b)-\alpha(a)} \le 4\varepsilon$, so (2) holds.

"(2) $\Rightarrow$ (3)": Suppose that (2) holds. Let $\varepsilon > 0$ and take $P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $0 \le U_P(f,\alpha) - L_P(f,\alpha) \le \varepsilon$ for all $P \supseteq P_\varepsilon$.

For all $P \supseteq P_\varepsilon$, $\overline{I}(f,\alpha) \le \underline{U_P(f,\alpha) \le L_P(f,\alpha) + \varepsilon} \le \underline{I}(f,\alpha) + \varepsilon$

As this is true $\forall \varepsilon$, thus $\overline{I}(f,\alpha) \le \underline{I}(f,\alpha)$, which implies $\overline{I}(f,\alpha) = \underline{I}(f,\alpha)$ ($\because$ "$\ge$" is trivially true)

"(3) $\Rightarrow$ (1)": Suppose that (3) holds. Let $L = \underline{I}(f,\alpha) = \overline{I}(f,\alpha)$.

Let $\varepsilon > 0$, take $P_\varepsilon^{(1)} \in \mathcal{P}([a,b])$ s.t. $L_P(f,\alpha) + \varepsilon \ge \underline{I}(f,\alpha) = L$ $\forall P \supseteq P_\varepsilon^{(1)}$

Take $P_\varepsilon^{(2)} \in \mathcal{P}([a,b])$, s.t. $U_P(f,\alpha) - \varepsilon \le \overline{I}(f,\alpha) = L$ $\forall P \supseteq P_\varepsilon^{(2)}$

Take $P_\varepsilon := P_\varepsilon^{(1)} \vee P_\varepsilon^{(2)}$, then $\forall P \supseteq P_\varepsilon$, tagged points t, we have:

$\hookrightarrow$ $S_{P,t}(f,\alpha) \le U_P(f,\alpha) \le L + \varepsilon$

$\hookrightarrow$ $S_{P,t}(f,\alpha) \ge L_P(f,\alpha) \ge L - \varepsilon$

$\therefore$ In other words, $|S_{P,t}(f,\alpha) - L| \le \varepsilon$. Namely, $f \in R(\alpha; a,b)$ and $\int_a^b f\,d\alpha = L$ □

PROPOSITION (APPLICATION)

Suppose that α is nondecreasing on $[a,b]$. If $f \in R(\alpha; a,b)$, then $f^2 \in R(\alpha; a,b)$

Proof

Let $f \in R(\alpha; a,b)$ and $P \in \mathcal{P}([a,b])$ be a partition.

For $1 \le k \le n$, we have $M_k(f^2) - m_k(f^2) = \overbrace{M_k(|f|)^2 - m_k(|f|)^2}^{\text{Key step}} = (M_k(|f|) + m_k(|f|))(M_k(|f|) - m_k(|f|)) \le 2M[M_k(|f|) - m_k(|f|)]$, where we have $M = \sup\{|f(x)| \mid x \in [a,b]\} < \infty$

This suffices because after one more step (i.e. Σ), we get $U_P - L_P$ again

$\because$ $|f|$ satisfies Riemann's condition ($U_P - L_P < \varepsilon$)

$\therefore$ f^2 satisfies Riemann's condition □

COROLLARY

Suppose that α is nondecreasing on $[a,b]$. If $f, g \in R(\alpha; a,b)$, then $fg \in R(\alpha; a,b)$

Proof

Use $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$, since $+/-$ operations preserve integrability, thus $fg \in R(\alpha; a,b)$ too. □

REMARK

We can use induction to further show that for nondecreasing α and $f \in R(\alpha; a,b)$, $\forall n \in \mathbb{N}_0$, then $f^n \in R(\alpha; a,b)$.

IMPORTANT

The converse of our proposition does not hold. Consider $f: [0,1] \longrightarrow \mathbb{R}$, $x \longmapsto \begin{cases} -1, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$. We have $f^2 \in R(\alpha; a,b)$ but $f \notin R(\alpha; a,b)$.

INTEGRATORS OF BOUNDED VARIATION

Shun/翔海 (@shun4midx)

GOAL

We want to use the decomposition theorem for functions of bounded variation to extend the previous results to general integrators, i.e. of bounded variation. We can mainly do so by recalling: $\alpha \in BV \Rightarrow \alpha = V - (V - \alpha)$, for a variation function V.

THEOREM

Let $\alpha \in BV([a,b])$ and V be its variation function. Then, $f \in R(\alpha; a, b) \Rightarrow f \in R(V; a, b)$

__Proof__

When α is a constant function, then $V \equiv 0$, and the theorem holds trivially. Now, let us assume $\alpha(a) < \alpha(b)$, so $V(b) > 0$

Let $\varepsilon > 0$. Take $P_\varepsilon^{(1)} \in \mathcal{P}([a,b])$ s.t. $\forall P \supseteq P_\varepsilon$ and tagged points t, then $\left|\sum_{k=1}^{n} [f(t_k) - f(t_k')] \Delta\alpha_k\right| \leq \varepsilon$.

Let $M = \sup|f|$. Take $P_\varepsilon^{(2)} \in \mathcal{P}([a,b])$, s.t. $\forall P \supseteq P_\varepsilon^{(2)}$, $V(b) = V_\alpha([a,b]) \leq V_P(\alpha) + \frac{\varepsilon}{M} = \sum_{k=1}^{n} |\Delta\alpha_k| + \frac{\varepsilon}{M}$ (Prop from before: $\forall \varepsilon > 0, \exists P_\varepsilon$, s.t. $P \supseteq P_\varepsilon \Rightarrow V_P(f) \leq V_f \leq V_P(f) + \varepsilon$)

We now factcheck that $f \in R(\alpha; a, b)$. Since V is non-decreasing, we only need to check that it satisfies Riemann's condition, i.e. we need to bound

$$\sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta V_k = \underbrace{\sum_{k=1}^{n} [M_k(f) - m_k(f)] |\Delta\alpha_k|}_{(**)} + \underbrace{\sum_{k=1}^{n} [M_k(f) - m_k(f)] [\Delta V_k - |\Delta\alpha_k|]}_{(*)}$$

Consider $(*)$.

Let $M = \sup|f|$, then $(*) \leq 2M \sum_{k=1}^{n} (\Delta V_k - |\Delta\alpha_k|) = 2M\left(V(b) - \sum_{k=1}^{n} |\Delta\alpha_k|\right) \leq 2\varepsilon$.

To bound $(**)$, we distinguish the indices k w.r.t. the sign of $\Delta\alpha_k$.

Let $K^+ := \{1 \leq k \leq n \mid \Delta\alpha_k \geq 0\}$ and $K^- := \{1 \leq k \leq n \mid \Delta\alpha_k < 0\}$, $\varepsilon' = \frac{\varepsilon}{V(b)}$ (really important tactic to deal with absolute values)

For $k \in K^+$, choose $t_k, t_k' \in [x_{k-1}, x_k]$, s.t. $f(t_k) - f(t_k') \geq M_k(f) - m_k(f) - \varepsilon'$

For $k \in K^-$, choose $t_k, t_k' \in [x_{k-1}, x_k]$, s.t. $f(t_k') - f(t_k) \geq M_k(f) - m_k(f) - \varepsilon'$

sub from above

Therefore, we find $(**) = \sum_{k=1}^{n} [M_k(f) - m_k(f)] |\Delta\alpha_k| = \sum_{k \in K^+} [M_k(f) - m_k(f)] \Delta\alpha_k + \sum_{k \in K^-} [M_k(f) - m_k(f)] (-\Delta\alpha_k)$

$\leq \sum_{k=1}^{n} [f(t_k) - f(t_k')] \Delta\alpha_k + \varepsilon' \sum_{k=1}^{n} |\Delta\alpha_k| \leq \varepsilon + \varepsilon' V(b) = 2\varepsilon$ □

COROLLARY (Formal restating of our goal)

Let $\alpha \in BV([a,b])$, bounded $f: [a,b] \to \mathbb{R}$, then $(1) \Leftrightarrow (2)$.

