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11-20-25 (WEEK 12)

THEOREM (RECALL)

$f(z)=\frac{az+b}{cz+d}$, $ad-bc\neq 0$ maps circles and lines to circles and lines

Proof

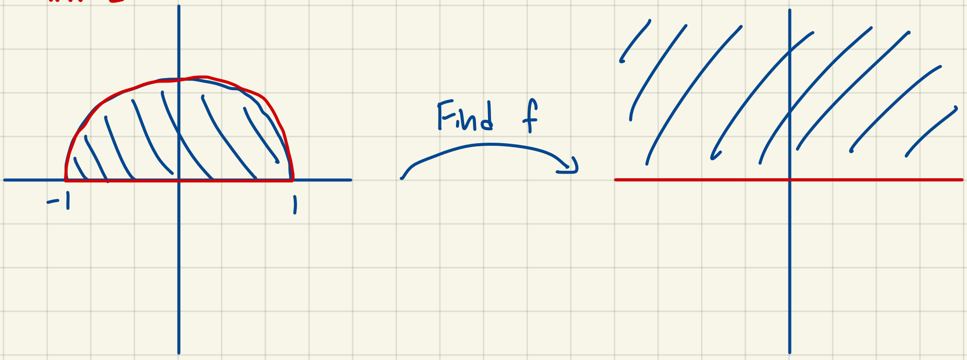
If $c=0$, then trivial.

If $c\neq 0$, $f(z)=\frac{1}{c}\left(a-\frac{ad-bc}{cz+d}\right)$ key transformation to only have $\frac{1}{\sim z+\sim}$

$\Rightarrow$ Consider $z\to cz+d\to\frac{1}{cz+d}\longrightarrow\frac{ad-bc}{cz+d}$

from lemma

EXAMPLE



$S=\{z\mid |z|<1,\ \mathrm{Im}\,z>0\}$

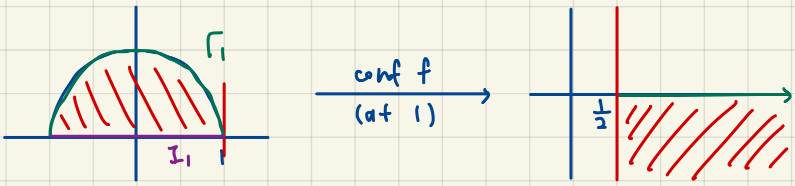
$\mathbb{H}=\{z\mid \mathrm{Im}\,z>0\}$

Intuition: Cut the ⌓, then map to straight line

So we set $-1\to\infty$ (i.e. a pole)

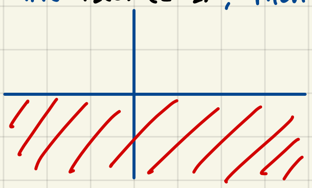
conformal $\forall z\in\mathbb{C}\setminus\{-1\}$

Consider $f_1=\frac{1}{z+1}$, then $f_1(1)=\frac{1}{2}$, $[-1,1)\xrightarrow{f}[\frac{1}{2},\infty)$. Notice, f_1 maps circles and lines to circles and lines.



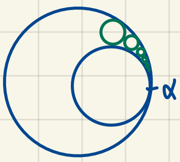
$\therefore f_1(\Gamma_1)=\{z\mid \mathrm{Im}\,z\leq 0,\ \mathrm{Re}\,z=\frac{1}{2}\}$, $f_1(I_1)=[\frac{1}{2},\infty)$

After $f_2(z)=(z-\frac{1}{2})^2$, then it will become



$\therefore f(z)=-\left(\frac{1}{z+1}-\frac{1}{2}\right)^2$

EXAMPLE



C_1, C_2: circles tangent at α

Find a chain of circles tangent to C_1 and C_2 and each other.

Then, the points of tangency lie on a circle or a line.

