Complex Analysis: Midterm Theorems

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Remark

I only will include theorems that are useful for me, i.e. things that I still find useful, or whose proofs I'm shaky on, after learning this course for 7 weeks. Otherwise, there will be too many theorems. For me, my dyslexic brain is only able to memorize something if I rewrite it because I can't really read, why I have to retype this many proofs.

Power Series

Analytic Polynomial

Proposition

A polynomial $P(x, y)$ is **analytic** $\Leftrightarrow P_y = iP_x$

Proof. "$\Rightarrow$": By def, $\exists\, \alpha_k \in \mathbb{C},\ N \in \mathbb{N}$, s.t. $P(x, y) = \sum_{k=0}^{N} \alpha_k (x + iy)^k$. Then, $P_y = \sum_{k=1}^{N} k\alpha_k (x + iy)^{k-1} i$, $P_x = \sum_{k=1}^{N} k\alpha_k (x + iy)^{k-1}$, so $P_y = iP_x$.

"$\Leftarrow$": We can rewrite $P(x, y) = \sum_{k=0}^{N} Q^k(x, y)$, where we have polynomials in the form $Q^k(x, y) = c_0 x^k + c_1 x^{k-1} y + \cdots + c_k y^k$.

From assumption, $Q^k_y = iQ^k_x\ \forall\, k$. Hence, $\sum_{p=1}^{k} pc_p x^{k-p} y^{p-1} = \sum_{p=0}^{k-1} (k - p)c_p x^{k-p-1} y^p i$

- $p = 1$: $1c_1 = ikc_0 \Rightarrow c_1 = \binom{k}{0} c_0$
- $p = 2$: $2c_2 = (k - 1)c_1 i \Rightarrow c_2 = i^2 \frac{k(k-1)}{2} c_0$
- For any $p > 1$, $pc_p = (k - p + 1)c_{p-1} i \Rightarrow c_p = i^p \binom{k}{p} c_0$

Hence, $Q^k = \sum_{p=0}^{k} i^p \binom{k}{p} c_0 x^{k-p} y^p = c_0 (x + iy)^k\ \forall k$, so P is **analytic**. $\square$

Radius of Convergence

Theorem

Given the power series $\sum_{k=0}^{\infty} c_k z^k = f(z)$, define $L := \limsup_{k\to\infty} |c_k|^{\frac{1}{k}}$, then we have:

1. $L = 0 \Rightarrow P(z)$ converges $\forall z \in \mathbb{C}$
2. $L = \infty \Rightarrow P(z)$ converges only at $z = 0$
3. $0 < L < \infty \Rightarrow P(z)$ converges on $|z| < \frac{1}{L}$ and diverges on $|z| > \frac{1}{L}$

Proof. We consider the three cases separately.

1. Hence, given any $z \in \mathbb{C}$, $\limsup_{k\to\infty} |c_k|^{\frac{1}{k}} = 0$. By def, take $\varepsilon = \frac{1}{2}$, $\exists\, N$, s.t. $k > N \Rightarrow |c_k|^{\frac{1}{k}}|z| < \frac{1}{2} \Rightarrow \sum |c_k z^k| = \sum(|c_k|^{\frac{1}{k}}|z|)^k < \sum(\frac{1}{2})^k = 1$
2. Consider small $|z|$, $\forall\, N \in \mathbb{N}$, $\exists\, k > N$, s.t. $|c_k|^{\frac{1}{k}} > \frac{1}{|z|} \Rightarrow |c_k z^k| > 1$, so $P(z)$ only converges when $z = 0$
3. Take $R = \frac{1}{L}$, $|z| = R(1-\delta)$, i.e. $1 > \delta > 0$ when $|z| < \frac{1}{L}$. Then, $\forall\, \varepsilon > 0, \exists N \in \mathbb{N}$, $n > N$, s.t. $|z|(L-\varepsilon) \le |z| \sup_{k\ge n} |c_k|^{\frac{1}{k}} \le |z|(\varepsilon + L) < 1 + \varepsilon R(1-\delta) - \delta < 1 - \frac{\delta}{2}$, so it is **abs conv**

   If $|z| > R$, $\limsup |c_k|^{\frac{1}{k}}|z| > 1 \Rightarrow$ for inf values of k, $|c_k z^k| > 1 \Rightarrow \sum c_k z^k$ div

□

Differentiation

Theorem

Given a power series $P(z) = \sum_{k=0}^{\infty} c_k z^k$ with radius of convergence R, then $P'(z)$ exists on $|z| < R$, $P'(z) = \sum_{k=1}^{\infty} k c_k z^{k-1}$

Proof. For $0 < R < \infty$, let $|z| = R - \delta$, $R \ge \delta > 0$. WLOG, consider $|h| < \frac{\delta}{2}$ and consider $\frac{P(z+h)-P(z)}{h}$.

$\frac{P(z+h)-P(z)}{h} = \frac{1}{h}\sum c_k((z+h)^k - z^k) = \sum_{k=1}^{\infty} k c_k z^{k-1} + \sum_{k=2}^{\infty} c_k b_k$, where $b_k = \sum_{p=2}^{k} \binom{k}{p} h^{p-1} z^{k-p}$

- If $|z| = 0$, $b_k = h^{k-1} \Rightarrow \sum c_k h^{k-1} < \infty \Rightarrow \sum c_k h^{k-1} \to 0$ as $h \to 0$
- If $|z| \neq 0$, $\binom{k}{p} \le \binom{k}{p-2} k^2$.

  Hence, $|b_k| \le \frac{|h|}{|z|^2} k^2 \sum_{p=2}^{k} \binom{k}{p-2} |h|^{p-2} |z|^{k-(p-2)} \le \frac{|h|}{|z|^2} k^2 \sum_{j=0}^{k} \binom{k}{j} |h|^j |z|^{k-j} = \frac{|h|}{|z|^2} k^2 (|z| + |h|)^k \le \frac{|h|}{|z|^2} k^2 (R - \frac{\delta}{2})^k$

  Thus, $|\sum_{k=2}^{\infty} c_k b_k| \le \frac{|h|}{|z|^2} k^2 |c_k| (R - \frac{\delta}{2})^k \to 0$

The remaining case is simple for $R = \infty$, we apply the case above and show it holds for any R. □

Corollary

Power series are **smooth in their domain of convergence**

Uniqueness

Theorem

If $\exists \{z_n\}_n \to 0$, and $\sum c_k z_n^k = 0$, then $c_k = 0\ \forall k$

Proof. As P is **conti**, thus $P(0) = \lim_{n\to\infty} P(z_n) = 0 \Rightarrow c_0 = 0$.

