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THEOREM (UNIQUENESS THEOREM)

Say D is a region (i.e. open connected) and f is an analytic function on D.
Suppose that $\exists$ seq of distinct zeros of D $\{z_n\}$, s.t. $\lim_{n\to\infty} z_n = z_0 \in D$, where we say the seq $\{z_n\}$ has an acc pt in D.
Then, $f \equiv 0$ on D.

Proof

f ana $\Rightarrow$ f conti

$\therefore$ By def, $f(z_0) = \lim_{n\to\infty} f(z_n) = 0$

We define $A := \{z \in D \mid z$ is an acc pt of zeros of f in $D\}$.

Claim: A is open

Proof

By uniqueness of power series, $f \equiv 0$ in some disk $D(z, \delta_z) \subseteq D$ $\forall z \in A$ ✓

Claim: $D \setminus A$ is open

Proof

z is NOT acc pt of zeros $\Rightarrow \exists$ open nbd U of z in D s.t. $f(z)$ has NO zeros in $U \setminus \{z\}$.
f conti $\Rightarrow \forall y \in U \setminus \{z\}$, $\exists$ open nbd $V_y \subseteq D$ of y', s.t. $f \neq 0$ on $V_y \Rightarrow y \in D \setminus A$ ✓

$\therefore$ $D = A \cup B$, A, B both open, $A \cap B = \emptyset$ $(B = D \setminus A)$
As $z_0 \in A$ and D is a region, $D = A$ □

COROLLARY

Say f, g are analytic on a region D.
If f and g agree at a set of pts with an acc pt, then $f \equiv g$ on D.

Proof

Set $h := f - g$, then apply thm above □

THEOREM

If f is entire and $f \to \infty$ as $z \to \infty$, then f is a polynomial

Proof

By def, $\forall M \in \mathbb{R}_{>0}$, $\exists \delta$, s.t. $\forall |z| > \delta$, $|f(z)| > M$

Let $M = 1$.
$\therefore$ $\exists \delta$, s.t. $\forall |z| > \delta$, $|f(z)| > 1$

By our assumption, f is NOT a constant

Claim: f has finitely many zeros

Proof

If NOT, by δ, all zeros in f are in $\overline{D(0,\delta)}$, otherwise, $|f(z)| \neq 0$.
As $\overline{D(0,\delta)}$ is compact, $\exists$ acc pt of zeros in $\overline{D(0,\delta)}$ ✓
↳ Suppose not. Then, $\forall x \in \overline{D(0,\delta)}$, $\exists$ an open nbd U_x, s.t. f has no zeros in $U_x \setminus \{x\}$
$U_x \setminus \{x\}$ is an open cover of $\overline{D(0,\delta)} \Rightarrow \exists x_1, \dots, x_n$ s.t. $\{U_{x_i}\}$ is an open cover of $\overline{D(0,\delta)}$ (by cpt)
However, each U_x has at most 1 zero $\Rightarrow \overline{D(0,\delta)}$ has at most n zeros ⁎

cover boundary at δ

By thm, $f \equiv 0$ on $D(0, \delta')$ for all $\delta' > \delta$
However, δ' can extend to ∞

We consider within $\bar{D}(0,\delta)$.
Let $\alpha_1, \dots, \alpha_n$ be zeros of f (counting by multiplicity)
Then, $g(z) := \frac{f(z)}{\prod_{i=1}^{n}(z-\alpha_i)}$ is entire and has no zeros on $\mathbb{C}$

Set $h := \frac{1}{g(z)}$, then h is entire, h has no zeros in $\mathbb{C} \Rightarrow h$ is bounded in disk

By Extended Liouville's Thm, $|h| < A + B|z|^n \ \forall |z| > \delta$ and $\forall |z| \leq \delta \Rightarrow h$ is a poly

However, h has no zeros in $\mathbb{C} \Rightarrow h = \text{const}$
$\therefore \exists c \in \mathbb{C}^\times$ s.t. $f(z) = c \prod_{i=1}^{n} (z - \alpha_i)$ □

REMARK
Say f, g are ana on region D, to check $f \equiv g$, we may apply the theorem above over $\mathbb{R}$ without needing to consider $\mathbb{C}$.

