THEOREM (GAUSS-LUCAS THEOREM)

The zeros of the derivative of a polynomial lie within the convex hull of the zeros of the polynomial

Proof

Let $p(x)$ be a nonconstant polynomial $\in \mathbb{C}[x]$, and $\alpha_1, \dots, \alpha_n$ be roots of p (counted by multiplicity)

Then, $p(x) = c\prod_{i=1}^{n}(x-\alpha_i)$. Moreover, $\frac{p'(x)}{p(x)} = \sum_{i=1}^{n}\frac{1}{x-\alpha_i}$.

otherwise it's trivial

Let a be a root of $p'(x)$ and $a \notin \{\alpha_1, \dots, \alpha_n\}$.

Then, $\frac{p'(a)}{p(a)} = \sum_{i=1}^{n}\frac{1}{a-\alpha_i} = \sum_{i=1}^{n}\frac{\bar{a}-\bar{\alpha}_i}{|a-\alpha_i|} \Rightarrow \bar{a} = \sum_{i=1}^{n} c_i\bar{\alpha}_i$, $c_i := \frac{1}{|a-\alpha_i|^2} \Big/ \sum_i \frac{1}{|a-\alpha_i|^2} \in \mathbb{R}_{\geq 0}$

by def

$\therefore a = \sum_{i=1}^{n} c_i\alpha_i$, $c_i \in \mathbb{R}_{\geq 0}$, $\sum c_i = 1$ $\square$

ANALYTIC FUNCTION ON A DISC

NOTATION

Assume open disc if disc

$D = D(\alpha; r)$ is an open disc $\subseteq \mathbb{C}$, then: analytic $f(z)$ in D, $\forall a \in D$, $g_a(z) = \begin{cases} (f(z)-f(a))/(z-a), & z \neq a \\ f'(a), & z = a \end{cases}$ is cont. on D (Notation for g_a)

THEOREM (RECTANGLE THEOREM)

For a closed rectangle $R \subseteq D$, define $\Gamma = \partial R$, then we proved $\int_\Gamma f(z)dz = \int_\Gamma \frac{f(z)-f(a)}{z-a}dz = \int_\Gamma g_a(z)dz = 0$ (They just both happen to equal 0, not they are directly related)

THEOREM

$\exists F, G_a$ analytic in D s.t. $f = F'$, $g_a = G_a'$

Proof

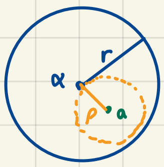
We consider $F := \int_\alpha^z f(z)dz$ and $G_a := \int_\alpha^z g_a(z)dz$

Define $C: \alpha \longrightarrow \alpha + \mathrm{Re}(z) \longrightarrow \alpha + z$, which is in D. $\square$

THEOREM (CLOSED CURVE THEOREM)

Let C be a closed curve $\subseteq D$. Then, $\int_C f dz = 0$

THEOREM (CAUCHY INTEGRAL FORMULA)



For some $0 < \rho < r$, $\forall |a - r| < \rho$, $f(a) = \frac{1}{2\pi i}\int_{C_\rho}\frac{f(z)}{z-a}dz$, where C_ρ is $\partial D(\alpha; \rho)$

THEOREM (TAYLOR EXPANSION)

$f(z) = f(\alpha) + f'(\alpha)(z-\alpha) + \frac{f''(\alpha)}{2}(z-\alpha)^2 + \dots$ holds $\forall |z-\alpha| < \rho$, for some ρ, s.t. $\exists a \in D$, $0 < \rho < r$, $|a - \alpha| < \rho$

Proof Sketch

$\forall |z-\alpha| < r$, $\frac{f^{(k)}(\alpha)}{k!} = \frac{1}{2\pi i}\int_{C_\rho}\frac{f(z)}{(z-\alpha)^{k+1}}dz$

THEOREM

Let $\tilde{D}$ be an arbitrary open set, f analytic on $\tilde{D}$. Then, fix $\alpha \in \tilde{D}$, we have $f(z) = \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ $\forall z \in D(\alpha, r) \subseteq \tilde{D}$

EXAMPLE

Let $f(z) = \frac{1}{z-i}$, analytic $\forall z \neq 1$.

Then, on $D=D(2;1)$, $f(z)=\frac{1}{1+(z-2)}=1-(z-2)+(z-2)^2-\dots \ \forall |z-2|<1$ Shun/翔海(@shun4midx)

In fact, the expression converges $\forall |z-2|<1$
diverges $\forall |z-2|\geq 1$

(However, $f(z)$ is analytic $\forall |z-2|=1$, so clearly its Taylor expansion is different)

PROPOSITION

$g_a(z)$ is analytic $\forall z,a \in D(\alpha; r)$

Proof

Use the thm above, in some neighborhood of α, $f(z)=f(\alpha)+f'(\alpha)(z-\alpha)+\frac{f''(\alpha)}{2}(z-\alpha)^2+\dots$

Then, g has the power series expansion $g(z)=f'(\alpha)+\frac{f''(\alpha)}{2!}(z-\alpha)+\frac{f^{(3)}(\alpha)}{3!}(z-\alpha)^2+\dots \Rightarrow g$ is analytic at α. □

THEOREM

If f is analytic at z, then f is infinitely diffable at z

Proof

We know from above, f may be expressed as a power series. Hence, it is infinitely diffable. □