RESIDUE AT INFINITY

DEFINITION

from the w-plane

$\mathrm{Res}(f,\infty) := -\frac{1}{2\pi i}\int_C f(z)\,dz$, $C := C_R(0)$

INTUITION/PROOF OF THEOREM

Say $f(z)$ is a poly of deg n, hence it is entire

Then, $z\to\infty \Rightarrow |f(z)|\to\infty$, i.e. $w=0$ is a pole ($w\to 0 \Rightarrow |f(w)|\to 0$) at N

Using $z=\frac{1}{w}$,

$$\int_{C_R(0)} f(z)\,dz = \int_{-C_{\frac{1}{R}}(0)} \underbrace{f\left(\tfrac{1}{w}\right)\cdot -\tfrac{1}{w^2}}_{=:\, g(w)}\,dw$$

Change of direction from change of variables

Notice, $g(w)$ is ana on $\mathbb{C}\setminus\{\frac{1}{z_i}, 0\}_{z_i\neq 0}$ on the w-plane. Define $\{w_j\} := \{\frac{1}{z_i}, 0\}_{z_i\neq 0}$

$|\frac{1}{z_i}| > \frac{1}{R}$

Then, $\int_{C_{\frac{1}{R}}(0)} f\left(\frac{1}{w}\right)\cdot\frac{1}{w^2}\,dw = -2\pi i \sum_{|w_j|<\frac{1}{R}} \mathrm{Res}\left(f\left(\frac{1}{w}\right)\frac{1}{w^2}, w_j\right)$

$= -2\pi i\,\mathrm{Res}\left(f\left(\frac{1}{w}\right)\frac{1}{w^2}, 0\right)$

THEOREM

$\mathrm{Res}(f,\infty) = -\mathrm{Res}\left(\frac{1}{w^2}f\left(\frac{1}{w}\right), 0\right)$

EXAMPLE

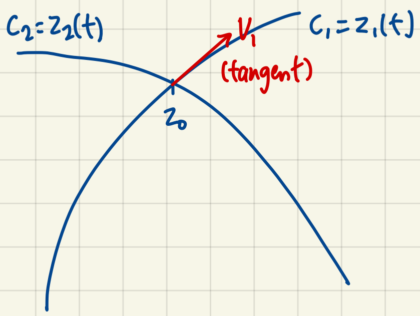
encompass all residues ($\because \frac{1}{0} = $"$\infty$")

Say $f(z) = \frac{5z-2}{z(z-1)}$, then $\int_{C_2(0)} f(z)\,dz = -2\pi i\,\mathrm{Res}\left(\frac{1}{w^2}f\left(\frac{1}{w}\right), 0\right) = 10\pi i$

CONFORMAL MAPPING (ANGLE-PRESERVING)

NOTES CONVENTION

All curves $z(t)$ here are assumed to have $\dot z(t)\neq 0$ $\forall t\in[a,b]$



Say $z_i(t) = x_i(t) + iy_i(t)$, $v_i :=$ tangent of C_i at $z=z_i$, $\angle C_1, C_2 =$ counterclockwise angle from v_1 to v_2

DEFINITION

For f: ana at z_0,

(i) f is locally 1-1 at z_0 if $\exists\,\delta>0$, s.t. $f|_{D(z_0,\delta)}$ is 1-1

(ii) For a region D, f is locally 1-1 throughout D if f is locally 1-1 at z $\forall z\in D$

(iii) f is 1-1 on D, if $f(z_1)\neq f(z_2)$ $\forall\, z_1\neq z_2\in D$

EXAMPLE

$f(z)=\exp(z)$ is
- Locally 1-1 on $\mathbb{C}$ ($\because$ we can always take a small enough ball s.t. it doesn't reach $2\pi i$ more)
- NOT 1-1 on $\mathbb{C}$

EXAMPLE

$f(z)=z^2$ is locally 1-1 $\forall z\neq 0$ ($\because f(z)=f(-z)$)

DEFINITION

For f: ana at $z=z_0$,
(i) f is conformal at z_0 if $\forall C_1, C_2$ passing through z_0, $\angle C_1, C_2 = \angle f(C_1), f(C_2)$
(ii) For a region D, f is conformal in D if f is conformal at z $\forall z\in D$

THEOREM

Say f is ana at z_0 and $f'(z_0)\neq 0$. Then, f is conformal and locally 1-1

REMARK/KEY EXAMPLE

Reflections aren't conformal mappings. In particular, $f(z)=\bar{z}$ is not conformal

PROOF OF THEOREM

For $C_i: z_i(t)=x_i(t)+iy_i(t)$, $z_i(t_0)=z_0$,

Define $v_i(t_0)=\dot{z}_i(t_0)$
We know $v_i(t)=|v_i|e^{i\theta_i(t)}$, $\theta(t)\in[0,2\pi)$
$\Rightarrow \angle C_1, C_2 = \theta_2(t)-\theta_1(t)$

Then, for $f(C_i)$,
$w_i(t)=f(z_i(t)) \Rightarrow \dot{w}_i(t)=f'(z_i(t))\,\dot{z}_i(t) \overset{f'(z_0)\neq 0}{=} |f'(z_i(t))|e^{i\varphi(t)}\,|\dot{z}_i(t)|e^{i\theta_i(t)} = |\cdot|\,e^{i(\varphi(t)+\theta_i(t))}$
$\therefore \angle f(C_1), f(C_2)\big|_{t=t_0} = (\varphi(t_0)+\theta_2(t_0))-(\varphi(t_0)+\theta_1(t_0)) = \angle C_1, C_2$

(Continue next time to have not just at $t=t_0$)