Consider, $g(z) = \frac{f(z)}{z}$ with the **same radius of convergence** as $f(z)$. Similarly, $g(0) = \lim_{n\to\infty} \frac{f(z_n)}{z_n} = 0 \Rightarrow c_1 = 0$. Note that this can be recusrively applied.

$\therefore$ By induction on n, $c_n = 0\ \forall n$. $\square$

Corollary

If $\sum a_k z^k$ and $\sum b_k z^k$ agree on $\{z_n\}_n$ as $n \to \infty$, then $a_k = b_k\ \forall k$

Analytic Functions

Proposition

If $f = u + iv$ is differentiable at z, then f_x and f_y exist and satisfy the CR-equation $f_y = if_x$

Proof. By def, f is diff $\Rightarrow \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists. Along the **real** axis, this limit is $\lim_{\xi\to 0} \frac{f(x+\xi,y)-f(x,y)}{\xi} = f_x$. Along the **imaginary** axis, this limit is $\lim_{\xi\to 0} \frac{f(x,y+\xi)-f(x,y)}{\xi i} = \frac{f_y}{i}$. Hence, $f_x = \frac{f_y}{i} \Rightarrow \boxed{f_y = if_x}$ $\square$

Counterexample for f_x and f_y exist at z and $f_y = if_x$, but f is not differentiable

$$f(z) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2}, & z \neq 0 \quad (\text{i.e. } xy \cdot \frac{z}{|z|}) \\ 0, & z = 0 \quad (\Leftrightarrow (x,y) = 0) \end{cases}$$

We notice, both on the **x-axis and y-axis**, we have $f(z) \equiv 0$, hence $f_x(0) = f_y(0) = 0$.

However, along $y = ax$ for $a \neq 0$, we get $f(x, ax) = \frac{a(1+ia)}{1+a^2} x \Rightarrow \lim_{x\to 0} \frac{f(x,ax)-f(0,0)}{x+axi} = \frac{a}{1+a^2} \neq 0$.

Hence, $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ does not exist, so f is NOT differentiable. $\square$

Proposition

Suppose that f_x and f_y exist in a **nbd of z** and are **conti** at z. If f satisfies the CR-eq, then f is **differentiable**.

Proof. Say $z = x + iy,\ h = \xi + i\eta$, and $f(z) = u(z) + iv(z)$.

Then, $\frac{f(z+h)-f(z)}{h} = \frac{[u(x+\xi,y+\eta)-u(x,y)]}{\xi+i\eta} + i\frac{[v(x+\xi,y+\eta)-v(x,y)]}{\xi+i\eta}$

By MVT with "$-u(x+\xi,y)+u(x+\xi,y)$" and $-v(x+\xi,y)+v(x+\xi,y)$, this equals:
$\frac{\eta}{\xi+i\eta}[\frac{u(x+\xi,y+\eta)-u(x+\xi,y)}{\eta} + i\frac{v(x+\xi,y+\eta)-v(x+\xi,y)}{\eta}] + \frac{\xi}{\xi+i\eta}[\frac{u(x+\xi,y)-u(x,y)}{\xi} + i\frac{v(x+\xi,y)-v(x,y)}{\eta}]$

$$= \frac{\eta}{\xi+i\eta}[u_y(x+\xi, y+\theta_1\eta) + iv_y(x+\xi, y+\theta_2\eta)] + \frac{\xi}{\xi+i\eta}[u_x(x+\theta_3\xi, y) + iv_x(x+\theta_4\xi, y)]$$

As we know, $0 < \theta_k < 1$, and $|\frac{\eta}{\xi+i\eta}| = |\frac{\Re(h)}{h}| \leq 1$, $|\frac{\xi}{\xi+i\eta}| \leq 1$.

Claim: $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f_x(z)$

Proof. We know, by CR-eq, $f_x(z) = \frac{\xi}{\xi+i\eta} f_x(z) + \frac{\eta}{\xi+i\eta} f_y(z)$

As f_x and f_y are conti, $\frac{f(z+h)-f(z)}{h} - f_x(z) = \frac{\eta}{\xi+i\eta}[(u_y(x+\xi, y+\theta_1\eta) - u_y(x,y)) + i(v_y(x+\xi, y+\theta_2\eta) - v_y(x,y))] + \frac{\xi}{\xi+i\eta}[(u_x(x+\theta_3\xi, y) - u_x(x,y)) + i(v_x(x+\theta_4\xi, y) - v_x(x,y))] \to 0$ as h, i.e. $\xi,\ \eta \to 0$. □

Hence, f is **diffable** and $\boxed{f'(z) = f_x(z)}$ □

Applications of CR Equation

Regions

Region Implies Connecting Vertical and Horizontal Line Segments

If D is a **region**, then $\forall x, y \in D, \exists$ a curve consisting of horizontal and vertical line segments that connect x and y

Proof. For any $x \in D$, say $U_x := \{y \in D \mid x$ connects to y via vertical/horizontal line segments that connect$\}$

1. "U_x is open": For all $y \in U_x \subseteq D$, as D is open, $\exists$ open disk $B_\delta(y) \subseteq D$. As $\forall a \in B_\delta(y)$, a can be connected to y by vertical/horizontal line segments, thus $x \to y \to a$ can be connected via these line segments, so $B_\delta(y) \subseteq U_x$.

2. "$D \setminus U_x$ is open": For $y \in D \setminus U_x$, D is open $\Rightarrow \exists$ open disk $B_\delta(y) \subseteq D \Rightarrow B_\delta(y) \cap U_x = \emptyset \Rightarrow B_\delta(y) \subseteq D \setminus U_x$

Combining (1), (2), and that D is connected, hence we get $\boxed{D = U_x}$. □

u is constant implies f is constant

If $f = u + iv$ is **analytic** on a region D and u is constant, then f is constant

Proof. u is constant $\Rightarrow u_x = u_y = 0$. By **CR-eq**, $v_x = v_y = 0$.

As D is a region, thus $\forall a, b \in D$, $\exists$ a horizontal/vertical connected path connecting a and b. Hence, $f(a) = f(b)$ (for all $a, b \in D$), so f is **constant**. □

Line Integrals

Smoothly Equivalent Integrals are Equivalent

If $\boxed{C_1 \sim_{sim} C_2}$, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

Proof. We set $f(z) = u(z) + iv(z)$ and $z = x(t) + iy(t)$.

