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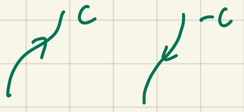
LINE INTEGRALS (CONTINUED)

DEFINITION

Let C be a curve defined by $z(t)=x(t)+iy(t)$, $t\in[a,b]$.
Then, $-C$ is a curve defined by $w(t)=z(a+b-t)$

In short, it is as follows:



PROPOSITION

$\int_{-C} f(z)dz = -\int_C f(z)dz$

PROPOSITION

Let C be a smooth curve, and f,g be continuous functions on C. Say, $\alpha\in\mathbb{C}$.

(i) $\int_C (f(z)+g(z))dz = \int_C f(z)dz + \int_C g(z)dz$

(ii) $\int_C \alpha f(z)dz = \alpha\int_C f(z)dz$

In other words, we say $\int_C (\cdot)dz$ is linear

EXAMPLE

Say $f(z)=\frac{1}{z}$, $C: R(\cos t + i\sin t)$, $t\in[0,2\pi]$

Then, $\int_C f(z)dz = \int_0^{2\pi}\frac{1}{\cancel{R}(\cos t + i\sin t)}\cancel{R}(-\sin t + i\cos t)\,dt$

$= \int_0^{2\pi} e^{-it}(-e^{i(t-\frac{\pi}{2})})\,dt$

$= \int_0^{2\pi} i\,dt$

$= 2\pi i$

LEMMA

Let $t\in\mathbb{R}$, $G(t)$ be a continuous complex-valued function. Then, $\left|\int_a^b G(t)dt\right| \le \int_a^b |G(t)|dt$

$(\alpha << \beta := |\alpha|\le|\beta|,\ \alpha,\beta\in\mathbb{C})$

Proof

Set $\int_a^b G(t)dt = Re^{i\theta}$, $\theta\in\mathbb{R}$, $R\in\mathbb{R}_{\ge 0}$

Then, $R = \left|\int_a^b G(t)dt\right| = \int_a^b e^{-i\theta}G(t)dt = \int_a^b A(t)dt + i\cancel{\int_a^b B(t)dt}^{\,0}$ $(e^{i\theta}G(t) = A(t)+iB(t))$

$\therefore R = \int_a^b A(t)dt \le \int_a^b |A(t)|dt \le \int_a^b |e^{-i\theta}G(t)|dt = \int_a^b |G(t)|dt$ □

PROPOSITION (ML-FORMULA)

Let C be a smooth curve of length L, and f be cont. on C and $\overset{|f|\le M}{f << M}$ throughout C. Then, $\int_C f(z)dz << ML$

Proof

Let C be $z(t)=x(t)+iy(t)$, $t\in[a,b]$.

Then, $\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt$

By the prev lemma, $\int_C f(z)dz << \int_a^b |f(z(t))|\overbrace{|\dot{z}(t)|}^{\text{arc length}}dt \le M\int_a^b|\dot{z}(t)|dt = ML$ □

EXAMPLE (FOR WHY ML IS THE TIGHT BOUND)

For $f(z)=\frac{1}{z}$, $C: \cos\theta + i\sin\theta$, $\int_C f(z) = 2\pi i \Rightarrow \left|\int_C f(z)\right| = 2\pi = ML$

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PROPOSITION

Suppose $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ unif on a smooth curve C. Then, $\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz$

Proof

$f_n \to f$ uniformly on C: "Given $\varepsilon>0$, $\exists N$, s.t. $\forall n \geq N$, $|f_n(z)-f(z)| < \varepsilon$ $\forall z \in C$."

So, $\left|\int_C f_n(z)dz - \int_C f(z)dz\right| = \left|\int_C (f_n-f)(z)dz\right| \overset{ML}{<} \varepsilon \cdot \text{len}(C)$ $\forall n \geq N$

$\therefore$ By def, $\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz$ □

PROPOSITION

Let F be an analytic function, $f = F'(z)$, and a smooth curve $C: z(t) = x(t)+iy(t)$, $t\in[a,b]$

Then, $\int_C f(z)dz = F(z(b)) - F(z(a))$

Proof

Let $\gamma(t) := F(z(t)) = A(t)+iB(t)$

Hence, $\dot\gamma(t) \overset{\text{smooth}}{=} \lim_{h\to 0} \frac{F(z(t+h))-F(z(t))}{z(t+h)-z(t)} \cdot \frac{z(t+h)-z(t)}{h} = F'(z(t))\,\dot z(t)$

Then, $\int_C f(z)dz = \int_a^b F'(z(t))\,\dot z(t)dt = \int_a^b \dot\gamma(t)dt = \gamma(b)-\gamma(a)$ □

DEFINITION

(i) A curve is closed if its initial and terminal points coincide.

