

CompAna Notes

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CompAna Midterm

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9-2-2S (WEEK 1)

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Can any two-variable function $f(x,y)$ be re-written into diffable $F(z)$ with $z=x+iy$? No. ($f(x,y)=x$)

POWER SERIES

DEFINITION OF ANALYTIC POLYNOMIAL

If $P(x,y) = \sum_{k=0}^N a_k z^k$ for some $a_k \in \mathbb{C}$, then it is an analytic polynomial

EXAMPLE

$x^2 - y^2 + 2xyi = (x+iy)^2 \Rightarrow$ analytic

However, $x^2 + y^2 - 2xyi$ is not (when we set $x^2 + y^2 - 2xyi = \sum a_k (x+iy)^k$, there is a contradiction)

DEFINITION OF PARTIAL DERIVATIVE

Let $f(x,y) = u(x,y) + iv(x,y)$, $u, v \in \mathbb{R}$.

If it exists, then $\begin{cases} f_x(x,y) = u_x(x,y) + iv_x(x,y) \\ f_y(x,y) = u_y(x,y) + iv_y(x,y) \end{cases}$

PROPOSITION

\neg differentiable (C-R eq)

A polynomial $P(x,y)$ is analytic $\Leftrightarrow P_y = iP_x$

Proof

$$\begin{aligned} \Rightarrow & \exists a_k \in \mathbb{C}, N \in \mathbb{N}, \text{s.t. } P(x,y) = \sum_{k=0}^N a_k (x+iy)^k \\ & \Rightarrow P_y = \sum_{k=1}^N k a_k (x+iy)^{k-1} i, P_x = \sum_{k=1}^N k a_k (x+iy)^{k-1} \\ & \Rightarrow \therefore P_y = iP_x \end{aligned}$$

\Leftarrow : With $Q^k(x,y) = c_0 x^k + c_1 x^{k-1} y + \dots + c_k y^k$, we can rewrite $P(x,y) = \sum_{k=0}^N Q^k(x,y)$

Notice, $Q_y^k = i Q_x^k \forall k$.

We know $Q^k = \sum_{p=0}^k C_p x^{k-p} y^p$

$$\therefore Q_y^k = \sum_{p=1}^k p C_p x^{k-p} y^{p-1} = \sum_{p=0}^{k-1} (k-p) C_p x^{k-p-1} y^p = i Q_x^k$$

In other words, $\sum_{p=0}^k p C_p x^{k-p} y^{p-1} = \sum_{p=0}^{k-1} (k-p+1) C_{p-1} x^{k-p} y^p$;

- $p=1$: $i k C_0 = C_1 \Rightarrow C_1 = \binom{k}{1} C_0$;
- $p=2$: $2 C_2 = (k-1) C_1 \Rightarrow C_2 = i^2 \frac{k(k-1)}{2} C_0$
- $p \geq 3$: $(k-p+1) C_{p-1} = i^p \binom{k}{p} C_0$

$$\therefore Q^k = \sum_{p=0}^k i^p \binom{k}{p} C_0 x^{k-p} y^p = (x+iy)^k \quad \forall k$$

$\therefore P$ is analytic. \square

REMARK

Usually we don't write " $P_y = iP_x$ ", rather:

$$\begin{cases} P_y = u_y + iv_y \\ P_x = u_x + iv_x \\ P_{xi} = -v_x + u_{xi} \end{cases} \Rightarrow \begin{cases} -v_x = u_y \\ u_x = v_y \end{cases}$$

REMARK

A nonconstant analytical polynomial can't be real (since we require $P_y = iP_x$)

DEFINITION

Consider f , a complex-valued function, defined on the neighborhood of $z=z_0$.

We say f is differentiable at $z=z_0$ if $\lim_{h \rightarrow 0} \frac{f(z_0+h) - f(z_0)}{h}$ exists, where it is denoted as $f'(z)$.

(Note: We must consider $|h| \rightarrow 0 \forall h \in \mathbb{C}$)

EXAMPLE $f(z) = \bar{z}$

$$\frac{f(z+h) - f(z)}{h} = \frac{\bar{h}}{h} \xrightarrow[h \rightarrow 0]{} 1 \text{ as } h \rightarrow 0.$$

$\therefore \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$ DNE, i.e. f is not diff

PROPOSITION

If f, g diff at $z=0$, $h_i = f+g \Rightarrow h'_i = f'_i + g'_i$. Product and quotient rules also hold. $(g(z_0) \neq 0)$

PROPOSITION

$P(z) = \sum_{k=0}^{\infty} a_k z^k$ is diff on \mathbb{C} , in fact: $P'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$

DEFINITION OF POWER SERIES

A power series is an infinite series in the form $\sum_{k=0}^{\infty} K_k z^k$

LIMSUP

$$\overline{\lim}_{n \rightarrow \infty} a_n = L \Leftrightarrow \forall \varepsilon > 0, \exists N > 0, \text{ s.t. } n \geq N \Rightarrow \left| \sup_{k \geq n} a_k - L \right| < \varepsilon$$

$\boxed{L - \varepsilon < \sup_{k \geq n} a_k < L + \varepsilon}$ $\forall \varepsilon > 0, \exists N, \text{ s.t. } n \geq N \Rightarrow a_k < L + \varepsilon$
 $\forall \varepsilon > 0, \exists N \exists k \geq N, \text{ s.t. } a_k > L - \varepsilon$

THEOREM

Given the power series $\sum_{k=0}^{\infty} c_k z^k = P(z)$, define $L := \overline{\lim}_{k \rightarrow \infty} |c_k|^{\frac{1}{k}}$, then we have:

- (1) $L = 0 \Rightarrow P(z)$ converges $\forall z \in \mathbb{C}$
- (2) $L = \infty \Rightarrow P(z)$ converges only at $z=0$
- (3) $0 < L < \infty \Rightarrow P(z)$ converges on $|z| < t$ and diverges on $|z| > t$

Proof

(i) Given any $z \in \mathbb{C}$, $\overline{\lim}_{n \rightarrow \infty} |c_n|^{\frac{1}{n}} z = 0$
 \therefore Take $\varepsilon = \frac{1}{2}$, $\exists N$ s.t. $k > N \Rightarrow |c_k|^{\frac{1}{k}} |z| < \frac{1}{2} \Rightarrow |c_k z^k| < (\frac{1}{2})^k$
 $\therefore \sum |c_k z^k| < \sum (\frac{1}{2})^k = 1 \quad \checkmark$

(ii) Consider small $|z|$, $\forall N \in \mathbb{N}, \exists k > N$, s.t. $|c_k|^{\frac{1}{k}} > \frac{1}{|z|} \therefore |c_k z^k| > 1$
 $\therefore P(z)$ does not converge at $z \checkmark$

(iii) Take $R = \frac{1}{L}$, $|z| = R(1-\delta)$, $1 > \delta > 0$ when $|z| < t$
We know $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall k > N$, s.t. $|z| (L - \varepsilon) < \sup_{k \geq N} |c_k|^{\frac{1}{k}} |z| \leq (\varepsilon + L) |z| = (1 + \varepsilon R)(1 - \delta) - \delta < 1 - \frac{\delta}{2}$
 \therefore It is abs conv

If $|z| > R$, $\overline{\lim} |c_k|^{\frac{1}{k}} |z| > 1 \Rightarrow$ for inf values of k , $|c_k z^k| > 1 \Rightarrow \sum c_k z^k$ div

REMARK

Let $t = R$ be the radius of convergence

Then, $\sum c_k z^k$ conv uni for $|z| < R - \delta$

$$\sum |c_k z^k| \leq \sum |c_k| (R - \delta)^k < \infty$$

\Rightarrow On $B(0, R - \delta)$, $\sum c_k z^k$ is conti. $\forall \delta > 0$

EXAMPLE (evaluating at R)

$$\sum_{n=0}^{\infty} n z^n$$

We know $\overline{\lim} n^{\frac{1}{n}} = 1 \Rightarrow R = 1$

When $|z| = 1$, $|n z^n| = n \Rightarrow$ diverge

$$\sum \frac{z^n}{n!} \Rightarrow R=1$$

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When $|z|=1$, it conv, similarly $|z|>1$ too.

$$\sum z^n, R=1$$

When $|z|=1, z \neq 1$ conv

$\sum z^n$ conv $\forall z \in \mathbb{C}$ by ratio test

CAUCHY PRODUCT

Given $P_1(z) = \sum a_k z^k, R=R_1$; $P_2(z) = \sum b_k z^k, R=R_2$. Then $P_1 P_2 = \sum c_k z^k$ where $c_k = \sum_{p+q=k} a_p b_q$

Then, $R_3 \geq \min(R_1, R_2)$

DIFFERENTIATION

THEOREM

Given $P(z) = \sum_{k=0}^{\infty} c_k z^k, R>0$, we know $\lim |c_k|^{\frac{1}{k}} = R$ and $\lim |k c_k|^{\frac{1}{k}} = R$ since $\lim |k|^{\frac{1}{k}} = 1$

Then, $P'(z) = \sum_{k=1}^{\infty} k c_k z^{k-1}$ with radius of convergence R

Proof

• $0 < R < \infty$: Take $|z| = R - \delta, R \geq \delta > 0$

$$\text{Then, } \frac{P(z+h) - P(z)}{h} = \frac{1}{h} \sum_{k=1}^{\infty} c_k [(z+h)^k - z^k] = \frac{1}{h} \left[\sum_{k=1}^{\infty} c_k k z^{k-1} h + h \sum_{k=2}^{\infty} c_k k z^{k-1} \right] = \sum_{k=1}^{\infty} k c_k z^{k-1} + \sum_{k=2}^{\infty} c_k k z^{k-1}$$

Proof continued next time!