Then, we know $\int_{C_1} f(z)dz = \int_a^b f(z(t))z'(t)dt$
$= \int_a^b [u(z(t))x'(t) - v(z(t))y'(t)]dt + i\int_a^b [u(z(t))y'(t) + v(z(t))x'(t)]dt$

With $\int_c^d u(z(\lambda(t)))x'(\lambda(t))\lambda'(t)dt = \int_a^b u(z(t))x'(t)dt$, substitute it back in and going in the opposite direction, we prove the equation. □

Lemma on Modulus of Integral

Let $t \in \mathbb{R}$, $G(t)$ be a continuous complex-valued function. Then, $|\int_a^b G(t)dt| \leq \int_a^b |G(t)|dt$

Proof. Set $R(t)e^{i\theta} := \int_a^b G(t)dt$, for some fixed $\theta \in \mathbb{R}$, $R \in \mathbb{R}_{\geq 0}$.

Then, $R = |\int_a^b G(t)dt| = \int_a^b e^{-i\theta}G(t)dt = \int_a^b A(t)dt + i\int_a^b B(t)dt$. By **comparing like terms**, we deduce $R = \int_a^b A(t)dt$. Hence, $R = \int_a^b A(t)dt \leq \int_a^b |A(t)|dt \leq \int_a^b |e^{-i\theta}G(t)|dt = \int_a^b |G(t)|dt$ □

ML-Formula

Let C be a **smooth** curve of length L, and f be conti on C and $f << M$ throughout C. Then, $\int_C f(z)dz << ML$

Proof. Let C be $z(t) = x(t) + iy(t)$ for some $t \in [a, b]$. Then, $\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$.

By the previous lemma, $\int_C f(z)dz << \int_a^b |f(z(t))||z'(t)|dt \leq M\int_a^b |z'(t)|dt = ML$ (integrating $|z'(t)|$ was the formula for arc-length) □

ML is the Tight Bound

For $f(z) = \frac{1}{z}$, $C : \cos\theta + i\sin\theta$, then $\int_C f(z)dz = 2\pi i \Rightarrow |\int_C f(z)dz| = 2\pi = ML$

Proposition (Limits)

Suppose $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ unif on a smooth curve C. Then, $\lim_{n\to\infty} \int_C f_n(z) = \int_C f(z)dz$

Proof. By def, $f_n \to f$ unif on C means "Given $\varepsilon > 0$, $\exists N$, s.t. $\forall n \geq N$, $|f_n(z) - f(z)| < \varepsilon\ \forall z \in C$"

Hence, $|\int_C f_n(z)dz - \int_C f(z)dz| = |\int_C (f_n - f)(z)dz| < \varepsilon \cdot \text{len}(C)\ \forall n \geq N$. (By ML-formula) □

FTC Variant

Let F be an analytic function, $f = F'(z)$ and a smooth curve $C : z(t) = x(t) + iy(t)$, $t \in [a, b]$. Then, $\int_C f(z)dz = F(z(b)) - F(z(a))$

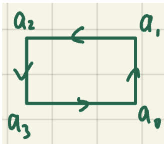
Proof. Let $\gamma(t) := F(z(t)) = A(t) + iB(t)$. As F is analytic, by the chain rule, $\gamma'(t) = F'(z(t))z'(t)$.

Then, $\int_C f(z)dz = \int_a^b F'(z(t))z'(t)dt = \int_a^b \gamma'(t)dt = \boxed{\gamma(b) - \gamma(a)}$ □

Rectangle Theorem

Lemma

If f is a **linear function**, i.e. $f = a + zb$, for $a, b \in \mathbb{C}$, and Γ is the **boundary of a rectangle**, then $\int_\Gamma f(z)dz = 0$

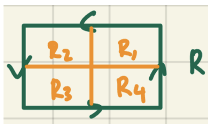


Proof. Say $\Gamma : z(t), a = a_0 \leq t \leq b = a_3$, and $f = F'(z) \Rightarrow F := \frac{a}{2}z^2 + bz$ (since f is ana on $\mathbb{C}$). Hence, we deduce $\int_\Gamma f(z)dz = \int_\Gamma F'(z)dz = F(z(b)) - F(z(a)) = 0$ (because $z(b) = z(a)$). □

Rectangle Theorem

Let f be an **entire function**, and Γ as above. Then, $\int_\Gamma f(z)dz = 0$

Proof. Let $I = \int_\Gamma f(z)dz$. Assume $f \neq 0$, otherwise $f = 0 \Rightarrow I = 0$. In this case, we divide R as follows:



Then, $\exists R_i$, s.t. $|\int_{\Gamma_i} f(z)dz| \geq \frac{|I|}{4}$, where Γ_i is the boundary of R_i. Define $R^{(1)}$ to be said R_i. Continuing this process, we get $R^{(1)} \supseteq R^{(2)} \supseteq \dots$. Let $z_0 \in \cap_{i=1}^{\infty} R^{(i)}$.

As f is entire, thus f is **analytic** at z_0. By def, $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$, s.t. $|h| < \delta \Rightarrow |\frac{f(z_0+h)-f(z_0)}{h} - f'(z_0)| < \varepsilon$. Hence, for some $|\varepsilon(z)| \leq \varepsilon$, we can have $f(z) = f(z_0) + f'(z_0)(z - z_0) + \varepsilon(z)(z - z_0)$.

Note that $f(z_0) + f'(z_0)(z - z_0)$ is **linear**. Thus, we can choose N s.t. $\forall n \geq N$, we have $|z - z_0| < \delta \Rightarrow \int_{\Gamma^{(n)}} f(z)dz = \int_{\Gamma^{(n)}} \epsilon(z)(z - z_0)dz$.

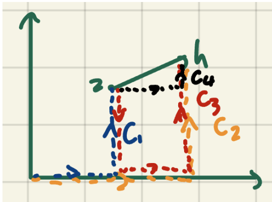
Let s be the length of the longest side of R. We know $|\Gamma^{(n)}| \leq \frac{4s}{2^n}$ and $|\varepsilon(z)(z - z_0)| < \varepsilon \cdot \frac{\sqrt{2}s}{2^n}$. Thus, by ML formula, $\int_{\Gamma^{(n)}} f(z)dz << \varepsilon \frac{4\sqrt{2}s^2}{4^n}$.

By our assumption, $|\int_{\Gamma^{(n)}} f(z)dz| \geq \frac{|I|}{4^n}$, hence $|I| \leq \varepsilon \cdot 4\sqrt{2}s^2\ \forall \varepsilon > 0$, hence $I = 0$. □

Integral Theorem

If f is entire, then f is everywhere the **derivative of an analytic function**. That is, $\exists$ an entire F s.t. $F'(z) = f(z)\ \forall z$.

