9-9-25 (WEEK 2) Shun/翔海(@shun4midx)

ANALYTIC FUNCTIONS

We write $z \in \mathbb{C}$, $z = x+iy$, $x, y \in \mathbb{R}$
Then, $f: \mathbb{C} \longrightarrow \mathbb{C}$ for $f(z) = u(z) + iv(z)$, $u, v: \mathbb{C} \longrightarrow \mathbb{R}$
$f(x,y) = u(x,y) + iv(x,y)$

PROPOSITION 1

If $f = u+iv$ is differentiable at z, then f_x, f_y exist and satisfy the Cauchy-Riemann Equation: $f_y = if_x$

Proof

By def, f is diff $\Rightarrow \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists.
(i) As $h \to 0$ along the real axis, limit $= \lim_{\xi\to 0} \frac{f(x+\xi,y)-f(x,y)}{\xi} = f_x$
(ii) As $h \to 0$ along the imaginary axis, limit $= \lim_{\xi\to 0} \frac{f(x,y+\xi)-f(x,y)}{\xi i} = \frac{f_y}{i}$ (since change in y as ξ means change in z as ξi)
$\therefore$ $f_y = if_x$ □

QUESTION: IF f_x, f_y EXIST AT A POINT z, AND $f_y = if_x$, DOES IT MEAN f IS DIFFERENTIABLE?

COUNTEREXAMPLE

$$f(z) = \begin{cases} \frac{xy(x+iy)}{x^2+y^2}, & z \neq 0 \quad (\text{i.e. } xy\cdot\frac{z}{|z|^2}) \\ 0, & z=0 \Leftrightarrow (x,y)=0 \end{cases}$$

We notice $f(z)=0$ on both x-axis and y-axis $\Rightarrow f_x(0) = f_y(0) = 0$
However, along $y=ax$ $(a\neq 0)$, we get: $f(x,ax) = \frac{a(1+ia)}{1+a^2}x \Rightarrow \lim_{x\to 0}\frac{f(x,ax)}{x+axi} = \frac{a}{1+a^2}$
$\therefore \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ DNE □

Note: If we require continuity, then the statement would have held true

PROPOSITION

Suppose that f_x, f_y exist in a nbd of z and are conti at z. If f satisfies the Cauchy-Riemann Equation, then f is differentiable

Proof

Say $z = x+iy$, $h = \xi + i\eta$, and $f(z) = u(z) + iv(z)$

$$\frac{f(z+h)-f(z)}{h} = \frac{[u(x+\xi, y+\eta) - u(x,y)]}{\xi+i\eta} + i\frac{[v(x+\xi,y+\eta) - v(x,y)]}{\xi+i\eta}$$

By MVT with "$-u(x+\xi,y)+u(x+\xi,y)$" and "$-v(x+\xi,y)+v(x+\xi,y)$",

$$= \frac{\eta}{\xi+i\eta}\left[\frac{u(x+\xi,y+\eta)-u(x+\xi,y)}{\eta} + i\frac{v(x+\xi,y+\eta)-v(x+\xi,y)}{\eta}\right] + \frac{\xi}{\xi+i\eta}\left[\frac{u(x+\xi,y)-u(x,y)}{\xi} + i\frac{v(x+\xi,y)-v(x,y)}{\xi}\right]$$

$$= \frac{\eta}{\xi+i\eta}[u_y(x+\xi, y+\theta_1\eta) + iv_y(x+\xi, y+\theta_2\eta)] + \frac{\xi}{\xi+i\eta}[u_x(x+\theta_3\xi,y) + iv_x(x+\theta_4\xi,y)]$$

We know $0<\theta_k<1$, $\left|\frac{\eta}{\xi+i\eta}\right| = \left|\frac{Re(h)}{h}\right| \leq 1$, $\left|\frac{\xi}{\xi+i\eta}\right| = \left|\frac{Im(h)}{h}\right| \leq 1$

Claim: $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = f_x(z)$

Proof

We know $f_x(z) = \frac{\xi+i\eta}{\xi+i\eta}f_x(z)$
By C-R eq, $f_x(z) = \frac{\xi}{\xi+i\eta}f_x(z) + \frac{\eta}{\xi+i\eta}f_y(z)$

As f_x, f_y are conti,

$$\frac{f(z+h)-f(z)}{h} - f_x(z) = \frac{\eta}{\xi+i\eta}[u_y(x+\xi, y+\theta_1\eta) - u_y(x,y) + i(v_y(x+\xi,y+\theta_2\eta) - v_y(x,y))] + \frac{\xi}{\xi+i\eta}[u_x(x+\theta_3\xi,y) - u_x(x,y) + i(v_x(x+\theta_4\xi,y) - v_x(x,y))]$$

$\longrightarrow 0$ as $h\to 0$, i.e. $\xi, \eta \longrightarrow 0$ □

$\therefore$ f is diffable and $f'(z) = f_x(z)$

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DEFINITION

f is analytic at z if f is diffable in a nbd of z

Similarly, f is analytic on a set S if f is diff at all pts of some open set containing S.

DEFINITION

Let S, T be open sets of $\mathbb{C}$, and f be a 1-1 function on S with $f(S)=T$

We say g is the inverse of f on T if $f(g(z))=z$ $\forall z \in T$.

