CompAna Notes

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9-2-25 (WEEK 1) Shun/翔海 (@shun4midx)

Can any two-variable function $f(x,y)$ be re-written into diffable $F(z)$ with $z=x+iy$? No. ($f(x,y)=x$)

POWER SERIES

DEFINITION OF ANALYTIC POLYNOMIAL

If $P(x,y)=\alpha_0+\alpha_1(x+iy)+\dots+\alpha_N(x+iy)^N=\sum_{k=0}^{N}\alpha_k z^k$ for some α_k's $\in\mathbb{C}$, then it is an analytic polynomial

EXAMPLE

$x^2-y^2+2xyi=(x+iy)^2$ is analytic

However, x^2+y^2-2xyi is not (when we set $x^2+y^2-2xyi=\sum\alpha_k(x+iy)^k$, there is a contradiction)

DEFINITION OF PARTIAL DERIVATIVE

Let $f(x,y)=u(x,y)+iv(x,y)$, $u,v\in\mathbb{R}$.

If it exists, then $\begin{cases} f_x(x,y)=u_x(x,y)+iv_x(x,y) \\ f_y(x,y)=u_y(x,y)+iv_y(x,y) \end{cases}$

PROPOSITION

A polynomial $P(x,y)$ is analytic $\Leftrightarrow$ $\overbrace{P_y=iP_x}^{\text{differentiable (C-R eq)}}$

Proof

"$\Rightarrow$": $\exists\alpha_k$'s $\in\mathbb{C}$, $N\in\mathbb{N}$, s.t. $P(x,y)=\sum_{k=0}^{N}\alpha_k(x+iy)^k$

$\Rightarrow P_y=\sum_{k=1}^{N}k\alpha_k(x+iy)^{k-1}i$, $P_x=\sum_{k=1}^{N}k\alpha_k(x+iy)^{k-1}$

$\Rightarrow \therefore P_y=iP_x$ ✓

"$\Leftarrow$": With $Q^k(x,y)=C_0x^k+C_1x^{k-1}y+\dots+C_ky^k$, we can rewrite $P(x,y)=\sum_{k=0}^{N}Q^k(x,y)$

Notice, $Q^k_y=iQ^k_x$ $\forall k$.

We know $Q^k=\sum_{p=0}^{k}C_px^{k-p}y^p$

$\therefore Q^k_y=\underline{\sum_{p=1}^{k}pC_px^{k-p}y^{p-1}=\sum_{p=0}^{k-1}(k-p)C_px^{k-p-1}y^p\,i}=iQ^k_x$

In other words, $\sum_{p=1}^{k}pC_px^{k-p}y^{p-1}=\sum_{p=1}^{k}(k-p+1)C_{p-1}x^{k-p}y^{p}i$

- $p=1$: $ikC_0=C_1\Rightarrow C_1=\binom{k}{1}C_0i$
- $p=2$: $2C_2=(k-1)C_1i\Rightarrow C_2=i^2\frac{k(k-1)}{2}C_0$
- $pC_p=(k-p+1)C_{p-1}i\Rightarrow C_p=i^p\binom{k}{p}C_0$

$\therefore Q^k=\sum_{p=0}^{k}i^p\binom{k}{p}C_0x^{k-p}y^p=(x+iy)^k$ $\forall k$

$\therefore P$ is analytic. □

REMARK

Usually we don't write "$P_y=iP_x$", rather:

$\begin{cases} P_y=u_y+iv_y \\ P_x=u_x+iv_x \\ P_xi=-v_x+u_xi \end{cases}\Rightarrow\begin{cases} -v_x=u_y \\ u_x=v_y \end{cases}$

REMARK

A nonconstant analytical polynomial can't be real (since we require $P_y=iP_x$)

DEFINITION

Consider f, a complex-valued function, defined on the neighborhood of $z=z_0$.

We say f is differentiable at $z=z_0$ if $\lim_{h\to0}\frac{f(z_0+h)-f(z_0)}{h}$ exists, where it is denoted as $f'(z)$

(Note: We must consider $|h|\to0$ $\forall h\in\mathbb{C}$)

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EXAMPLE

$f(z)=\bar{z}$

$\frac{f(z+h)-f(z)}{h}=\frac{\bar{h}}{h} \xrightarrow{\mathbb{R}} 1$, $\xrightarrow{i\mathbb{R}} -1$ as $h\to 0$.

$\therefore \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ DNE, i.e. f is not diff

PROPOSITION

If f,g diff at $z=0$, $h:=f+g \Rightarrow h'=f'+g'$. Product and quotient $(g(z_0)\neq 0)$ rules also hold.

PROPOSITION

$P(z)=\sum_{k=0}^{N}\alpha_k z^k$ is diff on $\mathbb{C}$, in fact: $P'(z)=\sum_{k=1}^{N}k\alpha_k z^{k-1}$

DEFINITION OF POWER SERIES

A power series is an infinite series in the form $\sum_{k=0}^{\infty}\alpha_k z^k$

LIMSUP

"$\overline{\lim_{n\to\infty}} a_n=L$" $\Leftrightarrow \forall\varepsilon>0, \exists N>0$, s.t. $n\geq N\Rightarrow |\sup_{k\geq n}a_k-L|<\varepsilon$

$\Updownarrow$

$L-\varepsilon<\sup_{k\geq n}a_k<L+\varepsilon$

$\forall\varepsilon>0, \exists N$, s.t. $n\geq N\Rightarrow a_k<L+\varepsilon$

$\forall\varepsilon>0, \forall N\ \exists k>N$, s.t. $a_k>L-\varepsilon$

THEOREM

Given the power series $\sum_{k=0}^{\infty}c_k z^k=P(z)$, define $L:=\overline{\lim_{k\to\infty}}|c_k|^{\frac{1}{k}}$, then we have:

(1) $L=0\Rightarrow P(z)$ converges $\forall z\in\mathbb{C}$

(2) $L=\infty\Rightarrow P(z)$ converges only at $z=0$

(3) $0<L<\infty\Rightarrow P(z)$ converges on $|z|<\frac{1}{L}$ and diverges on $|z|>\frac{1}{L}$

Proof

(i) Given any $z\in\mathbb{C}$, $\overline{\lim_{n\to\infty}}|c_k|^{\frac{1}{k}}z=0$

$\therefore$ Take $\varepsilon=\frac{1}{2}$, $\exists N$ s.t. $k>N\Rightarrow |c_k|^{\frac{1}{k}}|z|<\frac{1}{2}\Rightarrow |c_kz^k|<(\frac{1}{2})^k$

$\therefore \sum|c_kz^k|<\sum(\frac{1}{2})^k=1$ ✓

(ii) Consider small $|z|$, $\forall N\in\mathbb{N}, \exists k>N$, s.t. $|c_k|^{\frac{1}{k}}>\frac{1}{|z|}$ $\therefore |c_kz^k|>1$

$\therefore P(z)$ does not converge at z ✓

(iii) Take $R=\frac{1}{L}$, $|z|=R(1-\delta)$, $1>\delta>0$ when $|z|<\frac{1}{L}$

We know $\forall\varepsilon>0, \exists N\in\mathbb{N}$, $n>N$, s.t. $|z|(L-\varepsilon)<\sup_{k\geq n}|c_k|^{\frac{1}{k}}|z|\leq(\varepsilon+L)|z|=1+\varepsilon R(1-\delta)-\delta<1-\frac{\delta}{2}$

$\therefore$ It is abs conv

If $|z|>R$, $\overline{\lim}|c_k|^{\frac{1}{k}}|z|>1\Rightarrow$ for inf values of k, $|c_kz^k|>1\Rightarrow\sum c_kz^k$ div

REMARK

Let $\frac{1}{L}=R$ be the radius of convergence

Then, $\sum c_kz^k$ conv uni for $|z|<R-\delta$

- $\sum|c_kz^k|\leq\sum|c_k|(R-\delta)^k<\infty$

$\Rightarrow$ On $B(0,R-\delta)$, $\sum c_kz^k$ is conti $\forall\delta>0$

EXAMPLE (evaluating at R)

$\sum_{n=0}^{\infty}nz^n$

We know $\overline{\lim}\, n^{\frac{1}{n}}=1\Rightarrow R=1$

When $|z|=1$, $|nz^n|=n\Rightarrow$ diverge

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$\sum \frac{z^n}{n^2} \ni R=1$

When $z=1$, it conv, similarly $|z|=1$ too.

$\sum \frac{z^n}{n}$, $R=1$

When $|z|=1$, $z\neq 1$ conv

$\sum \frac{z^n}{n!}$ conv $\forall z\in\mathbb{C}$ by ratio test

CAUCHY PRODUCT

Given $P_1(z)=\sum a_k z^k$, $R=R_1$; $P_2(z)=\sum b_k z^k$, $R=R_2$. then $P_1P_2=\sum c_k z^k$ where $c_k=\sum_{p=0}^{k} a_p b_{k-p}$

Then, $R_3 \geq \min(R_1, R_2)$

DIFFERENTIATION

THEOREM

Given $P(z)=\sum_{k=0}^{\infty} c_k z^k$, $R>0$, we know $\overline{\lim}|c_k|^{\frac{1}{k}}=R$ and $\overline{\lim}|kc_k|^{\frac{1}{k}}=R$ since $\overline{\lim}|k|^{\frac{1}{k}}=1$

Then, $P'(z)=\sum_{k=1}^{\infty} kc_k z^{k-1}$ with radius of convergence R

Proof

- $0<R<\infty$: Take $|z|=R-\delta$, $R\geq\delta>0$

  Then, $\frac{P(z+h)-P(z)}{h}=\frac{1}{h}\sum_{k=1}^{\infty} c_k[(z+h)^k-z^k]=\frac{1}{h}\left[\sum_{k=1}^{\infty} c_k k z^{k-1}h+h\sum_{k=2}^{\infty} c_k b_k\right]=\sum_{k=1}^{\infty} kc_k z^{k-1}+\sum_{k=2}^{\infty} c_k b_k$

Proof continued next time!

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THEOREM

Given power series $P(z)=\sum c_k z^k$, radius of convergence R, then $P'(z)$ exists on $|z|<R$, $P'(z)=\sum k c_k z^{k-1}$

Proof

For $0<R<\infty$,

Let $|z|=R-\delta$, $R\geq\delta>0$. WLOG, consider $|h|<\frac{\delta}{2}$, and consider $\frac{P(z+h)-P(z)}{h}$ vs $\sum k c_k z^{k-1}$

Then, $\frac{P(z+h)-P(z)}{h}=\frac{1}{h}\sum c_k \underbrace{((z+h)^k-z^k)}_{\sum\binom{k}{p}h^{k-p}z^p}=\sum_{k=1}^{\infty}kc_kz^{k-1}+\underbrace{\sum_{k=2}^{\infty}c_kb_k}_{\star}$, $b_k=\sum_{p=2}^{k}\binom{k}{p}h^{p-1}z^{k-p}$

If $|z|=0$, $b_k=h^{k-1}\Rightarrow \star=\sum c_k h^{k-1}<\infty$, $\star\to 0$ as $h\mapsto 0$. ✓

If $|z|\neq 0$, $\binom{k}{p}=\binom{k}{p-2}\cdot\frac{(k-p+2)(k-p+1)}{p(p-1)}\leq\binom{k}{p-2}k^2$

$$\begin{aligned}\text{Then, } |b_k| &\leq \frac{|h|}{|z|^2}k^2\sum_{p=2}^{k}\binom{k}{p-2}|h|^{p-2}|z|^{k-(p-2)}, \text{ note } h\in\mathbb{C}\\ &\leq \frac{|h|}{|z|^2}k^2\sum_{j=0}^{k}\binom{k}{j}|h|^j|z|^{k-j}, \quad j=p-2\\ &= \frac{|h|}{|z|^2}k^2(|z|+|h|)^k\\ &\leq \frac{|h|}{|z|^2}k^2\left(R-\frac{\delta}{2}\right)^k\end{aligned}$$

Hence, $\left|\sum_{k=2}^{\infty}c_kb_k\right|\leq\sum_{k=2}^{\infty}|c_k||b_k|=\frac{|h|}{|z|^2}\underbrace{\sum_{k=2}^{\infty}k^2|c_k|\left(R-\frac{\delta}{2}\right)^k}_{<\infty}\longrightarrow 0$ ✓

Remaining case is simple for $R=\infty$.

EXAMPLE

$\sum_{k=0}^{\infty}\frac{z^k}{k!}$, $R=\infty$

$\downarrow \frac{d}{dz}$

$\sum_{k=1}^{\infty}\frac{z^{k-1}}{(k-1)!}=\sum_{k=0}^{\infty}\frac{z^k}{k!}$

$\because f'=f$

$\therefore \sum_{k=0}^{\infty}\frac{z^k}{k!}=ce^z$

COROLLARY

Power series are smooth in their domain of conv.

COROLLARY

$f(z)=\sum_{k=0}^{\infty}c_kz^k$, $R>0\Rightarrow c_k=\frac{f^{(k)}(0)}{k!}$

THEOREM (UNIQUENESS)

If $\exists\{z_n\}_n\longrightarrow 0$ and $\sum c_kz_n^k=0$, then $c_k=0\ \forall k$

Proof

$P(0)=\lim_{n\to\infty}P(z_n)=0\Rightarrow c_0=0$

Consider $g(z)=\frac{f(z)}{z}$, with the same radius of convergence as $f(z)$

Then, $g(0)=\lim_{n\to\infty}\frac{f(z_n)}{z_n}=0\Rightarrow c_1=0$

$\therefore$ By induction on n, $c_n=0\ \forall n$. □

COROLLARY

If $\sum a_kz^k$ and $\sum b_kz^k$ agree on $\{z_n\}_n$ as $n\to\infty$, then $a_k=b_k\ \forall k$

Proof

Simply consider $\sum(a_k-b_k)z^k=0$.

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ANALYTIC FUNCTIONS

We write $z \in \mathbb{C}$, $z = x+iy$, $x,y \in \mathbb{R}$

Then, $f: \mathbb{C} \longrightarrow \mathbb{C}$ for $\underset{f(x,y)=u(x,y)+iv(x,y)}{f(z)=u(z)+iv(z)}$, $u,v: \mathbb{C} \longrightarrow \mathbb{R}$

PROPOSITION 1

If $f = u+iv$ is differentiable at z, then f_x, f_y exist and satisfy the Cauchy-Riemann Equation: $f_y = if_x$

Proof

By def, f is diff $\Rightarrow \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists.

(i) As $h \to 0$ along the real axis, limit $= \lim_{\xi \to 0} \frac{f(x+\xi,y)-f(x,y)}{\xi} = f_x$

(ii) As $h \to 0$ along the imaginary axis, limit $= \lim_{\xi\to 0} \frac{f(x,y+\xi)-f(x,y)}{\xi i} = \frac{f_y}{i}$ (since change in y as ξ means change in z as ξi)

$\therefore$ $f_y = if_x$ □

QUESTION: IF f_x, f_y EXIST AT A POINT z, AND $f_y = if_x$, DOES IT MEAN f IS DIFFERENTIABLE?

COUNTEREXAMPLE

$$f(z) = \begin{cases} \frac{xy(x+iy)}{x^2+y^2}, & z \neq 0 \quad (\text{i.e. } xy \cdot \frac{z}{|z|^2}) \\ 0, & z=0 \Leftrightarrow (x,y)=0 \end{cases}$$

We notice $f(z)=0$ on both x-axis and y-axis $\Rightarrow f_x(0) = f_y(0) = 0$

However, along $y=ax$ $(a\neq 0)$, we get: $f(x,ax) = \frac{a(1+ia)}{1+a^2}x \Rightarrow \lim_{x\to 0} \frac{f(x,ax)}{x+axi} = \frac{a}{1+a^2}$

$\therefore \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ DNE □

Note: If we require continuity, then the statement would have held true

PROPOSITION

Suppose that f_x, f_y exist in a nbd of z and are conti at z. If f satisfies the Cauchy-Riemann Equation, then f is differentiable

Proof

Say $z = x+iy$, $h = \xi + i\eta$, and $f(z) = u(z)+iv(z)$

$$\frac{f(z+h)-f(z)}{h} = \frac{[u(x+\xi,y+\eta)-u(x,y)]}{\xi+i\eta} + i\frac{[v(x+\xi,y+\eta)-v(x,y)]}{\xi+i\eta}$$

By MVT with "$-u(x+\xi,y)+u(x+\xi,y)$" and "$-v(x+\xi,y)+v(x+\xi,y)$",

$$= \frac{\eta}{\xi+i\eta}\left[\frac{u(x+\xi,y+\eta)-u(x+\xi,y)}{\eta} + i\frac{v(x+\xi,y+\eta)-v(x+\xi,y)}{\eta}\right] + \frac{\xi}{\xi+i\eta}\left[\frac{u(x+\xi,y)-u(x,y)}{\xi} + i\frac{v(x+\xi,y)-v(x,y)}{\xi}\right]$$

$$= \frac{\eta}{\xi+i\eta}[u_y(x+\xi,y+\theta_1\eta)+iv_y(x+\xi,y+\theta_2\eta)] + \frac{\xi}{\xi+i\eta}[u_x(x+\theta_3\xi,y)+iv_x(x+\theta_4\xi,y)]$$

We know $0<\theta_k<1$, $\left|\frac{\eta}{\xi+i\eta}\right| = \left|\frac{\mathrm{Re}(h)}{h}\right| \le 1$, $\left|\frac{\xi}{\xi+i\eta}\right| = \left|\frac{\mathrm{Im}(h)}{h}\right| \le 1$

Claim: $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = f_x(z)$

Proof

We know $f_x(z) = \frac{\xi+i\eta}{\xi+i\eta}f_x(z)$

By C-R eq, $f_x(z) = \frac{\xi}{\xi+i\eta}f_x(z) + \frac{\eta}{\xi+i\eta}f_y(z)$

As f_x, f_y are conti,

$$\frac{f(z+h)-f(z)}{h} - f_x(z) = \frac{\eta}{\xi+i\eta}[u_y(x+\xi,y+\theta_1\eta)-u_y(x,y)+i(v_y(x+\xi,y+\theta_2\eta)-v_y(x,y))] + \frac{\xi}{\xi+i\eta}[u_x(x+\theta_3\xi,y)-u_x(x,y)+i(v_x(x+\theta_4\xi,y)-v_x(x,y))]$$

$\longrightarrow 0$ as $h\to 0$, i.e. $\xi,\eta \longrightarrow 0$ □

$\therefore$ f is diffable and $f'(z) = f_x(z)$

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DEFINITION

f is analytic at z if f is diffable in a nbd of z

Similarly, f is analytic on a set S if f is diff at all pts of some open set containing S.

DEFINITION

Let S, T be open sets of $\mathbb{C}$, and f be a 1-1 function on S with $f(S)=T$

We say g is the inverse of f on T if $f(g(z))=z$ $\forall z\in T$.

We say g is the inverse of f at z if $\exists$ open nbd U of z, s.t. g is the inverse of f on U

Remark: g is also 1-1

PROPOSITION

Suppose that g is the inverse of f at z_0 and g is continuous there. If f is diffable at $g(z_0)$ and if $f'(g(z_0))\neq 0$, then g is diffable at z_0 and $g'(z_0)=\frac{1}{f'(g(z_0))}$

Proof

$\frac{g(z+h)-g(z)}{h} = \frac{g(z+h)-g(z)}{f(g(z+h))-f(g(z))} = \left(\frac{f(g(z+h))-f(g(z))}{g(z+h)-g(z)}\right)^{-1} = \frac{1}{f'(g(z_0))}$ $\square$

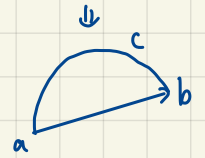
LINE INTEGRALS

Let $f(t)=u(t)+iv(t)$, $z(t)=x(t)+iy(t)$.

We consider curves as such:

$\underset{[a,b]}{\mathbb{R}^1} \xrightarrow{C} \mathbb{R}^n$

$\Downarrow$



We say $\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

DEFINITION

(i) Let $z(t)=x(t)+iy(t)$, $a\le t\le b$. The curve determined by $z(t)$ is called piecewise differentiable and we set $\dot z(t)=x'(t)+iy'(t)$ if x, y are continuous on $[a,b]$ and are continuously differentiable on each subinterval $[a,x_1], [x_1,x_2], \dots, [x_{n-1},b]$ of some partition of $[a,b]$

(ii) The curve is said to be smooth, if $\dot z(t)\neq 0$ except at finitely many points.

In the following, we assume our curves are smooth.

DEFINITION

Say C is a smooth curve $\subseteq\mathbb{C}$, where $z(t)=x(t)+iy(t)$

Then, $\int_C f(z)\,dz = \int_a^b f(z(t))\,dz = \int_a^b f(z(t))\dot z(t)\,dt$

DEFINITION

Let C_1, C_2 be smooth curves $\subseteq\mathbb{C}$, where $C_1: z(t)$, $a\le t\le b$ and $C_2: w(t)$, $c\le t\le d$.

C_1 and C_2 are smoothly equivalent if $\exists$ 1-1 C^1 mapping $\lambda:[c,d]\longrightarrow[a,b]$ s.t. $w(t)=z(\lambda(t))$

(By def, this is provably an equivalence relation)

We denote smoothly equivalent with $C_1 \underset{sm}{\sim} C_2$.

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PROPOSITION

If $C_1 \underset{sm}{\simeq} C_2$, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

Proof

We set $f(z) = u(z) + iv(z)$, $z = x(t) + iy(t)$

Then, $\int_{C_1} f\,dz = \int_a^b f(z(t))\dot{z}(t)dt = \int_a^b [u(z)x'(t) - v(z)y'(t)]dt + i\int_a^b [u(z)y'(t) + v(z)x'(t)]dt$

With $\int_c^d u(z(\lambda(t)))x'(\lambda(t))\lambda'(t) = \int_a^b u(z(t))x'(t)dt$, we can prove the equation □

APPLICATIONS OF CR-EQUATION

DEFINITION

$D \subseteq \mathbb{C}$ is called a region is open connected

Note, D: region $\Rightarrow x, y \in D$, $\exists$ a curve consisting of vertical and horizontal line segments that connect.

Proof

For $x \in D$, say $U_x := \{y \in D \mid x$ connects to y via vertical/horizontal line segments that connect$\}$

① "U_x is open":

For $y \in U_x \subseteq D$, D is open $\Rightarrow \exists$ open disk $B_\delta(y) \subseteq D$

$\because \forall a \in B_\delta(y)$, a can be connected to y by ⌐⌐

$\therefore x \sim y \sim a$ ✓

② "$D \setminus U_x$ is open":

For $y \in D \setminus U_x$, D is open $\Rightarrow \exists$ open disk $B_\delta(y) \subseteq D \Rightarrow B_\delta(y) \cap U_x = \emptyset \Rightarrow B_\delta(y) \subseteq D \setminus U_x$ ✓

$\therefore$ ① + ② + D is connected $\Rightarrow D = U_x$

PROPOSITION

If $f = u + iv$ is analytic on a region D and u is constant, then f is constant

Proof

u is const $\Rightarrow u_x = u_y = 0$

By CR-eq, $v_x = v_y = 0$

As D is a region, thus $\forall a, b \in D$, $\exists$ ⌐⌐ connecting a and b

$\Rightarrow f(a) = f(b) \Rightarrow f$ is const □

PROPOSITION

If f is analytic on a region D, and if $|f|$ is constant on D, then f is constant

Proof

$|f| = 0 \Rightarrow f = 0$ ✓

If $|f| \neq 0$, $|f| = C > 0 \Rightarrow u^2 + v^2 = C^2$

$\Rightarrow 2uu_x + 2vv_x = 0$; $2uu_y + 2vv_y = 0$

By CR-eq, $uu_x - vu_y = 0$; $uu_y + vu_x = 0 \Rightarrow (u^2+v^2)u_x = 0 \Rightarrow u_x = 0$. Similarly, we get $u_y = 0$.

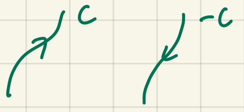
As ⌐⌐ + prop above, thus this prop is true. □

LINE INTEGRALS (CONTINUED)

DEFINITION

Let C be a curve defined by $z(t)=x(t)+iy(t)$, $t\in[a,b]$.
Then, $-C$ is a curve defined by $w(t)=z(a+b-t)$

In short, it is as follows:



PROPOSITION

$\int_{-C} f(z)dz = -\int_C f(z)dz$

PROPOSITION

Let C be a smooth curve, and f,g be continuous functions on C. Say, $\alpha\in\mathbb{C}$.
(i) $\int_C (f(z)+g(z))dz = \int_C f(z)dz + \int_C g(z)dz$
(ii) $\int_C \alpha f(z)dz = \alpha\int_C f(z)dz$

In other words, we say $\int_C(\cdot)dz$ is linear

EXAMPLE

Say $f(z)=\frac{1}{z}$, $C: R(\cos t + i\sin t)$, $t\in[0,2\pi]$
Then, $\int_C f(z)dz = \int_0^{2\pi} \frac{1}{\cancel{R}(\cos t + i\sin t)} \cancel{R}(-\sin t + i\cos t)\, dt$
$= \int_0^{2\pi} e^{-it}(-e^{i(t-\frac{\pi}{2})})\, dt$
$= \int_0^{2\pi} i\, dt$
$= 2\pi i$

LEMMA

Let $t\in\mathbb{R}$, $G(t)$ be a continuous complex-valued function. Then, $\left|\int_a^b G(t)dt\right| \le \int_a^b |G(t)|dt$
$(\alpha << \beta := |\alpha|\le|\beta|,\ \alpha,\beta\in\mathbb{C})$

Proof

Set $\int_a^b G(t)dt = Re^{i\theta}$, $\theta\in\mathbb{R}$, $R\in\mathbb{R}_{\ge 0}$
Then, $R = \left|\int_a^b G(t)dt\right| = \int_a^b e^{-i\theta}G(t)dt = \int_a^b A(t)dt + i\underbrace{\cancel{\int_a^b B(t)dt}}_{0}$ $(e^{i\theta}G(t) = A(t)+iB(t))$
$\therefore R = \int_a^b A(t)dt \le \int_a^b |A(t)|dt \le \int_a^b |e^{-i\theta}G(t)|dt = \int_a^b |G(t)|dt$ □

PROPOSITION (ML-FORMULA)

Let C be a smooth curve of length L, and f be cont. on C and $\overset{|f|\le M}{f << M}$ throughout C. Then, $\int_C f(z)dz << ML$

Proof

Let C be $z(t)=x(t)+iy(t)$, $t\in[a,b]$.
Then, $\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt$

By the prev lemma, $\int_C f(z)dz << \int_a^b |f(z(t))| \overset{\text{arc length}}{|\dot{z}(t)|} dt \le M\int_a^b |\dot{z}(t)|dt = ML$ □

EXAMPLE (FOR WHY ML IS THE TIGHT BOUND)

For $f(z)=\frac{1}{z}$, $C: \cos\theta + i\sin\theta$, $\int_C f(z) = 2\pi i \Rightarrow \left|\int_C f(z)\right| = 2\pi = ML$

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PROPOSITION

Suppose $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ unif on a smooth curve C. Then, $\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz$

Proof

$f_n \to f$ uniformly on C: "Given $\varepsilon>0$, $\exists N$, s.t. $\forall n \geq N$, $|f_n(z)-f(z)| < \varepsilon$ $\forall z \in C$."

So, $\left|\int_C f_n(z)dz - \int_C f(z)dz\right| = \left|\int_C (f_n-f)(z)dz\right| \overset{ML}{<} \varepsilon \cdot \text{len}(C)$ $\forall n \geq N$

$\therefore$ By def, $\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz$ □

PROPOSITION

Let F be an analytic function, $f=F'(z)$, and a smooth curve $C: z(t)=x(t)+iy(t)$, $t\in[a,b]$

Then, $\int_C f(z)dz = F(z(b)) - F(z(a))$

Proof

Let $\gamma(t) := F(z(t)) = A(t)+iB(t)$

Hence, $\dot{\gamma}(t) \overset{\text{smooth}}{=} \lim_{h\to 0} \frac{F(z(t+h))-F(z(t))}{z(t+h)-z(t)} \cdot \frac{z(t+h)-z(t)}{h} = F'(z(t))\,\dot{z}(t)$

Then, $\int_C f(z)dz = \int_a^b F'(z(t))\,\dot{z}(t)dt = \int_a^b \dot{\gamma}(t)dt = \gamma(b)-\gamma(a)$ □

DEFINITION

(i) A curve is closed if its initial and terminal points coincide.