Proof. Consider $F(z) = \int_C f(\eta)d\eta$, where $C : 0 \to \Re(z) \to z$.



Now, for $h \in \mathbb{C}$, we have $F(z+h) = \int_{C_1} f(z)dz$ and $F(z) = \int_{C_2} f(z)dz$.

This means, $F(z+h) - F(z) = \int_{C_1} f(\eta)d\eta + \int_{-C_2} f(\eta)d\eta = \int_{C_3} f(\eta)d\eta = \int_{C_4} f(\eta)d\eta$.

As $\frac{1}{h}\int_{C_4} dz = 1$, thus $(\frac{1}{h}\int_{C_4} f(\eta)d\eta) - f(z) = \frac{1}{h}\int_{C_4}(f(\eta) - f(z))d\eta = \frac{F(z+h)-F(z)}{h} - f(z)$

In other words, by ML-formula, $\frac{F(z+h)-F(z)}{h} << \frac{1}{|h|}\varepsilon \cdot 2|h| \Rightarrow \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$ □

Corollary

If f is entire and C is a **smooth closed curve**, then $\int_C f(z)dz = 0$

Rectangle Theorem II

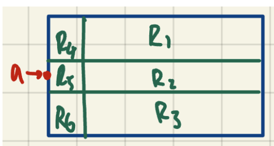
Let f be entire, and

$$g(z) := \begin{cases} \boxed{\dfrac{f(z)-f(a)}{z-a}}, & z \neq a, \\ f'(a), & z = a, \end{cases}$$ which is continuous (since f entire $\Rightarrow g$ continuous).

Then, $\int_\Gamma g(z)dz = 0$, where Γ is a boundary of a closed rectangle $R \subseteq \mathbb{C}$

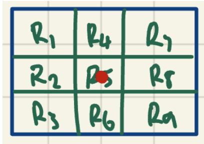
Proof. As g is conti, by def, $\exists M \in \mathbb{R}$, s.t. $|g(z)| < M\ \forall z \in \mathbb{R}$

1. If $a \in \mathbb{C} \setminus R$, then $g(z)$ is **analytic for all z** $\in$ **R**. Hence, by the argument of the Rectangle Thm, $\int_\Gamma g(z)dz = 0$
2. If $a \in \Gamma$, where $\Gamma_i :=$ boundary of R_i



Then, by case 1, $\int_\Gamma g(z)dz = \int_{\Gamma_s} g(z)dz << M \cdot 4\varepsilon$ by the ML-formula, with M indep of ε, where we define ε to be the length of the **longest side** of Γ_s. Hence, as $\varepsilon \to 0$, $\int_\Gamma g(z)dz = 0$

3. Otherwise, $a \in$ interior of R. Then we have:



Then, by case 1, $\int_\Gamma g(z)dz = \int_{\Gamma_s} g(z)dz << M \cdot 4\varepsilon \to 0$ as $\varepsilon \to 0$

□

Cauchy Integral Formula

Lemma

Define $C_\rho(\alpha) :=$ circle centered at α with radius ρ, where α may be omitted if there is no amibiguity. Then, $I := \int_{C_\rho(\alpha)} \frac{dz}{z-a} = 2\pi i$ $\forall |a-\alpha| < \rho$

Proof. If $a = \alpha$, then it's clear, since $C_\rho(\alpha) = \alpha + \rho e^{i\theta}$, $0 \leq \theta 2\pi$.

For $a \neq \alpha$, we know $\boxed{I = \displaystyle\int_{C_\rho(\alpha)} \frac{dz}{(z-\alpha)-(a-\alpha)} = \int_{C_\rho(\alpha)} \frac{1}{z-a} \cdot \frac{1}{1-\frac{a-\alpha}{z-\alpha}} dz}$

Notice, $\forall z \in C_\rho(\alpha)$, $|\frac{a-\alpha}{z-\alpha}| < 1$. Hence, we have unif conv: $(1-\frac{a-\alpha}{z-\alpha})^{-1} = 1 + (\frac{a-\alpha}{z-\alpha}) + (\frac{a-\alpha}{z-\alpha})^2 + \dots$

Hence, $I = \int_{C_\rho \alpha} \frac{1}{z-\alpha}(\sum_{k=0}^\infty (\frac{a-\alpha}{z-\alpha})^k) dz = \sum_{k=0}^\infty \int_{C_\rho(\alpha)} \frac{1}{z-\alpha}(\frac{a-\alpha}{z-\alpha})^k dz$.

We now consider $J_k := \int_{C_\rho(\alpha)} \frac{1}{(z-\alpha)^k} dz$. When $k = 1$, thus $J_1 = 2\pi i$. When $k > 1$, $J_k = \int_0^{2\pi} \frac{ie^{i\theta}}{\rho^k e^{ik\theta}} d\theta = \int_0^{2\pi} \frac{i}{\rho^k} e^{i\theta(k-1)} d\theta = 0$. Hence, $I = 2\pi i$. □

Cauchy Integral Formula

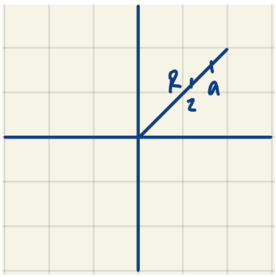
Given an entire f, $a \in \mathbb{C}$, $C = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$ with a within the unit disc of radius R, then we have $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$

Proof. By rectangle thm 2, we know $\int_C g(z)dz = \int_C \frac{f(z)}{z-a} - \frac{f(a)}{z-a} dz = 0$. Thus, $\int_C \frac{f(z)}{z-a} dz = \int_C \frac{f(a)}{z-a} dz = f(a) \int_C \frac{1}{z-a} dz = f(a) 2\pi i$ □

Taylor Expansion

Taylor Expansion for Entire Function

Given f is an entire function, then $f^{(k)}(0)$ exists $\forall k \in \mathbb{Z}_{>0}$ and $f(z) = \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!} z^k$ $\forall z \in \mathbb{C}$



Proof. Choose $a \in \mathbb{C}$, $|a| > |z|$, $\boxed{R := |a| + 1}$. By Cauchy Integral Formula, $f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{1}{1-\frac{z}{\omega}} \frac{f(\omega)}{\omega} d\omega$.

As $|\frac{z}{\omega}| < \frac{|a|}{1+|a|}$, then $f(z) = \sum_{k=0}^{\infty} \int_{C_R} \frac{f(\omega)}{\omega}(\frac{z}{\omega})^k d\omega = \sum_{k=0}^{\infty} z^k \int_{C_R} \frac{f(\omega)}{\omega^{k+1}} d\omega = \sum_{k=0}^{\infty} z^k C_k$.

