9-4-25 (WEEK 1)

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THEOREM

Given power series $P(z)=\sum c_k z^k$, radius of convergence R, then $P'(z)$ exists on $|z|<R$, $P'(z):=\sum k c_k z^{k-1}$

Proof

For $0<R<\infty$,

Let $|z|=R-\delta$, $R>\delta>0$. WLOG, consider $|h|<\frac{\delta}{2}$, and consider $\frac{P(z+h)-P(z)}{h}$ vs $\sum k c_k z^{k-1}$

Then, $\frac{P(z+h)-P(z)}{h} = \frac{1}{h}\sum c_k((z+h)^k - z^k) = \sum_{k=1}^{\infty} k c_k z^{k-1} + \sum_{k=2}^{\infty} c_k b_k$, $b_k = \sum_{p=2}^{k}\binom{k}{p} h^{p-1} z^{k-p}$

($\frac{1}{h}\sum\binom{k}{p}h^{k-p}z^p$; $\sum_{k=2}^{\infty} c_k b_k$ = ☆)

If $|z|=0$, $b_k = h^{k-1} \Rightarrow$ ☆ $= \sum c_k h^{k-1} < \infty$, ☆ $\to 0$ as $h \to 0$. ✓

If $|z| \neq 0$, $\binom{k}{p} := \binom{k}{p-2} \cdot \frac{(k-p+2)(k-p+1)}{p(p-1)} \leq \binom{k}{p-2} k^2$

Then, $|b_k| \leq \frac{|h|}{|z|^2} k^2 \sum_{p=2}^{k} \binom{k}{p-2} |h|^{p-2} |z|^{k-(p-2)}$, note $h \neq 0$

$\leq \frac{|h|}{|z|^2} k^2 \sum_{j=0}^{k} \binom{k}{j} |h|^j |z|^{k-j}$, $j = p-2$

$= \frac{|h|}{|z|^2} k^2 (|z|+|h|)^k$

$\leq \frac{|h|}{|z|^2} k^2 (R - \frac{\delta}{2})^k$

Hence, $|\sum_{k=2}^{\infty} c_k b_k| \leq \sum_{k=2}^{\infty} |c_k||b_k| = \frac{|h|}{|z|^2} \sum_{k=2}^{\infty} k^2 |c_k| (R-\frac{\delta}{2})^k \longrightarrow 0$ ✓

($\sum_{k=2}^{\infty} k^2 |c_k| (R-\frac{\delta}{2})^k < \infty$)

Remaining case is simple for $R=\infty$.

EXAMPLE

$\sum_{k=0}^{\infty} \frac{z^k}{k!}$, $R=\infty$

$\downarrow \frac{1}{dz}$

$\sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!}$

$\because f' = f$

$\therefore \sum_{k=0}^{\infty} \frac{z^k}{k!} = ce^z$

COROLLARY

Power series are smooth in their domain of conv.

COROLLARY

$f(z) = \sum_{k=0}^{\infty} c_k z^k$, $R>0 \Rightarrow c_k = \frac{f^{(k)}(0)}{k!}$

THEOREM (UNIQUENESS)

If $\exists \{z_n\}_n \longrightarrow 0$ and $\sum c_k z_n^k = 0$, then $c_k = 0$ $\forall k$

Proof

$P(0) := \lim_{n\to\infty} P(z_n) = 0 \Rightarrow c_0 = 0$

Consider $g(z) := \frac{f(z)}{z}$, with the same radius of convegence as $f(z)$

Then, $g(0) = \lim_{n\to\infty} \frac{f(z_n)}{z_n} = 0 \Rightarrow c_1 = 0$

$\therefore$ By induction on n, $c_n = 0$ $\forall n$. □

COROLLARY

If $\sum a_k z^k$ and $\sum b_k z^k$ agree on $\{z_n\}_n$ as $n\to\infty$, then $a_k = b_k$ $\forall k$

Proof

Simply consider $\sum (a_k - b_k) z^k = 0$.