12-9-25 (WEEK 15) Shun/翔海(@shun4midx)

THE METHOD OF MOMENT

Say $\sum_{n=0}^{\infty} c_n z^n =: f(z)$, how do we find an ana contn?

CASE 1

$c_n = \int_a^b g(t) t^n dt \Rightarrow f(z) = \sum_{n=0}^{\infty} \left(\int_a^b g(t) t^n dt\right) z^n \overset{|tz|<1:\text{ abs cnv}}{=} \int_a^b \sum_{n=0}^{\infty} g(t)(tz)^n dt \overset{|tz|<1}{=} \int_a^b \frac{g(t)}{1-tz} dt$

Say $h(z) := \int_a^b \frac{g(t)}{1-tz} dt$, then $z \notin [\frac{1}{b}, \frac{1}{a}] \Rightarrow 1-tz \neq 0$

Is $h(z)$ ana on $\mathbb{C} \setminus [\frac{1}{b}, \frac{1}{a}]$? (Morera. Prove cont. then Morera)

LEMMA

Suppose $\varphi(z,t)$ is a continuous function of t $\forall t \in [a,b]$ for fixed z and is analytic for $z \in D$ for fixed t.

Then, $f(t) = \int_a^b \varphi(z,t)\, dt$ is analytic $\forall z \in D$.

Proof

Notice, $\varphi(z,t)$: ana in $z \Rightarrow \varphi(z,t)$: cont. in $z \Rightarrow f(z)$: cont. in z

$\forall$ closed rectangle $\Gamma \subseteq D$,

$$\int_{\partial\Gamma} f(z) dz = \int_{\partial\Gamma} \int_a^b \varphi(z,t)\, dt\, dz \overset{\varphi:\text{ cont. on cpt set } \partial\Gamma \times [a,b]}{=} \int_a^b \underbrace{\int_{\partial\Gamma} \varphi(z,t)\, dz}_{=0\ (\because \varphi:\text{ ana})} dt = \int_a^b 0\, dt = 0$$

$\therefore$ By Morera's Thm, $f(z)$: ana in z. □

COROLLARY

$\int_a^b \frac{g(t)}{1-tz} dt$ is an analytic continuation of f

REMARK

We can replace $[a,b]$ with $[a,\infty]$: We check if we can $\int_{\partial\Gamma} \leftrightarrow \int_a^b$!

EXAMPLE

$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1}$, $|z|<1$

☆ $\frac{1}{n+1} = \int_0^1 t^n dt$

Then, $f(z) = \sum_{n=0}^{\infty} \left(\int_0^1 t^n dt\right) z^n = \int_0^1 \sum_{n=0}^{\infty} (tz)^n dt = \int_0^1 \frac{1}{1-tz} dt$: ana on $\mathbb{C} \setminus [1,\infty)$

CASE 2

$\int_0^{\infty} e^{-nt^2} dt = \frac{\sqrt{\pi}}{2\sqrt{n}}$

Then, $\underline{\sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}} = \frac{2}{\sqrt{\pi}} \left[\sum_{n=1}^{\infty} \left(\int_0^{\infty} e^{-nt^2} dt\right) z^n\right]} = \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \int_0^{\infty} (ze^{-t^2})^n dt \overset{|z|<1}{=} \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{z}{e^{t^2}-z} dt$

↳ Ana when $\underline{z \in \mathbb{C} \setminus [1,\infty] \Rightarrow e^{t^2} - z \neq 0}$

Here, $\int_{\partial\Gamma} \int_0^{\infty} \frac{z}{e^{t^2}-z} dt \overset{\text{If } \int_0^{\infty} \frac{|z|}{|e^{t^2}-z|} dt \text{ exists}}{=} \int_0^{\infty} \int_{\partial\Gamma} \frac{z}{e^{t^2}-z} dt$ $\therefore$ ana cont.

☆ $\frac{1}{n+a} = \int_0^{\infty} e^{-nt} e^{-at} dt$

ANALYTIC CONTINUATION FOR DIRICHLET SERIES (I.E. $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$)

Shun/翔海 (@shun4midx)

Recall Riemann-zeta function is a Dirichlet series, conv at $Re\, z > 2$ ("abscissa")

THEOREM (LANDAU'S THEOREM)

Suppose that $a_n \geq 0\ \forall n$, $b \in \mathbb{R}$ is a boundary point (i.e. f conv $\forall Re(z) > b$). Then, b is a singular point of f.

Proof Idea: If b: regular, then $\exists c < b$ s.t. $f(z)$ conv at $c \Rightarrow f(z)$ conv $\forall Re(z) > c \Rightarrow b$: not bdry pt

Proof

contin ana with b: ana

If b: regular, then $\exists \tilde{f}$: ana at b, s.t. $\tilde{f}|_{Re(z)>b} = f$

We can consider the Taylor expansion of $\tilde{f}$ at some $a > b$.