(1) $f \in R(\alpha; a, b)$

(2) $\exists$ nondecreasing α_1, α_2, s.t. $f \in R(\alpha_1) \cap R(\alpha_2)$ and $\alpha = \alpha_1 - \alpha_2$

__Proof__

"(2) ⇒ (1)": By linearity ✓

"(1) ⇒ (2)": Write $\alpha = V - (V - \alpha)$ and then use the theorem above. □

PROPOSITION

Let $\alpha \in BV([a,b])$ and $f \in R(\alpha; a, b)$. Then, for any $[c,d] \subseteq [a,b]$, we have $f \in R(\alpha; c, d)$

__Proof__

If only BV, we can use $\alpha = V - (V - \alpha) \Rightarrow f \in R(V; a, b) \cap R(V - \alpha; a, b) \Rightarrow f \in R(\alpha; a, b)$, as we have the prop above

Suppose α is non-decreasing on $[a,b]$. If we can check that $f \in R(\alpha; a, x)$ for all $x \in [a,b]$, then we get $f \in R(\alpha; a, c) \cap R(\alpha; a, d) = R(\alpha; c, d)$.

For $x \in [a,b]$ and a partition $P \in \mathcal{P}([a,b])$, define $\Delta_P(x) := U_P(f|_{[a,x]}, \alpha|_{[a,x]}) - L_P(f|_{[a,x]}, \alpha|_{[a,x]})$

Fix $x \in [a,b]$, let $\varepsilon > 0$, then take $P_\varepsilon \in \mathcal{P}([a,b])$, s.t. $\Delta_P(b) \leq \varepsilon$ $\forall P \supseteq P_\varepsilon$.

WLOG, we may assume $x \in \mathrm{Supp}(P_\varepsilon)$ ($\because$ "$\forall P \supseteq P_\varepsilon$" means any points chosen still make the inequality true).

Now, let $P_\varepsilon' := P_\varepsilon \cap [a, x]$. Now, $\forall P' \supseteq P_\varepsilon'$, define $P := P' \vee P_\varepsilon \supseteq P_\varepsilon$. Then, we have $\Delta_{P'}(x) \leq \Delta_P(b) \leq \varepsilon$. □

Shun/羊羽海 (@shun4midx)

THEOREM

Let $\alpha \in BV([a,b])$, $f, g \in R(\alpha; a, b)$.

Define for all $x \in [a,b]$, $F(x) = \int_a^x f(t)\,d\alpha(t)$, $G(x) = \int_a^x g(x)\,d\alpha(x)$. Then, $f \in R(G; a, b)$, $g \in R(F; a, b)$ and $fg \in R(\alpha; a, b)$, where:

$$\int_a^b f(x)g(x)\,d\alpha(x) = \int_a^b f(x)\,dG(x) = \int_a^b g(x)\,dF(x)$$

<u>Proof</u>

Similar to the above, it is sufficient to prove this is the case for non-increasing α.

Suppose α is non-decreasin, we already know that $fg \in R(\alpha)$. By symmetry, we only need to check $f \in R(G)$ and $\int_a^b fg\,dx = \int_a^b f\,dG$.

Let $P \in \mathcal{P}([a,b])$ and fix tagged points t, then $S_{P,t}(f,G) = \sum_{k=1}^n f(t_k)\Delta G_k = \sum_{k=1}^n f(t_k)\int_{x_{k-1}}^{x_k} g(t)\,d\alpha(t)$ and $\int_a^b fg\,d\alpha = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(t)g(t)\,d\alpha(t)$

Now, $|S_{P,t}(f,G) - \int_a^b fg\,d\alpha| = |\sum_{k=1}^n \int_{x_{k-1}}^{x_k} [f(t_k) - f(x)]\,g(x)\,d\alpha(x)|$

$\leq M\sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f(t_k) - f(x)|\,d\alpha(x)$, where $|g(x)| \leq M\ \forall x \in [a,b]$, since it is bounded.

$\leq M\sum_{k=1}^n \int_{x_{k-1}}^{x_k} [M_k(f) - m_k(f)]\,d\alpha(x)$

$= M[U_P(f,\alpha) - L_P(f,\alpha)]$

$\therefore$ By def, f satisfies Riemann's condition w.r.t. $\alpha \Rightarrow |S_{P,t}(f,G) - \int_a^b fg\,d\alpha| \leq M\varepsilon'$ $\square$

Shun / 羊羽海 (@shun4midx)

RIEMANN-STIELTJES INTEGRABILITY

THEOREM

Let $f, \alpha: [a,b] \longrightarrow \mathbb{R}$ be bounded. Then, $f \in \mathcal{R}(\alpha)$ if (1) or (2)

1) f is continuous and $\alpha \in BV$
2) α is continuous and $f \in BV$

COROLLARY

For $\alpha(x) = x$ and a continuous or a bounded variation f, then f is integrable

PROOF OF THEOREM

Integration by parts gives us $f \in \mathcal{R}(\alpha) \Leftrightarrow \alpha \in \mathcal{R}(f)$, therefore it is enough to show that when (1) holds with increasing α, $f \in \mathcal{R}(\alpha)$.

Let $\varepsilon > 0$. Since f is continuous on $[a,b]$ and $[a,b]$ is compact, we know that f is uniformly continuous

Take $\delta > 0$, s.t. $x, y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Let us take a partition $P \in \mathcal{P}([a,b])$ s.t. $\|P_\varepsilon\| < \delta$, then:

$U_P(f,\alpha) - L_P(f,\alpha) = \sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta\alpha_k \leq \varepsilon \sum_{k=1}^{n} \Delta\alpha_k = \varepsilon[\alpha(b) - \alpha(a)]$, thus Riemann's condition holds. □

THEOREM

Let $f, \alpha: [a,b] \to \mathbb{R}$ be bounded, α be nondecreasing. If (1) or (2) holds, then $f \notin \mathcal{R}(\alpha)$

1) $\exists c \in [a,b]$, s.t. f and α are not right-continuous at c
2) $\exists c \in [a,b]$, s.t. f and α are not left-continuous at c

Proof

By symmetry, we only need to check that $(1) \Rightarrow f \notin \mathcal{R}(\alpha)$

Suppose that $c \in [a,b]$, s.t. f and α are not right continuous at c

Let $\varepsilon > 0$, $\delta > 0$, s.t. $\exists x \in (c, c+\delta)$, $|f(x) - f(c)| > \varepsilon$ and $\exists y \in (c, c+\delta)$, s.t. $|\alpha(y) - \alpha(c)| > \varepsilon$

Let $P \in \mathcal{P}([a,b])$, s.t. $x_i = c$, $x_{i+1} = y$ for some $1 \leq i \leq n-1$.

Then, $U_P(f,\alpha) - L_P(f,\alpha) = \sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta\alpha_k \geq \varepsilon [M_{i+1}(f) - m_{i+1}(f)] \geq \varepsilon^2$ □

MEAN VALUE THEOREMS

FIRST MEAN VALUE THEOREM

Let $\alpha: [a,b] \to \mathbb{R}$ be a nondecreasing function and $f \in \mathcal{R}(\alpha; a, b)$. Let $M := \sup\{f(x) \mid x \in [a,b]\}$ and $m := \inf\{f(x) \mid x \in [a,b]\}$. Then, $\exists c \in [m, M]$, such that $\int_a^b f(x)\,d\alpha(x) = c\int_a^b d\alpha(x) = c[\alpha(b) - \alpha(a)]$. In particular, if f is continuous, then $c = f(x_0)$ for some $x_0 \in [a,b]$.

Proof

If α is a constant function, then of course $\int_a^b f\,d\alpha = 0$ and $\int_a^b d\alpha = 0 \Rightarrow$ The statement holds trivially

Suppose $\alpha(a) < \alpha(b)$. Then, for any given partition $P \in \mathcal{P}([a,b])$, we have $L_P(f,\alpha) \leq \underline{I}(f,\alpha) = \bar{I}(f,\alpha) = \int_a^b f\,d\alpha \leq U_P(f,\alpha)$

Moreover, $L_P(f,\alpha) = \sum_{k=1}^{n} m_k(f)\Delta\alpha_k \geq \sum_{k=1}^{n} m\Delta\alpha_k = m\int_a^b d\alpha$ and $U_P(f,\alpha) = \sum_{k=1}^{n} M_k(f)\Delta\alpha_k \leq \sum_{k=1}^{n} M\Delta\alpha_k = M\int_a^b f\,d\alpha$

Therefore, $m\int_a^b d\alpha \leq \int_a^b f\,d\alpha \leq M\int_a^b d\alpha$

$$\Rightarrow c = \frac{\int_a^b f\,d\alpha}{\int_a^b d\alpha} \in [m, M]$$ □

COROLLARY

Some assumptions give us $|\int_a^b f d\alpha| \leq M\int_a^b d\alpha = M[\alpha(b)-\alpha(a)]$, where $M = \sup\{f(x) \mid x\in[a,b]\}$
(This result could also be derived from the triangle inequality)

SECOND MEAN VALUE THEOREM

Let α be continuous and f be non-decreasing. Then, $\int_a^b \alpha df = f(a)\int_a^{x_0} d\alpha + f(b)\int_{x_0}^b d\alpha$ for some $x_0\in[a,b]$.

