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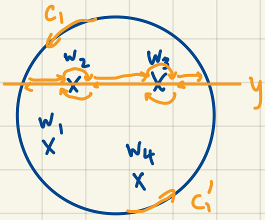
## PROPOSITION

For a region $D$, and $f$ ana on $D \setminus \{w_1, \dots, w_n\}$, with $w_i \in D(0,r)$, we have $\int_{C_r(0)} f(z)dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, w_i)$

**Proof**

Recall: $\int_{C_r(0)} \frac{1}{z} dz = 2\pi i$ (indep of $r$) $\Rightarrow \int_{C_r(0)} \frac{1}{z-a} dz = 2\pi i$




Let $Y := \{\mathrm{Im}(w_i) \mid 1 \le i \le n\}$, where we order $Y$ as $y_1 > y_2 > \dots > y_n$

Choose $\varepsilon$, s.t. $D(w_i, \varepsilon) \subseteq D(0,r)$ and $D(w_i, \varepsilon)$ are mutually disjoint

$\Rightarrow \int_{C_i + C_i'} f(z)dz = \int_{C_r(0)} f(z)dz - \sum_{i: \mathrm{Im}(w_i) = y_1} \int_{C_\varepsilon(w_i)} f(z)dz$

By Closed Curve Thm on simply connected domain, $\int_{C_i} f(z)dz = \int_{C_r(0)} f(z)dz - \sum_{i:\mathrm{Im}(w_i) = y_1} \int_{C_\varepsilon(w_i)} f(z)dz$

As we know, $C_2 + C_2' = C_i' - C_\varepsilon(w_1) \Rightarrow \int_{C_i'} f(z)dz = \cancel{\int_{C_2} \square}^{\,0} + \int_{C_2'} + \sum \int_{C_\varepsilon(w_i)}$

Inductively, we get: $\int_{C_r(0)} f(z)dz = \sum_{i=1}^{n} \int_{C_\varepsilon(w_i)} f(z)dz$

With the Laurent Expansion $f(z) = \sum_{k=-\infty}^{\infty} (z-w_i)^k$, hence $\int_{C_r(0)} f(z)dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, w_i)$ $\square$

## OBSERVATION

For a simple closed curve $\Gamma$, if we can define the "interior" (e.g. not ⓔ) of $\Gamma$, then we can replace the above proposition by $D \supseteq \Gamma \cup \mathrm{int}(\Gamma)$ with replacing $C_r(0)$ by $\Gamma$.

# WINDING NUMBER — CAUCHY RESIDUE THEOREM

## DEFINITION

(i) Suppose $\gamma$ is a closed curve and $a \notin \gamma$. We define the **winding number** of $\gamma$ around $a$ to be $\eta(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$

(E.g. $\gamma: C_r(a)$, $\eta(\gamma, b) = 1 \Leftrightarrow b \in D(a, r)$)

(ii) $\gamma$ is called a **regular closed curve** if $\gamma$ is a simple closed curve and $\eta(\gamma, a) \in \{0, 1\}$ $\forall a \notin \gamma$

## THEOREM

For any closed curve $\gamma$ and $a \notin \gamma$, $\eta(\gamma, a)$ is an integer

## REMARK

Fix $\gamma$, $\eta(\gamma, a)$ is cont. in $a \Rightarrow \eta(\gamma, a)$ is a locally constant function ($\eta(\gamma, a)$ is constant on each connected component of $\mathbb{C} \setminus \gamma$)

## PROOF OF THEOREM

For $\gamma: z(t)$, $t \in [0,1]$, define $F(s) = \int_0^s \frac{z'(t)}{z(t) - a} dt$ ($\int \frac{dz}{z}$) $\forall s \in [0,1]$.

Consider $g(z(s)) := (z(s) - a) \exp(-F(s))$

$\therefore g'(s) = z'(s) \exp(-F(s)) + (z(s) - a) \exp(-F(s))(-F'(s))$, where $F'(s) = \frac{z'(s)}{z(s) - a} \Rightarrow g'(s) = 0$

Thus, $g(s) = g(0) = z(0) - a$

$\quad \quad \quad \; \| \; g(1) = (z(1) - a) \exp(-F(1))$

$\therefore \exp(-F(1)) = \frac{z(1) - a}{z(0) - a}$

As $\gamma$: closed curve, thus $z(1) = z(0) \Rightarrow \exp(-F(1)) = 1 \Rightarrow F(1) = 2\pi i k$ for some $k \in \mathbb{Z}$.

$\therefore \eta(\gamma, a) = \frac{1}{2\pi i} F(1) = k \in \mathbb{Z}$ $\square$

## REMARK

Fix $\gamma$, $a \to \infty$. Then, $\left|\int_\gamma \frac{dz}{z-a}\right| \longrightarrow 0$

$\therefore \eta(\gamma, a) = 0$ when $a \to \infty \Rightarrow \eta(\gamma, a) = 0$ on unbounded connected component of $\mathbb{C} \setminus \gamma$.

## FACT (JORDAN CURVE THEOREM)

$\eta(\gamma, a) = 0 \Rightarrow a \in$ unbounded connected component of $\mathbb{C} \setminus \gamma$.

## THEOREM (CAUCHY RESIDUE THEOREM)

with winding #, we don't need to define inside or outside anymore

For $f$: ana on a s.c. domain $D$ except at $\underbrace{w_1, \ldots, w_n}_{\text{(distinct)}}$, $\gamma$: closed curve $\subseteq D \setminus \{w_1, \ldots, w_n\}$, then $\int_\gamma f(z)\,dz = 2\pi i \sum_i \text{Res}(f, w_i)\, \eta(\gamma, a)$

**Proof**

Considering the Laurent series around $w_i$,

$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-w_i)^k$ on $0 < |z-w_i| < \varepsilon$

$P_i\left(\frac{1}{z-w_i}\right) = \sum_{k=-\infty}^{-1} a_k (z-w_i)^k$

Notice, $P_i\left(\frac{1}{z-w_i}\right)$ is ana on $D \setminus \{w_i\}$.

$\therefore g(z) = f(z) - \sum_{i=1}^{n} P_i\left(\frac{1}{z-w_i}\right)$ is ana on $D \setminus \{w_1, \ldots, w_n\}$.

Around $w_i$ with $0 < |z-w_i| < \varepsilon$, $g(z) = \sum_{k=0}^{\infty} a_k (z-w_i)^k - \sum_{j \neq i} P_j\left(\frac{1}{z-w_j}\right) \Rightarrow g$ is ana at $z = w_i$

$\therefore w_1, \ldots, w_n$ are removable singularities at $w_i$.

By Closed Curve Thm, $\int_\gamma g(z)\,dz = 0 \Rightarrow \int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_\gamma P_i\left(\frac{1}{z-w_i}\right) dz = \sum_{i=1}^{n} 2\pi i\, \text{Res}(f, w_i) \int_\gamma \frac{1}{z-w_i}\,dz = 2\pi i \sum_i \text{Res}(f, w_i)\, \eta(\gamma, a)$ $\square$

## DEFINITION

$f$ is meromorphic on a domain $D$ if $f$ is ana on $D$ except at $\boxed{\text{isolated poles}}$