CompAna Midterm

Shun/翔海 (@shun4midx)

9-2-25 (WEEK 1) Shun/翔海 (@shun4midx)

Can any two-variable function $f(x,y)$ be re-written into diffable $F(z)$ with $z=x+iy$? No. ($f(x,y)=x$)

POWER SERIES

DEFINITION OF ANALYTIC POLYNOMIAL

If $P(x,y)=\alpha_0+\alpha_1(x+iy)+\ldots+\alpha_N(x+iy)^N=\sum_{k=0}^{N}\alpha_k z^k$ for some α_k's $\in\mathbb{C}$, then it is an analytic polynomial

EXAMPLE

$x^2-y^2+2xyi=(x+iy)^2$ is analytic

However, x^2+y^2-2xyi is not (when we set $x^2+y^2-2xyi=\sum\alpha_k(x+iy)^k$, there is a contradiction)

DEFINITION OF PARTIAL DERIVATIVE

Let $f(x,y)=u(x,y)+iv(x,y)$, $u,v\in\mathbb{R}$.

If it exists, then $\begin{cases} f_x(x,y)=u_x(x,y)+iv_x(x,y) \\ f_y(x,y)=u_y(x,y)+iv_y(x,y)\end{cases}$

PROPOSITION

A polynomial $P(x,y)$ is analytic $\Leftrightarrow$ $\underbrace{P_y=iP_x}_{}$ differentiable (C-R eq)

Proof

"$\Rightarrow$": $\exists\,\alpha_k$'s $\in\mathbb{C}$, $N\in\mathbb{N}$, s.t. $P(x,y)=\sum_{k=0}^{N}\alpha_k(x+iy)^k$

$\Rightarrow P_y=\sum_{k=1}^{N}k\alpha_k(x+iy)^{k-1}i$; $P_x=\sum_{k=1}^{N}k\alpha_k(x+iy)^{k-1}$

$\Rightarrow \therefore P_y=iP_x$ ✓

"$\Leftarrow$": With $Q^k(x,y)=c_0x^k+c_1x^{k-1}y+\ldots+c_ky^k$, we can rewrite $P(x,y)=\sum_{k=0}^{N}Q^k(x,y)$

Notice, $Q^k_y=iQ^k_x$ $\forall k$.

We know $Q^k=\sum_{p=0}^{k}c_px^{k-p}y^p$

$\therefore Q^k_y=\sum_{p=1}^{k}pc_px^{k-p}y^{p-1}=\sum_{p=0}^{k-1}(k-p)c_px^{k-p-1}y^pi=iQ^k_x$

In other words, $\sum_{p=1}^{k}pc_px^{k-p}y^{p-1}=\sum_{p=1}^{k}(k-p+1)c_{p-1}x^{k-p}y^pi$

- $p=1$: $ikc_0=c_1\Rightarrow c_1=\binom{k}{0}c_0i$
- $p=2$: $2c_2=(k-1)c_1i\Rightarrow c_2=i^2\frac{k(k-1)}{2}c_0$
- $pc_p=(k-p+1)c_{p-1}i\Rightarrow c_p=i^p\binom{k}{p}c_0$

$\therefore Q^k=\sum_{p=0}^{k}i^p\binom{k}{p}c_0x^{k-p}y^p=(x+iy)^k$ $\forall k$

$\therefore P$ is analytic. □

REMARK

Usually we don't write "$P_y=iP_x$", rather:

$\begin{cases} P_y=u_y+iv_y \\ P_x=u_x+iv_x \\ P_xi=-v_x+u_xi \end{cases}\Rightarrow\begin{cases}-v_x=u_y \\ u_x=v_y\end{cases}$

REMARK

A nonconstant analytical polynomial can't be real (since we require $P_y=iP_x$)

DEFINITION

Consider f, a complex-valued function, defined on the neighborhood of $z=z_0$.

We say f is differentiable at $z=z_0$ if $\lim_{h\to0}\frac{f(z_0+h)-f(z_0)}{h}$ exists, where it is denoted as $f'(z)$

(Note: We must consider $|h|\to0$ $\forall h\in\mathbb{C}$)

Shun/翔海(@shun4midx)

EXAMPLE

$f(z)=\bar{z}$

$\frac{f(z+h)-f(z)}{h}=\frac{\bar{h}}{h} \xrightarrow{\mathbb{R}} 1,\ \xrightarrow{i\mathbb{R}} -1$ as $h\to 0$.

$\therefore \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ DNE, i.e. f is not diff

PROPOSITION

If f,g diff at $z=0$, $h:=f+g \Rightarrow h'=f'+g'$. Product and quotient $(g(z_0)\neq 0)$ rules also hold.

PROPOSITION

$P(z)=\sum_{k=0}^{N}\alpha_k z^k$ is diff on $\mathbb{C}$, in fact: $P'(z)=\sum_{k=1}^{N}k\alpha_k z^{k-1}$

DEFINITION OF POWER SERIES

A power series is an infinite series in the form $\sum_{k=0}^{\infty}K_k z^k$

LIMSUP

"$\overline{\lim_{n\to\infty}}a_n=L$" $\Leftrightarrow \forall\varepsilon>0,\ \exists N>0$, s.t. $n\geq N\Rightarrow|\sup_{k\geq n}a_k-L|<\varepsilon$

$\Updownarrow$

$L-\varepsilon<\sup_{k\geq n}a_k<L+\varepsilon$

$\forall\varepsilon>0,\ \exists N$, s.t. $n\geq N\Rightarrow a_k<L+\varepsilon$

$\forall\varepsilon>0,\ \forall N\ \exists k>N$, s.t. $a_k>L-\varepsilon$

THEOREM

Given the power series $\sum_{k=0}^{\infty}c_k z^k=P(z)$, define $L:=\overline{\lim_{k\to\infty}}|c_k|^{\frac{1}{k}}$, then we have:

(1) $L=0\Rightarrow P(z)$ converges $\forall z\in\mathbb{C}$

(2) $L=\infty\Rightarrow P(z)$ converges only at $z=0$

(3) $0<L<\infty\Rightarrow P(z)$ converges on $|z|<\frac{1}{L}$ and diverges on $|z|>\frac{1}{L}$

Proof

(i) Given any $z\in\mathbb{C}$, $\overline{\lim_{n\to\infty}}|c_k|^{\frac{1}{k}}z=0$

$\therefore$ Take $\varepsilon=\frac{1}{2}$, $\exists N$ s.t. $k>N\Rightarrow|c_k|^{\frac{1}{k}}|z|<\frac{1}{2}\Rightarrow|c_kz^k|<(\frac{1}{2})^k$

$\therefore\ \sum|c_kz^k|<\sum(\frac{1}{2})^k=1$ ✓

(ii) Consider small $|z|$, $\forall N\in\mathbb{N}$, $\exists k>N$, s.t. $|c_k|^{\frac{1}{k}}>\frac{1}{|z|}$ $\therefore|c_kz^k|>1$

$\therefore$ $P(z)$ does not converge at z ✓

(iii) Take $R=\frac{1}{L}$, $|z|=R(1-\delta)$, $1>\delta>0$ when $|z|<\frac{1}{L}$

We know $\forall\varepsilon>0,\ \exists N\in\mathbb{N}$, $n>N$, s.t. $|z|(L-\varepsilon)<\sup_{k\geq n}|c_k|^{\frac{1}{k}}|z|\leq(\varepsilon+L)|z|=1+\varepsilon R(1-\delta)-\delta<1-\frac{\delta}{2}$

$\therefore$ It is abs conv

If $|z|>R$, $\overline{\lim}|c_k|^{\frac{1}{k}}|z|>1\Rightarrow$ for inf values of k, $|c_kz^k|>1\Rightarrow\sum c_kz^k$ div

REMARK

Let $\frac{1}{L}=R$ be the radius of convergence

Then, $\sum c_kz^k$ conv uni for $|z|<R-\delta$

$\cdot\ \sum|c_kz^k|\leq\sum|c_k|(R-\delta)^k<\infty$

$\Rightarrow$ On $B(0,R-\delta)$, $\sum c_kz^k$ is conti $\forall\delta>0$

EXAMPLE (evaluating at R)

$\sum_{n=0}^{\infty}nz^n$

We know $\overline{\lim}\,n^{\frac{1}{n}}=1\Rightarrow R=1$

When $|z|=1$, $|nz^n|=n\Rightarrow$ diverge

Shun/翔海(@shun4midx)

$\sum \frac{z^n}{n^2} \ni R=1$

When $z=1$, it conv, similarly $|z|=1$ too.

$\sum \frac{z^n}{n}$, $R=1$

When $|z|=1$, $z\neq 1$ conv

$\sum \frac{z^n}{n!}$ conv $\forall z \in \mathbb{C}$ by ratio test

CAUCHY PRODUCT

Given $P_1(z)=\sum a_k z^k$, $R=R_1$; $P_2(z)=\sum b_k z^k$, $R=R_2$. then $P_1P_2=\sum c_k z^k$ where $c_k=\sum_{p=0}^{k} a_p b_{k-p}$

Then, $R_3 \geq \min(R_1, R_2)$

DIFFERENTIATION

THEOREM

Given $P(z)=\sum_{k=0}^{\infty} c_k z^k$, $R>0$, we know $\overline{\lim}|c_k|^{\frac{1}{k}}=R$ and $\overline{\lim}|kc_k|^{\frac{1}{k}}=R$ since $\overline{\lim}|k|^{\frac{1}{k}}=1$

Then, $P'(z)=\sum_{k=1}^{\infty} kc_k z^{k-1}$ with radius of convergence R

Proof

- $0<R<\infty$: Take $|z|=R-\delta$, $R\geq\delta>0$

  Then, $\frac{P(z+h)-P(z)}{h}=\frac{1}{h}\sum_{k=1}^{\infty} c_k[(z+h)^k-z^k]=\frac{1}{h}\left[\sum_{k=1}^{\infty} c_k k z^{k-1} h + h\sum_{k=2}^{\infty} c_k b_k\right]=\sum_{k=1}^{\infty} kc_k z^{k-1}+\sum_{k=2}^{\infty} c_k b_k$

Proof continued next time!

Shun/翔海 (@shun4midx)

THEOREM

Given power series $P(z)=\sum c_k z^k$, radius of convergence R, then $P'(z)$ exists on $|z|<R$, $P'(z)=\sum k c_k z^{k-1}$

Proof

For $0<R<\infty$,

Let $|z|=R-\delta$, $R\geq\delta>0$. WLOG, consider $|h|<\frac{\delta}{2}$, and consider $\frac{P(z+h)-P(z)}{h}$ vs $\sum k c_k z^{k-1}$

Then, $\frac{P(z+h)-P(z)}{h}=\frac{1}{h}\sum c_k(\underbrace{(z+h)^k-z^k}_{\sum\binom{k}{p}h^{k-p}z^p})=\sum_{k=1}^{\infty}kc_kz^{k-1}+\underbrace{\sum_{k=2}^{\infty}c_kb_k}_{☆}$, $b_k=\sum_{p=2}^{k}\binom{k}{p}h^{p-1}z^{k-p}$

If $|z|=0$, $b_k=h^{k-1}\Rightarrow ☆=\sum c_k h^{k-1}<\infty$, $☆\to 0$ as $h\mapsto 0$. ✓

If $|z|\neq 0$, $\binom{k}{p}=\binom{k}{p-2}\cdot\frac{(k-p+2)(k-p+1)}{p(p-1)}\leq\binom{k}{p-2}k^2$

Then, $|b_k|\leq\frac{|h|}{|z|^2}k^2\sum_{p=2}^{k}\binom{k}{p-2}|h|^{p-2}|z|^{k-(p-2)}$, note $h\in\mathbb{C}$

$\leq\frac{|h|}{|z|^2}k^2\sum_{j=0}^{k}\binom{k}{j}|h|^j|z|^{k-j}$, $j=p-2$

$=\frac{|h|}{|z|^2}k^2(|z|+|h|)^k$

$\leq\frac{|h|}{|z|^2}k^2(R-\frac{\delta}{2})^k$

Hence, $\left|\sum_{k=2}^{\infty}c_kb_k\right|\leq\sum_{k=2}^{\infty}|c_k||b_k|=\frac{|h|}{|z|^2}\underbrace{\sum_{k=2}^{\infty}k^2|c_k|(R-\frac{\delta}{2})^k}_{<\infty}\longrightarrow 0$ ✓

Remaining case is simple for $R=\infty$.

EXAMPLE

$\sum_{k=0}^{\infty}\frac{z^k}{k!}$, $R=\infty$

$\downarrow\frac{d}{dz}$

$\sum_{k=1}^{\infty}\frac{z^{k-1}}{(k-1)!}=\sum_{k=0}^{\infty}\frac{z^k}{k!}$

$\because f'=f$

$\therefore\sum_{k=0}^{\infty}\frac{z^k}{k!}=ce^z$

COROLLARY

Power series are smooth in their domain of conv.

COROLLARY

$f(z)=\sum_{k=0}^{\infty}c_kz^k$, $R>0\Rightarrow c_k=\frac{f^{(k)}(0)}{k!}$

THEOREM (UNIQUENESS)

If $\exists\{z_n\}_n\longrightarrow 0$ and $\sum c_kz_n^k=0$, then $c_k=0\ \forall k$

Proof

$P(0)=\lim_{n\to\infty}P(z_n)=0\Rightarrow c_0=0$

Consider $g(z)=\frac{f(z)}{z}$, with the same radius of convergence as $f(z)$

Then, $g(0)=\lim_{n\to\infty}\frac{f(z_n)}{z_n}=0\Rightarrow c_1=0$

$\therefore$ By induction on n, $c_n=0\ \forall n$. □

COROLLARY

If $\sum a_kz^k$ and $\sum b_kz^k$ agree on $\{z_n\}_n$ as $n\to\infty$, then $a_k=b_k\ \forall k$

Proof

Simply consider $\sum(a_k-b_k)z^k=0$.