(ii) C is a simple closed curve with $t\in[a,b]$ if $z(t_1)=z(t_2)$ with $t_1<t_2$, then $t_1=a$ and $t_2=b$

DEFINITION

The boundary of a rectangle is the simple closed curve in the counterclockwise direction

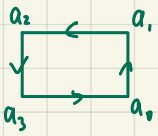
DEFINITION

f is an entire function if f is analytic on $\mathbb{C}$

LEMMA

If f is a linear function, i.e. $f=az+b$, $a,b\in\mathbb{C}$, Γ is the boundary of a rectangle, then $\int_\Gamma f(z)dz=0$

Proof



Say $\Gamma: z(t)$, $a=a_0 \leq t \leq b=a_3$, and $f=F'(z) \Rightarrow \underbrace{F := \frac{a}{2}z^2+bz}_{\text{analytic on } \mathbb{C}}$

Hence, we can deduce $\int_\Gamma f(z)dz = \int_\Gamma F'(z)dz = F(z(b)) - F(z(a)) = 0$ □ ($\because z(b)=z(a)$)

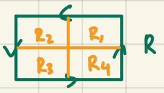
THEOREM (RECTANGLE THEOREM)

Let f be an entire function, and Γ as above, then $\int_\Gamma f(z)dz=0$

Proof

Let $I = \int_\Gamma f(z)dz$. Assume $f\not\equiv 0$, otherwise $f\equiv 0 \Rightarrow I=0$.

We divide R as follows:



Then, $\exists$ one of R_i s.t. $\left|\int_{\Gamma_i} f(z)dz\right| \geq \frac{|I|}{4}$, where Γ_i is the boundary of R_i.

Set $R^{(1)}$ to be such an R_i.

Continuing this process, we get $R^{(1)} \supseteq R^{(2)} \supseteq \dots$. Let $z_0 \in \bigcap_{i=1}^{\infty} R^{(i)}$.

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As f is an entire function, hence f is analytic at z_0

By def, $\forall \varepsilon>0, \exists \delta>0$, s.t. $|h|<\delta \Rightarrow \left|\frac{f(z_0+h)-f(z_0)}{h} - f'(z_0)\right| < \varepsilon$

$\therefore$ We see $f(z) = f(z_0) + f'(z_0)(z-z_0) + \varepsilon(z)(z-z_0)$, where $|\varepsilon(z)| \le \varepsilon$.

We choose N s.t. $\forall n \ge N$, $|z-z_0| < \delta \Rightarrow \int_{\Gamma^{(n)}} f(z)dz = \cancel{\int_{\Gamma^{(n)}} [f(z_0)+f'(z_0)(z-z_0)]dz} + \int_{\Gamma^{(n)}} \varepsilon(z)(z-z_0)dz$ $\to 0$ (from lemma since linear)

We know $|\Gamma^{(n)}| = \frac{4s}{2^n}$, so $|\varepsilon(z)(z-z_0)| < \varepsilon \cdot \frac{\sqrt{2}s}{2^n} \Rightarrow$ By ML formula, $\int_{\Gamma^{(n)}} f(z)dz << \varepsilon \frac{4\sqrt{2}s^2}{4^n}$

By our assumption, $\left|\int_{\Gamma^{(n)}} f(z)dz\right| \ge \frac{|I|}{4^n}$, hence $|I| \le \varepsilon \cdot 4\sqrt{2}s^2$ $\forall \varepsilon>0$, i.e. $I=0$ □

THEOREM (INTEGRAL THEOREM)

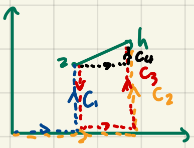
If f is entire, then f is everywhere the derivative of an analytic function. That is, $\exists$ an entire F, s.t. $F'(z) = f(z)$ $\forall z$

Proof

Consider $F(z) = \int_C f(\eta)d\eta$ where $C: 0 \to \operatorname{Re}(z) \to z$

Now, for $h \in \mathbb{C}$, $F(z+h) = \int_{C_1} f(z)dz$

$F(z) = \int_{C_2} f(z)dz$



Then, $F(z+h) - F(z) = \int_{C_1} f(\eta)d\eta + \int_{-C_2} f(\eta)d\eta = \int_{C_3} f(\eta)d\eta = \int_{C_4} f(\eta)d\eta$

Using $F(z+h) = F(z) + \int_{C_4} f(\eta)d\eta$, we get $\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{C_4} f(\eta)d\eta$

As $\frac{1}{h}\int_{C_4} dz = 1$, thus $\left(\frac{1}{h}\int_{C_4} f(\eta)d\eta\right) - f(z) = \frac{1}{h}\int_{C_4}(f(\eta)-f(z))d\eta = \frac{F(z+h)-F(z)}{h} - f(z)$

if h is small, $|f(\zeta)-f(z)| << \varepsilon$

In other words, by ML-formula, $\frac{F(z+h)-F(z)}{h} - f(z) << \frac{1}{\cancel{|h|}} \varepsilon\, 2\cancel{|h|} \Rightarrow \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$ □

THEOREM

If f is entire and if C is a smooth closed curve, then $\int_C f(z)dz = 0$

9-16-25 (WEEK 3)

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CAUCHY INTEGRAL FORMULA AND TAYLOR EXPANSION

THEOREM (RECTANGLE THEOREM II)

Let f be entire, and

$$g(z) := \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases},$$ which is continuous (f entire ⇒ g conti)

Then, $\int_\Gamma g(z)dz = 0$, Γ: a boundary of a rectangle (closed set) $R \subseteq \mathbb{C}$

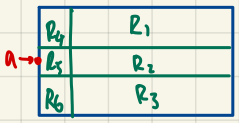
Proof

As g is conti, by def, $\exists M \in \mathbb{R}$, s.t. $|g(z)| < M$ $\forall z \in R$

(i) If $a \in \mathbb{C} \setminus R$, $g(z)$ is analytic $\forall z \in R$

$\therefore$ By the argument of Rectangle Thm, $\int_\Gamma g(z)dz = 0$

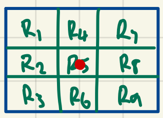
(ii) If $a \in \Gamma$, $\Gamma_i :=$ boundary of R_i:



Then, $\int_\Gamma g(z)dz = \sum_{i=1}^{6} \int_{\Gamma_i} g(z)dz \overset{(i)}{=} \int_{\Gamma_5} g(z)dz << M \cdot 4\varepsilon$ by ML-formula, with M indep of ε, where we define $\varepsilon :=$ length of longest side of Γ_5.

$\therefore$ As $\varepsilon \to 0$, $\int_\Gamma g(z)dz = 0$

(iii) Otherwise, $a \in$ interior of R



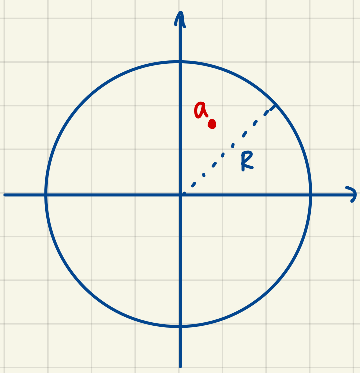
Then, $\int_\Gamma g(z)dz = \sum_{i=1}^{9} \int_{\Gamma_i} g(z)dz \overset{(i)}{=} \int_{\Gamma_5} g(z)dz << M \cdot 4\varepsilon \xrightarrow{\varepsilon \to 0} 0$

COROLLARY

The integral thm and closed curve thm apply to g (since g is cont.)

THEOREM (CAUCHY INTEGRAL FORMULA)

Given an entire f, $a \in \mathbb{C}$, $C = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$ with



Then, we have $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$

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LEMMA

Define $C_\rho(\alpha)$:= circle centered at α with radius ρ (α may be omitted if no ambiguity)

Then, $\int_{C_\rho(\alpha)} \frac{dz}{z-a} = 2\pi i \quad \forall |a-\alpha|<\rho$

Proof

If $a=\alpha$, then it's clear, since $C_\rho(\alpha): \alpha+\rho e^{i\theta}$, $0\le\theta\le 2\pi$.

For $a\neq\alpha$,

$$\int_{C_\rho(\alpha)} \frac{dz}{(z-\alpha)-(a-\alpha)} = \int_{C_\rho(\alpha)} \frac{1}{z-\alpha}\cdot\frac{1}{1-\frac{a-\alpha}{z-\alpha}}\,dz =: I$$

$\forall z\in C_\rho(\alpha)$, $\left|\frac{a-\alpha}{z-\alpha}\right|<1$. Hence, $\left(1-\frac{a-\alpha}{z-\alpha}\right)^{-1} = 1+\left(\frac{a-\alpha}{z-\alpha}\right)+\left(\frac{a-\alpha}{z-\alpha}\right)^2+\dots$ (unif conv, abs conv)

Hence,

$$I = \int_{C_\rho(\alpha)} \frac{1}{z-\alpha}\left(\sum_{k=0}^{\infty} \left(\frac{a-\alpha}{z-\alpha}\right)^k\right)dz = \boxed{\sum_{k=0}^{\infty} \int_{C_\rho(\alpha)}\frac{1}{z-\alpha}\left(\frac{a-\alpha}{z-\alpha}\right)^k dz}$$

We first consider the term $\int_{C_\rho(\alpha)} \frac{1}{(z-\alpha)^k}dz$ ($k=1 \Rightarrow 2\pi i$, since $\int_C \frac{1}{z}dz = 2\pi i$)

For $k>1$, $\int_{C_\rho(\alpha)}\frac{1}{(z-\alpha)^k}dz = \int_0^{2\pi}\frac{ie^{i\theta}}{\rho^k e^{ik\theta}}d\theta = \int_0^{2\pi}\frac{i}{\rho^k}e^{i\theta(k-1)}d\theta = 0$

$\therefore$ $I=2\pi i$ □

PROOF OF CAUCHY INTEGRAL FORMULA

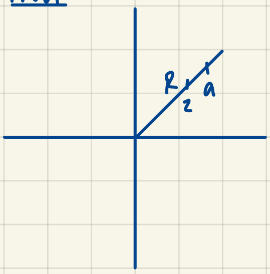
Now, we know by rectangle thm, $\int_C g(z)dz = \int_C \frac{f(z)}{z-a}-\frac{f(a)}{z-a}\,dz = 0$

$\therefore \int_C\frac{f(z)}{z-a}dz = \int_C\frac{f(a)}{z-a}dz = f(a)\int_C\frac{1}{z-a}dz = f(a)2\pi i$ □

THEOREM (TAYLOR EXPANSION FOR ENTIRE FUNCTION)

Given f is an entire function, then $f^{(k)}(0)$ exists $\forall k\in\mathbb{Z}_{\geq 0}$ and $f(z)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}z^k$ $\forall z\in\mathbb{C}$

Proof



Choose $a\in\mathbb{C}$, $|a|>|z|$, $R:=|a|+1$

Notice, by Cauchy Integral Formula,

$$f(z)=\frac{1}{2\pi i}\int_{C_R}\frac{f(\omega)}{\omega-z}d\omega = \frac{1}{2\pi i}\int_{C_R}\frac{1}{1-\frac{z}{\omega}}\,\frac{f(\omega)}{\omega}d\omega$$

As $\left|\frac{z}{\omega}\right|<\frac{|a|}{1+|a|}$, then $f(z)=\sum_{k=0}^{\infty}\int_{C_R}\frac{f(\omega)}{\omega}\left(\frac{z}{\omega}\right)^k d\omega = \sum_{k=0}^{\infty} z^k\int_{C_R}\frac{f(\omega)}{\omega^{k+1}}d\omega = \sum_{k=0}^{\infty} z^k C_k$

Notice, as $|z|<|a|$, then $f'(z)=\sum_{i=1}^{\infty} iz^{i-1}C_i \Rightarrow f'(0)=C_1$

If we continue this process, $f^{(k)}(0)$ exists $\forall k\in\mathbb{N}_{\geq 0}$ and $f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}z^k$ □

COROLLARY

An entire function is infinitely diff

Proof

Above thm. □

COROLLARY

If f is entire, then $f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2}(z-a)^2+\dots$

Proof

Let $h(z)=f(z+a)$.

f is entire $\Rightarrow$ z is entire.

Then, $h(z)=\sum_{k=0}^{\infty}\frac{h^{(k)}(0)}{k!}z^k \Rightarrow f(\omega)=f(a)+\sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}(\omega-a)^k$ □

PROPOSITION

If f is entire, then

$$g(z)=\begin{cases}\frac{f(z)-f(a)}{z-a}, & z\neq a\\ f'(a), & z=a\end{cases}$$ is entire

Proof

By corollary, $g(z)=\sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}(z-a)^{k-1}\Rightarrow g$ is entire □

COROLLARY

Let f be an entire function with zeros at $a_1,\dots,a_N$. Define $g(z)=\frac{f(z)}{(z-a_1)\cdots(z-a_N)}$, $z\notin\{a_1,\dots,a_N\}$. Then, $\lim_{z\to a_i}g(z)$ exists $\forall i$.

If we define $g(a_i):=\lim_{z\to a_i}g(z)$, then g is entire

Proof

Set $f_0=f$, $f_k:=\frac{f_{k-1}(z)-f_{k-1}(a_k)}{z-a_k}$ $\left(f_1(z)=\frac{f(z)}{z-a_1}\right)$

By proposition, we see f_1 is entire. By recurrence, g is entire □

THEOREM (LIOUVILLE'S THEOREM)

Entire bounded functions on $\mathbb{C}$ are constants

Proof

Let $a\in\mathbb{C}\setminus\{0\}$, $C_R(0)$, $R>|a|$.

Then, by Cauchy Integral Formula, $f(a)-f(0)=\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-a}dz-\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z}dz=\frac{1}{2\pi i}\int_{C_R}\frac{af(z)}{z(z-a)}dz$

As f is bounded, $\exists M\in\mathbb{R}_{>0}$, s.t. $|f(z)|<M$ $\forall z\in\mathbb{C}$

By ML-formula, $|f(a)-f(0)|<\frac{1}{2\pi i}\left(\frac{M\cdot|a|}{R(R-|a|)}\cdot 2\pi R\right)\xrightarrow{R\to\infty}0$ $\therefore f(a)=f(0)$ $\forall a\in\mathbb{C}$ □

THEOREM (EXTENDED LIOUVILLE'S THEOREM)

Given f is entire. Suppose $|f(z)|<A+B|z|^k$ for some constants $A,B\in\mathbb{R}_0$. Then, f is a polynomial with degree at most k

Proof

Consider induction on k,

- $k=0\Rightarrow$ True by Liouville's Thm.

Otherwise,

Define $g(0)=\begin{cases}\frac{f(z)-f(0)}{z}, & z\neq 0\\ f'(0), & z=0\end{cases}$, which we know is entire.

As $|f(z)|<A+B|z|^k$ is bounded, define $M_0:=\max_{z\in C_R}g(z)$

$$\left.\begin{array}{l}\text{For } z\in\mathbb{C}\setminus C_R,\ |g(z)|<A+B|z|^{k-1}\\ \forall z\in C_R,\ |g(z)|<M_0\end{array}\right\}\Rightarrow\exists D,E\in\mathbb{R}_{>0}\text{ s.t. }|g(z)|<D+E|z|^{k-1}$$

$\therefore g$ is poly of degree at most $k-1$, so f is poly of degree at most k. □

THEOREM (FUNDAMENTAL THEOREM OF ALGEBRA)

Shun/翔海 (@shun4midx)

Nonconstant polynomials have roots in $\mathbb{C}$.
in $\mathbb{C}[x]$

Proof

Consider poly $p(x)$.
Suppose p has no roots in $\mathbb{C}$.

We know $f(z) := \frac{1}{p(z)}$ is defined and differentiable on $\mathbb{C}$, so it is analytic.

As $z \to \infty$, $|f(z)| \longrightarrow 0$, then $|f(z)|$ is bounded
$\therefore$ By Liouville's Thm, $f(z)$ is const $\Rightarrow p(z)$ is const $\times$

DEFINITION

We say S is a convex set in $\mathbb{C}$ if $\forall x, y \in S$, $tx + (1-t)y \in S$ $\forall t \in [0,1]$
($\Rightarrow x_1, \dots, x_N \in S$ iff $\sum_{i=1}^{N} a_i x_i \in S$ $\forall \sum_{i=1}^{N} a_i = 1$ and $a_i \geq 0$)

9-18-25 (WEEK 3)

THEOREM (GAUSS-LUCAS THEOREM)

The zeros of the derivative of a polynomial lie within the convex hull of the zeros of the polynomial

Proof

Let $p(x)$ be a nonconstant polynomial $\in \mathbb{C}[x]$, and $\alpha_1, \dots, \alpha_n$ be roots of p (counted by multiplicity)

Then, $p(x) = c\prod_{i=1}^{n}(x-\alpha_i)$. Moreover, $\frac{p'(x)}{p(x)} = \sum_{i=1}^{n} \frac{1}{x-\alpha_i}$.

otherwise it's trivial

Let a be a root of $p'(x)$ and $a \notin \{\alpha_1, \dots, \alpha_n\}$.

Then, $\frac{p'(a)}{p(a)} = \sum_{i=1}^{n} \frac{1}{a-\alpha_i} = \sum_{i=1}^{n} \frac{\bar{a}-\bar{\alpha}_i}{|a-\alpha_i|} \Rightarrow \bar{a} = \sum_{i=1}^{n} c_i \bar{\alpha}_i$, $c_i := \frac{1}{|a-\alpha_i|^2} / \sum_i \frac{1}{|a-\alpha_i|^2} \in \mathbb{R}_{\geq 0}$

by def

$\therefore a = \sum_{i=1}^{n} c_i \alpha_i$, $c_i \in \mathbb{R}_{\geq 0}$, $\sum c_i = 1$ $\square$

ANALYTIC FUNCTION ON A DISC

NOTATION

Assume open disc if disc

$D = D(\alpha; r)$ is an open disc $\subseteq \mathbb{C}$, then: analytic $f(z)$ in D, $\forall a \in D$, $g_a(z) = \begin{cases} (f(z)-f(a))/(z-a) & , z \neq a \\ f'(a) & , z = a \end{cases}$ is cont. on D (Notation for g_a)

THEOREM (RECTANGLE THEOREM)

For a closed rectangle $R \subseteq D$, define $\Gamma = \partial R$, then we proved $\int_\Gamma f(z)dz = \int_\Gamma \frac{f(z)-f(a)}{z-a}dz = \int_\Gamma g_a(z)dz = 0$ (They just both happen to equal 0, not they are directly related)

THEOREM

$\exists F, G_a$ analytic in D s.t. $f = F'$, $g_a = G_a'$

Proof

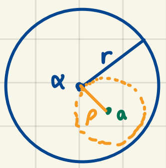
We consider $F := \int_\alpha^z f(z)dz$ and $G_a := \int_\alpha^z g_a(z)dz$

Define $C: \alpha \longrightarrow \alpha + \mathrm{Re}(z) \longrightarrow \alpha + z$, which is in D. $\square$

THEOREM (CLOSED CURVE THEOREM)

Let C be a closed curve $\subseteq D$. Then, $\int_C f dz = 0$

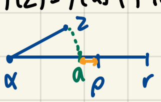
THEOREM (CAUCHY INTEGRAL FORMULA)



For some $0 < \rho < r$, $\forall |a-r| < \rho$, $f(a) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z-a}dz$, where C_ρ is $\partial D(\alpha; \rho)$

THEOREM (TAYLOR EXPANSION)

$f(z) = f(\alpha) + f'(\alpha)(z-\alpha) + \frac{f''(\alpha)}{2}(z-\alpha)^2 + \dots$ holds $\forall |z-\alpha| < \rho$, for some ρ, s.t. $\exists a \in D$, $0 < \rho < r$, $|a-\alpha| < \rho$



Proof Sketch

$\forall |z-\alpha| < r$, $\frac{f^{(k)}(\alpha)}{k!} = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{(z-\alpha)^{k+1}}dz$

THEOREM

Let $\tilde{D}$ be an arbitrary open set, f analytic on $\tilde{D}$. Then, fix $\alpha \in \tilde{D}$, we have $f(z) = \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ $\forall z \in D(\alpha, r) \subseteq \tilde{D}$

EXAMPLE

Let $f(z) = \frac{1}{z-i}$, analytic $\forall z \neq 1$.

Then, on $D=D(2;1)$, $f(z)=\frac{1}{1+(z-2)}=1-(z-2)+(z-2)^2-\dots\ \forall |z-2|<1$

In fact, the expression converges $\forall |z-2|<1$

diverges $\forall |z-2|\geq 1$

(However, $f(z)$ is analytic $\forall |z-2|=1$, so clearly its Taylor expansion is different)

PROPOSITION

$g_a(z)$ is analytic $\forall z,a\in D(\alpha;r)$

Proof

Use the thm above, in some neighborhood of α, $f(z)=f(\alpha)+f'(\alpha)(z-\alpha)+\frac{f''(\alpha)}{2}(z-\alpha)^2+\dots$

Then, g has the power series expansion $g(z)=f'(\alpha)+\frac{f''(\alpha)}{2!}(z-\alpha)+\frac{f^{(3)}(\alpha)}{3!}(z-\alpha)^2+\dots \Rightarrow g$ is analytic at α. □

THEOREM

If f is analytic at z, then f is infinitely diffable at z

Proof

We know from above, f may be expressed as a power series. Hence, it is infinitely diffable. □

Shun/翔海(@shun4midx)

THEOREM (UNIQUENESS THEOREM)

Say D is a region (i.e. open connected) and f is an analytic function on D.
Suppose that $\exists$ seq of distinct zeros of D $\{z_n\}$, s.t. $\lim_{n\to\infty} z_n = z_0 \in D$, where we say the seq $\{z_n\}$ has an acc pt in D.
Then, $f \equiv 0$ on D.

Proof

f ana $\Rightarrow$ f conti

$\therefore$ By def, $f(z_0) = \lim_{n\to\infty} f(z_n) = 0$

We define $A := \{z \in D \mid z \text{ is an acc pt of zeros of } f \text{ in } D\}$.

Claim: A is open

Proof

By uniqueness of power series, $f \equiv 0$ in some disk $D(z, \delta_z) \subseteq D$ $\forall z \in A$ ✓

Claim: $D \setminus A$ is open

Proof

z is NOT acc pt of zeros $\Rightarrow \exists$ open nbd U of z in D s.t. $f(z)$ has NO zeros in $U \setminus \{z\}$.
f conti $\Rightarrow \forall y \in U \setminus \{z\}$, $\exists$ open nbd $V_y \subseteq D$ of y', s.t. $f \neq 0$ on $V_y \Rightarrow y \in D \setminus A$ ✓

$\therefore D = A \cup B$, A, B both open, $A \cap B = \emptyset$ $(B = D \setminus A)$
As $z_0 \in A$ and D is a region, $D = A$ □

COROLLARY

Say f, g are analytic on a region D.
If f and g agree at a set of pts with an acc pt, then $f \equiv g$ on D.

Proof

Set $h := f - g$, then apply thm above □

THEOREM

If f is entire and $f \to \infty$ as $z \to \infty$, then f is a polynomial

Proof

By def, $\forall M \in \mathbb{R}_{>0}$, $\exists \delta$, s.t. $\forall |z| > \delta$, $|f(z)| > M$

Let $M = 1$.
$\therefore \exists \delta$, s.t. $\forall |z| > \delta$, $|f(z)| > 1$

By our assumption, f is NOT a constant

Claim: f has finitely many zeros

Proof

If NOT, by δ, all zeros in f are in $\overline{D(0,\delta)}$, otherwise, $|f(z)| \neq 0$.
As $\overline{D(0,\delta)}$ is compact, $\exists$ acc pt of zeros in $\overline{D}(0,\delta)$ ✓
↳ Suppose not. Then, $\forall x \in \overline{D}(0,\delta)$, $\exists$ an open nbd U_x, s.t. f has no zeros in $U_x \setminus \{x\}$
$U_x \setminus \{x\}$ is an open cover of $\overline{D(0,\delta)} \Rightarrow \exists x_1, \dots, x_n$ s.t. $\{U_{x_i}\}$ is an open cover of $\overline{D(0,\delta)}$ (by cpt)
However, each U_x has at most 1 zero $\Rightarrow \overline{D(0,\delta)}$ has at most n zeros ⁂

By thm, $f \equiv 0$ on $D(0, \delta')$ for all $\delta' > \delta$ (cover boundary at δ)
However, δ' can extend to ∞

Shun/翔海(@shun4midx)

We consider within $\bar{D}(0,\delta)$.
Let $\alpha_1,\dots,\alpha_n$ be zeros of f (counting by multiplicity)
Then, $g(z)=\frac{f(z)}{\prod_{i=1}^{n}(z-\alpha_i)}$ is entire and has no zeros on $\mathbb{C}$

Set $h:=\frac{1}{g(z)}$, then h is entire, h has no zeros in $\mathbb{C} \Rightarrow h$ is bounded in disk

By Extended Liouville's Thm, $|h|<A+B|z|^n$ $\forall |z|>\delta$ and $\forall |z|\leq\delta \Rightarrow h$ is a poly

However, h has no zeros in $\mathbb{C} \Rightarrow h=$ const
$\therefore \exists c\in\mathbb{C}^\times$ s.t. $f(z)=c\prod_{i=1}^{n}(z-\alpha_i)$ □

REMARK
Say f,g are ana on region D, to check $f\equiv g$, we may apply the theorem above over $\mathbb{R}$ without needing to consider $\mathbb{C}$.

THEOREM (MEAN VALUE THEOREM)
Let D be a region, f analytic on D, $\alpha\in D$.
Then $f(\alpha)=$ mean value of f taken around the boundary of any disk centered at α and contained at D
Proof
By Cauchy-Integral Formula, $f(\alpha)=\frac{1}{2\pi i}\int_{C_\delta(\alpha)}\frac{f(z)}{z-\alpha}dz$

Say $z=\alpha+\delta e^{i\theta}$, $\theta\in[0,2\pi]$, we get $f(\alpha)=\frac{1}{2\pi}\int_0^{2\pi}f(\alpha+\delta e^{i\theta})d\theta$ □

THEOREM (MAXIMUM MODULUS THEOREM)
Say f is noncont, ana on a region D. Then, $\forall z\in D$ and $\delta\in\mathbb{R}_{>0}$, $\exists$ some $w\in D(z,\delta)\cap D$, s.t. $f(w)>f(z)$
Proof
By MVT, $f(z)=\frac{1}{2\pi}\int_0^{2\pi}f(z+\delta e^{i\theta})d\theta$ for small enough δ s.t. $D(z,\delta)\subseteq D$
Then, $|f(z)|\leq\frac{1}{2\pi}\int_0^{2\pi}|f(z+\delta e^{i\theta})|d\theta \leq^{①} \frac{1}{2\pi}\max_{\theta\in[0,2\pi]}|f(z+\delta e^{i\theta})|\cdot 2\pi=\max|f(z+\delta e^{i\theta})|$

$\therefore$ When $\leq^{①}$ has equality, $|f(z+\delta e^{i\theta})|=\max_{\theta\in[0,2\pi]}|f(z+e^{i\theta})|$ $\forall\theta\in[0,2\pi] \Rightarrow f$ is const on $C_\delta(z)\subseteq D$
By coro, hence f is const on D

However, f is noncont.
$\therefore f(z)<\max_{\theta\in[0,2\pi]}|f(z+\delta e^{i\theta})|$ □

THEOREM (MINIMUM MODULUS THEOREM)
Say f is noncont, ana on a region D, $\forall z\in D$, $f(z)\neq 0$.
Then, f has no interior min points
Proof
$f(z)\neq 0$ $\forall z\in D \Rightarrow g(z)=\frac{1}{f(z)}$ is ana, noncont on D
Then, by max mod thm, we proved it. □

CAUTION
We can only apply uniqueness thm and its coro when its acc pts $\in D$ (Counterexample: $f(z)=\sin\frac{1}{z}$, $z_n=\frac{1}{n\pi}$)

THEOREM
CAN ONLY APPLY TO CIRCLES
Say $\bar{D}$ is a closed disk and f is analytic, noncont on $\bar{D}$. f assumes its max value at a boundary point z_0. Then, $f'(z_0)\neq 0$
Proof
conv rad
Suppose $f'(z_0)=z_0$. As f is ana on $\bar{D}$, $\exists\delta$ s.t. $\forall|\zeta|<\delta$, $f(z_0+\delta)=f(z_0)+f'(z_0)\zeta+\frac{1}{2}f''(z_0)\zeta^2+\dots$

Assume $f'(z_0)=0$

Then, $f(z_0+\delta) \approx f(z_0)+\frac{1}{2}f''(z_0)\zeta^2 \Rightarrow |f(z_0+\delta)| = f(z_0+\delta)\cdot\bar{f}(z_0+\delta) = |f(z_0)|^2+\frac{2}{k!}\mathrm{Re}(\bar{f}(z_0)f^{(k)}(z_0)\zeta^k)+\ldots$, for some $k\in\mathbb{N}$

From assumption, thus $k\geq 2$.