Notice, as $|z| < |a|$, then $f'(z) = \sum_{i=1}^{\infty} iz^{i-1}C_i \Rightarrow f'(0) = C_1$. If we continue this process, thus $f^{(k)}(0)$ exists $\forall k \in \mathbb{N}_{>0}$ and $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$ □

Corollary

Let f be an entire function with zeros at $a_1, \ldots, a_N$. Define $g(z) = \frac{f(z)}{(z-a_1)\cdots(z-a_N)}$ for $z \notin \{a_1, \ldots, a_N\}$. Then, $\lim_{z\to a_i} g(z)$ exists $\forall i$. If we define $g(a_i) := \lim_{z\to a_i} g(z)$, then g is **entire**.

Proof. Set $f_0 = f$, and $f_k := \frac{f_{k-1}(z) - f_{k-1}(a_k)}{z - a_k}$. Hence, f_1 is entire. By recurrence, g is entire. □

Liouville's Theorem

Liouville's Theorem

Entire bounded functions on $\mathbb{C}$ are **constants**.

Proof. Let $a \in \mathbb{C} \setminus \{0\}$ and consider $R > |a|$.

Then, by Cauchy Integral Formula, $f(a) - f(0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{z-a} - \frac{f(z)}{z} dz = \frac{1}{2\pi i}\int_{C_R} \frac{af(z)}{z(z-a)} dz$.

As f is **bounded**, $\exists M \in \mathbb{R}_{>0}$, s.t. $|f(z)| < M$ $\forall z \in \mathbb{C}$.

By ML-formula, $|f(a) - f(0)| < \frac{1}{2\pi i}(\frac{M\cdot|a|}{R(R-|a|)} \cdot 2\pi R) \to 0$ as $R \to \infty$. Hence, $\boxed{f(a) = f(0)}$ $\forall a \in \mathbb{C}$. □

Extended Liouville's Theorem

Given f is entire. Suppose $|f(z)| < A + B|z|^k$ for some constants $A, B \in \mathbb{R}_0$. Then, f is a polynomial with **degree at most k**

Proof. We consider induction on k. By Louville's Thm, we already know $k = 0$ is true.

For $k > 0$, define the **entire** function

$$g(z) := \begin{cases} \boxed{\dfrac{f(z) - f(0)}{z}}, & z \neq 0, \\ f'(0), & z = 0, \end{cases}$$

As $|f(z)| < A + B|z|^k$ is **bounded**, define $M_0 := \max_{z \in C_R} g(z)$ for some $R \geq 1$. Thus, we have for $z \in \mathbb{C} \setminus C_R, |g(z)| < A + B|z|^{k-1}$ and for $z \in C_R,\ |g(z)| < M_0$. This means, $\exists\, D, E \in \mathbb{R}_{>0}$, s.t. $\boxed{|g(z)| < D + E|z|^{k-1}}$

Hence, g is a poly of degree **at most k – 1**, i.e. f is a poly of degree **at most k**. □

Fundamental Theorem of Algebra

Nonconstant polynomials have roots in $\mathbb{C}$

Proof. Consider a polynomial $p(x)$. Suppose p has **no roots** in $\mathbb{C}$.

Then, $f(z) := \frac{1}{p(z)}$ is **entire**. Moreover, as $z \to \infty$, $|f(z)| \to 0$, so $|f(z)|$ is **bounded**.

Hence, by Louville's Thm, $f(z)$ is const $\Rightarrow p(z)$ is constant, which is a contradiction. □

Gauss-Lucas Theorem

The **zeros of the derivative of a polynomial** lie within the **convex hull** of the **zeros** of the polynomial.

Proof. Let $p(x)$ be a nonconstant polynomial in $\mathbb{C}[x]$, and $\alpha_1, \dots, \alpha_n$ be the **roots** of p counted by multiplicity. Then, $p(x) = c\Pi_{i=1}^n (x - \alpha_i)$. Moreover, $\frac{p'(x)}{p(x)} = \sum_{i=1}^n \frac{1}{x - \alpha_i}$.

Let a be a root of $p'(x)$ and $a \notin \{\alpha_1, \dots \alpha_n\}$. Then, $\frac{p'(a)}{p(a)} = \sum_{i=1}^n \frac{1}{a-\alpha_i} = \sum_{i=1}^n \frac{\bar{a} - \bar{\alpha}_i}{|a-\alpha_i|^2}$, so $\bar{a} = \sum_{i=1}^n c_i \bar{\alpha}_i$, where $c_i = \frac{1}{|a-\alpha_i|^2} / \sum_i \frac{1}{|a-\alpha_i|^2} \in \mathbb{R}_{\geq 0}$.

Hence, $a = \sum_{i=1}^n c_i \alpha_i$, $c_i \in \mathbb{R}_{\geq 0}$, $\sum c_i = 1$, so by def, this concludes the proof. □

Uniqueness, Mean Modulus, Max/Min Modulus Theorems

Uniqueness Theorem

Remark: We can only apply the following theorems below, including max/min modulus thm, only when its **acc points are in D**.

Uniqueness Theorem

Say D is a **region** and f is an **analytic region** on D. Suppose that $\exists$ seq of distinct **zeros** of D $\{z_n\}$, s.t. $\lim_{n\to\infty} z_n = z_0 \in D$, where we say the seq $\{z_n\}$ has an **acc pt** in D. Then, $f \equiv 0$ on D.

Proof. As we know, f is ana, so it is conti, i.e. $f(z_0) = \lim_{n\to\infty} f(z_n) = 0$.

Define $A := \{z \in D \mid z$ is an **acc point of zeros** of f in $D\}$

- "A is open": By uniqueness of power series, $f \equiv 0$ in some disk $D(z, \delta_z) \subseteq D\ \forall z \in A$, i.e. $D(z, \delta_z) \subseteq A\ \forall z \in A$.
- "$D \setminus A$ is open": z is **NOT an acc point of zeros** $\Rightarrow \exists$ open nbd U of z in D s.t. $f(z)$ has **NO zeros** in $U \setminus \{z\}$. As f is conti, $\forall y \in U \setminus \{z\}, \exists$ open nbd $V_y \subseteq D$ of y', s.t. $f \neq 0$ on $V_y \Rightarrow y \in D \setminus A$

Hence, $z_0 \in A$ and D is a region, with $D = A$ □

Polynomials

If f is **entire** and $f \to \infty$ as $z \to \infty$, then f is a **polynomial**.

