

## NOTICE

In  $\mathbb{C}$ ,  $\log a + \log b \neq \log(ab)$  in general

## ZETA FUNCTION

## AIM

Can we extend  $\zeta$  to a meromorphic function on  $\mathbb{C}$ ?

## LINK TO GAMMA FUNCTION

Recall  $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

Notice,  $\int_0^\infty e^{-nt} t^{z-1} dt \stackrel{\text{for } \zeta(z)}{=} \int_0^\infty e^{-s(\frac{s}{n})} (\frac{s}{n})^{z-1} \frac{ds}{n} = \frac{1}{n^z} \int_0^\infty e^{-s} s^{z-1} ds = \frac{1}{n^z} \Gamma(z)$

$\therefore$  For  $\operatorname{Re}(z) > 1$ ,  $\zeta(z) = \frac{1}{\Gamma(z)} \sum_{n=1}^\infty \int_0^\infty e^{-nt} t^{z-1} dt$

$\Gamma(z)$   $\left. \begin{array}{l} \textcircled{1} \text{ has no zeros} \\ \textcircled{2} \text{ has only poles as singularity} \end{array} \right\} \Rightarrow \frac{1}{\Gamma(z)} : \text{entire}$

Thus,  $\operatorname{Re}(z) > 1 \Rightarrow \sum_{n=1}^\infty \int_0^\infty e^{-nt} t^{z-1} dt = \int_0^\infty \left( \sum_{n=1}^\infty (e^{-nt}) \right) t^{z-1} dt = \int_0^1 (\sum_{n=1}^\infty e^{-nt}) t^{z-1} dt + \int_1^\infty (\sum_{n=1}^\infty e^{-nt}) t^{z-1} dt$   
(depends on z) holomorphic  $\forall z \in \mathbb{C}$

Notice,  $\int_0^1 (\sum_{n=1}^\infty e^{-nt}) t^{z-1} dt = \int_0^1 \frac{t^{z-1}}{e^t - 1} dt$   
 $e^t - 1 = t + \frac{t^2}{2} + \dots$

We know  $\frac{1}{e^t - 1}$  has a simple pole at  $t=0$ ,  $\operatorname{Res}(\frac{1}{e^t - 1}, 0) = 1$

Consider the Laurent Expansion around  $t=0$ ,

$\frac{1}{e^t - 1} = \frac{1}{t} + \sum_{i=0}^\infty A_i t^i$ . As  $\frac{1}{e^t - 1}$  defined  $\forall t \neq 0$ , thus  $\sum_{i=0}^\infty A_i t^i$  conv  $\forall t \in \mathbb{C}$ .

$\therefore$  For  $\operatorname{Re}(z) > 1$ ,  $\int_0^1 \frac{t^{z-1}}{e^t - 1} dt = \int_0^1 t^{z-2} + \sum_{i=0}^\infty A_i t^{i-2} dt = \frac{1}{z-1} + \sum_{i=0}^\infty \frac{A_i}{z+i}$   
conv  $\forall z \in \mathbb{C} \setminus \{z \leq 1\}$  because  $\textcircled{1} \sum_{i=0}^\infty A_i t^i$  conv  $\forall t \in \mathbb{C}$   
 $\textcircled{2} \frac{1}{|z+i|} < 1$  for  $i$  large enough

Thus,  $\operatorname{Re}(z) > 1 \Rightarrow \zeta(z) = \frac{1}{\Gamma(z)} \left( \frac{1}{z-1} + \sum_{i=0}^\infty \frac{A_i}{z+i} + g(z) \right)$   
def on  $\mathbb{C} \setminus \{z \leq 1\}$   
 $(\Delta)$  holds on  $\mathbb{C}$   
entire w/ zero of order 1 at  $z=0$

## THEOREM

$\zeta(z)$  can be analytically continued to  $\mathbb{C} \setminus \{z=1\}$

Proof

$\frac{1}{\Gamma(z)}$  has zero of order 1 at  $z=0$  and  $(\Delta)$  has simple poles at  $z \leq 1$

## SET-UP

Recall,  $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z} = \prod_{p: \text{prime}} (1 - p^{-z})^{-1}$ ,  $\operatorname{Re}(z) > 1$

$\therefore \zeta(z)$  has no zeros for  $\operatorname{Re}(z) > 1$

# THEOREM

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$\zeta(z)$  has no zeros for  $\text{Re}(z) \geq 1$  (i.e. by set-up,  $\zeta(z)$  has no zeros at  $x=1$ )

Proof Sketch

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{p:\text{prime}} \frac{[(1-p^{-z})^{-1}]'}{(1-p^{-z})^{-1}} = \sum_p \frac{(\log p)}{p^z-1} = \phi(z) + \underbrace{\sum_p \frac{\log p}{p^2(p^2-1)}}_{\text{holomorphic for } \text{Re}(z) > \frac{1}{2}}, \text{ where } \phi(z) = \sum_{z:\text{prime}} \frac{\log p}{p^z}$$

Use prime not log

Suppose  $\exists$  zero for  $\zeta$  at  $1+ia_0$  with  $a_0 \neq 0$ , of order  $\mu > 0$

We know  $z \in \mathbb{R}_{>1} \Rightarrow \zeta(z) \in \mathbb{R}$

$\therefore$  By Reflection Principle,  $\zeta(\bar{z}) = \overline{\zeta(z)}$

$\therefore 1-ia_0$  is a zero of order  $\mu$  of  $\zeta$

Say  $\zeta$  has a zero at  $1 \pm 2ia_0$  of order  $\nu$  ( $\nu=0$  if  $1 \pm 2ia_0$  is a "zero")

Thus,  $\phi(z)$  has poles of the same order as  $-\frac{\zeta'(z)}{\zeta(z)}$  = zeros and poles of  $\zeta(z)$

$$\begin{aligned} \downarrow \\ \lim_{\varepsilon \rightarrow 0} \varepsilon \phi(1+\varepsilon) &= 1 \quad (\zeta \text{ has simple pole at } z=1 \text{ with residue } 1) \\ \lim_{\varepsilon \rightarrow 0} \varepsilon \phi(1+ia_0+\varepsilon) &= \mu \\ \lim_{\varepsilon \rightarrow 0} \varepsilon \phi(1 \pm 2ia_0 + \varepsilon) &= \nu \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{-2 \leq k \leq 2} \binom{4}{2+k} \phi(1+\varepsilon + i k a_0) &= \sum_p \frac{\log p}{p^{1+\varepsilon}} \left( \sum_{-2 \leq k \leq 2} \binom{4}{2+k} \left( \frac{1}{p^{i \frac{k}{2}}} \right)^{1+\varepsilon} \left( \frac{1}{p^{-i \frac{k}{2}}} \right)^{-1+\varepsilon} \right) \\ &= \sum_p \frac{\log p}{p^{1+\varepsilon}} \underbrace{\left( p^{i \frac{\varepsilon}{2}} + p^{-i \frac{\varepsilon}{2}} \right)^4}_{\geq 0} \end{aligned}$$

$$\text{As } \varepsilon \rightarrow 0, -2\nu - 8\mu + 6 \geq 0 \Rightarrow \mu = 0 \quad \text{---}$$