Proof

The integration by parts gives us $\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df$
Now, applying 1st MVT, we obtain $\int_a^b \alpha df = \alpha(x_0)(f(b)-f(a))$ for some $x_0\in[a,b]$.
Hence, putting these statements together, we get $\int_a^b \alpha df = f(a)\int_a^{x_0} d\alpha + f(b)\int_{x_0}^b d\alpha$ □

FUNDAMENTAL THEOREMS OF CALCULUS

DEFINITION

Let $I\subseteq\mathbb{R}$ be an interval, $f, F: I\to\mathbb{R}$ be functions. If $F'(x) = f(x)\ \forall x\in \text{int}(I)$, we say F is a primitive or antiderivative of f

FIRST FUNDAMENTAL THEOREM OF CALCULUS

Let $\alpha\in BV([a,b])$ and $f\in R(\alpha; a,b)$. Define $F(x) = \int_a^x d\alpha\ \forall x\in[a,b]$. Then, we have:

(a) $F\in BV([a,b])$

(b) If α is continuous at some $c\in[a,b]$, then F is also continuous at c.

(c) If α is non-decreasing, then the derivative $F'(x)$ exists at $x\in(a,b)$ whenever $\alpha'(x)$ exists and f is continuous at x; additionally, for such x, we have $F'(x) = f(x)\alpha'(x)$

Proof

It is sufficient for us to prove the theorem for non-decreasing α.
Note that for $x<y$, $x,y\in[a,b]$, we have $|F(y)-F(x)| \leq M_{x,y}|\alpha(y)-\alpha(x)|$, where $M_{x,y} = \sup\{|f(t)| \mid t\in[x,y]\} \leq \sup\{|f(t)| \mid t\in[a,b]\} =: M$

(a) Given a partition $P\in\mathcal{P}([a,b])$, we write $V_P(F) = \sum_{k=1}^n |\Delta F_k| \leq \sum_{k=1}^n M|\Delta\alpha_k| = M[\alpha(b)-\alpha(a)] < \infty$ $\therefore$ We have $F\in BV([a,b])$ ✓

(b) Let $c\in[a,b]$, s.t. α is continuous at c. Let $\varepsilon>0$, take $\delta>0$, s.t. $\forall x\in[a,b]$, $|x-c|\leq\delta \Rightarrow |\alpha(x)-\alpha(c)|\leq\varepsilon$
Then, we can see $\forall x\in[a,b]$, $|x-c|\leq\delta\Rightarrow|F(x)-F(c)|\leq M\varepsilon$ ($\because \sum|\Delta F_k| \leq M\sum|\Delta\alpha_k|$)
By definition, F is continuous at c. ✓

(c) Let $x\in(a,b)$, s.t. $\alpha'(x)$ exists and f is continuous at x. By definition and MVT, $F'(x) = \lim_{y\to x}\frac{F(y)-F(x)}{y-x} = \lim_{y\to x}\frac{c[\alpha(y)-\alpha(x)]}{y-x}$
Since $c = c(x,y)\to f(x)$ when $y\to x$, by the continuity of f at x and $\alpha'(x)$ exists, we deduce $F'(x) = f(x)\alpha'(x)$ □

COROLLARY (Freshman Calculus I: Riemann Integrals)

Let $\alpha(x) = x$ and $f\in R(x; a,b)$ be a Riemann-integrable function.
Define $F(x) = \int_a^x f(t)dt$ and $G(x) = \int_a^x g(t)dt$
Then, the following properties hold:

(a) F and G are continuous functions of bounded variation on $[a,b]$.

(b) If f is continuous at $x\in(a,b)$, then $F'(x) = f(x)$

(c) For $f\in R(G; a,b)$, $g\in R(F; a,b)$ and $fg\in R(x; a,b)$, we have $\int_a^b f(x)g(x)dx = \int_a^b f(x)dG(x) = \int_a^b g(x)dF(x)$

SECOND FUNDAMENTAL THEOREM OF CALCULUS

Let $f\in R(x; a,b)$ and $F:[a,b]\to\mathbb{R}$, s.t. F is continuous and F' is well-defined on (a,b). Suppose that $F'(x) = f(x)$ for every $x\in(a,b)$. Then, we have $\int_a^b F'(x)dx = \int_a^b f(x)dx = F(b)-F(a)$

Proof

Let $\varepsilon>0$. Since $f\in R(x;a,b)$, we may find $P_\varepsilon\in\mathcal{P}([a,b])$, s.t. $|S_{P,t}(f,x) - \int_a^b f dx| < \varepsilon$ for any partition $P\supseteq P_\varepsilon$ and tagged points t.
For $P\supseteq P_\varepsilon$, and tagged points t, we have $S_{P,t}(f,x) = \sum_{k=1}^n f(t_k)\Delta x_k = \sum_{k=1}^n F'(t_k)\Delta x_k$

Why we need a special partition. It is valid too, since $t_k\in(x_{k-1}, x_k)$

Let us make a special choice of t, for a given partition $P\supseteq P_\varepsilon$, $\Delta F_k = F(x_k) - F(x_{k-1}) = F'(t_k)\Delta x_k = f(t_k)\Delta x_k$
Therefore, $S_{P,t}(f,x) = \sum_{k=1}^n f(t_k)\Delta x_k = \sum_{k=1}^n \Delta F_k = F(b)-F(a)$, this means $\int_a^b f(x)dx = F(b)-F(a)$

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COROLLARY

Let $f \in \mathcal{R}(x; a, b)$, $\alpha : [a,b] \to \mathbb{R}$ be continuous s.t. $\alpha' \in \mathcal{R}(x; a, b)$. Then, $\int_a^b f \, d\alpha = \int_a^b f \alpha' \, dx$

<u>Proof</u>

Just take $g = \alpha'$. $\square$

PROPOSITION (Change of Variables) Very important to check!

Let $g : [c,d] \to \mathbb{R}$ be $\mathcal{C}^1$. Let $f : g([c,d]) \to \mathbb{R}$ be continuous, and define $F(x) := \int_{g(c)}^{x} f(s) \, ds$ $\forall x \in g([c,d])$

Then, we have $\forall x \in [c,d]$, $\int_c^x f \circ g(t) g'(t) = F \circ g(x) = \int_{g(c)}^{g(x)} f(s) \, ds$

3-13-25 (WEEK 4)

Shun/翔海 (@shun4midx)

INTEGRALS DEPENDING ON A PARAMETER

(Sorry, today's notes would have terrible handwriting cuz lots of rain got into my stylus/iPad and it's so heavy to hold the pen so it hurts to even write, let alone neat qwq 本当にごめんねqwq orz 手がとても痛いTT)

QUESTION

Given $f: [0,1] \times [0,1] \longrightarrow \mathbb{R}$
$\qquad (x,y) \longmapsto f(x,y)$

- Integrate f w.r.t. x, how do we define the regularity of f and its integral?
- Interchange the order of integration?

PROPOSITION

Let $Q = [a,b] \times [c,d] \subseteq \mathbb{R}^2$.
Let $f: Q \to \mathbb{R}$ be continuous and $\alpha \in BV([a,b])$
Define $F: [c,d] \longrightarrow \mathbb{R}$
$\qquad y \longmapsto \int_a^b f(x,y)\,d\alpha(x)$
Then, F is continuous, that is $F(y_0) = \lim_{y \to y_0} F(y)$ $\forall y_0 \in [c,d]$.