Let $e^{i\theta}=\frac{\zeta}{|\zeta|}$. Then, $\bar{f}(z_0)f^{(k)}(z_0)=Ae^{i\alpha} \Rightarrow |f(z_0+\zeta)|^2=|f(z_0)|^2+\frac{2}{k!}A|\zeta|^k\cos(\alpha+k\theta)+\ldots$

For small enough ζ, $|f(z_0+\zeta)|-|f(z_0)|$ has the same sign as $\boxed{\cos(k\theta+\alpha)}$

As $|f(z_0)|$ is max, hence $|f(z_0+\zeta)|^2-|f(z_0)|^2\leq 0$ $\forall z_0+\zeta\in D$.

Notice, $\cos(k\theta+\alpha)\leq 0 \Leftrightarrow \frac{\pi}{2}+2\pi j\leq \alpha+k\theta\leq\frac{3\pi}{2}+2\pi j$ for $0\leq j\leq k-1 \Leftrightarrow \frac{(\frac{\pi}{2}-\alpha)}{k}+\frac{2\pi j}{k}\leq\theta\leq\frac{(\frac{3}{2}\pi-\alpha)}{k}+\frac{2\pi j}{k}$ (☆) ← Δ w/ angle $\frac{\pi}{k}$

kinda like they alternate

For a disc, $\exists\zeta$, s.t. $z_0+\zeta$ is NOT in any one of the cones (☆) since $k\geq 2$, $\frac{\pi}{k}\leq\frac{\pi}{2}$. $\times$

REMARK

This argument works for cpt $K\subseteq\mathbb{C}$ s.t. $\forall z_0$ on the boundary of K, K contains a cone $\{z_0+re^{i\theta} \mid \theta\in(\alpha,\beta), r\in(0,\varepsilon)\}$ with $\beta-\alpha>\frac{\pi}{2}$

Counterexample of squares:

$f(z)=z^2+i \Rightarrow |f(z)|$ has min 1 at $z=0$, but $f'(0)=0$

9-25-25 (WEEK 4) Shun/翔海 (@shun4midx)

DEFINITION

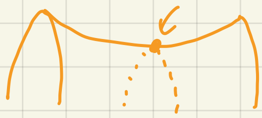
A function is C-analytic on a region D if it is analytic on D and continuous on $\bar{D}$

SADDLE POINTS

DEFINITION

z_0 is a saddle pt of an analytic function f (on a region D) if z_0 is a saddle pt of the real valued function $g(x,y)=|f(x,y)|$

In other words, $\underbrace{g \text{ is differentiable}}_{\lim_{h\to 0}\frac{\|g(x+h)-g(x)-L(x)\|}{\|h\|}=0}$ and $g_x(z_0)=g_y(z_0)=0$ but z_0 is NOT a local extremum



THEOREM

z_0 is a saddle pt of an analytic function f iff $f'(z_0)=0$ and $f(z_0)\neq 0$

Proof

We have $z=x+iy$, $f(z)=u(x,y)+iv(x,y)$, and $g(z)=(u^2+v^2)^{\frac{1}{2}}\geq 0$

"$\Rightarrow$": As $g(z_0)$ is not a local minimum, hence $g(z_0)\neq 0$, so $u(z_0)\neq 0$ or $v(z_0)\neq 0$

We know $g_x(z_0)=g_y(z_0)=0 \Rightarrow \frac{uu_x+vv_x}{g}\Big|_{z_0}=\frac{uu_y+vv_y}{g}\Big|_{z_0}=0$ (★)

$$\Rightarrow \begin{bmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0)\end{bmatrix}\underbrace{\begin{bmatrix} u(z_0) \\ v(x_0)\end{bmatrix}}_{\neq 0}=0$$

$$\therefore \det\begin{bmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0)\end{bmatrix}\overset{\text{CR-eq}}{=}u_x^2(z_0)+v_x^2(z_0)=0$$

$\therefore u_x(z_0)=v_x(z_0)=0$

As f is ana, hence $f'(z_0)=0$. From above with $g(z_0)\neq 0$, we know $f(z_0)\neq 0$. ✓

"$\Leftarrow$": Recall, $f'(z_0)=0$

Then, $u_x(z_0)=v_x(z_0)=0$ and $u_y(z_0)=v_y(z_0)=0$

$\therefore g_x(z_0)=g_y(z_0)=0$ as implied by (★)

As $f(z_0)\neq 0$, thus $|f(z_0)|$ is NOT a local extremum (excluding f is const) by the max and min modulus thms. ✓

OPEN MAPPING THM AND SCHWARTZ LEMMA

RECALL

$f:\mathbb{R}^2\to\mathbb{R}^2$ conti $\Leftrightarrow$ $U\subseteq\mathbb{R}^2$: open then $f^{-1}(U)$: open $\Leftrightarrow$ $\bar{U}\subseteq\mathbb{R}^2$: closed then $f^{-1}(\bar{U})$: closed

Then $K\subseteq\mathbb{R}^2$: cpt $\Rightarrow$ $f(K)$: cpt

THEOREM (OPEN MAPPING THEOREM)

$\forall$ open set $U\subseteq D$, $f(U)$: open in $\mathbb{C}$ for any nonconst ana f (Need not hold outside of $\mathbb{C}$, e.g. $U=(-1,1)$, $f(U)=[0,1)$ for $f(x)=x^2$)

Proof

It suffices to show $\forall \alpha\in D$, $\exists$ open disc $D(\alpha,\varepsilon)\subseteq D$ s.t. $f(D(\alpha,\varepsilon))$ is open.

(We want to show $\forall \beta=f(\alpha')\in f(D(\alpha,\varepsilon))$, $\exists D(\beta,\varepsilon')\subseteq f(D(\alpha,\beta))$)

WLOG, we can assume $f(\alpha)=0$, so we choose ε s.t. $\overline{D(\alpha,\varepsilon)}\subseteq D$

By uniqueness thm, $\exists\,\varepsilon$ s.t. f has no zeros in $\overline{D(\alpha,\varepsilon)}\setminus\{\alpha\}$ (or else $f\equiv 0$)

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Let $2\delta = \min_{z \in C_\varepsilon(\alpha)} |f(z)| > 0$

Claim: $D(f(\alpha)=0, \delta) \subseteq \mathrm{Im}(f)$

Proof

$\forall w \in D(0, \delta)$, consider $f(z) - w$

If $w \notin f(D(\alpha, \varepsilon))$, then $f(z) - w$ has no zeros on $D(\alpha, \varepsilon)$

$\therefore |f(z) - w| \geq |f(z)| - |w| \geq f(z) - \delta \geq \delta \quad \forall z \in C_\varepsilon(\alpha)$

However, we know $|f(\alpha) - w| < \delta$ $\times$

$\therefore w \in f(D(\alpha, \varepsilon)) \Rightarrow D(0, \delta) \subseteq \mathrm{Im}(f)$ $\square$

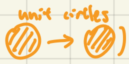
9-30-25 (WEEK 5) Shun/翔海(@shun4midx)

REMARK

We only have open mapping thm because extremum is not in interior pt.

SCHWARTZ'S LEMMA

THEOREM (SCHWARTZ'S LEMMA)

Suppose that f is analytic in an open unit disc D with $|f|\leq 1$ (f: ) and $f(0)=0$

Then, (i) $|f(z)|\leq |z|$

(ii) $|f'(0)|\leq 1$

with equality in either of the above iff $f(z)=e^{i\theta}z$

Proof

Define $g(z) := \begin{cases} \frac{f(z)}{z}, & z\neq 0 \\ f'(z), & z=0 \end{cases}$

$g(z)$ is ana on D since $f(z)$ is ana on D.

Consider $z\in C_r(0)$, $0<r<1$.

Then, $|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$

By max modulus thm, $\forall z\in \overline{D(0,r)}$, $|g(z)|\leq \frac{1}{r}$

As $r\to 1$, then $|g(z)|\leq 1 \ \forall z\in D$

By def of $g(z)$, $|f(z)|\leq |z|$ and $|f'(0)|\leq 1$ has either equality hold, when g is const and $|g|=1$ on D. $\therefore g=e^{i\theta}$ □

EXAMPLE (Removing $f(0)=0$ constraint)

Define $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha|<1$ — D

Then, (1) $B_\alpha(\alpha)=0$

(2) $B_\alpha(z)$ is ana on D, so $(B_\alpha(z))^2$ is ana on D. It is also conti on $\bar{D}$.

(3) $|B_\alpha(z)|^2\big|_{|z|=1}=1$, so by max modulus thm, $|B_\alpha(z)|\leq 1$ on D.

$\therefore$ We can use B_α for variations of Schwartz's lemma

EXAMPLE

Say f: ana on D, $|f(z)|\leq 1 \ \forall z\in D$ and $f(\frac{1}{2})=0$. Estimate $|f(\frac{3}{4})|$.

Consider $B_{\frac{1}{2}}(z) = \frac{z-\frac{1}{2}}{1-\frac{z}{2}}$.

We define $g(z) = \begin{cases} \frac{f(z)}{z-\frac{1}{2}}(1-\frac{1}{2}z), & z\neq \frac{1}{2} \\ \frac{3}{4}f'(\frac{1}{2}), & z=\frac{1}{2} \end{cases}$ $(B_\alpha(z))^{-1}$

Notice, $|B_{\frac{1}{2}}(z)|\leq 1$ on D, $|B_{\frac{1}{2}}(z)|=1$ on $C_1(0)$, and $B_{\frac{1}{2}}(z)$ conti on D.

$\therefore z\to 1$, $|B_{\frac{1}{2}}(z)| \longrightarrow 1$

$|g(z)|\leq 1/|B_{\frac{1}{2}}(z)| \Rightarrow |g(z)|\leq 1$ on D.

$\Rightarrow |f(z)|\leq |B_{\frac{1}{2}}(z)|$, $\frac{3}{4}|f'(\frac{1}{2})|\leq 1$

So, $|f(\frac{3}{4})| \leq |B_{\frac{1}{2}}(\frac{3}{4})| = \frac{2}{5}$

EXAMPLE

Say f is ana on D, $|f(z)|\leq 1$ on D. We claim: $|f'(\frac{1}{3})|$ is max when $f(\frac{1}{3})=0$ "your rate of improvement is max when your score is at its lowest" ≥≤? min modulus for $\mathbb{C}$.

<u>Proof</u>

Assume that $f(\frac{1}{3})\neq 0$.

$g(z) := \dfrac{f(z)-f(\frac{1}{3})}{1-\overline{f(\frac{1}{3})}f(z)} \Rightarrow g(z) = B_{f(\frac{1}{3})}(f(z))$. Note, g is bounded by 1, $g(\frac{1}{3})=0$. (Note it is ana too)

$\therefore g'(z)\big|_{z=\frac{1}{3}} = \dfrac{f'(\frac{1}{3})}{1-|f(\frac{1}{3})|^2} \Rightarrow |g'(\frac{1}{3})| > |f'(\frac{1}{3})|$ ($\because |f(\frac{1}{3})|>0$. Otherwise, then $|g'(\frac{1}{3})|$ can be larger, take $f:=g$ ✗)

$\therefore |f'(\frac{1}{3})|$ is max for $B_{\frac{1}{3}}(z)$.

PROPOSITION

Say f is entire. If $|f(z)| < \frac{1}{|\operatorname{Im} z|}$ $\forall z$, then $f\equiv 0$

<u>Proof</u>

Define $g(z) = (z^2-R^2)f(z)$, for some $R\in\mathbb{R}_{>0}$ (sufficiently large, e.g. $R\geq 1$, $R\to\infty$)

When $z\in C_R(0)$, $|z-R||z+R|\leq 2R|\operatorname{Im}(z)|$

$\therefore |g(z)| \leq \frac{2R}{|\operatorname{Im}(z)|^2} \leq 2R$ when $z\in C_R(0)$

By max modulus thm, $|g(z)|\leq 2R$ $\forall z\in D(0,R)$

$\Rightarrow |f(z)| \leq \frac{2R}{|z^2-R^2|}$ $\forall z\in D(0,R)$

As $R\to\infty$, $|f(z)|\to 0$

$\therefore f(z)=0$

MORERA'S THEOREM

THEOREM (MORERA'S THEOREM: CONVERSE OF RECTANGLE THEOREM)

Let f be continuous on an open set $D\subseteq\mathbb{C}$, and Γ be the boundary of a closed rectangle $R\subseteq D$.

If $\int_\Gamma f\,dz = 0$ $\forall\Gamma$ in $R\subseteq D$, then f is analytic in D.

<u>Proof</u>

Say $z_0\in D$, D: open.

Then, $\exists\varepsilon>0$, s.t. $D(z_0,\varepsilon)\subseteq D$.

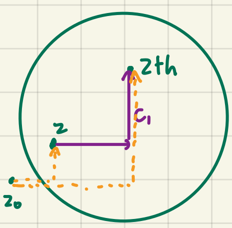
Define $F(z) := \int_C f(z)\,dz$ $\forall z\in D(z_0,\varepsilon)$, where $C: z_0 \to z_0+\operatorname{Re}(z-z_0)\to z$

For $z\in D(z_0,\varepsilon)$ and h small enough s.t. $z+h\in D(z_0,\varepsilon)$

Then,

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = \lim_{h\to 0}\int_{C_1} f(\omega)\,d\omega = f(z) \quad \square$$

f: cont.



EXAMPLE

Using $f(z)=\int_0^\infty \frac{e^{zt}}{t+1}\,dt$,

Claim: f is analytic $\forall z\in\{w\in\mathbb{C} \mid Re(w)<0\}$

Proof

We know for $z=x+iy$, $x<0$, $|e^{zt}|=e^{xt}$

Here, $\int_\Gamma\int_0^\infty \frac{|e^{zt}|}{t+1}\,dt\,dz < \int_\Gamma -\frac{1}{x}\,dz < \infty$ (ok for Fubini)

By Fubini's Thm, $\int_\Gamma\int_0^\infty \frac{e^{zt}}{t+1}\,dt\,dz=\int_0^\infty \underbrace{\boxed{\int_\Gamma \frac{e^{zt}}{t+1}\,dz}}_{=0\ (\because \text{By rectangle thm since } \frac{e^{zt}}{t+1}:\text{ ana})}\,dt=0$

$\therefore$ By Morera's Thm, $f(z)$ is analytic on $\{w\in\mathbb{C} \mid Re(w)<0\}$. □

DEFINITION

Let $\{f_n\}$ and f be defined on an open set D. We say that f_n converges uniformly on compacta if $f_n\to f$ uniformly on every compact subset $K\subseteq D$.

THEOREM

Let D be an open set in $\mathbb{C}$ and $\{f_n\}$ be a sequence of ana functions s.t. $f_n\to f$ unif on cpta. Then, f is also ana in D.

Proof

$\because$ f_n is conti, $\forall K\subseteq D$: cpt set we have $f_n\to f$ unif on K

$\therefore$ f is conti on K $\forall K$, i.e. f is conti on D

We hope "$\int_\Gamma f\,dz=0$", for Γ: boundary of a closed rectangle $R\subseteq D$

Hence, $\int_\Gamma f\,dz=\int_\Gamma \lim_{n\to\infty} f_n\,dz$

$$\| \ (f_n \text{ conti}, f_n\to f \text{ unif on } R)$$

$$\lim_{n\to\infty}\left(\int_\Gamma f_n\,dz\right)$$

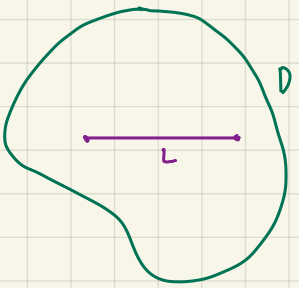
$$\| \ (\text{Rectangle thm} \because f_n:\text{ ana})$$

$$0$$

$\therefore$ By Morera's Thm, f is conti □

THEOREM

f is continuous on an open set $D\subseteq\mathbb{C}$ and analytic except on a line segment in D. Then, f is analytic throughout D.



Proof

Fixed in next PDF cuz I was def high when I wrote the proof for the wrong thm :: (yes, hence the reupload)

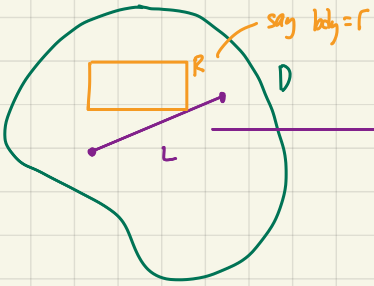
10-2-25 (WEEK 5) Shun/翔海 (@shun4midx)

REMARK

$f(z) = \frac{1}{z}$ is a counterexample as to why we cannot say "$\forall f$: ana exc on finitely many pts in a region $\Rightarrow f$: ana"

THEOREM

f is continuous on an open set $D \subseteq \mathbb{C}$ and analytic except on a line segment in D. Then, f is analytic throughout D.



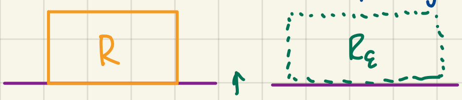
WLOG, $L \subseteq \mathbb{R}$-axis after a linear poly transformation :)

__Proof__

We know $f|_{D \setminus L}$: ana. Consider the following cases.

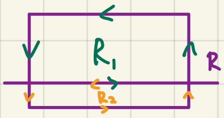
① $R \cap L = \emptyset \Rightarrow \int_\Gamma f dz = 0$ as f: ana on $D \setminus L$

② $R \cap L \subseteq \Gamma$, $R \cap L \neq \emptyset$: Lift one side, we get a rectangle $R_\varepsilon \subseteq R$, $R_\varepsilon \cap L = \emptyset$



By case ①, as f is cont., $\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} f(z) dz = 0 \Rightarrow \int_\Gamma f(z) dz = 0$

③ $R \cap L \not\subseteq \Gamma$



Then, $R = R_1 \cup R_2$, $\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f$

By cases ① and ②, hence $\int_\Gamma f(z) dz = 0$

$\therefore$ By Morera's Thm, f is analytic on D $\square$

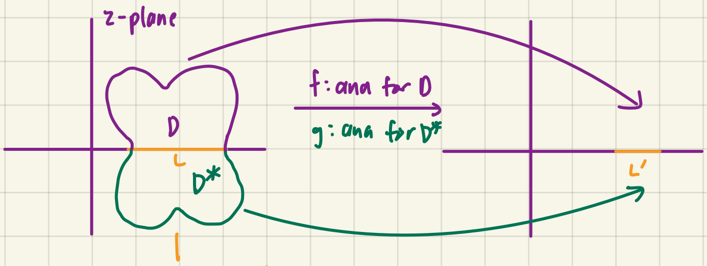
THEOREM (SCHWARZ REFLECTION PRINCIPLE)

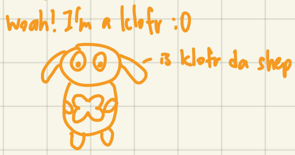
Suppose f is $\mathbb{C}$-analytic in a region D that is contained in either the upper or lower half plane and whose boundary contains a segment L on the real axis, and suppose f is real for real z.

Then, we can define an analytic "extension" g of f to the region $D \cup L \cup D^*$ that is symmetric with respect to the real axis by

$$g(z) = \begin{cases} f(z), & z \in D \cup L \\ \overline{f(\bar{z})}, & z \in D^* \end{cases}$$ where $D^* = \{z \mid \bar{z} \in D\}$

Graphically, we represent it as follows:





Shun/翔海(@shun4midx)

Proof

① $z \in D$, then $f|_D = g|_D$, f: ana $\Rightarrow g$: ana

② $z \in D^*$ and $z+h \in D^*$, then:

$$\lim_{h\to 0} \frac{g(z+h)-g(z)}{h} = \lim_{h\to 0} \overline{\left(\frac{f(\overline{z+h}) - f(\bar{z})}{\bar{h}}\right)} = \overline{f'(\bar{z})}$$

$\therefore$ g : ana

Since f : conti on $\mathbb{R}$-axis, so is g

$\therefore$ We can apply the thm above so g: ana throughout $D \cup L \cup D^* = U$ □

DEFINITION

A curve $\gamma : [a,b] \to \mathbb{C}$ is called a regular analytic arc if γ is an analytic map, 1-1 on $[a,b]$ with $\gamma' \neq 0$.

FACT (WILL PROVE IN THE FUTURE)

$\forall \alpha \in [a,b]$, $\exists D(\alpha, \varepsilon)$, s.t. γ: ana on $D(\alpha,\varepsilon)$, $\gamma'(z) \neq 0$ $\forall z \in D(\alpha, \varepsilon)$

In fact, $\exists \gamma^{-1}$: ana, $\gamma : D(\alpha, \varepsilon) \xrightarrow{\sim} \gamma(D(\alpha, \varepsilon))$

Proof Sketch

Map the boundary via γ to a real segment $[a,b]$, apply the thm, then apply γ^{-1}. Then, reflect similarly in image via λ.

Mathematically, $\lambda(\overline{\lambda^{-1}(f(\gamma(\overline{\gamma^{-1}(z)})))}$... XD (kill me)

10-7-25 (WEEK 6)

Shun/翔海 (@shun4midx)

SIMPLY CONNECTED DOMAIN

DEFINITIONS

Say $f: (S, \{U\}) \longrightarrow (S_1, \{N\})$ for two topological spaces

f is continuous if $\forall$ open $V \subseteq S_1$, $f^{-1}(V)$ is open in S

Path-connected: $\forall \alpha, \beta \in S$, $\exists \gamma: [0,1] \longrightarrow S$ continuously s.t. $\gamma(0)=\alpha$, $\gamma(1)=\beta$ (E.g. $\{x=0\} \cup \{\sin\frac{1}{x}\}$: connected but not path connected)

└ any open set cannot separate

IN $\mathbb{C}$ TOPOLOGICAL SPACE

We consider the topo space $\mathbb{C} \cong \mathbb{R}^2$

We say S is simply connected if it is:

① Path-connected ($\Rightarrow$ connected)

② For any continuous maps $f_i: [0,1] \longrightarrow S$ with $f_0(0)=f_1(0)$ and $f_0(1)=f_1(1)$, $\exists$ continuous $F: [0,1] \times [0,1] \longrightarrow S$ s.t. $F(t,0)=f_0(t)$, $F(t,1)=f_1(t)$ (intuitively: they are connected via shrinking a rubber band for any two pts)

Actually, $S=\mathbb{C} \cong \mathbb{R}^2$ is simply connected and a torus is not

└ can be seen as a disc with a hole taken out (disc裡挖出一個洞)

RECALL

f: ana on closed disc $D(0,1)$, then $\int_{C_1(0)} f\,dz = 0$

Torus

However, remember $\int_{C_1(0)} \frac{1}{z}\,dz = 2\pi i$, note that $\frac{1}{z}$: well-defined on $D(0,1) \setminus \{0\}$

Motivation: Simply via integration, we can determine the topo nature of subsets of $\mathbb{C}$, equal zero or nonzero?

↳ "Holomorphic simply connected" (= "topo simply connected")

DEFINITION

D is holomorphic simply connected (hsc) if $\forall f$: ana on D, $\int_\Gamma f\,dz = 0$ $\forall$ simple closed curve $\Gamma \subseteq D$

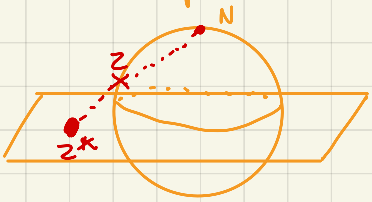
FACT: In $\mathbb{C}$, hsc $\Leftrightarrow$ sc

DEFINITION

We say the extended $\mathbb{C}$-plane is $\mathbb{C} \cup \{\infty\} \cong S^2$ (sphere)

┌ basically treat $\{\infty\}$ as one point, no more $\pm\infty$ as opposites, they are the same concept

∞ is the north pole in this stereographic projection



DEFINITION

If D is open connected $\subseteq \mathbb{C}$, then D is sc if $(\mathbb{C} \cup \{\infty\}) \setminus D$ is path connected

┌ simply connected

DEFINITION'

For a region $D \subseteq \mathbb{C}$, we say D is sc' if $\forall z_0 \in \mathbb{C} \setminus D$, $\forall \varepsilon > 0$, $\exists$ path $\gamma: [0,\infty) \longrightarrow \mathbb{C} \setminus D = \tilde{D}$, s.t.

① $d(\gamma, \tilde{D}) \leq \varepsilon$

② $\gamma(0) = z_0$

③ $\lim_{t\to\infty} \gamma(t) = \infty$

$\boxed{\text{Def} \Rightarrow \text{sc} \Rightarrow \text{hsc} \quad \text{Def}'}$

Shun/翔海(@shun4midx)

REMARK

For open $D \subseteq \mathbb{C}$, D is open connected $\Leftrightarrow$ D is connected

Proof

"$\Rightarrow$": Locally path connected (i.e. $\forall x \in D$, $\exists$ open nbd $U(x) \subseteq D$ s.t. $U(x)$: path-connected

DEFINITION

e.g. level 3: _‾

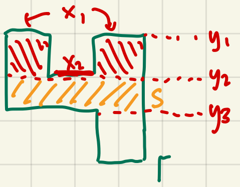
Given a polygonal path Γ ( ), we define the level of $\Gamma := \#$ diff values y_0 where the line $\operatorname{Im} z = y_0$ contains a horizontal segment of Γ

LEMMA

no intersections w/ itself

Γ: simple closed curve that is also a polygonal curve, say $\Gamma \subseteq D$, where D: SC region

Suppose the top level of Γ consists of points $y = y_1$, $x \in X_1$ and $y = y_2$, $x \in X_2$. Then the set $R := \{z = x + iy \mid y_2 \leq y \leq y_1, x \in X_1\}$ is contained in D



Proof

Consider induction on the level of Γ.

For $\operatorname{lev}(\Gamma) \geq 2$,

- $\operatorname{lev}(\Gamma) = 2$: $R = \bigcup_i R_i$, R_i: closed rectangle and $\bigcup \partial R_i = \Gamma$

  Let $z_0 \in R$.