$\hookrightarrow$ Choose $f(z)=\frac{1}{z-\alpha}$

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DEFINITION

1-1 + analytic   then bijective

A conformal mapping of a region D onto itself is called an **automorphism** of D, which we denote as $Aut(D)$

LEMMA

If $f: D_1 \to D_2$, D_1 and D_2 are regions, and f is a conformal mapping onto D_2,

Then, (i) For any other $h: D_1 \to D_2$, conformal onto D_2, then $\exists g \in Aut(D_2)$, s.t. $h = g \circ f$

(ii) $\forall h \in Aut(D_1), \exists g \in Aut(D_2)$, s.t. $h = f^{-1} \circ g \circ f$

Proof

by Thm

(i) For f: conformal and $f(D_1) = D_2$, we know $\exists f^{-1}: D_2 \to D_1$, conformally onto D_1

Then, $g := h \circ f^{-1} \in Aut(D_2) \Rightarrow h = g \circ f$

(ii) $\underbrace{f \circ h \circ f^{-1}}_{g} \in Aut(D_2) \Rightarrow h = f^{-1} \circ g \circ f$ □

LEMMA

The only automorphism of a unit disc with $f(0) = 0$ are given by $f(z) = e^{i\theta} z$

Proof

Let $D := D(0,1)$.

As $f \in Aut(D)$ and $f(0) = 0$, by Schwartz's Lemma, $|f(z)| \le |z|$ Schwartz's Lemma on f

By Thm, $\exists f^{-1}$ and $f^{-1} \in Aut(D) \Rightarrow$ By Schwartz's Lemma, $|f^{-1}(z)| \le |z|$ Schwartz's lemma on f^{-1}

$\Rightarrow |z| = |f^{-1}(f(z))| \le |f(z)| \le |z|$ $|f(z)| = |z|$

$\Rightarrow |f(z)| = |z|$

By Schwartz's Lemma, $f(z) = e^{i\theta} z$ □ (recall "equality $\Rightarrow f(z) = e^{i\theta} z$")

LEMMA

Let h be a bilinear transformation. If h maps D to D, where $D := D(0,1)$, and $h(\alpha) = 0$ for some $|\alpha| < 1$, then $h = e^{i\theta} \underbrace{\frac{z-\alpha}{1-\bar{z}\alpha}}_{B_\alpha(z)}$

Proof def of bilinear transformation

$h(z) = \frac{az+b}{cz+d}$ is globally 1-1, $h(\alpha) = a\left(\frac{z-\alpha}{cz+d}\right)$, $a \neq 0$

Then, $h(D) \subseteq D \Rightarrow h$: ana on D $h(D) \subseteq D \Rightarrow h$: ana on D

By Schwartz Reflection Principle, $h(\bar{\alpha}^{-1}) = \overline{(h(\alpha))}^{-1} = \infty$ Find $\to \infty$, so we can define a pole via Schwartz Reflection Principle

$\therefore h(z) = A\left(\frac{z-\alpha}{z-\bar{\alpha}^{-1}}\right)$, $A \neq 0$

However, by open mapping Thm, bdry $\to$ bdry

$\therefore |h(1)| = 1$

$\therefore \left|A \cdot \frac{1}{\bar{\alpha}}\right| \cdot \left|\cancel{\frac{1-\alpha}{\bar{\alpha}-1}}^{\,1}\right| = 1 \Rightarrow A = \bar{\alpha} e^{i\theta}$ □

THEOREM

$D := D(0,1)$.

Then, $Aut(D) = \{e^{i\theta}\left(\frac{z-\alpha}{1-z\bar{\alpha}}\right) \mid |\alpha| < 1, 0 \le \theta \le 2\pi\}$

Proof

$g \in Aut(D) \Rightarrow \alpha := g^{-1}(0) \in D \Rightarrow |\alpha| < 1$

Set $h = \frac{z-\alpha}{1-z\bar{\alpha}}$, then $g \circ h^{-1}(0) = 0$

$\therefore g \circ h^{-1} \in Aut(D)$, so by lemma, $g(h^{-1}(z)) = e^{i\theta} z \Rightarrow g(z) = e^{i\theta} h(z)$ □