Shun/翔海 (@shun4midx)

9-25-25 (WEEK 4)

DEFINITION

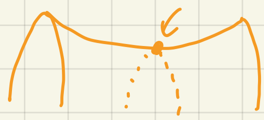
A function is C-analytic on a region D if it is analytic on D and continuous on $\bar{D}$

SADDLE POINTS

DEFINITION

z_0 is a saddle pt of an analytic function f (on a region D) if z_0 is a saddle pt of the real valued function $g(x,y)=|f(x,y)|$

In other words, $\underbrace{g \text{ is differentiable}}_{\lim_{h\to 0}\frac{\|g(x+h)-g(x)-L(x)\|}{\|h\|}=0}$ and $g_x(z_0)=g_y(z_0)=0$ but z_0 is NOT a local extremum



THEOREM

z_0 is a saddle pt of an analytic function f iff $f'(z_0)=0$ and $f(z_0)\neq 0$

Proof

We have $z=x+iy$, $f(z)=u(x,y)+iv(x,y)$, and $g(z)=(u^2+v^2)^{\frac{1}{2}}\geq 0$

"$\Rightarrow$": As $g(z_0)$ is not a local minimum, hence $g(z_0)\neq 0$, so $u(z_0)\neq 0$ or $v(z_0)\neq 0$

We know $g_x(z_0)=g_y(z_0)=0 \Rightarrow \frac{uu_x+vv_x}{g}\Big|_{z_0}=\frac{uu_y+vv_y}{g}\Big|_{z_0}=0$ (★)

$$\Rightarrow \begin{bmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0)\end{bmatrix}\underbrace{\begin{bmatrix} u(z_0) \\ v(x_0)\end{bmatrix}}_{\neq 0}=0$$

$$\therefore \det\begin{bmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0)\end{bmatrix} \overset{\text{CR-eq}}{=} u_x^2(z_0)+v_x^2(z_0)=0$$

$\therefore u_x(z_0)=v_x(z_0)=0$

As f is ana, hence $f'(z_0)=0$. From above with $g(z_0)\neq 0$, we know $f(z_0)\neq 0$. ✓

"$\Leftarrow$": Recall, $f'(z_0)=0$

Then, $u_x(z_0)=v_x(z_0)=0$ and $u_y(z_0)=v_y(z_0)=0$

$\therefore g_x(z_0)=g_y(z_0)=0$ as implied by (★)

As $f(z_0)\neq 0$, thus $|f(z_0)|$ is NOT a local extremum (excluding f is const) by the max and min modulus thms. ✓

OPEN MAPPING THM AND SCHWARTZ LEMMA

RECALL

$f:\mathbb{R}^2\to\mathbb{R}^2$ conti $\Leftrightarrow U\subseteq\mathbb{R}^2$: open then $f^{-1}(U)$: open $\Leftrightarrow \bar{U}\subseteq\mathbb{R}^2$: closed then $f^{-1}(\bar{U})$: closed

Then $K\subseteq\mathbb{R}^2$: cpt $\Rightarrow f(K)$: cpt

THEOREM (OPEN MAPPING THEOREM)

$\forall$ open set $U\subseteq D$, $f(U)$: open in $\mathbb{C}$ for any nonconst ana f (Need not hold outside of $\mathbb{C}$, e.g. $U=(-1,1)$, $f(U)=[0,1)$ for $f(x)=x^2$)

Proof

It suffices to show $\forall \alpha\in D$, $\exists$ open disc $D(\alpha,\varepsilon)\subseteq D$ s.t. $f(D(\alpha,\varepsilon))$ is open.

(We want to show $\forall \beta=f(\alpha')\in f(D(\alpha,\varepsilon))$, $\exists D(\beta,\varepsilon')\subseteq f(D(\alpha,\beta)))$

WLOG, we can assume $f(\alpha)=0$, so we choose ε s.t. $\overline{D(\alpha,\varepsilon)}\subseteq D$

By uniqueness thm, $\exists\,\varepsilon$ s.t. f has no zeros in $\overline{D(\alpha,\varepsilon)}\setminus\{\alpha\}$ (or else $f\equiv 0$)

Let $2\delta = \min_{z \in C_\varepsilon(\alpha)} |f(z)| > 0$

Shun/翔海(@shun4midx)

Claim: $D(f(\alpha)=0, \delta) \subseteq \mathrm{Im}(f)$

Proof

$\forall w \in D(0, \delta)$, consider $f(z) - w$

If $w \notin f(D(\alpha, \varepsilon))$, then $f(z) - w$ has no zeros on $D(\alpha, \varepsilon)$

$\therefore |f(z) - w| \geq |f(z)| - |w| \geq f(z) - \delta \geq \delta \quad \forall z \in C_\varepsilon(\alpha)$

However, we know $|f(\alpha) - w| < \delta$ ※

$\therefore w \in f(D(\alpha, \varepsilon)) \Rightarrow D(0, \delta) \subseteq \mathrm{Im}(f)$ $\square$