9-2-25 (WEEK 1) Shun/翔海(@shun4midx)

Can any two-variable function $f(x,y)$ be re-written into diffable $F(z)$ with $z=x+iy$? No. ($f(x,y)=x$)

POWER SERIES

DEFINITION OF ANALYTIC POLYNOMIAL

If $P(x,y)=\alpha_0+\alpha_1(x+iy)+\dots+\alpha_N(x+iy)^N=\sum_{k=0}^{N}\alpha_k z^k$ for some α_k's $\in\mathbb{C}$, then it is an analytic polynomial

EXAMPLE

$x^2-y^2+2xyi=(x+iy)^2$ is analytic

However, x^2+y^2-2xyi is not (when we set $x^2+y^2-2xyi=\sum\alpha_k(x+iy)^k$, there is a contradiction)

DEFINITION OF PARTIAL DERIVATIVE

Let $f(x,y)=u(x,y)+iv(x,y)$, $u,v\in\mathbb{R}$.

If it exists, then $\begin{cases} f_x(x,y)=u_x(x,y)+iv_x(x,y) \\ f_y(x,y)=u_y(x,y)+iv_y(x,y)\end{cases}$

PROPOSITION

A polynomial $P(x,y)$ is analytic $\Leftrightarrow$ $\underbrace{P_y=iP_x}_{\text{differentiable (C-R eq)}}$

Proof

"$\Rightarrow$": $\exists\alpha_k$'s $\in\mathbb{C}$, $N\in\mathbb{N}$, s.t. $P(x,y)=\sum_{k=0}^{N}\alpha_k(x+iy)^k$

$\Rightarrow P_y=\sum_{k=1}^{N}k\alpha_k(x+iy)^{k-1}i$, $P_x=\sum_{k=1}^{N}k\alpha_k(x+iy)^{k-1}$

$\Rightarrow \therefore P_y=iP_x$ ✓

"$\Leftarrow$": With $Q^k(x,y)=c_0x^k+c_1x^{k-1}y+\dots+c_ky^k$, we can rewrite $P(x,y)=\sum_{k=0}^{N}Q^k(x,y)$

Notice, $Q^k_y=iQ^k_x$ $\forall k$.

We know $Q^k=\sum_{p=0}^{k}c_px^{k-p}y^p$

$\therefore Q^k_y=\underline{\sum_{p=1}^{k}pc_px^{k-p}y^{p-1}=\sum_{p=0}^{k-1}(k-p)c_px^{k-p-1}y^p\,i}=iQ^k_x$

In other words, $\sum_{p=1}^{k}pc_px^{k-p}y^{p-1}=\sum_{p=1}^{k}(k-p+1)c_{p-1}x^{k-p}y^p\,i$

- $p=1$: $ikc_0=c_1\Rightarrow c_1=\binom{k}{1}c_0\,i$
- $p=2$: $2c_2=(k-1)c_1i\Rightarrow c_2=i^2\frac{k(k-1)}{2}c_0$
- $pc_p=(k-p+1)c_{p-1}i\Rightarrow c_p=i^p\binom{k}{p}c_0$

$\therefore Q^k=\sum_{p=0}^{k}i^p\binom{k}{p}c_0x^{k-p}y^p=(x+iy)^k$ $\forall k$

$\therefore P$ is analytic. □

REMARK

Usually we don't write "$P_y=iP_x$", rather:

$\begin{cases} P_y=u_y+iv_y \\ P_x=u_x+iv_x \\ P_xi=-v_x+u_xi\end{cases} \Rightarrow \boxed{\begin{cases} -v_x=u_y \\ u_x=v_y\end{cases}}$

REMARK

A nonconstant analytical polynomial can't be real (since we require $P_y=iP_x$)

DEFINITION

Consider f, a complex-valued function, defined on the neighborhood of $z=z_0$.

We say f is differentiable at $z=z_0$ if $\lim_{h\to0}\frac{f(z_0+h)-f(z_0)}{h}$ exists, where it is denoted as $f'(z)$

(Note: We must consider $|h|\to0$ $\forall h\in\mathbb{C}$)

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EXAMPLE
$f(z)=\bar{z}$

$\frac{f(z+h)-f(z)}{h}=\frac{\bar{h}}{h} \xrightarrow{\mathbb{R}} 1$, $\xrightarrow{i\mathbb{R}} -1$ as $h\to 0$.

$\therefore \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ DNE, i.e. f is not diff

PROPOSITION
If f,g diff at $z=0$, $h:=f+g \Rightarrow h'=f'+g'$. Product and quotient ($g(z_0)\neq 0$) rules also hold.