$\Rightarrow \tilde{f} = \sum_{k=0}^{\infty} c_k (z-a)^k$ (☆), $c_k = \frac{\tilde{f}^{(k)}(a)}{k!} = \frac{f^{(k)}(a)}{k!}$ $\quad \therefore \tilde{f} = \sum_{n=1}^{\infty} \frac{a_n (\log n)^k}{n^a k!}$

As $\tilde{f}$: ana at b, radius of conv of (☆) $> a-b$

In particular, $\exists\, \varepsilon > 0$, s.t.

$$\sum_{k=0}^{\infty} |c_k| |a-b+\varepsilon|^k \overset{\text{conv}}{\underset{a_n \geq 0}{=}} \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{a_n (\log n)^k}{n^a k!} \right) (a-b+\varepsilon)^k$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{a_n}{n^a} \Big[\underbrace{\frac{(\log n)^k}{k!} (a-b+\varepsilon)^k}_{\exp(\log n \cdot (a-b+\varepsilon)) = n^{a-b+\varepsilon}} \Big] = \sum_{n=1}^{\infty} \frac{a_n}{n^{b-\varepsilon}}$$

$\therefore f$: conv at $b - \varepsilon$ □

COROLLARY

$\exists \tilde{f}$: entire, $\tilde{f}|_{Re(z)>b} = f$ $\qquad$ f: conv throughout

If $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ with $a_n \geq 0$ and can be analytically continued to the entire complex plane, then it converges throughout the complex plane

GAMMA FUNCTION

CONSTRUCTION

$$f(z) = \int_0^{\infty} e^{-t} t^z \, dt \overset{\text{IBP}}{=} z f(z-1)$$

Notice, $|f(z)| \leq \int_0^{\infty} |e^{-t} t^z| \, dt = \int_0^{\infty} e^{-t} t^{Re(z)} \Rightarrow f(z)$: well-def $\forall Re(z) > -1$

$\therefore \Gamma(z) := f(z-1)$ is well-def $\forall Re(z) > 0$

ANALYTIC CONTINUATION OF Γ

As $\Gamma(z+1) = z\Gamma(z)$, "ana" can be checked via diffability.

① Note, $\Gamma(z+1) = z\Gamma(z) \Rightarrow \Gamma(z) = \frac{\Gamma(z+1)}{z} \Rightarrow$ simple pole at $z=0$

② $\Gamma(z+1)$ is defined $\forall Re(z) > -1$

$\therefore$ Define $\tilde{\Gamma}_1(z) := \begin{cases} \frac{\Gamma(z+1)}{z}, & -1 < Re(z) < 0 \\ \Gamma(z), & Re(z) > 0 \end{cases}$ analytic

Check: "$\tilde{\Gamma}_1(z)$: ana except for $z=0$"

By Morera, it suffices to check $\tilde{\Gamma}_1(z)$ is cont:

For $y \neq 0$, $\lim_{z \to iy} \tilde{\Gamma}_1(z) = \lim_{z \to iy} \frac{\Gamma(z+1)}{z} = \frac{\Gamma(iy+1)}{iy} = \Gamma(iy)$

$\therefore \tilde{\Gamma}_1(z)$ is cont. at $z = iy$, $\forall y \neq 0$ ✓

Continue this process, e.g.

$\tilde{\Gamma}_2(z) = \begin{cases} \tilde{\Gamma}_1(z), & Re(z) > -1 \\ \frac{\tilde{\Gamma}_1(z+1)}{z}, & -2 < Re(z) < -1 \end{cases}$ → simple pole at $z=-1$

etc...

In this way, Γ can be extended analytically to $\mathbb{C} \setminus \{n \mid n \in \mathbb{Z}_{\leq 0}\}$

RESIDUE OF Γ

EXAMPLE

$$\mathrm{Res}(\Gamma(z), -1) = \mathrm{Res}(\tilde{\Gamma}_2(z), -1) = \lim_{z \to -1} (z+1)\tilde{\Gamma}_2(z) = \lim_{z \to -1} \cancel{(z+1)} \frac{\Gamma(z+2)}{z\cancel{(z+1)}} = -1$$

Moreover, $\mathrm{Res}(\Gamma(z), -n) = \frac{(-1)^k}{k!}$

SPECIAL EQUALITIES

$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ (i.e. Γ has no zeros)

ZETA FUNCTION (VERY HANDWAVY)

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \Rightarrow \frac{1}{2^z}\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{(2n)^z} \Rightarrow \left(1 - \frac{1}{2^z}\right)\zeta = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^z} \Rightarrow \text{etc...}$$

$$\zeta(z) = \prod_{p:\text{prime}} \left(1 - \frac{1}{p^z}\right)^{-1} = \prod_{p:\text{prime}} \left(1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \dots\right)$$

EQUATIONS

$\exists\, g(z)$: entire, s.t. $\zeta(z) = \frac{1}{\Gamma(z)}\left[g(z) + \frac{1}{z-1} + \frac{A_0}{z} + \frac{A_1}{z+1} + \dots\right]$

80%~90% OF QUESTIONS IN THE FINAL WILL BE FROM THE HWs THIS SEM