ANALYTIC FUNCTIONS

We write $z \in \mathbb{C}$, $z = x+iy$, $x, y \in \mathbb{R}$

Then, $f: \mathbb{C} \longrightarrow \mathbb{C}$ for $f(z) = u(z) + iv(z)$, $u, v: \mathbb{C} \longrightarrow \mathbb{R}$
$f(x,y) = u(x,y) + iv(x,y)$

PROPOSITION 1

If $f = u + iv$ is differentiable at z, then f_x, f_y exist and satisfy the Cauchy-Riemann Equation: $f_y = if_x$

Proof

By def, f is diff $\Rightarrow \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists.

(i) As $h \to 0$ along the real axis, limit $= \lim_{\xi \to 0} \frac{f(x+\xi, y) - f(x,y)}{\xi} = f_x$

(ii) As $h \to 0$ along the imaginary axis, limit $= \lim_{\xi \to 0} \frac{f(x, y+\xi) - f(x,y)}{\xi i} = \frac{f_y}{i}$ (since change in y as ξ means change in z as ξi)

$\therefore$ $f_y = if_x$ □

QUESTION: IF f_x, f_y EXIST AT A POINT z, AND $f_y = if_x$, DOES IT MEAN f IS DIFFERENTIABLE?

COUNTEREXAMPLE

$$f(z) = \begin{cases} \frac{xy(x+iy)}{x^2+y^2}, & z \neq 0 \quad \left(\text{i.e. } xy \cdot \frac{z}{|z|^2}\right) \\ 0, & z = 0 \Leftrightarrow (x,y) = 0 \end{cases}$$

We notice $f(z) = 0$ on both x-axis and y-axis $\Rightarrow f_x(0) = f_y(0) = 0$

However, along $y = ax$ $(a \neq 0)$, we get: $f(x, ax) = \frac{a(1+ia)}{1+a^2} x \Rightarrow \lim_{x \to 0} \frac{f(x, ax)}{x + axi} = \frac{a}{1+a^2}$

$\therefore \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ DNE □

Note: If we require continuity, then the statement would have held true

PROPOSITION

Suppose that f_x, f_y exist in a nbd of z and are conti at z. If f satisfies the Cauchy-Riemann Equation, then f is differentiable

Proof

Say $z = x+iy$, $h = \xi + i\eta$, and $f(z) = u(z) + iv(z)$

$$\frac{f(z+h) - f(z)}{h} = \frac{[u(x+\xi, y+\eta) - u(x,y)]}{\xi + i\eta} + i\frac{[v(x+\xi, y+\eta) - v(x,y)]}{\xi + i\eta}$$

By MVT with "$-u(x+\xi, y) + u(x+\xi, y)$" and "$-v(x+\xi, y) + v(x+\xi, y)$",

$$= \frac{\eta}{\xi + i\eta}\left[\frac{u(x+\xi, y+\eta) - u(x+\xi, y)}{\eta} + i\frac{v(x+\xi, y+\eta) - v(x+\xi, y)}{\eta}\right] + \frac{\xi}{\xi + i\eta}\left[\frac{u(x+\xi, y) - u(x,y)}{\xi} + i\frac{v(x+\xi, y) - v(x,y)}{\xi}\right]$$

$$= \frac{\eta}{\xi + i\eta}[u_y(x+\xi, y+\theta_1\eta) + iv_y(x+\xi, y+\theta_2\eta)] + \frac{\xi}{\xi + i\eta}[u_x(x+\theta_3\xi, y) + iv_x(x+\theta_4\xi, y)]$$

We know $0 < \theta_k < 1$, $\left|\frac{\eta}{\xi + i\eta}\right| = \left|\frac{Re(h)}{h}\right| \leq 1$, $\left|\frac{\xi}{\xi + i\eta}\right| = \left|\frac{Im(h)}{h}\right| \leq 1$

Claim: $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f_x(z)$

Proof

We know $f_x(z) = \frac{\xi + i\eta}{\xi + i\eta} f_x(z)$

By C-R eq, $f_x(z) = \frac{\xi}{\xi + i\eta} f_x(z) + \frac{\eta}{\xi + i\eta} f_y(z)$

As f_x, f_y are conti,

$$\frac{f(z+h) - f(z)}{h} - f_x(z) = \frac{\eta}{\xi + i\eta}[u_y(x+\xi, y+\theta_1\eta) - u_y(x,y) + i(v_y(x+\xi, y+\theta_2\eta) - v_y(x,y))] + \frac{\xi}{\xi + i\eta}[u_x(x+\theta_3\xi, y) - u_x(x,y) + i(v_x(x+\theta_4\xi, y) - v_x(x,y))]$$

$\longrightarrow 0$ as $h \to 0$, i.e. $\xi, \eta \longrightarrow 0$ □

$\therefore$ f is diffable and $f'(z) = f_x(z)$

DEFINITION

f is analytic at z if f is diffable in a nbd of z.
Similarly, f is analytic on a set S if f is diff at all pts of some open set containing S.

DEFINITION

Let S, T be open sets of $\mathbb{C}$, and f be a 1-1 function on S with $f(S)=T$.
We say g is the inverse of f on T if $f(g(z))=z$ $\forall z\in T$.
We say g is the inverse of f at z if $\exists$ open nbd U of z, s.t. g is the inverse of f on U.

Remark: g is also 1-1

PROPOSITION

Suppose that g is the inverse of f at z_0 and g is continuous there. If f is diffable at $g(z_0)$ and if $f'(g(z_0))\neq 0$, then g is diffable at z_0 and $g'(z_0)=\frac{1}{f'(g(z_0))}$

Proof

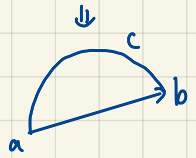
$$\frac{g(z+h)-g(z)}{h} = \frac{g(z+h)-g(z)}{f(g(z+h))-f(g(z))} = \left(\frac{f(g(z+h))-f(g(z))}{g(z+h)-g(z)}\right)^{-1} = \frac{1}{f'(g(z_0))} \quad \square$$

LINE INTEGRALS

Let $f(t)=u(t)+iv(t)$, $z(t)=x(t)+iy(t)$.
We consider curves as such:

$$\overset{t}{\mathbb{R}^1} \xrightarrow{C} \mathbb{R}^n$$

$\Downarrow$



We say $\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

DEFINITION

(i) Let $z(t)=x(t)+iy(t)$, $a\leq t\leq b$. The curve determined by $z(t)$ is called piecewise differentiable and we set $\dot{z}(t)=x'(t)+iy'(t)$ if x, y are continuous on $[a,b]$ and are continuously differentiable on each subinterval $[a,x_1], [x_1,x_2], \dots, [x_{n-1},b]$ of some partition of $[a,b]$

(ii) The curve is said to be smooth, if $\dot{z}(t)\neq 0$ except at finitely many points.

In the following, we assume our curves are smooth.

DEFINITION

Say C is a smooth curve $\subseteq \mathbb{C}$, where $z(t)=x(t)+iy(t)$
Then, $\int_C f(z)\,dz = \int_a^b f(z(t))\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt$

DEFINITION

Let C_1, C_2 be smooth curves $\subseteq \mathbb{C}$, where $C_1: z(t)$, $a\leq t\leq b$ and $C_2: w(t)$, $c\leq t\leq d$.
C_1 and C_2 are smoothly equivalent if $\exists$ 1-1 C^1 mapping $\lambda: [c,d] \longrightarrow [a,b]$ s.t. $w(t)=z(\lambda(t))$
(By def, this is provably an equivalence relation)

We denote smoothly equivalent with $C_1 \underset{sm}{\sim} C_2$.

Shun/翔海(@shun4midx)

PROPOSITION

If $C_1 \underset{sm}{\simeq} C_2$, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

Proof

We set $f(z)=u(z)+iv(z)$, $z=x(t)+iy(t)$

Then, $\int_{C_1} f\,dz = \int_a^b f(z(t))\dot{z}(t)dt = \int_a^b [u(z)x'(t)-v(z)y'(t)]dt + i\int_a^b [u(z)y'(t)+v(z)x'(t)]dt$

With $\int_c^d u(z(\lambda(t)))x'(\lambda(t))\lambda'(t) = \int_a^b u(z(t))x'(t)dt$, we can prove the equation □

APPLICATIONS OF CR-EQUATION

DEFINITION

$D\subseteq\mathbb{C}$ is called a region is open connected

Note, D: region $\Rightarrow x,y\in D$, $\exists$ a curve consisting of vertical and horizontal line segments that connect.

Proof

For $x\in D$, say $U_x := \{y\in D \mid x \text{ connects to } y \text{ via vertical/horizontal line segments that connect}\}$

① "U_x is open":

For $y\in U_x\subseteq D$, D is open $\Rightarrow \exists$ open disk $B_\delta(y)\subseteq D$

$\because \forall a\in B_\delta(y)$, a can be connected to y by

$\therefore x \sim y \sim a$ ✓

② "$D\setminus U_x$ is open":

For $y\in D\setminus U_x$, D is open $\Rightarrow \exists$ open disk $B_\delta(y)\subseteq D \Rightarrow B_\delta(y)\cap U_x=\emptyset \Rightarrow B_\delta(y)\subseteq D\setminus U_x$ ✓

$\therefore$ ① + ② + D is connected $\Rightarrow D=U_x$

PROPOSITION

If $f=u+iv$ is analytic on a region D and u is constant, then f is constant

Proof

u is const $\Rightarrow u_x=u_y=0$

By CR-eq, $v_x=v_y=0$

As D is a region, thus $\forall a,b\in D$, $\exists$ connecting a and b

$\Rightarrow f(a)=f(b) \Rightarrow f$ is const □

PROPOSITION

If f is analytic on a region D, and if $|f|$ is constant on D, then f is constant

Proof

$|f|=0 \Rightarrow f=0$ ✓

If $|f|\neq 0$, $|f|=C>0 \Rightarrow u^2+v^2=C^2$

$\Rightarrow 2uu_x+2vv_x=0$; $2uu_y+2vv_y=0$

By CR-eq, $uu_x-vu_y=0$; $uu_y+vu_x=0 \Rightarrow (u^2+v^2)u_x=0 \Rightarrow u_x=0$. Similarly, we get $u_y=0$.

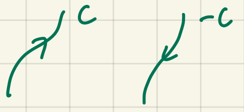
As + prop above, thus this prop is true. □

LINE INTEGRALS (CONTINUED)

DEFINITION

Let C be a curve defined by $z(t)=x(t)+iy(t)$, $t\in[a,b]$.
Then, $-C$ is a curve defined by $w(t)=z(a+b-t)$

In short, it is as follows:



PROPOSITION

$\int_{-C} f(z)dz = -\int_C f(z)dz$

PROPOSITION

Let C be a smooth curve, and f,g be continuous functions on C. Say, $\alpha\in\mathbb{C}$.
(i) $\int_C (f(z)+g(z))dz = \int_C f(z)dz + \int_C g(z)dz$
(ii) $\int_C \alpha f(z)dz = \alpha\int_C f(z)dz$

In other words, we say $\int_C(\cdot)dz$ is linear

EXAMPLE

Say $f(z)=\frac{1}{z}$, $C: R(\cos t+i\sin t)$, $t\in[0,2\pi]$
Then, $\int_C f(z)dz = \int_0^{2\pi}\frac{1}{\cancel{R}(\cos t+i\sin t)}\cancel{R}(-\sin t+i\cos t)\,dt$
$=\int_0^{2\pi} e^{-it}(-e^{i(t-\frac{\pi}{2})})\,dt$
$=\int_0^{2\pi} i\,dt$
$=2\pi i$

LEMMA

Let $t\in\mathbb{R}$, $G(t)$ be a continuous complex-valued function. Then, $\left|\int_a^b G(t)dt\right| \leq \int_a^b |G(t)|dt$
($\alpha<<\beta := |\alpha|\leq|\beta|$, $\alpha,\beta\in\mathbb{C}$)

Proof

Set $\int_a^b G(t)dt = Re^{i\theta}$, $\theta\in\mathbb{R}$, $R\in\mathbb{R}_{\geq 0}$
Then, $R=\left|\int_a^b G(t)dt\right| = \int_a^b e^{-i\theta}G(t)dt = \int_a^b A(t)dt + \cancelto{0}{i\int_a^b B(t)dt}$ $(e^{i\theta}G(t)=A(t)+iB(t))$
$\therefore R=\int_a^b A(t)dt \leq \int_a^b |A(t)|dt \leq \int_a^b |e^{-i\theta}G(t)|dt = \int_a^b |G(t)|dt$ □

PROPOSITION (ML-FORMULA)

Let C be a smooth curve of length L, and f be cont. on C and $\underline{f<<M}$ ($|f|\leq M$) throughout C. Then, $\int_C f(z)dz << ML$

Proof

Let C be $z(t)=x(t)+iy(t)$, $t\in[a,b]$.
Then, $\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt$

By the prev lemma, $\int_C f(z)dz << \int_a^b |f(z(t))|\overbrace{|\dot{z}(t)|}^{\text{arc length}}dt \leq M\int_a^b |\dot{z}(t)|dt = ML$ □

EXAMPLE (FOR WHY ML IS THE TIGHT BOUND)

For $f(z)=\frac{1}{z}$, $C: \cos\theta+i\sin\theta$, $\int_C f(z)=2\pi i \Rightarrow \left|\int_C f(z)\right| = 2\pi = ML$

Shun/翔海 (@shun4midx)

PROPOSITION

Suppose $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ unif on a smooth curve C. Then, $\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz$

Proof

$f_n \to f$ uniformly on C: "Given $\varepsilon>0$, $\exists N$, s.t. $\forall n \geq N$, $|f_n(z)-f(z)| < \varepsilon$ $\forall z \in C$."