<u>Proof</u>

Since Q is compact, f is uniformly continuous. Suppose α is non-increasing
Let $\varepsilon > 0$, and take $\delta > 0$, s.t. $\forall (x,y), (x',y') \in Q$, $\|(x,y) - (x',y')\|_1 \le \delta \Rightarrow |f(x,y) - f(x',y')| \le \varepsilon$

Fix $y_0 \in [c,d]$. For $y \in [c,d]$, s.t. $|y - y_0| \le \delta$, we have $|F(y) - F(y_0)| = |\int_a^b (f(x,y) - f(x,y_0))\,d\alpha(x)| \le \int_a^b |f(x,y) - f(x,y_0)|\,d\alpha(x) \overset{\text{requires uniform continuity}}{\le} \int_a^b \varepsilon\,d\alpha(x) = \varepsilon[\alpha(b) - \alpha(a)]$ □

COUNTEREXAMPLE FOR NONCONTINUOUS f

Let $f(x,y) = \begin{cases} \frac{4xy^2}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ (Behind the scenes: My hand is now screaming in agony from the pain of writing o_O)

It's not hard to see that $\forall x \in [0,1]$, $f(x,\cdot): y \mapsto f(x,y)$ is continuous. Similarly, $f(\cdot, y)$ is continuous $\forall y \in [0,1]$. However, f is not continuous on Q because $\lim_{\varepsilon \to 0^+} f(\varepsilon, \varepsilon) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} = +\infty$

Notice, $\forall y \in (0,1]$, we have $F(y) = \int_0^1 f(x,y)\,dx = \left[\frac{x^2 - y^2}{x^2 + y^2}\right]_{x=0}^1 = \frac{1-y^2}{1+y^2} + 1 = \frac{2}{1+y^2}$

Since $f(x,0) = 0$ $\forall x \in [0,1]$, we have $F(0) = \int_0^1 0\,dx = 0$. However, $\lim_{y \to 0^+} F(y) = 2 \neq 0$ $\therefore F$ is not continuous at 0.

COROLLARY

Let $f: Q \to \mathbb{R}$ be continuous and $g \in R(x; a, b)$.
Define $F: [c,d] \longrightarrow \mathbb{R}$
$\qquad y \longmapsto \int_a^b g(x) f(x,y)\,dx$
Then, F is continuous.

<u>Proof</u>

For $x \in [a,b]$, write $G(x) = \int_0^x g(t)\,dt$, and $G \in BV([a,b])$
Then, for $y \in [c,d]$, we have $F(y) = \int_a^b f(x,y)\,dG(x)$. The result follows from the proposition. □

PROPOSITION

Let $\alpha \in BV([a,b])$ and $f: Q = [a,b] \times [c,d] \longrightarrow \mathbb{R}$ be continuous.
Suppose that $\frac{\partial f}{\partial y}$ is continuous on Q. Then, for $y \in (c,d)$, $F'(y)$ exists and $\frac{d}{dy} \int_a^b f(x,y)\,d\alpha(x) = F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,d\alpha(x)$

<u>Proof</u>

Fix $y_0 \in (c,d)$. For $y \in (c,d) \setminus \{y_0\}$, write $\frac{F(y) - F(y_0)}{y - y_0} = \int_a^b \frac{f(x,y) - f(x,y_0)}{y - y_0}\,d\alpha(x)$, which MVT implies equals $\int_a^b \frac{\partial f}{\partial y}(x,y')\,d\alpha(x)$ for some $y' = y'(x,y)$ in between y and y_0.

When $y \to y_0$, we also have $y' \to y_0$. Also, the continuity of $\frac{\partial f}{\partial y}$ implies that $\lim_{y \to y_0} \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(x,y_0)$
Therefore, $F'(y_0) := \lim_{y \to y_0} \frac{F(y) - F(y_0)}{y - y_0} = \lim_{y \to y_0} \int_a^b \frac{\partial f}{\partial y}(x,y')\,d\alpha(x) \underset{\text{proposition above}}{=} \int_a^b \lim_{y \to y_0} \frac{\partial f}{\partial y}(x,y')\,d\alpha(x) = \int_a^b \frac{\partial f}{\partial y}(x,y_0)\,d\alpha(x)$ □

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THEOREM (FUBINI'S THEOREM)

Let $\alpha \in BV([a,b])$, $\beta \in BV([a,b])$, and $f: Q \to \mathbb{R}$ be continuous

Fix $(x,y) \in Q$, define $F(y) := \int_a^b f(x,y)\,d\alpha(x)$, $G(x) := \int_c^d f(x,y)\,d\beta(y)$.

Then, $F \in R(\beta)$ and $G \in R(\alpha)$ and $\int_c^d F(y)\,d\beta(y) = \int_a^b G(x)\,d\alpha(x) \Leftrightarrow \int_a^b \int_c^d f(x,y)\,d\beta(y)\,d\alpha(x) = \int_c^d \int_a^b f(x,y)\,d\alpha(x)\,d\beta(y)$

COUNTEREXAMPLE FOR NONCONTINUOUS f

Let $f(x,y) = \begin{cases} 1, & x \in \mathbb{Q} \\ 2y, & x \notin \mathbb{Q} \end{cases}$

- We know $\overline{\int_0^1} f(x,y)\,dy = \underline{\int_0^1} f(x,y)\,dy = \begin{cases} \int_0^1 1\,dy = 1, & x \in \mathbb{Q} \\ \int_0^1 2y\,dy = 1, & x \notin \mathbb{Q} \end{cases}$. Thus, $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \int_0^1 1\,dx = 1$
- Meanwhile, $\overline{\int_0^1} f(x,y)\,dx = \max(1, 2y)$ and $\underline{\int_0^1} f(x,y)\,dx = \min(1, 2y) \Rightarrow$ Other than $y = \frac{1}{2}$, $\int_0^1 f(x,y)\,dx$ is not well-defined.

PROOF OF FUBINI'S THEOREM (Behind the scenes: My hand is cramping so hard I'm using my elbow to control my pen movement)

WLOG, suppose α and β are nondecreasing. Since Q is compact, thus f is uniformly continuous

Let $\varepsilon > 0$, take $\delta > 0$, s.t. $\forall (x,y), (x',y') \in Q$, $\|(x,y) - (x',y')\|_\infty \le \delta \Rightarrow |f(x,y) - f(x',y')| \le \varepsilon$

Let $P_x = (x_k)_{0 \le k \le m} \in \mathcal{P}([a,b])$, $P_y = (y_\ell)_{0 \le \ell \le n} \in \mathcal{P}([c,d])$, s.t. $\|P_x\|, \|P_y\| < \delta$. Now, let us rewrite one of our integrals.

$\int_a^b \int_c^d f(x,y)\,d\beta(y)\,d\alpha(x) = \sum_{k=1}^m \sum_{\ell=1}^n \int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} f(x,y)\,d\beta(y)\,d\alpha(x) \overset{MVT}{=} \sum_{k=1}^m \sum_{\ell=1}^n \int_{x_{k-1}}^{x_k} \int_{y_{\ell-1}}^{y_\ell} f(x, y_\ell')\,d\beta(y)\,d\alpha(x)$ for some $y_\ell' \in [y_{\ell-1}, y_\ell]$.

By MVT, $= \sum_{k=1}^m \sum_{\ell=1}^n \int_{x_{k-1}}^{x_k} f(x, y_\ell')(\beta(y_\ell) - \beta(y_{\ell-1}))\,d\alpha(x)$

By MVT, $= \sum_{k=1}^m \sum_{\ell=1}^n f(x_k', y_\ell')[\alpha(x_k) - \alpha(x_{k-1})][\beta(y_\ell) - \beta(y_{\ell-1})]$, for some $(x_k', y_\ell') \in [x_{k-1}, x_k] \times [y_{\ell-1}, y_\ell]$

Similarly, we have $\int_c^d \int_a^b f(x,y)\,d\alpha(x)\,d\beta(y) = \sum_{\ell=1}^n \sum_{k=1}^m f(x_k'', y_\ell'')[\alpha(x_k) - \alpha(x_{k-1})][\beta(y_\ell) - \beta(y_{\ell-1})]$

Taking their difference, we find: $\left|\int_a^b \int_c^d f(x,y)\,d\beta(y)\,d\alpha(x) - \int_c^d \int_a^b f(x,y)\,d\alpha(x)\,d\beta(y)\right| \le \sum_{k=1}^m \sum_{\ell=1}^n \underbrace{|f(x_k', y_\ell') - f(x_k'', y_\ell'')|}_{\le \varepsilon \text{ as assumed}} \Delta\alpha_k \Delta\beta_\ell = \varepsilon[\alpha(b) - \alpha(a)][\beta(d) - \beta(c)]$

RIEMANN INTEGRALS

DEFINITION

Let $S \subseteq \mathbb{R}$ be a subset. We may say that S has **measure zero** if for every $\varepsilon > 0$, $\exists$ a countable family $\{U_i := (a_i, b_i) \mid i \in I\}$ of open intervals such that:

(i) $S \subseteq \bigcup_{i \in I} (a_i, b_i)$ ("S can be covered by these open intervals")

(ii) The sum of lengths satisfy $\sum_{i \in I} |U_i| = \sum_{i \in I} (b_i - a_i) \le \varepsilon$

where $|U_i| = b_i - a_i$ denotes the length of the open interval U_i for $i \in I$.

