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REMARK

We only have open mapping thm because extremum is not in interior pt.

SCHWARTZ'S LEMMA

THEOREM (SCHWARTZ'S LEMMA)

Suppose that f is analytic in an open unit disc D with $|f| \leq 1$ (f: unit circles $\to$) and $f(0)=0$

Then, (i) $|f(z)| \leq |z|$

(ii) $|f'(0)| \leq 1$

with equality in either of the above iff $f(z) = e^{i\theta} z$

Proof

Define $g(z) := \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(z), & z = 0 \end{cases}$

$g(z)$ is ana on D since $f(z)$ is ana on D.

Consider $z \in C_r(0)$, $0 < r < 1$.

Then, $|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$

By max modulus thm, $\forall z \in \overline{D(0,r)}$, $|g(z)| \leq \frac{1}{r}$

As $r \to 1$, then $|g(z)| \leq 1 \ \forall z \in D$

By def of $g(z)$, $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$ has either equality hold, when g is const and $|g| = 1$ on D. $\therefore g = e^{i\theta}$ $\square$

EXAMPLE (Removing $f(0)=0$ constraint)

Define $B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha| < 1 - D$

Then, (1) $B_\alpha(\alpha) = 0$

(2) $B_\alpha(z)$ is ana on D, so $(B_\alpha(z))^2$ is ana on D. It is also conti on $\bar{D}$.

(3) $|B_\alpha(z)|^2\big|_{|z|=1} = 1$, so by max modulus thm, $|B_\alpha(z)| \leq 1$ on D.

$\therefore$ We can use B_α for variations of Schwartz's lemma

EXAMPLE

Say f: ana on D, $|f(z)| \leq 1 \ \forall z \in D$ and $f(\frac{1}{2}) = 0$. Estimate $|f(\frac{3}{4})|$.

Consider $B_{\frac{1}{2}}(z) = \frac{z - \frac{1}{2}}{1 - \frac{z}{2}}$.

We define $g(z) = \begin{cases} \frac{f(z)}{z-\frac{1}{2}}(1 - \frac{1}{2}z), & z \neq \frac{1}{2} \\ \frac{3}{4}f'(\frac{1}{2}), & z = \frac{1}{2} \end{cases}$ $(B_\alpha(z))^{-1}$

Notice, $|B_{\frac{1}{2}}(z)| \leq 1$ on D, $|B_{\frac{1}{2}}(z)| = 1$ on $C_1(0)$, and $B_{\frac{1}{2}}(z)$ conti on D.

$\therefore z \to 1$, $|B_{\frac{1}{2}}(z)| \longrightarrow 1$

$|g(z)| \leq 1/|B_{\frac{1}{2}}(z)| \Rightarrow |g(z)| \leq 1$ on D.

$\Rightarrow |f(z)| \leq |B_{\frac{1}{2}}(z)|$, $\frac{3}{4}|f'(\frac{1}{2})| \leq 1$

So, $|f(\frac{3}{4})| \leq |B_{\frac{1}{2}}(\frac{3}{4})| = \frac{2}{5}$

EXAMPLE

Say f is ana on D, $|f(z)|\leq 1$ on D. We claim: $|f'(\frac{1}{3})|$ is max when $f(\frac{1}{3})=0$

Proof

Assume that $f(\frac{1}{3})\neq 0$.

$g(z) := \dfrac{f(z)-f(\frac{1}{3})}{1-\overline{f(\frac{1}{3})}\,f(z)} \Rightarrow g(z)=B_{f(\frac{1}{3})}(f(z))$. Note, g is bounded by 1, $g(\frac{1}{3})=0$.

$\therefore g'(z)\big|_{z=\frac{1}{3}} = \dfrac{f'(\frac{1}{3})}{1-|f(\frac{1}{3})|^2} \Rightarrow |g'(\frac{1}{3})| > |f'(\frac{1}{3})|$

$\therefore |f'(\frac{1}{3})|$ is max for $B_{\frac{1}{3}}(z)$.

