Analysis II Theorems

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Statements

Notice: I have briefly mentioned this in my README.md document, but by "Theorems", I refer to not just Theorems, but also Lemmas, Propositions, Corollaries and other things of the sort.

2-18-25 (Week 1): Riemann-Stieltjes Integrals (Functions of Bounded Variation)

Proposition 1.1. Let $f : [a, b] \to \mathbb{R}$ be a **monotonic function**. Then, the set of its **discontinuities** D is a **countable** set.

Remark 1.1. For any $P = (x_k)_{0 \leq k \leq n} \in \mathcal{P}([a, b])$, we have $b - a = \sum_{k=1}^{n} \Delta x_k$.

Proposition 1.2. Let $f \in \mathcal{BV}([a, b], \mathbb{R})$, then

(1) For any partitions $P \subseteq P'$, we have $\underline{V_P(f) \leq V_{P'}(f)}$

(2) $\forall\, \varepsilon > 0$, $\exists$ partition $P_\varepsilon \in \mathcal{P}([a, b])$, s.t. $\forall$ **finer partition** $P \supseteq P_\varepsilon$, we have $\underline{V_P(f) \leq V_f \leq V_P(f) + \varepsilon}$

Proposition 1.3. If $f : [a, b] \to \mathbb{R}$ is **monotonic**, then $\underline{f \in \mathcal{BV}([a, b])}$ and $\underline{V_f = |f(b) - f(a)|}$

Proposition 1.4. If $f : [a, b] \to \mathbb{R}$ is **continuous** and **differentiable** on (a, b) with **bounded derivative**, then $\underline{f \in \mathcal{BV}([a, b])}$.

Statements and Proof Outlines

Notice: I would give enough detail but not write my proofs formally here. They are proof **outlines** so it is easier to recall **for me**, not actual proofs nor in complete English. It may look messy to you, that's understandable. I have a certain chronic eye condition and am dyslexic, so I need such "messiness" to understand what I write. When I can't color-code stuff like in my notes, this is the best alternative.

2-18-25 (Week 1): Riemann-Stieltjes Integrals (Functions of Bounded Variation)

Proposition 1.1. Let $f : [a, b] \to \mathbb{R}$ be a **monotonic function**. Then, the set of its **discontinuities** D is a **countable** set.

Proof. $D := \{x \in I \mid f(x-) \neq f(x+)\}$. WLOG, assume f incr $\Rightarrow$ $f(x-) \lneq f(x+)\ \forall x \in D$. $\mathbb{Q}$ dense in $\mathbb{R}$ $\Rightarrow$ $\exists\ q_x \in \mathbb{Q} \cap (f(x-), f(x+))$. $\therefore \exists$ map $D \to \mathbb{Q}$, $x \mapsto q_x$, **inj** cuz $\forall x_1, x_2 \in D$, $\underline{x_1 < x_2 \Rightarrow f(x_1) < f(x_2)}$. $\because D$ **inj** in **countable** $\mathbb{Q}$ $\therefore D$ is **countable**. □

Proposition 1.2. Let $f \in \mathcal{BV}([a, b], \mathbb{R})$, then

(1) For any partitions $P \subseteq P'$, we have $\underline{V_P(f) \leq V_{P'}(f)}$

(2) $\forall\ \varepsilon > 0$, $\exists$ partition $P_\varepsilon \in \mathcal{P}([a, b])$, s.t. $\forall$ **finer partition** $P \supseteq P_\varepsilon$, we have $\underline{V_P(f) \leq V_f \leq V_P(f) + \varepsilon}$

Proof.

(1) By MI, only need to prove when $|\text{Supp}(P')| = |\text{Supp}(P)| + 1$. Let P, P, k s.t. $\text{Supp}(P') = \text{Supp}(P) \cup \{c\}$, $x_{k-1} < c < x_k$. Then, by $\triangle$ ineq, $V_{P'}(f) = \sum_{k=1, k\neq i}^n |f(x_k) - f(x_{k-1})| + |f(c) - f(x_{i-1})| + |f(x_i) - f(c)| \geq \sum_{k=1}^n |f(x_k) - f(x_{k-1})| = V_P(f)$. Now by MI, done.

(2) By def, $V_f = \sup_{P\in\mathcal{P}([a,b])} V_P(f) \Rightarrow \forall\ \varepsilon > 0,\ \exists\ P_\varepsilon \in \mathcal{P}([a, b])$, s.t. $V_f \leq V_{P_\varepsilon}(f) + \varepsilon$. $\therefore \forall P \supseteq P_\varepsilon$, by (1), $\underline{V_f \leq V_{P_\varepsilon}(f) + \varepsilon \leq V_P(f) + \varepsilon}$ □

Proposition 1.3. If $f : [a, b] \to \mathbb{R}$ is **monotonic**, then $\underline{f \in \mathcal{BV}([a, b])}$ and $\underline{V_f = |f(b) - f(a)|}$

Proof. WLOG, assume f incr $\Rightarrow \forall\ P \in \mathcal{P}([a, b]),\ V_P(f) = \sum_{k=1}^n |\Delta f_k| = \sum_{k=1}^n \Delta f_k = f(b) - f(a)$, which is **indep** of P. $\therefore f \in \mathcal{BV}([a, b]),\ V_f = |f(b) - f(a)|$ □

Proposition 1.4. If $f : [a, b] \to \mathbb{R}$ is **continuous** and **differentiable** on (a, b) with **bounded derivative**, then $\underline{f \in \mathcal{BV}([a, b])}$.

Proof. Let $P = (x_k)_{0\leq k\leq n}$, then by MVT,
$V_P(f) = \sum_{k=1}^n |\Delta f_k| = \sum_{k=1}^n |f'(t_k)|\Delta x_k \leq \sup_{t\in[a,b]} |f'(t)| \sum_{k=1}^n \Delta x_k = \sup_{t\in[a,b]} |f'(t)||(b - a)$ □