  Suppose $z_0 \notin R$. As D is simply connected, $\exists \gamma: [0, \infty] \longrightarrow (\mathbb{C} \cup \{\infty\}) \setminus D$ s.t. $\gamma(0) = z_0$ and $\lim_{t \to \infty} \gamma(t) = \infty$, $t_0 := \sup\{t \mid \gamma(t) \in R\}$

  Claim: $\gamma(t_0) \in \Gamma \subseteq D$ ($\Rightarrow$ $\divideontimes$) [annotation: $\setminus D$ $\divideontimes$]

  Proof

  (i) $\gamma(t_0) \notin R \setminus \Gamma$: open (If no, then $\gamma(t_0) \in \Gamma$ or $\gamma(t_0) \in \mathbb{C} \setminus R$)

  If yes, as $R \setminus \Gamma$: open, $\exists D(\gamma(t_0), \varepsilon) \subseteq R \setminus \Gamma$, $t_0 \in \gamma^{-1}(D(\gamma(t_0), \varepsilon)) \subseteq [0, \infty)$ (as $t_0 = \sup\{t \mid \gamma(t) \in R\}$) $\divideontimes$ [annotation: $\lim_{t \to \infty} \gamma(t) = \infty$]

  (ii) $\gamma(t_0) \notin \mathbb{C} \setminus R$: Similarly, $\divideontimes$

  $\therefore$ $\gamma(t_0) \in \Gamma \subseteq D$ $\divideontimes$ [annotation: $\setminus D$ $\divideontimes$]

- $\operatorname{lev}(\Gamma) > 2$: Note: $\bigcup(\partial R_i)$ not necessarily equal to Γ

  By the same argument as base case, $t_0 = \sup\{t \mid \gamma(t) \notin R\}$, $t_0 \in \partial R_i$ for some R_i

  If $t_0 \in \Gamma$, as in the base case, $\divideontimes$

  Def $\tilde{R} := \mathbb{C} \setminus R$, $\Gamma' := (\Gamma \cap \tilde{R}) \cup \bar{L}$, where $L := \{x + iy \mid y = y_2, x \in X_1 \setminus X_2\}$

  For small h, $\gamma(t_0 + h)$ is between the top two levels of Γ'

  As $\gamma(t_0 + h) \notin D$ and $\gamma(t_0 + h) \in S$, $\operatorname{level}(\Gamma') < \operatorname{level}(\Gamma)$

  $\therefore$ By induction, $\gamma: [t_0 + h, \infty] \to \infty$ intersects $\Gamma' \setminus \bar{L}$ $\square$

  as $\forall t > t_0$, $\gamma(t) \notin R$

10-9-25 (WEEK 6) Shun/翔海 (@shun4midx)

USED DEFINITION

For a region $D \subseteq \mathbb{C}$, D is simply connected if $(\mathbb{C} \cup \{\infty\}) \setminus D$ is path connected

(We want to consider domain D s.t. $\int_\Gamma f(z)\,dz = 0$, f: analytic over D, $\Gamma \subseteq D$: simple closed curve)

THEOREM

f: ana in a s.c. region D and $\Gamma \subseteq D$ simple closed polygonal path. Then, $\int_\Gamma f\,dz = 0$

Proof

Lemma from prev note $\Rightarrow R \subseteq D \Rightarrow \partial R \subseteq D$

As $\Gamma = \partial R + \Gamma'$, $\int_\Gamma f\,dz = \int_{\partial R} f\,dz + \int_{\Gamma'} f\,dz$

By rectangle thm, $R \subseteq D$, f: ana in $D \Rightarrow f$: ana on $R \Rightarrow \int_{\partial R} f\,dz = 0$

$\therefore$ By induction on lev(Γ), we get $\int_\Gamma f\,dz = 0$ □

THEOREM

f: ana on a s.c. region $D \Rightarrow \exists$ primitive F, $F' = f$

Proof

Fix $z_0 \in D$, define $F(z) = \int_\Gamma f(\zeta)\,d\zeta$, where Γ = any polygonal path from z to $z_0 \subseteq D$

- F is well-defined: Suppose Γ_1, Γ_2 satisfy the polygonal path condition

  Then, $\Gamma_1 - \Gamma_2 = \bigcup_{i=1}^{k} C_i$, C_i: simple closed polygonal curve $\subseteq D$

  $\Rightarrow \int_{\Gamma_1} f(\zeta)\,d\zeta - \int_{\Gamma_2} f(\zeta)\,d\zeta = \sum_{i=1}^{k} \int_{C_i} f(\zeta)\,d\zeta \overset{\text{by thm above}}{=} 0$ ✓

Now, let h be small enough s.t. $z+h \in D$

$\Rightarrow \frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left[\int_{\Gamma_1} f(\zeta)\,d\zeta - \int_{\Gamma_2} f(\zeta)\,d\zeta\right] = \frac{1}{h}\int_{\Gamma_3} f(\zeta)\,d\zeta$, where

Γ_1: any poly path $z_0 \to z+h \subseteq D$

Γ_2: any poly path $z_0 \to z \subseteq D$

Choose Γ_2 first, then $\Gamma_1 = \Gamma_2 + \Gamma_3$, where Γ_3: any poly path $z \to z+h \subseteq D$

Then, $\lim_{h \to 0} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \lim_{h \to 0} \frac{1}{|h|} \left| \int_{\Gamma_3} \underbrace{[f(\zeta) - f(z)]}_{z \to z+h \subseteq D}\,d\zeta \right| = 0$ □

THEOREM (CLOSED CURVE THEOREM)

Let f: ana on a s.c. region D. Then, $\forall$ simple closed curve $C \subseteq D$, $\int_C f(z)\,dz = 0$

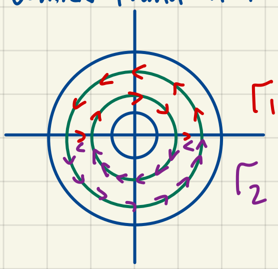
Proof

By Thm 3, $f = F'$ for some ana F

$\therefore$ For any $C: \gamma(t): [0,1] \to D$, $\int_C f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) = 0$ $\because$ closed $\Rightarrow \gamma(1) = \gamma(0)$ □

EXAMPLE

Consider f: ana on $1 < |z| < 4$



Claim: $\int_{C_2(0)} f(z)dz = \int_{C_3(0)} f(z)dz$

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Proof

We have $\int_{C_3(0)} f(z)dz - \int_{C_2(0)} f(z)dz = \boxed{\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz} = 0$ by closed curve thm $\square$

THE PROBLEM WITH DEFINING LOG

$\log z := u(z) + iv(z) \Rightarrow \underbrace{z = e^{u(z)}e^{iv(z)}}_{|z| = |e^{u(z)}|}$, but $\theta = v(z) + 2\pi k$, $k \in \mathbb{Z}$ all are fine, so how do we fix a value so log is well-def?

DEFINITION

We say f is an analytic branch of $\log z$ in a domain D if:

(i) f is analytic

(ii) $e^{\log z} = z$, candidate: $f(z) = \underbrace{\log}_{\log \text{ over } \mathbb{R}} |z| + i\underbrace{\operatorname{Arg} z}_{\in [0, 2\pi)}$

10-14-25 (WEEK 7)

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RESOLVING THE WELL-DEFINEDNESS OF LOG OVER $\mathbb{C}$

We can consider approaching $\log z$ via integration.

We want $\log z$ s.t. ① f: ana
② $\exp(f(z)) = z$

$\therefore$ If $f(z) = \log z$, we want it to satisfy $f'(z) = \frac{1}{z}$

We can $\boxed{\text{fix } z_0 \in \mathbb{R}^+}$, so $f(z) := \int_C \frac{1}{\zeta} d\zeta + \boxed{\log(z_0)}$ where $C: z_0 \to z$

However, we need $\int_C \frac{1}{\zeta} d\zeta$ to be well-defined indep of path...

$\therefore$ Choose a s.c. region D, then $\forall C \subseteq D$, $\int_C \frac{1}{\zeta} d\zeta$ is well-defined (details in proof below)

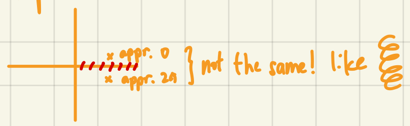
Proof (Sketch)

$\forall C_1, C_2 \subseteq D$ with the same endpoints, $C_1 - C_2$ forms a closed path in D

$\therefore \int_{C_1 - C_2} \frac{1}{\zeta} d\zeta = 0 \Rightarrow \int_{C_1} \frac{1}{\zeta} d\zeta = \int_{C_2} \frac{1}{\zeta} d\zeta$

THEOREM

Example for D



Set $f(z) := \int_{z_0}^{z} \frac{1}{\zeta} d\zeta + \log z_0$ on a s.c. region $D \subseteq \mathbb{C} \setminus \{0\}$, we fix a $z_0 \in D$ and choose $\log z_0$

Then, f is an analytic branch of $\log z$ in D.

Proof

As D: s.c., $C_1 - C_2$: closed curve

$\therefore$ By closed curve thm, $\int_{C_1 - C_2} \frac{1}{\zeta} d\zeta = \int_{C_1} \frac{1}{\zeta} d\zeta - \int_{C_2} \frac{1}{\zeta} d\zeta = 0$

$\therefore f$ is analytic

Moreover, we want "$\exp(f(z)) = z$" $\Leftrightarrow$ "$ze^{-f(z)} = 1$"

Set $g(z) := ze^{-f(z)} \Rightarrow g'(z) = e^{-f(z)} - zf'(z)e^{-f(z)} = \underline{\underline{0}}$

$\therefore g(z) = \text{const} = g(z_0) = z_0 e^{-\log z_0} = 1$

APPLICATION

Instead of this directly only used for log, we can use analytic branch to define $\sqrt{z}$.

Reason:

Say $z = re^{i\theta}$, then $(\sqrt{r}e^{i(\frac{\theta}{2} + \pi k)})^2 = re^{i\theta} = z$ $\forall k \in \mathbb{Z}$, so $\sqrt{z}$ is not uniquely defined

$\therefore$ For $f(z) = \sqrt[n]{z}$, we can define an analytic branch for $\log z$ by $\sqrt[n]{z} = \frac{1}{n}\exp(\log z + 2\pi n k)$

SINGULARITY

DEFINITION

A deleted neighborhood of z_0 is an open set of $\{z \mid \underline{0 < |z - z_0|} < \delta\}$

the actual deleted thing lmao

DEFINITION

f is said to have an isolated singularity at z_0 if f is analytic in a deleted neighborhood D of z_0 but is not analytic in z_0

EXAMPLES (Intuition, formal names given later)

① "Artificial" singularity: $f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$

② "Fixable by multiplying a polynomial": $\frac{1}{z}$ at $z = 0$

③ "Unfixable": $\exp(\frac{1}{z})$ at $z = 0$

DEFINITION

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Say z_0 is a singularity of f, we can classify it as follows:

① If $\exists\, g$: ana at z_0 and $f(z)=g(z)$ in some deleted nbd of z_0, we say f has a removable singularity at z_0

② If for $z\neq z_0$, f can be written as $f(z)=\frac{A(z)}{B(z)}$ where A and B are analytic at z_0, $A(z_0)\neq 0$, $B(z_0)=0$, we say f has a pole at z_0. In particular, if B has a zero of order k at z_0, then we say z_0 is a pole of f of order k

③ f has neither a removable singularity nor a pole at z_0, then we call z_0 an essential singularity of f (not the focus of this course)

THEOREM (RIEMANN'S PRINCIPLE OF REMOVABLE SINGULARITIES)

If f has an isolated singularity at z_0 and if $\lim_{z\to z_0}(z-z_0)f(z)=0$, then the singularity is removable

Proof

Define $D'(z_0,\delta) := D(z_0,\delta)\setminus\{z_0\}$, $\exists\,\delta$, s.t. f: ana on $D'(z_0,\delta)$

Set $g(z) := \begin{cases}(z-z_0)f(z), & z\in D'(z_0,\delta)\\ 0, & z=z_0\end{cases}$

Since $\lim_{z\to z_0}(z-z_0)f(z)=0=g(z_0)$, hence g is conti at z_0.

$\because$ f: ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D'(z_0,\delta)$

Morera needs it to be conti on the whole domain.

$\therefore$ g: conti on $D(z_0,\delta)$ + ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D(z_0,\delta)$ (apply the conti except on a line segment thing)

Now, set:

$h(z) := \begin{cases}\frac{g(z)-\cancel{g(z_0)}^{\,0}}{z-z_0}, & z\in D'(z_0,\delta)\\ g'(z), & z=z_0\end{cases}$

h is ana because g is ana.

Moreover, as $f(z)=h(z)$ on $D'(z_0,\delta)$, thus z_0 is a removable singularity

COROLLARY

f has an isolated singularity at z_0. If f is bounded on some deleted nbd of z_0, then z_0 is a removable singularity

Proof

$\exists\,\delta$, s.t. f: ana and bounded on $D'(z_0,\delta)$

Given $\varepsilon>0$, $\forall\, 0<|z-z_0|<\frac{\varepsilon}{M}$, $|(z-z_0)f(z)|<\varepsilon \Rightarrow \lim_{z\to z_0}(z-z_0)f(z)=0$

$\therefore$ Conclude with thm above. $\square$

THEOREM 3

Say f has an isolated singularity at z_0.

If $\exists\, k\in\mathbb{Z}_{\geq 0}$, s.t. $\lim_{z\to z_0}(z-z_0)^k f(z)\neq 0$ but $\lim_{z\to z_0}(z-z_0)^{k+1}f(z)=0$, then f has a pole of order k at z_0 (rem. sing = pole of order 0)

Proof

Set $g(z)=\begin{cases}(z-z_0)^{k+1}f(z), & z\in D'(z_0,\delta)\\ 0, & z=z_0\end{cases}$

$\because \lim_{z\to z_0}(z-z_0)^{k+1}=0$ $\therefore$ g: conti at z_0

$\because$ f: ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D'(z_0,\delta)$

$\therefore$ g: conti on $D(z_0,\delta)$ + ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D(z_0,\delta)$

Set $h(z) := \begin{cases} \frac{g(z)-g(z_0)}{z-z_0} = (z-z_0)^k f(z), & z \in D'(z_0, \delta) \\ g'(z), & z = z_0 \end{cases}$

$\therefore$ h: ana on $D(z_0, \delta)$.

As we know, by assumption, $\lim_{z \to z_0} h(z) \neq 0 \Rightarrow h(z_0) \neq 0$ ($\because h$: ana)

$\therefore$ $f(z) = \frac{h(z)}{(z-z_0)^k} \Rightarrow f$ has a pole of order k at z_0 □

REMARK

$|f(z)| < \frac{1}{\sqrt{|z|}}$ in a deleted nbd of 0 and f has an isolated singularity at $0 \Rightarrow 0$ is a removable singularity

($\because$ There exists nonbounded removable singularity)

Proof

Actually, $|zf(z)| < \sqrt{|z|} \Rightarrow \lim_{z \to z_0} zf(z) = 0 \Rightarrow 0$ is a removable singularity □

REMARK

Similarly, if we have $|f(z)| < \frac{1}{|z|^{1.5}}$, then we know $|z^2 f(z)| < \sqrt{|z|} \Rightarrow \lim_{z \to z_0} z^2 f(z) = 0$

$\therefore \begin{cases} \text{Case 1: } \lim_{z \to z_0} zf(z) = 0, \text{ then removable singularity (pole of order 0)} \\ \text{Case 2. } \lim_{z \to z_0} zf(z) \neq 0, \text{ then pole of order 1} \end{cases}$

$\Rightarrow$ It has a pole of at most order 1 (higher the order $\Rightarrow$ the worse the pole)

THEOREM (CASORATI – WEIERSTRASS THEOREM)

If f has an essential singularity at z_0 and D is a deleted neighborhood of z_0, where f is analytic, then the range $R := \{f(z) | z \in D\}$ is dense in $\mathbb{C}$

Proof

Suppose not, then $\exists w \in \mathbb{C}$ and $\delta > 0$, s.t. open $D(w, \delta) \cap R = \phi$

In other words, $\forall z \in D$, $|f(z) - w| \geq \delta \Rightarrow \left|\frac{1}{f(z)-w}\right| \leq \frac{1}{\delta}$ $\forall z \in D$ $\Rightarrow \frac{1}{f(z)-w}$ is bounded in the del nbd

By coro, $\frac{1}{f(z)-w}$ has a removable singularity at z_0

$\therefore \exists g$: ana on $D' \cup \{z_0\}$, s.t. $g(z) = \frac{1}{f(z)-w} \Rightarrow f(z) = w + \frac{1}{g(z)}$ $\forall z \in D'$

$\therefore$ z_0 is a zero of $g(z)$ of $\underbrace{\text{finite order}}_{= n}$ or $g(z_0) \neq 0$

$\therefore$ $f(z)$ has a pole of order $\leq n$ at z_0, so not an essential singularity ※

10-16-25 (WEEK 7) Shun/翔海 (@shun4midx)

DEFINITION

We say $\sum_{i=-\infty}^{\infty} \mu_i = L$ if $\sum_{i=0}^{\infty} \mu_i$ and $\sum_{j=1}^{\infty} \mu_{-j}$ both converge and their sum is L

THEOREM

We dk $R_1 ? R_2$, so cannot say $R_1 < |z| < R_2$

$f(z) = \sum_{-\infty}^{\infty} a_k z^k$ is convergent in the domain $D = \{z : R_1 < |z| \text{ and } |z| < R_2\}$, where $R_2 = \left(\limsup_{k\to\infty} |a_k|^{\frac{1}{k}}\right)^{-1}$, $R_1 = \limsup_{k\to\infty} |a_{-k}|^{\frac{1}{k}}$

Proof

$\sum_{k=0}^{\infty} a_k z^k$ converges when $|z| < \left(\limsup_{k\to\infty} |a_k|^{\frac{1}{k}}\right)^{-1}$

$\sum_{k=1}^{\infty} a_{-k} (z^{-1})^k$ coneges when $|z^{-1}| < \left(\limsup_{k\to\infty} |a_{-k}|^{\frac{1}{k}}\right)^{-1} \Rightarrow |z| > \limsup_{k\to\infty} |a_{-k}|^{\frac{1}{k}}$

$\therefore f$: conv $\forall z \in D$.

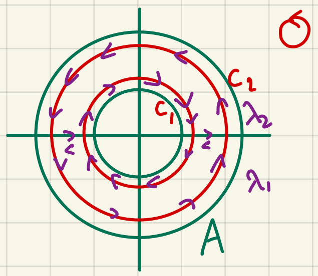
THEOREM

Laurent is when you expand around z, f at z: not ana (Taylor $\Rightarrow$ ana)

Let $A := \{z \mid R_1 < |z| < R_2\}$. If f is analytic in A, then f has a Laurent expansion $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ in A

Proof

Say $C_i := R_{r_i}(0)$, $R_1 < r_1 < r_2 < R_2$, $A = A_1 \cup A_2$, $A_1 = \{z \mid z \in A \text{ and } \operatorname{Im} z > -1\}$, $A_2 = \{z \mid z \in A \text{ and } \operatorname{Im} z < 1\}$.



$\therefore C_2 - C_1 = \lambda_1 + \lambda_2$

$\therefore \underbrace{\int_{C_2 - C_1} \frac{f(\omega) - f(z)}{\omega - z} d\omega}_{(\star)} = \int_{\lambda_1 + \lambda_2} \frac{f(\omega) - f(z)}{\omega - z} d\omega \quad \forall \omega \in A$

As f is ana in A,

$$g(\omega) := \begin{cases} \frac{f(\omega) - f(z)}{\omega - z}, & \omega \in A \setminus \{z\} \\ f'(\omega), & \omega = z \end{cases} \text{ is ana}$$

$\Rightarrow (\star) = \int_{\lambda_1} \frac{f(\omega) - f(z)}{\omega - z} d\omega + \int_{\lambda_2} \frac{f(\omega) - f(z)}{\omega - z} d\omega = 0$ by closed curve thm,

Moreover, by closed curve thm, $\int_{C_1} \frac{d\omega}{\omega - z} = 0$

As $\underbrace{\int_{C_2} \frac{d\omega}{\omega - z} = 2\pi i}_{\text{not } A, \text{ so why}}$, $f(z) \int_{C_2 - C_1} \frac{d\omega}{\omega - z} = 2\pi i f(z)$

$$\therefore 2\pi i f(z) = \int_{C_2 - C_1} \frac{f(\omega)}{\omega - z} d\omega = \int_{C_2} \frac{f(\omega)}{\omega - z} d\omega - \int_{C_1} \frac{f(\omega)}{\omega - z} d\omega$$

$$= \int_{C_2} \frac{f(\omega)}{\omega(1 - \frac{z}{\omega})} d\omega + \int_{C_1} \frac{f(\omega)}{z(1 - \frac{\omega}{z})} d\omega$$

$$= \boxed{\int_{C_2} \left[\frac{f(\omega)}{\omega} \sum_{i=0}^{\infty} \left(\frac{z}{\omega}\right)^i\right] d\omega}^{②} + \boxed{\int_{C_1} \left[\frac{f(\omega)}{z} \sum_{i=0}^{\infty} \left(\frac{\omega}{z}\right)^i\right] d\omega}^{①}$$

We know $\sum_{i=0}^{\infty} \left(\frac{z}{\omega}\right)^2$ conv abs unif on C_2.

Hence, ② $= \sum_{i=0}^{\infty} \left[\int_{C_2} \frac{f(\omega)}{\omega^{i+1}} d\omega\right] z^i$

① $= \sum_{i=0}^{\infty} \left[\int_{C_1} f(\omega) \omega^i d\omega\right] z^{-(i+1)}$

$\therefore f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \quad \forall R_1 < |z| < R_2$ □

Uniqueness

$\int_C z^{i-(k+1)}\,dz = \begin{cases} 0, & i-k\neq 0 \\ 2\pi i, & i-k=0 \end{cases}$

If $f(z)=\sum_{k=-\infty}^{\infty} c_k z^k \ \forall R_1<|z|<R_2$, then $\int_C \frac{f(z)}{z^{k+1}}\,dz = \int_C \sum_{i=-\infty}^{\infty} c_i z^{i-(k+1)}\,dz = \sum_{i=-\infty}^{\infty} c_i z^{i-(k+1)}\,dz$ for $C=C_r(0)\subseteq A$.

$\therefore \int_C \frac{f(z)}{z^{k+1}}\,dz = c_k 2\pi i \ \forall C=C_r(0)\subseteq A$

Thus, (i) $\int_C \frac{f(z)}{z^{k+1}}\,dz = c_k$ is indep of r for $C=C_r(0)\subseteq A$

(ii) $a_k = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{i+1}}\,dw$ is indep of C_1 and C_2

$\therefore$ The Laurent expansion of f in A is unique $\square$

REMARK

Taking $R_1=0$, we can take Laurent expansion at a pole or removable singularity.

10-23-25 (WEEK 8) (Sorry if the notes are shit, I'm very fucking upset at myself and attended CompAna today just trying not to cry lol, DM me if any part is unclear, thx. Why am I even fucking offering my notes as such a shit and lazy student anyway, it's not like my notes are beneficial)

Shun/翔海 (@shun4midx)

EXAMPLES OF LAURRENT EXPANSIONS

Around $z=0$,

$\frac{(z+1)^2}{z} = z + \frac{1}{z} + 2$

$\exp(\frac{1}{z}) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \dots$

$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}(1+z+z^2+\dots) = \frac{1}{z^2} + \frac{1}{z} + 1 + z + \dots$

DEFINITION

If $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ is a Laurent expansion of f around an isolated singularity z_0. Then, $\sum_{-\infty}^{-1} a_k(z-z_0)^k$ is called the principal part of f at z_0 and $\sum_{i=0}^{\infty} a_i(z-z_0)^i$ is called the analytic part of f

PROPOSITION

(i) f has a removable singularity at z_0. Then, $a_k = 0 \ \forall k<0$

(ii) f has a pole of order k at z_0. Then, $a_i = 0 \ \forall i < -k$ but $a_{-k} \neq 0$

(iii) f has an essential singularity at z_0. Then, it must have inf many nonzero terms in its principal part

Proof

(i) $\exists D'(z_0, \delta)$ and $\exists$ ana g on $D(z_0, \delta)$, s.t. $g = f$ on $D'(z_0, \delta)$

$\because$ g is ana at z_0

$\therefore$ $g = \sum_{i=0}^{\infty} b_i(z-z_0)^i$, $|z-z_0|<\delta$

$\because$ f: ana on $D'(z_0, \delta)$

$\therefore$ $f(z) = \sum_{i=-\infty}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} b_k(z-z_0)^k$ on $D'(z_0, \delta)$

By uniqueness of Laurent expansion, $a_k = 0 \ \forall k<0$ □

(ii) $f = \frac{A(z)}{B(z)}$, A, B ana on $D(z_0, \delta)$ with $A(z_0) \neq 0$ and $B(z)$ has a zero of order k at z_0.

$\therefore$ $B(z) = C_k(z-z_0)^k + C_{k+1}(z-z_0)^{k+1} + \dots$, $C_k \neq 0$

Thus, $B(z) = (z-z_0)^k \underbrace{(C_k + C_{k+1}(z-z_0)^1 + \dots)}_{=: H(z)}$

$\because$ $H(z)$ is ana on $D(z_0, \delta)$ and $H(z_0) \neq 0$

$\therefore$ $f(z) = \frac{1}{(z-z_0)^k} \frac{A(z)}{H(z)}$

As H: cont, $\exists D(z_0, \delta_1)$, $\delta_1 \leq \delta$, s.t. H has no zeros on $D(z_0, \delta_1)$.

$\therefore$ $f = \frac{1}{(z-z_0)^k} \frac{A(z)}{H(z)}$ on $D'(z_0, \delta_1)$ with $\frac{A(z)}{H(z)}$ ana on $D(z_0, \delta_1)$

We consider Taylor expansion of $\frac{A}{H}$ around z_0.

Then, $\frac{A(z)}{H(z)} = \sum_{i=0}^{\infty} e_i(z-z_0)^i$

$\because$ $A(z_0) \neq 0$, $H(z_0) \neq 0$

$\therefore$ $e_0 \neq 0 \Rightarrow a_i = 0 \ \forall i < -k$, $a_{-k} \neq 0$ □

(iii) z_0 is an essential singularity, $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ on $D'(z_0, \delta)$

If only finitely many $a_k \neq 0$ for $k>0$, then $\lim_{z \to z_0} f(z)(z-z_0)^N = 0$ for big enough N $\rightarrow\!\!\leftarrow$

PROPOSITION 10

P, Q: poly with $\deg P < \deg Q$.

Say $Q(z) = \prod_{i=1}^{n} (z-z_i)^{e_i}$ with distinct z_i. Then, $R(z) = \frac{P(z)}{Q(z)}$ is a sum of polynomials in $\frac{1}{z-z_i}$ with $1 \leq i \leq n$

Proof

$R(z)$ is ana on $\mathbb{C} \setminus \{z_1, \dots, z_n\}$.

z_i is an isolated singularity of R which is a pole of order at most e_i

Then, $R(z) = \sum_{i=-e_1}^{\infty} (z-z_1)^i = \underbrace{\sum_{i=-e_1}^{-1} (z-z_1)^i}_{P_1\left(\frac{1}{z-z_1}\right)} + \underbrace{\sum_{i=0}^{\infty} c_i (z-z_1)^i}_{A_1(z)}$

Then, $A_1(z) = R(z) - P_1\left(\frac{1}{z-z_1}\right)$ is ana on U.
$\lim_{z\to z_1} (z-z_1) A_1(z) = 0 \Rightarrow z_1$ is a removable singularity at $A_1(z) \Rightarrow A_1(z_1) := \lim_{z\to z_1} A_1(z)$
$\therefore A_1(z)$ ana on $\mathbb{C}\setminus\{z_1, \dots, z_n\} \Rightarrow$ Inductively, we get $A_n(z) := A_{n-1}(z) - P_n\left(\frac{1}{z-z_n}\right) \rightsquigarrow$ principal part of $R(z)$

However, A_n is bounded since $R, P_n \xrightarrow{z\to\infty} 0$.
$\therefore$ By Liouville's Thm, A_n is const, so $A_n \equiv 0$
$\therefore R(z) = P_1\left(\frac{1}{z-z_1}\right) + \dots + P_n\left(\frac{1}{z-z_n}\right)$ □

THE RESIDUE THEOREM

KEY POINT

$C_1(0) \Rightarrow \int_{C_1(0)} z^k \, dz = \begin{cases} 2\pi i, & k=-1 \\ 0, & \text{otherwise} \end{cases}$

$(\star)$

f is ana on $D'(z_0, \delta) \Rightarrow \boxed{f = \sum_{i=-\infty}^{\infty} a_i (z-z_0)^i, \quad 0<|z-z_0|<\delta.}$
As $C_r(z_0) \subseteq D'(z_0, \delta) \Rightarrow \int_{C_r(z_0)} f \, dz = a_{-1}(2\pi i)$

DEFINITION

$(\star)$, then we define $\mathrm{Res}(f, z_0) := -1$

PROPOSITION

Given $(\star)$,
(i) z_0 is a simple pole (pole of order 1) then $a_{-1} = \lim_{z\to z_0} (z-z_0) f(z) = \frac{A(z_0)}{B'(z_0)}$
Proof
$(z-z_0) f(z) = A(z) \div \frac{B(z) - \cancelto{0}{B(z_0)}}{z-z_0}$. A, B : ana $\Rightarrow$ done

(ii) If f has a pole of order k, then $a_{-1} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-z_0)^k f(z)\right)\Big|_{z=z_0}$

EXAMPLE

$\mathrm{Res}(\csc z, 0) = \frac{1}{\cos z}\Big|_{z=0} = 1$
$\llcorner$ $\frac{1}{\sin z}$ has a simple pole at $z=0$.