PROPOSITION

Say f is entire. If $|f(z)| < \frac{1}{|\mathrm{Im}\, z|}$ $\forall z$, then $f\equiv 0$

Proof

Define $g(z)=(z^2-R^2)f(z)$, for some $R\in\mathbb{R}_{>0}$

When $z\in C_R(0)$, $|z-R||z+R|\leq 2R|\mathrm{Im}(z)|$

$\therefore |g(z)| \leq \frac{2R}{|\mathrm{Im}(z)|^2} \leq 2R$ when $z\in C_R(0)$

By max modulus thm, $|g(z)|\leq 2R$ $\forall z\in D(0,R)$

$\Rightarrow |f(z)| \leq \frac{2R}{|z^2-R^2|}$ $\forall z\in D(0,R)$

As $R\to\infty$, $|f(z)|\to 0$

$\therefore f(z)=0$

MORERA'S THEOREM

THEOREM (MORERA'S THEOREM: CONVERSE OF RECTANGLE THEOREM)

Let f be continuous on an open set $D\subseteq\mathbb{C}$, and Γ be the boundary of a closed rectangle $R\subseteq D$.
If $\int_\Gamma f\,dz=0$ $\forall\Gamma$ in $R\subseteq D$, then f is analytic in D.

Proof

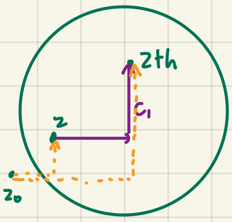
Say $z_0\in D$, D: open.
Then, $\exists\,\varepsilon>0$, s.t. $D(z_0,\varepsilon)\subseteq D$.

Define $F(z) := \int_C f(z)\,dz$ $\forall z\in D(z_0,\varepsilon)$, where $C: z_0 \to z_0+\mathrm{Re}(z-z_0)\to z$

For $z\in D(z_0,\varepsilon)$ and k small enough s.t. $z+h\in D(z_0,\varepsilon)$
Then,

f: cont.

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = \lim_{h\to 0}\int_{C_1} f(w)\,dw = f(z) \quad \square$$



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EXAMPLE

Using $f(z) = \int_0^\infty \frac{e^{zt}}{t+1}\,dt$,

Claim: f is analytic $\forall z \in \{w \in \mathbb{C} \mid Re(w) < 0\}$

Proof

We know for $z = x+iy$, $x<0$, $|e^{zt}| = e^{xt}$

Here, $\int_\Gamma \int_0^\infty \frac{|e^{zt}|}{t+1}\,dt\,dz < \int_\Gamma -\frac{1}{x}\,dz < \infty$ (ok for Fubini)

By Fubini's Thm, $\int_\Gamma \int_0^\infty \frac{e^{zt}}{t+1}\,dt\,dz = \int_0^\infty \underbrace{\boxed{\int_\Gamma \frac{e^{zt}}{t+1}\,dz}}_{}\,dt = 0$ $=0$ ($\because$ By rectangle thm since $\frac{e^{zt}}{t+1}$: ana)

$\therefore$ By Morera's Thm, $f(z)$ is analytic on $\{w \in \mathbb{C} \mid Re(w) < 0\}$. $\square$

DEFINITION

Let $\{f_n\}$ and f be defined on an open set D. We say that f_n **converges uniformly on compacta** if $f_n \to f$ uniformly on every compact subset $K \subseteq D$.

THEOREM

Let D be an open set in $\mathbb{C}$ and $\{f_n\}$ be a sequence of ana functions s.t. $f_n \to f$ unif on cpta. Then, f is also ana in D.

Proof

$\because$ f_n is conti, $\forall K \subseteq D$: cpt set we have $f_n \to f$ unif on K

$\therefore$ f is conti on K $\forall K$, i.e. f is conti on D

We hope "$\int_\Gamma f\,dz = 0$", for Γ : boundary of a closed rectangle $R \subseteq D$

Hence, $\int_\Gamma f\,dz = \int_\Gamma \lim_{n\to\infty} f_n\,dz$

$\quad \parallel$ (f_n conti, $f_n \to f$ unif on R)

$\lim_{n\to\infty} \left(\int_\Gamma f_n\,dz\right)$

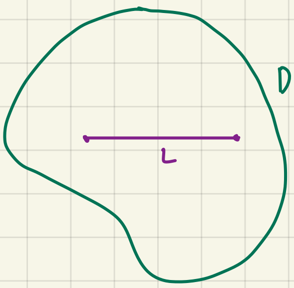
$\quad \parallel$ (Rectangle thm $\because f_n$: ana)

$\quad 0$

$\therefore$ By Morera's Thm, f is conti $\square$

THEOREM

f is continuous on an open set $D \subseteq \mathbb{C}$ and **analytic except on a line segment** in D. Then, f is **analytic throughout D**.



Proof

In some cpt nbd K of each pt z_0, f: unif limit of conti f_n.

$\therefore$ f: conti in D

Also, $\forall R \subseteq K$, $\int_\Gamma f = \int_\Gamma \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Gamma f_n = 0$ since $f_n \to f$ unif on Γ.

$\therefore$ By Morera's Thm, f: analytic in D. $\square$