11-27-25 (WEEK 13)

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RIEMANN-MAPPING THEOREM PROOF CONTINUED

CLAIM C: $\exists f\in\mathcal{F}_{z_0}$, s.t. f: onto to U

To do this, we find $f\in\mathcal{F}_{z_0}$, s.t. $f'(z_0)$: max

Idea: $\boxed{\text{Suppose}}$ $f\in\mathcal{F}_{z_0}$, s.t. $f'(z)$ is maximum

(i) $f(z)=0$

Proof

If $f(z_0)=\alpha\neq 0\in U$, then $B_\alpha\circ f(z_0)=0$ (shifted) up to $e^{i\theta}$ and $B_\alpha\circ f\in\mathcal{F}_{z_0}$

$\Rightarrow g'(z_0)=\dfrac{f'(z_0)}{1-|\alpha|^2} >> f'(z_0)$ —✕—

shift center to equal 0 then derivative results in a larger one ✱

(ii) f is onto

Proof

If not, $\alpha\neq 0$, $\alpha\in U\setminus \mathrm{Im}(f)$

Replace f by some $e^{i\theta}f$ ($|f'(z_0)|$ still max), then we can assume $\alpha\in\mathbb{R}_{<0}\Rightarrow$ Set $t\in\mathbb{R}_{>0}$, $\alpha=-t^2$

shift so $f(z_0)\in\mathbb{R}_{<0}$

$\therefore f_1(z)=\frac{f(z)-\alpha}{1-\alpha f(z)}=\frac{f(z)+t^2}{1+t^2 f(z)}$

$0\notin\mathrm{Im}(f_1)$, $f_1: R\to U$, 1-1, analytic

$\therefore \frac{1}{f_1(z)}$, $f_1'(z)$ · ana on R

must check $\frac{f_1'}{f_1}$ ana

As R: s.c., by closed curve thm, $\forall$ closed curve $C\subseteq R$, $\int_C \frac{f_1'(z)}{f_1(z)}dz=0$

$\therefore$ We can define $\log f_1(z)$ on R

want to define $\log z$ $\int_C \frac{f_1'(z)}{f_1(z)}dz$

In particular, we can choose a branch of $f_1(z)$ on R, s.t. $\sqrt{f_1(z_0)}=t$

So we can have $\sqrt{f_1(z_0)}=t$

Let $f_2(z):=\sqrt{f_1(z)}$, $f_3(z):=\frac{f_2(z)-t}{1-tf_2(z)}$, f_3: 1-1.

Notice, $\left.\begin{aligned} f_1'(z_0)&=f'(z_0)(1-t^4)\\ f_2'(z_0)&=\frac{f_1'(z_0)}{2t}\\ f_3'(z_0)&=\frac{f_2'(z_0)}{1-t^2}\end{aligned}\right\}\Rightarrow f_3'(z_0)=\frac{f'(z_0)(1+t^2)}{2t}>>f'(z_0)$

If we set $g(z):=e^{i\theta}f_3(z)$, then $g\in\mathcal{F}_{z_0}$, $g'(z_0)>f'(z_0)$ —✕—

$\therefore f$ must be onto ✓

☆ Conclusion: $|f'(z_0)|$ max $\Rightarrow f(z_0)=0$ and f: onto □

CLAIM B: $\exists \varphi\in\mathcal{F}_{z_0}$, s.t. $\varphi'(z_0)=M$, where $M=\sup_{f\in\mathcal{F}_{z_0}} f'(z_0)<\infty$

① $M<\infty$

Choose r, s.t. $\overline{D(z_0,r)}\subseteq R$

$\therefore$ By Cauchy Integral Formula, $f'(z_0)=\frac{1}{2\pi i}\int_{C_r(z_0)}\frac{f(z)}{(z-z_0)^2}dz$

$\therefore |f'(z_0)|\le\frac{1}{2\pi}\cdot\frac{1}{r^2}\cdot 2\pi r=\frac{1}{r}$

② $\{\zeta_i\}_{i=1}^\infty$ is a countable dense subset in R, where we choose $\{f_n\}_{n=1}^\infty\subseteq\mathcal{F}_{z_0}$, s.t. $\lim_{n\to\infty}f_n'(z_0)=M$

Find a subseq $\{f_{1n}\}$ s.t. $\{f_{1n}(\zeta_1)\}$ conv, find $\{f_{2n}\}$ subseq of $\{f_{1n}\}$ s.t. $\{f_{2n}(\zeta_2)\}$ conv $\Rightarrow \{f_{2n}(\zeta_1)\}$ and $\{f_{2n}(\zeta_2)\}$ conv

Continuing this process, $\{f_{in}\}_{n=1}$ conv at $\zeta_1,\dots,\zeta_i\Rightarrow\{f_{nn}\}$ conv at all ζ_i

Find all f_{in} via recursion :)

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Claim: $\{f_{nn}\} := \{\varphi_n\}$ conv to an analytic function

<u>Proof</u>

(i) $\varphi_i \in \mathcal{F}_{z_0} \Rightarrow |\varphi_i(z)| < 1 \quad \forall z \in R \Rightarrow \{\varphi_i\}$: uniformly bounded (indep of i, z) ✓

(ii) $\{\varphi_n\}$ conv unif on $\overline{D(w_0, r)} \subseteq R$

$\mathbb{C} \setminus R$: closed, so $2d := d(\tilde{R}, \overline{D(w_0, r)}) = \inf \overline{d(z_1, z_2)}$, $z_1 \in \tilde{R}$, $z_2 \in \overline{D(w_0, r)}$

$\forall z \in \overline{D(w_0, r)} \Rightarrow \overline{D(z, d)} \subseteq R$, so $|\varphi_n(z)| = \frac{1}{2\pi} \left| \int_{C_d(z)} \frac{\varphi_n(w)}{(w-z)^2} dw \right| \leq \frac{1}{d}$

$\Rightarrow \forall z_1, z_2 \in \overline{D(w_0, r)}$, $|\varphi_n(z_1) - \varphi_n(z_2)| = \left| \int_{z_1}^{z_2} \varphi_n'(z) dz \right| = \frac{|z_2 - z_1|}{d}$

Given ε, $\forall n$, $\forall |z_1 - z_2| < \varepsilon d$, we have $|\varphi_n(z_1) - \varphi_n(z_2)| < \varepsilon$

$\therefore \varphi$: uniformly equicontinuous (indep of n, z_1, z_2) ✓