10-14-25 (WEEK 7) Shun/翔海 (@shun4midx)

RESOLVING THE WELL-DEFINEDNESS OF LOG OVER $\mathbb{C}$

We can consider approaching $\log z$ via integration.

We want $\log z$ s.t. ① f: ana
② $\exp(f(z)) = z$

$\therefore$ If $f(z) = \log z$, we want it to satisfy $f'(z) = \frac{1}{z}$

We can $\boxed{\text{fix } z_0 \in \mathbb{R}^+}$, so $f(z) := \int_C \frac{1}{\zeta} d\zeta + \boxed{\log(z_0)}$ where $C: z_0 \to z$

However, we need $\int_C \frac{1}{\zeta} d\zeta$ to be well-defined indep of path...

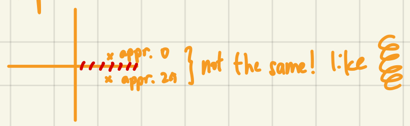
$\therefore$ Choose a s.c. region D, then $\forall C \subseteq D$, $\int_C \frac{1}{\zeta} d\zeta$ is well-defined (details in proof below)

<u>Proof</u> (Sketch)

$\forall C_1, C_2 \subseteq D$ with the same endpoints, $C_1 - C_2$ forms a closed path in D

$\therefore \int_{C_1 - C_2} \frac{1}{\zeta} d\zeta = 0 \Rightarrow \int_{C_1} \frac{1}{\zeta} d\zeta = \int_{C_2} \frac{1}{\zeta} d\zeta$

Example for D



THEOREM

Set $f(z) := \int_{z_0}^{z} \frac{1}{\zeta} d\zeta + \log z_0$ on a s.c. region $D \subseteq \mathbb{C} \setminus \{0\}$, we fix a $z_0 \in D$ and choose $\log z_0$

Then, f is an analytic branch of $\log z$ in D.

<u>Proof</u>

As D: s.c., $C_1 - C_2$: closed curve

$\therefore$ By closed curve thm, $\int_{C_1 - C_2} \frac{1}{\zeta} d\zeta = \int_{C_1} \frac{1}{\zeta} d\zeta - \int_{C_2} \frac{1}{\zeta} d\zeta = 0$

$\therefore f$ is analytic

Moreover, we want "$\exp(f(z)) = z$" $\Leftrightarrow$ "$z e^{-f(z)} = 1$"

Set $g(z) := z e^{-f(z)} \Rightarrow g'(z) = e^{-f(z)} - z f'(z) e^{-f(z)} = \underline{\underline{0}}$

$\therefore g(z) = \text{const} = g(z_0) = z_0 e^{-\log z_0} = 1$

APPLICATION

Instead of this directly only used for log, we can use analytic branch to define $\sqrt{z}$.

Reason:

Say $z = re^{i\theta}$, then $(\sqrt{r} e^{i(\frac{\theta}{2} + \pi k)})^2 = re^{i\theta} = z$ $\forall k \in \mathbb{Z}$, so $\sqrt{z}$ is not uniquely defined

$\therefore$ For $f(z) = \sqrt[n]{z}$, we can define an analytic branch for $\log z$ by $\sqrt[n]{z} = \frac{1}{n} \exp(\log z + 2\pi n k)$

SINGULARITY

DEFINITION

A deleted neighborhood of z_0 is an open set of $\{z \mid \underline{0 <} |z - z_0| < \delta\}$

the actual deleted thing lmao

DEFINITION

f is said to have an isolated singularity at z_0 if f is analytic in a deleted neighborhood D of z_0 but is not analytic in z_0

EXAMPLES (Intuition, formal names given later)

① "Artificial" singularity: $f(z) = \begin{cases} \sin z, & z \neq 0 \\ 1, & z = 0 \end{cases}$

② "Fixable by multiplying a polynomial": $\frac{1}{z}$ at $z = 0$

③ "Unfixable": $\exp(\frac{1}{z})$ at $z = 0$

Shun/翔海(@shun4midx)

DEFINITION

Say z_0 is a singularity of f, we can classify it as follows:

① If $\exists\ g$: ana at z_0 and $f(z)=g(z)$ in some deleted nbd of z_0, we say f has a **removable singularity** at z_0

② If for $z\neq z_0$, f can be written as $f(z)=\frac{A(z)}{B(z)}$ where A and B are analytic at z_0, $A(z_0)\neq 0$, $B(z_0)=0$, we say f has a **pole** at z_0. In particular, if B has a zero of order k at z_0, then we say z_0 is a pole of f of order k

③ f has neither a removable singularity nor a pole at z_0, then we call z_0 an essential singularity of f (not the focus of this course)

THEOREM (RIEMANN'S PRINCIPLE OF REMOVABLE SINGULARITIES)

If f has an isolated singularity at z_0 and if $\lim_{z\to z_0}(z-z_0)f(z)=0$, then the singularity is **removable**

Proof

Define $D'(z_0,\delta):=D(z_0,\delta)\setminus\{z_0\}$, $\exists\,\delta$, s.t. f: ana on $D'(z_0,\delta)$

Set $g(z):=\begin{cases}(z-z_0)f(z), & z\in D'(z_0,\delta)\\ 0, & z=z_0\end{cases}$

Since $\lim_{z\to z_0}(z-z_0)f(z)=0=g(z_0)$, hence g is conti at z_0.

