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CONTINUED RIEMANN MAPPING THEOREM PROOF

As S is dense in D, $\exists \xi_i \in S$, s.t. $|z-\xi_i| < \varepsilon d$

$\because \lim_{n\to\infty} \varphi_n(\xi_i)$ exists
$\therefore \exists N>0$, s.t. $|\varphi_n(\xi_i) - \varphi_m(\xi_i)| < \frac{\varepsilon}{3}$ $\forall n,m>N$

Hence, $|\varphi_n(z) - \varphi_m(z)| \leq |\varphi_n(z) - \varphi_n(\xi_i)| + |\varphi_n(\xi_i) - \varphi_m(\xi_i)| + |\varphi_m(\xi_i) - \varphi_m(z)| < \varepsilon$
$\therefore \lim_{n\to\infty} f_n(z)$ exists $\forall z \in R$

(iii) Define $\varphi(z) := \lim_{n\to\infty} \varphi_n(z)$ $\forall z \in R$.
We want "$\varphi_n \to \varphi$ on cpta of R"

$\forall K \subseteq R$: cpt, given $\varepsilon>0$, $V_j := \{z\in K \mid |\varphi_n(z) - \varphi(z)| < \varepsilon \ \forall n \geq j\}$
Then, $K := \bigcup_{j=1}^{\infty} V_j \xRightarrow{K:\text{cpt}} K = \bigcup_{j=1}^{N} V_j \Rightarrow \varphi_n \to \varphi$ unif on K

∵ conti   by our choice
As φ_n: ana $\Rightarrow \varphi$ ana and $\varphi_n' \to \varphi'$, thus φ: ana and $\varphi'(z_0) = \lim_{z\to\infty} \varphi_n'(z_0) = M$ □

RIEMANN-ZETA FUNCTION

DEFINITION

notice, n is the base now, not z!!!

A Dirichlet series is in the form of $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$, where $n^z = \exp(z\log n)$ and we choose the branch s.t. $\log n \in \mathbb{R}$

In particular, the Riemann zeta function is $\sum_{n=1}^{\infty} \frac{1}{n^z}$

THEOREM

If $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges at $z=z_0$, then it converges at all $z \in \{\eta \mid Re(\eta) > Re(z_0)\} =: H_0$. Moreover, it conv unif on cpta $\forall K \subseteq H_0$

Proof

Fix $z \in H_0$, we want "$\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ conv"

Claim: Given $\varepsilon>0$, $\exists n_0$, s.t. $N,M>n_0 \Rightarrow \left|\sum_{n=M}^{N} \frac{a_n}{n^z}\right| < \varepsilon$

Proof

Define $A_i := \sum_{n=1}^{i} \frac{a_n}{n^{z_0}}$, $b_i := \frac{1}{n^{z-z_0}}$. Then, $\sum_{n=M}^{\infty} \overset{(A_n - A_{n-1})b_n}{\frac{a_n}{n^z}} = A_{M-1}b_M + \underbrace{A_M(b_M - b_{M+1}) + \dots + A_{N-1}(b_{N-1} - b_N)}_{(☆)} + A_N b_N$

Since $\sum_{n=0}^{\infty} \frac{a_n}{n^{z_0}}$ conv, $\exists A>0$, s.t. $|A_i| < A$, $b_n - b_{n+1} = \frac{1}{n^\omega} - \frac{1}{(n+1)^\omega} = [-t^{\omega}]_n^{n+1} = \int_n^{n+1} \omega t^{-\omega-1}\,dt \Rightarrow |b_n - b_{n+1}| < \frac{|\omega|}{n^{1+\delta}}$, $\delta = Re(z-z_0) > 0$

Moreover, $☆ \leq A \cdot \sum_{n=M}^{N-1} \frac{|\omega|}{n^{1+\delta}}$, where $\sum_{n=1}^{\infty} \frac{|\omega|}{n^{1+\delta}}$ conv $\forall \delta>0$

$\therefore \exists C>0$, s.t. $\forall n, n_1, n_2 > C$, $\left|\sum_{n=n_1}^{n_2} \frac{|\omega|}{n^{1+\delta}}\right| < \frac{\varepsilon}{A}$, so $☆ < \varepsilon$
$\Rightarrow \forall N,M>C$, $\left|\sum_{n=M}^{N} \frac{a_n}{n^z}\right| < 3\varepsilon$ □

REMARK / COUNTEREXAMPLE

$A_i = \sum_{n=1}^{i} \frac{a_n}{n^{z_0}}$, indep of $z \in H_0$. $(\triangle)$
In particular, $\exists A$, s.t. $|A_i| < A$ $\forall i$. Then, $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges? (No!)

