9-16-25 (WEEK 3)

Shun/翔海 (@shun4midx)

CAUCHY INTEGRAL FORMULA AND TAYLOR EXPANSION

THEOREM (RECTANGLE THEOREM II)

Let f be entire, and

$$g(z) := \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases},$$ which is continuous (f entire ⇒ g conti)

Then, $\int_\Gamma g(z)\,dz = 0$, Γ: a boundary of a rectangle (closed set) $R \subseteq \mathbb{C}$

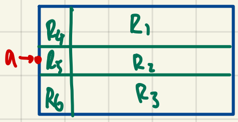
Proof

As g is cont., by def, $\exists M \in \mathbb{R}$, s.t. $|g(z)| < M$ $\forall z \in R$

(i) If $a \in \mathbb{C} \setminus R$, $g(z)$ is analytic $\forall z \in R$

$\therefore$ By the argument of Rectangle Thm, $\int_\Gamma g(z)\,dz = 0$

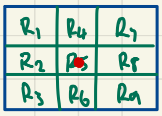
(ii) If $a \in \Gamma$, $\Gamma_i :=$ boundary of R_i:



Then, $\int_\Gamma g(z)\,dz = \sum_{i=1}^{6} \int_{\Gamma_i} g(z)\,dz \overset{(i)}{=} \int_{\Gamma_5} g(z)\,dz << M \cdot 4\varepsilon$ by ML-formula, with M indep of ε, where we define $\varepsilon :=$ length of longest side of Γ_5.

$\therefore$ As $\varepsilon \to 0$, $\int_\Gamma g(z)\,dz = 0$

(iii) Otherwise, $a \in$ interior of R



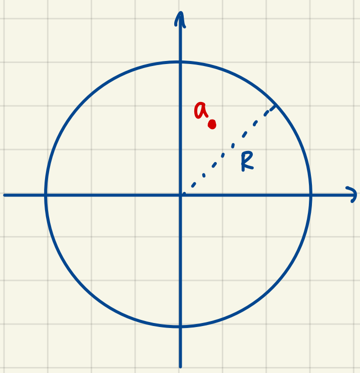
Then, $\int_\Gamma g(z)\,dz = \sum_{i=1}^{9} \int_{\Gamma_i} g(z)\,dz \overset{(i)}{=} \int_{\Gamma_5} g(z)\,dz << M \cdot 4\varepsilon \xrightarrow{\varepsilon \to 0} 0$

COROLLARY

The integral thm and closed curve thm apply to g (since g is cont.)

THEOREM (CAUCHY INTEGRAL FORMULA)

Given an entire f, $a \in \mathbb{C}$, $C = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$ with



Then, we have $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\,dz$

Shun/翔海(@shun4midx)

LEMMA

Define $C_\rho(\alpha) :=$ circle centered at α with radius ρ (α may be omitted if no ambiguity)

Then, $\int_{C_\rho(\alpha)} \frac{dz}{z-a} = 2\pi i \quad \forall |a-\alpha| < \rho$

<u>Proof</u>

If $a = \alpha$, then it's clear, since $C_\rho(\alpha): \alpha + \rho e^{i\theta}$, $0 \le \theta \le 2\pi$.

For $a \neq \alpha$,

$$\int_{C_\rho(\alpha)} \frac{dz}{(z-\alpha)-(a-\alpha)} = \int_{C_\rho(\alpha)} \frac{1}{z-\alpha} \cdot \frac{1}{1-\frac{a-\alpha}{z-\alpha}} \, dz =: I$$

$\forall z \in C_\rho(\alpha)$, $\left|\frac{a-\alpha}{z-\alpha}\right| < 1$. Hence, $\left(1-\frac{a-\alpha}{z-\alpha}\right)^{-1} = 1 + \left(\frac{a-\alpha}{z-\alpha}\right) + \left(\frac{a-\alpha}{z-\alpha}\right)^2 + \ldots$ (unif conv, abs conv)

