# 11-19-25 (WEEK 12)

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## THEOREM

If $f$ is ana at $z_0$ and $f'(z_0)\neq 0$, $f$ is conformal and locally 1-1 at $z_0$

**Proof**

"$f$ is locally 1-1 at $z_0$": Set $h(z):=f(z)-f(z_0)=\alpha$

└ conformal proved last time

Then, $h(z_0)=0$, $h'(z_0)\neq 0 \Rightarrow h(z)$ is not const in a nbd of $z_0$

$\Rightarrow \exists\, D'(z_0,\delta)$ s.t. $\forall z\in \underbrace{D'(z_0,\delta)}_{= D(z_0,\delta)\setminus\{z_0\}}$, $h(z)\neq 0$

Prove loc const

$\forall\, \delta_1<\delta$, $C_1:=C(z_0,\delta_1)\subseteq D(z_0,\delta)$,

$\underbrace{n_z(h)}_{\overset{\text{in } C_1}{=} 1} = \frac{1}{2\pi i}\int_{C_1}\frac{h'}{h}dz$, $h$: ana in $D(z_0,\delta)$

Turn into integral

Define $w:=h(z)$,

$\int_{C_1}\frac{h'}{h}dz=\int_{h(C_1)}\frac{dw}{w}=\underbrace{\eta}_{\text{winding number}}(h(C_1),0)$

$w:=h(z)$

winding # ⇒ shrink to show $\eta$ = loc const 1 means locally 1-1

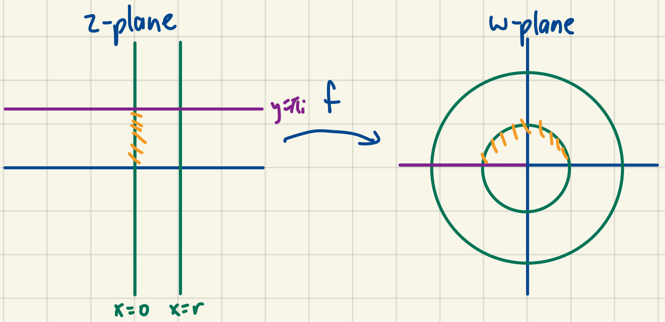
As $\eta(h(C_1),\alpha)$ is locally const in $\alpha$, $\exists\,\varepsilon$, s.t. $\overbrace{\eta(h(C_1),\alpha)}^{\frac{1}{2\pi i}\int_{h(C_1)}\frac{1}{w-\alpha}dw}=1$ $\forall \alpha\in D(0,\varepsilon)$

Consider $\delta_2<\delta$, s.t. $D(z_0,\delta_2)\subseteq h^{-1}(D(0,\varepsilon))$, i.e. $\forall z_1\in D(z_0,\delta_2)$, $1=\frac{1}{2\pi i}\int_{C_1}\frac{d(h(z)-h(z_1))}{h(z)-h(z_1)}=n_z(h(z)-h(z_1))$

$\Rightarrow$ only $z_1\in D(z_0,\delta_2)$, s.t. $h(z_1)=h(z_1)$ ← as in only $z_1$ is s.t. $h(\cdot)=h(z_1)$ □

## EXAMPLE

$f(z):=e^z$: entire, $f'(z)\neq 0\ \forall z\in\mathbb{C}$ ⇒ everywhere conformal + locally 1-1




## DEFINITION

Let $k\in\mathbb{Z}_{>0}$. $f$ is a $k$-to-1 mapping of $D_1$ onto $D_2$ if $\forall\alpha\in D_2$, $f(z)=\alpha$ has $k$ roots in $D_2$ counting by multiplicity

## LEMMA

Let $f(z)=z^k$, $k\in\mathbb{Z}_{>0}$. Then, $f$ magnifies angles at 0 by a factor of $k$ and $f$ maps $D(0,\delta)$ onto $D(0,\delta^k)$ in a $k$-to-1 manner.