So, $\left|\int_C f_n(z)dz - \int_C f(z)dz\right| = \left|\int_C (f_n-f)(z)dz\right| \overset{ML}{<} \varepsilon \cdot \text{len}(C)$ $\forall n \geq N$

$\therefore$ By def, $\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz$ □

PROPOSITION

Let F be an analytic function, $f = F'(z)$, and a smooth curve $C: z(t) = x(t) + iy(t)$, $t \in [a,b]$

Then, $\int_C f(z)dz = F(z(b)) - F(z(a))$

Proof

Let $\gamma(t) := F(z(t)) = A(t) + iB(t)$

Hence, $\dot{\gamma}(t) \overset{\text{smooth}}{=} \lim_{h\to 0} \frac{F(z(t+h))-F(z(t))}{z(t+h)-z(t)} \cdot \frac{z(t+h)-z(t)}{h} = F'(z(t))\,\dot{z}(t)$

Then, $\int_C f(z)dz = \int_a^b F'(z(t))\,\dot{z}(t)dt = \int_a^b \dot{\gamma}(t)dt = \gamma(b) - \gamma(a)$ □

DEFINITION

(i) A curve is closed if its initial and terminal points coincide.

(ii) C is a simple closed curve with $t \in [a,b]$ if $z(t_1) = z(t_2)$ with $t_1 < t_2$, then $t_1 = a$ and $t_2 = b$

DEFINITION

The boundary of a rectangle is the simple closed curve in the counterclockwise direction

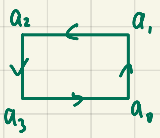
DEFINITION

f is an entire function if f is analytic on $\mathbb{C}$

LEMMA

If f is a linear function, i.e. $f = az+b$, $a,b \in \mathbb{C}$, Γ is the boundary of a rectangle, then $\int_\Gamma f(z)dz = 0$

Proof



Say $\Gamma: z(t)$, $a = a_0 \leq t \leq b = a_3$, and $f = F'(z) \Rightarrow F := \frac{a}{2}z^2 + bz$ (analytic on $\mathbb{C}$)

Hence, we can deduce $\int_\Gamma f(z)dz = \int_\Gamma F'(z)dz = F(z(b)) - F(z(a)) = 0$ □ ($\because z(b) = z(a)$)

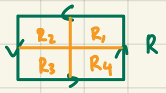
THEOREM (RECTANGLE THEOREM)

Let f be an entire function, and Γ as above, then $\int_\Gamma f(z)dz = 0$

Proof

Let $I = \int_\Gamma f(z)dz$. Assume $f \neq 0$, otherwise $f \equiv 0 \Rightarrow I = 0$.

We divide R as follows:



Then, $\exists$ one of R_i s.t. $\left|\int_{\Gamma_i} f(z)dz\right| \geq \frac{|I|}{4}$, where Γ_i is the boundary of R_i.

Set $R^{(1)}$ to be such an R_i.

Continuing this process, we get $R^{(1)} \supseteq R^{(2)} \supseteq \ldots$ Let $z_0 \in \bigcap_{i=1}^{\infty} R^{(i)}$.

As f is an entire function, hence f is analytic at z_0

By def, $\forall \varepsilon>0, \exists \delta>0$, s.t. $|h|<\delta \Rightarrow \left|\frac{f(z_0+h)-f(z_0)}{h} - f'(z_0)\right| < \varepsilon$

$\therefore$ We see $f(z) = f(z_0) + f'(z_0)(z-z_0) + \varepsilon(z)(z-z_0)$, where $|\varepsilon(z)| \le \varepsilon$.

$\to 0$ (from lemma since linear)

We choose N s.t. $\forall n \ge N$, $|z-z_0|<\delta \Rightarrow \int_{\Gamma^{(n)}} f(z)\,dz = \cancel{\int_{\Gamma^{(n)}} [f(z_0)+f'(z_0)(z-z_0)]\,dz} + \int_{\Gamma^{(n)}} \varepsilon(z)(z-z_0)\,dz$

We know $|\Gamma^{(n)}| = \frac{4s}{2^n}$, so $|\varepsilon(z)(z-z_0)| < \varepsilon \cdot \frac{\sqrt{2}s}{2^n} \Rightarrow$ By ML formula, $\int_{\Gamma^{(n)}} f(z)\,dz << \varepsilon \frac{4\sqrt{2}s^2}{4^n}$

By our assumption, $\left|\int_{\Gamma^{(n)}} f(z)\,dz\right| \ge \frac{|I|}{4^n}$, hence $|I| \le \varepsilon \cdot 4\sqrt{2}s^2 \ \forall \varepsilon>0$, i.e. $I=0$ □

THEOREM (INTEGRAL THEOREM)

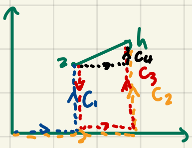
If f is entire, then f is everywhere the derivative of an analytic function. That is, $\exists$ an entire F, s.t. $F'(z)=f(z)\ \forall z$

Proof

Consider $F(z) = \int_C f(\eta)\,d\eta$ where C: $0 \to \mathrm{Re}(z) \to z$

Now, for $h \in \mathbb{C}$, $F(z+h) = \int_{C_1} f(z)\,dz$

$F(z) = \int_{C_2} f(z)\,dz$



Then, $F(z+h) - F(z) = \int_{C_1} f(\eta)\,d\eta + \int_{-C_2} f(\eta)\,d\eta = \int_{C_3} f(\eta)\,d\eta = \int_{C_4} f(\eta)\,d\eta$

Using $F(z+h) = F(z) + \int_{C_4} f(\eta)\,d\eta$, we get $\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{C_4} f(\eta)\,d\eta$

As $\frac{1}{h}\int_{C_4} dz = 1$, thus $\left(\frac{1}{h}\int_{C_4} f(\eta)\,d\eta\right) - f(z) = \frac{1}{h}\int_{C_4} (f(\eta)-f(z))\,d\eta = \frac{F(z+h)-F(z)}{h} - f(z)$

if h is small, $|f(\zeta)-f(z)| << \varepsilon$

In other words, by ML-formula, $\frac{F(z+h)-F(z)}{h} - f(z) << \frac{1}{\cancel{|h|}}\,\varepsilon\, 2\cancel{|h|} \Rightarrow \lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z)$ □

THEOREM

If f is entire and if C is a smooth closed curve, then $\int_C f(z)\,dz = 0$

9-16-25 (WEEK 3)

Shun/翔海 (@shun4midx)

CAUCHY INTEGRAL FORMULA AND TAYLOR EXPANSION

THEOREM (RECTANGLE THEOREM II)

Let f be entire, and

$$g(z) := \begin{cases} \frac{f(z)-f(a)}{z-a} & , z \neq a \\ f'(a) & , z = a \end{cases},$$ which is continuous (f entire $\Rightarrow$ g conti.)

Then, $\int_\Gamma g(z)dz = 0$, Γ: a boundary of a rectangle $R \subseteq \mathbb{C}$ (closed set)

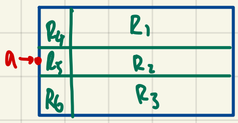
Proof

As g is cont., by def, $\exists M \in \mathbb{R}$, s.t. $|g(z)| < M$ $\forall z \in R$

(i) If $a \in \mathbb{C} \setminus R$, $g(z)$ is analytic $\forall z \in R$

$\therefore$ By the argument of Rectangle Thm, $\int_\Gamma g(z)dz = 0$

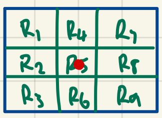
(ii) If $a \in \Gamma$, $\Gamma_i :=$ boundary of R_i:



Then, $\int_\Gamma g(z)dz = \sum_{i=1}^{6} \int_{\Gamma_i} g(z)dz \overset{(i)}{=} \int_{\Gamma_5} g(z)dz << M \cdot 4\varepsilon$ by ML-formula, with M indep of ε, where we define $\varepsilon :=$ length of longest side of Γ_5.

$\therefore$ As $\varepsilon \to 0$, $\int_\Gamma g(z)dz = 0$

(iii) Otherwise, $a \in$ interior of R



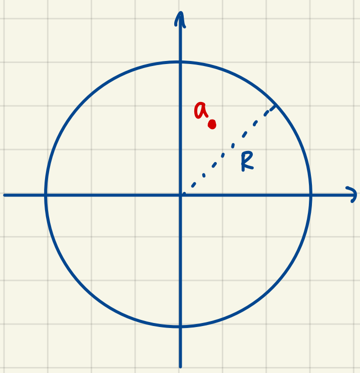
Then, $\int_\Gamma g(z)dz = \sum_{i=1}^{9} \int_{\Gamma_i} g(z)dz \overset{(i)}{=} \int_{\Gamma_5} g(z)dz << M \cdot 4\varepsilon \xrightarrow{\varepsilon \to 0} 0$

COROLLARY

The integral thm and closed curve thm apply to g (since g is cont.)

THEOREM (CAUCHY INTEGRAL FORMULA)

Given an entire f, $a \in \mathbb{C}$, $C = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$ with



Then, we have $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$

LEMMA

Define $C_\rho(\alpha) :=$ circle centered at α with radius ρ (α may be omitted if no amiguity)

Then, $\int_{C_\rho(\alpha)} \frac{dz}{z-a} = 2\pi i \quad \forall |a-\alpha| < \rho$

Proof

If $a = \alpha$, then it's clear, since $C_\rho(\alpha): \alpha + \rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

For $a \neq \alpha$,

$$\int_{C_\rho(\alpha)} \frac{dz}{(z-\alpha)-(a-\alpha)} = \int_{C_\rho(\alpha)} \frac{1}{z-\alpha} \cdot \frac{1}{1-\frac{a-\alpha}{z-\alpha}} \, dz =: I$$

$\forall z \in C_\rho(\alpha)$, $\left|\frac{a-\alpha}{z-\alpha}\right| < 1$. Hence, $\left(1-\frac{a-\alpha}{z-\alpha}\right)^{-1} = 1 + \left(\frac{a-\alpha}{z-\alpha}\right) + \left(\frac{a-\alpha}{z-\alpha}\right)^2 + \ldots$ (unif conv) , abs conv

Hence,

$$I = \int_{C_\rho(\alpha)} \frac{1}{z-\alpha} \left(\sum_{k=0}^{\infty} \left(\frac{a-\alpha}{z-\alpha}\right)^k \right) dz = \boxed{\sum_{k=0}^{\infty} \int_{C_\rho(\alpha)} \frac{1}{z-\alpha} \left(\frac{a-\alpha}{z-\alpha}\right)^k dz}$$

We first consider the term $\int_{C_\rho(\alpha)} \frac{1}{(z-\alpha)^k} dz$ ($k=1 \Rightarrow 2\pi i$, since $\int_C \frac{1}{z} dz = 2\pi i$)

For $k > 1$, $\int_{C_\rho(\alpha)} \frac{1}{(z-\alpha)^k} dz = \int_0^{2\pi} \frac{i e^{i\theta}}{\rho^k e^{ik\theta}} d\theta = \int_0^{2\pi} \frac{i}{\rho^k} e^{-i\theta(k-1)} d\theta = 0$

$\therefore I = 2\pi i$ □

PROOF OF CAUCHY INTEGRAL FORMULA

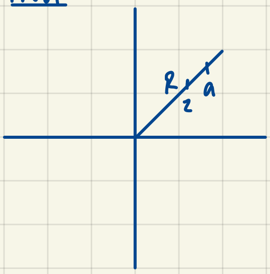
Now, we know by rectangle thm, $\int_C g(z) dz = \int_C \frac{f(z)}{z-a} - \frac{f(a)}{z-a} dz = 0$

$\therefore \int_C \frac{f(z)}{z-a} dz = \int_C \frac{f(a)}{z-a} dz = f(a) \int_C \frac{1}{z-a} dz = f(a) 2\pi i$ □

THEOREM (TAYLOR EXPANSION FOR ENTIRE FUNCTION)

Given f is an entire function, then $f^{(k)}(0)$ exists $\forall k \in \mathbb{Z}_{\geq 0}$ and $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \quad \forall z \in \mathbb{C}$

Proof



Choose $a \in \mathbb{C}$, $|a| > |z|$, $R := |a|+1$

Notice, by Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\omega)}{\omega - z} d\omega = \frac{1}{2\pi i} \int_{C_R} \frac{1}{1-\frac{z}{\omega}} \frac{f(\omega)}{\omega} d\omega$$

As $\left|\frac{z}{\omega}\right| < \frac{|a|}{1+|a|}$, then $f(z) = \sum_{k=0}^{\infty} \int_{C_R} \frac{f(\omega)}{\omega} \left(\frac{z}{\omega}\right)^k d\omega = \sum_{k=0}^{\infty} z^k \int_{C_R} \frac{f(\omega)}{\omega^{k+1}} d\omega = \sum_{k=0}^{\infty} z^k C_k$

Notice, as $|z| < |a|$, then $f'(z) = \sum_{i=1}^{\infty} i z^{i-1} C_i \Rightarrow f'(0) = C_1$

If we continue this process, $f^{(k)}(0)$ exists $\forall k \in \mathbb{N}_{\geq 0}$ and $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$ □

COROLLARY

An entire function is infinitely diff

Proof

Above thm. □

Shun/翔海(@shun4midx)

COROLLARY

If f is entire, then $f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2}(z-a)^2+\dots$

Proof

Let $h(z)=f(z+a)$.

f is entire $\Rightarrow z$ is entire.

