APPLICATIONS OF RESIDUE THEOREM

THEOREM (ROUCHÉ'S THEOREM)

Suppose f, g are analytic inside and on a regular closed curve γ, and suppose $|f(z)| > |g(z)| \ \forall z \in \gamma$, then $n_z(f+g) = n_z(f)$ inside γ

Proof

As $|f(z)| > |g(z)| \geq 0 \ \forall z \in \gamma$, $f+g$ has no zeros on γ (and f has no zeros on γ)

$f+g$ has no poles since f, g: ana.

$f, 1 + \frac{g}{f}$ have no zeros or poles on γ

$$\therefore n_z(f+g) + 0 = \int_\gamma \frac{(f+g)'}{f+g} \, dz = \int_\gamma \frac{[f(1+\frac{g}{f})]'}{f(1+\frac{g}{f})} \, dz = \int_\gamma \frac{f'}{f} dz + \int_\gamma \frac{(1+\frac{g}{f})'}{1+\frac{g}{f}} \, dz = n_z(f) + n_z(1+\frac{g}{f})$$

Notice, $1+\frac{g}{f}$ will never equal zero, since that will need $|\frac{g(z)}{f(z)}| = 1$, but by max mod thm, $\forall z \in$ inside, $|\frac{g}{f}| < 1$

$\therefore n_z(1+\frac{g}{f}) = 0$

$\therefore n_z(f+g) = n_z(f)$ □

EXAMPLE

For $f = 2z^{10} + 4z^2 + 1$, how many zeros of f are inside $D(0,1)$?

Take $g = 4z^2$, $h = 2z^{10} + 1 \Rightarrow \forall |z| = 1, |g(z)| > |h(z)|$

By Rouché's Thm, $n_z(g+h) = n_z(g)$ in $D(0,1) \Rightarrow f$ has two zeros in $D(0,1)$.

THEOREM (GENERALIZED CAUCHY INTEGRAL FORMULA)

Let f be analytic on a s.c. domain D, and γ be a regular closed curve contained in D. Then, for each z inside γ, $k \in \mathbb{Z}_{\geq 0}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\omega)}{(\omega - z)^{k+1}} \, d\omega$$

Proof

Around z, we have $f(\omega) = f(z) + f'(z)(\omega - z) + \ldots + \frac{f^{(k)}(z)}{k!}(\omega - z)^k + \ldots$

Now, we know $\text{Res}(\frac{f(\omega)}{(\omega - z)^{k+1}}, z) = \frac{f^{(k)}(z)}{k!}$

$$\therefore \frac{1}{2\pi i} \int_\gamma \frac{f(\omega)}{(\omega - z)^{k+1}} \, d\omega = \frac{f^{(k)}(z)}{k!} \quad \square$$

RECALL

For ana $\{f_n\}_n$, if $f_n \to f$ uniformly on compacta of D, it means "$\forall V \subseteq D$: cpt, $f_n \to f$ uniformly on V"

Moreover, by Morera's Thm, we had f_n: ana $\Rightarrow f$: ana

THEOREM

Let $\{f_n\}$ be a sequence of analytic functions that converges uniformly on compacta of region D

Then, f is analytic and $f_n' \to f'$ uniformly on compacta of D

Proof

$\forall z_0 \in D$, choose $r_0 > 1$, s.t. $\overline{D(z_0, r_0)} \subseteq D$, $\gamma := C_{r_0}(z_0)$

Then, by Generalized Cauchy integral formula,

$$f_n'(z) - f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(\omega) - f(\omega)}{(\omega - z)^2} \, d\omega$$

Recall, $f_n \to f$ uniformly on $\overline{D(z_0, r_0)}$. By def, given $\varepsilon > 0$, $\exists N$, s.t. $|f_n(z) - f(z)| < \frac{\varepsilon r_0^2}{4} \ \forall z \in D(z_0, r_0)$

In particular, $\forall z \in D(z_0, \frac{1}{2}r_0), |f_n(z) - f(z)| < \frac{\varepsilon r_0^2}{4}$

By ML-formula, $f_n'(z) - f(z) << \frac{1}{2\pi} \frac{\frac{\varepsilon r_0^2}{4}}{(\frac{r_0}{2})^2} \cdot 2\pi r_0 = \varepsilon$

Now, $\forall$ compact $V \subseteq D$, $\forall z \in V$, $\exists r < 1$, s.t. $f_n' \to f'$ unif on $\overline{D(z, \frac{r}{2})}$

By def, $V \subseteq \bigcup_{z \in V} (D, \frac{r_z}{2}) \Rightarrow \exists z_i, r_i$, s.t. $V =: \bigcup_{i=1}^{Q} D(z_i, \frac{r_i}{2})$

$\therefore f_n' \to f$ uniformly on V. $\square$

THEOREM (HURWITZ'S THEOREM)

Let $\{f_n\}$ be a sequence of non-vanishing analytic functions in a region D.

