

RIEMANN-STIELTJES INTEGRALS

FUNCTIONS OF BOUNDED VARIATION

DEFINITION

Let $I \subseteq \mathbb{R}$ be an interval, $f: I \rightarrow \mathbb{R}$ be a function.

(1) f is non-increasing/decreasing if $f(x) \geq f(y) \forall x \leq y, x, y \in I$

(2) f is non-decreasing/increasing if $f(x) \leq f(y) \forall x \leq y, x, y \in I$

(3) f is monotonic if (1) or (2) holds

DEFINITION

Let $f: I \rightarrow \mathbb{R}$ be monotonic

For $x \in I$, define:

↳ The left limit at x to be $f(x-) = \lim_{y \nearrow x} f(y)$ if $(x-\varepsilon, x) \cap I \neq \emptyset$ for $\varepsilon > 0$

↳ The right limit at x to be $f(x+) = \lim_{y \searrow x} f(y)$ if $(x, x+\varepsilon) \cap I \neq \emptyset$ for $\varepsilon > 0$

— E.g. we can't just pick a point at the boundary

PROPOSITION

Let $f: [a, b] \rightarrow \mathbb{R}$ be a monotonic function. Then, the set of its discontinuities D is a countable set.

Proof

— monotonic \Rightarrow left/right limits are well-defined

Define $D = \{x \in I \mid f(x-) \neq f(x+)\}$

By symmetry, WLOG, assume f is increasing, then $f(x-) \leq f(x+) \forall x \in D$

— Key! This is since $x \in D$, i.e. it is discontinuous

As \mathbb{Q} is dense in \mathbb{R} , we know $\exists q_x \in \mathbb{Q} \cap (f(x-), f(x+))$

\therefore This gives us a map $\{x \mapsto q_x\}$, which is injective because $\forall x_1, x_2 \in D, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$\therefore D$ can be injected in the countable set \mathbb{Q}

$\therefore D$ is countable. \square

DEFINITION (PARTITIONS)

Let $a < b$ and $[a, b] \in \mathbb{R}$ be a segment.

A partition or a subdivision of $[a, b]$ is a finite sequence $P = (x_k)_{0 \leq k \leq n}$ satisfying $a = x_0 < x_1 < \dots < x_n = b$, where n is the length of P

We denote $\text{Supp}(P) := \{x_k \mid 0 \leq k \leq n\}$ as the support of P

For a finite subset $A \subseteq [a, b]$ with $a, b \in A$, we may find a partition P of $[a, b]$ s.t. $\text{Supp}(P) = A$. This is called the partition corresponding to A .

We say $[x_{k-1}, x_k]$ is the k^{th} subinterval of P , $\Delta x_k := x_k - x_{k-1}$, $1 \leq k \leq n$. Then, we say the mesh size of P is $\|P\| := \max_{1 \leq k \leq n} \Delta x_k$

— This is not a norm!

Let P, P' be partitions. If $\text{Supp}(P) \subseteq \text{Supp}(P')$, then we say P' is finer than P , and we say $P \subseteq P'$. This also implies $\|P\| \leq \|P'\|$

Let P_1, P_2 be partitions. Define their joint partition or smallest common refinement to be $P := P_1 \vee P_2$, which is the partition P with $\text{support} = \text{Supp}(P_1) \cup \text{Supp}(P_2)$

We denote $\mathcal{P}([a, b])$ as the collection of all possible partitions of $[a, b]$

REMARK

For any $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a, b])$, we have $b - a = \sum_{k=1}^n \Delta x_k$

DEFINITION (BOUNDED VARIATIONS)

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Let $f: [a, b] \rightarrow \mathbb{R}$ be a function, $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a, b])$, define $\Delta f_k := f(x_k) - f(x_{k-1})$ for $1 \leq k \leq n$.

Define $V_P(f) := \sum_{k=1}^n |\Delta f_k|$ and $V_f := V_f([a, b]) = \sup_{P \in \mathcal{P}([a, b])} V_P(f) \in [0, \infty]$ to be the total variation of f . We say that f is of bounded variation if $V_P < +\infty$.

We write $BV([a, b]) = BV([a, b], \mathbb{R})$ for the collection of such functions defined on $[a, b]$.