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THEOREM

Given power series $P(z) = \sum c_k z^k$, radius of convergence R , then $P'(z)$ exists on $|z| < R$, $P'(z) = \sum k c_k z^{k-1}$

Proof

For $0 < R < \infty$,

Let $|z| = R - \frac{h}{2}$, $R > h > 0$. WLOG, consider $|h| < \frac{R}{2}$, and consider $\frac{P(z+h) - P(z)}{h}$ vs $\sum k c_k z^{k-1}$
 Then, $\frac{P(z+h) - P(z)}{h} = \frac{1}{h} \sum c_k (z+h)^{k-1} - \sum c_k z^{k-1} = \sum_{k=1}^{\infty} k c_k z^{k-1} + \sum_{k=2}^{\infty} c_k h^{k-1}$, $b_k = \sum_{p=2}^k \binom{k}{p} h^{p-1} z^{k-p}$
 $\star \sum_{p=2}^k \binom{k}{p} h^{p-1} z^{k-p}$

If $|z| = 0$, $b_k = h^{k-1} \Rightarrow \star = \sum c_k h^{k-1} < \infty$, $\star \rightarrow 0$ as $h \rightarrow 0$. ✓

$$\text{If } |z| \neq 0, \binom{k}{p} = \binom{k}{p-2} \cdot \frac{(k-p+2)(k-p+1)}{p(p-1)} \leq \binom{k}{p-2} k^2$$

$$\begin{aligned} \text{Then, } |b_k| &\leq \frac{|h|^k}{(2\pi)^k} k^2 \sum_{p=2}^k \binom{k}{p-2} (|h|)^{p-2} (|z|)^{k-(p-2)} \\ &\leq \frac{|h|^k}{(2\pi)^k} k^2 \sum_{j=0}^k \binom{k}{j} (|h|)^j (|z|)^{k-j}, j=p-2 \\ &= \frac{|h|^k}{(2\pi)^k} k^2 (|z| + |h|)^k \\ &\leq \frac{|h|^k}{(2\pi)^k} k^2 (R - \frac{h}{2})^k \end{aligned}$$

$$\text{Hence, } \left| \sum_{k=2}^{\infty} c_k b_k \right| \leq \sum_{k=2}^{\infty} |c_k| |b_k| = \frac{|h|^k}{(2\pi)^k} \sum_{k=2}^{\infty} k^2 |c_k| (R - \frac{h}{2})^k \xrightarrow[<\infty]{} 0$$

Remaining case is simple for $R = \infty$.

EXAMPLE

$$\sum_{k=0}^{\infty} \frac{z^k}{k!}, R = \infty$$

$$\downarrow \frac{1}{k!}$$

$$\sum_{k=1}^{\infty} \frac{z^k}{(k-1)!} = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!}$$

$$\begin{aligned} \therefore f' &= f \\ \therefore \sum_{k=0}^{\infty} \frac{z^k}{k!} &= ce^z \end{aligned}$$

COROLLARY

Power series are smooth in their domain of conv.

COROLLARY

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, R > 0 \Rightarrow c_k = \frac{f^{(k)}(0)}{k!}$$

THEOREM (UNIQUENESS)

If $\exists \{z_n\}_n \xrightarrow{n \rightarrow \infty} 0$ and $\sum c_k z_n^k = 0$, then $c_k = 0 \forall k$

Proof

$$P(0) = \lim_{n \rightarrow \infty} P(z_n) = 0 \Rightarrow c_0 = 0$$

Consider $g(z) = \frac{f(z)}{z}$, with the same radius of convergence as $f(z)$

$$\text{Then, } g(0) = \lim_{n \rightarrow \infty} \frac{f(z_n)}{z_n} = 0 \Rightarrow c_1 = 0$$

\therefore By induction on n , $c_n = 0 \forall n$. \square

COROLLARY

If $\sum a_k z^k$ and $\sum b_k z^k$ agree on $\{z_n\}_n$ as $n \rightarrow \infty$, then $a_k = b_k \forall k$

Proof

Simply consider $\sum (a_k - b_k) z^k = 0$.

ANALYTIC FUNCTIONS

We write $z \in \mathbb{C}$, $z = x + iy$, $x, y \in \mathbb{R}$

Then, $f: \mathbb{C} \rightarrow \mathbb{C}$ for $f(z) = u(z) + iv(z)$, $u, v: \mathbb{C} \rightarrow \mathbb{R}$
 $\text{f}(x,y) = u(x,y) + i v(x,y)$

PROPOSITION 1

If $f = u + iv$ is differentiable at z , then f_x, f_y exist and satisfy the Cauchy-Riemann Equation: $f_y = if_x$

Proof

By def, f is diff $\Rightarrow \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$ exists.

$$(i) \text{ As } h \rightarrow 0 \text{ along the real axis, } \lim_{h \rightarrow 0} \frac{f(x+\xi, y) - f(x, y)}{\xi} = f_x$$

$$(ii) \text{ As } h \rightarrow 0 \text{ along the imaginary axis, } \lim_{h \rightarrow 0} \frac{f(x, y+\xi) - f(x, y)}{\xi} = f_y \quad (\text{since change in } y \text{ as } \xi \text{ means change in } z \text{ as } \xi)$$

$$\therefore f_y = if_x \quad \square$$

QUESTION: IF f_x, f_y EXIST AT A POINT z , AND $f_y = if_x$, DOES IT MEAN f IS DIFFERENTIABLE?

COUNTEREXAMPLE

$$f(z) = \begin{cases} \frac{xy(x+i\bar{y})}{x^2+y^2}, & z \neq 0 \\ 0, & z=0 \Leftrightarrow (x, y)=0 \end{cases} \quad (\text{i.e. } xy \cdot \frac{z}{|z|^2})$$

We notice $f(z) = 0$ on both x -axis and y -axis $\Rightarrow f_x(0) = f_y(0) = 0$

However, along $y=x$ ($a \neq 0$), we get: $f(x, ax) = \frac{a(1+ia)}{1+a^2} x \Rightarrow \lim_{x \rightarrow 0} \frac{f(x, ax)}{x} = \frac{a}{1+a^2}$
 $\therefore \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} \text{ DNE} \quad \square$

Note: If we require continuity, then the statement would have held true

PROPOSITION

Suppose that f_x, f_y exist in a nbd of z and are conti at z . If f satisfies the Cauchy-Riemann Equation, then f is differentiable

Proof

Say $z = x + iy$, $h = \xi + i\eta$, and $f(z) = u(z) + iv(z)$

$$\frac{f(z+h) - f(z)}{h} = \frac{[u(x+\xi, y+\eta) - u(x, y)]}{\xi + i\eta} + i \frac{[v(x+\xi, y+\eta) - v(x, y)]}{\xi + i\eta}$$

By MVT with " $-u(x+\xi, y) + u(x+\xi, y)$ " and " $-v(x+\xi, y) + v(x+\xi, y)$ ",

$$\begin{aligned} &= \frac{\eta}{\xi + i\eta} \left[\frac{u(x+\xi, y+\eta) - u(x+\xi, y)}{\eta} + i \frac{v(x+\xi, y+\eta) - v(x+\xi, y)}{\eta} \right] + \frac{\xi}{\xi + i\eta} \left[\frac{u(x+\xi, y) - u(x, y)}{\xi} + i \frac{v(x+\xi, y) - v(x, y)}{\xi} \right] \\ &= \frac{\eta}{\xi + i\eta} [u_y(x+\xi, y+\theta_1\eta) + iv_y(x+\xi, y+\theta_2\eta)] + \frac{\xi}{\xi + i\eta} [u_x(x+\theta_3\xi, y) + iv_x(x+\theta_4\xi, y)] \end{aligned}$$

