11-6-25 (WEEK 10)

ok, elephant in the room: I think I'll still wanna cry every CompAna lecture so I'll stop venting about it in my comments but yeah feel free to not use my notes anymore due to the lower quality. Don't ask why I wanna cry, and before you wonder, no, I didn't fail any exam; yes, I still love CompAna.

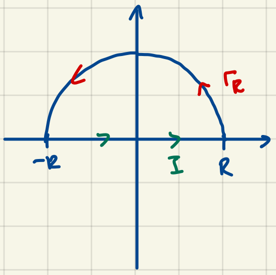
Shun/翔海 (@shun4midx)

TYPE II INTEGRALS

no need to +2

$$\boxed{f(x)=\frac{P(x)}{Q(x)},\ \deg Q \geq \deg P + 1,\ \int_{-\infty}^{\infty} f(x)\cos x\,dx \text{ or } \int_{-\infty}^{\infty} f(x)\sin x\,dx}$$

If we want to apply the route in (I), we must check "$\lim_{R\to\infty}\int_{\Gamma_R} f(x)\cos x\,dx$" $=0$ (or the equivalent for $\sin x$)



The key is in $e^{iz}=\cos z + i\sin z$ (However, note $|e^{iz}| = e^{R\cos\theta}$ may not vanish!)

$N(m) := \int_{\Gamma_M + I} f(z)e^{iz}dz \Rightarrow$ We have $\int_{-\infty}^{\infty} f(x)\sin x\,dx = \lim_{M\to\infty} \operatorname{Im}(N(m))$, $\int_{-\infty}^{\infty} f(x)\cos x\,dx = \lim_{M\to\infty}\operatorname{Re}(N(m))$

☆ Caution: $f(z)e^{iz}$ may have singularities on the real axis, we may need to consider $e^{iz}-c$, $c\in\mathbb{C}^\times$

EXAMPLE

$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = ?$

We CANNOT consider $\int_I \frac{e^{iz}}{z}dz \Rightarrow$ consider $\int_I \frac{e^{iz}-1}{z}dz$ ($z=0$: removable singularity)

Then, $\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \operatorname{Im}\left(\lim_{M\to\infty}\int_{\Gamma_M+I}\frac{e^{iz}-1}{z}dz\right)$

LEMMA 1 ☆ Key lemma for cool complex analysis integral types

Let f be an analytic function defined on a sector of the upper-half plane

Suppose $|f(z)| \longrightarrow 0$ as $z\to\infty$

Then, we have

$\left|\int_{\Gamma_R} f(z)e^{iz}dz\right| \longrightarrow 0$ as $R\to\infty$, where $\Gamma_R: Re^{i\theta}$, $0\leq\theta_1\leq\theta\leq\theta_2\leq\pi$

Proof

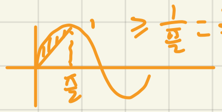
Define $M := \max_{z\in\Gamma_R}|f(z)|$, notice $M\to 0$ as $R\to\infty$

Then, $\left|\int_{\Gamma_R} f(z)e^{iz}dz\right| \leq M\int_{\Gamma_R}|e^{iz}||dz| = M\int_{\theta_1}^{\theta_2} e^{-R\sin\theta}R\,d\theta \leq M\int_0^{\pi} e^{-R\sin\theta}R\,d\theta = 2\pi\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}R\,d\theta$

$(\sin x - x)' \leq 0$

Notice, $\frac{2}{\pi}\leq\frac{\sin\theta}{\theta}\leq 1$

$\frac{\sin\theta}{\theta}$ concave down:



$\geq \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

Thus, $\left|\int_{\Gamma_R} f(z)e^{iz}dz\right| \leq 2M\int_0^{\frac{\pi}{2}} Re^{-R\theta\cdot\frac{2}{\pi}}d\theta \leq M\pi$

$\therefore$ As $R\to\infty$, $\left|\int_{\Gamma_R} f(z)e^{iz}dz\right| \longrightarrow 0$ □

APPLICATION TO CASE (II) INTEGRALS

For $f(x)=\frac{P(x)}{Q(x)}$, for big enough R, $f(x)e^{iz}$ has no poles on Γ_R, hence: $\lim_{R\to\infty}\int_{\Gamma_R+I} f(z)e^{iz}dz = \lim_{R\to\infty}\int_{-R}^{R} f(z)dz$

Shun/翔海(@shun4midx)

CALCULATED EXAMPLE

find it using Taylor's expansion of e^{iz}.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \mathrm{Im}\left(\underbrace{\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iz}-1}{z}\,dz}\right) = \mathrm{Im}\left(2\pi i\, \mathrm{Res}\left(\frac{e^{iz}-1}{z}, 0\right)\right) = \pi$$

Do not separate this into $\int \frac{e^{iz}}{z}$ and $\int \frac{1}{z}$, they both are divergent.