THEOREM (MEAN VALUE THEOREM)
Let D be a region, f analytic on D, $\alpha \in D$.
Then $f(\alpha) =$ mean value of f taken around the boundary of any disk centered at α and contained at D
Proof
By Cauchy-Integral Formula, $f(\alpha) = \frac{1}{2\pi i} \int_{C_\delta(\alpha)} \frac{f(z)}{z-\alpha} dz$

Say $z = \alpha + \delta e^{i\theta}$, $\theta \in [0, 2\pi]$, we get $f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + \delta e^{i\theta}) d\theta$ □

THEOREM (MAXIMUM MODULUS THEOREM)
Say f is nonconst, ana on a region D. Then, $\forall z \in D$ and $\delta \in \mathbb{R}_{\geq 0}$, $\exists$ some $w \in D(z, \delta) \cap D$, s.t. $f(w) > f(z)$
Proof
By MVT, $f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + \delta e^{i\theta}) d\theta$ for small enough δ s.t. $D(z, \delta) \subseteq D$
Then, $|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + \delta e^{i\theta})| d\theta \leq^{①} \frac{1}{2\pi} \max_{\theta \in [0, 2\pi]} |f(z + \delta e^{i\theta})| \cdot 2\pi = \max |f(z + \delta e^{i\theta})|$

$\therefore$ When $\leq^{①}$ has equality, $|f(z + \delta e^{i\theta})| = \max_{\theta \in [0, 2\pi]} |f(z + e^{i\theta})| \ \forall \theta \in [0, 2\pi] \Rightarrow f$ is const on $C_\delta(z) \subseteq D$
By coro, hence f is const on D

However, f is nonconst.
$\therefore f(z) < \max_{\theta \in [0, 2\pi]} |f(z + \delta e^{i\theta})|$ □

THEOREM (MINIMUM MODULUS THEOREM)
Say f is nonconst, ana on a region D, $\forall z \in D$, $f(z) \neq 0$.
Then, f has no interior min points
Proof
$f(z) \neq 0 \ \forall z \in D \Rightarrow g(z) = \frac{1}{f(z)}$ is ana, nonconst on D
Then, by max mod thm, we proved it. □

CAUTION
We can only apply uniqueness thm and its coro when its acc pts $\in D$ (Counterexample: $f(z) = \sin \frac{1}{z}$, $z_n = \frac{1}{n\pi}$)

THEOREM CAN ONLY APPLY TO CIRCLES
Say $\bar{D}$ is a closed disk and f is analytic, nonconst on $\bar{D}$. f assumes its max value at a boundary point z_0. Then, $f'(z_0) \neq 0$
Proof
Suppose $f'(z_0) = z_0$. As f is ana on $\bar{D}$, $\exists \delta$ (conv rad) s.t. $\forall |\zeta| < \delta$, $f(z_0 + \delta) = f(z_0) + f'(z_0)\zeta + \frac{1}{2} f''(z_0) \zeta^2 + \dots$

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Assume $f'(z_0)=0$

Then, $f(z_0+\delta)\approx f(z_0)+\frac{1}{2}f''(z_0)\zeta^2 \Rightarrow |f(z_0+\delta)|=f(z_0+\delta)\cdot\bar{f}(z_0+\delta)=|f(z_0)|^2+\frac{2}{k!}\mathrm{Re}(\bar{f}(z_0)f^{(k)}(z_0)\zeta^k)+\dots$ for some $k\in\mathbb{N}$

From assumption, thus $k\geq 2$.

Let $e^{i\theta}=\frac{\zeta}{|\zeta|}$. Then, $\bar{f}(z_0)f^{(k)}(z_0)=Ae^{i\alpha}\Rightarrow |f(z_0+\zeta)|^2=|f(z_0)|^2+\frac{2}{k!}A|\zeta|^k\cos(\alpha+k\theta)+\dots$

For small enough ζ, $|f(z_0+\zeta)|-|f(z_0)|$ has the same sign as $\boxed{\cos(k\theta+\alpha)}$

As $|f(z_0)|$ is max, hence $|f(z_0+\zeta)|^2-|f(z_0)|^2\leq 0\ \forall z_0+\zeta\in D$.

Notice, $\cos(k\theta+\alpha)\leq 0 \Leftrightarrow \frac{\pi}{2}+2\pi j\leq \alpha+k\theta\leq\frac{3\pi}{2}+2\pi j$ for $0\leq j\leq k-1 \Leftrightarrow \frac{(\frac{\pi}{2}-\alpha)}{k}+\frac{2\pi j}{k}\leq\theta\leq\frac{(\frac{3}{2}\pi-\alpha)}{k}+\frac{2\pi j}{k}$ (☆) $\leftarrow \triangle$ w/ angle $\frac{\pi}{k}$

kinda like they alternate

For a disc, $\exists\zeta$, s.t. $z_0+\zeta$ is NOT in any one of the cones (☆) since $k\geq 2$, $\frac{\pi}{k}\leq\frac{\pi}{2}$. $\times$

REMARK

This argument works for cpt $K\subseteq\mathbb{C}$ s.t. $\forall z_0$ on the boundary of K, K contains a cone $\{z_0+re^{i\theta}\mid\theta\in(\alpha,\beta),\ r\in(0,\varepsilon)\}$ with $\beta-\alpha>\frac{\pi}{2}$

Counterexample of squares:

$f(z)=z^2+i\Rightarrow|f(z)|$ has min 1 at $z=0$, but $f'(0)=0$