PROPOSITION
$P(z)=\sum_{k=0}^{N}\alpha_k z^k$ is diff on $\mathbb{C}$, in fact: $P'(z)=\sum_{k=1}^{N}k\alpha_k z^{k-1}$

DEFINITION OF POWER SERIES
A power series is an infinite series in the form $\sum_{k=0}^{\infty}K_k z^k$

LIMSUP
"$\overline{\lim_{n\to\infty}}\, a_n=L$" $\Leftrightarrow \forall \varepsilon>0, \exists N>0$, s.t. $n\geq N \Rightarrow |\sup_{k\geq n} a_k - L|<\varepsilon$

$\Updownarrow$

$L-\varepsilon<\sup_{k\geq n} a_k<L+\varepsilon$

$\forall \varepsilon>0, \exists N$, s.t. $n\geq N \Rightarrow a_k<L+\varepsilon$

$\forall\varepsilon>0, \forall N\ \exists k>N$, s.t. $a_k>L-\varepsilon$

THEOREM
Given the power series $\sum_{k=0}^{\infty}C_k z^k=P(z)$, define $L:=\overline{\lim_{k\to\infty}}|C_k|^{\frac{1}{k}}$, then we have:

(1) $L=0\Rightarrow P(z)$ converges $\forall z\in\mathbb{C}$

(2) $L=\infty\Rightarrow P(z)$ converges only at $z=0$

(3) $0<L<\infty \Rightarrow P(z)$ converges on $|z|<\frac{1}{L}$ and diverges on $|z|>\frac{1}{L}$

Proof

(i) Given any $z\in\mathbb{C}$, $\overline{\lim_{n\to\infty}}|C_k|^{\frac{1}{k}}z=0$

$\therefore$ Take $\varepsilon=\frac{1}{2}$, $\exists N$ s.t. $k>N\Rightarrow |C_k|^{\frac{1}{k}}|z|<\frac{1}{2}\Rightarrow |C_k z^k|<(\frac{1}{2})^k$

$\therefore \sum|C_k z^k|<\sum(\frac{1}{2})^k=1$ ✓

(ii) Consider small $|z|$, $\forall N\in\mathbb{N}, \exists k>N$, s.t. $|C_k|^{\frac{1}{k}}>\frac{1}{|z|}$ $\therefore |C_k z^k|>1$

$\therefore P(z)$ does not converge at z ✓

(iii) Take $R=\frac{1}{L}$, $|z|=R(1-\delta)$, $1>\delta>0$ when $|z|<\frac{1}{L}$

We know $\forall\varepsilon>0, \exists N\in\mathbb{N}$, $n>N$, s.t. $|z|(L-\varepsilon)<\sup_{k\geq n}|C_k|^{\frac{1}{k}}|z|\leq(\varepsilon+L)|z|=1+\varepsilon R(1-\delta)-\delta<1-\frac{\delta}{2}$

$\therefore$ It is abs conv

If $|z|>R$, $\overline{\lim}|C_k|^{\frac{1}{k}}|z|>1\Rightarrow$ for inf values of k, $|C_k z^k|>1\Rightarrow \sum C_k z^k$ div

REMARK
Let $\frac{1}{L}=R$ be the radius of convergence

Then, $\sum C_k z^k$ conv uni for $|z|<R-\delta$

$\because \sum|C_k z^k|\leq\sum|C_k|(R-\delta)^k<\infty$

$\Rightarrow$ On $B(0,R-\delta)$, $\sum C_k z^k$ is conti $\forall\delta>0$

EXAMPLE (evaluating at R)
$\sum_{n=0}^{\infty}nz^n$

We know $\overline{\lim}\, n^{\frac{1}{n}}=1\Rightarrow R=1$

When $|z|=1$, $|nz^n|=n\Rightarrow$ diverge

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$\sum \frac{z^n}{n^2} \ni R=1$

When $z=1$, it conv, similarly $|z|=1$ too.

$\sum \frac{z^n}{n}$, $R=1$

When $|z|=1$, $z\neq 1$ conv

$\sum \frac{z^n}{n!}$ conv $\forall z \in \mathbb{C}$ by ratio test

CAUCHY PRODUCT

Given $P_1(z)=\sum a_k z^k$, $R=R_1$; $P_2(z)=\sum b_k z^k$, $R=R_2$. then $P_1 P_2 = \sum c_k z^k$ where $c_k = \sum_{p=0}^{k} a_p b_{k-p}$

Then, $R_3 \geq \min(R_1, R_2)$

DIFFERENTIATION

THEOREM

Given $P(z)=\sum_{k=0}^{\infty} c_k z^k$, $R>0$, we know $\overline{\lim}|c_k|^{\frac{1}{k}} = R$ and $\overline{\lim}|kc_k|^{\frac{1}{k}} = R$ since $\overline{\lim}|k|^{\frac{1}{k}} = 1$

Then, $P'(z) = \sum_{k=1}^{\infty} kc_k z^{k-1}$ with radius of convergence R

<u>Proof</u>

- $0<R<\infty$: Take $|z|=R-\delta$, $R \geq \delta > 0$

  Then, $\frac{P(z+h)-P(z)}{h} = \frac{1}{h}\sum_{k=1}^{\infty} c_k[(z+h)^k - z^k] = \frac{1}{h}\left[\sum_{k=1}^{\infty} c_k k z^{k-1} h + h \sum_{k=2}^{\infty} c_k b_k\right] = \sum_{k=1}^{\infty} kc_k z^{k-1} + \sum_{k=2}^{\infty} c_k b_k$

Proof continued next time!