

REMARK

We only have open mapping thm because extremum is not an interior pt.

SCHWARTZ'S LEMMA

THEOREM (SCHWARTZ'S LEMMA)

Suppose that f is analytic in an open unit disc D with $|f| \leq 1$ ($f: \text{unit circles} \rightarrow \text{unit circles}$) and $f(0)=0$

Then, (i) $|f(z)| \leq |z|$

(ii) $|f'(0)| \leq 1$

with equality in either of the above iff $f(z) = e^{i\theta} z$

Proof

$$\text{Define } g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

$g(z)$ is ana on D since $f(z)$ is ana on D .

Consider $z \in C_r(0)$, $0 < r < 1$.

$$\text{Then, } |g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

By max modulus thm, $\forall z \in \overline{D(0, r)}$, $|g(z)| \leq \frac{1}{r}$

As $r \rightarrow 1$, then $|g(z)| \leq 1 \quad \forall z \in D$

By def of $g(z)$, $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$ has either equality hold, when g is const and $|g|=1$ on D . $\therefore g = e^{i\theta}$ \square

EXAMPLE (Removing $f(0)=0$ constraint)

$$\text{Define } B_a(z) = \frac{z-a}{1-\bar{a}z}, \quad |a| < 1 \rightarrow D$$

Then, (1) $B_a(a) = 0$

(2) $B_a(z)$ is ana on D , so $(B_a(z))^{-1}$ is ana on D . It is also conti on \bar{D} .

(3) $|B_a(z)|^2|_{z=1} = 1$, so by max modulus thm, $|B_a(z)| \leq 1$ on D .

\therefore We can use B_a for variations of Schwartz's lemma

EXAMPLE

Sup f : ana on D , $|f(z)| \leq 1 \quad \forall z \in D$ and $f(\frac{1}{2}) = 0$. Estimate $|f(\frac{3}{4})|$.

$$\text{Consider } B_{\frac{1}{2}}(z) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z} \quad (B_a(z))^{-1}$$

$$\text{We define } g(z) = \begin{cases} \frac{f(z)}{B_{\frac{1}{2}}(z)}, & z \neq \frac{1}{2} \\ \frac{3}{4} f'(\frac{1}{2}), & z = \frac{1}{2} \end{cases}$$

Notice, $|B_{\frac{1}{2}}(z)| \leq 1$ on D , $|B_{\frac{1}{2}}(z)| = 1$ on $C_1(0)$, and $B_{\frac{1}{2}}(z)$ conti on D .

$\therefore z \rightarrow 1, |B_{\frac{1}{2}}(z)| \rightarrow 1$

$$\begin{aligned} |g(z)| \leq 1/|B_{\frac{1}{2}}(z)| &\Rightarrow |g(z)| \leq 1 \text{ on } D. \\ &\Rightarrow |f(z)| \leq |B_{\frac{1}{2}}(z)|, \quad \frac{3}{4}|f'(\frac{1}{2})| \leq 1 \end{aligned}$$

$$\text{So, } |f(\frac{3}{4})| \leq |B_{\frac{1}{2}}(\frac{3}{4})| = \frac{2}{5}$$

EXAMPLE

Say f is ana on D , $|f(z)| \leq 1$ on D . We claim: $|f'(\frac{1}{3})| \geq \max$ when $|f(\frac{1}{3})| = 0$ is at its lowest "is"?
 "your rate of improvement is max when your score is at its lowest" \geq ?
 min modulus for \mathbb{C} .

Proof

Assume that $f(\frac{1}{3}) \neq 0$.

$g(z) := \frac{f(z) - f(\frac{1}{3})}{1 - \overline{f(\frac{1}{3})}f(z)} \Rightarrow g(z) = B_{f(\frac{1}{3})}(f(z))$. Note, g is bounded by 1, $g(\frac{1}{3}) = 0$. (Note it is ana too)

$\therefore g'(z)|_{z=\frac{1}{3}} = \frac{f'(\frac{1}{3})}{1 - |f(\frac{1}{3})|^2} \Rightarrow |g'(\frac{1}{3})| > |f'(\frac{1}{3})|$ ($\because |f(\frac{1}{3})| > 0$. Otherwise, then $|g'(\frac{1}{3})|$ can be larger, take $f := g - x$)

$\therefore |f'(\frac{1}{3})|$ is max for $B_{\frac{1}{3}}(z)$.