We know $0 < \theta_k < 1$, $\left| \frac{\eta}{\xi + i\eta} \right| = \left| \frac{Re(h)}{h} \right| \leq 1$, $\left| \frac{\xi}{\xi + i\eta} \right| = \left| \frac{Im(h)}{h} \right| \leq 1$

$$\text{Claim: } \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

Proof

$$\text{We know } f_x(z) = \frac{\xi}{\xi + i\eta} f_x(z)$$

$$\text{By C-R eq, } f_x(z) = \frac{\xi}{\xi + i\eta} f_x(z) + \frac{\eta}{\xi + i\eta} f_y(z)$$

As f_x, f_y are conti,

$$\frac{f(z+h) - f(z)}{h} - f'(z) = \frac{\eta}{\xi + i\eta} [u_y(x+\xi, y+\theta_1\eta) - u_y(x, y) + i(v_y(x+\xi, y+\theta_2\eta) - v_y(x, y))] + \frac{\xi}{\xi + i\eta} [u_x(x+\theta_3\xi, y) - u_x(x, y) + i(v_x(x+\theta_4\xi, y) - v_x(x, y))] \longrightarrow 0 \text{ as } h \rightarrow 0, \text{ i.e. } \xi, \eta \rightarrow 0 \quad \square$$

$\therefore f$ is differentiable and $f'(z) = f_x(z)$

DEFINITION

f is analytic at z if f is diffable in a nbd of z .

Similarly, f is analytic on a set S if f is diff at all pts of some open set containing S .

DEFINITION

Let S, T be open sets of \mathbb{C} , and f be a 1-1 function on S with $f(S)=T$

We say g is the inverse of f on T if $f(g(z))=z \forall z \in T$.

We say g is the inverse of f at z if \exists open nbd V of z , s.t. g is the inverse of f on V .

Remark: g is also 1-1

PROPOSITION

Suppose that g is the inverse of f at z_0 and g is continuous there. If f is diffable at $g(z_0)$ and $f'(g(z_0)) \neq 0$, then g is diffable at z_0 and $g'(z_0) = \frac{1}{f'(g(z_0))}$

Proof

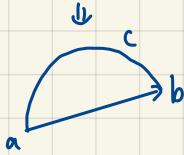
$$\frac{g(z+h)-g(z)}{h} = \frac{\frac{g(z+h)-g(z)}{f(g(z+h))-f(g(z))}}{\frac{f(g(z+h))-f(g(z))}{h}} = \frac{(f(g(z+h))-f(g(z)))^{-1}}{g(z+h)-g(z)} = \frac{1}{f'(g(z))} \quad \square$$

LINE INTEGRALS

Let $f(t) = u(t) + iv(t)$, $z(t) = x(t) + iy(t)$.

We consider curves as such:

$$\mathbb{R}^1 \xrightarrow{C} \mathbb{R}^n$$



We say $\int_a^b f(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt$

DEFINITION

(i) Let $z(t) = x(t) + iy(t)$, $a \leq t \leq b$. The curve determined by $z(t)$ is called piecewise differentiable and we set $\dot{z}(t) = x'(t) + iy'(t)$ if x, y are continuous on $[a, b]$ and are continuously differentiable on each subinterval $[a, x_1], [x_1, x_2], \dots, [x_{n-1}, b]$ of some partition of $[a, b]$.

(ii) The curve is said to be smooth, if $\dot{z}(t) \neq 0$ except at finitely many points.

In the following, we assume our curves are smooth.

DEFINITION

Say C is a smooth curve $\subseteq \mathbb{C}$, where $z(t) = x(t) + iy(t)$

Then, $\int_C f(z) dz = \int_a^b f(z(t)) dz = \int_a^b f(z(t)) \dot{z}(t) dt$

DEFINITION

Let C_1, C_2 be smooth curves $\subseteq \mathbb{C}$, where $C_1: z(t), a \leq t \leq b$ and $C_2: w(t), c \leq t \leq d$.

C_1 and C_2 are smoothly equivalent if \exists 1-1 C' mapping $\lambda: [c, d] \rightarrow [a, b]$ s.t. $w(t) = z(\lambda(t))$

(By def, this is probably an equivalence relation)

We denote smoothly equivalent with $C_1 \sim C_2$.

PROPOSITION

If $C_1 \tilde{=} C_2$, then $\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$

Proof

We set $f(z) = u(z) + iv(z)$, $z = x(t) + iy(t)$

Then, $\int_{C_1} f dz = \int_a^b f(z(t)) \dot{z}(t) dt = \int_a^b [u(z)x'(t) - v(z)y'(t)] dt + i \int_a^b [u(z)y'(t) + v(z)x'(t)] dt$
 With $\int_C u(z(\lambda(t))) x'(\lambda(t)) \lambda'(t) = \int_a^b u(z(t)) x'(t) dt$, we can prove the equation \square

APPLICATIONS OF CR-EQUATION

DEFINITION

$D \subseteq \mathbb{C}$ is called a **region** if open connected

Note, D : region $\Rightarrow x, y \in D, \exists$ a curve consisting of vertical and horizontal line segments that connect.

Proof

For $x \in D$, say $U_x := \{y \in D \mid x \text{ connects to } y \text{ via vertical/horizontal line segments that connect}\}$

① " U_x is open":

For $y \in U_x \cap D$, D is open $\Rightarrow \exists$ open disk $B(y) \subseteq D$

$\because \forall a \in B(y)$, a can be connected to y by \curvearrowright

$\therefore x \overset{\curvearrowright}{\sim} y \overset{\curvearrowright}{\sim} a \checkmark$

② " $D \setminus U_x$ is open":

For $y \in D \setminus U_x$, D is open $\Rightarrow \exists$ open disk $B(y) \subseteq D \Rightarrow B(y) \cap U_x = \emptyset \Rightarrow B(y) \subseteq D \setminus U_x \checkmark$

$\therefore ① + ② + D \Rightarrow$ connected $\Rightarrow D = U_x$

PROPOSITION

If $f = u + iv$ is analytic on a region D and u is constant, then f is constant

Proof

u is const $\Rightarrow u_x = u_y = 0$

By CR-eq, $v_x = v_y = 0$

As D is a region, thus $\forall a, b \in D, \exists \curvearrowright$ connecting a and b

$\Rightarrow f(a) = f(b) \Rightarrow f$ is const \square

PROPOSITION

If f is analytic on a region D , and $|f|$ is constant on D , then f is constant

Proof

$|f| = 0 \Rightarrow f = 0 \checkmark$

If $|f| \neq 0$, $|f| = C > 0 \Rightarrow u^2 + v^2 = C^2$

$\Rightarrow 2u u_x + 2v v_x = 0; 2u u_y + 2v v_y = 0$

By CR-eq, $u u_x - v u_y = 0; u u_y + v u_x = 0 \Rightarrow (u^2 + v^2) u_x = 0 \Rightarrow u_x = 0$. Similarly, we get $u_y = 0$.

