11-25-25 (WEEK 13)

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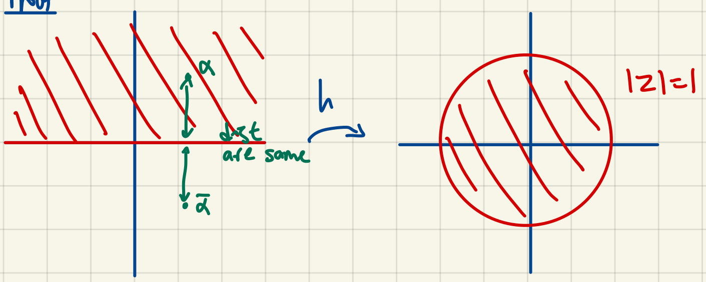
THEOREM

The conformal mapping $h: \mathbb{H} := \{z \mid \operatorname{Im} z > 0\} \xrightarrow{\text{onto}} D(0,1)$ are of the form $h(z) = e^{i\theta}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$ for some $\alpha \in \mathbb{H}$

Proof



same dist of $\bar{\alpha}$

As we want $|z|=1$, consider $f(z) = \frac{z-\alpha}{z-\bar{\alpha}}$, $\alpha \in \mathbb{H}$, $z \in \mathbb{R} \Rightarrow |f(z)| = 1$

check $f(z) = \frac{z-\alpha}{z-\bar{\alpha}}$ on real plane

As f is globally 1-1 and ana on $\mathbb{C} \setminus \{\bar{\alpha}\} \supseteq \mathbb{H}$, and e.g. $\left|\frac{i-\alpha}{i-\bar{\alpha}}\right| < 1$, $i \in \mathbb{H}$

$\therefore \forall z \in \mathbb{H}$, $|f(z)| \leq 1$, i.e. $f(\mathbb{H}) \subseteq D(0,1)$

find the inside, so we need:

1. $|f(z)| \leq 1 \ \forall z \in \mathbb{H}$
2. $f^{-1}(D(0,1)) \subseteq \mathbb{H}$

Then, check f^{-1} to see $f^{-1}(D(0,1)) \subseteq \mathbb{H}$ (trivial)

By lemma, $\forall h$ satisfying $\mathbb{H} \to D(0,1)$ that are onto, $h = g \circ f$, $g \in \operatorname{Aut}(D(0,1)) = e^{i\theta} B_\alpha \Rightarrow h = (e^{i\theta} B) \circ f$ □

THEOREM

$h \in \operatorname{Aut}(\mathbb{H}) \Rightarrow h = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$

Proof Sketch

$\forall h \in \operatorname{Aut}(\mathbb{H})$, $\exists g \in \operatorname{Aut}(D(0,1))$, s.t. $h = f^{-1} \circ g \circ f = f^{-1} \circ (e^{i\theta} B_\beta) \circ f$, then apply Thm above for f

DEFINITION

For f: func on $\mathbb{C}$, z_0 is a fixed point of f if $f(z_0) = z_0$

PROPOSITION

A bilinear transformation f other than identity has ≤ 2 fixed points. If we regard f as a function on $\mathbb{C} \cup \{\infty\}$, then it has 2 fixed pts counted by multiplicity

Proof

For $f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$, $f(z_0) = z_0 \Rightarrow az_0 + b = z_0(cz_0 + d)$

Consider two cases,

① $c \neq 0$, then z_0 is a root of $cx^2 + (d-a)x + b = 0 \Rightarrow f$ has ≤ 2 fixed pts $\left(f(\infty) = \frac{a}{c}\right)$

② $c = 0$, then $f(z)$ is a linear function $\Rightarrow f$ has one fixed pt $(f(\infty) = \infty)$

LEMMA

Let z_1, z_2, z_3 be distinct pts in $\mathbb{C}$.

The unique bilinear transformation sending z_1, z_2, z_3 to $\infty, 0, 1$ respectively (☆) is given by $T_{(z_1, z_2, z_3)}(z) = \frac{z-z_2}{z-z_1} \cdot \frac{z_3-z_1}{z_3-z_2}$

Proof

- Uniqueness

  Let S be a bilinear transformation satisfying ☆

  Then, $S^{-1} \circ T(z_i) = z_i$, $i = 1, 2, 3$, so by prop above, $S^{-1} \circ T = \text{id}$ ($\because$ 3 fixed pts)

Consider another S, then $S^{-1} \circ T$ we say it's $S^{-1} \circ T = \text{id}$.