10-28-25 (WEEK 9)

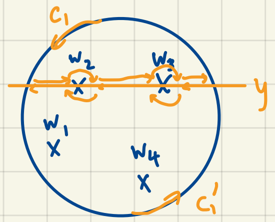
Shun/翔海 (@shun4midx)

PROPOSITION

For a region D, and f ana on $D \setminus \{w_1, \dots, w_n\}$, with $w_i \in D(0,r)$, we have $\int_{C_r(0)} f(z)dz = 2\pi i \sum_{i=1}^{n} \text{Res}(f, w_i)$

Proof

Recall: $\int_{C_r(0)} \frac{1}{z} dz = 2\pi i$ (indep of r) $\Rightarrow \int_{C_r(0)} \frac{1}{z-a} dz = 2\pi i$



Let $Y := \{\text{Im}(w_i) \mid 1 \leq i \leq n\}$, where we order Y as $y_1 > y_2 > \dots > y_n$

Choose ε, s.t. $D(w_i, \varepsilon) \subseteq D(0,r)$ and $D(w_i, \varepsilon)$ are mutually disjoint

$\Rightarrow \int_{C_1 + C_1'} f(z)dz = \int_{C_r(0)} f(z)dz - \sum_{i:\text{Im}(w_i)=y_1} \int_{C_\varepsilon(w_i)} f(z)dz$

By Closed Curve Thm on simply connected domain, $\int_{C_1'} f(z)dz = \int_{C_r(0)} f(z)dz - \sum_{i:\text{Im}(w_i)=y_1} \int_{C_\varepsilon(w_i)} f(z)dz$

As we know, $C_2 + C_2' = C_1' - C_\varepsilon(w_1) \Rightarrow \int_{C_1'} f(z)dz = \cancel{\int_{C_2} \square}^{\,0} + \int_{C_2'} + \sum \int_{C_\varepsilon(w_i)}$

Inductively, we get: $\int_{C_r(0)} f(z)dz = \sum_{i=1}^{n} \int_{C_\varepsilon(w_i)} f(z)dz$

With the Laurent Expansion $f(z) = \sum_{k=-\infty}^{\infty} (z-w_i)^k$, hence $\int_{C_r(0)} f(z)dz = 2\pi i \sum_{i=1}^{n} \text{Res}(f, w_i)$ □

OBSERVATION

For a simple closed curve Γ, if we can define the "interior" (e.g. not ⓔ) of Γ, then we can replace the above proposition by $D \supseteq \Gamma \cup \text{int}(\Gamma)$ with replacing $C_r(0)$ by Γ.

WINDING NUMBER — CAUCHY RESIDUE THEOREM

DEFINITION

(i) Suppose γ is a closed curve and $a \notin \gamma$. We define the **winding number** of γ around a to be $\eta(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$

(E.g. $\gamma: C_r(a)$, $\eta(\gamma, b) = 1 \Leftrightarrow b \in D(a,r)$)

(ii) γ is called a **regular closed curve** if γ is a simple closed curve and $\eta(\gamma, a) \in \{0, 1\}$ $\forall a \notin \gamma$

THEOREM

For any closed curve γ and $a \notin \gamma$, $\eta(\gamma, a)$ is an integer

REMARK

Fix γ, $\eta(\gamma, a)$ is cont. in $a \Rightarrow \eta(\gamma, a)$ is a locally constant function ($\eta(\gamma, a)$ is constant on each connected component of $\mathbb{C} \setminus \gamma$)

PROOF OF THEOREM

For $\gamma: z(t)$, $t \in [0,1]$, define $F(s) = \int_0^s \frac{z'(t)}{z(t)-a} dt$ ($\int \frac{dz}{z}$) $\forall s \in [0,1]$.

Consider $g(z(s)) := (z(s)-a)\exp(-F(s))$

$\therefore g'(s) = z'(s)\exp(-F(s)) + (z(s)-a)\exp(-F(s))(-F'(s))$, where $F'(s) = \frac{z'(s)}{z(s)-a} \Rightarrow g'(s) = 0$

Thus, $g(s) = g(0) = z(0) - a$

$\quad\quad = g(1) = (z(1)-a)\exp(-F(1))$

$\therefore \exp(-F(1)) = \frac{z(1)-a}{z(0)-a}$

As γ: closed curve, thus $z(1) = z(0) \Rightarrow \exp(-F(1)) = 1 \Rightarrow F(1) = 2\pi i k$ for some $k \in \mathbb{Z}$.

$\therefore \eta(\gamma, a) = \frac{1}{2\pi i} F(1) = k \in \mathbb{Z}$ □

REMARK

Fix γ, $a \to \infty$. Then, $\left|\int_\gamma \frac{dz}{z-a}\right| \longrightarrow 0$

$\therefore \eta(\gamma, a) = 0$ when $a \to \infty \Rightarrow \eta(\gamma, a) = 0$ on unbounded connected component of $\mathbb{C} \setminus \gamma$.

FACT (JORDAN CURVE THEOREM)

$\eta(\gamma, a) = 0 \Rightarrow a \in$ unbounded connected component of $\mathbb{C} \setminus \gamma$.

with winding #, we don't need to define inside or outside anymore

THEOREM (CAUCHY RESIDUE THEOREM)

For f: ana on a s.c. domain D except at $\underbrace{w_1, \dots, w_n}_{\text{(distinct)}}$, γ: closed curve $\subseteq D \setminus \{w_1, \dots, w_n\}$, then $\int_\gamma f(z)\,dz = 2\pi i \sum_i \operatorname{Res}(f, w_i)\,\eta(\gamma, a)$

Proof

Considering the Laurent series around w_i,

$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-w_i)^k$ on $0 < |z-w_i| < \varepsilon$

$P_i\left(\frac{1}{z-w_i}\right) = \sum_{k=-\infty}^{-1} a_k (z-w_i)^k$

Notice, $P_i\left(\frac{1}{z-w_i}\right)$ is ana on $D \setminus \{w_i\}$.

$\therefore g(z) = f(z) - \sum_{i=1}^{n} P_i\left(\frac{1}{z-w_i}\right)$ is ana on $D \setminus \{w_1, \dots, w_n\}$.

Around w_i with $0 < |z-w_i| < \varepsilon$, $g(z) = \sum_{k=0}^{\infty} a_k (z-w_i)^k - \sum_{j \neq i} P_j\left(\frac{1}{z-w_j}\right) \Rightarrow g$ is ana at $z = w_i$

$\therefore w_1, \dots, w_n$ are removable singularities at w_i.

By Closed Curve Thm, $\int_\gamma g(z)\,dz = 0 \Rightarrow \int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_\gamma P_i\left(\frac{1}{z-w_i}\right) dz = \sum_{i=1}^{n} 2\pi i \operatorname{Res}(f, w_i) \int_\gamma \frac{1}{z-w_i}\,dz = 2\pi i \sum_i \operatorname{Res}(f, w_i)\,\eta(\gamma, a)$ □

DEFINITION

f is meromorphic on a domain D if f is ana on D except at isolated poles

10-30-25 (WEEK 9) Shun/翔海 (@shun4midx)

THEOREM

Let γ be a regular closed curve and f be a meromorphic function on and inside γ.
Suppose f has no poles nor zeros of f inside γ.

Set n_z = #zeros inside γ, n_p = #poles inside γ
Then, $\int_\gamma \frac{f'}{f}dz = (n_z - n_p)2\pi i$ ☆ $\int$ rly fits winding number

Proof

① f has a zero of order k at ω, around ω, we have $f(z) = \frac{f^{(k)}(\omega)}{k!}(z-\omega)^k + \dots = (z-\omega)^k g(z)$ and $g(z) \neq 0$ ana around ω

$$\Rightarrow \frac{f'}{f} = \frac{k(z-\omega)^{k-1}g(z) + (z-\omega)^k g'(z)}{(z-\omega)^k g(z)} = \boxed{\frac{k}{z-\omega} + \frac{g'(z)}{g(z)}} \text{ around } \omega$$

$\therefore \frac{f'}{f}$ has a simple pole at ω with radius k

② f has a pole at ω of order k around ω.

$\therefore f = (z-\omega)^{-k}\sum_{i=0}^{\infty} a_i (z-\omega)^{k+i} = (z-\omega)^{-k}g(z) \Rightarrow \boxed{\frac{f'}{f} = -k\frac{1}{z-\omega} + \frac{g'(z)}{g(z)}}$ around ω, $\frac{f'}{f}$ has a simple pole with residue $-k$

③ ω is not a pole nor a zero, then $\text{Res}(\frac{f'}{f}, \omega) = 0$ as $\frac{f'}{f}$ is ana at ω

By Cauchy Residue Thm, $\boxed{\int_\gamma \frac{f'}{f}dz = 2\pi i \sum_{i=1}^{n} n(\gamma, \omega_i)\,\text{Res}(\frac{f'}{f}, \omega_i)}$

Now, we know:

$n(\gamma, \omega_i) = 1$ for ω_i inside γ

$$\text{Res}\left(\frac{f'}{f}, \omega_i\right) = \begin{cases} k, & \omega_i: \text{zero of order } k \\ -k, & \omega_i: \text{pole of order } k \end{cases}$$

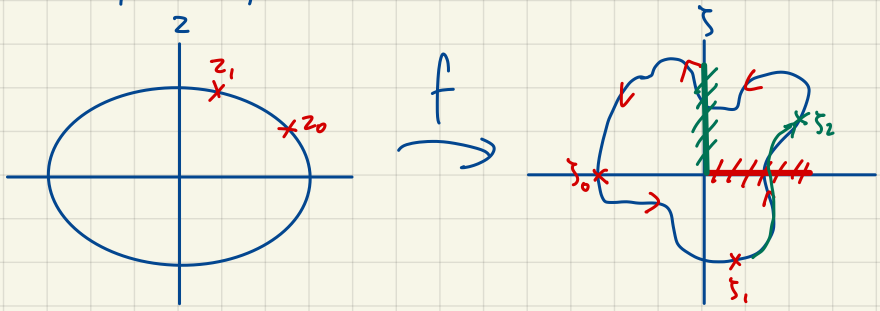
$\therefore \int_\gamma \frac{f'}{f}dz = (n_z - n_p)2\pi i$ □

COROLLARY (ARGUMENT PRINCIPLE) ☆ IMPORTANT TO KNOW THE INTUITION

If f is ana inside and on a closed curve γ and f has no zeros on γ, then $n_z = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz$

REMARK ON COROLLARY

For $\gamma: z(t)$, $\gamma' = f(z(t))$,



We can integrate over these curves:

$\log(\zeta) = \int_{\zeta_0}^{\zeta} \frac{d\zeta}{\zeta} + \log(\zeta_0)$

$\log(\zeta_1) = \int_{\zeta_0}^{\zeta_1} \frac{d\zeta}{\zeta} + \log(\zeta_0) \leftarrow \arg \in (\pi, 2\pi]$

$\underbrace{\log(\zeta_2)}_{\arg \in (\frac{\pi}{2}, \frac{5\pi}{2})} = \int_{\zeta_1}^{\zeta_2} \frac{d\zeta}{\zeta} + \log(\zeta_1) \leftarrow \arg \in [\pi, 2\pi]$

} Different analytic branches (need to choose one where the entire path is cont:)

(If we want to find a whole loop's worth of integration, we can do so by finding enough ana branches)

$\Rightarrow \int_{\gamma'} \frac{d\zeta}{\zeta} = \log(\zeta_n) - \log(\zeta_0)$ where ζ_n is the "overlapping" ζ_0 point on a new ana branch

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APPLICATIONS OF RESIDUE THEOREM

THEOREM (ROUCHÉ'S THEOREM)

Suppose f, g are analytic inside and on a regular closed curve γ, and suppose $|f(z)|>|g(z)|\ \forall z\in\gamma$, then $n_z(f+g)=n_z(f)$ inside γ

Proof

As $|f(z)|>|g(z)|\geq 0\ \forall z\in\gamma$, $f+g$ has no zeros on γ (and f has no zeros on γ)

$f+g$ has no poles since f, g: ana.

$f, 1+\frac{g}{f}$ have no zeros or poles on γ

$$\therefore n_z(f+g)+0=\int_\gamma \frac{(f+g)'}{f+g}dz=\int_\gamma \frac{[f(1+\frac{g}{f})]'}{f(1+\frac{g}{f})}dz=\int_\gamma \frac{f'}{f}dz+\int_\gamma \frac{(1+\frac{g}{f})'}{1+\frac{g}{f}}dz=n_z(f)+n_z(1+\tfrac{g}{f})$$

Notice, $1+\frac{g}{f}$ will never equal zero, since that will need $|\frac{g(z)}{f(z)}|=1$, but by max mod thm, $\forall z\in$ inside, $|\frac{g}{f}|<1$

$\therefore n_z(1+\frac{g}{f})=0$

$\therefore n_z(f+g)=n_z(f)$ □

EXAMPLE

For $f=2z^{10}+4z^2+1$, how many zeros of f are inside $D(0,1)$?

Take $g=4z^2$, $h=2z^{10}+1 \Rightarrow \forall |z|=1, |g(z)|>|h(z)|$

By Rouché's Thm, $n_z(g+h)=n_z(g)$ in $D(0,1) \Rightarrow f$ has two zeros in $D(0,1)$.

THEOREM (GENERALIZED CAUCHY INTEGRAL FORMULA)

Let f be analytic on a s.c. domain D, and γ be a regular closed curve contained in D. Then, for each z inside γ, $k\in\mathbb{Z}_{\geq 0}$,

$$f^{(k)}(z)=\frac{k!}{2\pi i}\int_\gamma \frac{f(\omega)}{(\omega-z)^{k+1}}d\omega$$

Proof

Around z, we have $f(\omega)=f(z)+f'(z)(\omega-z)+\ldots+\boxed{\frac{f^{(k)}(z)}{k!}(\omega-z)^k}+\ldots$

Now, we know $\text{Res}\left(\frac{f(\omega)}{(\omega-z)^{k+1}}, z\right)=\frac{f^{(k)}(z)}{k!}$

$\therefore \frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{(\omega-z)^{k+1}}d\omega=\frac{f^{(k)}(z)}{k!}$ □

RECALL

For ana $\{f_n\}_n$, if $f_n\to f$ uniformly on compacta of D, it means "$\forall V\subseteq D$: cpt, $f_n\to f$ uniformly on V"

Moreover, by Morera's Thm, we had f_n: ana $\Rightarrow f$: ana

THEOREM

Let $\{f_n\}$ be a sequence of analytic functions that converges uniformly on compacta of region D

Then, f is analytic and $f_n'\to f'$ uniformly on compacta of D

Proof

$\forall z_0\in D$, choose $r_0>1$, s.t. $\overline{D(z_0,r_0)}\subseteq D$, $\gamma:=C_{r_0}(z_0)$

Then, by Generalized Cauchy integral formula,

$$f_n'(z)-f'(z)=\frac{1}{2\pi i}\int_\gamma \frac{f_n(\omega)-f(\omega)}{(\omega-z)^2}d\omega$$

Recall, $f_n\to f$ uniformly on $\overline{D(z_0,r_0)}$. By def, given $\varepsilon>0$, $\exists N$, s.t. $|f_n(z)-f(z)|<\frac{\varepsilon r_0^2}{4}\ \forall z\in D(z_0,r_0)$

In particular, $\forall z\in D(z_0,\frac{1}{2}r_0)$, $|f_n(z)-f(z)|<\frac{\varepsilon r_0^2}{4}$

By ML-formula, $f_n'(z)-f(z) << \frac{1}{2\pi}\frac{\frac{\varepsilon r_0^2}{4}}{(\frac{r_0}{2})^2}\cdot 2\pi r_0=\varepsilon$

Now, $\forall$ compact $V \subseteq D$, $\forall z \in V$, $\exists r < 1$, s.t. $f_n' \to f'$ unif on $\overline{D(z, \frac{r}{2})}$

By def, $V \subseteq \bigcup_{z \in V} D(z, \frac{r}{2}) \Rightarrow \exists z_i, r_i$, s.t. $V = \bigcup_{i=1}^{n} D(z_i, \frac{r_i}{2})$

$\therefore$ $f_n' \to f$ uniformly on V. $\square$

THEOREM (HURWITZ'S THEOREM)

Let $\{f_n\}$ be a sequence of non-vanishing analytic functions in a region D.

Suppose that $f_n \to f$ uniformly on compacta of D. Then, either $f \equiv 0$ on D or $f(z) \neq 0$ $\forall z \in D$

Proof

Suppose $\exists z \in D$, s.t. $f(z) = 0$

Claim: $f \equiv 0$ on D

Proof

As f_n: analytic and has no zero on D, choose r, s.t. $\overline{D(z,r)} \subseteq D$ and f has no zero on $C_r(z)$ (otherwise, f has an acc point of zero + D: region $\Rightarrow f \equiv 0$ on D)

Now, $\frac{1}{2\pi i} \int_{C_r(z)} \frac{f'}{f} dz = n_z(f)$ in $D(z,r) \geq 1$

We know $f_n' \to f'$ unif on $D(z,r)$, so $\frac{f_n'}{f_n} \to \frac{f'}{f}$ unif on $C_r(z)$, since f_n, f have no zeros on $C_r(z)$

However, $\int_{C_r(z)} \frac{f'}{f} dz = \lim_{n \to \infty} \int_{C_r(z)} \frac{f_n'}{f_n} dz = 0$ (since f_n has no zeros in D) $\rightarrow\!\!\times\!\!\leftarrow$

COROLLARY

Replace "0" with "a" in Hurwitz's Thm

Proof

Consider $g_n(z) = f_n(z) - a$. $\square$

EXAMPLE

We know $\sin z = \sum_{i=0}^{\infty} (-1)^i \frac{z^{2i+1}}{(2i+1)!}$, take $f_n(z) = \sum_{i=0}^{n} (-1)^i \frac{z^{2i+1}}{(2i+1)!}$

$f_n \to \sin z$ uniformly on any compact set $V \subseteq \mathbb{C}$.

$\therefore$ By Hurwitz's Thm, $f_n(z)$ has a zero in $D(0, 2\pi)$ $\forall n \geq N$ for some $N > 0$. !!

THEOREM

Let $\{f_n\}$ be a seq of ana functions, and $f_n \to f$ unif on cpta of a region D. If f_n is 1-1 $\forall n$, then either f: const or 1-1

Proof

Suppose $\exists z_1, z_2 \in D$, $z_1 \neq z_2$ s.t. $f(z_1) = f(z_2) = a$

$\exists r$, s.t. $D(z_1, r) \subseteq D$ and $D(z_1, r) \cap D(z_2, r) = \emptyset$

On $D(z_1, r)$, $f_n \to f$ on cpta and $f(z_1) = a$. Hence, if $f_n(z) \neq a$ $\forall z \in D(z_1, r)$, then $f \equiv a$ on $D(z_1, r)$.

If $f \equiv a$ on $D(z_1, r)$, as D is a region, $f \equiv a$ on D.

If $f \not\equiv a$ on D, then by coro, $\exists N$, s.t. $f_n(z) = a$ has a zero on $D(z_1, r)$ for $n \geq N$.

f_n are 1-1 $\Rightarrow \forall n \geq N$, $f_n - a$ has no zeros on $D(z_2, r)$

$\therefore$ $f \equiv a$ on D $\rightarrow\!\!\times\!\!\leftarrow$

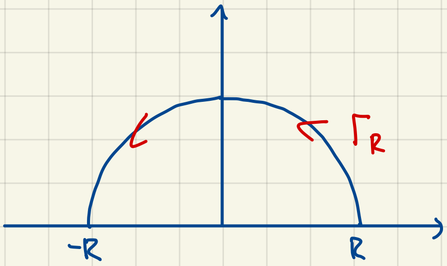
RESIDUE THEOREM AND EVALUATION OF INTEGRALS AND SUMS

Shun/翔海(@shun4midx)

TYPE I INTEGRALS

$$\boxed{\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx}$$

If P, Q: poly and $\deg P \leq \deg Q + 2$, $\gcd(P,Q)=1$, and $Q(x)$ has no real roots



Define $\gamma: \Gamma_R + (-R \to R)$

Then, $\displaystyle\int_\gamma \frac{P(x)}{Q(x)}\,dx = \int_{\Gamma_R} \frac{P(x)}{Q(x)}\,dx + \boxed{\int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx} \overset{\text{Residue Thm}}{=} 2\pi i \left(\sum_{\substack{\omega_i:\text{ root of } Q(x) \\ \text{in upper-half plane}}} \mathrm{Res}\left(\frac{P(x)}{Q(x)}, \omega_i\right)\right)$

When R is big enough,

Claim: $\displaystyle\lim_{R\to\infty} \left|\int_{\Gamma_R} \frac{P(z)}{Q(z)}\,dz\right| = 0$

Proof

Notice, for $R \gg 0$, $\displaystyle\left|\frac{P(z)}{Q(z)}\right| \leq \frac{A}{|z|^2} \leq \frac{A}{R^2} \quad \forall |z| \geq R$

By ML-Formula,

$$\int_{\Gamma_R} \frac{P(z)}{Q(z)}\,dz \ll \frac{A}{R^2} \cdot \pi R = \frac{\pi A}{R} \xrightarrow{R\to\infty} 0$$

$$\therefore \int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx = 2\pi i \left(\sum_{\substack{\omega_i:\text{ root of } Q(x) \\ \text{in upper-half plane}}} \mathrm{Res}\left(\frac{P(x)}{Q(x)}, \omega_i\right)\right)$$

EXAMPLE

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i\left(\mathrm{Res}\left(\frac{1}{x^4+1}, e^{\frac{\pi}{4}i}\right) + \mathrm{Res}\left(\frac{1}{x^4+1}, e^{\frac{3\pi}{4}i}\right)\right) = \frac{\sqrt{2}}{2}\pi$$

11-6-25 (WEEK 10)

ok, elephant in the room: I think I'll still wanna cry every CompAna lecture so I'll stop venting about it in my comments but yeah feel free to not use my notes anymore due to the lower quality. Don't ask why I wanna cry, and before you wonder, no, I didn't fail any exam; yes, I still love CompAna.

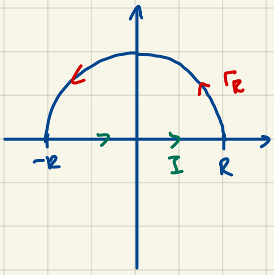
Shun/翔海 (@shun4midx)

TYPE II INTEGRALS

no need to +2

$$\boxed{f(x)=\frac{P(x)}{Q(x)},\ \deg Q \geq \deg P+1,\ \int_{-\infty}^{\infty} f(x)\cos x\,dx \text{ or } \int_{-\infty}^{\infty} f(x)\sin x\,dx}$$

If we want to apply the route in (I), we must check "$\lim_{R\to\infty}\int_{\Gamma_R} f(x)\cos x\,dx$" $=0$ (or the equivalent for $\sin x$)



The key is in $e^{iz}=\cos z+i\sin z$ (However, note $|e^{iz}|=e^{R\cos\theta}$ may not vanish!)

$N(m):=\int_{\Gamma_{M+I}} f(z)e^{iz}dz \Rightarrow$ We have $\int_{-\infty}^{\infty} f(x)\sin x\,dx=\lim_{M\to\infty}\mathrm{Im}(N(m))$, $\int_{-\infty}^{\infty} f(x)\cos x\,dx=\lim_{M\to\infty}\mathrm{Re}(N(m))$

☆ Caution: $f(z)e^{iz}$ may have singularities on the real axis, we may need to consider $e^{iz}-c$, $c\in\mathbb{C}^\times$

EXAMPLE

$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=?$

We CANNOT consider $\int_I \frac{e^{iz}}{z}dz \Rightarrow$ consider $\int_I \frac{e^{iz}-1}{z}dz$ ($z=0$: removable singularity)

Then, $\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\mathrm{Im}\left(\lim_{M\to\infty}\int_{\Gamma_{M+I}}\frac{e^{iz}-1}{z}dz\right)$

LEMMA 1 ☆ Key lemma for cool complex analysis integral types

Let f be an analytic function defined on a sector of the upper-half plane

Suppose $|f(z)|\longrightarrow 0$ as $z\to\infty$

Then, we have

$\left|\int_{\Gamma_R} f(z)e^{iz}dz\right|\longrightarrow 0$ as $R\to\infty$, where $\Gamma_R: Re^{i\theta}$, $0\leq\theta_1\leq\theta\leq\theta_2\leq\pi$

<u>Proof</u>

Define $M:=\max_{z\in\Gamma_R}|f(z)|$, notice $M\to 0$ as $R\to\infty$

Then, $\left|\int_{\Gamma_R} f(z)e^{iz}dz\right|\leq M\int_{\Gamma_R}|e^{iz}|dz=M\int_{\theta_1}^{\theta_2}e^{-R\sin\theta}R\,d\theta\leq M\int_0^{\pi}e^{-R\sin\theta}R\,d\theta=2\pi\int_0^{\frac{\pi}{2}}e^{-R\sin\theta}R\,d\theta$

$(\sin x-x)'\leq 0$

Notice, $\frac{2}{\pi}\leq\frac{\sin\theta}{\theta}\leq 1$

$\frac{\sin\theta}{\theta}$ concave down:

$\geq\frac{1}{\frac{\pi}{2}}=\frac{2}{\pi}$

Thus, $\left|\int_{\Gamma_R} f(z)e^{iz}dz\right|\leq 2M\int_0^{\frac{\pi}{2}}Re^{-R\theta\cdot\frac{2}{\pi}}d\theta\leq M\pi$

$\therefore$ As $R\to\infty$, $\left|\int_{\Gamma_R} f(z)e^{iz}dz\right|\longrightarrow 0$ □

APPLICATION TO CASE (II) INTEGRALS

For $f(x)=\frac{P(x)}{Q(x)}$, for big enough R, $f(x)e^{iz}$ has no poles on Γ_R, hence: $\lim_{R\to\infty}\int_{\Gamma_{R+I}} f(z)e^{iz}dz=\lim_{R\to\infty}\int_{-R}^{R} f(z)dz$

Shun/翔海(@shun4midx)

CALCULATED EXAMPLE

find it using Taylor's expansion of e^{iz}.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \operatorname{Im}\left(\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{iz}-1}{z}\,dz\right) = \operatorname{Im}\left(2\pi i \operatorname{Res}\left(\frac{e^{iz}-1}{z}, 0\right)\right) = \pi$$

Do not separate this into $\int \frac{e^{iz}}{z}$ and $\int \frac{1}{z}$, they both are divergent.

11-11-25 (WEEK 11)

Shun/翔海 (@shun4midx)

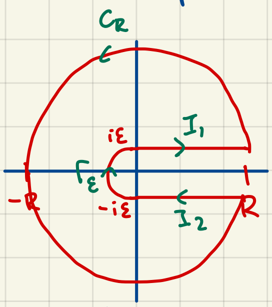
TYPE (III) INTEGRALS

$\boxed{\int_0^\infty \frac{P(x)}{Q(x)}\,dx}$ for $\gcd(P,Q)=1$, $\deg Q \geq \deg P+2$, $Q(x)\neq 0\ \forall x\geq 0$

If $\frac{P(x)}{Q(x)}$ is even, we should just do $\frac{1}{2}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$

Let $f(x) := \frac{P(x)}{Q(x)}$. Consider if $f(x)$ is NOT even.

Consider this path:



Abuse of notation to write C_R, Γ_ε. Consider $R\to\infty$, $\varepsilon\to 0$.

Consider $\log z$ on $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$, $\arg\in(0,2\pi)$

For $f(z)\log z$,

I_1: $\log(t+i\varepsilon) = \log|t+i\varepsilon| + \theta(t)i$;
I_2: $\log(t-i\varepsilon) = \log|t-i\varepsilon| + [2\pi-\theta(t)]i$; $\Big\}$ difference in arg due to symmetry

Notice, $\theta(t) \xrightarrow{\varepsilon\to 0} 0$ rapidly

Rmk

If $f(0)\neq 0$, then $f(z)\log z$ is not defined at $z=0$, so don't write $\lim_{\substack{M\to\infty\\ \varepsilon\to 0}}\int_\varepsilon^M f(x)\log x\,dx = \int_0^\infty f(x)\log x\,dx$

We consider the paths I_1 and I_2,

I_1: $\int_0^R f(t+i\varepsilon)\log(t+i\varepsilon)\,dt$

I_2: $\int_0^R f(t-i\varepsilon)\log(t-i\varepsilon)\,dt$

$\Rightarrow$ Total: $\int_0^R [f(t+i\varepsilon)\log(t+i\varepsilon) - f(t-i\varepsilon)\log(t-i\varepsilon)]\,dt$

$$= \int_0^R \underbrace{[f(t+i\varepsilon)-f(t+i\varepsilon)]}_{\substack{c_1\varepsilon+c_2\varepsilon^2+\dots\\ \text{begins at } \varepsilon \text{ term}}}\underbrace{\log(t+i\varepsilon)}_{\log|t+i\varepsilon|+\theta(t)i}\,dt + \int_0^R f(t+i\varepsilon)\underbrace{[\log(t+i\varepsilon)-\log(t-i\varepsilon)]}_{[-2\pi+2\theta(t)]i}\,dt$$

For $(x,\varepsilon)\in[0,\varepsilon_0]\times[0,\varepsilon_0]$,

$[f(x+i\varepsilon)-f(x-i\varepsilon)]\log(x+i\varepsilon) =: g(x,\varepsilon)$. Let $\alpha := \max_{x,\varepsilon}(f(x+i\varepsilon)-f(x-i\varepsilon))\log(x+i\varepsilon)$

Given $R, \delta_1, d>0$, $\exists\,\varepsilon$, s.t. ① $\theta(x+i\varepsilon)<\delta_1\ \forall x\geq d$

② $|f(x+i\varepsilon)-f(x-i\varepsilon)| < \frac{\delta_1}{M}$

Then, as $\varepsilon\to 0$,

$$\int_0^R f(t+i\varepsilon)\log(t+i\varepsilon)\,dt - \int_0^R f(t-i\varepsilon)\log(t-i\varepsilon)\,dt = \int_0^R f(t+i\varepsilon)\,2\pi i\,dt \xrightarrow{\varepsilon\to 0} \int_0^R f(t)\,2\pi i\,dt \quad ①$$

Now, we know $\left|\frac{P(x)}{Q(x)}\right| \leq \frac{A}{|R|^2}$

So, as $\varepsilon\to 0$, $\left|\int_{C_R} f(z)\log z\,dz\right| \leq \frac{A}{R^2}(\log R+2\pi)\cdot 2\pi R \xrightarrow{R\to\infty} 0$ ②

Choose a small ε, s.t. $Q(z)\neq 0\ \forall z\in D(0,\varepsilon)$.