EXAMPLE

1) If $S \subseteq \mathbb{R}$ is a finite set, then S has measure zero
2) If $S = \{s_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ is a countable set, then S has measure zero

   For $\varepsilon > 0$, take $U_i := \left(s_i - \frac{\varepsilon}{2^{i+1}}, s_i + \frac{\varepsilon}{2^{i+1}}\right)$. Then, $\sum_{i=1}^\infty |U_i| = \sum_{i=1}^\infty \frac{\varepsilon}{2^i} = \varepsilon$

THEOREM (LEBESGUE'S CRITERION) (Sneak peak to next main section of notes)

Let $f: [a,b] \to \mathbb{R}$ be bounded, and D be the set of its discontinuities. Then, $f \in R(x; a, b) \Leftrightarrow D$ has measure zero

(Sorry no proof today qwq my hand is hurting too much)

3-25-25 (WEEK 6) Shun/翔海 (@shun4midx)

LESBEGUE'S CRITERION

(Too tired and depressed to make my own notes for the main thm proof, sorry... I'll include all the other relevant notation and lemmas here... short notes for a reason. :( )

PROPOSITION

Let $S=(S_n)_{n\geq 1}$ be a sequence of measure zero subsets. Then, their union $S=\bigcup_{n\geq 1} S_n$ also has measure zero

Proof

Let $\varepsilon>0$. $\forall n\geq 1$, since S_n has measure zero, we may find a countable family $\{U_{n,i} \mid i\geq 1\}$ of open intervals covering U_n, and such that $\sum_{i\geq 1}|U_{n,i}|\leq\frac{\varepsilon}{2^n}$

Notice, $\mathcal{U}:=\{U_{n,i} \mid i,n\geq 1\}$ is still countable, since it is a countable union of countable families.
It also covers S, and $\sum_{n,i\geq 1}|U_{n,i}|=\sum_{n\geq 1}\sum_{i\geq 1}|U_{n,i}|=\sum_{n\geq 1}\frac{\varepsilon}{2^n}=\varepsilon$ □

DEFINITION

Let $f:[a,b]\to\mathbb{R}$ be a bounded function. For any subset $A\subseteq[a,b]$, define the oscillation of f on A to be $\Omega_f(A):=\sup\{f(x)-f(y) \mid x,y\in A\}$
For $x\in[a,b]$, define the oscillation of f at x to be $\omega_f(x):=\lim_{h\to 0^+}\Omega_f(B(x,h)\cap[a,b])$ ← The idea is to view the point as an infinitely small ball

REMARK

"$\Omega_f(A)$" has actually appeared before already in Darboux sums

Notice, $\forall A\subseteq B\subseteq[a,b]$, we have $\Omega_f(A)\leq\Omega_f(B)$. Then, the function $h\longmapsto\Omega_f(B(x,h)\cap[a,b])$ is nondecreasing and the limit as $h\to 0^+$ is always well-defined since it is also bounded.

PROPOSITION

f is continuous at $x \iff \omega_f(x)=0$

Proof

"$\Rightarrow$": By def, fix $\varepsilon>0$, then $\exists\delta>0$, s.t. $\forall y\in B(x,\delta)$, $|f(x)-f(y)|<\varepsilon$ ← Main idea
$\therefore \forall h\in(0,\delta)$, we have $\Omega_f(B(x,h)\cap[a,b])\leq 2\varepsilon$
As $\varepsilon>0$ can be arbitrarily small, thus $\omega_f(x)=0$

"$\Leftarrow$": Fix $\varepsilon>0$. By def, $\exists\delta>0$, s.t. $h<\delta\Rightarrow\Omega_f(B(x,h)\cap[a,b])<\varepsilon$ (expand the limit def of ω_f)
$\therefore \forall y\in B(x,\delta)$, $|f(x)-f(y)|<\varepsilon$, so by def, f is cont. at x. □

PROPOSITION

$\forall\varepsilon>0$, $J_\varepsilon:=\{x\in[a,b] \mid \omega_f(x)\geq\varepsilon\}$ is a closed set

Proof

Assume not. Let $\varepsilon>0$ be s.t. J_ε is not closed. Let $x\in\overline{J_\varepsilon}\setminus J_\varepsilon$, i.e. $\omega_f(x)<\varepsilon$ ← Take opposite
By def of limit, $\exists\delta>0$, s.t. $h\in(0,\delta)\Rightarrow\Omega_f(B(x,h)\cap[a,b])<\varepsilon$
$\therefore \forall y\in B(x,h)$, we also have $\omega_f(y)<\varepsilon$, i.e. $B(x,h)\cap J_\varepsilon=\emptyset$, which contradicts "$x\in\overline{J_\varepsilon}$". →←

LEMMA

Suppose $\omega_f(x)<\varepsilon$ $\forall x\in[a,b]$. Then, $\exists\delta>0$, s.t. $\forall[c,d]\subseteq[a,b]$ with $|d-c|<\delta$, we have $\Omega_f([c,d])<\varepsilon$

Proof

$\forall x\in[a,b]$, by def of limit, $\exists\delta_x>0$, s.t. $\Omega_f(B(x,\delta_x)\cap[a,b])<\varepsilon$

$\because \{B(x,\frac{\delta_x}{2}) \mid x\in[a,b]\}$ is an open covering of the compact set $[a,b]$ ← closed + bounded for over $\mathbb{R}$
$\therefore$ By Borel-Lebesgue property, we can extract a finite subcovering

Let $x_1,\dots,x_n\in[a,b]$ be s.t. $\{B(x_i,\frac{\delta_i}{2}) \mid 1\leq i\leq n\}$ covers $[a,b]$. Take $\delta=\min\{\frac{\delta_i}{2} \mid 1\leq i\leq n\}$. ← We only need to prove "$\exists$" δ anyway
For any segment $[c,d]\subseteq[a,b]$ with $|d-c|<\delta$, $\exists 1\leq i\leq n$, s.t. $[c,d]\cap B(x_i,\frac{\delta_i}{2})\neq\emptyset$ (obv, but it is a covering, so...)
$\therefore [c,d]\subseteq B(x_i,\frac{\delta_i}{2}+\delta)\cap[a,b]\subseteq B(x_i,\delta_i)\cap[a,b]$ ← strategic choice of $\delta=\min\{\frac{\delta_i}{2}\}$
$\therefore$ From our remark, $\Omega_f([c,d])\leq\Omega_f(B(x_i,\delta_i)\cap[a,b])<\varepsilon$. □

3-27-25 (WEEK 6)

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SEQUENCES AND SERIES

We take sequences in metric spaces

We take series in normed vector spaces in order to take summations

↳ Sometimes, we also need completeness, i.e. a Banach space

BASIC NOTATIONS

REMINDERS ($\mathbb{R}$-valued sequences)

DEFINITION

Let $(a_n)_{n\geq 1}$ be a real-valued sequence

We say $(a_n)_{n\geq 1}$ converges to $\ell \in \mathbb{R}$, i.e. $x_n \xrightarrow[n\to\infty]{} \ell$ if $\forall \varepsilon > 0, \exists N \geq 1$, s.t. $|x_n - \ell| \leq \varepsilon$ $\forall n \geq N$

CAUCHY'S CONDITION

In a complete vector space, to check for convergence, it is enough to check: $\forall \varepsilon > 0, \exists N \geq 1$, s.t. $\forall m, n \geq N$, $|a_m - a_n| < \varepsilon$ (no need to know the limit ℓ to compute)

PROPOSITION

(1) If $(a_n)_{n\geq 1}$ is nondecreasing and bounded above by some $M < \infty$, then $(a_n)_{n\geq 1}$ converges to a limit $\ell \leq M$

(2) If $(a_n)_{n\geq 1}$ is nonincreasing and bounded below by some $M > -\infty$, then $(a_n)_{n\geq 1}$ converges to a limit $\ell \geq M$

DEFINITION

Given two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of real numbers, we say they are adjacent if one is increasing and the other is decreasing with $a_n - b_n \xrightarrow[n\to\infty]{} 0$

PROPOSITION

If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are adjacent, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

DEFINITION

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two real sequences. Here are some asymptotic notations (CSを勉強するボクは見って、笑った)