Suppose that $f_n \to f$ uniformly on compacta of D. Then, either $f \equiv 0$ on D or $f(z) \neq 0 \ \forall z \in D$

Proof

Suppose $\exists z \in D$, s.t. $f(z) = 0$

Claim: $f \equiv 0$ on D

Proof

As f_n: analytic and has no zero on D, choose r, s.t. $\overline{D(z,r)} \subseteq D$ and f has no zero on $C_r(z)$ (otherwise, f has an acc point of zero $+ D$: region $\Rightarrow f \equiv 0$ on D)

Now, $\frac{1}{2\pi i} \int_{C_r(z)} \frac{f'}{f} dz = n_z(f)$ in $D(z, r) \geq 1$

We know $f_n' \to f'$ unif on $D(z,r)$, so $\frac{f_n'}{f_n} \to \frac{f'}{f}$ unif on $C_r(z)$, since f_n, f have no zeros on $C_r(z)$

However, $\int_{C_r(z)} \frac{f'}{f} dz = \lim_{n \to \infty} \int_{C_r(z)} \frac{f_n'}{f_n} dz = 0$ (since f_n has no zeros in D) —✗—

COROLLARY

Replace "0" with "a" in Hurwitz's Thm

Proof

Consider $g_n(z) = f_n(z) - a$. $\square$

EXAMPLE

We know $\sin z = \sum_{i=0}^{\infty} (-1)^i \frac{z^{2i+1}}{(2i+1)!}$, take $f_n(z) = \sum_{i=0}^{n} (-1)^i \frac{z^{2i+1}}{(2i+1)!}$

$f_n \to \sin z$ uniformly on any compact set $V \subseteq \mathbb{C}$.

$\therefore$ By Hurwitz's Thm, $f_n(z)$ has a zero in $D(0, 2\pi)$ $\forall n \geq N$ for some $N > 0$. !!

THEOREM

Let $\{f_n\}$ be a seq of ana functions, and $f_n \to f$ unif on cpta of a region D. If f_n is 1-1 $\forall n$, then either f: const or 1-1

Proof

Suppose $\exists z_1, z_2 \in D$, $z_1 \neq z_2$ s.t. $f(z_1) = f(z_2) = a$

$\exists r$, s.t. $D(z_1, r) \subseteq D$ and $D(z_1, r) \cap D(z_2, r) = \emptyset$

On $D(z_1, r)$, $f_n \to f$ on cpta and $f(z_1) = a$. Hence, if $f_n(z) \neq a$ $\forall z \in D(z_1, r)$, then $f \equiv a$ on $D(z_1, r)$.

If $f \equiv a$ on $D(z_1, r)$, as D is a region, $f \equiv a$ on D.

If $f \not\equiv a$ on D, then by coro, $\exists N$, s.t. $f_n(z) = a$ has a zero on $D(z_1, r)$ for $n \geq N$.

f_n are 1-1 $\Rightarrow \forall n \geq N$, $f_n - a$ has no zeros on $D(z_2, r)$
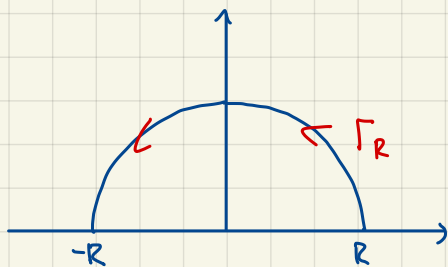
$\therefore f \equiv a$ on D —✗—

RESIDUE THEOREM AND EVALUATION OF INTEGRALS AND SUMS

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TYPE I INTEGRALS

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} \, dx$$

If P, Q: poly and $\deg P \leq \deg Q + 2$, $\gcd(P, Q) = 1$, and $Q(x)$ has no real roots

Define $\gamma: \Gamma_R + (-R \to R)$

Then, $\displaystyle \int_{\gamma} \frac{P(x)}{Q(x)} \, dx = \int_{\Gamma_R} \frac{P(x)}{Q(x)} \, dx + \boxed{\int_{-R}^{R} \frac{P(x)}{Q(x)} \, dx} = 2\pi i \left(\sum_{\substack{\omega_i \text{ root of } Q(x) \\ \text{in upper-half plane}}} \text{Res}\left(\frac{P(x)}{Q(x)}, \omega_i \right) \right)$

↑ Residue Thm

When R is big enough,

Claim: $\displaystyle \lim_{R \to \infty} \left| \int_{\Gamma_R} \frac{P(z)}{Q(z)} \, dz \right| = 0$

Proof

Notice, for $R \gg 0$, $\displaystyle \left| \frac{P(z)}{Q(z)} \right| \leq \frac{A}{|z|^2} \leq \frac{A}{R^2}$ $\forall |z| \geq R$

By ML-Formula,

$$\int_{\Gamma_R} \frac{P(z)}{Q(z)} \, dz \ll \frac{A}{R^2} \cdot \pi R = \frac{\pi A}{R} \xrightarrow{R \to \infty} 0$$

$\therefore \displaystyle \int_{-R}^{R} \frac{P(x)}{Q(x)} \, dx = 2\pi i \left(\sum_{\substack{\omega_i \text{ root of } Q(x) \\ \text{in upper-half plane}}} \text{Res}\left(\frac{P(x)}{Q(x)}, \omega_i \right) \right)$

EXAMPLE

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} \, dx = 2\pi i \cdot \left(\text{Res}\left(\frac{1}{x^4+1}, e^{\frac{\pi}{4}i} \right) + \text{Res}\left(\frac{1}{x^4+1}, e^{\frac{3\pi}{4}i} \right) \right) = \frac{\sqrt{2}}{2}\pi$$