Proof. By def, $\forall M \in \mathbb{R}_{>0}, \exists \delta$, s.t. $\forall |z| > \delta$, $|f(z)| > M$.

Let $M = 1$. Thus, $\exists\ \delta$, s.t. $\forall |z| > \delta, |f(z)| > 1$. By assumption, f is **NOT constant**.

Claim: f has **finitely many zeros**

Proof. By δ, all zeros in f are in $\overline{D(0,\delta)}$, otherwise, $|f(z)| \neq 0$. As $\overline{D(0,\delta)}$ is cpt, $\exists$ acc pt of zeros in $\overline{D}(0, \delta) \Rightarrow f \equiv 0$ on $D(0, \delta')$ for all $\delta' > \delta$, which is a contradiction. □

Now consider within $\bar{D}(0, \delta)$. Let $\alpha_1, \ldots, \alpha_N$ be the zeros of f. Then, $g(z) = f(z)/\Pi_{i=1}^n (z - \alpha_i)$ is **entire** and has **no zeros** in $\mathbb{C}$.

Set $h(z) := \frac{1}{g(z)}$, then h is **entire** and is **bounded** in the disk.

By Extended Liouville's Thm, thus $|h| < A + B|z|^n$ for all $|z| > \delta$ (By $|f(z)| > 1 \Rightarrow |h(z)| < \Pi_{i=1}^n (z - \alpha i)$) and $|z| \leq \delta \Rightarrow h$ is a poly.

However, h has **no zeros** in $\mathbb{C}$. Thus, h is **const**.

$\therefore \exists\ c \in \mathbb{C}^\times$, s.t. $f(z) = c\Pi_{i=1}^n (z - \alpha_i)$ □

Mean Modulus Theorem

Mean Value Theorem

Let D be a region, f ana on D, $\alpha \in D$. Then, $f(\alpha)$ = mean value of f taken around the **boundary** of **any disk centered at α and contained at D**

Proof. By Cauchy-Integral Formula, $f(\alpha) = \frac{1}{2\pi i} \int_{C_\delta(\alpha)} \frac{f(z)}{z-\alpha} dz$. Say $z = \alpha + \delta e^{i\theta}$ for $\theta \in [0, 2\pi]$, we get $f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + \delta e^{i\theta}) d\theta$ □

Max/Min Modulus Theorem

Maximum Modulus Theorem

Say f is **nonconst**, ana on a region D. Then, $\forall z \in D$ and $\delta \in \mathbb{R}_{\geq 0}$, $\exists$ some $\omega \in D(z,\delta) \cap D$, s.t. $|f(\omega)| > |f(z)|$

Proof. By MVT, $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+\delta e^{i\theta})d\theta$ for small enough δ s.t. $D(z,\delta) \subseteq D$.

Then, by ML-formula $|f(z)| \leq \frac{1}{2\pi}\max_{\theta\in[0,2\pi]}|f(z+\delta e^{i\theta})| \cdot 2\pi = \max|f(z+\delta e^{i\theta})|$

When $\leq$ has equality, then $|f(z+\delta e^{i\theta})| = \max|f(z+\delta e^{i\theta})| \Rightarrow f$ is **const** on $C_\delta(z) \subseteq D$. As f agrees with $g \equiv$ const **on a set of points with acc point**, f is const on D.

However, f is nonconst. Hence, $|f(z)| < \max_{\theta\in[0,2\pi]}|f(z+\delta e^{i\theta})|$ □

Minimum Modulus Theorem

Say f is nonconst and ana on a region D, $\forall\ z \in D$, $f(z) \neq 0$. Then, f has **no interior min points**.

Proof. Let $g(z) := \frac{1}{f(z)}$. Observe, g is ana and nonconst on D. By max modulus thm, we proved it. □

Saddle Points

Theorem on Maxima

Say $\bar{D}$ is a **closed disk (circle)** and f is analytic and nonconst on $\bar{D}$. f assumes its **max value** at a boundary point z_0, then $f'(z_0) \neq 0$

Proof. Suppose $f'(z_0) = 0$.

Then, $f(z_0+\delta) \approx f(z_0) + \frac{1}{2}f''(z_0)\xi^2 \Rightarrow |f(z_0+\delta)|^2 = |f(z_0)|^2 + \frac{2}{k!}\Re(\bar{f}(z_0)f^{(k)}(z_0)x^k) + \dots$ for some $k \geq 2$.

Let $e^{i\theta} = \frac{\xi}{|\xi|}$. Then, $\bar{f}(z_0)f^{(k)}(z_0) = Ae^{i\alpha} \Rightarrow |f(z_0+\xi)|^2 - |f(z_0)|^2 = \frac{2}{k!}A|\xi|^k\cos(\alpha+k\theta) + \dots$.

As $|f(z_0)|$ is **max**, hence $|f(z_0+\xi)|^2 - |f(z_0)|^2 \leq 0\ \forall z_0+\xi \in D$, in other words, for small enough ξ,

$\cos(k\theta+\alpha) \leq 0 \Rightarrow \frac{\frac{\pi}{2}-\alpha}{k} + \frac{2\pi j}{k} \leq \theta \leq \frac{\frac{3}{2}\pi-\alpha}{k} + \frac{2\pi j}{k}$ for $0 \leq j \leq k-1$.

However, for a disc, $\exists\ \xi$, s.t. $z_0+\xi$ is NOT in any one of the cones, since $\frac{\pi}{k} \leq \frac{\pi}{2}$. Thus, there is a contradiction. □

Theorem on Saddle Points

z_0 is a **saddle point** of an analytic function f iff $f'(z_0) = 0$ and $f(z_0) \neq 0$

Proof. We have $z = x + iy$, $f(z) = u(x, y) + iv(x, y)$, and $g(z) = \sqrt{u^2 + v^2} \geq 0$

- "$\Rightarrow$": As $g(z_0)$ is not a local minimum, hence $g(z_0) \neq 0$, so $u(z_0) \neq 0$ or $v(z_0) \neq 0$

  We know $g_x(z_0) = g_y(z_0) = 0 \Rightarrow \frac{uu_x + vv_x}{g}|_{z_0} = \frac{uu_y + vv_y}{g}|_{z_0} = 0$, i.e. $\begin{bmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0) \end{bmatrix} \begin{bmatrix} u(z_0) \\ v(z_0) \end{bmatrix} = 0$

  However, by CR-eq, $\det \begin{bmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0) \end{bmatrix} = u_x^2(z_0) + v_x^2(z_0)$. Hence, $u_x(z_0) = v_x(z_0) = 0$

  As f is ana, hence $f'(z_0) = 0$. From above with $g(z_0) \neq 0$, we also know $f(z_0) \neq 0$.