$\because$ f: ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D'(z_0,\delta)$

Morera needs it to be conti on the whole domain.

$\therefore$ g: conti on $D(z_0,\delta)$ + ana on $D'(z_0,\delta)$

$\therefore$ **g: ana on $D(z_0,\delta)$** (apply the conti except on a line segment thing)

Now, set:

$$h(z):=\begin{cases}\frac{g(z)-g(z_0)}{z-z_0}, & z\in D'(z_0,\delta)\\ g'(z), & z=z_0\end{cases}$$

h is ana because g is ana.

Moreover, as $f(z)=h(z)$ on $D'(z_0,\delta)$, thus z_0 is a removable singularity

COROLLARY

f has an isolated singularity at z_0. If f is **bounded on some deleted nbd of z_0, then z_0 is a removable singularity**

Proof

$\exists\,\delta$, s.t. f: ana and bounded on $D'(z_0,\delta)$

Given $\varepsilon>0$, $\forall\, 0<|z-z_0|<\frac{\varepsilon}{M}$, $|(z-z_0)f(z)|<\varepsilon \Rightarrow \lim_{z\to z_0}(z-z_0)f(z)=0$

$\therefore$ Conclude with thm above. $\square$

THEOREM 3

Say f has an isolated singularity at z_0.

If $\exists\, k\in\mathbb{Z}_{\geq 0}$, s.t. $\lim_{z\to z_0}(z-z_0)^k f(z)\neq 0$ but $\lim_{z\to z_0}(z-z_0)^{k+1}f(z)=0$, then f has a **pole of order k** at z_0 (rem. sing = pole of order 0)

Proof

Set $g(z)=\begin{cases}(z-z_0)^{k+1}f(z), & z\in D'(z_0,\delta)\\ 0, & z=z_0\end{cases}$

$\because \lim_{z\to z_0}(z-z_0)^{k+1}=0$ $\therefore$ g: conti at z_0

$\because$ f: ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D'(z_0,\delta)$

$\therefore$ g: conti on $D(z_0,\delta)$ + ana on $D'(z_0,\delta)$

$\therefore$ **g: ana on $D(z_0,\delta)$**

Set $h(z) := \begin{cases} \frac{g(z) - g(z_0)}{z - z_0} = (z-z_0)^k f(z), & z \in D'(z_0, \delta) \\ g'(z), & z = z_0 \end{cases}$ Shun/翔海(@shun4midx)

$\therefore$ h: ana on $D(z_0, \delta)$.

As we know, by assumption, $\lim_{z \to z_0} h(z) \neq 0 \Rightarrow h(z_0) \neq 0$ ($\because h$: ana)

$\therefore$ $f(z) = \frac{h(z)}{(z-z_0)^k} \Rightarrow f$ has a pole of order k at z_0 □

REMARK

$|f(z)| < \frac{1}{\sqrt{|z|}}$ in a deleted nbd of 0 and f has an isolated singularity at $0 \Rightarrow 0$ is a removable singularity

($\because$ There exists nonbounded removable singularity)

Proof

Actually, $|zf(z)| < \sqrt{|z|} \Rightarrow \lim_{z \to z_0} zf(z) = 0 \Rightarrow 0$ is a removable singularity □

REMARK

Similarly, if we have $|f(z)| < \frac{1}{|z|^{1.5}}$, then we know $|z^2 f(z)| < \sqrt{|z|} \Rightarrow \lim_{z \to z_0} z^2 f(z) = 0$

$\therefore \begin{cases} \text{Case 1: } \lim_{z \to z_0} zf(z) = 0, \text{ then removable singularity (pole of order 0)} \\ \text{Case 2: } \lim_{z \to z_0} zf(z) \neq 0, \text{ then pole of order 1} \end{cases}$

$\Rightarrow$ It has a pole of at most order 1 (higher the order $\Rightarrow$ the worse the pole)

THEOREM (CASORATI – WEIERSTRASS THEOREM)

If f has an essential singularity at z_0 and D is a deleted neighborhood of z_0, where f is analytic, then the range $R := \{f(z) \mid z \in D\}$ is dense in $\mathbb{C}$

Proof

Suppose not, then $\exists w \in \mathbb{C}$ and $\delta > 0$, s.t. open $D(w, \delta) \cap R = \phi$

In other words, $\forall z \in D$, $|f(z) - w| \geq \delta \Rightarrow \left|\frac{1}{f(z) - w}\right| \leq \frac{1}{\delta}$ $\forall z \in D$ $\Rightarrow \frac{1}{f(z)-w}$ is bounded in the del nbd

By coro, $\frac{1}{f(z)-w}$ has a removable singularity at z_0

$\therefore \exists$ g: ana on $D' \cup \{z_0\}$, s.t. $g(z) = \frac{1}{f(z) - w} \Rightarrow f(z) = w + \frac{1}{g(z)}$ $\forall z \in D'$

$\therefore$ z_0 is a zero of $g(z)$ of finite order ($\overset{!!}{n}$) or $g(z_0) \neq 0$

$\therefore$ $f(z)$ has a pole of order $\leq n$ at z_0, so not an essential singularity ※