Let $m:=\max_{z\in \overline{D(0,\varepsilon)}}|f(z)|$.

$\therefore \int_{\gamma_\varepsilon} f(z)\log z\,dz \ll m(|\log\varepsilon|+c)\cdot\pi\varepsilon \xrightarrow{\varepsilon\to 0} 0$ ③

$$\therefore -2\pi i\int_0^\infty f(t)\,dt = 2\pi i\sum_{\omega_k: Q\text{'s roots}} \operatorname{Res}(f(z)\log z, \omega_k)$$

$$\Rightarrow \int_0^\infty f(t)\,dt = -\sum_{\omega_k: Q\text{'s roots}} \operatorname{Res}(f(z)\log z, \omega_k)$$

EXAMPLE

$$\int_0^\infty \frac{1}{1+x^3}\,dx = -\sum_{k=1,3,5}\operatorname{Res}\left(\frac{1}{1+z^3}\log z, \omega_k\right)$$

ALTERNATE FORM

$\int_a^\infty \frac{P(x)}{Q(x)}\,dx$ can be evaluated by considering $\int_{C_M}\log(z-a)\frac{P(z)}{Q(z)}\,dz$. In fact, $\int_0^a = \int_0^\infty - \int_a^\infty$

ALTERNATE FORM

$\int_0^\infty \frac{x^{\alpha-1}}{Q(x)}\,dx$, $0<\alpha<1$ and Q: poly w/ deg 1

Notice, $z^{\alpha-1}:=\exp[(\alpha-1)\log z]$

Then,

$\int_{C_R}\longrightarrow 0$ as $R\longrightarrow\infty$

$\int_{\gamma_\varepsilon}\longrightarrow 0$ as $\varepsilon\longrightarrow 0$

I_1, $\varepsilon\to 0$, $\int_0^\infty \frac{x^{\alpha-1}}{Q(x)}\,dx$

I_2, $\varepsilon\to 0$, $\int_{I_2}\frac{z^{\alpha-1}}{Q(z)}\,dz = \int_0^\infty \frac{x^{\alpha-1}e^{2\pi i(\alpha-1)}}{Q(x)}\,dx$

$$\therefore [1-e^{2\pi i(\alpha-1)}]\int_0^\infty \frac{x^{\alpha-1}}{Q(x)}\,dx = 2\pi i\sum_{\omega_k:\text{ roots of }Q}\operatorname{Res}\left(\frac{z^{\alpha-1}}{Q(z)}, \omega_k\right)$$

TYPE (IV) INTEGRALS

$$\boxed{\text{For } R(x,y)=\frac{P(x,y)}{Q(x,y)},\ P,Q\in\mathbb{C}[x,y],\ \int_0^{2\pi} R(\cos\theta,\sin\theta)\,d\theta}$$

We take $z=\sin\theta+i\cos\theta$, $d\theta=\frac{dz}{iz}$.

$\cos\theta=\frac{z+z^{-1}}{2}$, $\sin\theta=\frac{z-z^{-1}}{2i}$

$$\Rightarrow \text{Change of variables gives us } \int_{|z|=1}\underbrace{\frac{P\left(\frac{z+z^{-1}}{2},\frac{z-z^{-1}}{2i}\right)}{Q\left(\frac{z+z^{-1}}{2},\frac{z-z^{-1}}{2i}\right)}\frac{dz}{iz}}_{f(z)} = 2\pi i\sum_{\omega_i:\text{ poles of }f(z)}\operatorname{Res}(f(z),\omega_i)$$

EXAMPLE

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \int_{|z|=1}\frac{-2i}{z^2+4z+1}\,dz = 4\pi\operatorname{Res}\left(\frac{1}{z^2+4z+1}, \sqrt{3}-i\right)$$

ESTIMATING SUMS

Shun/翔海 (@shun4midx)

TYPE (I)

$\boxed{\sum_{n=-\infty}^{\infty} f(n) \; Ⓐ, \; \sum_{n=-\infty}^{\infty} (-1)^n f(n)} \; Ⓑ$

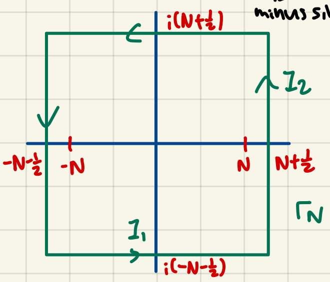
WANT: Find an analytic $g(z)$ on $\mathbb{C} \setminus \{n \in \mathbb{Z}\}$, s.t. $\text{Res}(g(z), n) = c f(n)$, $c \in \mathbb{C}^\times$

↳ $\frac{1}{\sin \pi z}$ has simple roots at $z \in \mathbb{Z}$ and ana elsewhere

We have: $\text{Res}\left(\frac{1}{\sin \pi z}, n\right) = \frac{1}{\pi}(-1)^n$

Let $g(z) = f(z) \frac{\pi}{\sin \pi z} \Rightarrow \text{Res}(g(z), n) = (-1)^n f(n)$

$\therefore \sum_{n=-\infty}^{\infty} f(n)(-1)^n = \sum_{\substack{n=-\infty \\ \text{minus sing}}}^{\infty} \text{Res}(g(z), n) + \sum_{\substack{w_k: \text{sing} \\ \text{of } f}} \text{Res}(g(z), w_k) = \frac{1}{2\pi i} \int_{\Gamma_N} g(z)\, dz$, where Γ_N is as follows:



Ensure $\sum f(n)$, $\sum (-1)^n f(n)$ conv, we assume $|f(z)| \leq \frac{A}{|z|^2}$

Here, $\frac{1}{\sin \pi z} = \frac{2i}{e^{i\pi z} - e^{-i\pi z}} = \frac{2i e^{i\pi z}}{e^{2i\pi z} - 1}$ ☆ key term w/ poles

$I_1: \left|\frac{1}{\sin \pi z}\right| = \left|\frac{2e^{\pi(N+\frac{1}{2})}}{e^{2\pi(N+\frac{1}{2})} - 1}\right| < 1$

$I_2: \left|\frac{1}{\sin \pi z}\right| = \left|\frac{2e^{-y\pi}}{e^{-2\pi y} e^{(2N+1)\pi} - 1}\right| = \frac{e^{-y\pi}}{e^{-2\pi y} + 1}$

$\therefore \lim_{N \to \infty} \left| \int_{\Gamma_N} f(z) \frac{\pi}{\sin \pi z} dz \right| \leq \lim_{N \to \infty} 4(2N+1) c \cdot \frac{A}{|N+\frac{1}{2}|^2} = 0 \Rightarrow \sum_{\substack{n=-\infty \\ n \neq \text{sing}}}^{\infty} f(n) = -\sum_{w_k: \text{sing}} \text{Res}(g(z), w_k)$

$\cos \pi z$ gives the $(-1)^n$

Ⓐ: $g(z) = f(z) \cot \pi z$

Ⓑ: $g(z) = f(z) \csc \pi z$

WARNING

When evaluating $\sum_{n=1}^{\infty} \frac{1}{n^2}$, notice $\frac{1}{n^2}$ is undefined.

$\Rightarrow \int_{\Gamma_N} \frac{\pi}{z^2} \frac{2\cos \pi z}{\sin \pi z} dz = 2\pi i \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \text{Res}(g(z), n) + \text{Res}(g(z), 0)$

Then we evaluate as usual.

SUMMARY

For Ⓐ, consider $g(z) := f(z) \frac{\cos z\pi}{\sin z\pi} \pi$

For Ⓑ, consider $g(z) := f(z) \frac{1}{\sin z\pi} \pi$

TYPE (II) — BINOMIAL COEFFICIENTS

We know $\binom{n}{k} \rightsquigarrow$ coef of z^k in $(1+z)^n$, so $\binom{n}{k} = \frac{1}{2\pi i} \int \frac{(1+z)^n}{z^{k+1}} dz$

EXAMPLE

$\sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{(5z)^n}$ conv abs

$\left[\frac{(1+z)^2}{5z}\right]^n$ ← geom series

$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C_r(0)} \frac{(1+z)^{2n}}{(5z)^n} \frac{dz}{z} = \frac{1}{2\pi i} \int_{C_r(0)} \sum_{n=0}^{\infty} \underline{\frac{(1+z)^{2n}}{(5z)^n}} \frac{dz}{z} = \frac{5}{2\pi i} \int_{|z|=1} \frac{1}{3z-1-z^2} dz = 5\, \text{Res}\left(\frac{1}{3z-1-z^2}, \frac{3-\sqrt{5}}{2}\right)$

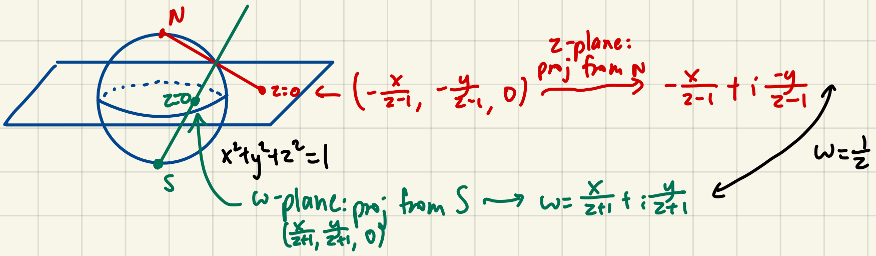
RESIDUE AT INFINITY

Shun/翔海(@shun4midx)

DEFINITION

For f: ana on $\mathbb{C}\setminus\{w_1, \dots, w_k\}$, $\mathrm{Res}(f,\infty) := -\int_{C_R} f(z)dz$

For $R>>0$, s.t. $|w_i|<R$, we have "∞" as follows:



$\rightsquigarrow$ We can switch to the w-plane, to only have one residue (N) remaining
└ change of coords lol

11-13-25 (WEEK 11) Shun/翔海(@shun4midx)

RESIDUE AT INFINITY

DEFINITION

from the w-plane.

$$\operatorname{Res}(f,\infty) := -\frac{1}{2\pi i}\int_C f(z)\,dz\,,\quad C := C_R(0)$$

INTUITION/PROOF OF THEOREM

Say $f(z)$ is a poly of deg n, hence it is entire

Then, $z\to\infty \Rightarrow |f(z)|\to\infty$, i.e. $w=0$ is a pole ($w\to 0 \Rightarrow |f(w)|\to 0$) at N

Using $z=\frac{1}{w}$,

$$\int_{C_R(0)} f(z)\,dz = \int_{-C_{\frac{1}{R}}(0)} \underbrace{f\left(\tfrac{1}{w}\right)\cdot -\tfrac{1}{w^2}}_{=:\,g(w)}\,dw$$

└ Change of direction from change of variables

Notice, $g(w)$ is ana on $\mathbb{C}\setminus\{\frac{1}{z_i},0\}_{z_i\neq 0}$ on the w-plane. Define $\{w_j\} := \{\frac{1}{z_i},0\}_{z_i\neq 0}$ └ $|\frac{1}{z_i}| > \frac{1}{R}$

Then, $\displaystyle\int_{C_{\frac{1}{R}}(0)} f\left(\tfrac{1}{w}\right)\cdot\tfrac{1}{w^2}\,dw = -2\pi i \sum_{|w_j|<\frac{1}{R}} \operatorname{Res}\left(f\left(\tfrac{1}{w}\right)\tfrac{1}{w^2},\, w_j\right)$

$\displaystyle = -2\pi i\operatorname{Res}\left(f\left(\tfrac{1}{w}\right)\tfrac{1}{w^2},\, 0\right)$

THEOREM

$$\operatorname{Res}(f,\infty) = -\operatorname{Res}\left(\frac{1}{w^2}f\left(\frac{1}{w}\right),\,0\right)$$

EXAMPLE

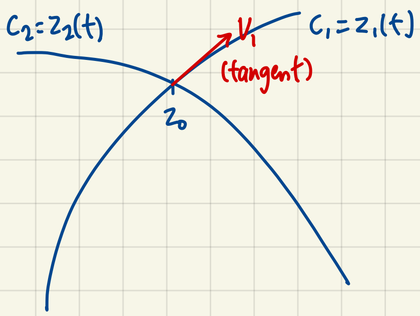
encompass all residues ($\because \frac{1}{0} =$ "∞")

Say $f(z) = \frac{5z-2}{z(z-1)}$, then $\displaystyle\int_{C_2(0)} f(z)\,dz = -2\pi i\operatorname{Res}\left(\tfrac{1}{w^2}f\left(\tfrac{1}{w}\right),\,0\right) = 10\pi i$

CONFORMAL MAPPING (ANGLE-PRESERVING)

NOTES CONVENTION

All curves $z(t)$ here are assumed to have $\dot{z}(t)\neq 0\ \forall t\in[a,b]$



Say $z_i(t) = x_i(t)+iy_i(t)$, $v_i :=$ tangent of C_i at $z=z_i$, $\angle C_1,C_2 =$ counterclockwise angle from v_1 to v_2

DEFINITION

For f: ana at z_0,

(i) f is locally 1-1 at z_0 if $\exists\,\delta>0$, s.t. $f|_{D(z_0,\delta)}$ is 1-1

(ii) For a region D, f is locally 1-1 throughout D if f is locally 1-1 at z $\forall z\in D$

(iii) f is 1-1 on D, if $f(z_1)\neq f(z_2)$ $\forall\, z_1\neq z_2\in D$

EXAMPLE

$f(z)=\exp(z)$ is

- Locally 1-1 on $\mathbb{C}$ ($\because$ we can always take a small enough ball s.t. it doesn't reach $2\pi i$ more)
- NOT 1-1 on $\mathbb{C}$

EXAMPLE

$f(z)=z^2$ is locally 1-1 $\forall z\neq 0$ ($\because f(z)=f(-z)$)

DEFINITION

For f: ana at $z=z_0$,

(i) f is conformal at z_0 if $\forall C_1, C_2$ passing through z_0, $\angle C_1, C_2 = \angle f(C_1), f(C_2)$

(ii) For a region D, f is conformal in D if f is conformal at z $\forall z\in D$

THEOREM

Say f is ana at z_0 and $f'(z_0)\neq 0$. Then, f is conformal and locally 1-1

REMARK/KEY EXAMPLE

Reflections aren't conformal mappings. In particular, $f(z)=\bar{z}$ is not conformal

PROOF OF THEOREM

For $C_i: z_i(t)=x_i(t)+iy_i(t)$, $z_i(t_0)=z_0$,

Define $v_i(t_0)=\dot{z}_i(t_0)$

We know $v_i(t)=|v_i|e^{i\theta_i(t)}$, $\theta(t)\in[0,2\pi)$

$\Rightarrow \angle C_1, C_2 = \theta_2(t)-\theta_1(t)$

Then, for $f(C_i)$,

$w_i(t)=f(z_i(t)) \Rightarrow \dot{w}_i(t)=f'(z_i(t))\dot{z}_i(t) \overset{f'(z_0)\neq 0}{=} |f'(z_i(t))|e^{i\varphi(t)}|\dot{z}_i(t)|e^{i\theta_i(t)} = |\cdot|e^{i(\varphi(t)+\theta_i(t))}$

$\therefore \angle f(C_1), f(C_2)|_{t=t_0} = (\varphi(t_0)+\theta_2(t_0))-(\varphi(t_0)+\theta_1(t_0)) = \angle C_1, C_2$

(Continue next time to have not just at $t=t_0$)

11-18-25 (WEEK 12)

Shun/翔海(@shun4midx)

THEOREM

If f is ana at z_0 and $f'(z_0)\neq 0$, f is conformal and locally 1-1 at z_0

Proof

"f is locally 1-1 at z_0": Set $h(z):=f(z)-f(z_0)=\alpha$

└ conformal proved last time

Then, $h(z_0)=0$, $h'(z_0)\neq 0 \Rightarrow h(z)$ is not const in a nbd of z_0

$\Rightarrow \exists D'(z_0,\delta)$ s.t. $\forall z\in \underset{D(z_0,\delta)\setminus\{z_0\}}{\underset{\shortparallel}{D'(z_0,\delta)}}$, $h(z)\neq 0$

Prove loc const

$\forall \delta_1<\delta$, $C_1:=C(z_0,\delta_1)\subseteq D(z_0,\delta)$,

$\underset{1}{\underset{\shortparallel \text{ in } C_1}{n_z(h)}} = \frac{1}{2\pi i}\int_{C_1}\frac{h'}{h}dz$, h: ana in $D(z_0,\delta)$

Turn into integral

Define $\omega:=h(z)$,

$$\int_{C_1}\frac{h'}{h}dz=\int_{h(C_1)}\frac{d\omega}{\omega}=\overbrace{\eta}^{\text{winding number}}(h(C_1),0)$$

$\omega:=h(z)$

winding # $\Rightarrow$ shrink to show η = loc const 1 means locally 1-1

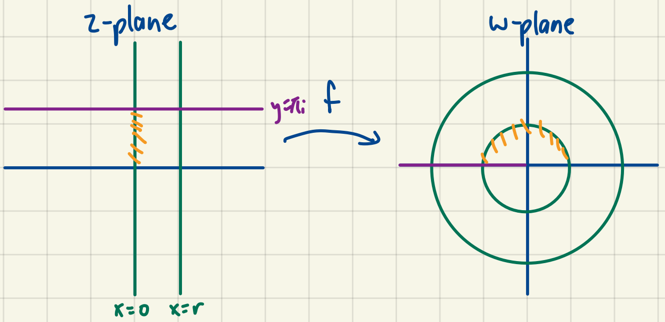
As $\eta(h(C_1),\alpha)$ is locally const in α, $\exists\ \varepsilon$, s.t. $\overbrace{\eta(h(C_1),\alpha)}^{\frac{1}{2\pi i}\int_{h(C_1)}\frac{1}{w-\alpha}dw}=1\ \forall\alpha\in D(0,\varepsilon)$

Consider $\delta_2<\delta$, s.t. $D(z_0,\delta_2)\subseteq h^{-1}(D(0,\varepsilon))$, i.e. $\forall z_1\in D(z_0,\delta_2)$, $1=\frac{1}{2\pi i}\int_{C_1}\frac{d(h(z)-h(z_1))}{h(z)-h(z_1)}=n_z(h(z)-h(z_1))$

$\Rightarrow$ only $z_1\in D(z_0,\delta_2)$, s.t. $h(z_1)=h(z_1)$ ← as in only z_1 is s.t. $h(\cdot)=h(z_1)$ □

EXAMPLE

$f(z):=e^z$: entire, $f'(z)\neq 0\ \forall z\in\mathbb{C}$ $\Rightarrow$ everywhere conformal + locally 1-1



DEFINITION

Let $k\in\mathbb{Z}_{>0}$. f is a k-to-1 mapping of D_1 onto D_2 if $\forall\alpha\in D_2$, $f(z)=\alpha$ has k roots in D_2 counting by multiplicity

LEMMA

Let $f(z)=z^k$, $k\in\mathbb{Z}_{>0}$. Then, f magnifies angles at 0 by a factor of k and f maps $D(0,\delta)$ onto $D(0,\delta^k)$ in a k-to-1 manner.

Proof

For $z\neq 0$, we have $z=|z|e^{i\theta}$, $\theta\in[0,2\pi)$, $z^k=|z|^ke^{ik\theta}$

For $\alpha=|\alpha|e^{i\theta}\neq 0$, (Case 1)

$f(z)=\alpha\Rightarrow z=|\alpha|^{\frac{1}{k}}e^{\frac{i(\theta+2\pi n)}{k}}$, $0\leq n\leq k-1$ are the roots of $f(z)=\alpha$

For $\alpha=0$, (Case 2)

$f(z)=0$ has a zero at $z=0$ with multiplicity k □

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THEOREM

$\begin{cases} f: \text{ana at } z_0 \\ f'(z_0) \neq 0 \end{cases}$ (Δ)

Suppose f is not const. Then, $\exists$ nbd U of z_0 s.t. $f|_U$ is a k-to-1 mapping and magnifies angles at z_0 by a factor of k, where k is the least positive integer s.t. $f^{(k)}(z_0) \neq 0$ (☆)

└ Δ ⇒ $k \geq 2$

Proof

After replacing f by $f(z)-f(z_0)$, we can assume $f(z_0)=0$

As f is ana at $z=z_0$ + (☆), $\exists D(z_0,\delta_0)$, s.t. $f(z)=\sum_{i=k}^{\infty} a_i(z-z_0)^i$ (Taylor expansion) with $a_k \neq 0$ — Taylor expansion + factor

$\Rightarrow f(z) = (z-z_0)^k \underbrace{(a_k + a_{k+1}(z-z_0)^1 + \dots)}_{g(z)}$, with $g(z_0)\neq 0$ ($\because a_k \neq 0$)

$\exists \varepsilon$, s.t. $0 \notin D(a_k,\varepsilon) \Rightarrow$ we can choose a branch of log defined on $D(a_k,\varepsilon)$ — Choose a branch to k^{th} root

$\therefore \forall z \in \underbrace{g^{-1}(D(a_k,\varepsilon))}_{D_1}$, we can define $\exp(\frac{1}{k}\log g(z)) =: h$ (k^{th} root)

$\therefore f(z) = [(z-z_0)h]^k$, define $H := (z-z_0)h \Rightarrow f = m_k \circ H$ on D_1, where $m_k(z) := z^k$ — Apply root + unroot

As H is ana on D_1, and $H'(z_0) = h(z_0) \neq 0$ ($\because g(z_0)\neq 0$), thus H: locally 1-1 + conformal — Prove root: conformal

$\therefore$ By lemma, we have f's angles magnified by k and f: k-to-1 □

└ H preserves angles, then m_k magnifies

THEOREM

Say f: 1-1 ana on a region D.

Then, (i) f^{-1} exists and is ana on $f(D)$

(ii) f and f^{-1} are conformal in D and $f(D)$ respectively

Proof

As f: 1-1, on $f(D)$, $\forall y \in f(D)$, $g(y)=x$ s.t. $f(x)=y$ is an inverse of f

Claim: g is cont. ("g^{-1}(open) = open")

Proof

$\forall$ open $U \subseteq D$, notice: $g^{-1}(U) := \{y \in f(D) \mid g(y) \in U\} = \{f(x) \mid x \in U\} = f(U)$ — Define g as f^{-1}, prove it's cont. w/ "g^{-1}(open) = open"

By open mapping Thm, $g^{-1}(U) = f(U)$ is open ✓

As $f \circ g = id_{f(D)}$, $g \circ f = id_D$, g: cont., $f'(z)\neq 0$ $\forall z \in D$ (by Thm above, $f'=0 \Rightarrow f$: not 1-1.), thus f: ana $\Rightarrow g$: ana, $g'(f(z)) = \frac{1}{f'(z)}$ — We get g is ana and $g'(z)\neq 0$ $\forall z$

As $f'(z)\neq 0$, thus $g'(f(z))\neq 0$ and is well-def $\Rightarrow g$: locally 1-1 and conformal on $f(D)$

As f: 1-1, thus g: 1-1 □

DEFINITION

① A 1-1 analytic mapping is called a conformal mapping

② Two regions D_1 and D_2 are conformally equivalent if $\exists$ conformal mapping from D_1 onto D_2

FACT

Conformal equivalence is an equivalence relation

SPECIAL MAPPINGS (EXAMPLES OF CONFORMAL MAPPINGS)

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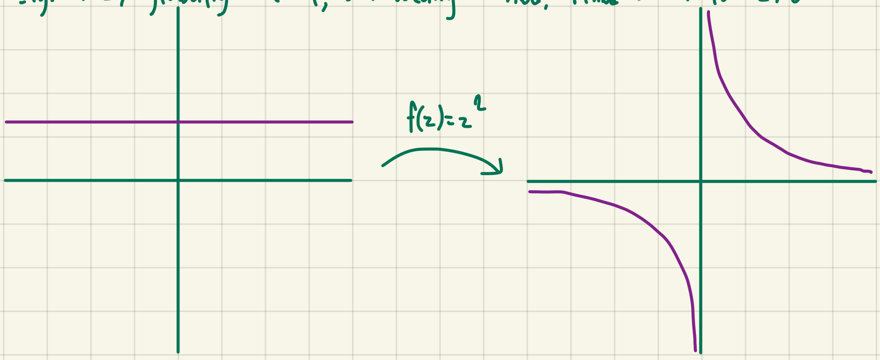
(i) $f(z)=az+b$, $a\neq 0$ (rotation + shrinkage)

(ii) $f(z)=z^\alpha$, locally at $z\neq 0$, for $\alpha>0$

E.g. $\alpha=2$, globally 2-to-1, but locally $\exists$ nbd, $f|_{\text{nbd}}$ is 1-1 for $z\neq 0$



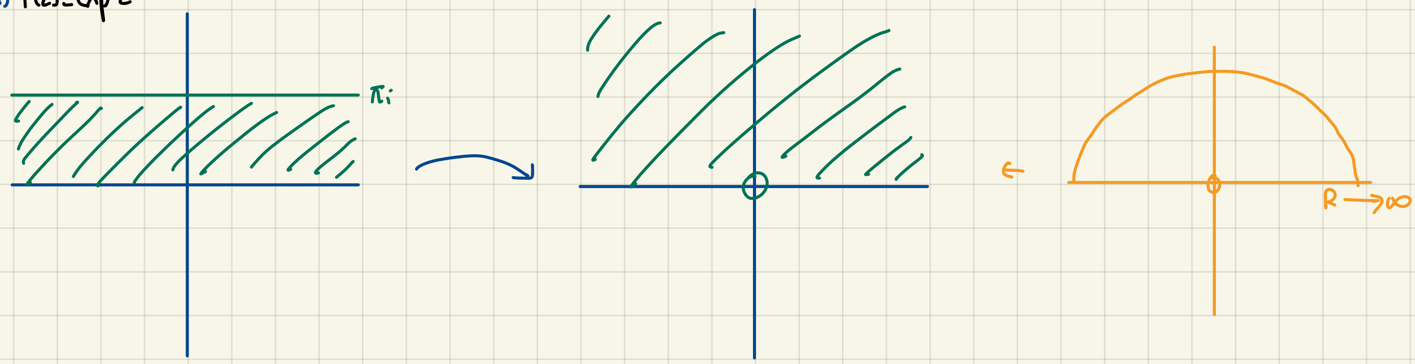
$(x+iy)^2=(x^2-y^2)+2ixy$

For $z\neq 0$, $z=re^{i\theta}\Rightarrow f(z)=r^\alpha e^{i\alpha\theta}$

$\{z \mid \theta_1\le \operatorname{Arg} z\le\theta_2\}\xrightarrow{f(z)}\{z\mid \alpha\theta_1\le\operatorname{Arg} z\le\alpha\theta_2\}$

If $\alpha\theta_2-\alpha\theta_1<2\pi$, then f: 1-1 in a nbd of z and conformal

(iii) $f(z)=\exp z$



(iv) Bilinear transformation

$f(z)=\dfrac{az+b}{cz+d}$, $\underbrace{ad-bc}_{\text{det}}\neq 0$, ana on $\mathbb{C}\setminus -\frac{d}{c}$

$f'(z)=\dfrac{ad-bc}{(cz+d)^2}\neq 0$

Then, f is locally 1-1 + conformal in $\mathbb{C}\setminus -\frac{d}{c}$

As $f(z_1)=f(z_2)\Rightarrow (ad-bc)(z_1-z_2)=0$, thus f is globally 1-1 ($\frac{a}{c}\notin \operatorname{Im} f$)

If we extend f as a function on the Riemann sphere $\mathbb{C}\cup\{\infty\}$, then f: conformal + globally 1-1

LEMMA

If S is a circle or a line, and $f(z)=\frac{1}{z}$, then $f(S)$ is a circle or a line

Proof

Wasn't this a HW problem lmfao

Rmk: This extends to $f(z)=\frac{az+b}{cz+d}$

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THEOREM (RECALL)

$f(z)=\frac{az+b}{cz+d}$, $ad-bc\neq 0$ maps circles and lines to circles and lines

Proof

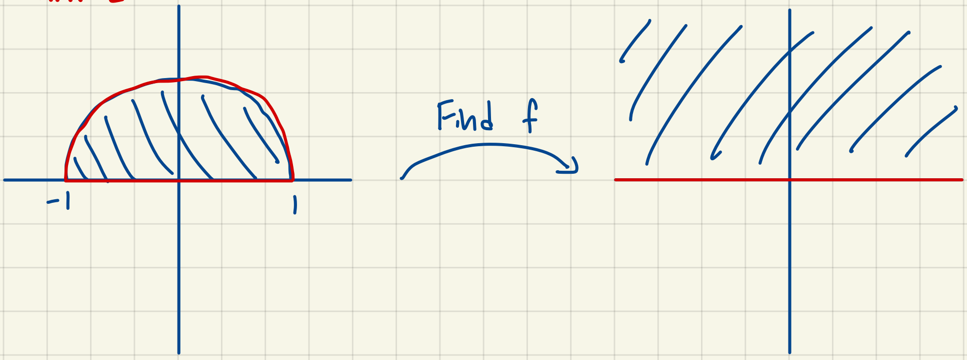
If $c=0$, then trivial.