As $\curvearrowright +$ prop above, thus this prop is true. \square

LINE INTEGRALS (CONTINUED)

DEFINITION

Let C be a curve defined by $z(t) = x(t) + iy(t)$, $t \in [a, b]$.
 Then, $-C$ is a curve defined by $w(t) = z(a+b-t)$

In short, it is as follows:



PROPOSITION

$$\int_{-C} f(z) dz = - \int_C f(z) dz$$

PROPOSITION

Let C be a smooth curve, and f, g be continuous functions on C . Say, $\alpha \in \mathbb{C}$.

$$(i) \int_C (f(z)g(z)) dz = \int_C f(z) dz + \int_C g(z) dz$$

$$(ii) \int_C \alpha f(z) dz = \alpha \int_C f(z) dz$$

In other words, we say $\int_C (\cdot) dz$ is linear

EXAMPLE

$$\text{Say } f(z) = \frac{1}{z}, C: R(\cos t + i \sin t), t \in [0, 2\pi]$$

$$\begin{aligned} \text{Then, } \int_C f(z) dz &= \int_0^{2\pi} \frac{1}{R(\cos t + i \sin t)} (-\sin t + i \cos t) dt \\ &= \int_0^{2\pi} e^{-it} (-e^{i(t-\frac{\pi}{2})}) dt \\ &= \int_0^{2\pi} i dt \\ &= 2\pi i \end{aligned}$$

LEMMA

Let $t \in \mathbb{R}$, $G(t)$ be a continuous complex-valued function. Then, $|\int_a^b G(t) dt| \leq \int_a^b |G(t)| dt$

$$(\alpha < \beta := |\alpha| \leq |\beta|, \alpha, \beta \in \mathbb{C})$$

Proof

$$\text{Set } \int_a^b G(t) dt = Re^{i\theta}, \theta \in \mathbb{R}, R \in \mathbb{R}_{\geq 0}$$

$$\text{Then, } R = |\int_a^b G(t) dt| = \int_a^b e^{-i\theta} G(t) dt = \int_a^b A(t) dt + i \int_a^b B(t) dt \quad (e^{i\theta} G(t) = A(t) + iB(t))$$

$$\therefore R = \int_a^b A(t) dt \leq \int_a^b |A(t)| dt \leq \int_a^b |e^{-i\theta} G(t)| dt = \int_a^b |G(t)| dt \quad \square$$

PROPOSITION (ML-FORMULA)

Let C be a smooth curve of length L , and f be continuous on C and $|f| \leq M$ throughout C . Then, $|\int_C f(z) dz| \leq ML$

Proof

$$\text{Let } C \text{ be } z(t) = x(t) + iy(t), t \in [a, b].$$

$$\text{Then, } \int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$$

$$\text{By the prev lemma, } \int_C f(z) dz \leq \int_a^b |f(z(t))| |z'(t)| dt \leq M \int_a^b |z'(t)| dt = ML \quad \square$$

call length

EXAMPLE (FOR WHY ML IS THE TIGHT BOUND)

$$\text{For } f(z) = \frac{1}{z}, C: \cos \theta + i \sin \theta, \int_C f(z) dz = 2\pi i \Rightarrow |\int_C f(z) dz| = 2\pi = ML$$

PROPOSITION

Suppose $\{f_n\}$ is a sequence of continuous functions and $f_n \rightarrow f$ uniformly on a smooth curve C . Then, $\lim_{n \rightarrow \infty} \int_C f_n(z) dz = \int_C f(z) dz$

Proof

$f_n \rightarrow f$ uniformly on C : "Given $\epsilon > 0$, $\exists N$, s.t. $\forall n \geq N$, $|f_n(z) - f(z)| < \epsilon \ \forall z \in C$."

$$\text{So, } \left| \int_C f_n(z) dz - \int_C f(z) dz \right| = \left| \int_C (f_n - f)(z) dz \right| < \epsilon \cdot \text{len}(C) \quad \forall n \geq N$$

C ML

\therefore By def., $\lim_{n \rightarrow \infty} \int_C f_n(z) dz = \int_C f(z) dz \quad \square$

PROPOSITION

Let F be an analytic function, $f = F'(z)$, and a smooth curve C : $z(t) = x(t) + iy(t)$, $t \in [a, b]$

Then, $\int_C f(z) dz = F(z(b)) - F(z(a))$

Proof

Let $\gamma(t) := f(z(t)) = A(t) + iB(t)$

$$\text{Hence, } \dot{\gamma}(t) = \lim_{h \rightarrow 0} \frac{F(z(t+h)) - F(z(t))}{z(t+h) - z(t)} \cdot \frac{z(t+h) - z(t)}{h} = F'(z(t)) \dot{z}(t)$$

$$\text{Then, } \int_C f(z) dz = \int_a^b F'(z(t)) \dot{z}(t) dt = \int_a^b \dot{\gamma}(t) dt = \gamma(b) - \gamma(a) \quad \square$$

DEFINITION

(i) A curve is closed if its initial and terminal points coincide.

(ii) C is a simple closed curve with $t \in [a, b]$ if $z(t_1) = z(t_2)$ with $t_1 < t_2$, then $t_1 = a$ and $t_2 = b$

DEFINITION

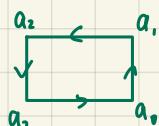
The boundary of a rectangle is the simple closed curve in the counterclockwise direction

DEFINITION

f is an entire function $\Leftrightarrow f$ is analytic on \mathbb{C}

LEMMA

If f is a linear function, i.e. $f = az + b$, $a, b \in \mathbb{C}$, Γ is the boundary of a rectangle, then $\int_\Gamma f(z) dz = 0$

Proof

f analytic on \mathbb{C}

Say $\Gamma: z(t)$, $a = a_0 \leq t \leq b = a_3$, and $f = F'(z) \Rightarrow F := \frac{1}{2}z^2 + bz$

Hence, we can deduce $\int_\Gamma f(z) dz = \int_\Gamma F'(z) dz = F(z(b)) - F(z(a)) = 0 \quad (\because z(b) = z(a))$

THEOREM (RECTANGLE THEOREM)

Let f be an entire function, and Γ as above, then $\int_\Gamma f(z) dz = 0$

Proof

Let $I = \int_\Gamma f(z) dz$. Assume $f \neq 0$, otherwise $f = 0 \Rightarrow I = 0$.

We divide R as follows:



Then, \exists one of R_i s.t. $|\int_{\Gamma_i} f(z) dz| \geq \frac{|I|}{4}$, where Γ_i is the boundary of R_i .

Set $R^{(1)}$ to be such an R_i :

Continuing this process, we get $R^{(1)} \supseteq R^{(2)} \supseteq \dots$. Let $z_0 \in \bigcap_{i=1}^{\infty} R^{(i)}$.

As f is an entire function, hence f is analytic at z_0
 By def., $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $|h| < \delta \Rightarrow |\frac{f(z_0+h) - f(z_0)}{h} - f'(z_0)| < \varepsilon$

\therefore We see $f(z) = f(z_0) + f'(z_0)(z-z_0) + \varepsilon(z)(z-z_0)$, where $|\varepsilon(z)| \leq \varepsilon$.

We choose N s.t. $\forall n \geq N$, $|z-z_0| < \delta \Rightarrow \int_{\Gamma^{(n)}} f(z) dz = \int_{\Gamma^{(n)}} [f(z_0) + f'(z_0)(z-z_0)] dz + \int_{\Gamma^{(n)}} \varepsilon(z)(z-z_0) dz$ $\rightarrow 0$ (from lemma since linear)

We know $|\Gamma^{(n)}| = \frac{4\pi}{2^n}$, so $|\varepsilon(z)(z-z_0)| \leq \varepsilon \cdot \frac{4\pi}{2^n} \Rightarrow$ By ML formula, $\int_{\Gamma^{(n)}} f(z) dz \leq \varepsilon \frac{4\pi \delta^2}{4^n}$

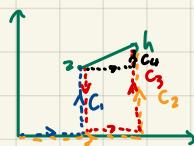
By our assumption, $|\int_{\Gamma^{(n)}} f(z) dz| \geq \frac{|I|}{4^n}$, hence $|I| \leq \varepsilon \cdot 4\sqrt{\varepsilon}^2 \forall \varepsilon > 0$, i.e. $I=0$ \square

THEOREM (INTEGRAL THEOREM)

If f is entire, then f is everywhere the derivative of an analytic function. That is, \exists an entire F , s.t. $F'(z) = f(z) \forall z$
Proof

Consider $F(z) = \int_C f(\eta) d\eta$ where $C: 0 \rightarrow \text{Re}(z) \rightarrow z$

Now, for $h \in \mathbb{C}$, $F(z+h) = \int_{C_h} f(z) dz$



Then, $F(z+h) - F(z) = \int_{C_h} f(\eta) d\eta + \int_{-C_2} f(\eta) d\eta = \int_{C_1} f(\eta) d\eta = \int_{C_1} f(\eta) d\eta$

Using $F(z+h) = F(z) + \int_{C_1} f(\eta) d\eta$, we get $\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_{C_1} f(\eta) d\eta$

As $\frac{1}{h} \int_{C_1} dz = 1$, thus $(\frac{1}{h} \int_{C_1} f(\eta) d\eta) - f(z) = \frac{1}{h} \int_{C_1} (f(\eta) - f(z)) d\eta = \frac{F(z+h) - F(z)}{h} - f(z)$

In other words, by ML-formula, $\frac{F(z+h) - F(z)}{h} - f(z) \ll \frac{1}{h} \int_{C_1} |\varepsilon| dz$ if h is small, $|f(z+h) - f(z)| \ll \varepsilon$ $\Rightarrow \lim_{h \rightarrow 0} \frac{F(z+h) - F(z)}{h} = f(z)$ \square