PROPOSITION

Say f is entire. If $|f(z)| < \frac{1}{|Im z|} \forall z$, then $f \equiv 0$

Proof

Define $g(z) = (z^2 - R^2)f(z)$, for some $R \in \mathbb{R}_{>0}$ (sufficiently large, e.g. $R \geq 1, R \rightarrow \infty$)

When $z \in \mathbb{C}_{\neq 0}$, $|z - R|z + R| \leq 2R|Im(z)|$

$\therefore |g(z)| \leq \frac{2R}{|Im(z)|^2} \leq 2R$ when $z \in \mathbb{C}_{\neq 0}$

By max modulus thm, $|g(z)| \leq 2R \forall z \in D(0, R)$

$\Rightarrow |f(z)| \leq \frac{2R}{|z|^2 - R^2} \forall z \in D(0, R)$

As $R \rightarrow \infty, |f(z)| \rightarrow 0$

$\therefore f(z) = 0$

MORERA'S THEOREM

THEOREM (MORERA'S THEOREM: CONVERSE OF RECTANGLE THEOREM)

Let f be continuous on an open set $D \subseteq \mathbb{C}$, and Γ be the boundary of a closed rectangle $R \subseteq D$.

If $\int_{\Gamma} f dz = 0 \forall \Gamma$ in $R \subseteq D$, then f is analytic in D .

Proof

Say $z_0 \in D$, D : open.

Then, $\exists \varepsilon > 0$, s.t. $D(z_0, \varepsilon) \subseteq D$.

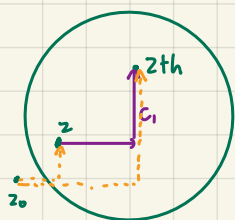
Define $F(z) := \int_C f(z) dz \forall z \in D(z_0, \varepsilon)$, where $C: z_0 \rightarrow z_0 + Re(z - z_0) \rightarrow z$

For $z \in D(z_0, \varepsilon)$ and h small enough s.t. $z+h \in D(z_0, \varepsilon)$

Then,

$$\lim_{h \rightarrow 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \rightarrow 0} \int_C f(\omega) d\omega = f(z) \quad \square$$

f : cont.



EXAMPLE

Using $f(z) = \int_0^\infty \frac{e^{zt}}{t+1} dt$,

Claim: f is analytic $\forall z \in \{w \in \mathbb{C} \mid \operatorname{Re}(w) < 0\}$

Proof

We know for $z = x+iy$, $x < 0$, $|e^{zt}| = e^{xt}$

Here, $\int_\Gamma \int_0^\infty \frac{|e^{zt}|}{t+1} dt dz < \int_\Gamma -\frac{1}{x} dz < \infty$ (ok for Fubini)

By Fubini's Thm, $\int_\Gamma \int_0^\infty \frac{e^{zt}}{t+1} dt dz = \int_0^\infty \int_\Gamma \frac{e^{zt}}{t+1} dz dt = 0$ (By rectangle thm since $\frac{e^{zt}}{t+1}$ is ana)

\therefore By Morera's Thm, $f(z)$ is analytic on $\{w \in \mathbb{C} \mid \operatorname{Re}(w) < 0\}$. \square

DEFINITION

Let $\{f_n\}$ and f be defined on an open set D . We say that f_n converges uniformly on compacta if $f_n \rightarrow f$ uniformly on every compact subset $K \subset D$.

THEOREM

Let D be an open set in \mathbb{C} and $\{f_n\}$ be a sequence of ana functions s.t. $f_n \rightarrow f$ unif on cpta. Then, f is also ana in D .

Proof

$\therefore f_n$ is conti, $\forall K \subset D$: cpta set we have $f_n \rightarrow f$ unif on K

$\therefore f$ is conti on $K \forall K$, i.e. f is conti on D

We hope " $\int_\Gamma f dz = 0$ ", for Γ : boundary of a closed rectangle $R \subset D$

Hence, $\int_\Gamma f dz = \int_\Gamma \lim_{n \rightarrow \infty} f_n dz$

|| (f_n conti, $f_n \rightarrow f$ unif on R)

$\lim_{n \rightarrow \infty} (\int_\Gamma f_n dz)$

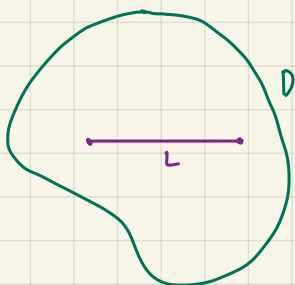
|| (Rectangle thm $\therefore f_n$ ana)

0

\therefore By Morera's Thm, f is conti: \square

THEOREM

f is continuous on an open set $D \subset \mathbb{C}$ and analytic except on a line segment in D . Then, f is analytic throughout D .



Proof

In some cpta nbd K of each pt z_0 , f : unif limit of conti f_n .

$\therefore f$: conti in D

Also, $\forall R \subset K$, $\int_R f = \int_R \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int_R f_n = 0$ since $f_n \rightarrow f$ unif on Γ .

\therefore By Morera's Thm, f : analytic in D . \square