If $c\neq 0$, $\underline{f(z)=\frac{1}{c}\left(a-\frac{ad-bc}{cz+d}\right)}$ key transformation to only have $\frac{1}{\sim z+\sim}$

$\Rightarrow$ Consider $z\to cz+d \to \frac{1}{cz+d} \longrightarrow \frac{ad-bc}{cz+d}$

from lemma

EXAMPLE

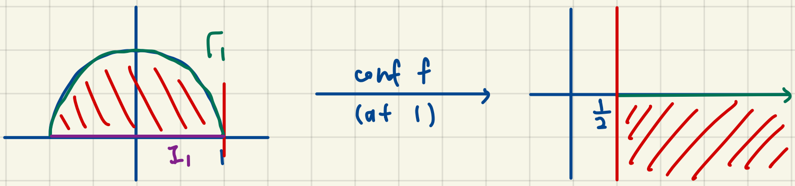


$S=\{z \mid |z|<1, \operatorname{Im} z>0\}$ $\mathbb{H}=\{z \mid \operatorname{Im} z>0\}$

Intuition: Cut the ⌓, then map to straight line

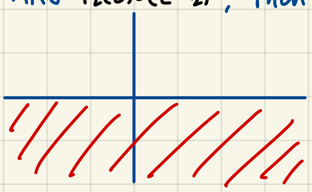
So we set $-1\to\infty$ (i.e. a pole)

Consider $f_1=\frac{1}{z+1}$ (conformal $\forall z\in\mathbb{C}\setminus\{-1\}$), then $f_1(1)=\frac{1}{2}$, $[-1,1)\xrightarrow{f}[\frac{1}{2},\infty)$. Notice, f_1 maps circles and lines to circles and lines.



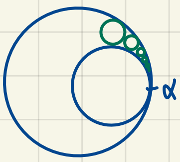
$\therefore f_1(\Gamma_1)=\{z \mid \operatorname{Im} z\leq 0, \operatorname{Re} z=\frac{1}{2}\}$, $f_1(I_1)=[\frac{1}{2},\infty)$

After $f_2(z)=(z-\frac{1}{2})^2$, then it will become



$\therefore f(z)=-\left(\frac{1}{z+1}-\frac{1}{2}\right)^2$

EXAMPLE



C_1, C_2: circles tangent at α

Find a chain of circles tangent to C_1 and C_2 and each other.

Then, the points of tangency lie on a circle or a line.

$\hookrightarrow$ Choose $f(z)=\frac{1}{z-\alpha}$

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DEFINITION

A conformal mapping (1-1 + analytic) of a region D onto (then bijective) itself is called an **automorphism** of D, which we denote as $\mathrm{Aut}(D)$

LEMMA

If $f: D_1 \to D_2$, D_1 and D_2 are regions, and f is a conformal mapping onto D_2,
Then, (i) For any other $h: D_1 \to D_2$, conformal onto D_2, then $\exists g \in \mathrm{Aut}(D_2)$, s.t. $h = g \circ f$
(ii) $\forall h \in \mathrm{Aut}(D_1)$, $\exists g \in \mathrm{Aut}(D_2)$, s.t. $h = f^{-1} \circ g \circ f$

Proof

(i) For f: conformal and $f(D_1) = D_2$, we know $\exists f^{-1}: D_2 \to D_1$ (by Thm), conformally onto D_1
Then, $g := h \circ f^{-1} \in \mathrm{Aut}(D_2) \Rightarrow h = g \circ f$

(ii) $\underbrace{f \circ h \circ f^{-1}}_{g} \in \mathrm{Aut}(D_2) \Rightarrow h = f^{-1} \circ g \circ f$ □

LEMMA

The only automorphism of a unit disc with $f(0) = 0$ are given by $f(z) = e^{i\theta} z$

Proof

Let $D := D(0,1)$.

As $f \in \mathrm{Aut}(D)$ and $f(0) = 0$, by Schwartz's Lemma, $|f(z)| \leq |z|$ — Schwartz's Lemma on f

By Thm, $\exists f^{-1}$ and $f^{-1} \in \mathrm{Aut}(D) \Rightarrow$ By Schwartz's Lemma, $|f^{-1}(z)| \leq |z|$ — Schwartz's lemma on f^{-1}

$\Rightarrow |z| = |f^{-1}(f(z))| \leq |f(z)| \leq |z|$ — $|f(z)| = |z|$

$\Rightarrow |f(z)| = |z|$

By Schwartz's Lemma, $f(z) = e^{i\theta} z$ □ (recall "equality $\Rightarrow f(z) = e^{i\theta} z$")

LEMMA

Let h be a bilinear transformation. If h maps D to D, where $D := D(0,1)$, and $h(\alpha) = 0$ for some $|\alpha| < 1$, then $h = e^{i\theta} \underbrace{\frac{z-\alpha}{1-z\bar{\alpha}}}_{B_\alpha(z)}$

Proof (def of bilinear transformation)

$h(z) = \frac{az+b}{cz+d}$ is globally 1-1, $h(\alpha) = a\left(\frac{z-\alpha}{cz+d}\right)$, $a \neq 0$

Then, $h(D) \subseteq D \Rightarrow h$: ana on D — $h(D) \subseteq D \Rightarrow h$: ana on D

By Schwartz Reflection Principle, $h(\bar{\alpha}^{-1}) = \overline{(h(\alpha))}^{-1} = \infty$ — Find $\to \infty$, so we can define a pole via Schwartz Reflection Principle

$\therefore h(z) = A\left(\frac{z-\alpha}{z-\bar{\alpha}^{-1}}\right)$, $A \neq 0$

However, by open mapping Thm, bdry $\longrightarrow$ bdry

$\therefore |h(1)| = 1$

$\therefore \left|A \cdot \frac{1}{\bar{\alpha}}\right| \cdot \left|\cancel{\frac{1-\alpha}{\bar{\alpha}-1}}\right| = 1 \Rightarrow A = \bar{\alpha} e^{i\theta}$ □

THEOREM

$D := D(0,1)$.

Then, $\mathrm{Aut}(D) = \{e^{i\theta}\left(\frac{z-\alpha}{1-z\bar{\alpha}}\right) \mid |\alpha| < 1, 0 \leq \theta \leq 2\pi\}$

Proof

$g \in \mathrm{Aut}(D) \Rightarrow \alpha := g^{-1}(0) \in D \Rightarrow |\alpha| < 1$

Set $h = \frac{z-\alpha}{1-z\bar{\alpha}}$, then $g \circ h^{-1}(0) = 0$

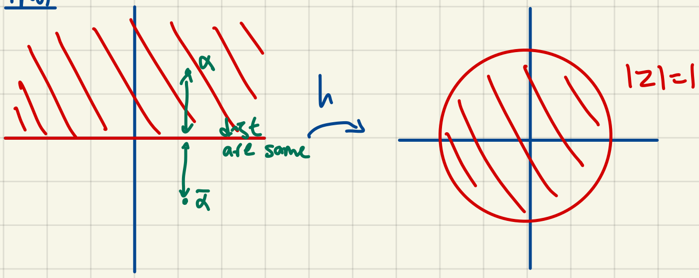
$\therefore g \circ h^{-1} \in \mathrm{Aut}(D)$, so by lemma, $g(h^{-1}(z)) = e^{i\theta} z \Rightarrow g(z) = e^{i\theta} h(z)$ □

11-25-25 (WEEK 13) Shun/翔海(@shun4midx)

THEOREM

The conformal mapping $h: \mathbb{H} := \{z \mid \operatorname{Im} z > 0\} \xrightarrow{\text{onto}} D(0,1)$ are of the form $h(z) = e^{i\theta}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$ for some $\alpha \in \mathbb{H}$

Proof



same dist if $\bar{\alpha}$

As we want $|z|=1$, consider $f(z) = \frac{z-\alpha}{z-\bar{\alpha}}$, $\alpha \in \mathbb{H}$, $z \in \mathbb{R} \Rightarrow |f(z)| = 1$

check $f(z) = \frac{z-\alpha}{z-\bar{\alpha}}$ on real plane

As f is globally 1-1 and ana on $\mathbb{C} \setminus \{\bar{\alpha}\} \supseteq \mathbb{H}$, and e.g. $\left|\frac{i-\alpha}{i-\bar{\alpha}}\right| < 1$, $i \in \mathbb{H}$
$\therefore \forall z \in \mathbb{H}$, $|f(z)| \le 1$, i.e. $f(\mathbb{H}) \subseteq D(0,1)$

find the inside, so we need:
1. $|f(z)| \le 1 \ \forall z \in \mathbb{H}$
2. $f^{-1}(D(0,1)) \subseteq \mathbb{H}$

Then, check f^{-1} to see $f^{-1}(D(0,1)) \subseteq \mathbb{H}$ (trivial)

By lemma, $\forall h$ satisfying $\mathbb{H} \to D(0,1)$ that are onto, $h = g \circ f$, $g \in \operatorname{Aut}(D(0,1)) = e^{i\theta} B_\alpha \Rightarrow h = (e^{i\theta} B) \circ f$ □

THEOREM

$h \in \operatorname{Aut}(\mathbb{H}) \Rightarrow h = \frac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{R}$, $ad-bc > 0$

Proof Sketch

$\forall h \in \operatorname{Aut}(\mathbb{H})$, $\exists g \in \operatorname{Aut}(D(0,1))$, s.t. $h = f^{-1} \circ g \circ f = f^{-1} \circ (e^{i\theta} B_\beta) \circ f$, then apply Thm above for f

DEFINITION

For f: func on $\mathbb{C}$, z_0 is a fixed point of f if $f(z_0) = z_0$

PROPOSITION

A bilinear transformation f other than identity has ≤ 2 fixed points. If we regard f as a function on $\mathbb{C} \cup \{\infty\}$, then it has 2 fixed pts counted by multiplicity

Proof

For $f(z) = \frac{az+b}{cz+d}$, $ad-bc \ne 0$, $f(z_0) = z_0 \Rightarrow az_0 + b = z_0(cz_0 + d)$

Consider two cases,
① $c \ne 0$, then z_0 is a root of $cx^2 + (d-a)x + b = 0 \Rightarrow f$ has ≤ 2 fixed pts $\left(f(\infty) = \frac{a}{c}\right)$
② $c = 0$, then $f(z)$ is a linear function $\Rightarrow f$ has one fixed pt $(f(\infty) = \infty)$

LEMMA

Let z_1, z_2, z_3 be distinct pts in $\mathbb{C}$.
The unique bilinear transformation sending z_1, z_2, z_3 to $\infty, 0, 1$ respectively (☆) is given by $T_{(z_1,z_2,z_3)}(z) = \frac{z-z_2}{z-z_1} \cdot \frac{z_3-z_1}{z_3-z_2}$

Proof

- Uniqueness
  Let S be a bilinear transformation satisfying ☆
  Then, $S^{-1} \circ T(z_i) = z_i$, $i = 1,2,3$, so by prop above, $S^{-1} \circ T = id$ ($\because$ 3 fixed pts)
- Existence: Trivial. □

Consider another S, then $S^{-1} \circ T$ we say it's $S^{-1} \circ T = id$.

REMARK

For the lemma above, $z_1 = \infty \Rightarrow \frac{z-z_2}{z_3-z_2}$, $z_2 = \infty \Rightarrow \frac{z_3-z_1}{z_3-z}$, $z_3 = \infty \Rightarrow \frac{z-z_2}{z-z_1}$ (just consider dominating terms, don't memorize)

DEFINITION

For distinct $z_1, z_2, z_3, z_4 \in \mathbb{C}$, the cross-ratio is defined as $(z_1, z_2, z_3, z_4) := T_{(z_1,z_2,z_3)}(z_4)$

PROPOSITION

The cross-ratio is invariant under bilinear transformation S, i.e. $(z_1, z_2, z_3, z_4) = (Sz_1, Sz_2, Sz_3, Sz_4)$

Proof

$$z_1, z_2, z_3 \xrightarrow{T_{(z_1,z_2,z_3)}} \infty, 0, 1$$
$$\downarrow S$$
$$Sz_1, Sz_2, Sz_3 \xrightarrow{T_{(Sz_1,Sz_2,Sz_3)}} \infty, 0, 1$$

Then, $T^{-1}_{(Sz_1,Sz_2,Sz_3)} \circ T_{(z_1,z_2,z_3)}(z_i) = S(z_i)$ for $i=1,2,3$

By prop, $T^{-1}_{(Sz_1,Sz_2,Sz_3)} \circ T_{(z_1,z_2,z_3)} = S$ (3 fixed pts)

$\Rightarrow T^{-1}_{(Sz_1,Sz_2,Sz_3)} \circ T_{(z_1,z_2,z_3)}(z_4) = S(z_4) \Rightarrow T_{(z_1,z_2,z_3)}(z_4) = T_{(Sz_1,Sz_2,Sz_3)}(S(z_4))$ □

RIEMANN MAPPING THEOREM

OVERVIEW

"holomorphically simply connected D" (D: open + connected)

$\forall f$: ana on D, C: simple closed curve $\subseteq D$, and $\int_C f\,dz = 0$ (☆)

We define "s.c." in $\mathbb{C}$ as $(\mathbb{C} \cup \{\infty\}) \setminus D$: path-connected, then we say D is s.c.

$\Rightarrow$ (Already Proved) D satisfies (☆)

With Riemann Mapping Thm, we get $D \overset{\text{conformally equiv}}{\sim} D(0,1) \Rightarrow D$ is s.c. in topological sense

RIEMANN MAPPING THEOREM

Let $R :=$ s.c. region $\subsetneq \mathbb{C}$, $U := D(0,1)$.

Then, given $z_0 \in R$, $\exists$ a unique conformal mapping φ of R onto U s.t. $\varphi'(z_0) > 0$ and $\varphi(z_0) = 0$ (△)

Intuitively, use Schwarz Lemma, $\because 0 \to 0$

Proof

- Uniqueness

  Let φ_1, φ_2 satisfy (△).

  Then, $\Phi := \varphi_2 \circ \varphi_1^{-1} \in Aut(U)$ with $\Phi(0) = 0$

  $\therefore$ By lemma, $\Phi(z) = e^{i\theta} z \Rightarrow \Phi'(z) = e^{i\theta}$

  However, $\Phi'(0) = \varphi_2'(z_0)(\overset{>0}{\varphi_1'(z_0)})^{-1} \Rightarrow \theta = 0$

Uniqueness

$\Phi := \varphi_2^{-1} \circ \varphi_1$, $\Phi(0) = 0$, $\Phi \in Aut(U)$

$\therefore \Phi = e^{i\theta} z$

- Existence

  Let $\mathcal{F}_{z_0} := \{ f \mid f: R \to U, \text{ conformal}, f'(z_0) > 0 \}$

  ($|f| \le 1$; conformal: 1-1 + ana)

CLAIM A: $\mathcal{F}_{z_0} \neq \emptyset$

Consider $p_0 \in \mathbb{C} \setminus R$. Then, $\frac{1}{z - p_0}$: 1-1 + analytic

Case ①: $\exists r$, s.t. $D(p_0, r) \cap R = \emptyset \Rightarrow \frac{1}{|z - p_0|} \le \frac{1}{r}$ (bounded $\Rightarrow$ can normalize to $\le 1 \rightsquigarrow U$)

Case ②: $\exists \{\zeta_i\} \to p_0$, $\zeta_i \in R$, $\boxed{R: \text{s.c.}}$

We can define a branch of log on R

$\Rightarrow$ We can choose a branch of log of $f(z) := \sqrt{\frac{z - p_0}{z_0 - p_0}}$ s.t. $f(z_0) = 1$ ($\Rightarrow$ We chose s.t. -1 would not appear!)

Claim: $|f(z) - (-1)| > \delta$ for some $\delta > 0$ $\forall z \in R$ (universal bound)

Proof

Suppose not, $\exists \{\zeta_n\}$ with $\lim_{n\to\infty} f(\zeta_n) = -1 \Rightarrow \frac{\zeta_n - p_0}{z_0 - p_0} \to 1$ as $n \to \infty \Rightarrow \zeta_n \to z_0$

However, f: conti $\Rightarrow \lim_{n\to\infty} f(\zeta_n) = f(z_0) = 1$ ※

Set $g(z)=\frac{1}{f(z)+i}$, then $|g(z)|\le\frac{1}{\delta}$ $\forall z\in R$ (bounded $\Rightarrow$ can normalize to $\le 1 \rightsquigarrow U$)

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For Cases ① and ②, (let $g(z):=\frac{1}{z-p_0}$ for ①), $g'(z_0)\neq 0 \Rightarrow \exists\,\theta$, s.t. $e^{i\theta}g'(z_0)=|g'(z_0)|>0$
$\Rightarrow \mathcal{F}_{z_0}\neq\emptyset$ $\square$

Next time: Claim B+C.

11-27-25 (WEEK 13)

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RIEMANN-MAPPING THEOREM PROOF CONTINUED

CLAIM C: $\exists f \in \mathcal{F}_{z_0}$, s.t. f: onto to U

To do this, we find $f \in \mathcal{F}_{z_0}$, s.t. $f'(z_0)$: max

Idea: $\boxed{\text{Suppose}}$ $f \in \mathcal{F}_{z_0}$, s.t. $f'(z)$ is maximum

(i) $f(z_0)=0$

Proof

If $f(z_0)=\alpha \neq 0 \in U$, then $B_\alpha \circ f(z_0)=0$ (shifted) up to $e^{i\theta}$ and $B_\alpha \circ f \in \mathcal{F}_{z_0}$

$\Rightarrow g'(z_0) = \frac{f'(z_0)}{1-|\alpha|^2} >> f'(z_0)$ ✗

shift center to equal 0 then derivative results in a larger one ✗

(ii) f is onto

Proof

If not, $\alpha \neq 0$, $\alpha \in U \setminus \mathrm{Im}(f)$

Replace f by some $e^{i\theta}f$ ($|f'(z_0)|$ still max), then we can assume $\alpha \in \mathbb{R}_{<0} \Rightarrow$ Set $t \in \mathbb{R}_{>0}$, $\alpha = -t^2$

shift so $f(z_0) \in \mathbb{R}_{<0}$

$\therefore f_1(z) = \frac{f(z)-\alpha}{1-\alpha f(z)} = \frac{f(z)+t^2}{1+t^2 f(z)}$

$0 \notin \mathrm{Im}(f_1)$, $f_1: R \to U$, 1-1, analytic

$\therefore \frac{1}{f_1(z)}$, $f_1'(z)$: ana on R

must check $\frac{f_1'}{f_1}$ ana

As R: s.c., by closed curve thm, $\forall$ closed curve $C \subseteq R$, $\int_C \frac{f_1'(z)}{f_1(z)} dz = 0$

$\therefore$ We can define $\log f_1(z)$ on R

want to define $\log z$ $\int_C \frac{f_1'(z)}{f_1(z)} dz$

In particular, we can choose a branch of $f_1(z)$ on R, s.t. $\sqrt{f_1(z_0)} = t$

So we can have $\sqrt{f_1(z_0)} = t$

Let $f_2(z) := \sqrt{f_1(z)}$, $f_3(z) := \frac{f_2(z)-t}{1-tf_2(z)}$, f_3: 1-1.

Notice, $f_1'(z_0) = f'(z_0)(1-t^4)$, $f_2'(z_0) = \frac{f_1'(z_0)}{2t}$, $f_3'(z_0) = \frac{f_2'(z_0)}{1-t^2}$ $\Rightarrow f_3'(z_0) = \frac{f'(z_0)(1+t^2)}{2t} >> f'(z_0)$

If we set $g(z) := e^{i\theta} f_3(z)$, then $g \in \mathcal{F}_{z_0}$, $g'(z_0) > f'(z_0)$ ✗

$\therefore f$ must be onto ✓

☆ Conclusion: $|f'(z_0)|$ max $\Rightarrow f(z_0)=0$ and f: onto □

CLAIM B: $\exists \varphi \in \mathcal{F}_{z_0}$, s.t. $\varphi'(z_0)=M$, where $M = \sup_{f \in \mathcal{F}_{z_0}} f'(z_0) < \infty$

① $M < \infty$

Choose r, s.t. $\overline{D(z_0, r)} \subseteq R$

$\therefore$ By Cauchy Integral Formula, $f'(z_0) = \frac{-1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{(z-z_0)^2} dz$ ($f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$)

$\therefore |f'(z_0)| \leq \frac{1}{2\pi} \cdot \frac{1}{r^2} \cdot 2\pi r = \frac{1}{r}$

② $\{\zeta_i\}_{i=1}^\infty$ is a countable dense subset in R, where we choose $\{f_n\}_{n=1}^\infty \subseteq \mathcal{F}_{z_0}$, s.t. $\lim_{n\to\infty} f_n'(z_0) = M$

Find a subseq $\{f_{1n}\}$ s.t. $\{f_{1n}(\zeta_1)\}$ conv, find $\{f_{2n}\}$ subseq of $\{f_{1n}\}$ s.t. $\{f_{2n}(\zeta_2)\}$ conv $\Rightarrow$ $\{f_{2n}(\zeta_1)\}$ and $\{f_{2n}(\zeta_2)\}$ conv

Continuing this process, $\{f_{in}\}_{n=1}$ conv at $\zeta_1, \dots, \zeta_i$ $\Rightarrow$ $\{f_{nn}\}$ conv at all ζ_i

Find all f_{in} via recursion :)

Claim: $\{f_{nn}\} := \{\varphi_n\}$ conv to an analytic function

<u>Proof</u>

(i) $\varphi_i \in \mathcal{F}_{z_0} \Rightarrow |\varphi_i(z)| < 1 \quad \forall z \in R \Rightarrow \{\varphi_i\}$: uniformly bounded (indep of i, z) ✓

(ii) $\{\varphi_n\}$ conv unif on $\overline{D(w_0, r)} \subseteq R$

$\mathbb{C} \setminus R$: closed, so $2d := d(\tilde{R}, \overline{D(w_0, r)}) = \inf \overline{d(z_1, z_2)}$, $z_1 \in \tilde{R}$, $z_2 \in \overline{D(w_0, r)}$

$\forall z \in \overline{D(w_0, r)} \Rightarrow \overline{D(z, d)} \subseteq R$, so $|\varphi_n(z)| = \frac{1}{2\pi} \left| \int_{C_d(z)} \frac{\varphi_n(w)}{(w-z)^2} dw \right| \leq \frac{1}{d}$

$\Rightarrow \forall z_1, z_2 \in \overline{D(w_0, r)}$, $|\varphi_n(z_1) - \varphi_n(z_2)| = \left| \int_{z_1}^{z_2} \varphi_n'(z) dz \right| = \frac{|z_2 - z_1|}{d}$

Given ε, $\forall n$, $\forall |z_1 - z_2| < \varepsilon d$, we have $|\varphi_n(z_1) - \varphi_n(z_2)| < \varepsilon$

$\therefore \varphi$: uniformly equicontinuous (indep of n, z_1, z_2) ✓

12-2-25 (WEEK 14) Shun/翔海 (@shun4midx)

CONTINUED RIEMANN MAPPING THEOREM PROOF

As S is dense in D, $\exists \xi_i \in S$, s.t. $|z-\xi_i|<\varepsilon d$

$\because \lim_{n\to\infty} \varphi_n(\xi_i)$ exists

$\therefore \exists N>0$, s.t. $|\varphi_n(\xi_i)-\varphi_m(\xi_i)|<\frac{\varepsilon}{3}$ $\forall n,m>N$

Hence, $|\varphi_n(z)-\varphi_m(z)| \le |\varphi_n(z)-\varphi_n(\xi_i)|+|\varphi_n(\xi_i)-\varphi_m(\xi_i)|+|\varphi_m(\xi_i)-\varphi_m(z)|<\varepsilon$

$\therefore \lim_{n\to\infty} f_n(z)$ exists $\forall z\in R$

(iii) Define $\varphi(z):=\lim_{n\to\infty}\varphi_n(z)$ $\forall z\in R$.

We want "$\varphi_n \to \varphi$ on cpta of R"

$\forall K\subseteq R$: cpt, given $\varepsilon>0$, $V_j:=\{z\in K \mid |\varphi_n(z)-\varphi(z)|<\varepsilon \ \forall n\ge j\}$

Then, $K:=\bigcup_{j=1}^{\infty} V_j \xRightarrow{K:\text{cpt}} K=\bigcup_{j=1}^{N} V_j \Rightarrow \varphi_n\to\varphi$ unif on K

$\because$ conti / by our choice

As φ_n: ana $\Rightarrow \varphi$ ana and $\varphi_n'\to\varphi'$, thus φ: ana and $\varphi'(z_0)=\lim_{n\to\infty}\varphi_n'(z_0)=M$ □

RIEMANN-ZETA FUNCTION

DEFINITION

notice, n is the base now, not z!!!

A Dirichlet series is in the form of $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$, where $n^z=\exp(z\log n)$ and we choose the branch s.t. $\log n\in\mathbb{R}$

In particular, the Riemann zeta function is $\sum_{n=1}^{\infty}\frac{1}{n^z}$

THEOREM

If $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ converges at $z=z_0$, then it converges at all $z\in\{\eta \mid \operatorname{Re}(\eta)>\operatorname{Re}(z_0)\}=:H_0$. Moreover, it conv unif on cpta $\forall K\subseteq H_0$

Proof

Fix $z\in H_0$, we want "$\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ conv"

Claim: Given $\varepsilon>0$, $\exists n_0$, s.t. $N,M>n_0 \Rightarrow \left|\sum_{n=M}^{N}\frac{a_n}{n^z}\right|<\varepsilon$

Proof

$(A_n-A_{n-1})b_n$

Define $A_i:=\sum_{n=1}^{i}\frac{a_n}{n^{z_0}}$, $b_i:=\frac{1}{n^{z-z_0}}$. Then, $\sum_{n=M}^{\infty}\frac{a_n}{n^z}=A_{M-1}b_M+\underbrace{A_M(b_M-b_{M+1})+\dots+A_{N-1}(b_{N-1}-b_N)}_{(\star)}+A_Nb_N$

Since $\sum_{n=0}^{\infty}\frac{a_n}{n^{z_0}}$ conv, $\exists A>0$, s.t. $|A_i|<A$, $b_n-b_{n+1}=\frac{1}{n^{\omega}}-\frac{1}{(n+1)^{\omega}}=\left[-t^{-\omega}\right]_n^{n+1}=\int_n^{n+1}\omega t^{-\omega-1}dt \Rightarrow |b_n-b_{n+1}|<\frac{|\omega|}{n^{1+\delta}}$, $\delta=\operatorname{Re}(z-z_0)>0$

Moreover, $\star\le A\cdot\sum_{n=M}^{N-1}\frac{|\omega|}{n^{1+\delta}}$, where $\sum_{n=1}^{\infty}\frac{|\omega|}{n^{1+\delta}}$ conv $\forall\delta>0$

$\therefore \exists C>0$, s.t. $\forall n, n_1, n_2>C$, $\left|\sum_{n=n_1}^{n_2}\frac{|\omega|}{n^{1+\delta}}\right|<\frac{\varepsilon}{A}$, so $\star<\varepsilon$

$\Rightarrow \forall N,M>C$, $\left|\sum_{n=M}^{N}\frac{a_n}{n^z}\right|<3\varepsilon$ □

REMARK / COUNTEREXAMPLE

$A_i=\sum_{n=1}^{i}\frac{a_n}{n^{z_0}}$, indep of $z\in H_0$. $(\triangle)$

In particular, $\exists A$, s.t. $|A_i|<A$ $\forall i$. Then, $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ converges? (No!)