1) We say that a is dominated by b, denoted by $a_n = O(b_n)$, if $\exists$ bounded sequence $c = (c_n)_{n\geq 1}$ and $N \in \mathbb{N}$, s.t. $a_n = c_n b_n$ $\forall n \geq N$
2) We say that a is negligible compared to b, i.e. $a_n = o(b_n)$, if $\exists$ sequence $\varepsilon = (\varepsilon_n)_{n\geq 1}$ that converges to 0 and $N \in \mathbb{N}$, s.t. $a_n = \varepsilon_n b_n$ $\forall n \in \mathbb{N}$
3) We say that a is equivalent to b, i.e. $a_n \sim b_n$ if $\exists$ sequence $c = (c_n)_{n\geq 1}$ that converges to 1 and $n \in \mathbb{N}$, s.t. $a_n = c_n b_n$ $\forall n \in \mathbb{N}$

Remark: $\sim$ is an equivalence relation in $\mathbb{R}^{\mathbb{N}}$, but

EXAMPLES

1) Define $a_n = \frac{1}{n}$, $b_n = \frac{1}{n} + \frac{1}{n^2}$ for $n \geq 1$, then $a_n = O(b_n)$ and $a_n \sim b_n$
2) Let $(a_n)_{n\geq 1} = (0, 1, 1, \ldots)$ and $(b_n)_{n\geq 1} = (1, 1, \ldots)$. Then, $a_n = O(b_n)$ and $a_n \sim b_n$
3) Let $a_n = n^2$, $b_n = 2^n$ for $n \geq 1$

DEFINITIONS

Let $(u_n)_{n\geq 0}$ be a sequence in a normed vector space $(W, \|\cdot\|)$

- Define $S_0 := 0$, $S_n := u_1 + \ldots + u_n$ for $n \geq 1$
- The series with general term u_n is the sequence $(S_n)_{n\geq 1}$, denoted as $\sum_{n\geq 1} u_n$ — same notation but different meaning
- For $n \geq 0$, S_n is called the n^{th} partial sum of $\sum u_n$
- We say that the series $\sum u_n$ converges if the sequence $(S_n)_{n\geq 0}$ converges in $(W, \|\cdot\|)$. In this case, we write $\sum_{n\geq 1} u_n$ for the limit
- In the case that $\sum_{n\geq 1} u_n$ converges, we define its n^{th} remainder by $R_n = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{n} u_k = \sum_{k=n+1}^{\infty} u_k$

REMARK
$(S_n)_{n\geq 0}$ converges $\Leftrightarrow \sum(S_{n+1}-S_n)$ converges, since $\underbrace{\sum_{n=0}^{N-1}(S_{n+1}-S_n)=S_N-S_0}_{\text{Telescoping series}}=S_N$

PROPOSITION
(1) If the series $\sum u_n$ converges, then $(S_n)_{n\geq 1}$ is a Cauchy sequence
(2) If $(W,\|\cdot\|)$ is a Banach space, then the series $\sum u_n$ converges iff $(S_n)_{n\geq 1}$ is Cauchy

Proof

CVG ⇒ Cauchy in general metric space
Cauchy ⇒ CVG in complete space

COROLLARY (CAUCHY'S CRITERION)
Suppose that $(W,\|\cdot\|)$ is a Banach space. The series $\sum u_n$ converges iff $\forall \varepsilon>0, \exists N\geq 1$, s.t. $\forall n\geq N, \forall k\geq 1$, $\underbrace{\|u_{n+1}+\dots+u_{n+k}\|<\varepsilon}_{\text{Use norms}}$

Proof

For $n\geq 1, k\geq 1$, $S_{n+k}-S_n=u_{n+1}+\dots+u_{n+k}$. Then, by the above proposition, QED.

COROLLARY
If $\sum_{n\geq 1} u_n$ is a convergent series, then $\lim_{n\to\infty} u_n=0$

Proof

It is a satisfaction of the fact that $(S_n)_{n\geq 0}$ is a Cauchy sequence. □

REMARK
The converse does not hold, $\sum\frac{1}{n}=\infty$

DEFINITION
Suppose that $(W,\|\cdot\|)$ is a Banach space, and let $\sum u_n$ be a series with general terms in W
- If the series $\sum\|u_n\|$ converges, we say that the series $\sum u_n$ converges absolutely (w/o norm)
- If the series $\sum u_n$ converges but not absolutely, then we say $\sum u_n$ converges conditionally

EXAMPLE
$\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}=\ln 2$ is convergent but not absolutely convergent

THEOREM
For a Banach space $(W,\|\cdot\|)$, if $\sum u_n$ converges absolutely, then $\sum u_n$ converges

Proof

$\forall n,k\geq 1$, we have $\|u_{n+1}+\dots+u_{n+k}\|\leq\|u_{n+1}\|+\dots+\|u_{n+k}\|$
$\therefore$ Cauchy's condition for $\sum\|u_n\|$ ⇒ Cauchy's condition for $\sum u_n$ (Shows how useful Cauchy's criterion is)

APPLICATIONS
Useful in metric spaces like vector spaces of matrices or function spaces, we only need to examine numbers due to the norm.

SERIES WITH NONNEGATIVE TERMS
COMPARISON BETWEEN SERIES

PROPOSITION
Let $\sum u_n$ be a series with nonnegative terms, then $\sum_{n\geq 1} u_n$ converges $\Leftrightarrow$ $\underbrace{(S_n)_{n\geq 0}}_{\text{sequence of partial sums}}$ is bounded from above

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PROPOSITION (COMPARISON TEST)

We consider two nonnegative series $\sum u_n$ and $\sum v_n$ satisfying $\forall n \geqslant 1$, $0 \leqslant u_n \leqslant v_n$

(1) If $\sum v_n$ converges, then $\sum u_n$ converges

(2) If $\sum u_n$ diverges, then $\sum v_n$ diverges

Proof

Let $(S_n)_{n\geqslant 0}$ be the partial sums of $\sum u_n$ and $(T_n)_{n\geqslant 0}$ be the partial sums of $\sum v_n$. Then, $\forall n \geqslant 0$, $S_n \leqslant T_n$. Conclude by prop above. □

THEOREM

Let $\sum u_n$ and $\sum v_n$ be series with nonnegative terms

(1) If $v_n = O(u_n)$, and $\sum u_n$ converges, then $\sum v_n$ converges

(2) If $u_n \sim v_n$, then $\sum u_n$ and $\sum v_n$ either both converge or both diverge

Proof

(2) is a direct consequence of (1), $\because v(n) = O(u_n)$ and $u_n = O(v_n) \Leftrightarrow u_n \sim v_n$

$\therefore$ It suffices to prove (1)

Suppose $v_n = O(u_n)$

Let $M > 0$ and $N \geqslant 1$, s.t. $v_n \leqslant M u_n$ $\forall n \geqslant N$.

Then, $\forall n \geqslant N$, $\sum_{k=1}^{n} v_k = \sum_{v=1}^{N-1} v_k + \sum_{k=N}^{n} v_k \leqslant \sum_{k=1}^{N-1} v_k + M \sum_{k=N}^{n} v_k$

Since $\sum u_n$ converges and $(\sum_{k=N}^{n} u_k)_{n\geqslant N}$ is bounded from above $\therefore \sum v_n$ converges □

REMARK

Define $u_n = \frac{(-1)^n}{n}$ and $v_n = \frac{1}{n}$, $n \geqslant 1$. It is clear that $u_n = O(v_n)$, but $\sum u_n$ converges and $\sum v_n$ diverges $\therefore$ "non-negative" is a really important

Same with $u_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}$ and $v_n = \frac{(-1)^n}{\sqrt{n}}$, with $u_n \sim v_n$.

EXAMPLE

Let's study the behavior of $\sum \frac{1}{n^2}$.