Hence,

$$I = \int_{C_\rho(\alpha)} \frac{1}{z-\alpha} \left(\sum_{k=0}^{\infty} \left(\frac{a-\alpha}{z-\alpha}\right)^k \right) dz = \boxed{\sum_{k=0}^{\infty} \int_{C_\rho(\alpha)} \frac{1}{z-\alpha} \left(\frac{a-\alpha}{z-\alpha}\right)^k dz}$$

We first consider the term $\int_{C_\rho(\alpha)} \frac{1}{(z-\alpha)^k} dz$ ($k=1 \Rightarrow 2\pi i$, since $\int_C \frac{1}{z} dz = 2\pi i$)

For $k > 1$, $\int_{C_\rho(\alpha)} \frac{1}{(z-\alpha)^k} dz = \int_0^{2\pi} \frac{i e^{i\theta}}{\rho^k e^{ik\theta}} d\theta = \int_0^{2\pi} \frac{i}{\rho^k} e^{-i\theta(k-1)} d\theta = 0$

$\therefore I = 2\pi i$ □

PROOF OF CAUCHY INTEGRAL FORMULA

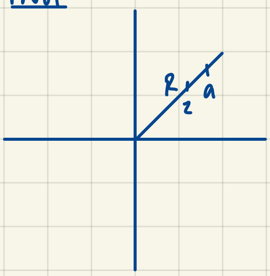
Now, we know by rectangle thm, $\int_C g(z) dz = \int_C \frac{f(z)}{z-a} - \frac{f(a)}{z-a} dz = 0$

$\therefore \int_C \frac{f(z)}{z-a} dz = \int_C \frac{f(a)}{z-a} dz = f(a) \int_C \frac{1}{z-a} dz = f(a) 2\pi i$ □

THEOREM (TAYLOR EXPANSION FOR ENTIRE FUNCTION)

Given f is an entire function, then $f^{(k)}(0)$ exists $\forall k \in \mathbb{Z}_{\ge 0}$ and $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \quad \forall z \in \mathbb{C}$

<u>Proof</u>



Choose $a \in \mathbb{C}$, $|a| > |z|$, $R := |a| + 1$

Notice, by Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\omega)}{\omega - z} d\omega = \frac{1}{2\pi i} \int_{C_R} \frac{1}{1-\frac{z}{\omega}} \frac{f(\omega)}{\omega} d\omega$$

As $\left|\frac{z}{\omega}\right| < \frac{|a|}{1+|a|}$, then $f(z) = \sum_{k=0}^{\infty} \int_{C_R} \frac{f(\omega)}{\omega} \left(\frac{z}{\omega}\right)^k d\omega = \sum_{k=0}^{\infty} z^k \int_{C_R} \frac{f(\omega)}{\omega^{k+1}} d\omega = \sum_{k=0}^{\infty} z^k C_k$

Notice, as $|z| < |a|$, then $f'(z) = \sum_{i=1}^{\infty} i z^{i-1} C_i \Rightarrow f'(0) = C_1$

If we continue this process, $f^{(k)}(0)$ exists $\forall k \in \mathbb{N}_{\ge 0}$ and $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$ □

COROLLARY

An entire function is infinitely diff

<u>Proof</u>

Above thm. □

COROLLARY

If f is entire, then $f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2}(z-a)^2+\dots$

Proof

Let $h(z)=f(z+a)$.

f is entire $\Rightarrow$ z is entire.