Then, $h(z)=\sum_{k=0}^{\infty}\frac{h^{(k)}(0)}{k!}z^k \Rightarrow f(\omega)=f(a)+\sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}(\omega-a)^k$ □

PROPOSITION

If f is entire, then

$$g(z)=\begin{cases}\frac{f(z)-f(a)}{z-a}, & z\neq a\\ f'(a), & z=a\end{cases}$$ is entire

Proof

By corollary, $g(z)=\sum_{k\geq 1}^{\infty}\frac{f^{(k)}(a)}{k!}(z-a)^{k-1} \Rightarrow g$ is entire □

COROLLARY

Let f be an entire function with zeros at $a_1,\dots,a_N$. Define $g(z)=\frac{f(z)}{(z-a_1)\cdots(z-a_N)}$, $z\notin\{a_1,\dots,a_N\}$. Then, $\lim_{z\to a_i}g(z)$ exists $\forall i$.
If we define $g(a_i):=\lim_{z\to a_i}g(z)$, then g is entire

Proof

Set $f_0=f$, $f_k:=\frac{f_{k-1}(z)-f_{k-1}(a_k)}{z-a_k}$ $\left(f_1(z)=\frac{f(z)}{z-a_1}\right)$

By proposition, we see f_1 is entire. By recurrence, g is entire □

THEOREM (LIOUVILLE'S THEOREM)

Entire bounded functions on $\mathbb{C}$ are constants

Proof

Let $a\in\mathbb{C}\setminus\{0\}$, $C_R(0)$, $R>|a|$.

Then, by Cauchy Integral Formula, $f(a)-f(0)=\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-a}dz-\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z}dz=\frac{1}{2\pi i}\int_{C_R}\frac{af(z)}{z(z-a)}dz$

As f is bounded, $\exists M\in\mathbb{R}_{>0}$, s.t. $|f(z)|<M$ $\forall z\in\mathbb{C}$

By ML-formula, $|f(a)-f(0)|<\frac{1}{2\pi i}\left(\frac{M\cdot|a|}{R(R-|a|)}\cdot 2\pi R\right)\xrightarrow{R\to\infty}0$ $\therefore f(a)=f(0)$ $\forall a\in\mathbb{C}$ □

THEOREM (EXTENDED LIOUVILLE'S THEOREM)

Given f is entire. Suppose $|f(z)|<A+B|z|^k$ for some constants $A,B\in\mathbb{R}_0$. Then, f is a polynomial with degree at most k

Proof

Consider induction on k,

- $k=0\Rightarrow$ True by Liouville's Thm.

Otherwise,

Define $g(0)=\begin{cases}\frac{f(z)-f(0)}{z}, & z\neq 0\\ f'(0), & z=0\end{cases}$, which we know is entire.

As $|f(z)|<A+B|z|^k$ is bounded, define $M_0:=\max_{z\in C_R}g(z)$

$$\left.\begin{array}{l}\text{For } z\in\mathbb{C}\setminus C_R,\ |g(z)|<A+B|z|^{k-1}\\ \forall z\in C_R,\ |g(z)|<M_0\end{array}\right\}\Rightarrow\exists D,E\in\mathbb{R}_{>0} \text{ s.t. } |g(z)|<D+E|z|^{k-1}$$

$\therefore g$ is poly of degree at most $k-1$, so f is poly of degree at most k. □

Shun/翔海(@shun4midx)

THEOREM (FUNDAMENTAL THEOREM OF ALGEBRA)

Nonconstant polynomials have roots in $\mathbb{C}$.

in $\mathbb{C}[x]$

Proof

Consider poly $p(x)$.

Suppose p has no roots in $\mathbb{C}$.

We know $f(z) := \frac{1}{p(z)}$ is defined and differentiable on $\mathbb{C}$, so it is analytic.

As $z \to \infty$, $|f(z)| \longrightarrow 0$, then $|f(z)|$ is bounded

$\therefore$ By Liouville's Thm, $f(z)$ is const $\Rightarrow p(z)$ is const $\times$

DEFINITION

We say S is a convex set in $\mathbb{C}$ if $\forall x, y \in S$, $tx + (1-t)y \in S$ $\forall t \in [0,1]$

($\Rightarrow$ $x_1, \ldots, x_N \in S$ iff $\sum_{i=1}^{N} a_i x_i \in S$ $\forall \sum_{i=1}^{N} a_i = 1$ and $a_i \geq 0$)

Shun/翔海 (@shun4midx)

9-18-25 (WEEK 3)

THEOREM (GAUSS-LUCAS THEOREM)

The zeros of the derivative of a polynomial lie within the convex hull of the zeros of the polynomial

Proof

Let $p(x)$ be a nonconstant polynomial $\in \mathbb{C}[x]$, and $\alpha_1, \dots, \alpha_n$ be roots of p (counted by multiplicity)

Then, $p(x) = c\prod_{i=1}^{n}(x-\alpha_i)$. Moreover, $\frac{p'(x)}{p(x)} = \sum_{i=1}^{n}\frac{1}{x-\alpha_i}$.

otherwise it's trivial

Let a be a root of $p'(x)$ and $a \notin \{\alpha_1, \dots, \alpha_n\}$.

Then, $\frac{p'(a)}{p(a)} = \sum_{i=1}^{n}\frac{1}{a-\alpha_i} = \sum_{i=1}^{n}\frac{\bar{a}-\bar{\alpha}_i}{|a-\alpha_i|^2} \Rightarrow \bar{a} = \sum_{i=1}^{n} c_i\bar{\alpha}_i$, $c_i := \frac{1}{|a-\alpha_i|^2} \Big/ \sum_i \frac{1}{|a-\alpha_i|^2} \in \mathbb{R}_{\geq 0}$

$\therefore a = \sum_{i=1}^{n} c_i\alpha_i$, $c_i \in \mathbb{R}_{\geq 0}$, $\sum c_i = 1$ $\square$ by def

ANALYTIC FUNCTION ON A DISC

NOTATION

Assume open disc if disc

$D = D(\alpha; r)$ is an open disc $\subseteq \mathbb{C}$, then: analytic $f(z)$ in D, $\forall a \in D$, $g_a(z) = \begin{cases} (f(z)-f(a))/(z-a), & z \neq a \\ f'(a), & z = a \end{cases}$ is cont. on D (Notation for g_a)

THEOREM (RECTANGLE THEOREM)

For a closed rectangle $R \subseteq D$, define $\Gamma = \partial R$, then we proved $\int_\Gamma f(z)dz = \int_\Gamma \frac{f(z)-f(a)}{z-a}dz = \int_\Gamma g_a(z)dz = 0$ (They just both happen to equal 0, not they are directly related)

THEOREM

$\exists F, G_a$ analytic in D s.t. $f = F'$, $g_a = G_a'$

Proof

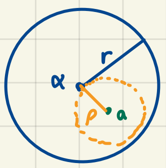
We consider $F := \int_\alpha^z f(z)dz$ and $G_a := \int_\alpha^z g_a(z)dz$

Define $C: \alpha \longrightarrow \alpha + \mathrm{Re}(z) \longrightarrow \alpha + z$, which is in D. $\square$

THEOREM (CLOSED CURVE THEOREM)

Let C be a closed curve $\subseteq D$. Then, $\int_C f dz = 0$

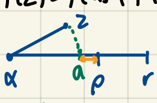
THEOREM (CAUCHY INTEGRAL FORMULA)



For some $0 < \rho < r$, $\forall |a-r| < \rho$, $f(a) = \frac{1}{2\pi i}\int_{C_\rho}\frac{f(z)}{z-a}dz$, where C_ρ is $\partial D(\alpha; \rho)$

THEOREM (TAYLOR EXPANSION)

$f(z) = f(\alpha) + f'(\alpha)(z-\alpha) + \frac{f''(\alpha)}{2}(z-\alpha)^2 + \dots$ holds $\forall |z-\alpha| < \rho$, for some ρ, s.t. $\exists a \in D$, $0 < \rho < r$, $|a-\alpha| < \rho$



Proof Sketch

$\forall |z-\alpha| < r$, $\frac{f^{(k)}(\alpha)}{k!} = \frac{1}{2\pi i}\int_{C_\rho}\frac{f(z)}{(z-\alpha)^{k+1}}dz$

THEOREM

Let $\tilde{D}$ be an arbitrary open set, f analytic on $\tilde{D}$. Then, fix $\alpha \in \tilde{D}$, we have $f(z) = \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ $\forall z \in D(\alpha, r) \subseteq \tilde{D}$

EXAMPLE

Let $f(z) = \frac{1}{z-1}$, analytic $\forall z \neq 1$.

Then, on $D=D(2;1)$, $f(z)=\frac{1}{1+(z-2)}=1-(z-2)+(z-2)^2-\dots\ \forall |z-2|<1$

In fact, the expression converges $\forall |z-2|<1$
diverges $\forall |z-2|\geq 1$

(However, $f(z)$ is analytic $\forall |z-2|=1$, so clearly its Taylor expansion is different)

PROPOSITION

$g_a(z)$ is analytic $\forall z,a\in D(\alpha;r)$

Proof

Use the thm above, in some neighborhood of α, $f(z)=f(\alpha)+f'(\alpha)(z-\alpha)+\frac{f''(\alpha)}{2}(z-\alpha)^2+\dots$

Then, g has the power series expansion $g(z)=f'(\alpha)+\frac{f''(\alpha)}{2!}(z-\alpha)+\frac{f^{(3)}(\alpha)}{3!}(z-\alpha)^2+\dots \Rightarrow g$ is analytic at α. □

THEOREM

If f is analytic at z, then f is infinitely diffable at z

Proof

We know from above, f may be expressed as a power series. Hence, it is infinitely diffable. □

 Shun/翔海(@shun4midx)

THEOREM (UNIQUENESS THEOREM)

Say D is a region (i.e. open connected) and f is an analytic function on D.
Suppose that $\exists$ seq of distinct zeros of D $\{z_n\}$, s.t. $\lim_{n\to\infty} z_n = z_0 \in D$, where we say the seq $\{z_n\}$ has an acc pt in D.
Then, $f \equiv 0$ on D.

Proof

f ann $\Rightarrow$ f conti

$\therefore$ By def, $f(z_0) = \lim_{n\to\infty} f(z_n) = 0$

We define $A := \{z \in D \mid z \text{ is an acc pt of zeros of } f \text{ in } D\}$.

Claim: A is open

Proof

By uniqueness of power series, $f \equiv 0$ in some disk $D(z, \delta_z) \subseteq D$ $\forall z \in A$ ✓

Claim: $D \setminus A$ is open

Proof

z is NOT acc pt of zeros $\Rightarrow \exists$ open nbd U of z in D s.t. $f(z)$ has NO zeros in $U \setminus \{z\}$.
f conti $\Rightarrow \forall y \in U \setminus \{z\}$, $\exists$ open nbd $V_y \subseteq D$ of y', s.t. $f \neq 0$ on $V_y \Rightarrow y \in D \setminus A$ ✓

$\therefore D = A \cup B$, A, B both open, $A \cap B = \emptyset$ $(B = D \setminus A)$
As $z_0 \in A$ and D is a region, $D = A$ □

COROLLARY

Say f, g are analytic on a region D.
If f and g agree at a set of pts with an acc pt, then $f \equiv g$ on D.

Proof

Set $h := f - g$, then apply thm above □

THEOREM

If f is entire and $f \to \infty$ as $z \to \infty$, then f is a polynomial

Proof

By def, $\forall M \in \mathbb{R}_{>0}$, $\exists \delta$, s.t. $\forall |z| > \delta$, $|f(z)| > M$

Let $M = 1$.
$\therefore \exists \delta$, s.t. $\forall |z| > \delta$, $|f(z)| > 1$

By our assumption, f is NOT a constant

Claim: f has finitely many zeros

Proof

If NOT, by δ, all zeros in f are in $\overline{D(0,\delta)}$, otherwise, $|f(z)| \neq 0$.
As $\overline{D(0,\delta)}$ is compact, $\exists$ acc pt of zeros in $\overline{D}(0,\delta)$ ✓
↳ Suppose not. Then, $\forall x \in \overline{D}(0,\delta)$, $\exists$ an open nbd U_x, s.t. f has no zeros in $U_x \setminus \{x\}$
$U_x \setminus \{x\}$ is an open cover of $\overline{D(0,\delta)} \Rightarrow \exists x_1, \dots, x_n$ s.t. $\{U_{x_i}\}$ is an open cover of $\overline{D(0,\delta)}$ (by cpt)
However, each U_x has at most 1 zero $\Rightarrow \overline{D(0,\delta)}$ has at most n zeros ※

By thm, $f \equiv 0$ on $D(0, \delta')$ for all $\delta' > \delta$ ← cover boundary at δ
However, δ' can extend to ∞

We consider within $\bar{D}(0,\delta)$.
Let $\alpha_1, \dots, \alpha_n$ be zeros of f (counting by multiplicity)
Then, $g(z) := \frac{f(z)}{\prod_{i=1}^{n}(z-\alpha_i)}$ is entire and has no zeros on $\mathbb{C}$

Set $h := \frac{1}{g(z)}$, then h is entire, h has no zeros in $\mathbb{C} \Rightarrow h$ is bounded in disk

By Extended Liouville's Thm, $|h| < A + B|z|^n$ $\forall |z| > \delta$ and $\forall |z| \le \delta \Rightarrow h$ is a poly

However, h has no zeros in $\mathbb{C} \Rightarrow h = \text{const}$
$\therefore \exists c \in \mathbb{C}^\times$ s.t. $f(z) = c\prod_{i=1}^{n}(z-\alpha_i)$ □

REMARK
Say f, g are ana on region D, to check $f \equiv g$, we may apply the theorem above over $\mathbb{R}$ without needing to consider $\mathbb{C}$.