THEOREM

If f is entire and if C is a smooth closed curve, then $\int_C f(z) dz = 0$

CAUCHY INTEGRAL FORMULA AND TAYLOR EXPANSION

THEOREM (RECTANGLE THEOREM II)

Let f be entire, and

$$g(z) := \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z=a \end{cases}, \text{ which is continuous (} f \text{ entire} \Rightarrow g \text{ conti})$$

Then, $\int_{\Gamma} g(z) dz = 0$, Γ : a boundary of a rectangle $R \subseteq \mathbb{C}$

Proof

As g is conti., by def., $\exists M \in \mathbb{R}$, s.t. $|g(z)| < M \forall z \in R$

(i) If $a \in \mathbb{C} \setminus R$, $g(z)$ is analytic $\forall z \in R$

\therefore By the argument of Rectangle Thm, $\int_{\Gamma} g(z) dz = 0$

(ii) If $a \in \Gamma$, Γ := boundary of R :

R_1	R_2
R_3	R_4
R_5	R_6

Then, $\int_{\Gamma} g(z) dz = \sum_{i=1}^6 \int_{\Gamma_i} g(z) dz \stackrel{(i)}{=} \int_{\Gamma_5} g(z) dz \ll M \cdot 4\epsilon$ by ML-formula, with M indep of ϵ , where we define $\epsilon := \text{length of longest side of } \Gamma_5$.

\therefore As $\epsilon \rightarrow 0$, $\int_{\Gamma} g(z) dz = 0$

(iii) Otherwise, $a \in \text{interior of } R$

R_1	R_4	R_7
R_2	R_5	R_8
R_3	R_6	R_9

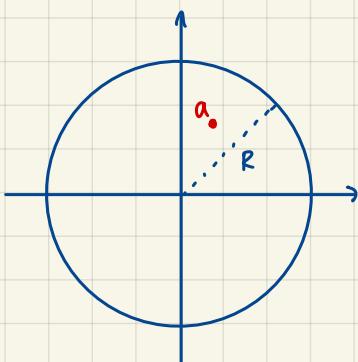
Then, $\int_{\Gamma} g(z) dz = \sum_{i=1}^9 \int_{\Gamma_i} g(z) dz \stackrel{(i)}{=} \int_{\Gamma_5} g(z) dz \ll M \cdot 4\epsilon \xrightarrow{\epsilon \rightarrow 0} 0$

COROLLARY

The integral thm and closed curve thm apply to g (since g is conti.)

THEOREM (CAUCHY INTEGRAL FORMULA)

Given an entire f , $a \in \mathbb{C}$, $C = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$ with



Then, we have $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$

LEMMA

Define $C_p(\alpha) :=$ circle centered at α with radius p (α may be omitted if no ambiguity)

Then, $\int_{C_p(\alpha)} \frac{dz}{z-\alpha} = 2\pi i$ $\forall |\alpha - \alpha| < p$

Proof

If $\alpha \in \mathbb{R}$, then it's clear since $C_p(\alpha) = \alpha + pe^{i\theta}, 0 \leq \theta \leq 2\pi$.

For $\alpha \notin \mathbb{R}$,

$$\int_{C_p(\alpha)} \frac{dz}{(z-\alpha)(z-\bar{\alpha})} = \int_{C_p(\alpha)} \frac{1}{z-\alpha} \cdot \frac{1}{1-\frac{\alpha-\bar{\alpha}}{z-\alpha}} dz =: I$$

$\forall z \in C_p(\alpha)$, $|\frac{\alpha-\bar{\alpha}}{z-\alpha}| < 1$. Hence, $(1-\frac{\alpha-\bar{\alpha}}{z-\alpha})^{-1} = 1 + (\frac{\alpha-\bar{\alpha}}{z-\alpha}) + (\frac{\alpha-\bar{\alpha}}{z-\alpha})^2 + \dots$ (unit conv) , abs conv

Hence,

$$I = \int_{C_p(\alpha)} \frac{1}{z-\alpha} \left(\sum_{k=0}^{\infty} \frac{(\alpha-\bar{\alpha})^k}{z-\alpha} \right) dz = \sum_{k=0}^{\infty} \int_{C_p(\alpha)} \frac{1}{z-\alpha} \frac{(\alpha-\bar{\alpha})^k}{z-\alpha} dz$$

We first consider the term $\int_{C_p(\alpha)} \frac{1}{(z-\alpha)^k} dz$ ($k=1 \Rightarrow 2\pi i$; since $\int_C \frac{1}{z} dz = 2\pi i$)

For $k > 1$, $\int_{C_p(\alpha)} \frac{1}{(z-\alpha)^k} dz = \int_0^{2\pi} \frac{e^{ik\theta}}{p^k e^{ik\theta}} d\theta = \int_0^{2\pi} \frac{1}{p^k} e^{i\theta(k-1)} d\theta = 0$

$\therefore I = 2\pi i$ \square

PROOF OF CAUCHY INTEGRAL FORMULA

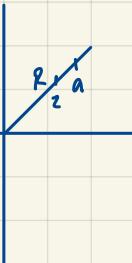
Now, we know by rectangle thm, $\int_C g(z) dz = \int_C \frac{f(z)}{z-a} - \frac{f(a)}{z-a} dz = 0$

$\therefore \int_C \frac{f(z)}{z-a} dz = \int_C \frac{f(a)}{z-a} dz = f(a) \int_C \frac{1}{z-a} dz = f(a) 2\pi i$ \square

THEOREM (TAYLOR EXPANSION FOR ENTIRE FUNCTION)

Given f is an entire function, then $f^{(k)}(0)$ exists $\forall k \in \mathbb{Z}_{\geq 0}$ and $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \quad \forall z \in \mathbb{C}$

Proof



Choose $a \in \mathbb{C}$, $|a| > |z|$, $R := |a| + 1$

Notice, by Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \int_{C_R} \frac{1}{1-\frac{z}{w}} \frac{f(w)}{w} dw$$

As $|\frac{z}{w}| < \frac{|a|}{|a|+1}$, then $f(z) = \sum_{k=0}^{\infty} \int_{C_R} \frac{f(w)}{w} \left(\frac{z}{w}\right)^k dw = \sum_{k=0}^{\infty} z^k \int_{C_R} \frac{f(w)}{w^{k+1}} dw = \sum_{k=0}^{\infty} z^k C_k$

Notice, as $|z| < |a|$, then $f'(z) = \sum_{i=1}^{\infty} iz^{i-1} C_i \Rightarrow f'(0) = C_1$

If we continue this process, $f^{(k)}(0)$ exists $\forall k \in \mathbb{N}_{\geq 0}$ and $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$ \square

COROLLARY

An entire function is infinitely diff

Proof

Above thm. \square

COROLLARY

If f is entire, then $f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \dots$

Proof

Let $h(z) = f(z+a)$.

f is entire $\Rightarrow h$ is entire.

$$\text{Then, } h(z) = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} z^k \Rightarrow f(w) = f(a) + \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} (w-a)^k \quad \square$$

PROPOSITION

If f is entire, then

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z=a \end{cases} \text{ is entire}$$

Proof

$$\text{By corollary, } g(z) = \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^{k-1} \Rightarrow g \text{ is entire} \quad \square$$

COROLLARY

Let f be an entire function with zeros at a_1, \dots, a_n . Define $g(z) = \frac{f(z)}{(z-a_1)\dots(z-a_n)}$, $z \notin \{a_1, \dots, a_n\}$. Then, $\lim_{z \rightarrow a_i} g(z)$ exists $\forall i$.

If we define $g(a_i) := \lim_{z \rightarrow a_i} g(z)$, then g is entire.

Proof

$$\text{Set } f_0 = f, f_k := \frac{f_{k-1}(z) - f_{k-1}(a_k)}{z-a_k} \quad (f_k(z) = \frac{f(z)}{z-a_k})$$

By proposition, we see f_k is entire. By recurrence, g is entire \square

THEOREM (LIOUVILLE'S THEOREM)

Entire bounded functions on \mathbb{C} are constants

Proof

Let $a \in \mathbb{C} \setminus \{0\}$, $C_R(0)$, $R > |a|$.

