10-23-25 (WEEK 8) (Sorry if the notes are shit, I'm very fucking upset at myself and attended CompAna today just trying not to cry lol, DM me if any part is unclear, thx. Why am I even fucking offering my notes as such a shit and lazy student anyway, it's not like my notes are beneficial) Shun/翔海 (@shun4midx)

EXAMPLES OF LAURRENT EXPANSIONS

Around $z=0$,

$\frac{(z+1)^2}{z} = z+\frac{1}{z}+2$

$\exp(\frac{1}{z}) = 1+\frac{1}{z}+\frac{1}{2z^2}+\dots$

$f(z)=\frac{1}{z^2(1-z)}=\frac{1}{z^2}(1+z+z^2+\dots)=\frac{1}{z^2}+\frac{1}{z}+1+z+\dots$

DEFINITION

If $f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k$ is a Laurent expansion of f around an isolated singularity z_0. Then, $\sum_{-\infty}^{-1}a_k(z-z_0)^k$ is called the principal part of f at z_0 and $\sum_{i=0}^{\infty}a_i(z-z_0)^i$ is called the analytic part of f

PROPOSITION

(i) f has a removable singularity at z_0. Then, $a_k=0\ \forall k<0$

(ii) f has a pole of order k at z_0. Then, $a_i=0\ \forall i<-k$ but $a_{-k}\neq 0$

(iii) f has an essential singularity at z_0. Then, it must have inf many nonzero terms in its principal part

Proof

(i) $\exists D'(z_0,\delta)$ and $\exists$ ana g on $D(z_0,\delta)$, s.t. $g=f$ on $D'(z_0,\delta)$

$\because$ g is ana at z_0

$\therefore$ $g=\sum_{i=0}^{\infty}b_i(z-z_0)^i$, $|z-z_0|<\delta$

$\because$ f: ana on $D'(z_0,\delta)$

$\therefore$ $f(z)=\sum_{i=-\infty}^{\infty}a_k(z-z_0)^k=\sum_{k=0}^{\infty}b_k(z-z_0)^k$ on $D'(z_0,\delta)$

By uniqueness of Laurent expansion, $a_k=0\ \forall k<0$ □

(ii) $f=\frac{A(z)}{B(z)}$, A,B ana on $D(z_0,\delta)$ with $A(z_0)\neq 0$ and $B(z)$ has a zero of order k at z_0.

$\therefore$ $B(z)=C_k(z-z_0)^k+C_{k+1}(z-z_0)^{k+1}+\dots$, $C_k\neq 0$

Thus, $B(z)=(z-z_0)^k\underbrace{(C_k+C_{k+1}(z-z_0)^1+\dots)}_{=:H(z)}$

$\because$ $H(z)$ is ana on $D(z_0,\delta)$ and $H(z_0)\neq 0$

$\therefore$ $f(z)=\frac{1}{(z-z_0)^k}\frac{A(z)}{H(z)}$

As H: cont:, $\exists D(z_0,\delta_1)$, $\delta_1\leq\delta$, s.t. H has no zeros on $D(z_0,\delta_1)$.

$\therefore$ $f=\frac{1}{(z-z_0)^k}\frac{A(z)}{H(z)}$ on $D'(z_0,\delta_1)$ with $\frac{A(z)}{H(z)}$ ana on $D(z_0,\delta_1)$

We consider Taylor expansion of $\frac{A}{H}$ around z_0.

Then, $\frac{A(z)}{H(z)}=\sum_{i=0}^{\infty}e_i(z-z_0)^i$

$\because$ $A(z_0)\neq 0$, $H(z_0)\neq 0$

$\therefore$ $e_0\neq 0\Rightarrow a_i=0\ \forall i<-k$, $a_{-k}\neq 0$ □

(iii) z_0 is an essential singularity, $f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k$ on $D'(z_0,\delta)$

If only finitely many $a_k\neq 0$ for $k>0$, then $\lim_{z\to z_0}f(z)(z-z_0)^N=0$ for big enough N $\to\!\times$

PROPOSITION 10

P,Q: poly with $\deg P<\deg Q$.

Say $Q(z)=\prod_{i=1}^{n}(z-z_i)^{e_i}$ with distinct z_i. Then, $R(z)=\frac{P(z)}{Q(z)}$ is a sum of polynomials in $\frac{1}{z-z_i}$ with $1\leq i\leq n$

Proof

$R(z)$ is ana on $\mathbb{C}\setminus\{z_1,\dots,z_n\}$.

z_1 is an isolated singularity of R which is a pole of order at most e_1

Then, $R(z)=\sum_{i=-e_1}^{\infty}(z-z_1)^i = \underbrace{\sum_{i=-e_1}^{-1}(z-z_1)^i}_{P_1\left(\frac{1}{z-z_1}\right)} + \underbrace{\sum_{i=0}^{\infty} c_i(z-z_1)^i}_{A_1(z)}$

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Then, $A_1(z)=R(z)-P_1\left(\frac{1}{z-z_1}\right)$ is ana on U.

$\lim_{z\to z_1}(z-z_1)A_1(z)=0 \Rightarrow z_1$ is a removable singularity at $A_1(z) \Rightarrow A_1(z_1):=\lim_{z\to z_1}A_1(z)$

$\therefore A_1(z)$ ana on $\mathbb{C}\setminus\{z_1,\dots,z_n\} \Rightarrow$ Inductively, we get $A_n(z):=A_{n-1}(z)-P_n\left(\frac{1}{z-z_n}\right)$ ⇝ principal part of $R(z)$

However, A_n is bounded since $R, P_n \xrightarrow{z\to\infty} 0$.

$\therefore$ By Liouville's Thm, A_n is const, so $A_n\equiv 0$

$\therefore R(z)=P_1\left(\frac{1}{z-z_1}\right)+\dots+P_n\left(\frac{1}{z-z_n}\right)$ □

THE RESIDUE THEOREM

KEY POINT

$C_1(0) \Rightarrow \int_{C_1(0)} z^k\,dz = \begin{cases} 2\pi i, & k=-1 \\ 0, & \text{otherwise} \end{cases}$

(☆)

f is ana on $D'(z_0,\delta) \Rightarrow \boxed{f=\sum_{i=-\infty}^{\infty} a_i(z-z_0)^i,\ 0<|z-z_0|<\delta.}$

As $C_r(z_0)\subseteq D'(z_0,\delta) \Rightarrow \int_{C_r(z_0)} f\,dz = a_{-1}(2\pi i)$

DEFINITION

(☆), then we define $\text{Res}(f,z_0):=a_{-1}$

PROPOSITION

Given (☆),

(i) z_0 is a simple pole (pole of order 1) then $a_{-1}=\lim_{z\to z_0}(z-z_0)f(z)=\frac{A(z_0)}{B'(z_0)}$

Proof

$(z-z_0)f(z)=A(z)\div\frac{B(z)-\cancel{B(z_0)}^{0}}{z-z_0}$. A,B : ana $\Rightarrow$ done

(ii) If f has a pole of order k, then $a_{-1}=\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k f(z)\right)\Big|_{z=z_0}$

EXAMPLE

$\text{Res}(\csc z, 0)=\frac{1}{\cos z}\Big|_{z=0}=1$

↳ $\frac{1}{\sin z}$ has a simple pole at $z=0$.