**Proof**

For $z\neq 0$, we have $z=|z|e^{i\theta}$, $\theta\in[0,2\pi)$, $z^k=|z|^k e^{ik\theta}$

For $\alpha=|\alpha|e^{i\theta}\neq 0$, (Case 1)

$f(z)=\alpha \Rightarrow z=|\alpha|^{\frac{1}{k}}e^{\frac{i(\theta+2\pi n)}{k}}$, $0\leq n\leq k-1$ are the roots of $f(z)=\alpha$

For $\alpha=0$, (Case 2)

$f(z)=0$ has a zero at $z=0$ with multiplicity $k$ □

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## THEOREM

$\begin{cases} f: \text{ana at } z_0 \\ f'(z_0) \neq 0 \end{cases}$ $(\Delta)$

Suppose $f$ is not const. Then, $\exists$ nbd $U$ of $z_0$ s.t. $f|_U$ is a k-to-1 mapping and magnifies angles at $z_0$ by a factor of $k$, where $k$ is the least positive integer s.t. $f^{(k)}(z_0) \neq 0$ $(\star)$

$\Delta \Rightarrow k \geq 2$

### Proof

After replacing $f$ by $f(z)-f(z_0)$, we can assume $f(z_0)=0$

Taylor expansion

As $f$ is ana at $z=z_0 + (\star)$, $\exists D(z_0, \delta_0)$, s.t. $f(z) = \sum_{i=k}^{\infty} a_i(z-z_0)^i$ with $a_k \neq 0$

$\Rightarrow f(z) = (z-z_0)^k \underbrace{(a_k + a_{k+1}(z-z_0)^1 + \dots)}_{g(z)}$, with $g(z_0) \neq 0$ $(\because a_k \neq 0)$

Taylor expansion + factor

$\exists \varepsilon$, s.t. $D(a_k, \varepsilon) \not\ni 0 \Rightarrow$ we can choose a branch of log defined on $D(a_k, \varepsilon)$

$\therefore \forall z \in \underbrace{g^{-1}(D(a_k, \varepsilon))}_{=: D_1}$, we can define $\exp(\frac{1}{k}\log g(z)) =: h$ ($k^{th}$ root)

Choose a branch to $k^{th}$ root

$\therefore f(z) = [(z-z_0)h]^k$, define $H := (z-z_0)h \Rightarrow f = m_k \circ H$ on $D_1$, where $m_k(z) := z^k$

Apply root + unroot

As $H$ is ana on $D_1$, and $H'(z_0) = h(z_0) \neq 0$ $(\because g(z_0) \neq 0)$, thus $H$: locally 1-1 + conformal

Prove root: conformal

$\therefore$ By lemma, we have $f$'s angles magnified by $k$ and $f$: k-to-1 $\square$

H preserves angles, then $m_k$ magnifies

## THEOREM

Say $f$: 1-1 ana on a region $D$.

Then, (i) $f^{-1}$ exists and is ana on $f(D)$

(ii) $f$ and $f^{-1}$ are conformal in $D$ and $f(D)$ respectively

### Proof

As $f$: 1-1, on $f(D)$, $\forall y \in f(D)$, $g(y) = x$ s.t. $f(x) = y$ is an inverse of $f$

Claim: $g$ is cont: ("$g^{-1}$(open) = open")

#### Proof

$\forall$ open $U \subseteq D$, notice: $g^{-1}(U) := \{y \in f(D) \mid g(y) \in U\} = \{f(x) \mid x \in U\} = f(U)$

By open mapping Thm, $g^{-1}(U) = f(U)$ is open ✓

Define $g$ as $f^{-1}$, prove it's cont: w/ "$g^{-1}$(open) = open"

by Thm above, $f' = 0 \Rightarrow f$: not 1-1.

As $f \circ g = id_{f(D)}$, $g \circ f = id_D$, $g$: cont; $f'(z) \neq 0$ $\forall z \in D$, thus $f$: ana $\Rightarrow g$: ana, $g'(f(z)) = \frac{1}{f'(z)}$

We get $g$ is ana and $g'(z) \neq 0$ $\forall z$

As $f'(z) \neq 0$, thus $g'(f(z)) \neq 0$ and is well-def $\Rightarrow g$: locally 1-1 and conformal on $f(D)$

As $f$: 1-1, thus $g$: 1-1 $\square$

## DEFINITION

① A 1-1 analytic mapping is called a conformal mapping

② Two regions $D_1$ and $D_2$ are conformally equivalent if $\exists$ conformal mapping from $D_1$ onto $D_2$