Then, $h(z)=\sum_{k=0}^{\infty}\frac{h^{(k)}(0)}{k!}z^k \Rightarrow f(\omega)=f(a)+\sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}(\omega-a)^k$ □

PROPOSITION

If f is entire, then

$$g(z)=\begin{cases}\frac{f(z)-f(a)}{z-a}, & z\neq a\\ f'(a), & z=a\end{cases}$$ is entire

Proof

By corollary, $g(z)=\sum_{k\geq 1}^{\infty}\frac{f^{(k)}(a)}{k!}(z-a)^{k-1} \Rightarrow g$ is entire □

COROLLARY

Let f be an entire function with zeros at $a_1,\dots,a_N$. Define $g(z)=\frac{f(z)}{(z-a_1)\cdots(z-a_N)}$, $z\notin\{a_1,\dots,a_N\}$. Then, $\lim_{z\to a_i}g(z)$ exists $\forall i$.

If we define $g(a_i):=\lim_{z\to a_i}g(z)$, then g is entire

Proof

Set $f_0=f$, $f_k:=\frac{f_{k-1}(z)-f_{k-1}(a_k)}{z-a_k}$ $\left(f_1(z)=\frac{f(z)}{z-a_1}\right)$

By proposition, we see f_1 is entire. By recurrence, g is entire □

THEOREM (LIOUVILLE'S THEOREM)

Entire bounded functions on $\mathbb{C}$ are constants

Proof

Let $a\in\mathbb{C}\setminus\{0\}$, $C_R(0)$, $R>|a|$.

Then, by Cauchy Integral Formula, $f(a)-f(0)=\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-a}dz-\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z}dz=\frac{1}{2\pi i}\int_{C_R}\frac{af(z)}{z(z-a)}dz$

As f is bounded, $\exists M\in\mathbb{R}_{>0}$, s.t. $|f(z)|<M$ $\forall z\in\mathbb{C}$

By ML-formula, $|f(a)-f(0)|<\frac{1}{2\pi i}\left(\frac{M\cdot|a|}{R(R-|a|)}\cdot 2\pi R\right)\xrightarrow{R\to\infty}0$ $\therefore f(a)=f(0)$ $\forall a\in\mathbb{C}$ □

THEOREM (EXTENDED LIOUVILLE'S THEOREM)

Given f is entire. Suppose $|f(z)|<A+B|z|^k$ for some constants $A,B\in\mathbb{R}_0$. Then, f is a polynomial with degree at most k

Proof

Consider induction on k,

- $k=0\Rightarrow$ True by Liouville's Thm.

Otherwise,

Define $g(0)=\begin{cases}\frac{f(z)-f(0)}{z}, & z\neq 0\\ f'(0), & z=0\end{cases}$, which we know is entire.

As $|f(z)|<A+B|z|^k$ is bounded, define $M_0:=\max_{z\in C_R}g(z)$

For $z\in\mathbb{C}\setminus C_R$, $|g(z)|<A+B|z|^{k-1}$
$\forall z\in C_R$, $|g(z)|<M_0$
$\Bigg\}\Rightarrow\exists D,E\in\mathbb{R}_{>0}$ s.t. $|g(z)|<D+E|z|^{k-1}$

$\therefore g$ is poly of degree at most $k-1$, so f is poly of degree at most k. □

THEOREM (FUNDAMENTAL THEOREM OF ALGEBRA)

Nonconstant polynomials have roots in $\mathbb{C}$.
in $\mathbb{C}[x]$

Proof

Consider poly $p(x)$.
Suppose p has no roots in $\mathbb{C}$.

We know $f(z) := \frac{1}{p(z)}$ is defined and differentiable on $\mathbb{C}$, so it is analytic.

As $z \to \infty$, $|f(z)| \longrightarrow 0$, then $|f(z)|$ is bounded
$\therefore$ By Louville's Thm, $f(z)$ is const $\Rightarrow p(z)$ is const $\times$

DEFINITION

We say S is a convex set in $\mathbb{C}$ if $\forall x, y \in S$, $tx + (1-t)y \in S$ $\forall t \in [0,1]$
($\Rightarrow$ $x_1, \ldots, x_N \in S$ iff $\sum_{i=1}^{N} a_i x_i \in S$ $\forall \sum_{i=1}^{N} a_i = 1$ and $a_i \geq 0$)