- "$\Leftarrow$": As $f'(z_0) = 0$, thus $u_x(z_0) = v_x(z_0) = u_y(z_0) = v_y(z_0) = 0$, i.e. $g_x(z_0) = g_y(z_0) = 0$. However, by the **max and min modulus thms**, $|f(z_0)|$ is NOT a local extremum.

$\square$

Oppen Mapping Theorem and Schwarz Lemma

Open Mapping Theorem

Open Mapping Theorem

$\forall$ **open set U $\subseteq$ D, f(U) is also open** in $\mathbb{C}$ for any nonconst **ana** f.

Proof. It suffices to show $\forall \beta = f(\alpha') \in f(D(\alpha, \varepsilon)), \exists D(\beta, \varepsilon') \subseteq f(D(\alpha, \beta))$.

WLOG, we can assume $f(\alpha) = 0$, so we choose ε s.t. $\overline{D(\alpha, \varepsilon)} \subseteq D$. By uniqueness thm, $\exists s$, s.t. f has **no zeros** in $\overline{D(\alpha, \varepsilon)} \setminus \{\alpha\}$ or else $f \equiv 0$.

Let $2\delta = \min_{z \in C_\varepsilon(\alpha)} |f(z)| > 0$

Claim: $D(f(\alpha) = 0, \delta) \subseteq Im(f)$

Proof. $\forall w \in D(0, \delta)$, consider $f(z) - w$.

If $w \notin f(D(\alpha, \varepsilon))$, then $f(z) - w$ **has no zeros** on $D(\alpha, \varepsilon)$.

Hence, $|f(z) - w| \geq |f(z)| - |w| \geq \delta$ $\forall z \in C_\varepsilon(\alpha)$. However, we know $|f(\alpha) - w| < \delta$, which is a contradiction. $\square$

Thus, $w \in f(D(\alpha, \varepsilon)) \Rightarrow D(0, \delta) \subseteq Im(f)$ $\square$

Schwarz Lemma

Schwarz Lemma

Suppose that f is analytic in an **open unit disc** D with $|f| \leq 1$ and $f(0) = 0$. Then,

1. $|f(z)| \leq |z|$
2. $|f'(0)| \leq 1$

with equality in either of the above iff $f(z) = e^{i\theta} z$

Proof. Define $g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(z), & z = 0 \end{cases}$
$g(z)$ is ana on D since $f(z)$ is ana on D.

Consider $z \in C_r(0)$ for $0 < r < 1$, then $|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$

By **max modulus thm**, $\forall z \in \overline{D(0,r)}, |g(z)| \leq \frac{1}{r}$. As $r \to 1$, then $|g(z)| \leq 1 \, \forall z \in D$.

By def of $g(z)$, $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$ has either equality hold, when g is const and $|g| = 1$ on D. Hence, $g = e^{i\theta}$. $\square$

Proposition

Say f is **entire**. If $|f(z)| < \frac{1}{|\Im(z)|}$ for all z, then $f \equiv 0$

Proof. Define $g(z) = (z^2 - R^2) f(z)$, for some $R \in \mathbb{R}_{>0}$ (sufficiently large, e.g. $R \geq 1, R \to \infty$).

When $z \in C_R(0)$, $|z - R||z + R| \leq 2R|\Im(z)|$. Hence, $|g(z)| \leq \frac{2R}{|\Im(z)|^2} \leq 2R$ when $z \in C_R(0)$

By max modulus thm, $|g(z)| \leq 2R \, \forall z \in D(0, R)$. Hence, $\forall z \in D(0,R), |f(z)| \leq \frac{2R}{|z^2 - R^2|} \to 0$ as $R \to \infty$. Thus, $f(z) = 0$ $\square$

Morera's Theorem

Morera's Theorem

Morera's Theorem

Let f be **continuous** on an open set $D \subseteq \mathbb{C}$ and Γ be the boundary of a **closed rectangle** $R \subseteq D$. If $\int_\Gamma f dz = 0 \, \forall \, \Gamma \in R \subseteq D$, then f is **analytic** in D.

Proof. Say $z_0 \in D$, where D is open. Then, $\exists \varepsilon > 0$, s.t. $D(z_0, \varepsilon) \subseteq D$.

Define $F(z) := \int_C f(z)dz$ $\forall z \in D(z_0, \varepsilon)$, where $C : z_0 \to z_0 + \Re(z - z_0) \to z$.

For $z \in D(z_0, \varepsilon)$ and k small enough s.t. $z + h \in D(z_0, \varepsilon)$, then:

$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = \lim_{h\to 0} \int_{C_1} f(\omega)d\omega = f(z)$

$\square$

Corollary

Let D be an open set in $\mathbb{C}$ and $\{f_n\}$ be a sequence of ana functions s.t. $f_n \to f$ unif on cpta. Then, f is also **ana in D**.

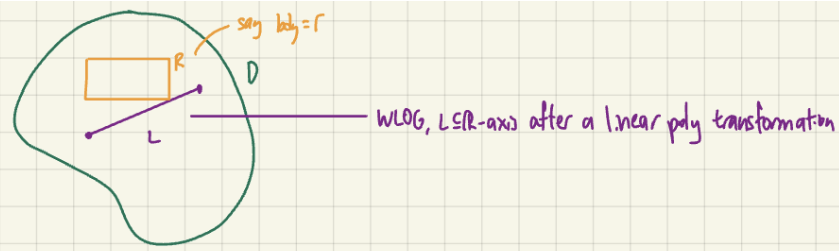
Proof. As f_n is conti $\forall K \subseteq D$ that is a cpt set, we have $f_n \to f$ unif on K. Thus, f is conti on K for all K, i.e. f is conti on D.

Notice, $\int_\Gamma f dz = \int_\Gamma \lim_{n\to\infty} f_n dz = \lim_{n\to\infty}(\int_\Gamma f_n dz) = 0$. Thus, by Morera's Thm, f is conti. $\square$

Schwarz Reflection Principle

Analytic Except on a Line Segment

f is continuous on an open set $D \subseteq \mathbb{C}$ and **analytic except on a line segment** in D. Then, f is **analytic throughout D**.



Proof

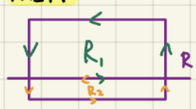
We know $f|_{D\setminus L}$: ana. Consider the following cases.

