

## NOTICE

In  $\mathbb{C}$ ,  $\log a + \log b \neq \log(ab)$  in general

## ZETA FUNCTION

## AIM

Can we extend  $\zeta$  to a meromorphic function on  $\mathbb{C}$ ?

## LINK TO GAMMA FUNCTION

$$\text{Recall } \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

$$\text{Notice, } \int_0^\infty e^{-nt} t^{z-1} dt \stackrel{s=nt}{=} \int_0^\infty e^{-s} \left(\frac{s}{n}\right)^{z-1} \frac{ds}{n} = \underbrace{\frac{1}{n^z}}_{\text{for } \zeta(z)} \int_0^\infty e^{-s} s^{z-1} ds \approx \Gamma(z)$$

$$\therefore \text{For } \operatorname{Re}(z) > 1, \quad \zeta(z) = \frac{1}{\Gamma(z)} \sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^{z-1} dt$$

$$\begin{aligned} \Gamma(z) \quad &\text{(1) has no zeros} \\ &\text{(2) has only poles as singularity} \end{aligned} \quad \left. \begin{aligned} &\Rightarrow \frac{1}{\Gamma(z)} : \text{entire} \\ &\text{depends on } z \quad \text{holomorphic } \forall z \in \mathbb{C} \end{aligned} \right\}$$

$$\text{Thus, } \operatorname{Re}(z) > 1 \Rightarrow \sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^{z-1} dt = \int_0^\infty \left( \sum_{n=1}^{\infty} (e^{-nt}) \right) t^{z-1} dt = \int_0^\infty (\sum_{n=1}^{\infty} e^{-nt}) t^{z-1} dt + \int_0^\infty (\sum_{n=1}^{\infty} e^{-nt}) t^{z-1} dt$$

$$\text{Notice, } \int_0^1 (\sum_{n=1}^{\infty} e^{-nt}) t^{z-1} dt = \int_0^1 \frac{t^{z-1}}{e^{t-1}} dt \quad \text{e}^{t-1} = x + tx^2 + \dots$$

$$\text{We know } \frac{1}{e^{t-1}} \text{ has a simple pole at } t=0, \quad \operatorname{Res}\left(\frac{1}{e^{t-1}}, 0\right) = 1$$

Consider the Laurent Expansion around  $t=0$ ,

$$\frac{1}{e^{t-1}} = \frac{1}{t} + \sum_{i=0}^{\infty} A_i t^i. \quad \text{As } \frac{1}{e^{t-1}} \text{ defined } \forall t \neq 0, \text{ thus } \sum_{i=0}^{\infty} A_i t^i \text{ conv } \forall t \in \mathbb{C}.$$

$$\therefore \text{For } \operatorname{Re}(z) > 1, \quad \int_0^1 \frac{t^{z-1}}{e^{t-1}} dt = \int_0^1 t^{z-2} + \sum_{i=0}^{\infty} A_i t^{i-2} dt = \frac{1}{z-1} + \sum_{i=0}^{\infty} \frac{A_i}{z+i} \quad \text{conv } \forall z \in \mathbb{C} \setminus \{z \leq 1\} \text{ because } \begin{aligned} &\text{(1) } \sum_{i=0}^{\infty} A_i t^i \text{ conv } \forall t \in \mathbb{C} \\ &\text{(2) } \frac{1}{z+i} \in \mathbb{C} \text{ for } i \text{ large enough} \end{aligned}$$

$$\text{Thus, } \operatorname{Re}(z) > 1 \Rightarrow \zeta(z) = \frac{1}{\Gamma(z)} \left( \frac{1}{z-1} + \sum_{i=0}^{\infty} \frac{A_i}{z+i} + g(z) \right) \quad \begin{aligned} &\text{def on } \mathbb{C} \setminus \{z \leq 1\} \\ &\text{holo in } \mathbb{C} \\ &\text{entire w/ zero of order 1 at } z=0 \end{aligned}$$

## THEOREM

$\zeta(z)$  can be analytically continued to  $\mathbb{C} \setminus \{z=1\}$

## Proof

$\frac{1}{z-1}$  has zero of order 1 at  $z=0$  and (a) has simple poles at  $z \leq 1$

## SET-UP

$$\text{Recall, } \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p: \text{prime}} (1 - p^{-z})^{-1}, \quad \operatorname{Re}(z) > 1$$

$\therefore \zeta(z)$  has no zeros for  $\operatorname{Re}(z) > 1$

**THEOREM**

$\zeta(z)$  has no zeros for  $\operatorname{Re}(z) \geq 1$  (i.e. by set-up,  $\zeta(z)$  has no zeros at  $x=1$ )

**Proof Sketch**

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{p: \text{prime}} \frac{[(1-p^{-z})^{-1}]}{(1-p^{-z})^{-1}} = \sum_p \frac{\log p}{p^{z-1}} = \phi(z) + \underbrace{\sum_p \frac{\log p}{p^z(p^z-1)}}_{\text{holomorphic for } \operatorname{Re}(z) > \frac{1}{2}}, \text{ where } \phi(z) = \sum_{p: \text{prime}} \frac{\log p}{p^z}$$

Use prime not log

Suppose  $\exists$  zero for  $\zeta$  at  $1+ia_0$  with  $a_0 \neq 0$ , of order  $\mu > 0$

We know  $z \in \mathbb{R}_{>1} \Rightarrow \zeta(z) \in \mathbb{R}$

$\therefore$  By Reflection Principle,  $\zeta(\bar{z}) = \overline{\zeta(z)}$

$\therefore 1-ia_0$  is a zero of order  $\mu$  of  $\zeta$

Say  $\zeta$  has a zero at  $1 \pm 2ia_0$  of order  $\nu$  ( $\nu = 0$  if  $1 \pm 2ia_0$  is a "zero")

Thus,  $\phi(z)$  has poles of the same order as  $-\frac{\zeta'(z)}{\zeta(z)}$  = zeros and poles of  $\zeta(z)$

↓

$$\lim_{\varepsilon \rightarrow 0} \sum \varepsilon \phi(1+\varepsilon) = 1 \quad (\zeta \text{ has simple pole at } z=1 \text{ with residue 1})$$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \phi(1+ia_0 + \varepsilon) = \mu$$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \phi(1+2ia_0 + \varepsilon) = \nu$$

Finally,

$$\begin{aligned} \sum_{-2 \leq k \leq 2} \binom{4}{2+k} \phi(1+\varepsilon+i\lambda a_0) &= \sum_p \frac{\log p}{p^{1+\varepsilon}} \left( \sum_{-2 \leq k \leq 2} \binom{4}{2+k} \left( \frac{1}{p^{1+\frac{i\varepsilon}{2}}} \right)^{k+2} \left( \frac{1}{p^{-\frac{i\lambda a_0}{2}}} \right)^{-k-2} \right) \\ &= \sum_p \frac{\log p}{p^{1+\varepsilon}} \underbrace{\left( p^{\frac{i\varepsilon}{2}} + p^{-\frac{i\lambda a_0}{2}} \right)^4}_{\geq 0} \end{aligned}$$

$$\text{As } \varepsilon \rightarrow 0, -2\nu - 8\mu + 6 \geq 0 \Rightarrow \mu = 0 \rightarrow$$