12-4-25 (WEEK 14) (I'm so fucking stressed but we don't talk about that... hope I'm gonna be okay...) Shun/翔海(@shun4midx)

DEFINITION

Suppose that f is analytic in a disc D and $z_0 \in \partial D$. Then, f is said to be regular at z_0 if f can be continued analytically to a region D_1 with $z \in D_1$. Otherwise, f is said to be a singularity at z_0.

CAUTION

$z \in \partial D$ is a singularity does NOT depend on if f is continuous at z

Example 1: $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ for $|z|<1$, but $f(z)$ is cont. on $|z|=1$

Example 2: $f(z) = \sum_{i=0}^{\infty} z^i$ for $|z|<1$, $\lim_{z \to e^{i0}} f(z)$ DNE but $\frac{1}{1-z}$ is defined $\forall z \in \mathbb{C} \setminus \{1\} \Rightarrow \forall e^{i\theta} \neq 1$, $e^{i\theta}$: regular pt of f

THEOREM

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a radius of conv $R<\infty$ and $\boxed{a_n \geq 0 \ \forall n}$. Then, $f(z)$ becomes a singularity at some $z=R$

Proof

By Thm last time, f has a singularity at some $|z|=R$

Let $z = Re^{i\theta}$ be such a singularity.

Consider f at $z = \rho e^{i\theta}$, $0<\rho<R$

radius of conv $R-\rho$

By Taylor expansion at $z = \rho e^{i\theta}$, $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\rho e^{i\theta})}{n!} (z-\rho e^{i\theta})^n$ in $|z-\rho e^{i\theta}| < R-\rho$ as $Re^{i\theta}$ is a singularity of f

Consider $\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(\rho e^{i\theta})}{n!} \right| |z-\rho e^{i\theta}|^n$, where $f^{(n)}(\rho e^{i\theta}) = \sum_{i=n}^{\infty} \underbrace{n(n-1)\cdots(a_i)}_{\geq 0} (\rho e^{i\theta})^{i-n}$

$\therefore |f^{(n)}(\rho e^{i\theta})| \leq f^{(n)}(\rho)$

$\therefore$ At $z=\rho$, $f(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(\rho)}{n!} (z-\rho)^n$, moreover, radius of conv at $z=\rho e^{i\theta} \geq$ radius of conv at $z=\rho$

$\therefore z=R$ is a singular point of f.

DEFINITION

If $\sum_{n=0}^{\infty} a_n z^n$ has a singularity at every point on its circle of convergence, then the circle is called a natural boundary

(e.g. $\sum_{i=0}^{\infty} z^{i!}$, $|z|=1$: natural boundary)

THEOREM

If $a_k \neq 0$, $\sum a_k z^k \leadsto \lim_{k\to\infty} \frac{k+1}{k} = 1$, so "$>1$" means the spacing between nonzero terms is large.

Let $f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$ with $c_k \neq 0 \ \forall k$. Suppose $\lim_{k\to\infty} \frac{n_{k+1}}{n_k} > 1$. Then, the circle of convergence of f is a natural boundary

Proof

As the assumption is indep of c_k ($c_k \neq 0$), we can replace z by Rz and assume the radius of convegence $=1$

If $f(z)$ is singular at some $z=e^{i\theta}$, replace z with $ze^{i\theta}$ so $f(z)$ is singular at $z=1$

$\therefore$ It suffices to prove, $\forall \theta$, $f(z)$ has a singularity at $z=1$

Consider the map $w \xrightarrow{n} w$ $(n>1)$, it fixes $|w|=1$, $|w|<1$, $|w|>1$ to their corresponding (in)equality with 1

For $h: w \xrightarrow{\ \ } \frac{w^n + w^{n+1}}{2}$

Define $g(z) := f \circ h$

As $|h(w)|<1 \ \forall |w|=1$ but $w \neq 1$, thus g is regular at $|w|=1$ but $w \neq 1$.

If we can show that the radius of conv of g is 1, by Thm, we know that $w=1$ is a singular point of $g \Rightarrow f$: singular at $h(1)=1$

Claim: Conv radius of $g=1$ for some n as m, exponent of w.

Proof

$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1+\delta$, $\delta>0 \Rightarrow$ Choose m s.t. $\frac{m+1}{m} < 1+\delta$, as $k>>0$, set $n=m$. We can even assume $n_0 >> 0$.

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Then, $g(\omega)=f\left(\frac{\omega^m+\omega^{m+1}}{2}\right)=c_0\left(\frac{\omega^m+\omega^{m+1}}{2}\right)^{n_0}+c_1\left(\frac{\omega^m+\omega^{m+1}}{2}\right)^{n_1}+\dots$

$$=\frac{c_0\omega^{mn_0}}{2^{n_0}}+\frac{c_0n_0\omega^{mn_0+1}}{2^{n_0}}+\dots+\frac{c_0}{2^{n_0}}\omega^{mn_0+n_0}+\frac{c_1}{2^{n_1}}\omega^{mn_1}+\frac{c_1n_1}{2^{n_1}}\omega^{mn_1+1}+\dots+\frac{c_1}{2^{n_1}}\omega^{mn_1+n_1}+\dots$$

If conv radius of $g=r>1$, then

$$\left|\frac{c_0\omega^{mn_0}}{2^{n_0}}\right|+\left|\frac{c_0n_0\omega^{mn_0+1}}{2^{n_0}}\right|+\dots+\left|\frac{c_0}{2^{n_0}}\omega^{mn_0+n_0}\right|+\left|\frac{c_1}{2^{n_1}}\omega^{mn_1}\right|+\left|\frac{c_1n_1}{2^{n_1}}\omega^{mn_1+1}\right|+\dots+\left|\frac{c_1}{2^{n_1}}\omega^{mn_1+n_1}\right|+\dots \quad (\star)$$

conv $\forall |\omega|<1$. In particular, $\exists\, 1<\alpha<r$, s.t. $|\omega|=\alpha$, s.t $(\star)$ conv and $(\star)=f\left(\frac{\alpha^m+\alpha^{m+1}}{2}\right)$ $\times$ $\because$ f has radius of conv 1

$\therefore$ Radius of conv of $g=1$ $\square$