THEOREM (MEAN VALUE THEOREM)
Let D be a region, f analytic on D, $\alpha \in D$.
Then $f(\alpha)$ = mean value of f taken around the boundary of any disk centered at α and contained at D

Proof

By Cauchy-Integral Formula, $f(\alpha) = \frac{1}{2\pi i}\int_{C_\delta(\alpha)} \frac{f(z)}{z-\alpha}\,dz$

Say $z = \alpha + \delta e^{i\theta}$, $\theta \in [0, 2\pi]$, we get $f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + \delta e^{i\theta})\,d\theta$ □

THEOREM (MAXIMUM MODULUS THEOREM)
Say f is noncont, ana on a region D. Then, $\forall z \in D$ and $\delta \in \mathbb{R}_{\ge 0}$, $\exists$ some $w \in D(z,\delta) \cap D$, s.t. $f(w) > f(z)$

Proof

By MVT, $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+\delta e^{i\theta})\,d\theta$ for small enough δ s.t. $D(z,\delta) \subseteq D$
Then, $|f(z)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z+\delta e^{i\theta})|\,d\theta \le^{①} \frac{1}{2\pi} \max_{\theta \in [0,2\pi]} |f(z+\delta e^{i\theta})| \cdot 2\pi = \max |f(z+\delta e^{i\theta})|$

$\therefore$ When $\le^{①}$ has equality, $|f(z+\delta e^{i\theta})| = \max_{\theta \in [0,2\pi]} |f(z+e^{i\theta})|$ $\forall \theta \in [0, 2\pi] \Rightarrow f$ is const on $C_\delta(z) \subseteq D$
By coro, hence f is const on D

However, f is noncont.
$\therefore f(z) < \max_{\theta \in [0,2\pi]} |f(z+\delta e^{i\theta})|$ □

THEOREM (MINIMUM MODULUS THEOREM)
Say f is noncont, ana on a region D, $\forall z \in D$, $f(z) \neq 0$.
Then, f has no interior min points

Proof

$f(z) \neq 0$ $\forall z \in D \Rightarrow g(z) = \frac{1}{f(z)}$ is ana, noncont on D
Then, by max mod thm, we proved it. □

CAUTION
We can only apply uniqueness thm and its coro when its acc pts $\in D$ (Counterexample: $f(z) = \sin\frac{1}{z}$, $z_n = \frac{1}{n\pi}$)

THEOREM
CAN ONLY APPLY TO CIRCLES

Say $\bar{D}$ is a closed disk and f is analytic, noncont on $\bar{D}$. f assumes its max value at a boundary point z_0. Then, $f'(z_0) \neq 0$

Proof

Suppose $f'(z_0) = z_0$. As f is ana on $\bar{D}$, $\exists \delta$ (conv rad) s.t. $\forall |\zeta| < \delta$, $f(z_0+\delta) = f(z_0) + f'(z_0)\zeta + \frac{1}{2}f''(z_0)\zeta^2 + \dots$

Shun/翔海(@shun4midx)

Assume $f'(z_0)=0$

Then, $f(z_0+\delta)\approx f(z_0)+\frac{1}{2}f''(z_0)\zeta^2 \Rightarrow |f(z_0+\delta)|=f(z_0+\delta)\cdot\bar{f}(z_0+\delta)=|f(z_0)|^2+\frac{2}{k!}\mathrm{Re}(\bar{f}(z_0)f^{(k)}(z_0)\zeta^k)+\ldots$, for some $k\in\mathbb{N}$

From assumption, thus $k\geq 2$.

Let $e^{i\theta}=\frac{\zeta}{|\zeta|}$. Then, $\bar{f}(z_0)f^{(k)}(z_0)=Ae^{i\alpha} \Rightarrow |f(z_0+\zeta)|^2=|f(z_0)|^2+\frac{2}{k!}A|\zeta|^k\cos(\alpha+k\theta)+\ldots$

For small enough ζ, $|f(z_0+\zeta)|-|f(z_0)|$ has the same sign as $\boxed{\cos(k\theta+\alpha)}$

As $|f(z_0)|$ is max, hence $|f(z_0+\zeta)|^2-|f(z_0)|^2\leq 0$ $\forall z_0+\zeta\in D$.

Notice, $\cos(k\theta+\alpha)\leq 0 \Leftrightarrow \frac{\pi}{2}+2\pi j\leq\alpha+k\theta\leq\frac{3\pi}{2}+2\pi j$ for $0\leq j\leq k-1 \Leftrightarrow \frac{(\frac{\pi}{2}-\alpha)}{k}+\frac{2\pi j}{k}\leq\theta\leq\frac{(\frac{3}{2}\pi-\alpha)}{k}+\frac{2\pi j}{k}$ $(\star)$ $\leftarrow$ Δ w/ angle $\frac{\pi}{k}$

kinda like they alternate

For a disc, $\exists\zeta$, s.t. $z_0+\zeta$ is NOT in any one of the cones $(\star)$ since $k\geq 2$, $\frac{\pi}{k}\leq\frac{\pi}{2}$. $\times$

REMARK

This argument works for cpt $K\subseteq\mathbb{C}$ s.t. $\forall z_0$ on the boundary of K, K contains a cone $\{z_0+re^{i\theta} \mid \theta\in(\alpha,\beta), r\in(0,\varepsilon)\}$ with $\beta-\alpha>\frac{\pi}{2}$

Counterexample of squares:

$f(z)=z^2+i \Rightarrow |f(z)|$ has min 1 at $z=0$, but $f'(0)=0$

9-25-25 (WEEK 4) Shun/翔海 (@shun4midx)

DEFINITION

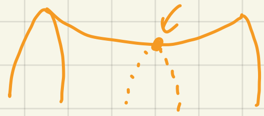
A function is C-analytic on a region D if it is analytic on D and continuous on $\bar{D}$

SADDLE POINTS

DEFINITION

z_0 is a saddle pt of an analytic function f (on a region D) if z_0 is a saddle pt of the real valued function $g(x,y) = |f(x,y)|$

In other words, $\underbrace{g \text{ is differentiable}}_{\lim_{h\to 0} \frac{\|g(x+h)-g(x)-L(x)\|}{\|h\|}=0}$ and $g_x(z_0) = g_y(z_0) = 0$ but z_0 is NOT a local extremum



THEOREM

z_0 is a saddle pt of an analytic function f iff $f'(z_0) = 0$ and $f(z_0) \neq 0$

Proof

We have $z = x+iy$, $f(z) = u(x,y) + iv(x,y)$, and $g(z) = (u^2+v^2)^{\frac{1}{2}} \geq 0$

"$\Rightarrow$": As $g(z_0)$ is not a local minimum, hence $g(z_0) \neq 0$, so $u(z_0) \neq 0$ or $v(z_0) \neq 0$

We know $g_x(z_0) = g_y(z_0) = 0 \Rightarrow \frac{uu_x+vv_x}{g}\Big|_{z_0} = \frac{uu_y+vv_y}{g}\Big|_{z_0} = 0$ (★)

$$\Rightarrow \begin{bmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0) \end{bmatrix} \underbrace{\begin{bmatrix} u(z_0) \\ v(x_0) \end{bmatrix}}_{\neq 0} = 0$$

$$\therefore \det \begin{bmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0) \end{bmatrix} \overset{\text{CR-eq}}{=} u_x^2(z_0) + v_x^2(z_0) = 0$$

$\therefore u_x(z_0) = v_x(z_0) = 0$

As f is ana, hence $f'(z_0) = 0$. From above with $g(z_0) \neq 0$, we know $f(z_0) \neq 0$. ✓

"$\Leftarrow$": Recall, $f'(z_0) = 0$

Then, $u_x(z_0) = v_x(z_0) = 0$ and $u_y(z_0) = v_y(z_0) = 0$

$\therefore g_x(z_0) = g_y(z_0) = 0$ as implied by (★)

As $f(z_0) \neq 0$, thus $|f(z_0)|$ is NOT a local extremum (excluding f is const) by the max and min modulus thms. ✓

OPEN MAPPING THM AND SCHWARTZ LEMMA

RECALL

$f: \mathbb{R}^2 \to \mathbb{R}^2$ conti $\Leftrightarrow$ $U \subseteq \mathbb{R}^2$: open then $f^{-1}(U)$: open $\Leftrightarrow$ $\bar{U} \subseteq \mathbb{R}^2$: closed then $f^{-1}(\bar{U})$: closed

Then $K \subseteq \mathbb{R}^2$: cpt $\Rightarrow$ $f(K)$: cpt

THEOREM (OPEN MAPPING THEOREM)

$\forall$ open set $U \subseteq D$, $f(U)$: open in $\mathbb{C}$ for any nonconst ana f (Need not hold outside of $\mathbb{C}$, e.g. $U = (-1,1)$, $f(U) = [0,1)$ for $f(x) = x^2$)

Proof

It suffices to show $\forall \alpha \in D$, $\exists$ open disc $D(\alpha, \varepsilon) \subseteq D$ s.t. $f(D(\alpha,\varepsilon))$ is open.

(We want to show $\forall \beta = f(\alpha') \in f(D(\alpha,\varepsilon))$, $\exists D(\beta, \varepsilon') \subseteq f(D(\alpha, \beta))$)

WLOG, we can assume $f(\alpha) = 0$, so we choose ε s.t. $\overline{D(\alpha,\varepsilon)} \subseteq D$

By uniqueness thm, $\exists \varepsilon$ s.t. f has no zeros in $\overline{D(\alpha,\varepsilon)} \setminus \{\alpha\}$ (or else $f \equiv 0$)

Let $2\delta = \min_{z \in C_\varepsilon(\alpha)} |f(z)| > 0$

Claim: $D(f(\alpha)=0, \delta) \subseteq \text{Im}(f)$

Proof

$\forall w \in D(0, \delta)$, consider $f(z) - w$

If $w \notin f(D(\alpha, \varepsilon))$, then $f(z) - w$ has no zeros on $D(\alpha, \varepsilon)$

$\therefore |f(z) - w| \geq |f(z)| - |w| \geq f(z) - \delta \geq \delta \quad \forall z \in C_\varepsilon(\alpha)$

However, we know $|f(\alpha) - w| < \delta \rightarrow\!\leftarrow$

$\therefore w \in f(D(\alpha, \varepsilon)) \Rightarrow D(0, \delta) \subseteq \text{Im}(f)$ $\square$

9-30-25 (WEEK 5)

Shun/翔海(@shun4midx)

REMARK

We only have open mapping thm because extremum is not in interior pt.

SCHWARTZ'S LEMMA

THEOREM (SCHWARTZ'S LEMMA)

Suppose that f is analytic in an open unit disc D with $|f|\leq 1$ (f: unit circles $\to$) and $f(0)=0$

Then, (i) $|f(z)|\leq |z|$

(ii) $|f'(0)|\leq 1$

with equality in either of the above iff $f(z)=e^{i\theta}z$

Proof

Define $g(z)=\begin{cases} \frac{f(z)}{z}, & z\neq 0 \\ f'(z), & z=0\end{cases}$

$g(z)$ is ana on D since $f(z)$ is ana on D.

Consider $z\in C_r(0)$, $0<r<1$.

Then, $|g(z)|=\frac{|f(z)|}{|z|}\leq \frac{1}{r}$

By max modulus thm, $\forall z\in \overline{D(0,r)}$, $|g(z)|\leq \frac{1}{r}$

As $r\to 1$, then $|g(z)|\leq 1\ \forall z\in D$

By def of $g(z)$, $|f(z)|\leq |z|$ and $|f'(0)|\leq 1$ has either equality hold, when g is const and $|g|=1$ on D. $\therefore g=e^{i\theta}$ □

EXAMPLE (Removing $f(0)=0$ constraint)

Define $B_\alpha(z)=\frac{z-\alpha}{1-\bar{\alpha}z}$, $|\alpha|<1$ — D

Then, (1) $B_\alpha(\alpha)=0$

(2) $B_\alpha(z)$ is ana on D, so $(B_\alpha(z))^2$ is ana on D. It is also conti on $\bar{D}$.

(3) $|B_\alpha(z)|^2\Big|_{|z|=1}=1$, so by max modulus thm, $|B_\alpha(z)|\leq 1$ on D.

$\therefore$ We can use B_α for variations of Schwartz's lemma

EXAMPLE

Say f: ana on D, $|f(z)|\leq 1\ \forall z\in D$ and $f(\frac{1}{2})=0$. Estimate $|f(\frac{3}{4})|$.

Consider $B_{\frac{1}{2}}(z)=\dfrac{z-\frac{1}{2}}{1-\frac{z}{2}}$.

