10-9-25 (WEEK 6)

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USED DEFINITION

For a region $D \subseteq \mathbb{C}$, D is simply connected if $(\mathbb{C} \cup \{\infty\}) \setminus D$ is path connected

(We want to consider domain D s.t. $\int_\Gamma f(z)\,dz = 0$, f: analytic over D, $\Gamma \subseteq D$: simple closed curve

THEOREM

f: ana in a s.c. region D and $\Gamma \subseteq D$ simple closed polygonal path. Then, $\int_\Gamma f\,dz = 0$

Proof

Lemma from prev note $\Rightarrow R \subseteq D \Rightarrow \partial R \subseteq D$

As $\Gamma = \partial R + \Gamma'$, $\int_\Gamma f\,dz = \int_{\partial R} f\,dz + \int_{\Gamma'} f\,dz$

By rectangle thm, $R \subseteq D$, f: ana in $D \Rightarrow f$: ana on $R \Rightarrow \int_{\partial R} f\,dz = 0$

$\therefore$ By induction on $\operatorname{lev}(\Gamma)$, we get $\int_\Gamma f\,dz = 0$ □

THEOREM

f: ana on a s.c. region $D \Rightarrow \exists$ primitive F, $F' = f$

Proof

Fix $z_0 \in D$, define $F(z) = \int_\Gamma f(\zeta)\,d\zeta$, where Γ = any polygonal path from z to $z_0 \subseteq D$

- F is well-defined: Suppose Γ_1, Γ_2 satisfy the polygonal path condition

  Then, $\Gamma_1 - \Gamma_2 = \bigcup_{i=1}^{k} C_i$, C_i: simple closed polygonal curve $\subseteq D$

  $\Rightarrow \int_{\Gamma_1} f(\zeta)\,d\zeta - \int_{\Gamma_2} f(\zeta)\,d\zeta = \sum_{i=1}^{k} \int_{C_i} f(\zeta)\,d\zeta \overset{\text{by thm above}}{=} 0$ ✓

Now, let h be small enough s.t. $z+h \in D$

$\Rightarrow \dfrac{F(z+h) - F(z)}{h} = \dfrac{1}{h}\left[\int_{\Gamma_1} f(\zeta)\,d\zeta - \int_{\Gamma_2} f(\zeta)\,d\zeta\right] = \dfrac{1}{h}\int_{\Gamma_3} f(\zeta)\,d\zeta$, where

Γ_1: any poly path $z_0 \to z+h \subseteq D$

Γ_2: any poly path $z_0 \to z \subseteq D$

Choose Γ_2 first, then $\Gamma_1 = \Gamma_2 + \Gamma_3$, where Γ_3: any poly path $z \to z+h \subseteq D$

Then, $\lim_{h \to 0}\left|\dfrac{F(z+h) - F(z)}{h} - f(z)\right| = \lim_{h\to 0} \dfrac{1}{|h|}\left|\int_{\Gamma_3} [f(\zeta) - f(z)]\,d\zeta\right| = 0$ □

Γ_3: $z \to z+h \subseteq D$

THEOREM (CLOSED CURVE THEOREM)

Let f: ana on a s.c. region D. Then, $\forall$ simple closed curve $C \subseteq D$, $\int_C f(z)\,dz = 0$

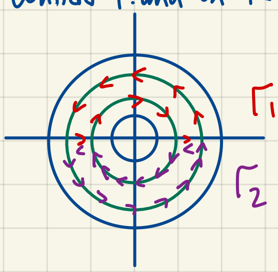
Proof

By Thm 3, $f = F'$ for some ana F

$\therefore$ For any $C: \gamma(t): [0,1] \to D$, $\int_C f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) = 0$ $\because$ closed $\Rightarrow \gamma(1) = \gamma(0)$ □

EXAMPLE

Consider f: ana on $1 < |z| < 4$



Claim: $\int_{C_2(0)} f(z)dz = \int_{C_3(0)} f(z)dz$

Proof

We have $\int_{C_3(0)} f(z)dz - \int_{C_2(0)} f(z)dz = \boxed{\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz} = 0$ by closed curve thm $\square$

THE PROBLEM WITH DEFINING LOG

$\log z := u(z) + iv(z) \Rightarrow \underbrace{z = e^{u(z)} e^{iv(z)}}_{|z| = |e^{u(z)}|}$, but $\theta = v(z) + 2\pi k, k \in \mathbb{Z}$ all are fine, so how do we fix a value so log is well-def?

DEFINITION

We say f is an analytic branch of $\log z$ in a domain D if:

(i) f is analytic

(ii) $e^{\log z} = z$, candidate: $f(z) = \overset{\log \text{ over } \mathbb{R}}{\log |z|} + i \operatorname{Arg} z \in [0, 2\pi)$

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