We say g is the inverse of f at z if $\exists$ open nbd U of z, s.t. g is the inverse of f on U

Remark: g is also 1-1

PROPOSITION

Suppose that g is the inverse of f at z_0 and g is continuous there. If f is diffable at $g(z_0)$ and if $f'(g(z_0)) \neq 0$, then g is diffable at z_0 and $g'(z_0) = \frac{1}{f'(g(z_0))}$

Proof

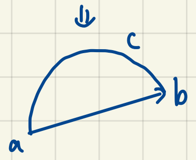
$\frac{g(z+h)-g(z)}{h} = \frac{g(z+h)-g(z)}{f(g(z+h))-f(g(z))} = \left(\frac{f(g(z+h))-f(g(z))}{g(z+h)-g(z)}\right)^{-1} = \frac{1}{f'(g(z_0))}$ $\square$

LINE INTEGRALS

Let $f(t)=u(t)+iv(t)$, $z(t)=x(t)+iy(t)$.

We consider curves as such:

$\overset{[t]}{\mathbb{R}^1} \xrightarrow{C} \mathbb{R}^n$



We say $\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

DEFINITION

(i) Let $z(t)=x(t)+iy(t)$, $a \leq t \leq b$. The curve determined by $z(t)$ is called piecewise differentiable and we set $\dot{z}(t)=x'(t)+iy'(t)$ if x, y are continuous on $[a,b]$ and are continuously differentiable on each subinterval $[a,x_1], [x_1,x_2], \dots, [x_{n-1},b]$ of some partition of $[a,b]$

(ii) The curve is said to be smooth, if $\dot{z}(t) \neq 0$ except at finitely many points.

In the following, we assume our curves are smooth.

DEFINITION

Say C is a smooth curve $\subseteq \mathbb{C}$, where $z(t)=x(t)+iy(t)$

Then, $\int_C f(z)\,dz = \int_a^b f(z(t))\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt$

DEFINITION

Let C_1, C_2 be smooth curves $\subseteq \mathbb{C}$, where $C_1: z(t)$, $a \leq t \leq b$ and $C_2: w(t)$, $c \leq t \leq d$.

C_1 and C_2 are smoothly equivalent if $\exists$ 1-1 C^1 mapping $\lambda: [c,d] \longrightarrow [a,b]$ s.t. $w(t)=z(\lambda(t))$

(By def, this is provably an equivalence relation)

We denote smoothly equivalent with $C_1 \underset{sm}{\sim} C_2$.

PROPOSITION

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If $C_1 \underset{sm}{\sim} C_2$, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

Proof

We set $f(z) = u(z) + iv(z)$, $z = x(t) + iy(t)$

Then, $\int_{C_1} f\,dz = \int_a^b f(z(t))\dot{z}(t)dt = \int_a^b [u(z)x'(t) - v(z)y'(t)]dt + i\int_a^b [u(z)y'(t) + v(z)x'(t)]dt$

With $\int_c^d u(z(\lambda(t)))x'(\lambda(t))\lambda'(t) = \int_a^b u(z(t))x'(t)dt$, we can prove the equation $\square$

APPLICATIONS OF CR-EQUATION

DEFINITION

$D \subseteq \mathbb{C}$ is called a region is open connected

Note, D: region $\Rightarrow x, y \in D$, $\exists$ a curve consisting of vertical and horizontal line segments that connect.

Proof

For $x \in D$, say $U_x := \{y \in D \mid x$ connects to y via vertical/horizontal line segments that connect$\}$

① "U_x is open":

For $y \in U_x \subseteq D$, D is open $\Rightarrow \exists$ open disk $B_\delta(y) \subseteq D$

$\because \forall a \in B_\delta(y)$, a can be connected to y by

$\therefore x \sim y \sim a$ ✓

② "$D \setminus U_x$ is open":

For $y \in D \setminus U_x$, D is open $\Rightarrow \exists$ open disk $B_\delta(y) \subseteq D \Rightarrow B_\delta(y) \cap U_x = \emptyset \Rightarrow B_\delta(y) \subseteq D \setminus U_x$ ✓

$\therefore$ ① + ② + D is connected $\Rightarrow D = U_x$

PROPOSITION

If $f = u + iv$ is analytic on a region D and u is constant, then f is constant

Proof

u is const $\Rightarrow u_x = u_y = 0$

By CR-eq, $v_x = v_y = 0$

As D is a region, thus $\forall a, b \in D$, $\exists$ connecting a and b

$\Rightarrow f(a) = f(b) \Rightarrow f$ is const $\square$

PROPOSITION

If f is analytic on a region D, and if $|f|$ is constant on D, then f is constant

Proof

$|f| = 0 \Rightarrow f = 0$ ✓

If $|f| \neq 0$, $|f| = C > 0 \Rightarrow u^2 + v^2 = C^2$

$\Rightarrow 2uu_x + 2vv_x = 0$; $2uu_y + 2vv_y = 0$

By CR-eq, $uu_x - vu_y = 0$; $uu_y + vu_x = 0 \Rightarrow (u^2+v^2)u_x = 0 \Rightarrow u_x = 0$. Similarly, we get $u_y = 0$.

As + prop above, thus this prop is true. $\square$