For $k \geqslant 2$, $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \leqslant \frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$

$\therefore \sum_{k=n}^{N} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{n-1} - \frac{1}{N} \xrightarrow[N\to\infty]{} \frac{1}{n-1}$ $\forall n \geqslant 2$

$\therefore \sum \frac{1}{n^2}$ converges

Moreover, $\sum_{k\geqslant 1} \frac{1}{k^2} \geqslant 1 + \sum_{k\geqslant 2} \frac{1}{k^2} \geqslant 1 + \overbrace{1}^{\frac{1}{2-1}} = 2$

Consider $R_n := \sum_{k\geqslant n+1} \frac{1}{k^2}$. We know that $\sum_{k\geqslant n+1} \left(\frac{1}{k} - \frac{1}{k-1}\right) = \frac{1}{n+1} \leqslant R_n \leqslant \sum_{k\geqslant n+1} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{n}$

$\therefore R_n \sim \frac{1}{n}$ as $n \to +\infty$

We can try and consider the denominator

PROPOSITION (RIEMANN SERIES)

Let $\alpha \in \mathbb{R}$. The Riemann series $\sum \frac{1}{n^\alpha}$. We note $\sum \frac{1}{n^\alpha}$ converges $\Leftrightarrow \alpha > 1$

Proof

For $\alpha > \beta$, $n \geqslant 1$, define $\frac{1}{\alpha^n} \geqslant \frac{1}{\beta^n}$

- $\alpha = 1$: $\sum \frac{1}{n}$ is divergent, so $\forall \alpha < 1$, $\sum \frac{1}{n^\alpha}$ is divergent

  To check divergence, see $\frac{1}{k} = \int_k^{k+1} \frac{dt}{k} \geqslant \int_k^{k+1} \frac{dt}{t} = \ln(k+1) - \ln(k)$ $\forall k \geqslant 1$

  $\therefore \forall n \geqslant 1$, $\sum_{k=1}^{n} \frac{1}{k} \geqslant \sum_{k=1}^{n} (\ln(k+1) - \ln(k)) = \ln(n+1) \xrightarrow{n\to\infty} +\infty$

  $\therefore \sum \frac{1}{n}$ diverges

- $\alpha > 1$. For $k \geqslant 2$, we have $\frac{1}{k^\alpha} \leqslant \int_{k-1}^{k} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\left(\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}}\right)$

  Moreover, $\sum_{k\geqslant 2} \frac{1}{1-\alpha}\left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}}\right]$ is convergent, so $\sum_{k\geqslant 2} \frac{1}{k^\alpha}$ is convergent. More specifically, $\sum_{k\geqslant 2} \frac{1}{k^\alpha} \leqslant \sum_{k\geqslant 2} \frac{1}{1-\alpha}\left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}}\right] = \frac{1}{\alpha-1}$

REMARK (Studying the remainder)

integral trick

The integral trick again from the previous part

Let us study the remainder $\sum_{k\geqslant n} \frac{1}{k^\alpha} = \frac{1}{n^\alpha} + \sum_{k\geqslant n+1} \frac{1}{k^\alpha} \leqslant \frac{1}{n^\alpha} + \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}$

Similarly, $\forall n \geqslant 2$, $\frac{1}{k^\alpha} \geqslant \int_k^{k+1} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\left[\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{k^\alpha}\right] \Rightarrow \sum_{k\geqslant n} \frac{1}{k^\alpha} \geqslant \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}$

$\therefore R_n \sim \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}$

4-1-25 (WEEK 7)

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PARTIAL SUMS AND REMAINDERS

(Sorry for the bad handwriting... once again, rain got into my stylus and I forgot to zip my bag in heavy rain, so...)

THEOREM

Let $\sum_n u_n$ and $\sum_n v_n$ be two series with nonnegative terms

Suppose that $u_n \sim v_n$. Then,

1) If $\sum u_n$ converges, then $\sum v_n$ converges, and $\sum_{k=n+1}^{\infty} u_k \sim \sum_{k=n+1}^{\infty} v_k$ (remainders, start from n+1 to ∞)
2) If $\sum u_n$ diverges, then $\sum v_n$ diverges, and $\sum_{k=1}^{n} u_k \sim \sum_{k=1}^{n} v_k$ (partial sums, start from 1 to n)

Proof

We have seen that $\sum u_n$ and $\sum v_n$ have the same behavior

1) Let $\varepsilon>0$, take $N>0$, s.t. $(1-\varepsilon)u_n \le v_n \le (1+\varepsilon)u_n \ \forall n\ge N$

   $\forall M \ge n \ge N$, we have $\sum_{k=n+1}^{M} v_k \le \sum_{k=n+1}^{n} (1+\varepsilon)u_k \le (1+\varepsilon)\sum_{k=n+1}^{\infty} u_k \Rightarrow \sum_{k=n+1}^{\infty} v_k \le (1+\varepsilon)\sum_{k=n+1}^{\infty} u_k$

   Similarly, we also have $\sum_{k=n+1}^{\infty} v_k \ge (1-\varepsilon)\sum_{k=n+1}^{\infty} u_k$, which gets us $\sum_{k=n+1}^{\infty} v_k \sim \sum_{k=n+1}^{\infty} u_k$

2) Let $\varepsilon>0$, take $N>0$, s.t. $(1-\varepsilon)u_n \le v_n \le (1+\varepsilon)u_n \ \forall n \ge N$.

   Write $\sum_{k=1}^{n} v_k = \sum_{k=1}^{N-1} v_k + \sum_{k=N}^{n} v_k$ for $n \ge N$.

   We want to show "$(1-2\varepsilon)\sum_{k=1}^{n} u_k \overset{(b)}{\le} \sum_{k=1}^{n} v_k \overset{(a)}{\le} (1+2\varepsilon)\sum_{k=1}^{n} u_k$" for large enough n.

   "(a)": $\sum_{k=1}^{n} v_k \le \sum_{k=1}^{N-1} v_k + (1+\varepsilon)\sum_{k=N}^{n} v_k$

   $\therefore \dfrac{\sum_{k=1}^{N} v_k \text{ (const)}}{\sum_{k=1}^{n} u_k \text{ (div series)}} \xrightarrow[n\to\infty]{} 0$, thus $\exists N' \ge N$, s.t. $\sum_{k=1}^{N} v_k \le \varepsilon \sum_{k=1}^{n} u_k$

   Therefore, $\sum_{k=1}^{n} v_k \le \varepsilon \sum_{k=1}^{n} u_k + (1+\varepsilon)\sum_{k=1}^{n} u_k = (1+2\varepsilon)\sum_{k=1}^{n} u_k$ for $n \ge N'$

   "(b)": For $n\ge N$, we have $\sum_{k=1}^{n} v_k \ge \sum_{k=1}^{N-1} v_k + (1-\varepsilon)\sum_{k=N}^{n} u_k$

   Now, take large $N' \ge N$, s.t. $\varepsilon \sum_{k=N}^{n} u_k \ge (1-2\varepsilon)\sum_{k=1}^{N-1} u_k - \sum_{k=1}^{N-1} v_k \ \forall n \ge N$

   Then, we find $\sum_{k=1}^{n} v_k \ge (1-2\varepsilon)\sum_{k=1}^{n} u_k$ □

   (Main idea: It diverges, so abuse the fact it's very large)

EXAMPLE (Algorithmic approach of analyzing remainders)

Let's study the asymptotic behavior of the harmonic series

Define $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \ \forall n \ge 1$

1) Note that $\ln(1+x) \sim \infty$ when $x \to \infty$. We have $\frac{1}{n} \sim \ln(1+\frac{1}{n}) = \ln(n+1) - \ln(n)$

   The series $\sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)]$ diverges, so $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

   Moreover, we know that $H_n \sim \sum_{k=1}^{n} (\ln(k+1) - \ln(k)) = \ln(n+1) \sim \ln(n)$

   $\therefore \frac{H_n}{\ln n} \to 1$ as $n \to \infty$, i.e. $H_n = \ln n + o(\ln n)$

2) Let's understand the term $o(\ln n)$.

   Let $A_n := H_n - \ln n$ for $n \ge 1$

   $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots$

   For $n \ge 2$, $A_n - A_{n-1} = H_n - H_{n-1} - \ln n + \ln(n-1) = \frac{1}{n} + \ln(1-\frac{1}{n}) = \frac{1}{n} + (-\frac{1}{n} - \frac{1}{2n^2} + o(\frac{1}{n^2})) = -\frac{1}{2n^2} + o(\frac{1}{n^2})$

   $\Rightarrow A_n - A_{n-1} \sim -\frac{1}{2n^2}$

   By the thm above, hence $\sum (A_n - A_{n-1})$ converges, i.e. $(A_n)_{n\ge1}$ converges □

   Now, let $\gamma := \lim_{n\to\infty} A_n$, and call it Euler's constant

   Then, we get $A_n = H_n - \ln n = \gamma + o(1) \Rightarrow H_n = \ln n + \gamma + o(1)$

3) Compare the partial sums for $\sum_{k=n+1}^{\infty} (A_k - A_{k+1}) = \gamma - A_n$ and $\sum_{k=n+1}^{\infty} \frac{1}{k^2} \sim \frac{1}{n}$ by Riemann series.