Then, by Cauchy Integral Formula, $|f(a) - f(0)| = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \int_{C_R} \frac{af(z)}{z(z-a)} dz$

As f is bounded, $\exists M \in \mathbb{R}_{>0}$, s.t. $|f(z)| \leq M \forall z \in \mathbb{C}$

By ML-formula, $|f(a) - f(0)| \leq \frac{1}{2\pi i} \left(\frac{M \cdot |a|}{|a(a-|a|)|} \cdot 2\pi R \right) \xrightarrow{R \rightarrow \infty} 0 \quad \therefore f(a) = f(0) \quad \forall a \in \mathbb{C} \quad \square$

THEOREM (EXTENDED LIOUVILLE'S THEOREM)

Given f is entire. Suppose $|f(z)| \leq A + B|z|^k$ for some constants $A, B \in \mathbb{R}_0$. Then, f is a polynomial with degree at most k

Proof

Consider induction on k ,

- $k=0 \Rightarrow$ True by Liouville's Thm.

Otherwise,

Define $g(0) = \begin{cases} \frac{f(z)-f(0)}{z}, & z \neq 0 \\ f'(0), & z=0 \end{cases}$, which we know is entire.

As $|f(z)| \leq A + B|z|^k$ is bounded, define $M_0 := \max_{z \in C_R} g(z)$

For $z \in \mathbb{C} \setminus C_R$, $|g(z)| \leq A + B|z|^{k-1}$ $\left\{ \Rightarrow \exists D, E \in \mathbb{R}_{>0} \text{ s.t. } |g(z)| \leq D + E|z|^{k-1} \right.$
 $\forall z \in C_R$, $|g(z)| \leq M_0$

$\therefore g$ is poly of degree at most $k-1$, so f is poly of degree at most k . \square

THEOREM (FUNDAMENTAL THEOREM OF ALGEBRA)

Shun/翔海 (@shun4midx)

Nonconstant polynomials have roots in \mathbb{C} .

Proof $\mathbb{C} \in \mathbb{C}[x]$

Consider poly $p(x)$.

Suppose p has no roots in \mathbb{C} .

We know $f(z) := \frac{1}{p(z)}$ is defined and differentiable on \mathbb{C} , so it is analytic.

As $z \rightarrow \infty$, $|f(z)| \rightarrow 0$, then $|f(z)|$ is bounded

\therefore By Cauchy's Thm, $f(z)$ is const $\Rightarrow p(z)$ is const \rightarrow

DEFINITION

We say S is a convex set in \mathbb{C} if $\forall x, y \in S$, $tx + (1-t)y \in S \quad \forall t \in [0, 1]$

($\Rightarrow x_1, \dots, x_n \in S$ iff $\sum_{i=1}^n a_i x_i \in S \quad \forall \sum_{i=1}^n a_i = 1$ and $a_i \geq 0$)

9-18-25 (WEEK 3)

Shun/舒海 (@shun4midx)

THEOREM (GAUSS-LUCAS THEOREM)

The zeros of the derivative of a polynomial lie within the convex hull of the zeros of the polynomial

Proof

Let $p(x)$ be a nonconstant polynomial $\in \mathbb{C}[x]$, and $\alpha_1, \dots, \alpha_n$ be roots of p (counted by multiplicity)

Then, $p(x) = c \prod_{i=1}^n (x - \alpha_i)$. Moreover, $\frac{p'(x)}{p(x)} = \sum_{i=1}^n \frac{1}{x - \alpha_i}$.

[otherwise it's trivial]

Let a be a root of $p'(x)$ and $a \notin \{\alpha_1, \dots, \alpha_n\}$.

Then, $\frac{p'(a)}{p(a)} = \sum_{i=1}^n \frac{1}{a - \alpha_i} = \sum_{i=1}^n \frac{1}{|a - \alpha_i|} \Rightarrow a = \sum_{i=1}^n c_i \bar{\alpha}_i$, $c_i = \frac{1}{|a - \alpha_i|^2} / \sum_{i=1}^n \frac{1}{|a - \alpha_i|^2} \in \mathbb{R}_{\geq 0}$

$\therefore a = \sum_{i=1}^n c_i \alpha_i$, $c_i \in \mathbb{R}_{\geq 0}$, $\sum c_i = 1$ \square [by def]

ANALYTIC FUNCTION ON A DISC

NOTATION

[Assume open disc if disc]

$D = D(a; r) \rightarrow$ an open disc $\subseteq \mathbb{C}$, then: analytic $f(z)$ in D , $\forall a \in D$, $g_a(z) = \begin{cases} (f(z) - f(a))/(z - a), & z \neq a \\ f'(a), & z = a \end{cases}$ is conti on D (Notation for g_a)

THEOREM (RECTANGLE THEOREM)

For a closed rectangle $R \subseteq \mathbb{C}$, define $\Gamma = \partial R$, then we proved $\int_R f(z) dz = \int_\Gamma \frac{f(z) - f(a)}{z - a} dz = \int_\Gamma g_a(z) dz = 0$ (They just both happen to equal 0, not they are directly related)

THEOREM

$\exists F, G_a$ analytic in D s.t. $f = F'$, $g_a = G'_a$

Proof

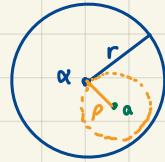
We consider $F = \int_a^z f(z) dz$ and $G_a = \int_a^z g_a(z) dz$

Define $C: \alpha \rightarrow \alpha + Re(z) \rightarrow \alpha + z$, which is in D . \square

THEOREM (CLOSED CURVE THEOREM)

Let C be a closed curve $\subseteq D$. Then, $\int_C f dz = 0$

THEOREM (CAUCHY INTEGRAL FORMULA)



For some $0 < p < r$, $\forall |a - r| < p$, $f(a) = \frac{1}{2\pi i} \int_{C_p} \frac{f(z)}{z - a} dz$, where C_p is $\partial D(\alpha; p)$

THEOREM (TAYLOR EXPANSION)

$f(z) = f(a) + f'(a)(z - a) + \frac{f''(a)}{2!}(z - a)^2 + \dots$ holds $\forall |z - a| < p$, for some p , s.t. $\exists a \in D$, $0 < p < r$, $|a - a| < p$



Proof Sketch

$$\forall |z - a| < r, \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{C_p} \frac{f(z)}{(z - a)^{n+1}} dz$$

THEOREM

Let \tilde{D} be an arbitrary open set, f analytic on \tilde{D} . Then, fix $x \in \tilde{D}$, we have $f(z) = \sum_{k=0}^{\infty} c_k (z - x)^k \quad \forall z \in D(x, r) \subseteq \tilde{D}$

EXAMPLE

Let $f(z) = \frac{1}{z - i}$, analytic $\forall z \neq 1$.

Then, on $D = D(z; 1)$, $f(z) = \frac{1}{1+(z-2)} = 1 - (z-2) + (z-2)^2 - \dots$ if $|z-2| < 1$

In fact, the expression converges if $|z-2| < 1$

diverges if $|z-2| \geq 1$

(However, $f(z)$ is analytic if $|z-2|=1$, so clearly its Taylor expansion is different)

PROPOSITION

$g(z)$ is analytic if $z, a \in D(\alpha; r)$

Proof

Use the than above, in some neighborhood of α , $f(z) = f(\alpha) + f'(\alpha)(z-\alpha) + \frac{f''(\alpha)}{2!} (z-\alpha)^2 + \dots$

Then, g has the power series expansion $g(z) = f'(z) + \frac{f'''(z)}{2!} (z-\alpha) + \frac{f^{(13)}(z)}{3!} (z-\alpha)^2 + \dots \Rightarrow g$ is analytic at α . \square

THEOREM

If f is analytic at z , then f is infinitely diffable at z

Proof

We know from above, f may be expressed as a power series. Hence, it is infinitely diffable. \square

9-23-25 (WEEK 4)

THEOREM (UNIQUENESS THEOREM)

Say D is a region (i.e. open connected) and f is an analytic function on D .

Suppose that \exists seq of distinct zeros of D $\{z_n\}$, s.t. $\lim_{n \rightarrow \infty} z_n = z_0 \in D$, where we say the seq $\{z_n\}$ has an acc pt in D .

Then, $f \equiv 0$ on D .

Proof

f ann $\Rightarrow f$ conti

$$\therefore \text{By def, } f(z_0) = \lim_{n \rightarrow \infty} f(z_n) = 0$$

We define $A := \{z \in D \mid z \text{ is an acc pt of zeros of } f \text{ in } D\}$.

