9-30-25 (WEEK 5)

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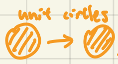
REMARK

We only have open mapping thm because extremum is not in interior pt.

SCHWARTZ'S LEMMA

THEOREM (SCHWARTZ'S LEMMA)

Suppose that f is analytic in an open unit disc D with $|f|\le 1$ (f: 

) and $f(0)=0$

Then, (i) $|f(z)|\le|z|$

(ii) $|f'(0)|\le 1$

with equality in either of the above iff $f(z)=e^{i\theta}z$

Proof

Define $g(z):=\begin{cases}\frac{f(z)}{z}, & z\neq 0\\ f'(z), & z=0\end{cases}$

$g(z)$ is ana on D since $f(z)$ is ana on D.

Consider $z\in C_r(0)$, $0<r<1$.

Then, $|g(z)|=\frac{|f(z)|}{|z|}\le\frac{1}{r}$

By max modulus thm, $\forall z\in\overline{D(0,r)}$, $|g(z)|\le\frac{1}{r}$

As $r\to 1$, then $|g(z)|\le 1$ $\forall z\in D$

By def of $g(z)$, $|f(z)|\le|z|$ and $|f'(0)|\le 1$ has either equality hold, when g is const and $|g|=1$ on D. $\therefore g=e^{i\theta}$ □

EXAMPLE (Removing $f(0)=0$ constraint)

Define $B_\alpha(z)=\frac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha|<1$ — D

Then, (1) $B_\alpha(\alpha)=0$

(2) $B_\alpha(z)$ is ana on D, so $(B_\alpha(z))^2$ is ana on D. It is also conti on $\bar{D}$.

(3) $|B_\alpha(z)|^2\big|_{|z|=1}=1$, so by max modulus thm, $|B_\alpha(z)|\le 1$ on D.

$\therefore$ We can use B_α for variations of Schwartz's lemma

EXAMPLE

Say f: ana on D, $|f(z)|\le 1$ $\forall z\in D$ and $f(\frac{1}{2})=0$. Estimate $|f(\frac{3}{4})|$.

Consider $B_{\frac{1}{2}}(z)=\dfrac{z-\frac{1}{2}}{1-\frac{z}{2}}$.

$(B_\alpha(z))^{-1}$

We define $g(z)=\begin{cases}\frac{f(z)}{z-\frac{1}{2}}(1-\frac{1}{2}z), & z\neq\frac{1}{2}\\ \frac{3}{4}f'(\frac{1}{2}), & z=\frac{1}{2}\end{cases}$

Notice, $|B_{\frac{1}{2}}(z)|\le 1$ on D, $|B_{\frac{1}{2}}(z)|=1$ on $C_1(0)$, and $B_{\frac{1}{2}}(z)$ conti on D.

$\therefore z\to 1$, $|B_{\frac{1}{2}}(z)|\longrightarrow 1$

$|g(z)|\le 1/|B_{\frac{1}{2}}(z)|\Rightarrow|g(z)|\le 1$ on D.

$\Rightarrow|f(z)|\le|B_{\frac{1}{2}}(z)|$, $\frac{3}{4}|f'(\frac{1}{2})|\le 1$

So, $|f(\frac{3}{4})|\le|B_{\frac{1}{2}}(\frac{3}{4})|=\frac{2}{5}$

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EXAMPLE

Say f is ana on D, $|f(z)| \le 1$ on D. We claim: $|f'(\frac{1}{3})|$ is max when $f(\frac{1}{3}) = 0$ "your rate of improvement is max when your score is at its lowest" ≧?

min modulus for $\mathbb{C}$.

Proof

Assume that $f(\frac{1}{3}) \neq 0$.

$g(z) := \dfrac{f(z) - f(\frac{1}{3})}{1 - \overline{f(\frac{1}{3})} f(z)} \Rightarrow g(z) = B_{f(\frac{1}{3})}(f(z))$. Note, g is bounded by 1, $g(\frac{1}{3}) = 0$. (Note it is ana too)

$\therefore g'(z)\big|_{z=\frac{1}{3}} = \dfrac{f'(\frac{1}{3})}{1 - |f(\frac{1}{3})|^2} \Rightarrow |g'(\frac{1}{3})| > |f'(\frac{1}{3})|$ ($\because |f(\frac{1}{3})| > 0$. Otherwise, then $|g'(\frac{1}{3})|$ can be larger, take $f := g$ $\times$)

$\therefore |f'(\frac{1}{3})|$ is max for $B_{\frac{1}{3}}(z)$.