For example, let $a_n := (-1)^n$, then for $z=0$, $\sum_{n=1}^{\infty} (-1)^n$ does not converge, but it satisfies $(\triangle)$. Hence, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}$ conv $\forall Re(z)>0$

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THEOREM

If $\sum_{n=0}^{\infty}\frac{a_n}{n^z}$ converges (abs conv) for some z but not all $z_0\in\mathbb{C}$, then $\exists x_0\in\mathbb{R}$ (called abscissa of convergence (abs conv)) s.t. $\sum_{n=1}^{\infty}\frac{a_n}{n^z}$ converges (conv abs) if $Re(z)>x_0$ and $\sum_{n=0}^{\infty}\frac{a_n}{n^z}$ diverges (not abs conv) $\forall Re(z)<x_0$

Proof

Consider the Thm, we know $\underbrace{[\inf\{Re(w) \mid \sum\frac{a_n}{n^w}\text{ conv}\}]}_{=:C}>-\infty$ as $f(z):=\sum\frac{a_n}{n^z}$ not conv $\forall z\in\mathbb{C}$.

def for inf

$\forall z'$ with $Re(z')>C$, $\exists z_0$ with $C<Re(z_0)<Re(z')$, s.t. $f(z)$: conv at z_0 $\Rightarrow$ $f(z')$: conv

By Thm, $x_0=C$

conv at z_0

Now, $\sum\left|\frac{a_n}{n^z}\right| = \sum\frac{|a_n|}{n^{Re(z)}}$

$\forall z$ with $Re(z)>Re(z_0)$, $\sum\frac{|a_n|}{n^{Re(z)}} < \sum\frac{|a_n|}{n^{Re(z_0)}}$ □

EXAMPLE

$\zeta(z):=\sum\frac{1}{n^z} \Rightarrow$ abscissa of ζ is $z=1$

square, i.e. $\zeta\cdot\zeta$.

$\zeta^2(z)=\sum\frac{c_n}{n^z}$, $c_n=\sum_{d\mid n}1$

In general, $f(z):=\sum\frac{a_n}{n^z}$, $g(z):=\sum\frac{b_n}{n^z}$: conv $\forall Re(z)>C$

Then, we can define $f(z)+g(z)$, $f(z)g(z)=\sum\frac{c_n}{n^z}$, $c_n=\sum_{d\mid n}a_d b_{\frac{n}{d}}$ for $Re(z)>C$

ANALYTIC CONTINUATION

MAIN QUESTION

any arbitrary $g(z)$: ana on $D(0,R)$

Given $\sum_{i=0}^{\infty}a_n z^i$ conv abs for $|z|<R$, $\exists?$ ana f s.t. $f|_{D(0,R)}=\sum_{i=0}^{\infty}a_n z^i$ but f: defined on $D\supsetneq D(0,R)$?

↳ We have seen examples such as Schwarz Lemma and removable singularities: $\frac{1}{1-z}$: ana cont. of $\sum_{i=1}^{\infty}z^n$, $|z|<1$

(I) POWER SERIES

THEOREM

If $f(z)=\sum_{n=0}^{\infty}a_n z^n$ has a positive radius of conv R, then f has a singularity at $|z|=R$

EXAMPLE

$\sum_{n=1}^{\infty}\frac{z^n}{n^2}$ conv at $z=1$, ana at $|z|<1$

If f: ana in a nbd of $z=1$ and $f(z)=\sum\frac{z^n}{n^2}$ at $z=1$, $f(z)|_{D(1,\varepsilon)}=\sum\frac{z^n}{n^2}$,

Then, $f''(z)$: ana on $D(1,\varepsilon)$

However, $f''(z)=\sum_{n=1}^{\infty}\frac{n(n-1)}{n^2}z^n$ div at $z=1$ ⨳

$\Rightarrow$ $z=1$ is a singular point of $\sum_{n=1}^{\infty}\frac{z^n}{n^2}$

PROOF OF THEOREM I

If not, $\forall |z|=R$, $\exists$ max $\varepsilon_z>0$, s.t. f can be continued analytically on $D(z,\varepsilon_z)$ to $\tilde{f}$, where ε_z varies cont. in z.

$|z|=R$: cpt $\Rightarrow$ $\varepsilon=\min_{|z|=R}\varepsilon_z$, $\exists |z_0|=R$, $\varepsilon_{z_0}=\varepsilon$ $>0 \Leftarrow z_0$: regular

Then, $\tilde{f}$ can be defined on $D(0,R+\varepsilon)$ analytically $\Rightarrow \tilde{f}=\sum_{n=0}^{\infty}b_n z^n$, $\tilde{f}|_{D(0,R)}=f \overset{\text{Uniqueness}}{\Longrightarrow} a_n=b_n$, so radius of conv $=R$ ⨳