① $R \cap L = \emptyset \Rightarrow \int_\Gamma f dz = 0$ as f : ana on $D\setminus L$

② $R \cap L \subseteq \Gamma$, $R \cap L \neq \emptyset$: Lift one side, we get a rectangle $R_\varepsilon \subseteq R$, $R_\varepsilon \cap L = \emptyset$

By case ①, as f is cont., $\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} f(z)dz = 0 \Rightarrow \int_\Gamma f(z)dz = 0$

③ $R \cap L \not\subseteq \Gamma$



Then, $R = R_1 \cup R_2$, $\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f$

By cases ① and ②, hence $\int_\Gamma f(z)dz = 0$

$\therefore$ By Morera's Thm, f is analytic on D $\square$

Proof. (Note the image has a small typo, it should be "by cases (1) and (2)", not "by cases (1) and (3)", but I find it way easier to use that photo than to retype a mainly visual proof.) □

Schwarz Reflection Principle

Suppose f is C-analytic on a region D that is contained in either the upper or lower half plane and whose boundary contains a segment L on the real axis, and suppose f is real for real z.

Then, we can define an analytic "extension" g of f to the region $D \cup L \cup D*$ that is **symmetric** w.r.t. the real axis by:

$$g(z) = \begin{cases} f(z), & z \in D \cup L \\ \overline{f(\overline{z})}, & z \in D* \end{cases}, \quad \text{where } D* = \{z \mid \bar{z} \in D\}$$

Proof. We consider the proof for two main cases:

1. For any $z \in D$, then $f|_D = g|_D$, so f ana implies g is ana
2. For any $z \in D*$ and $z + h \in D*$, we have
   $\lim_{h\to 0} \frac{g(z+h)-g(z)}{h} = \lim_{h\to 0} \overline{\left(\frac{f(\overline{z+h})-f(\overline{z})}{\overline{h}}\right)} = \overline{f'(\overline{z})}$, so g is ana

As f is conti on the $\mathbb{R}$-axis, so is g, so we can apply the theorem of "analytic except for a line segment", to determine g is ana throughout $D \cup L \cup D* = U$ □

Simply Connected Domain

Although there are other theorems, I won't spend time proving them here because they all derive from this key lemma anyway.

Key Lemma for Polygonal Curves

Let Γ be a simple closed polygonal curve bounding D (simply connected).

Then any **horizontal segment** joining two consecutive "top-level" intersection points of Γ lies entirely inside D.

Proof. Here is a very rough proof that won't get you marks, I just don't have the time to summarize this proof, and I don't think it's tested (?).
Induct on the **level** of Γ:

Base: Γ a rectangle, trivial.

Inductive step: split Γ into lower-level subcurves, show that the horizontal strip between consecutive top levels stays inside D (using path-connectedness and openness arguments). □

Analytic Branch

Analytic Branch for log

Set $f(z) := \int_{z_0}^{z} \frac{1}{\xi} d\xi + \log z_0$ on a s.c. region $D \subseteq \mathbb{C} \setminus \{0\}$, we fix a $z_0 \in D$ and choose $\log z_0$. Then, f is an **analytic branch** of $\log z$ in D.

Proof. As D is s.c., choose a closed curve $C_1 - C_2$ (i.e. choose the endpoints and have two different paths).

By closed curve thm, $\int_{C_1 - C_2} \frac{1}{\xi} d\xi = 0 \Rightarrow \int_{C_1} \frac{1}{\xi} d\xi = \int_{C_2} \frac{1}{\xi} d\xi$, thus f is **analytic**.

Moreover, we want "$e^{f(z)} = z$" $\Leftrightarrow$ "$ze^{-f(z)} = 1$". TL;DR, set $g(z) := ze^{-f(z)}$, then we get $g'(z) = 0$, so $g \equiv 1$. $\square$

Singularity

Riemann's Principle of Removable Singularities

(I don't want to waste time here, so let's just say it's the same as the poles of order k thing below, except $k = 0$.)

Poles of Order k

Say f has an **isolated singularity** at z_0. If $\exists k \in \mathbb{Z}_{>0}$, s.t. $\lim_{z \to z_0} (z - z_0)^k f(z) \neq 0$ but $\lim_{z \to z_0} (z - z_0)^{k+1} f(z) = 0$, then f has a pole of order k at z_0.

Proof. Set $g(z) = \begin{cases} \boxed{(z - z_0)^{k+1} f(z)}, & z \in D'(z_0, \delta) = D(z_0, \delta) \setminus \{z_0\} \\ 0, & z = z_0 \end{cases}$

As $\lim_{z \to z_0} (z - z_0)^{k+1} = 0$, thus g is conti at $\mathbf{z_0}$.

As f is ana on $D'(z_0, \delta)$, g is also ana on $D'(z_0, \delta)$.

Hence, using the "analytic except for a line" theorem, we get g is ana on $D(z_0, \delta)$

Thus, set $h(z) := \begin{cases} \boxed{\frac{g(z) - g(z_0)}{z - z_0} = (z - z_0)^k f(z)}, & z \in D'(z_0, \delta) \\ g'(z), & z = z_0 \end{cases}$, hence h is ana on $D(z_0, \delta)$.

As we know, $\lim_{z \to z_0} h(z) = h(z_0) \neq 0$. Thus, $f(z) = \frac{h(z)}{(z - z_0)^k}$ has a pole of order k at z_0. $\square$

Casorati-Weierstrass Theorem

If f has an **essential singularity** at z_0 and D is a **deleted neighborhood** of z_0, whre f is **analytic**, then the range $R := \{f(z) \mid z \in D\}$ **is dense** in $\mathbb{C}$.

Proof. Suppose not, then $\exists \omega \in \mathbb{C}$ and $\delta > 0$, s.t. **open** $D(\omega, \delta) \cap R = \emptyset$.

I.e., $\forall z \in D, |f(z) - \omega| \geq \delta \Rightarrow |\frac{1}{f(z) - \omega}| \leq \frac{1}{\delta} \forall\, z \in D$. Thus, $\frac{1}{f(z) - \omega}$ has a **removable singularity** at z_0.

Hence, $\exists$ ana g on $D' \cup \{z_0\}$, s.t. $g(z) = \frac{1}{f(z)-\omega} \Rightarrow f(z) = \omega + \frac{1}{g(z)}$ $\forall\, z \in D'$.

Hence, z_0 is a zero of $g(z)$ of finite order n or $g(z_0) \neq 0$. Thus, $f(z)$ has a pole of order $\geq n$ at z_0, so it is not an **essential singularity**. $\square$