We define $g(z)=\begin{cases} \frac{f(z)}{z-\frac{1}{2}}(1-\frac{1}{2}z), & z\neq \frac{1}{2} \\ \frac{3}{4}f'(\frac{1}{2}), & z=\frac{1}{2}\end{cases}$ $(B_\alpha(z))^{-1}$

Notice, $|B_{\frac{1}{2}}(z)|\leq 1$ on D, $|B_{\frac{1}{2}}(z)|=1$ on $C_1(0)$, and $B_{\frac{1}{2}}(z)$ conti on D.

$\therefore z\to 1$, $|B_{\frac{1}{2}}(z)|\longrightarrow 1$

$|g(z)|\leq 1/|B_{\frac{1}{2}}(z)|\Rightarrow |g(z)|\leq 1$ on D.

$\Rightarrow |f(z)|\leq |B_{\frac{1}{2}}(z)|$, $\frac{3}{4}|f'(\frac{1}{2})|\leq 1$

So, $|f(\frac{3}{4})|\leq |B_{\frac{1}{2}}(\frac{3}{4})|=\frac{2}{5}$

Shun/翔海(@shun4midx)

EXAMPLE

Say f is ana on D, $|f(z)|\leq 1$ on D. We claim: $|f'(\frac{1}{3})|$ is max when $f(\frac{1}{3})=0$

"your rate of improvement is max when your score is at its lowest" ≥≤?

min modulus for $\mathbb{C}$.

Proof

Assume that $f(\frac{1}{3})\neq 0$.

$g(z):=\frac{f(z)-f(\frac{1}{3})}{1-\overline{f(\frac{1}{3})}f(z)} \Rightarrow g(z)=B_{f(\frac{1}{3})}(f(z))$. Note, g is bounded by 1, $g(\frac{1}{3})=0$. (Note it is ana too)

$\therefore g'(z)\Big|_{z=\frac{1}{3}}=\frac{f'(\frac{1}{3})}{1-|f(\frac{1}{3})|^2} \Rightarrow |g'(\frac{1}{3})|>|f'(\frac{1}{3})|$ ($\because |f(\frac{1}{3})|>0$. Otherwise, then $|g'(\frac{1}{3})|$ can be larger, take $f:=g$ ~~×~~)

$\therefore |f'(\frac{1}{3})|$ is max for $B_{\frac{1}{3}}(z)$.

PROPOSITION

Say f is entire. If $|f(z)|<\frac{1}{|\text{Im}\, z|}$ $\forall z$, then $f\equiv 0$

Proof

Define $g(z)=(z^2-R^2)f(z)$, for some $R\in\mathbb{R}_{>0}$ (sufficiently large, e.g. $R\geq 1$, $R\to\infty$)

When $z\in C_R(0)$, $|z-R||z+R|\leq 2R|\text{Im}(z)|$

$\therefore |g(z)|\leq\frac{2R}{|\text{Im}(z)|^2}\leq 2R$ when $z\in C_R(0)$

By max modulus thm, $|g(z)|\leq 2R$ $\forall z\in D(0,R)$

$\Rightarrow |f(z)|\leq\frac{2R}{|z^2-R^2|}$ $\forall z\in D(0,R)$

As $R\to\infty$, $|f(z)|\longrightarrow 0$

$\therefore f(z)=0$

MORERA'S THEOREM

THEOREM (MORERA'S THEOREM: CONVERSE OF RECTANGLE THEOREM)

Let f be continuous on an open set $D\subseteq\mathbb{C}$, and Γ be the boundary of a closed rectangle $R\subseteq D$.

If $\int_\Gamma f\,dz=0$ $\forall\Gamma$ in $R\subseteq D$, then f is analytic in D.

Proof

Say $z_0\in D$, D: open.

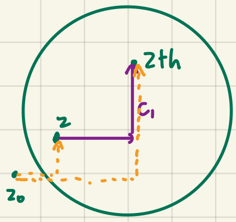
Then, $\exists\,\varepsilon>0$, s.t. $D(z_0,\varepsilon)\subseteq D$.

Define $F(z):=\int_C f(z)\,dz$ $\forall z\in D(z_0,\varepsilon)$, where $C: z_0\to z_0+\text{Re}(z-z_0)\to z$

For $z\in D(z_0,\varepsilon)$ and k small enough s.t. $z+h\in D(z_0,\varepsilon)$

Then,

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=\lim_{h\to 0}\int_{C_1}f(\omega)\,d\omega\overset{f\text{: cont.}}{=}f(z)\ \square$$



Shun/翔海 (@shun4midx)

EXAMPLE

Using $f(z)=\int_0^\infty \frac{e^{zt}}{t+1}\,dt$,

Claim: f is analytic $\forall z\in\{w\in\mathbb{C} \mid Re(w)<0\}$

Proof

We know for $z=x+iy$, $x<0$, $|e^{zt}|=e^{xt}$

Here, $\int_\Gamma\int_0^\infty \frac{|e^{zt}|}{t+1}\,dt\,dz < \int_\Gamma -\frac{1}{x}\,dz<\infty$ (ok for Fubini)

$=0$ (∵ By rectangle thm since $\frac{e^{zt}}{t+1}$: ana)

By Fubini's Thm, $\int_\Gamma\int_0^\infty \frac{e^{zt}}{t+1}\,dt\,dz=\int_0^\infty \boxed{\int_\Gamma \frac{e^{zt}}{t+1}\,dz}\,dt=0$

∴ By Morera's Thm, $f(z)$ is analytic on $\{w\in\mathbb{C} \mid Re(w)<0\}$. □

DEFINITION

Let $\{f_n\}$ and f be defined on an open set D. We say that f_n **converges uniformly on compacta** if $f_n\to f$ uniformly on every compact subset $K\subseteq D$.

THEOREM

Let D be an open set in $\mathbb{C}$ and $\{f_n\}$ be a sequence of ana functions s.t. $f_n\to f$ unif on cpta. Then, f is also ana in D.

Proof

∵ f_n is conti, $\forall K\subseteq D$: cpt set we have $f_n\to f$ unif on K

∴ f is conti on K $\forall K$, i.e. f is conti on D

We hope "$\int_\Gamma f\,dz=0$", for Γ : boundary of a closed rectangle $R\subseteq D$

Hence, $\int_\Gamma f\,dz=\int_\Gamma \lim_{n\to\infty} f_n\,dz$

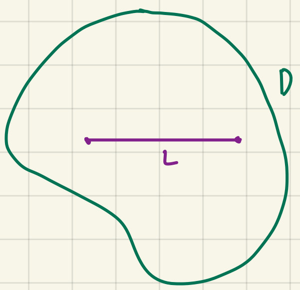
$$= \lim_{n\to\infty}\left(\int_\Gamma f_n\,dz\right) \quad (f_n \text{ conti., } f_n\to f \text{ unif on } R)$$

$$= 0 \quad (\text{Rectangle thm } \because f_n : \text{ana})$$

∴ By Morera's Thm, f is conti. □

THEOREM

f is continuous on an open set $D\subseteq\mathbb{C}$ and **analytic except on a line segment** in D. Then, f is **analytic throughout D**.



Proof

Fixed in next PDF cuz I was def high when I wrote the proof for the wrong thm ∵ (yes, hence the reupload)

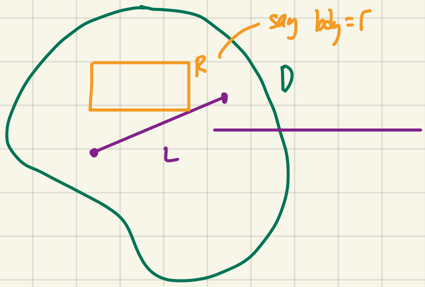
Shun/翔海 (@shun4midx)

REMARK

$f(z) := \frac{1}{z}$ is a counterexample as to why we cannot say "$\forall f$: ana exc on finitely many pts in a region $\Rightarrow f$: ana"

THEOREM

f is continuous on an open set $D \subseteq \mathbb{C}$ and analytic except on a line segment in D. Then, f is analytic throughout D.



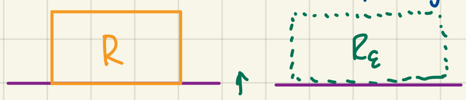
WLOG, $L \subseteq \mathbb{R}$-axis after a linear poly transformation :)

Proof

We know $f|_{D \setminus L}$: ana. Consider the following cases.

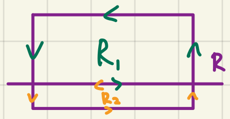
① $R \cap L = \emptyset \Rightarrow \int_\Gamma f dz = 0$ as f: ana on $D \setminus L$

② $R \cap L \subseteq \Gamma$, $R \cap L \neq \emptyset$: Lift one side, we get a rectangle $R_\varepsilon \subseteq R$, $R_\varepsilon \cap L = \emptyset$



By case ①, as f is cont., $\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} f(z)dz = 0 \Rightarrow \int_\Gamma f(z)dz = 0$

③ $R \cap L \not\subseteq \Gamma$



Then, $R = R_1 \cup R_2$, $\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f$

By cases ① and ②, hence $\int_\Gamma f(z)dz = 0$

$\therefore$ By Morera's Thm, f is analytic on D □

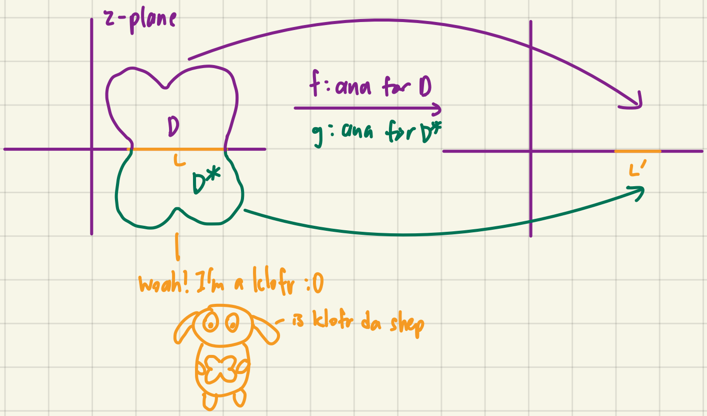
THEOREM (SCHWARZ REFLECTION PRINCIPLE)

Suppose f is $\mathbb{C}$-analytic in a region D that is contained in either the upper or lower half plane and whose boundary contains a segment L on the real axis, and suppose f is real for real z.

Then, we can define an analytic "extension" g of f to the region $D \cup L \cup D^*$ that is symmetric with respect to the real axis by

$g(z) = \begin{cases} f(z), & z \in D \cup L \\ \overline{f(\bar{z})}, & z \in D^* \end{cases}$ where $D^* = \{z \mid \bar{z} \in D\}$

Graphically, we represent it as follows:



Proof

① $z \in D$, then $f|_D = g|_D$, f: ana $\Rightarrow g$: ana

② $z \in D^*$ and $z+h \in D^*$, then:

$$\lim_{h\to 0} \frac{g(z+h)-g(z)}{h} = \lim_{h\to 0} \overline{\left(\frac{f(\overline{z+h}) - f(\overline{z})}{\overline{h}} \right)} = \overline{f'(\overline{z})}$$

$\therefore$ g : ana

Since f : conti on $\mathbb{R}$-axis, so is g

$\therefore$ We can apply the thm above so g: ana throughout $D \cup L \cup D^* = U$ □

DEFINITION

A curve $\gamma: [a,b] \to \mathbb{C}$ is called a regular analytic arc if γ is an analytic map, 1-1 on $[a,b]$ with $\gamma' \neq 0$.

FACT (WILL PROVE IN THE FUTURE)

$\forall \alpha \in [a,b]$, $\exists D(\alpha, \varepsilon)$, s.t. γ: ana on $D(\alpha,\varepsilon)$, $\gamma'(z) \neq 0$ $\forall z \in D(\alpha, \varepsilon)$

In fact, $\exists \gamma^{-1}$: ana, $\gamma: D(\alpha,\varepsilon) \xrightarrow{\sim} \gamma(D(\alpha,\varepsilon))$

Proof Sketch

Map the boundary via γ to a real segment $[a,b]$, apply the thm, then apply γ^{-1}. Then, reflect similarly in image via λ.

Mathematically, $\lambda(\overline{\lambda^{-1}(f(\gamma(\overline{\gamma^{-1}(z)})))})$... XD (kill me)

10-7-25 (WEEK 6)

Shun/翔海 (@shun4midx)

SIMPLY CONNECTED DOMAIN

DEFINITIONS

Say $f: (S, \{U\}) \longrightarrow (S_1, \{N\})$ for two topological spaces

f is continuous if $\forall$ open $V \subseteq S_1$, $f^{-1}(V)$ is open in S

Path-connected: $\forall \alpha, \beta \in S$, $\exists \gamma: [0,1] \longrightarrow S$ continuously s.t. $\gamma(0)=\alpha$, $\gamma(1)=\beta$ (E.g. $\{x=0\} \cup \{\sin\frac{1}{x}\}$: connected but not path connected)

any open set cannot separate

IN $\mathbb{C}$ TOPOLOGICAL SPACE

We consider the topo space $\mathbb{C} \cong \mathbb{R}^2$

We say S is simply connected if it is:

① Path-connected ($\Rightarrow$ connected)

② For any continuous maps $f_i: [0,1] \longrightarrow S$ with $f_0(0)=f_1(0)$ and $f_0(1)=f_1(1)$, $\exists$ continuous $F: [0,1] \times [0,1] \longrightarrow S$ s.t. $F(t,0)=f_0(t)$, $F(t,1)=f_1(t)$ (intuitively: they are connected via shrinking a rubber band for any two pts)

Actually, $S=\mathbb{C} \cong \mathbb{R}^2$ is simply connected and a torus is not

can be seen as a disc with a hole taken out (disc裡挖出一個洞)

RECALL

f: ana on closed disc $D(0,1)$, then $\int_{C_1(0)} f dz = 0$

However, remember $\int_{C_1(0)} \frac{1}{z} dz = 2\pi i$, note that $\frac{1}{z}$: well-defined on $D(0,1) \setminus \{0\}$ Torus

Motivation: Simply via integration, we can determine the topo nature of subsets of $\mathbb{C}$, equal zero or nonzero?