Claim: A is open

Proof

By uniqueness of power series, $f \equiv 0$ in some disk $D(z, \delta_2) \subseteq D \ \forall z \in A$ ✓

Claim: $D \setminus A$ is open

Proof

$z \Rightarrow \text{NOT acc pt of zeros} \Rightarrow \exists$ open nbd U of z in D s.t. $f(z)$ has NO zeros in $U \setminus \{z\}$.

f conti $\Rightarrow \forall y \in U \setminus \{z\}$, \exists open nbd $U_y \subseteq D$ of y , s.t. $f \neq 0$ on $U_y \Rightarrow y \in D \setminus A$ ✓

$$\therefore D = A \cup (D \setminus A), A, (D \setminus A) \text{ both open, } A \cap (D \setminus A) = \emptyset \quad (B = D \setminus A)$$

As $z_0 \in A$ and D is a region, $D = A$ □

COROLLARY

Say f, g are analytic on a region D .

If f and g agree at a set of pts with an acc pt, then $f \equiv g$ on D .

Proof

Set $h = f - g$, then apply them above □

THEOREM

If f is entire and $f \rightarrow \infty$ as $z \rightarrow \infty$, then f is a polynomial

Proof

By def, $\forall M \in \mathbb{R}_{>0}, \exists \delta, \text{s.t. } \forall |z| > \delta, |f(z)| > M$

Let $M = 1$.

$\therefore \exists \delta, \text{s.t. } \forall |z| > \delta, |f(z)| > 1$

By our assumption, f is NOT a constant

Claim: f has finitely many zeros

Proof

If NOT, by δ , all zeros of f are in $\overline{D(0, \delta)}$, otherwise, $|f(z)| \neq 0$.

As $D(0, \delta)$ is compact, \exists acc pt of zeros in $\overline{D(0, \delta)}$ ✓

↪ Suppose not. Then, $\forall x \in \overline{D(0, \delta)}$, \exists an open nbd U_x , s.t. f has no zeros in $U_x \setminus \{x\}$

$U_x \setminus \{x\}$ is an open cover of $\overline{D(0, \delta)}$ $\Rightarrow \exists x_1, \dots, x_n$ s.t. $\{U_{x_i}\}$ is an open cover of $\overline{D(0, \delta)}$ (by cpt)

However, each U_{x_i} has at most 1 zero $\Rightarrow \overline{D(0, \delta)}$ has at most n zeros *

cover boundary at δ

By thm, $f \equiv 0$ on $D(0, \delta')$ for all $\delta' > \delta$

However, δ' can extend to ∞

We consider within $\bar{D}(0, \delta)$.

Let $\alpha_1, \dots, \alpha_n$ be zeros of f (counting by multiplicity).

Then, $g(z) = \frac{f(z)}{\prod_{i=1}^n (z - \alpha_i)}$ is entire and has no zeros on \mathbb{C}

Set $h := \frac{1}{g(z)}$, then h is entire, h has no zeros in $\mathbb{C} \Rightarrow h$ is bounded in disk

By Extended Liouville's Thm, $|h| < A + B|z|^n$ $|h(z)| \leq \delta$ and $|h| \leq \delta \Rightarrow h$ is a poly

However, h has no zeros in $\mathbb{C} \Rightarrow h = \text{const}$

$\therefore \exists c \in \mathbb{C}^* \text{ s.t. } f(z) = c \prod_{i=1}^n (z - \alpha_i) \quad \square$

REMARK

Say f, g are ana on region D , to check $f \equiv g$, we may apply the theorem above over \mathbb{R} without needing to consider \mathbb{C} .

THEOREM (MEAN VALUE THEOREM)

Let D be a region, f analyt. on D , $\forall z \in D$.

Then $f(z) = \text{mean value of } f \text{ taken around the boundary of any disk centred at } z \text{ and contained in } D$

Proof

By Cauchy - Integral Formula, $f(z) = \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{f(z)}{z - \omega} dz$

Say $z = \alpha + \delta e^{i\theta}$, $\theta \in [0, 2\pi]$, we get $f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + \delta e^{i\theta}) d\theta \quad \square$

THEOREM (MAXIMUM MODULUS THEOREM)

Say f is nonconst, ana on a region D . Then, $\forall z \in D$ and $\delta > 0$, \exists some $w \in D(z, \delta) \cap D$, s.t. $|f(w)| > |f(z)|$

Proof

By MVT, $f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + \delta e^{i\theta}) d\theta$ for small enough δ s.t. $D(z, \delta) \subseteq D$

Then, $|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + \delta e^{i\theta})| d\theta \leq \frac{1}{2\pi} \max_{\theta \in [0, 2\pi]} |f(z + \delta e^{i\theta})| \cdot 2\pi = \max |f(z + \delta e^{i\theta})|$

\therefore When \leq has equality, $|f(z + \delta e^{i\theta})| = \max_{\theta \in [0, 2\pi]} |f(z + \delta e^{i\theta})| \quad \forall \theta \in [0, 2\pi] \Rightarrow f$ is const on $C_\delta(z) \cap D$

By coro, hence f is const on D

However, f is nonconst.

$\therefore f(z) < \max_{\theta \in [0, 2\pi]} |f(z + \delta e^{i\theta})| \quad \square$

THEOREM (MINIMUM MODULUS THEOREM)

Say f is nonconst, ana on a region D , $\forall z \in D$, $f(z) \neq 0$.

Then, f has no interior min points

Proof

$f(z) \neq 0 \quad \forall z \in D \Rightarrow g(z) = \frac{1}{f(z)}$ is ana, nonconst on D

Then, by max mod thm, we proved it. \square

CAUTION

We can only apply uniqueness thm and its coro when its acc pts $\in D$ (Counterexample: $f(z) = \sin \frac{1}{z}$, $z_n = \frac{1}{n\pi}$)

THEOREM CAN ONLY APPLY TO CIRCLES

Say \bar{D} is a closed disk and f is analytic, nonconst on \bar{D} . f assumes its max value at a boundary point z_0 . Then, $f'(z_0) \neq 0$

Proof

conv rad

Suppose $f'(z_0) = 0$. As f is ana on \bar{D} , $\exists \delta$ s.t. $\forall |z| < \delta$, $f(z_0 + \delta) = f(z_0) + f'(z_0)\delta + \frac{1}{2} f''(z_0)\delta^2 + \dots$

Assume $f'(z_0) = 0$

Then, $f(z_0 + \delta) \approx f(z_0) + \frac{1}{2}f''(z_0)\delta^2 \Rightarrow |f(z_0 + \delta)| = |f(z_0 + \delta)| \cdot F(z_0 + \delta) = |f(z_0)|^2 + \frac{2}{k!} \operatorname{Re}(\bar{f}(z_0) f^{(k)}(z_0)) \delta^k + \dots$ for some $k \in \mathbb{N}$

From assumption, thus $k \geq 2$.

Let $e^{i\theta} = \frac{\delta}{|f(z_0)|}$. Then, $\bar{f}(z_0) f^{(k)}(z_0) = A e^{i\alpha} \Rightarrow |f(z_0 + \delta)|^2 = |f(z_0)|^2 + \frac{2}{k!} A |\delta|^k \cos(\alpha + k\theta) + \dots$

For small enough δ , $|f(z_0 + \delta)| - |f(z_0)|$ has the same sign as $\cos(k\theta + \alpha)$

As $|f(z_0)|$ is max, hence $|f(z_0 + \delta)|^2 - |f(z_0)|^2 \leq 0 \forall z_0 + \delta \in D$.

Notice, $\cos(k\theta + \alpha) \leq 0 \Leftrightarrow \frac{\pi}{2} + 2j\pi \leq \alpha + k\theta \leq \frac{3\pi}{2} + 2\pi j$; for $0 \leq j \leq k-1 \Leftrightarrow \frac{(\pi-\alpha)}{k} + \frac{2\pi j}{k} \leq \theta \leq \frac{(\frac{3}{2}\pi-\alpha)}{k} + \frac{2\pi j}{k}$ (★) $\subset \Delta$ w/ angle $\frac{\pi}{k}$

kinda like they alternate

For a disc, $\exists \delta$, s.t. $z_0 + \delta \in \text{NOT in any one of the cones } (\star)$ since $k \geq 2$, $\frac{\pi}{k} \leq \frac{\pi}{2}$. ——*

REMARK

This argument works for cpt $K \subseteq \mathbb{C}$ s.t. $\forall z_0$ on the boundary of K , K contains a cone $\{z_0 + r e^{i\theta} \mid \theta \in [\alpha, \beta], r \in (0, \varepsilon)\}$ with $\beta - \alpha > \frac{\pi}{2}$

Counterexample of squares:

$f(z) = z^2 + i \Rightarrow |f(z)|$ has min 1 at $z=0$, but $f'(0) = 0$

DEFINITION

A function is C-analytic on a region D if it is analytic on D and continuous on \bar{D}

SADDLE POINTS**DEFINITION**

z_0 is a saddle pt of an analytic function f (on a region D) if z_0 is a saddle pt of the real valued function $g(x, y) = |f(x, y)|$

In other words, g is differentiable and $g_x(z_0) = g_y(z_0) = 0$ but z_0 is NOT a local extremum.