PROPOSITION

Say f is entire. If $|f(z)| < \frac{1}{|\mathrm{Im}\, z|}$ $\forall z$, then $f \equiv 0$

Proof

Define $g(z) = (z^2 - R^2) f(z)$, for some $R \in \mathbb{R}_{>0}$ (sufficiently large, e.g. $R \ge 1$, $R \to \infty$)

When $z \in C_R(0)$, $|z - R||z + R| \le 2R\,|\mathrm{Im}(z)|$

$\therefore |g(z)| \le \frac{2R}{|\mathrm{Im}(z)|^2} \le 2R$ when $z \in C_R(0)$

By max modulus thm, $|g(z)| \le 2R$ $\forall z \in D(0, R)$

$\Rightarrow |f(z)| \le \frac{2R}{|z^2 - R^2|}$ $\forall z \in D(0, R)$

As $R \to \infty$, $|f(z)| \longrightarrow 0$

$\therefore f(z) = 0$

MORERA'S THEOREM

THEOREM (MORERA'S THEOREM: CONVERSE OF RECTANGLE THEOREM)

Let f be continuous on an open set $D \subseteq \mathbb{C}$, and Γ be the boundary of a closed rectangle $R \subseteq D$.
If $\int_\Gamma f\, dz = 0$ $\forall \Gamma$ in $R \subseteq D$, then f is analytic in D.

Proof

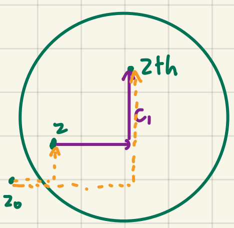
Say $z_0 \in D$, D: open.
Then, $\exists \varepsilon > 0$, s.t. $D(z_0, \varepsilon) \subseteq D$.

Define $F(z) := \int_C f(z)\, dz$ $\forall z \in D(z_0, \varepsilon)$, where $C: z_0 \longrightarrow z_0 + \mathrm{Re}(z - z_0) \rightarrow z$

For $z \in D(z_0, \varepsilon)$ and h small enough s.t. $z + h \in D(z_0, \varepsilon)$
Then,

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \to 0} \int_{C_1} f(\omega)\, d\omega = f(z) \quad \square$$

f: cont.



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EXAMPLE

Using $f(z)=\int_0^\infty \frac{e^{zt}}{t+1}\,dt$,

Claim: f is analytic $\forall z\in\{w\in\mathbb{C}\mid Re(w)<0\}$

Proof

We know for $z=x+iy$, $x<0$, $|e^{zt}|=e^{xt}$

Here, $\int_\Gamma\int_0^\infty \frac{|e^{zt}|}{t+1}\,dt\,dz<\int_\Gamma -\frac{1}{x}\,dz<\infty$ (ok for Fubini)

By Fubini's Thm, $\int_\Gamma\int_0^\infty \frac{e^{zt}}{t+1}\,dt\,dz=\int_0^\infty \underbrace{\boxed{\int_\Gamma \frac{e^{zt}}{t+1}\,dz}}_{=0\ (\because \text{By rectangle thm since } \frac{e^{zt}}{t+1}:\text{ana})}dt=0$

$\therefore$ By Morera's Thm, $f(z)$ is analytic on $\{w\in\mathbb{C}\mid Re(w)<0\}$. □

DEFINITION

Let $\{f_n\}$ and f be defined on an open set D. We say that f_n **converges uniformly on compacta** if $f_n\to f$ uniformly on every compact subset $K\subseteq D$.

THEOREM

Let D be an open set in $\mathbb{C}$ and $\{f_n\}$ be a sequence of ana functions s.t. $f_n\to f$ unif on cpta. Then, f is also ana in D.

Proof

$\because$ f_n is conti, $\forall K\subseteq D$: cpt set we have $f_n\to f$ unif on K

$\therefore$ f is conti on K $\forall K$, i.e. f is conti on D

We hope "$\int_\Gamma f\,dz=0$", for Γ: boundary of a closed rectangle $R\subseteq D$

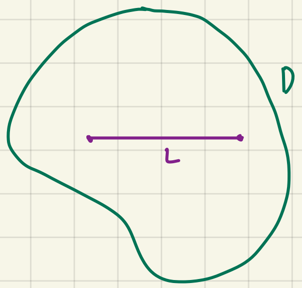
Hence, $\int_\Gamma f\,dz=\int_\Gamma \lim_{n\to\infty} f_n\,dz$

$$\overset{(f_n \text{ conti},\ f_n\to f \text{ unif on } R)}{=} \lim_{n\to\infty}\left(\int_\Gamma f_n\,dz\right) \overset{(\text{Rectangle thm } \because f_n:\text{ ana})}{=} 0$$

$\therefore$ By Morera's Thm, f is conti □

THEOREM

f is continuous on an open set $D\subseteq\mathbb{C}$ and **analytic except on a line segment** in D. Then, f is **analytic throughout D**.



Proof

Fixed in next PDF cuz I was def high when I wrote the proof for the wrong thm :: (yes, hence the reupload)