EXAMPLE

Consider the function $f: [0, 1] \rightarrow \mathbb{R}$, $f(x) = \begin{cases} x \cos(\frac{1}{x}) & x \in (0, 2\pi] \\ 0 & x = 0 \end{cases}$

For $n \geq 1$, consider the partition P with support $\{0, \frac{1}{2n}, \frac{1}{2n-1}, \dots, \frac{1}{2}, 1\}$, i.e. $x_0 = 0, x_k = \frac{1}{2n+1-k} \forall 1 \leq k \leq 2n$

By completeness, we find $V_P(f) = \sum_{k=1}^{2n} |\Delta f_k| = \left| \frac{(-1)^{2n}}{2n} - 0 \right| + \sum_{k=2}^{2n} \left| \frac{(-1)^{k-1}}{2n+1-k} - \frac{(-1)^k}{2n+2-k} \right| = \frac{1}{2n} + \sum_{k=2}^{2n} \left(\frac{1}{2n+1-k} + \frac{1}{2n+2-k} \right) = 1 + \sum_{k=2}^{2n} \frac{2}{k} + \frac{1}{2n}$, which is not bounded for $n \geq 1$. So f is not of bounded variation.

PROPOSITION

Let $f \in BV([a, b], \mathbb{R})$, then

(1) For any partitions $P \leq P'$, we have $V_P(f) \leq V_{P'}(f)$

(2) $\forall \varepsilon > 0, \exists$ partition $P_\varepsilon \in \mathcal{P}([a, b])$ s.t. \forall finer partition $P \geq P_\varepsilon$, we have $V_P(f) \leq V_{P_\varepsilon}(f) + \varepsilon$

Proof

(1) By induction, we only need to prove this is true whenever $|\text{Supp}(P')| = |\text{Supp}(P)| + 1$

Let $P, P' \in \mathcal{P}([a, b])$ be partitions with support s.t. $\text{Supp}(P') = \text{Supp}(P) \cup \{c\}$, $x_{k-1} < c < x_k$ for some $1 \leq k \leq n$.

$$\begin{aligned} \text{Then, } V_{P'}(f) &= \sum_{k=1, k \neq i}^n |f(x_k) - f(x_{k-1})| + |f(c) - f(x_{i-1})| + |f(x_i) - f(c)| \\ &\geq \sum_{k=1, k \neq i}^n |f(x_k) - f(x_{k-1})| + |f(x_i) - f(x_{i-1})| \\ &= \sum_{k=1}^n |f(x_k) - f(x_{k-1})| = V_P(f) \quad \checkmark \end{aligned}$$

\therefore By induction, the statement holds. \square

(2) Let $\varepsilon > 0$, by the characterization of supremum, we can find $P_\varepsilon \in \mathcal{P}([a, b])$ s.t. $V_f \leq V_{P_\varepsilon}(f) + \varepsilon$ "directly follows from $V_f = \sup_{P \in \mathcal{P}([a, b])} V_P(f)$ ".

$\therefore \forall$ finer partitions $P \geq P_\varepsilon$, from (1), $V_f \leq V_{P_\varepsilon}(f) + \varepsilon \leq V_P(f) + \varepsilon \quad \square$

PROPOSITION

If $f: [a, b] \rightarrow \mathbb{R}$ is monotonic, then $f \in BV([a, b])$ and $V_f = |f(b) - f(a)|$

Proof

WLOG, assume that f is increasing.

Then, $\forall P \in \mathcal{P}([a, b])$, $V_P(f) = \sum_{k=1}^n |\Delta f_k| = \sum_{k=1}^n \Delta f_k = f(b) - f(a)$, which is independent of P

$\therefore f \in BV([a, b])$ and $V_f = |f(b) - f(a)| \quad \square$

PROPOSITION

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and differentiable on (a, b) with bounded derivative, then $f \in BV([a, b])$

Proof

Let $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a, b])$ be a partition, then $V_P(f) = \sum_{k=1}^n |\Delta f_k| \stackrel{\text{MVT}}{=} \sum_{k=1}^n |f'(t_k)| \Delta x_k \leq \sup_{t \in [a, b]} |f'(t)| \sum_{k=1}^n \Delta x_k = \sup_{t \in [a, b]} |f'(t)| (b-a) \quad \square$