## FACT

Conformal equivalence is an equivalence relation

# SPECIAL MAPPINGS (EXAMPLES OF CONFORMAL MAPPINGS)

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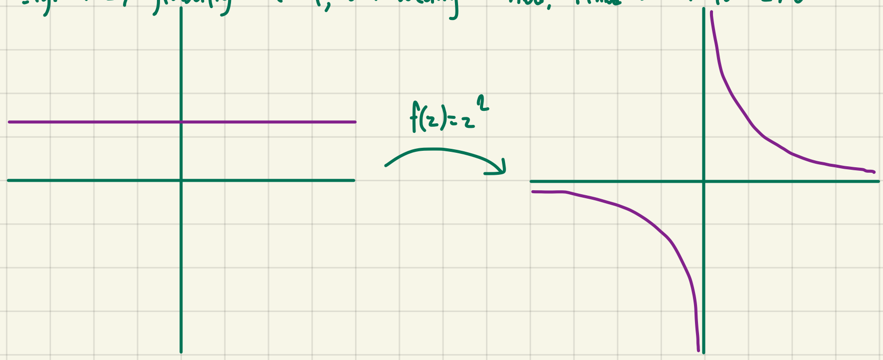
(i) $f(z)=az+b$, $a\neq 0$ (rotation + shrinkage)

(ii) $f(z)=z^\alpha$, locally at $z\neq 0$, for $\alpha>0$

E.g. $\alpha=2$, globally 2-to-1, but locally $\exists$ nbd, $f|_{nbd}$ is 1-1 for $z\neq 0$



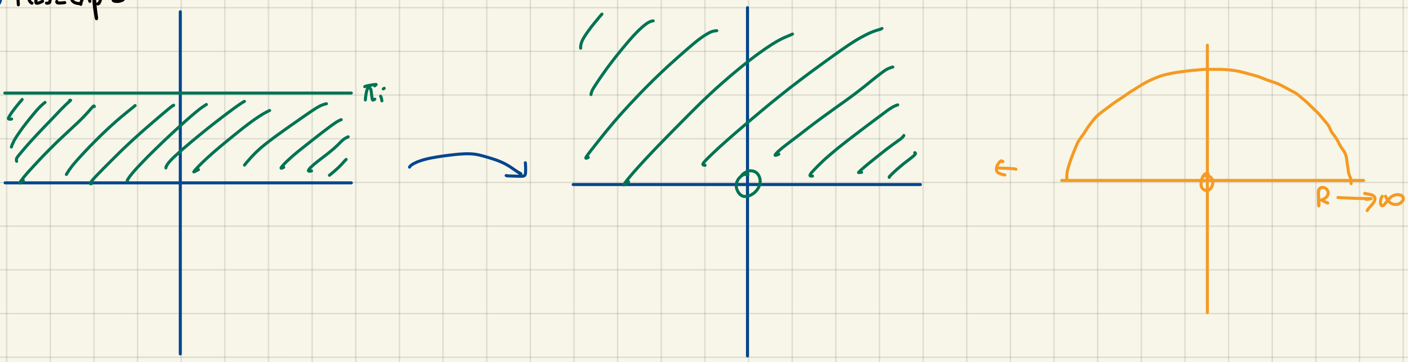

$(x+iy)^2=(x^2-y^2)+2ixy$

For $z\neq 0$, $z=re^{i\theta}\Rightarrow f(z)=r^\alpha e^{i\alpha\theta}$

$\{z \mid \theta_1\le \operatorname{Arg} z\le\theta_2\}\xrightarrow{f(z)}\{z \mid \alpha\theta_1\le \operatorname{Arg} z\le \alpha\theta_2\}$

If $\alpha\theta_2-\alpha\theta_1<2\pi$, then $f$: 1-1 in a nbd of $z$ and conformal

(iii) $f(z)=\exp z$




(iv) Bilinear transformation

$f(z)=\frac{az+b}{cz+d}$, $\underbrace{ad-bc}_{\det}\neq 0$, ana on $\mathbb{C}\setminus -\frac{d}{c}$

$f'(z)=\frac{ad-bc}{(cz+d)^2}\neq 0$

Then, $f$ is locally 1-1 + conformal in $\mathbb{C}\setminus -\frac{d}{c}$

As $f(z_1)=f(z_2)\Rightarrow(ad-bc)(z_1-z_2)=0$, thus $f$ is globally 1-1 ($\frac{a}{c}\notin \operatorname{Im} f$)

If we extend $f$ as a function on the Riemann sphere $\mathbb{C}\cup\{\infty\}$, then $f$: conformal + globally 1-1

**LEMMA**

If $S$ is a circle or a line, and $f(z)=\frac{1}{z}$, then $f(S)$ is a circle or a line

Proof

Wasn't this a HW problem lmfao

Rmk: This extends to $f(z)=\frac{az+b}{cz+d}$