↳ "Holomorphic simply connected" (= "topo simply connected")

DEFINITION

D is holomorphic simply connected (hsc) if $\forall f$: ana on D, $\int_\Gamma f dz = 0$ $\forall$ simple closed curve $\Gamma \subseteq D$

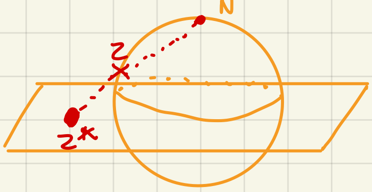
FACT: In $\mathbb{C}$, hsc $\Leftrightarrow$ sc

DEFINITION

We say the extended $\mathbb{C}$-plane is $\mathbb{C} \cup \{\infty\} \cong S^2$ (sphere)

basically treat $\{\infty\}$ as one point, no more $\pm\infty$ as opposites, they are the same concept

∞ is the north pole in this stereographic projection



DEFINITION

If D is open connected $\subseteq \mathbb{C}$, then D is sc if $(\mathbb{C} \cup \{\infty\}) \setminus D$ is path connected

simply connected

DEFINITION'

For a region $D \subseteq \mathbb{C}$, we say D is sc' if $\forall z_0 \in \mathbb{C} \setminus D$, $\forall \varepsilon > 0$, $\exists$ path $\gamma: [0,\infty) \longrightarrow \mathbb{C} \setminus D = \tilde{D}$, s.t.

① $d(\gamma, \tilde{D}) \leq \varepsilon$

② $\gamma(0) = z_0$

③ $\lim_{t \to \infty} \gamma(t) = \infty$

$\boxed{\text{Def} \Rightarrow \text{sc} \Rightarrow \text{hsc}, \quad \text{Def}'}$

REMARK

For $\boxed{\text{open}}$ $D \subseteq \mathbb{C}$, D is open connected $\Leftrightarrow$ D is connected

Proof

"$\Rightarrow$": Locally path connected (i.e. $\forall x \in D$, $\exists$ open nbd $U(x) \subseteq D$ s.t. $U(x)$: path-connected

DEFINITION

e.g. level 3: _-‾

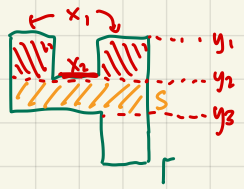
Given a polygonal path Γ (⌐‾), we define the level of $\Gamma :=$ # diff values y_0 where the line $\operatorname{Im} z = y_0$ contains a horizontal segment of Γ

LEMMA

no intersections w/ itself

Γ: simple closed curve that is also a polygonal curve, say $\Gamma \subseteq D$, where D: SC region

Suppose the top level of Γ consists of points $y=y_1$, $x \in X_1$ and $y=y_2$, $x \in X_2$. Then the set $R := \{z = x+iy \mid y_2 \leq y \leq y_1, x \in X_1\}$ is contained in D



Proof

Consider induction on the level of Γ.

For $\operatorname{lev}(\Gamma) \geq 2$,

- $\operatorname{lev}(\Gamma) = 2$: $R = \bigcup_{i=1}^{n} R_i$, R_i: closed rectangle and $\bigcup \partial R_i := \Gamma$

  Let $z_0 \in R$.

  Suppose $z_0 \notin R$. As D is simply connected, $\exists \gamma : [0, \infty] \longrightarrow (\mathbb{C} \cup \{\infty\}) \setminus D$ s.t. $\gamma(0) = z_0$ and $\lim_{t \to \infty} \gamma(t) = \infty$, $t_0 := \sup\{t \mid \gamma(t) \in R\}$

  $\setminus D$ ✱

  Claim: $\gamma(t_0) \in \Gamma \subseteq D$ ($\Rightarrow$ ✱)

  Proof

  (i) $\gamma(t_0) \overset{?}{\in} R \setminus \Gamma$: open (If no, then $\gamma(t_0) \in \Gamma$ or $\gamma(t_0) \in \mathbb{C} \setminus R$)

  $\lim_{t \to \infty} \gamma(t) = \infty$

  If yes, as $R \setminus \Gamma$: open, $\exists D(\gamma(t_0), \varepsilon) \subseteq R \setminus \Gamma$, $t_0 \in \gamma^{-1}(D(\gamma(t_0), \varepsilon)) \subseteq [0, \infty)$ (as $t_0 = \sup\{t \mid \gamma(t) \in R\}$) ✱

  (ii) $\gamma(t_0) \notin \mathbb{C} \setminus R$: Similarly, ✱

  $\setminus D$ ✱

  $\therefore$ $\gamma(t_0) \in \Gamma \subseteq D$ ✱

- $\operatorname{lev}(\Gamma) > 2$: Note: $\bigcup(\partial R_i)$ not necessarily equal to Γ

  By the same argument as base case, $t_0 = \sup\{t \mid \gamma(t) \notin R\}$, $t_0 \in \partial R_i$ for some R_i

  If $t_0 \in \Gamma$, as in the base case, ✱

  Def $\tilde{R} := \mathbb{C} \setminus R$, $\Gamma' := (\Gamma \cap \tilde{R}) \cup \bar{L}$, where $L := \{x+iy \mid y = y_2, x \in X_1 \setminus X_2\}$

  For small h, $\gamma(t_0+h)$ is between the top two levels of Γ'

  As $\gamma(t_0+h) \notin D$ and $\gamma(t_0+h) \in S$, $\operatorname{level}(\Gamma') < \operatorname{level}(\Gamma)$

  $\therefore$ By induction, $\gamma : [t_0+h, \infty] \to \infty$ intersects $\Gamma' \setminus \bar{L}$ $\square$

  as $\forall t > t_0$, $\gamma(t) \notin R$

Shun/翔海 (@shun4midx)

10-9-25 (WEEK 6)

USED DEFINITION

For a region $D \subseteq \mathbb{C}$, D is simply connected if $(\mathbb{C} \cup \{\infty\}) \setminus D$ is path connected

(We want to consider domain D s.t. $\int_\Gamma f(z)\,dz = 0$, f: analytic over D, $\Gamma \subseteq D$: simple closed curve)

THEOREM

f: ana in a s.c. region D and $\Gamma \subseteq D$ simple closed polygonal path. Then, $\int_\Gamma f\,dz = 0$

Proof

Lemma from prev note $\Rightarrow R \subseteq D \Rightarrow \partial R \subseteq D$

As $\Gamma = \partial R + \Gamma'$, $\int_\Gamma f\,dz = \int_{\partial R} f\,dz + \int_{\Gamma'} f\,dz$

By rectangle thm, $R \subseteq D$, f: ana in $D \Rightarrow f$: ana on $R \Rightarrow \int_{\partial R} f\,dz = 0$

$\therefore$ By induction on $\mathrm{lev}(\Gamma)$, we get $\int_\Gamma f\,dz = 0$ □

THEOREM

f: ana on a s.c. region $D \Rightarrow \exists$ primitive F, $F' = f$

Proof

Fix $z_0 \in D$, define $F(z) = \int_\Gamma f(\zeta)\,d\zeta$, where Γ = any polygonal path from z to $z_0 \subseteq D$

- F is well-defined: Suppose Γ_1, Γ_2 satisfy the polygonal path condition

  Then, $\Gamma_1 - \Gamma_2 = \bigcup_{i=1}^{r} C_i$, C_i: simple closed polygonal curve $\subseteq D$

  $$\Rightarrow \int_{\Gamma_1} f(\zeta)\,d\zeta - \int_{\Gamma_2} f(\zeta)\,d\zeta = \sum_{i=1}^{r} \int_{C_i} f(\zeta)\,d\zeta \overset{\text{by thm above}}{=} 0 \checkmark$$

Now, let h be small enough s.t. $z+h \in D$

$$\Rightarrow \frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left[\int_{\Gamma_1} f(\zeta)\,d\zeta - \int_{\Gamma_2} f(\zeta)\,d\zeta\right] = \frac{1}{h}\int_{\Gamma_3} f(\zeta)\,d\zeta, \text{ where}$$

Γ_1: any poly path $z_0 \to z+h \subseteq D$

Γ_2: any poly path $z_0 \to z \subseteq D$

Choose Γ_2 first, then $\Gamma_1 = \Gamma_2 + \Gamma_3$, where Γ_3: any poly path $z \to z+h \subseteq D$

Then, $\lim_{h \to 0} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \lim_{h \to 0} \frac{1}{|h|} \left| \int_{\Gamma_3} [f(\zeta) - f(z)]\,d\zeta \right| = 0$ □

Γ_3: $z \to z+h \subseteq D$

THEOREM (CLOSED CURVE THEOREM)

Let f: ana on a s.c. region D. Then, $\forall$ simple closed curve $C \subseteq D$, $\int_C f(z)\,dz = 0$

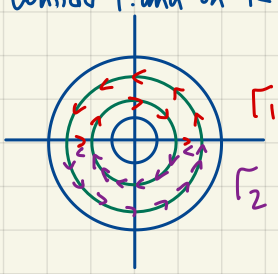
Proof

By Thm 3, $f = F'$ for some ana F

$\therefore$ For any $C: \gamma(t): [0,1] \to D$, $\int_C f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) = 0$ $\because$ closed $\Rightarrow \gamma(1) = \gamma(0)$ □

EXAMPLE

Consider f: ana on $1 < |z| < 4$



Shun/翔海(@shun4midx)

Claim: $\int_{C_2(0)} f(z)\,dz = \int_{C_3(0)} f(z)\,dz$

Proof

We have $\int_{C_3(0)} f(z)\,dz - \int_{C_2(0)} f(z)\,dz = \boxed{\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz} = 0$ by closed curve thm $\square$

THE PROBLEM WITH DEFINING LOG

$\log z := u(z) + i v(z) \Rightarrow \underbrace{z = e^{u(z)} e^{i v(z)}}_{|z| = |e^{u(z)}|}$, but $\theta = v(z) + 2\pi k$, $k \in \mathbb{Z}$ all are fine, so how do we fix a value so log is well-def?

DEFINITION

We say f is an analytic branch of $\log z$ in a domain D if:

(i) f is analytic

(ii) $e^{\log z} = z$, candidate: $f(z) = \overbrace{\log}^{\text{log over } \mathbb{R}} |z| + i \underline{\operatorname{Arg} z}^{\in [0, 2\pi)}$

Shun/翔海 (@shun4midx)

10-14-25 (WEEK 7)

RESOLVING THE WELL-DEFINEDNESS OF LOG OVER $\mathbb{C}$

We can consider approaching $\log z$ via integration.
We want $\log z$ s.t. ① f: ana
② $\exp(f(z)) = z$

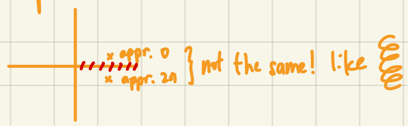
$\therefore$ If $f(z) = \log z$, we want it to satisfy $f'(z) = \frac{1}{z}$

We can $\boxed{\text{fix } z_0 \in \mathbb{R}^+}$, so $f(z) := \int_C \frac{1}{\zeta} d\zeta + \boxed{\log(z_0)}$ where $C: z_0 \to z$

However, we need $\int_C \frac{1}{\zeta} d\zeta$ to be well-defined indep of path...
$\therefore$ Choose a s.c. region D, then $\forall C \subseteq D$, $\int_C \frac{1}{\zeta} d\zeta$ is well-defined (details in proof below)

Proof (Sketch)
$\forall C_1, C_2 \subseteq D$ with the same endpoints, $C_1 - C_2$ forms a closed path in D
$\therefore \int_{C_1 - C_2} \frac{1}{\zeta} d\zeta = 0 \Rightarrow \int_{C_1} \frac{1}{\zeta} d\zeta = \int_{C_2} \frac{1}{\zeta} d\zeta$

Example for D



THEOREM
Set $f(z) := \int_{z_0}^{z} \frac{1}{\zeta} d\zeta + \log z_0$ on a s.c. region $D \subseteq \mathbb{C} \setminus \{0\}$, we fix a $z_0 \in D$ and choose $\log z_0$
Then, f is an analytic branch of $\log z$ in D.