(ii) C is a simple closed curve with $t\in[a,b]$ if $z(t_1)=z(t_2)$ with $t_1<t_2$, then $t_1=a$ and $t_2=b$

DEFINITION

The boundary of a rectangle is the simple closed curve in the counterclockwise direction

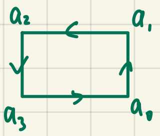
DEFINITION

f is an entire function if f is analytic on $\mathbb{C}$

LEMMA

If f is a linear function, i.e. $f=az+b$, $a,b\in\mathbb{C}$, Γ is the boundary of a rectangle, then $\int_\Gamma f(z)dz=0$

Proof



Say $\Gamma: z(t)$, $a=a_0 \leq t \leq b = a_3$, and $f = F'(z) \Rightarrow F := \frac{a}{2}z^2+bz$ (analytic on $\mathbb{C}$)

Hence, we can deduce $\int_\Gamma f(z)dz = \int_\Gamma F'(z)dz = F(z(b)) - F(z(a)) = 0$ □ ($\because z(b)=z(a)$)

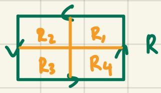
THEOREM (RECTANGLE THEOREM)

Let f be an entire function, and Γ as above, then $\int_\Gamma f(z)dz=0$

Proof

Let $I = \int_\Gamma f(z)dz$. Assume $f\neq 0$, otherwise $f\equiv 0 \Rightarrow I=0$.

We divide R as follows:



Then, $\exists$ one of R_i s.t. $\left|\int_{\Gamma_i} f(z)dz\right| \geq \frac{|I|}{4}$, where Γ_i is the boundary of R_i.

Set $R^{(1)}$ to be such an R_i.

Continuing this process, we get $R^{(1)} \supseteq R^{(2)} \supseteq \dots$ Let $z_0 \in \bigcap_{i=1}^{\infty} R^{(i)}$.

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As f is an entire function, hence f is analytic at z_0

By def, $\forall \varepsilon>0, \exists \delta>0$, s.t. $|h|<\delta \Rightarrow \left|\frac{f(z_0+h)-f(z_0)}{h} - f'(z_0)\right| < \varepsilon$

$\therefore$ We see $f(z) = f(z_0) + f'(z_0)(z-z_0) + \varepsilon(z)(z-z_0)$, where $|\varepsilon(z)| \leq \varepsilon$.

$\to 0$ (from lemma since linear)

We choose N s.t. $\forall n \geq N$, $|z-z_0|<\delta \Rightarrow \int_{\Gamma^{(n)}} f(z)dz = \cancel{\int_{\Gamma^{(n)}} [f(z_0)+f'(z_0)(z-z_0)]dz} + \int_{\Gamma^{(n)}} \varepsilon(z)(z-z_0)dz$

We know $|\Gamma^{(n)}| = \frac{4s}{2^n}$, so $|\varepsilon(z)(z-z_0)| < \varepsilon \cdot \frac{\sqrt{2}s}{2^n} \Rightarrow$ By ML formula, $\int_{\Gamma^{(n)}} f(z)dz << \varepsilon \frac{4\sqrt{2}s^2}{4^n}$

By our assumption, $\left|\int_{\Gamma^{(n)}} f(z)dz\right| \geq \frac{|I|}{4^n}$, hence $|I| \leq \varepsilon \cdot 4\sqrt{2}s^2$ $\forall \varepsilon>0$, i.e. $I=0$ □

THEOREM (INTEGRAL THEOREM)

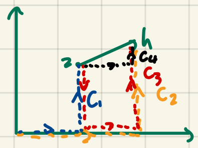
If f is entire, then f is everywhere the derivative of an analytic function. That is, $\exists$ an entire F, s.t. $F'(z)=f(z)$ $\forall z$

Proof

Consider $F(z) = \int_C f(\eta)d\eta$ where $C: 0 \to \mathrm{Re}(z) \to z$

Now, for $h \in \mathbb{C}$, $F(z+h) = \int_{C_1} f(z)dz$

$F(z) = \int_{C_2} f(z)dz$



Then, $F(z+h) - F(z) = \int_{C_1} f(\eta)d\eta + \int_{-C_2} f(\eta)d\eta = \int_{C_3} f(\eta)d\eta = \int_{C_4} f(\eta)d\eta$

Using $F(z+h) = F(z) + \int_{C_4} f(\eta)d\eta$, we get $\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{C_4} f(\eta)d\eta$

As $\frac{1}{h}\int_{C_4} dz = 1$, thus $\left(\frac{1}{h}\int_{C_4} f(\eta)d\eta\right) - f(z) = \frac{1}{h}\int_{C_4}(f(\eta)-f(z))d\eta = \frac{F(z+h)-F(z)}{h} - f(z)$

if h is small, $|f(\zeta)-f(z)| << \varepsilon$

In other words, by ML-formula, $\frac{F(z+h)-F(z)}{h} - f(z) << \cancel{\frac{1}{|h|}}\, \varepsilon\, 2\cancel{|h|} \Rightarrow \lim_{h \to 0} \frac{F(z+h)-F(z)}{h} = f(z)$ □

THEOREM

If f is entire and if C is a smooth closed curve, then $\int_C f(z)dz = 0$