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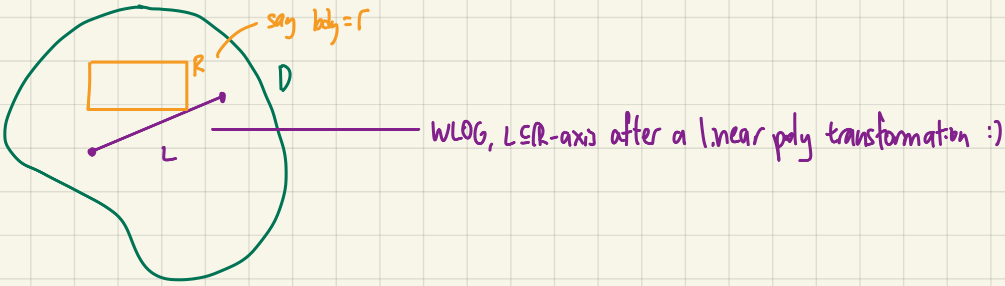
REMARK

$f(z) := \frac{1}{z}$ is a counterexample as to why we cannot say "$\forall f$: ana exc on finitely many pts in a region $\Rightarrow f$: ana"

THEOREM

f is continuous on an open set $D \subseteq \mathbb{C}$ and analytic except on a line segment in D. Then, f is analytic throughout D.

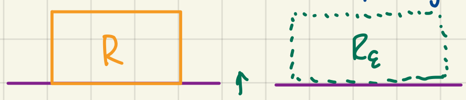


Proof

We know $f|_{D \setminus L}$: ana. Consider the following cases.

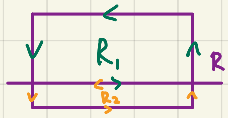
① $R \cap L = \emptyset \Rightarrow \int_\Gamma f\,dz = 0$ as f: ana on $D \setminus L$

② $R \cap L \subseteq \Gamma$, $R \cap L \neq \emptyset$: Lift one side, we get a rectangle $R_\varepsilon \subseteq R$, $R_\varepsilon \cap L = \emptyset$



By case ①, as f is cont., $\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} f(z)\,dz = 0 \Rightarrow \int_\Gamma f(z)\,dz = 0$

③ $R \cap L \not\subseteq \Gamma$



Then, $R = R_1 \cup R_2$, $\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f$

By cases ① and ②, hence $\int_\Gamma f(z)\,dz = 0$

$\therefore$ By Morera's Thm, f is analytic on D $\square$

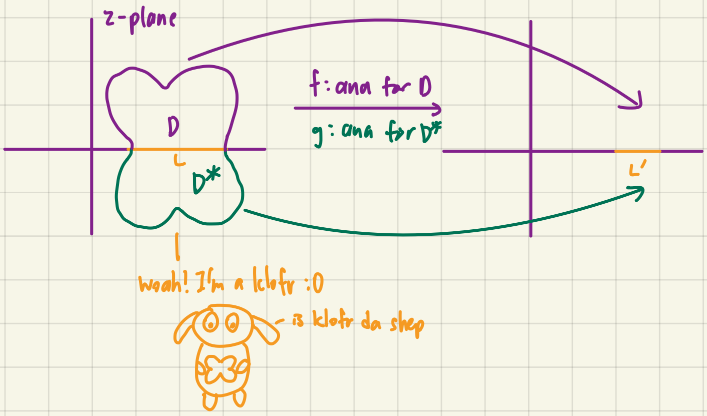
THEOREM (SCHWARZ REFLECTION PRINCIPLE)

Suppose f is $\mathbb{C}$-analytic in a region D that is contained in either the upper or lower half plane and whose boundary contains a segment L on the real axis, and suppose f is real for real z.

Then, we can define an analytic "extension" g of f to the region $D \cup L \cup D^*$ that is symmetric with respect to the real axis by

$$g(z) = \begin{cases} f(z), & z \in D \cup L \\ \overline{f(\bar{z})}, & z \in D^* \end{cases} \quad \text{where } D^* = \{z \mid \bar{z} \in D\}$$

Graphically, we represent it as follows:



Proof

① $z \in D$, then $f|_D = g|_D$, f: ana $\Rightarrow$ g: ana

② $z \in D^*$ and $z+h \in D^*$, then:

$$\lim_{h\to 0} \frac{g(z+h)-g(z)}{h} = \lim_{h\to 0} \overline{\left(\frac{f(\overline{z+h})-f(\bar{z})}{\bar{h}}\right)} = \overline{f'(\bar{z})}$$

$\therefore$ g : ana

Since f : conti on $\mathbb{R}$-axis, so is g

$\therefore$ We can apply the thm above so g: ana throughout $D \cup L \cup D^* = U$ □

DEFINITION

A curve $\gamma:[a,b]\to\mathbb{C}$ is called a regular analytic arc if γ is an analytic map, 1-1 on $[a,b]$ with $\gamma' \neq 0$.

FACT (WILL PROVE IN THE FUTURE)

$\forall \alpha \in [a,b]$, $\exists D(\alpha,\varepsilon)$, s.t. γ: ana on $D(\alpha,\varepsilon)$, $\gamma'(z)\neq 0$ $\forall z \in D(\alpha,\varepsilon)$

In fact, $\exists \gamma^{-1}$: ana, $\gamma: D(\alpha,\varepsilon) \xrightarrow{\sim} \gamma(D(\alpha,\varepsilon))$

Proof Sketch

Map the boundary via γ to a real segment $[a,b]$, apply the thm, then apply γ^{-1}. Then, reflect similarly in image via λ.

Mathematically, $\lambda(\overline{\lambda^{-1}(f(\gamma(\overline{\gamma^{-1}(z)})))})$... XD (kill me)