10-16-25 (WEEK 7)

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DEFINITION

We say $\sum_{i=-\infty}^{\infty} \mu_i = L$ if $\sum_{i=0}^{\infty} \mu_i$ and $\sum_{j=1}^{\infty} \mu_{-j}$ both converge and their sum is L

THEOREM

We dk $R_1 ? R_2$, so cannot say $R_1 < |z| < R_2$

$f(z) = \sum_{-\infty}^{\infty} a_k z^k$ is convergent in the domain $D = \{z : R_1 < |z| \text{ and } |z| < R_2\}$, where $R_2 = (\limsup_{k\to\infty} |a_k|^{\frac{1}{k}})^{-1}$, $R_1 = \limsup_{k\to\infty} |a_{-k}|^{\frac{1}{k}}$

Proof

$\sum_{k=0}^{\infty} a_k z^k$ converges when $|z| < (\limsup_{k\to\infty} |a_k|^{\frac{1}{k}})^{-1}$

$\sum_{k=1}^{\infty} a_{-k} (z^{-1})^k$ converges when $|z^{-1}| < (\limsup_{k\to\infty} |a_{-k}|^{\frac{1}{k}})^{-1} \Rightarrow |z| > \limsup_{k\to\infty} |a_{-k}|^{\frac{1}{k}}$

$\therefore f$: conv $\forall z \in D$.

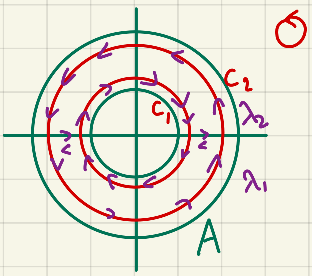
THEOREM

Laurent is when you expand around z, f at z: not ana (Taylor $\Rightarrow$ ana)

Let $A := \{z \mid R_1 < |z| < R_2\}$. If f is analytic in A, then f has a $\boxed{\text{Laurent expansion } f(z) = \sum_{k=-\infty}^{\infty} a_k z^k}$ in A

Proof

Say $C_i := R_i(0)$, $R_1 < r_1 < r_2 < R_2$, $A = A_1 \cup A_2$, $A_1 = \{z \mid z \in A \text{ and } \operatorname{Im} z > -1\}$, $A_2 = \{z \mid z \in A \text{ and } \operatorname{Im} z < 1\}$.



$\therefore C_2 - C_1 = \lambda_1 + \lambda_2$

$\therefore \underbrace{\int_{C_2 - C_1} \frac{f(\omega) - f(z)}{\omega - z} d\omega}_{(\star)} = \int_{\lambda_1 + \lambda_2} \frac{f(\omega) - f(z)}{\omega - z} d\omega \quad \forall \omega \in A$

As f is ana in A,

$$g(\omega) := \begin{cases} \frac{f(\omega) - f(z)}{\omega - z}, & \omega \in A \setminus \{z\} \\ f'(\omega), & \omega = z \end{cases} \text{ is ana}$$

$\Rightarrow (\star) = \int_{\lambda_1} \frac{f(\omega) - f(z)}{\omega - z} d\omega + \int_{\lambda_2} \frac{f(\omega) - f(z)}{\omega - z} d\omega = 0$ by closed curve thm,

Moreover, by closed curve thm, $\int_{C_1} \frac{d\omega}{\omega - z} = 0$

As $\int_{C_2} \frac{d\omega}{\omega - z} = 2\pi i$, $f(z) \int_{C_2 - C_1} \frac{d\omega}{\omega - z} = 2\pi i f(z)$

not A, so why

$\therefore 2\pi i f(z) = \int_{C_2 - C_1} \frac{f(\omega)}{\omega - z} d\omega = \int_{C_2} \frac{f(\omega)}{\omega - z} d\omega - \int_{C_1} \frac{f(\omega)}{\omega - z} d\omega$

$$= \int_{C_2} \frac{f(\omega)}{\omega(1 - \frac{z}{\omega})} d\omega + \int_{C_1} \frac{f(\omega)}{z(1 - \frac{\omega}{z})} d\omega$$

$$= \boxed{\int_{C_2} \left[\frac{f(\omega)}{\omega} \sum_{i=0}^{\infty} \left(\frac{z}{\omega}\right)^i \right] d\omega}^{②} + \boxed{\int_{C_1} \left[\frac{f(\omega)}{z} \sum_{i=0}^{\infty} \left(\frac{\omega}{z}\right)^i \right] d\omega}^{①}$$

We know $\sum_{i=0}^{\infty} \left(\frac{z}{\omega}\right)^2$ conv abs unif on C_2.

Hence, ② $= \sum_{i=0}^{\infty} \left[\int_{C_2} \frac{f(\omega)}{\omega^{i+1}} d\omega\right] z^i$

① $= \sum_{i=0}^{\infty} \left[\int_{C_1} f(\omega) \omega^i d\omega\right] z^{-(i+1)}$

$\therefore f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \quad \forall R_1 < |z| < R_2$ □

Uniqueness

$\int_C z^{i-(k+1)}\,dz = \begin{cases} 0, & i-k \neq 0 \\ 2\pi i, & i-k = 0 \end{cases}$

If $f(z)=\sum_{k=-\infty}^{\infty} c_k z^k \ \forall R_1<|z|<R_2$, then $\int_C \frac{f(z)}{z^{k+1}}\,dz = \int_C \sum_{i=-\infty}^{\infty} c_i z^{i-(k+1)}\,dz = \sum_{i=-\infty}^{\infty} c_i z^{i-(k+1)}\,dz$ for $C=C_r(0)\subseteq A$.

$\therefore \int_C \frac{f(z)}{z^{k+1}}\,dz = c_k 2\pi i \ \forall C=C_r(0)\subseteq A$

Thus, (i) $\int_C \frac{f(z)}{z^{k+1}}\,dz = c_k$ is indep of r for $C=C_r(0)\subseteq A$

(ii) $a_k = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{i+1}}\,dw$ is indep of C_1 and C_2

$\therefore$ The Laurent expansion of f in A is unique $\square$

REMARK

Taking $R_1=0$, we can take Laurent expansion at a pole or removable singularity.