   We get: $\gamma - A_n = \gamma - H_n + \ln n \sim -\frac{1}{2n} \Rightarrow \gamma - H_n + \ln n = -\frac{1}{2n} + o(\frac{1}{n}) \Rightarrow H_n = \ln n + \gamma + \frac{1}{2n} + o(\frac{1}{n})$

4) Let $D_n = H_n - \ln n - \gamma - \frac{1}{2n}$ for $n \ge 1$.

   $D_n - D_{n-1} = H_n - H_{n-1} - \ln n + \ln(n-1) - \frac{1}{2n} + \frac{1}{2(n-1)}$

   $= \frac{1}{n} + \ln(1-\frac{1}{n}) + \frac{1}{2n} - \frac{1}{2n}\left(\frac{1}{1-\frac{1}{n}}\right)$ $\quad \frac{1}{1-x} = 1 + x + x^2 + \dots$

   $= \frac{1}{n} - (\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + o(\frac{1}{n^3})) - \frac{1}{2n} + \frac{1}{2n}(1 + \frac{1}{n} + \frac{1}{n^2} + o(\frac{1}{n^2})) = \frac{1}{6n^3} + o(\frac{1}{n^3})$

   Again, by Riemann series, $\sum \frac{1}{n^3}$ converges $\Rightarrow \sum (D_n - D_{n-1})$ converges, moreover, $\sum_{k=n+1}^{\infty} (D_k - D_{k-1}) \sim \sum_{k=n+1}^{\infty} \frac{1}{k^3} \sim \frac{1}{2n^2} \Rightarrow D_n \sim \frac{1}{2n^2} \Rightarrow H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + o(\frac{1}{n^2})$

5) We have the following expression: $H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{N} \frac{B_{2k}}{2k n^{2k}} + o\left(\frac{1}{n^{2N}}\right)$ when $n \to \infty$, where $(B_{2k})_{k\geq 1}$ are Bernoulli numbers

REMARK

Using this similar approach, we can derive the infamous Stirling's formula ~~(i.e. every CS student's nightmare when learning about asymptotic notation :clown:)~~

COMPARISON BETWEEN SERIES AND INTEGRALS

PROPOSITION

Let $f: [1, +\infty) \to \mathbb{R}_+$ be a nondecreasing function with $\lim_{x\to\infty} f(x) = 0$. $\forall n \geq 1$, define $S_n = \sum_{k=1}^{n} f(k)$, $I_n = \int_1^n f(t)\,dt$, $D_n = S_n - I_n$

Then, the following properties hold:

(1) For $n \geq 1$, we have $0 \leq f(n+1) \leq D_{n+1} \leq D_n \leq f(1)$

(2) The sequence $(D_n)_{n\geq 1}$ converges, and denote $D := \lim_{n\to\infty} D_n$

(3) The series $\sum f(n)$ and the integral $\int_1^\infty f(t)\,dt := \lim_{x\to\infty} \int_1^x f(t)\,dt$ have the same behavior (both converge or both diverge)

(4) $\forall n \geq 1$, we have $0 \leq D_n - D \leq f(n)$

Proof

1) For $k \geq 1$, we have $f(k+1) \leq \int_k^{k+1} f(t)\,dt \leq f(k)$

Then, $I_{n+1} = \sum_{k=1}^{n} \int_k^{k+1} f(t)\,dt \leq \sum_{k=1}^{n} f(k) = S_n \Rightarrow f(n+1) = S_{n+1} - S_n \leq S_{n+1} - I_{n+1} = D_{n+1}$

$D_{n+1} - D_n = (S_{n+1} - S_n) - (I_{n+1} - I_n) = f(n+1) - \int_n^{n+1} f(t)\,dt \leq 0$ $\therefore (D_n)_{n\geq 1}$ is decreasing

$\therefore$ $D_{n+1} \leq D_n \leq \ldots \leq D_1 = S_1 - I_1 = f(1)$ ✓

2) We know $(D_n)_{n\geq 1}$ is decreasing and bounded from below by 0, hence it converges

3) We know $D = \lim_{n\to\infty} D_n = \lim_{n\to\infty} (S_n - I_n)$ converges

If $\lim_{n\to\infty} S_n$ exists, then $\lim_{n\to\infty} I_n = D - \lim_{n\to\infty} S_n$ exists. Same argument for "$\lim_{n\to\infty} I_n$ exists".

4) We know $D_n - D = \sum_{k\geq n} (D_k - D_{k+1}) \geq 0$. We have $D_k - D_{k+1} = \int_k^{k+1} f(t)\,dt - f(k+1) \leq f(k) - f(k+1)$ $\therefore \sum_{k\geq n} (D_k - D_{k+1}) \leq \sum_{k\geq n} (f(k) - f(k+1)) \Rightarrow D_n - D \leq f(n)$ □

REMARK

From (4), we find $0 \leq \sum_{k=1}^{n} f(k) - \int_1^n f(t)\,dt - D \leq f(n) \Rightarrow \sum_{k=1}^{n} f(k) = \int_1^n f(t)\,dt + D + O(f(n))$

If we take $f(x) = \frac{1}{x}$, we find $H_n = \int_1^n \frac{dx}{x} + D + O\left(\frac{1}{n}\right) = \ln n + D + O\left(\frac{1}{n}\right)$ $\left(O\left(\frac{1}{n}\right) < o(1)\right)$

EXAMPLE (Riemann-Zeta function)

Let $s \in \mathbb{R}$, $f(x) = x^{-s}$ $\forall x \geq 1$

Consider the series $\sum_{n\geq s} n^{-s} = \begin{cases} \text{convergent}, & s > 1 \\ \text{divergent}, & s \leq 1 \end{cases}$

We define the Riemann-Zeta function, i.e. $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$

Similarly, we can deduce $\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{s-1}\left(\frac{1}{n^{s-1}} - 1\right) + D(s) + O\left(\frac{1}{n^s}\right) = C(s) + \frac{1}{s-1} \frac{1}{n^{s-1}} + O\left(\frac{1}{n^s}\right)$

Since $\sum_{k=1}^{n} \frac{1}{k^s} \xrightarrow[n\to\infty]{} \zeta(s) = C(s)$, hence $\sum_{k=1}^{\infty} \frac{1}{k^s} = \zeta(s) = \sum_{k\leq n} \frac{1}{n^s} = \frac{1}{s-1} \frac{1}{n^{s-1}} + O\left(\frac{1}{n^s}\right)$

PROPOSITION (BERTAND'S SERIES)

For $\alpha, \beta \in \mathbb{R}$, consider the series $\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$

1) When $\alpha > 1$, the series converges

2) When $\alpha = 1, \beta > 1$, the series converges

3) Otherwise, it diverges

Proof

1) Let $\alpha > 1$ and $\beta \in \mathbb{R}$.

Notice, $\frac{1}{n^\alpha (\ln n)^\beta} = o\left(\frac{1}{n^{\frac{1+\alpha}{2}}}\right)$ $\left[\frac{1}{n^\alpha (\ln n)^\beta} \div \frac{1}{n^{\frac{1+\alpha}{2}}} = \frac{1}{n^{\frac{\alpha-1}{2}} (\ln \beta)} \to 0\right]$

Since $\sum \frac{1}{n^{\frac{1+\alpha}{2}}}$ converges $\therefore \sum \frac{1}{n^\alpha (\ln n)^\beta}$ also converges

2) Let $\alpha = 1, \beta > 1$, $f(x) = \frac{1}{x (\ln x)^\beta}$, $x \geq 2$.

$$\int_2^n f(x)\,dx = \int_2^n \frac{1}{x (\ln x)^\beta}\,dx = \int_{\ln 2}^{\ln n} \frac{1}{y^\beta}\,d\beta \text{ is convergent}$$

$\therefore \sum f(x)$ converges

3) Finally, when $\alpha=\beta=1$, $\int_2^n \frac{dx}{x\ln x} = \int_{\ln 2}^{\ln n} \frac{1}{y}\,dy$ is divergent

Now, for $\alpha=1$, $\beta<1$, we have $\frac{1}{x\ln x} \leqslant \frac{1}{x(\ln x)^\beta} \Rightarrow \sum \frac{1}{x(\ln x)^\beta}$ diverges

When $\alpha<1$, thus $\frac{1}{n^{(1+\alpha)/2}} = o\left(\frac{1}{n^\alpha(\ln n)^\beta}\right)$, since $\sum \frac{1}{n^{(1+\alpha)/2}}$ diverges, thus $\sum \frac{1}{n^\alpha(\ln n)^\beta}$ diverges too. $\square$