Proof
As D: s.c., $C_1 - C_2$: closed curve
$\therefore$ By closed curve thm, $\int_{C_1 - C_2} \frac{1}{\zeta} d\zeta = \int_{C_1} \frac{1}{\zeta} d\zeta - \int_{C_2} \frac{1}{\zeta} d\zeta = 0$
$\therefore f$ is analytic

Moreover, we want "$\exp(f(z)) = z$" $\Leftrightarrow$ "$ze^{-f(z)} = 1$"
Set $g(z) := ze^{-f(z)} \Rightarrow g'(z) = e^{-f(z)} - zf'(z)e^{-f(z)} = 0$

$\therefore g(z) = \text{const} = g(z_0) = z_0 e^{-\log z_0} = 1$

APPLICATION
Instead of this directly only used for log, we can use analytic branch to define $\sqrt{z}$.
Reason:
Say $z = re^{i\theta}$, then $(\sqrt{r}e^{i(\frac{\theta}{2} + \pi k)})^2 = re^{i\theta} = z \ \forall k \in \mathbb{Z}$, so $\sqrt{z}$ is not uniquely defined
$\therefore$ For $f(z) = \sqrt[n]{z}$, we can define an analytic branch for $\log z$ by $\sqrt[n]{z} = \frac{1}{n}\exp(\log z + 2\pi nk)$

SINGULARITY

DEFINITION
A deleted neighborhood of z_0 is an open set of $\{z \mid 0 < |z - z_0| < \delta\}$
the actual deleted thing lmao

DEFINITION
f is said to have an isolated singularity at z_0 if f is analytic in a deleted neighborhood D of z_0 but is not analytic in z_0

EXAMPLES (Intuition, formal names given later)
① "Artificial" singularity: $f(z) = \begin{cases} \sin z, & z \neq 0 \\ 1, & z = 0 \end{cases}$
② "Fixable by multiplying a polynomial": $\frac{1}{z}$ at $z = 0$
③ "Unfixable": $\exp(\frac{1}{z})$ at $z = 0$

Shun/翔海(@shun4midx)

DEFINITION

Say z_0 is a singularity of f, we can classify it as follows:

① If $\exists\, g$: ana at z_0 and $f(z)=g(z)$ in some deleted nbd of z_0, we say f has a removable singularity at z_0

② If for $z\neq z_0$, f can be written as $f(z)=\frac{A(z)}{B(z)}$ where A and B are analytic at z_0, $A(z_0)\neq 0$, $B(z_0)=0$, we say f has a pole at z_0. In particular, if B has a zero of order k at z_0, then we say z_0 is a pole of f of order k

③ f has neither a removable singularity nor a pole at z_0, then we call z_0 an essential singularity of f (not the focus of this course)

THEOREM (RIEMANN'S PRINCIPLE OF REMOVABLE SINGULARITIES)

If f has an isolated singularity at z_0 and if $\lim_{z\to z_0}(z-z_0)f(z)=0$, then the singularity is removable

Proof

Define $D'(z_0,\delta) := D(z_0,\delta)\setminus\{z_0\}$, $\exists\,\delta$, s.t. f: ana on $D'(z_0,\delta)$

Set $g(z) := \begin{cases} (z-z_0)f(z), & z\in D'(z_0,\delta) \\ 0, & z=z_0 \end{cases}$

Since $\lim_{z\to z_0}(z-z_0)f(z)=0=g(z_0)$, hence g is conti at z_0.

$\because$ f: ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D'(z_0,\delta)$

Morera needs it to be conti on the whole domain.

$\therefore$ g: conti on $D(z_0,\delta)$ + ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D(z_0,\delta)$ (apply the conti except on a line segment thing)

Now, set:

$h(z) := \begin{cases} \frac{g(z)-g(z_0)}{z-z_0}, & z\in D'(z_0,\delta) \\ g'(z), & z=z_0 \end{cases}$

h is ana because g is ana.

Moreover, as $f(z)=h(z)$ on $D'(z_0,\delta)$, thus z_0 is a removable singularity

COROLLARY

f has an isolated singularity at z_0. If f is bounded on some deleted nbd of z_0, then z_0 is a removable singularity

Proof

$\exists\,\delta$, s.t. f: ana and bounded on $D'(z_0,\delta)$

Given $\varepsilon>0$, $\forall\, 0<|z-z_0|<\frac{\varepsilon}{M}$, $|(z-z_0)f(z)|<\varepsilon \Rightarrow \lim_{z\to z_0}(z-z_0)f(z)=0$

$\therefore$ Conclude with thm above. □

THEOREM 3

Say f has an isolated singularity at z_0.

If $\exists\, k\in\mathbb{Z}_{\geq 0}$, s.t. $\lim_{z\to z_0}(z-z_0)^k f(z)\neq 0$ but $\lim_{z\to z_0}(z-z_0)^{k+1}f(z)=0$, then f has a pole of order k at z_0 (rem. sing = pole of order 0)

Proof

Set $g(z)=\begin{cases} (z-z_0)^{k+1}f(z), & z\in D'(z_0,\delta) \\ 0, & z=z_0 \end{cases}$

$\because \lim_{z\to z_0}(z-z_0)^{k+1}=0$ $\therefore$ g: conti at z_0

$\because$ f: ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D'(z_0,\delta)$

$\therefore$ g: conti on $D(z_0,\delta)$ + ana on $D'(z_0,\delta)$

$\therefore$ g: ana on $D(z_0,\delta)$

Set $h(z) := \begin{cases} \frac{g(z)-g(z_0)}{z-z_0} = (z-z_0)^k f(z), & z \in D'(z_0, \delta) \\ g'(z), & z = z_0 \end{cases}$ Shun/翔海 (@shun4midx)

$\therefore$ h: ana on $D(z_0, \delta)$.

As we know, by assumption, $\lim_{z \to z_0} h(z) \neq 0 \Rightarrow h(z_0) \neq 0$ ($\because$ h: ana)

$\therefore$ $f(z) = \frac{h(z)}{(z-z_0)^k} \Rightarrow f$ has a pole of order k at z_0 $\square$

REMARK

$|f(z)| < \frac{1}{\sqrt{|z|}}$ in a deleted nbd of 0 and f has an isolated singularity at $0 \Rightarrow 0$ is a removable singularity

($\because$ There exists nonbounded removable singularity)

Proof

Actually, $|zf(z)| < \sqrt{|z|} \Rightarrow \lim_{z \to z_0} zf(z) = 0 \Rightarrow 0$ is a removable singularity $\square$

REMARK

Similarly, if we have $|f(z)| < \frac{1}{|z|^{1.5}}$, then we know $|z^2 f(z)| < \sqrt{|z|} \Rightarrow \lim_{z \to z_0} z^2 f(z) = 0$

$\therefore \begin{cases} \text{Case 1: } \lim_{z \to z_0} zf(z) = 0, \text{ then removable singularity (pole of order 0)} \\ \text{Case 2: } \lim_{z \to z_0} zf(z) = 0, \text{ then pole of order 1} \end{cases}$

$\Rightarrow$ It has a pole of at most order 1 (higher the order $\Rightarrow$ the worse the pole)

THEOREM (CASORATI – WEIERSTRASS THEOREM)

If f has an essential singularity at z_0 and D is a deleted neighborhood of z_0, where f is analytic, then the range $R := \{f(z) \mid z \in D\}$ is dense in $\mathbb{C}$

Proof

Suppose not, then $\exists w \in \mathbb{C}$ and $\delta > 0$, s.t. open $D(w, \delta) \cap R = \phi$

In other words, $\forall z \in D$, $|f(z) - w| \geq \delta \Rightarrow \left|\frac{1}{f(z)-w}\right| \leq \frac{1}{\delta}$ $\forall z \in D \Rightarrow \frac{1}{f(z)-w}$ is bounded in the del nbd

By coro, $\frac{1}{f(z)-w}$ has a removable singularity at z_0

$\therefore \exists$ g: ana on $D' \cup \{z_0\}$, s.t. $g(z) = \frac{1}{f(z)-w} \Rightarrow f(z) = w + \frac{1}{g(z)}$ $\forall z \in D'$

$\therefore$ z_0 is a zero of $g(z)$ of $\underbrace{\text{finite order}}_{n}$ or $g(z_0) \neq 0$

$\therefore$ $f(z)$ has a pole of order $\leq n$ at z_0, so not an essential singularity $\times$

10-16-25 (WEEK 7) Shun/翔海 (@shun4midx)

DEFINITION

We say $\sum_{i=-\infty}^{\infty} \mu_i = L$ if $\sum_{i=0}^{\infty} \mu_i$ and $\sum_{j=1}^{\infty} \mu_{-j}$ both converge and their sum is L

THEOREM

$f(z)=\sum_{-\infty}^{\infty} a_k z^k$ is convergent in the domain $D=\{z: R_1<|z| \text{ and } |z|<R_2\}$, where $R_2 = \left(\limsup_{k\to\infty} |a_k|^{\frac{1}{k}}\right)^{-1}$, $R_1 = \limsup_{k\to\infty} |a_{-k}|^{\frac{1}{k}}$

We dk $R_1 ? R_2$, so cannot say $R_1<|z|<R_2$

Proof

$\sum_{k=0}^{\infty} a_k z^k$ converges when $|z| < \left(\limsup_{k\to\infty} |a_k|^{\frac{1}{k}}\right)^{-1}$

$\sum_{k=1}^{\infty} a_{-k}(z^{-1})^k$ converges when $|z^{-1}| < \left(\limsup_{k\to\infty} |a_{-k}|^{\frac{1}{k}}\right)^{-1} \Rightarrow |z| > \limsup_{k\to\infty} |a_{-k}|^{\frac{1}{k}}$

$\therefore f$: conv $\forall z \in D$.

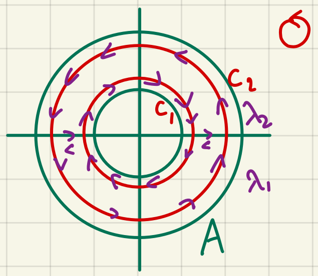
THEOREM

Laurent is when you expand around z_i f at z_i not ana (Taylor $\Rightarrow$ ana)

Let $A := \{z \mid R_1<|z|<R_2\}$. If f is analytic in A, then f has a $\boxed{\text{Laurent expansion } f(z)=\sum_{k=-\infty}^{\infty} a_k z^k}$ in A

Proof

Say $C_i := \partial B_{r_i}(0)$, $R_1<r_1<r_2<R_2$, $A = A_1 \cup A_2$, $A_1 = \{z \mid z\in A \text{ and } \operatorname{Im} z > -1\}$, $A_2 = \{z \mid z \in A \text{ and } \operatorname{Im} z < 1\}$.



$\therefore C_2 - C_1 = \lambda_1 + \lambda_2$

$\therefore \underbrace{\int_{C_2-C_1} \frac{f(\omega)-f(z)}{\omega - z}\, d\omega}_{(\star)} = \int_{\lambda_1+\lambda_2} \frac{f(\omega)-f(z)}{\omega-z}\, d\omega \quad \forall \omega \in A$

As f is ana in A,

$$g(\omega) := \begin{cases} \frac{f(\omega)-f(z)}{\omega - z}, & \omega \in A\setminus\{z\} \\ f'(\omega), & \omega = z \end{cases} \text{ is ana}$$

$\Rightarrow (\star) = \int_{\lambda_1} \frac{f(\omega)-f(z)}{\omega-z}\, d\omega + \int_{\lambda_2} \frac{f(\omega)-f(z)}{\omega - z}\, d\omega = 0$ by closed curve thm,

Moreover, by closed curve thm, $\int_{C_1} \frac{d\omega}{\omega - z} = 0$

As $\underbrace{\int_{C_2} \frac{d\omega}{\omega-z} = 2\pi i}_{\text{not } A, \text{ so why}}$, $f(z)\int_{C_2-C_1} \frac{d\omega}{\omega - z} = 2\pi i\, f(z)$

$$\begin{aligned} \therefore 2\pi i f(z) &= \int_{C_2 - C_1} \frac{f(\omega)}{\omega - z}\, d\omega = \int_{C_2} \frac{f(\omega)}{\omega-z}\, d\omega - \int_{C_1} \frac{f(\omega)}{\omega - z}\, d\omega \\ &= \int_{C_2} \frac{f(\omega)}{\omega(1-\frac{z}{\omega})}\, d\omega + \int_{C_1} \frac{f(\omega)}{z(1-\frac{\omega}{z})}\, d\omega \\ &= \boxed{\int_{C_2} \left[\frac{f(\omega)}{\omega} \sum_{i=0}^{\infty} \left(\frac{z}{\omega}\right)^i\right] d\omega}^{②} + \boxed{\int_{C_1} \left[\frac{f(\omega)}{z} \sum_{i=0}^{\infty} \left(\frac{\omega}{z}\right)^i\right] d\omega}^{①} \end{aligned}$$

We know $\sum_{i=0}^{\infty} \left(\frac{z}{\omega}\right)^2$ conv abs unif on C_2.

Hence, ② $= \sum_{i=0}^{\infty} \left[\int_{C_2} \frac{f(\omega)}{\omega^{i+1}}\, d\omega\right] z^i$

① $= \sum_{i=0}^{\infty} \left[\int_{C_1} f(\omega)\omega^i\, d\omega\right] z^{-(i+1)}$

$\therefore f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \quad \forall R_1<|z|<R_2$ □

Shun/翔海(@shun4midx)

Uniqueness

$\int_C z^{i-(k+1)} dz = \begin{cases} 0, & i-k \neq 0 \\ 2\pi i, & i-k=0 \end{cases}$

If $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k \ \forall R_1 < |z| < R_2$, then $\int_C \frac{f(z)}{z^{k+1}} dz = \int_C \sum_{i=-\infty}^{\infty} c_i z^{i-(k+1)} dz = \sum_{i=-\infty}^{\infty} c_i z^{i-(k+1)} dz$ for $C = C_r(0) \subseteq A$.

$\therefore \int_C \frac{f(z)}{z^{k+1}} dz = c_k 2\pi i \ \forall C = C_r(0) \subseteq A$

Thus, (i) $\int_C \frac{f(z)}{z^{k+1}} dz = c_k$ is indep of r for $C = C_r(0) \subseteq A$

(ii) $a_k = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{\omega^{i+1}} d\omega$ is indep of C_1 and C_2

$\therefore$ The Laurent expansion of f in A is unique $\square$

REMARK

Taking $R_1 = 0$, we can take Laurent expansion at a pole or removable singularity.