- Existence: Trivial. □

REMARK

For the lemma above, $z_1 = \infty \Rightarrow \frac{z-z_2}{z_3-z_2}$, $z_2 = \infty \Rightarrow \frac{z_3-z_1}{z_3-z}$, $z_3 = \infty \Rightarrow \frac{z-z_2}{z-z_1}$ (just consider dominating terms, don't memorize)

DEFINITION

For distinct $z_1, z_2, z_3, z_4 \in \mathbb{C}$, the cross-ratio is defined as $(z_1, z_2, z_3, z_4) := T_{(z_1,z_2,z_3)}(z_4)$

PROPOSITION

The cross-ratio is invariant under bilinear transformation S, i.e. $(z_1, z_2, z_3, z_4) = (Sz_1, Sz_2, Sz_3, Sz_4)$

Proof

$$\begin{tikzcd} z_1, z_2, z_3 \arrow[r, "T_{(z_1,z_2,z_3)}"] \arrow[d, "S"] & \infty, 0, 1 \\ Sz_1, Sz_2, Sz_3 \arrow[r, "T_{(Sz_1,Sz_2,Sz_3)}"] & \infty, 0, 1 \end{tikzcd}$$

Then, $T^{-1}_{(Sz_1,Sz_2,Sz_3)} \circ T_{(z_1,z_2,z_3)}(z_i) = S(z_i)$ for $i = 1,2,3$

By prop, $T^{-1}_{(Sz_1,Sz_2,Sz_3)} \circ T_{(z_1,z_2,z_3)} = S$ (3 fixed pts)

$\Rightarrow T^{-1}_{(Sz_1,Sz_2,Sz_3)} \circ T_{(z_1,z_2,z_3)}(z_4) = S(z_4) \Rightarrow T_{(z_1,z_2,z_3)}(z_4) = T_{(Sz_1,Sz_2,Sz_3)}(S(z_4))$ $\square$

RIEMANN MAPPING THEOREM

OVERVIEW

"holomorphically simply connected D" (D: open + connected)

$\forall f$: ana on D, C: simple closed curve $\subseteq D$, and $\int_C f\,dz = 0$ (☆)

We define "s.c." in $\mathbb{C}$ as $(\mathbb{C} \cup \{\infty\}) \setminus D$: path-connected, then we say D is s.c.

$\Rightarrow$ (Already Proved) D satisfies (☆)

With Riemann Mapping Thm, we get $D \overset{\text{conformally equiv}}{\sim} D(0,1) \Rightarrow D$ is s.c. in topological sense

RIEMANN MAPPING THEOREM

Let $R :=$ s.c. region $\subsetneq \mathbb{C}$, $U := D(0,1)$.

Then, given $z_0 \in R$, $\exists$ a unique conformal mapping φ of R onto U s.t. $\varphi'(z_0) > 0$ and $\varphi(z_0) = 0$ (△)

Intuitively, use Schwarz Lemma, $\because 0 \to 0$

Proof

- Uniqueness

  Let φ_1, φ_2 satisfy (△).

  Then, $\Phi := \varphi_2 \circ \varphi_1^{-1} \in \mathrm{Aut}(U)$ with $\Phi(0) = 0$

  $\therefore$ By Lemma, $\Phi(z) = e^{i\theta} z \Rightarrow \Phi'(z) = e^{i\theta}$

  However, $\Phi'(0) = \varphi_2'(z_0)(\varphi_1'(z_0))^{-1} > 0 \Rightarrow \theta = 0$

  Uniqueness

  $\Phi := \varphi_2^{-1} \circ \varphi_1$, $\Phi(0) = 0$, $\Phi \in \mathrm{Aut}(U)$

  $\therefore \Phi = e^{i\theta} z$

- Existence

  Let $\mathcal{F}_{z_0} := \{f \mid f: R \to U,\ \text{conformal},\ f'(z_0) > 0\}$ ($|f| \le 1$; 1-1 + ana)

CLAIM A: $\mathcal{F}_{z_0} \neq \emptyset$

Consider $p_0 \in \mathbb{C} \setminus R$. Then, $\frac{1}{z - p_0}$: 1-1 + analytic

Case ①: $\exists r$, s.t. $D(p_0, r) \cap R = \emptyset \Rightarrow \frac{1}{z - p_0} \le \frac{1}{r}$ (bounded $\Rightarrow$ can normalize to $\le 1 \rightsquigarrow U$)

Case ②: $\exists \{\zeta_i\} \to p_0$, $\zeta_i \in R$, $\boxed{R\text{: s.c.}}$

We can define a branch of log on R

$\Rightarrow$ We can choose a branch of log of $f(z) := \sqrt{\frac{z - p_0}{z_0 - p_0}}$ s.t. $f(z_0) = 1$ ($\Rightarrow$ We chose s.t. -1 would not appear!)

Claim: $|f(z) - (-1)| > \delta$ for some $\delta > 0$ $\forall z \in R$ (universal bound)

Proof

Suppose not, $\exists \{\zeta_n\}$ with $\lim_{n\to\infty} f(\zeta_n) = -1 \Rightarrow \frac{\zeta_n - p_0}{z_0 - p_0} \to 1$ as $n \to \infty \Rightarrow \zeta_n \to z_0$

However, f: conti $\Rightarrow \lim_{n\to\infty} f(\zeta_n) = f(z_0) = 1$ $\times$

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Set $g(z) = \frac{1}{f(z)+i}$, then $|g(z)| \leq \frac{1}{\delta}$ $\forall z \in R$ (bounded $\Rightarrow$ can normalize to $\leq 1 \rightsquigarrow U$)

For Cases ① and ②, (let $g(z) := \frac{1}{z-p_0}$ for ①), $g'(z_0) \neq 0 \Rightarrow \exists \theta$, s.t. $e^{i\theta} g'(z_0) = |g'(z_0)| > 0$
$\Rightarrow \mathcal{F}_{z_0} \neq \emptyset$ $\square$

Next time: Claim B+C.