Complex Analysis: Midterm Definitions

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Remark

I only will include definitions that are useful for me, i.e. things that I still find useful after learning this course for 7 weeks. Otherwise, there will be too many definitions.

Power Series

Analytic Polynomial

Analytic Polynomial

If $P(x,y) = \alpha_0 + \alpha_1(x+iy) + \cdots + \alpha_N(x+iy)^N = \sum_{k=0}^{N} \alpha_k z^k$ for some $\alpha_k \in \mathbb{C}$, then it is an **analytic polynomial**.

Cauchy-Riemann Equations

For $P(x,y) = u(x,y) + iv(x,y)$, the **Cauchy-Riemann equations** are: $u_x = v_y$ and $u_y = -v_x$. Another way to view it is $P_y = iP_x$.

Radius of Convergence

Cauchy Product

Given $P_1(z) = \sum a_k z^k$, $R = R_1$; $P_2(z) = \sum b_k z^k$, $R = R_2$. Then, $P_1 P_2 = \sum c_k z^k$, where $c_k = \sum_{p=0}^{k} a_p b_{k-p}$, and $\boxed{R \geq \min(R_1, R_2)}$

Analytic Functions

Analytic Functions

Analytic

f is **analytic** at z if f is **differentiable in a neighborhood** of z. Similarly, f is **analytic** on a set S if f is **diff at all points** of some open set containing S.

Line Integrals

Smooth Curves

Smooth

The curve $z(t) = x(t) + iy(t)$ is said to be **smooth** if $z'(t) \neq 0$ except at *finitely many points*.

Line Integral

Say C is a **smooth** curve in $\mathbb{C}$, where $z(t) = x(t) + iy(t)$. Then, $\int_C f(z)dz = \int_a^b f(z(t))dz = \int_a^b f(z(t))z'(t)dt$

Smoothly Equivalent

Let C_1 and C_1 be **smooth curves** in $\mathbb{C}$, where $C_1 : z(t), a \leq t \leq b$ and $C_2 : w(t), c \leq t \leq d$. C_1 and C_2 are said to be **smoothly equivalent** if $\exists$ 1-1 $\mathcal{C}^1$ mapping $\lambda : [c, d] \to [a, b]$, s.t. $w(t) = z(\lambda(t))$.

As this is an equivalence relation, we denote **smoothly equivalent** with $C_1 \sim_{sim} C_2$

Rectangle Theorem

Simple Closed Curve

- A curve is **closed** if its internal and terminal points coincide
- C is a **simple closed curve** with $t \in [a, b]$ if $z(t_1) = z(t_2)$ with $t_1 < t_2$ implies $t_1 = a$ and $t_2 = b$
- The **boundary of a rectangle** is the simple closed curve in the **counterclockwise direction**

Liouville's Theorem

Convex Set

We say S is a **convex set** in $\mathbb{C}$ if $\forall x, y \in S$, $tx + (1-t)y \in S$ $\forall t \in [0, 1]$.

Note, this implies $x_1, \dots, x_N \in S \Leftrightarrow \sum_{i=1}^N a_i x_i \in S$ $\forall \sum_{i=1}^N a_i = 1$ and $a_i \geq 0$

Saddle Points

C-analytic

A function is **C-analytic** on a region D if it is analytic on D and **continuous** on $\boxed{\bar{D}}$

Saddle Point

z_0 is a **saddle point** of an analytic function f on a region D if z_0 is a saddle point on the real valued function $g(x, y) = |f(x, y)|$. In other words, g **is differentiable** and $g_x(z_0) = g_y(z_0) = 0$ but z_0 is **NOT a local extremum**.

Schwarz Lemma

$B_\alpha(z)$

Define $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$, for $|\alpha| < 1$. Then,

1. $B_\alpha(\alpha) = 0$
2. $B_\alpha(z)$ is **ana on D**, and it is **conti on D̄**.
3. $|B_\alpha(z)|^2|_{z=1} = 1$, so by **max modulus thm,** $|B_\alpha(z)| \leq 1$ on D

Morera's Theorem

Converges Uniformly on Compacta

Let $\{f_n\}$ and f be defined on an open set D. We say that f_n **converges uniformly on compacta** if $f_n \to f$ uniformly on **every compact subset** $K \subseteq D$.

Regular Analytic

A curve $\gamma : [a, b] \to \mathbb{C}$ is called a **regular analytic** arc if γ is an **analytic map, 1–1** on $[a, b]$ with $\gamma' \neq 0$

Simply Connected Domain

Simply Connected (Book)

We say S is **simply connected** if it is **path connected** and for any conti maps $f_1 : [0, 1] \to S$ with $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$, $\exists$ conti $F : [0, 1] \times [0, 1] \to S$, s.t. $F(t, 0) = f_0(t),\ F(t, 1) = f_1(t)$.

Simply Connected (Lecture)

For a region $D \subseteq \mathbb{C}$, D is **simply connected** if $(\mathbb{C} \cup \{\infty\}) \setminus D$ is **path connected**

Holomorphic Simply Connected (HSC)

D is hsc if $\forall f$: ana on D, $\int_\Gamma f dz = 0$ for all simple closed curve $\Gamma \subseteq D$

Singularity

Deleted Neighborhood

A **deleted neighborhood** of z is an open set of $\{z \mid 0 < |z - z_0| < \delta\}$

Definition of Singularities

Say z_0 is a **singularity** of f, we can classify it as follows:

1. If $\exists g$ that is **ana** at z_0 and $f(z) = g(z)$ in some **deleted nbd** of z_0, we say f has a **removable singularity**

2. If for $z \neq z_0$, f can be written as $f(z) = \frac{A(z)}{B(z)}$, where A and B are analytic at z_0, $A(z_0) \neq 0, B(z_0) = 0$, we say f has a **pole** at z_0. In particular, if B has a zero of order k at z_0, then we say z_0 is a pole of f of order k

3. If f has neither a removable singularity nor a pole at z_0, then we call z_0 an **essential singularity** of f