For example, let $a_n:=(-1)^n$, then for $z=0$, $\sum_{n=1}^{\infty}(-1)^n$ does not converge, but it satisfies $(\triangle)$. Hence, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^z}$ conv $\forall \operatorname{Re}(z)>0$

THEOREM

Shun/翔海(@shun4midx)

If $\sum_{n=0}^{\infty}\frac{a_n}{n^z}$ converges abs conv for some z but not all $z_0\in\mathbb{C}$, then $\exists x_0\in\mathbb{R}$ (called abscissa of convergence abs conv) s.t. $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ converges conv abs if $Re(z)>x_0$ and $\sum_{n=0}^{\infty}\frac{a_n}{n^z}$ diverges not abs conv $\forall Re(z)<x_0$

Proof

Consider the Thm, we know $[\underbrace{\inf\{Re(w)\mid\sum\frac{a_n}{n^w}\text{ conv}\}}_{=:C}]>-\infty$ as $f(z):=\sum\frac{a_n}{n^z}$ not conv $\forall z\in\mathbb{C}$.

$\forall z'$ with $Re(z')>C$, $\exists z_0$ with $\underbrace{c<Re(z_0)<Re(z')}_{\text{def for inf}}$, s.t. $f(z)$: conv at $z_0\Rightarrow f(z')$: conv

By Thm, $x_0=C$

Now, $\underbrace{\sum\left|\frac{a_n}{n^z}\right|}_{\text{conv at } z_0}=\sum\frac{|a_n|}{n^{Re(z)}}$

$\forall z$ with $Re(z)>Re(z_0)$, $\sum\frac{|a_n|}{n^{Re(z)}}<\sum\frac{|a_n|}{n^{Re(z_0)}}$ □

EXAMPLE

$\zeta(z):=\sum\frac{1}{n^z}\Rightarrow$ abscissa of ζ is $z=1$

$\underbrace{\zeta^2(z)}_{\text{square, i.e. } \zeta\cdot\zeta}=\sum\frac{C_n}{n^z}$, $C_n=\sum_{d\mid n}1$

In general, $f(z):=\sum\frac{a_n}{n^z}$, $g(z):=\sum\frac{b_n}{n^z}$: conv $\forall Re(z)>C$

Then, we can define $f(z)+g(z)$, $f(z)g(z)=\sum\frac{C_n}{n^z}$, $C_n=\sum_{d\mid n}a_d b_{\frac{n}{d}}$ for $Re(z)>C$

ANALYTIC CONTINUATION

MAIN QUESTION

Given $\underbrace{\sum_{i=0}^{\infty}a_n z^i}_{\text{any arbitrary } g(z)\text{: ana on } D(0,R)}$ conv abs for $|z|<R$, $\exists$? ana f s.t. $f|_{D(0,R)}=\sum_{i=0}^{\infty}a_n z^i$ but f: defined on $D\supsetneq D(0,R)$?

↪ We have seen examples such as Schwarz Lemma and removable singularities: $\frac{1}{1-z}$: ana cont. of $\sum_{i=1}^{\infty}z^n$, $|z|<1$

(I) POWER SERIES

THEOREM

If $f(z)=\sum_{n=0}^{\infty}a_n z^n$ has a positive radius of conv R, then f has a singularity at $|z|=R$

EXAMPLE

$\sum_{n=1}^{\infty}\frac{z^n}{n^2}$ conv at $z=1$, ana at $|z|<1$

If f: ana in a nbd of $z=1$ and $f(z)=\sum\frac{z^n}{n^2}$ at $z=1$, $f(z)|_{D(1,\varepsilon)}=\sum\frac{z^n}{n^2}$,
Then, $f''(z)$: ana on $D(1,\varepsilon)$

However, $f''(z)=\sum_{n=1}^{\infty}\frac{n(n-1)}{n^2}z^n$ div at $z=1$ ⨯

$\Rightarrow z=1$ is a singular point of $\sum_{n=1}^{\infty}\frac{z^n}{n^2}$

PROOF OF THEOREM

If not, $\forall|z|=R$, $\exists$ max $\varepsilon_z>0$, s.t. f can be continued analytically on $D(z,\varepsilon_z)$ to $\tilde{f}$, where ε_z varies cont. in z.

$|z|=R$: cpt $\Rightarrow\varepsilon=\min_{|z|=R}\varepsilon_z$, $\exists|z_0|=R$, $\underbrace{\varepsilon_{z_0}=\varepsilon}_{>0 \;\leftarrow z_0\text{: regular}}$

Then, $\tilde{f}$ can be defined on $D(0,R+\varepsilon)$ analytically $\Rightarrow\tilde{f}=\sum_{n=0}^{\infty}b_n z^n$, $\tilde{f}|_{D(0,R)}=f\xRightarrow{\text{Uniqueness}}a_n=b_n$, so radius of conv $=R$ ⨯

12-4-25 (WEEK 14) (I'm so fucking stressed but we don't talk about that... hope I'm gonna be okay...) Shun/翔海 (@shun4midx)

DEFINITION

Suppose that f is analytic in a disc D and $z_0 \in \partial D$. Then, f is said to be regular at z_0 if f can be continued analytically to a region D_1 with $z \in D_1$. Otherwise, f is said to be a singularity at z_0.

CAUTION

$z \in \partial D$ is a singularity does NOT depend on if f is continuous at z

Example 1: $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ for $|z|<1$, but $f(z)$ is cont. on $|z|=1$

Example 2: $f(z) = \sum_{i=0}^{\infty} z^i$ for $|z|<1$, $\lim_{z \to e^{i\theta}} f(z)$ DNE but $\frac{1}{1-z}$ is defined $\forall z \in \mathbb{C} \setminus \{1\} \Rightarrow \forall e^{i\theta} \neq 1$, $e^{i\theta}$: regular pt of f

THEOREM

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a radius of conv $R < \infty$ and $\boxed{a_n \geq 0 \ \forall n}$. Then, $f(z)$ becomes a singularity at some $z=R$

Proof

By Thm last time, f has a singularity at some $|z|=R$

Let $z = Re^{i\theta}$ be such a singularity.

Consider f at $z = \rho e^{i\theta}$, $0<\rho<R$

By Taylor expansion at $z = \rho e^{i\theta}$, $\underbrace{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\rho e^{i\theta})}{n!} (z-\rho e^{i\theta})^n}$ radius of conv $R-\rho$ in $|z-\rho e^{i\theta}| < R-\rho$ as $Re^{i\theta}$ is a singularity of f

Consider $\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(\rho e^{i\theta})}{n!} \right| |z-\rho e^{i\theta}|^n$, where $f^{(n)}(\rho e^{i\theta}) = \sum_{i=n}^{\infty} \underbrace{n(n-1)\cdots(a_i)}_{\geq 0} (\rho e^{i\theta})^{i-n}$

$\therefore |f^{(n)}(\rho e^{i\theta})| \leq f^{(n)}(\rho)$

$\therefore$ At $z=\rho$, $f(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(\rho)}{n!} (z-\rho)^n$, moreover, radius of conv at $z=\rho e^{i\theta} \geq$ radius of conv at $z=\rho$

$\therefore$ $z=R$ is a singular point of f.

DEFINITION

If $\sum_{n=0}^{\infty} a_n z^n$ has a singularity at every point on its circle of convergence, then the circle is called a natural boundary

(e.g. $\sum_{i=0}^{\infty} z^i$, $|z|=1$: natural boundary)

THEOREM

Let $f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$ with $c_k \neq 0 \ \forall k$. Suppose $\lim_{k \to \infty} \frac{n_{k+1}}{n_k} > 1$. Then, the circle of convergence of f is a natural boundary

If $a_k \neq 0$, $\sum a_k z^k \rightsquigarrow \lim_{k \to \infty} \frac{k+1}{k} = 1$, so "$>1$" means the spacing between nonzero terms is large.

Proof

As the assumption is indep of c_k ($c_k \neq 0$), we can replace z by Rz and assume the radius of convergence $=1$

If $f(z)$ is singular at some $z=e^{i\theta}$, replace z with $ze^{i\theta}$ so $f(z)$ is singular at $z=1$

$\therefore$ It suffices to prove, $\forall \theta$, $f(z)$ has a singularity at $z=1$

Consider the map $\omega \xrightarrow{n} \omega$ $(n>1)$, it fixes $|\omega|=1$, $|\omega|<1$, $|\omega|>1$ to their corresponding (in)equality with 1

For $h: \omega \xrightarrow{\wedge} \frac{\omega^n + \omega^{n+1}}{2}$

Define $g(z) := f \circ h$

As $|h(\omega)|<1 \ \forall |\omega|=1$ but $\omega \neq 1$, thus g is regular at $|\omega|=1$ but $\omega \neq 1$.

If we can show that the radius of conv of g is 1, by Thm, we know that $\omega=1$ is a singular point of $g \Rightarrow f$: singular at $h(1)=1$

Claim: Conv radius of $g=1$ for some n — as m, exponent of ω.

Proof

$\lim_{k \to \infty} \frac{n_{k+1}}{n_k} = 1+\delta$, $\delta>0 \Rightarrow$ Choose m s.t. $\frac{m+1}{m} < 1+\delta$, as $k >> 0$, set $n=m$. We can even assume $n_0 >> 0$.

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Then, $g(\omega) = f\left(\frac{\omega^m+\omega^{m+1}}{2}\right) = c_0\left(\frac{\omega^m+\omega^{m+1}}{2}\right)^{n_0} + c_1\left(\frac{\omega^m+\omega^{m+1}}{2}\right)^{n_1} + \dots$

$$= \frac{c_0\omega^{mn_0}}{2^{n_0}} + \frac{c_0 n_0 \omega^{mn_0+1}}{2^{n_0}} + \dots + \frac{c_0}{2^{n_0}}\omega^{mn_0+n_0} + \frac{c_1}{2^{n_1}}\omega^{mn_1} + \frac{c_1 n_1}{2^{n_1}}\omega^{mn_1+1} + \dots + \frac{c_1}{2^{n_1}}\omega^{mn_1+n_1} + \dots$$

If conv radius of $g = r > 1$, then

$$\left|\frac{c_0\omega^{mn_0}}{2^{n_0}}\right| + \left|\frac{c_0 n_0\omega^{mn_0+1}}{2^{n_0}}\right| + \dots + \left|\frac{c_0}{2^{n_0}}\omega^{mn_0+n_0}\right| + \left|\frac{c_1}{2^{n_1}}\omega^{mn_1}\right| + \left|\frac{c_1 n_1}{2^{n_1}}\omega^{mn_1+1}\right| + \dots + \left|\frac{c_1}{2^{n_1}}\omega^{mn_1+n_1}\right| + \dots \quad (\star)$$

conv $\forall |\omega| < 1$. In particular, $\exists 1 < \alpha < r$, s.t. $|\omega| = \alpha$, s.t $(\star)$ conv and $(\star) = f\left(\frac{\alpha^m+\alpha^{m+1}}{2}\right)$ $\times$ $\because f$ has radius of conv 1

$\therefore$ Radius of conv of $g = 1$ $\square$

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THE METHOD OF MOMENT

Say $\sum_{n=0}^{\infty} c_n z^n =: f(z)$, how do we find an ana contn?

CASE 1

$c_n = \int_a^b g(t)t^n dt \Rightarrow f(z) = \sum_{n=0}^{\infty}\left(\int_a^b g(t)t^n dt\right)z^n \overset{|tz|<1:\text{ abs cnv}}{=} \int_a^b \sum_{n=0}^{\infty} g(t)(tz)^n dt \overset{|tz|<1}{=} \int_a^b \frac{g(t)}{1-tz}dt$

Say $h(z) := \int_a^b \frac{g(t)}{1-tz}dt$, then $z \notin [\frac{1}{b}, \frac{1}{a}] \Rightarrow 1-tz \neq 0$

Is $h(z)$ ana on $\mathbb{C}\setminus[\frac{1}{b}, \frac{1}{a}]$? (Morera. Prove cont. then Morera)

LEMMA

Suppose $\varphi(z,t)$ is a continuous function of t $\forall t \in [a,b]$ for fixed z and is analytic for $z \in D$ for fixed t.

Then, $f(t) = \int_a^b \varphi(z,t)\,dt$ is analytic $\forall z \in D$.

Proof

Notice, $\varphi(z,t)$: ana in $z \Rightarrow \varphi(z,t)$: cont. in $z \Rightarrow f(z)$: cont. in z

$\forall$ closed rectangle $\Gamma \subseteq D$,

$$\int_{\partial\Gamma} f(z)dz = \int_{\partial\Gamma}\int_a^b \varphi(z,t)\,dt\,dz \overset{\varphi:\text{ cont. on cpt set } \partial\Gamma\times[a,b]}{=} \int_a^b \underbrace{\int_{\partial\Gamma} \varphi(z,t)\,dz}_{=0\ (\because \varphi:\text{ana})}dt = \int_a^b 0\,dt = 0$$

$\therefore$ By Morera's Thm, $f(z)$: ana in z. $\square$

COROLLARY

$\int_a^b \frac{g(t)}{1-tz}dt$ is an analytic continuation of f

REMARK

We can replace $[a,b]$ with $[a,\infty]$: We check if we can $\int_{\partial\Gamma} \leftrightarrow \int_a^b$!

EXAMPLE

$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1}$, $|z|<1$

☆ $\frac{1}{n+1} = \int_0^1 t^n dt$

Then, $f(z) = \sum_{n=0}^{\infty}\left(\int_0^1 t^n dt\right)z^n = \int_0^1 \sum_{n=0}^{\infty}(tz)^n dt = \int_0^1 \frac{1}{1-tz}dt$: ana on $\mathbb{C}\setminus[1,\infty)$

CASE 2

$\int_0^\infty e^{-nt^2}dt = \frac{\sqrt{\pi}}{2\sqrt{n}}$

Then, $\underline{\sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}} = \frac{2}{\sqrt{\pi}}\left[\sum_{n=1}^{\infty}\left(\int_0^\infty e^{-nt^2}dt\right)z^n\right]} = \frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\int_0^\infty (ze^{-t^2})^n dt \overset{|z|<1}{=} \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{z}{e^{t^2}-z}dt$

↳ Ana when $\underline{z \in \mathbb{C}\setminus[1,\infty] \Rightarrow e^{t^2}-z \neq 0}$

Here, $\int_{\partial\Gamma}\int_0^\infty \frac{z}{e^{t^2}-z}dt \overset{\text{If } \int_0^\infty \frac{|z|}{|e^{t^2}-z|}dt \text{ exists}}{=} \int_0^\infty\int_{\partial\Gamma} \frac{z}{e^{t^2}-z}dt$ $\therefore$ ana cont.

☆ $\frac{1}{n+a} = \int_0^\infty e^{-nt}e^{-at}dt$

ANALYTIC CONTINUATION FOR DIRICHLET SERIES (I.E. $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$)

Shun/翔海(@shun4midx)

Recall Riemann-zeta function is a Dirichlet series, conv at $Re\, z>2$ ("abscissa")

Will test in exam

THEOREM (LANDAU'S THEOREM)

Suppose that $a_n \geq 0\ \forall n$, $b \in \mathbb{R}$ is a boundary point (i.e. f conv $\forall Re(z)>b$). Then, b is a singular point of f.

Proof Idea: If b: regular, then $\exists c<b$ s.t. $f(z)$ conv at $c \Rightarrow f(z)$ conv $\forall Re(z)>c \Rightarrow b$: not bdry pt

Proof

contin ana with b: ana

If b: regular, then $\exists \tilde{f}$: ana at b, s.t. $\tilde{f}|_{Re(z)>b} = f$

We can consider the Taylor expansion of $\tilde{f}$ at some $a>b$.

$\Rightarrow \tilde{f} = \sum_{k=0}^{\infty} c_k (z-a)^k$ (☆), $c_k = \frac{\tilde{f}^{(k)}(a)}{k!} = \frac{f^{(k)}(a)}{k!}$ $\therefore \tilde{f} = \sum_{n=1}^{\infty} \frac{a_n (\log n)^k}{n^a k!}$

As $\tilde{f}$: ana at b, radius of conv of (☆) $> a-b$

In particular, $\exists \varepsilon > 0$, s.t.

$$\sum_{k=0}^{\infty} |c_k| |a-b+\varepsilon|^k \underset{a_n \geq 0}{\overset{\text{conv}}{=}} \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{a_n (\log n)^k}{n^a k!} \right) (a-b+\varepsilon)^k$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{a_n}{n^a} \left[\underbrace{\frac{(\log n)^k}{k!} (a-b+\varepsilon)^k}_{\exp(\log n \cdot a-b+\varepsilon) = n^{a-b+\varepsilon}} \right] = \sum_{n=1}^{\infty} \frac{a_n}{n^{b-\varepsilon}}$$

$\therefore f$: conv at $b-\varepsilon$ □

COROLLARY

If $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ with $a_n \geq 0$ and can be analytically continued to the entire complex plane ($\exists \tilde{f}$: entire, $\tilde{f}|_{Re(z)>b} = f$), then it converges throughout the complex plane (f: conv throughout)

GAMMA FUNCTION

CONSTRUCTION

$f(z) = \int_0^{\infty} e^{-t} t^z \, dt \overset{\text{IBP}}{=} z f(z-1)$

Notice, $|f(z)| \leq \int_0^{\infty} |e^{-t} t^z| \, dt = \int_0^{\infty} e^{-t} t^{Re(z)} \Rightarrow f(z)$: well-def $\forall Re(z) > -1$

$\therefore \Gamma(z) := f(z-1)$ is well-def $\forall Re(z) > 0$

ANALYTIC CONTINUATION OF Γ

As $\Gamma(z+1) = z\Gamma(z)$, "ana" can be checked via diffability.

① Note, $\Gamma(z+1) = z\Gamma(z) \Rightarrow \Gamma(z) = \frac{\Gamma(z+1)}{z} \Rightarrow$ simple pole at $z=0$

② $\Gamma(z+1)$ is defined $\forall Re(z) > -1$

$\therefore$ Define $\tilde{F}_1(z) := \begin{cases} \frac{\Gamma(z+1)}{z}, & -1 < Re(z) < 0 \\ \Gamma(z), & Re(z) > 0 \end{cases}$ analytic

Check: "$\tilde{F}_1(z)$: ana except for $z=0$"

By Morera, it suffices to check $\tilde{F}_1(z)$ is cont:

For $y \neq 0$, $\lim_{z \to iy} \tilde{F}_1(z) = \lim_{z \to iy} \frac{\Gamma(z+1)}{z} = \frac{\Gamma(iy+1)}{iy} = \Gamma(iy)$

$\therefore \tilde{F}_1(z)$ is cont at $z = iy$, $\forall y \neq 0$ ✓

Continue this process, e.g.

$\tilde{F}_2(z) = \begin{cases} \tilde{F}_1(z), & Re(z) > -1 \\ \frac{\tilde{F}_1(z+1)}{z}, & -2 < Re(z) < -1 \end{cases} \rightarrow$ simple pole at $z=-1$

etc...

In this way, Γ can be extended analytically to $\mathbb{C}\setminus\{n \mid n\in\mathbb{Z}_{\le 0}\}$ Shun/翔海(@shun4midx)

RESIDUE OF Γ

EXAMPLE

$\mathrm{Res}(\Gamma(z),-1)=\mathrm{Res}(\tilde{\Gamma}_2(z),-1)=\lim_{z\to-1}(z+1)\tilde{\Gamma}_2(z)=\lim_{z\to-1}\cancel{(z+1)}\frac{\Gamma(z+2)}{z\cancel{(z+1)}}=-1$

Moreover, $\mathrm{Res}(\Gamma(z),-n)=\frac{(-1)^k}{k!}$

SPECIAL EQUALITIES

$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$ (i.e. Γ has no zeros)

ZETA FUNCTION (VERY HANDWAVY)

$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}\Rightarrow\frac{1}{2^z}\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{(2n)^z}\Rightarrow\left(1-\frac{1}{2^z}\right)\zeta=\sum_{n=1}^{\infty}\frac{1}{(2n+1)^z}\Rightarrow$ etc...

$\zeta(z)=\prod_{p:\text{prime}}\left(1-\frac{1}{p^z}\right)^{-1}=\prod_{p:\text{prime}}\left(1+\frac{1}{p^z}+\frac{1}{p^{2z}}+\dots\right)$

EQUATIONS

$\exists\ g(z)$: entire, s.t. $\zeta(z)=\frac{1}{\Gamma(z)}\left[g(z)+\frac{1}{z-1}+\frac{A_0}{z}+\frac{A_1}{z+1}+\dots\right]$

80%~90% OF QUESTIONS IN THE FINAL WILL BE FROM THE HWs THIS SEM

12-11-25 (WEEK 15) — FINAL LECTURE

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NOTICE

In $\mathbb{C}$, $\log a + \log b \neq \log(ab)$ in general

ZETA FUNCTION

AIM

Can we extend ζ to a meromorphic function on $\mathbb{C}$?

LINK TO GAMMA FUNCTION

Recall $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt$

Notice, $\int_0^\infty e^{-nt} t^{z-1}\,dt \overset{s=nt}{=} \int_0^\infty e^{-s}\left(\frac{s}{n}\right)^{z-1}\frac{ds}{n} = \overbrace{\frac{1}{n^z}}^{\text{for } \zeta(z)} \underbrace{\int_0^\infty e^{-s} s^{z-1}\,ds}_{=\Gamma(z)}$

$\therefore$ For $Re(z)>1$, $\zeta(z) = \frac{1}{\Gamma(z)} \sum_{n=1}^\infty \int_0^\infty e^{-nt} t^{z-1}\,dt$

$\Gamma(z)$ ① has no zeros
② has only poles as singularity
$\Rightarrow \frac{1}{\Gamma(z)}$: entire

Thus,
$Re(z)>1 \Rightarrow \sum_{n=1}^\infty \int_0^\infty e^{-nt} t^{z-1}\,dt = \int_0^\infty \left(\sum_{n=1}^\infty (e^{-nt})\right) t^{z-1}\,dt = \overbrace{\int_0^1 \left(\sum e^{-nt}\right) t^{z-1}\,dt}^{\text{depends on } z} + \overbrace{\int_1^\infty \left(\sum e^{-nt}\right) t^{z-1}\,dt}^{\text{holomorphic } \forall z \in \mathbb{C}}$

Notice, $\int_0^1 \left(\sum_n e^{-nt}\right) t^{z-1}\,dt = \int_0^1 \frac{t^{z-1}}{e^t - 1}\,dt$ $\leftarrow e^x - 1 = x + \frac{1}{2}x^2 + \dots$

We know $\frac{1}{e^t-1}$ has a simple pole at $t=0$, $Res\left(\frac{1}{e^t-1}, 0\right) = 1$

Consider the Laurent Expansion around $t=0$,

$\frac{1}{e^t-1} = \frac{1}{t} + \sum_{i=0}^\infty A_i t^i$. As $\frac{1}{e^t-1}$: defined $\forall t \neq 0$, thus $\sum_{i=0}^\infty A_i t^i$ conv $\forall t \in \mathbb{C}$.

$\therefore$ For $Re(z)>1$, $\int_0^1 \frac{t^{z-1}}{e^t-1}\,dt = \int_0^1 t^{z-2} + \sum_{n=0}^\infty A_i t^{i-2}\,dt = \frac{1}{z-1} + \sum_{n=0}^\infty \frac{A_i}{z+i}$ — conv $\forall z \in \mathbb{C} \setminus \{\mathbb{Z}_{\leq 1}\}$ because ① $\sum_{i=0}^\infty A_i t^i$ conv $\forall t \in \mathbb{C}$ ② $\frac{1}{|z+i|} < 1$ for i large enough

Thus, $Re\, z > 1 \Rightarrow \zeta(z) = \underbrace{\frac{1}{\Gamma(z)}}_{\text{entire w/ zero of order 1 at } \mathbb{Z}_{\leq 0}} \Big(\overbrace{\frac{1}{z-1} + \sum_{i=0}^\infty \frac{A_i}{z+i}}^{(\Delta)\ \text{def on } \mathbb{C} \setminus \{\mathbb{Z}_{\leq 1}\}} + \overbrace{g(z)}^{\text{holo in } \mathbb{C}} \Big)$

THEOREM

$\zeta(z)$ can be analytically continued to $\mathbb{C} \setminus \{z=1\}$

<u>Proof</u>

$\frac{1}{\Gamma(z)}$ has zero of order 1 at $\mathbb{Z}_{\leq 0}$ and (Δ) has simple poles at $\mathbb{Z}_{\leq 1}$

SET-UP

Recall, $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z} = \prod_{p:\text{prime}} (1-p^{-z})^{-1}$, $Re(z) > 1$

$\therefore$ $\zeta(z)$ has no zeros for $Re(z) > 1$

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THEOREM

$\zeta(z)$ has no zeros for $Re(z) \geq 1$ (i.e. by set-up, $\zeta(z)$ has no zeros at $x=1$)

Proof Sketch

holomorphic for $Re(z) > \frac{1}{2}$

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{p:\text{prime}} \frac{[(1-p^{-z})^{-1}]'}{(1-p^{-z})^{-1}} = \sum_p \frac{(\log p)}{p^z - 1} = \phi(z) + \sum_p \frac{\log p}{p^z(p^z-1)}, \text{ where } \phi(z) = \sum_{z:\text{prime}} \frac{\log p}{p^z}$$

Use prime not log

Suppose $\exists$ zero for ζ at $1+ia_0$ with $a_0 \neq 0$, of order $\mu > 0$

We know $z \in \mathbb{R}_{>1} \Rightarrow \zeta(z) \in \mathbb{R}$

$\therefore$ By Reflection Principle, $\zeta(\bar{z}) = \overline{\zeta(z)}$

$\therefore$ $1-ia_0$ is a zero of order μ of ζ

Say ζ has a zero at $1 \pm 2ia_0$ of order ν ($\nu = 0$ if $1 \pm 2ia_0$ is a "zero")

Thus, $\phi(z)$ has poles of the same order as $-\frac{\zeta'(z)}{\zeta(z)}$ = zeros and poles of $\zeta(z)$

$\Downarrow$

$\lim_{\varepsilon \to 0} \varepsilon \phi(1+\varepsilon) = 1$ (ζ has simple pole at $z=1$ with residue 1)

$\lim_{\varepsilon \to 0} \varepsilon \phi(1 \pm ia_0 + \varepsilon) = \mu$

$\lim_{\varepsilon \to 0} \varepsilon \phi(1 \pm 2ia_0 + \varepsilon) = \nu$

Finally,

$$\sum_{-2 \leq \ell \leq 2} \binom{4}{2+\ell} \phi(1+\varepsilon+i\ell a_0) = \sum_p \frac{\log p}{p^{1+\varepsilon}} \left(\sum_{-2 \leq \ell \leq 2} \binom{4}{2+\ell} \left(\frac{1}{p^{\frac{ia_0}{2}}}\right)^{\ell+2} \left(\frac{1}{p^{-\frac{ia_0}{2}}}\right)^{-\ell+2} \right)$$

$$= \underbrace{\sum_p \frac{\log p}{p^{1+\varepsilon}} \underline{\left(p^{\frac{ia_0}{2}} + p^{-\frac{ia_0}{2}}\right)^4}}_{\geq 0} > 0$$

As $\varepsilon \to 0$, $-2\nu - 8\mu + 6 \geq 0 \Rightarrow \mu = 0$ $\times$