**THEOREM**

z_0 is a saddle pt of an analytic function f iff $f'(z_0) = 0$ and $f''(z_0) \neq 0$

Proof

We have $z = x + iy$, $f(z) = u(x, y) + iv(x, y)$, and $g(z) = (u^2 + v^2)^{\frac{1}{2}} \geq 0$

" \Rightarrow ": As $g(z_0)$ is not a local minimum, hence $g'(z_0) \neq 0$, so $u(z_0) \neq 0$ or $v(z_0) \neq 0$

$$\begin{aligned} \text{We know } g_x(z_0) = g_y(z_0) = 0 &\Rightarrow \frac{u u_x + v v_x}{g} \Big|_{z_0} = \frac{u u_y + v v_y}{g} \Big|_{z_0} = 0 \quad (\star) \\ &\Rightarrow \begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix} \begin{bmatrix} u(z_0) \\ v(z_0) \end{bmatrix} = 0 \end{aligned}$$

$$\begin{aligned} \therefore \det \begin{bmatrix} u_x(z_0) & v_x(z_0) \\ u_y(z_0) & v_y(z_0) \end{bmatrix} &\stackrel{(\star)}{=} u_x^2(z_0) + v_x^2(z_0) = 0 \\ \therefore u_x(z_0) = v_x(z_0) &= 0 \end{aligned}$$

As f is ana, hence $f''(z_0) = 0$. From above with $g'(z_0) \neq 0$, we know $f'(z_0) \neq 0$. ✓

" \Leftarrow ": Recall, $f'(z_0) = 0$

Then, $u_x(z_0) = v_x(z_0) = 0$ and $u_y(z_0) = v_y(z_0) = 0$

$\therefore g_x(z_0) = g_y(z_0) = 0$ as implied by (\star)

As $f'(z_0) \neq 0$, thus $|f(z_0)|$ is NOT a local extremum (excluding f is const) by the max and min modulus thms. ✓

OPEN MAPPING THM AND SCHWARTZ LEMMA**RECALL**

$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ conti $\Leftrightarrow U \subseteq \mathbb{R}^2$: open then $f^{-1}(U)$: open $\Leftrightarrow \bar{U} \subseteq \mathbb{R}^2$: closed then $f^{-1}(\bar{U})$: closed

Then $K \subseteq \mathbb{R}^2$: cpt $\Rightarrow f(K)$: cpt

THEOREM (OPEN MAPPING THEOREM)

V open set $U \subseteq D$, $f(U)$: open in C for any nonconst ana f (Need not hold outside of C , e.g. $U = (-1, 1)$, $f(U) = [0, 1]$ for $f(x) = x^2$)

Proof

It suffices to show $\forall \alpha \in D$, \exists open disc $D(\alpha, \epsilon) \subseteq D$ s.t. $f(D(\alpha, \epsilon))$ is open.

(We want to show $\forall \beta = f(\alpha') \in f(D(\alpha, \epsilon))$, $\exists D(\beta, \epsilon') \subseteq f(D(\alpha, \epsilon))$)

WLOG, we can assume $f(\alpha) = 0$, so we choose ϵ s.t. $\overline{D(\alpha, \epsilon)} \subseteq D$

By uniqueness thm, $\exists \epsilon' \text{ s.t. } f \text{ has no zeros in } \overline{D(\alpha, \epsilon)} \setminus \{\alpha\}$ (or else $f \equiv 0$)

Let $\exists \delta = \min_{z \in C(\alpha)} |f(z)| > 0$

Shun/翔海 (@shun4midx)

Claim: $D(f(\alpha) = 0, \delta) \subseteq \text{Im}(f)$

Proof

$\forall w \in D(0, \delta)$, consider $f(z) - w$

If $w \notin f(D(\alpha, \varepsilon))$, then $f(z) - w$ has no zeros on $D(\alpha, \varepsilon)$

$$\therefore |f(z) - w| \geq |f(z)| - |w| \geq f(z) - \delta \geq \delta \quad \forall z \in C(\alpha)$$

However, we know $|f(z) - w| < \delta \times$

$$\therefore w \notin f(D(\alpha, \varepsilon)) \Rightarrow D(0, \delta) \subseteq \text{Im}(f) \quad \square$$

REMARK

We only have open mapping thm because extremum is not in interior pt.

SCHWARTZ'S LEMMA

THEOREM (SCHWARTZ'S LEMMA)

Suppose that f is analytic in an open unit disc D with $|f| \leq 1$ ($f: \text{unit circles} \rightarrow \text{unit circles}$) and $f(0)=0$

Then, (i) $|f(z)| \leq |z|$

(ii) $|f'(0)| \leq 1$

with equality in either of the above iff $f(z) = e^{i\theta} z$

Proof

$$\text{Define } g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

$g(z)$ is ana on D since $f(z)$ is ana on D .

Consider $z \in C_r(0)$, $0 < r < 1$.

$$\text{Then, } |g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

By max modulus thm, $\forall z \in \overline{D(0,r)}$, $|g(z)| \leq \frac{1}{r}$

As $r \rightarrow 1$, then $|g(z)| \leq 1 \quad \forall z \in D$

By def of $g(z)$, $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$ has either equality hold, when g is const and $|g|=1$ on D . $\therefore g = e^{i\theta}$ □

EXAMPLE (Removing $f(0)=0$ constraint)

$$\text{Define } B_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}, \quad |\alpha| < 1 - D$$

Then, (1) $B_\alpha(\alpha) = 0$

(2) $B_\alpha(z)$ is ana on D , so $(B_\alpha(z))^2$ is ana on D . It is also conti on \overline{D} .

(3) $|B_\alpha(z)|^2|_{z=1} = 1$, so by max modulus thm, $|B_\alpha(z)| \leq 1$ on D .

\therefore We can use B_α for variations of Schwartz's lemma

EXAMPLE

Suppose f is ana on D , $|f(z)| \leq 1 \quad \forall z \in D$ and $f\left(\frac{1}{2}\right) = 0$. Estimate $|f\left(\frac{3}{4}\right)|$.

$$\text{Consider } B_{\frac{1}{2}}(z) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}.$$

$\curvearrowleft (B_{\frac{1}{2}}(z))^{-1}$

$$\text{We define } g(z) = \begin{cases} \frac{f(z)}{z - \frac{1}{2}}, & z \neq \frac{1}{2} \\ \frac{3}{4}f'\left(\frac{1}{2}\right), & z = \frac{1}{2} \end{cases}$$

Notice, $|B_{\frac{1}{2}}(z)| \leq 1$ on D , $|B_{\frac{1}{2}}(z)| = 1$ on $C_1(0)$, and $B_{\frac{1}{2}}(z)$ conti on D .

$\therefore z \rightarrow 1$, $|B_{\frac{1}{2}}(z)| \rightarrow 1$

$$|g(z)| \leq 1 / |B_{\frac{1}{2}}(z)| \Rightarrow |g(z)| \leq 1 \text{ on } D.$$

$$\Rightarrow |f(z)| \leq |B_{\frac{1}{2}}(z)|, \frac{3}{4}|f'\left(\frac{1}{2}\right)| \leq 1$$

$$\text{So, } |f\left(\frac{3}{4}\right)| \leq |B_{\frac{1}{2}}\left(\frac{3}{4}\right)| = \frac{2}{3}$$

EXAMPLE

Say f is ana on D , $|f(z)| \leq 1$ on D . We claim: $|f'(\frac{1}{z})| \geq \max_{z \in D} |f(z)|$ when $f(\frac{1}{z}) = 0$ is at its lowest ≥ 1 .

Shun/海 (@shun4mida)

Proof

Assume that $f(\frac{1}{z}) \neq 0$.

$$g(z) := \frac{f(z) - f(\frac{1}{z})}{1 - \overline{f(\frac{1}{z})} f(z)} \Rightarrow g(z) = B_{f(\frac{1}{z})}(f(z)). \text{ Note, } g \text{ is bounded by } 1, g(\frac{1}{z}) = 0. \text{ (Note it is ana too)}$$

$$\therefore |g'(z)|_{z=\frac{1}{z}} = \frac{|f'(\frac{1}{z})|}{|1 - \overline{f(\frac{1}{z})} f(z)|^2} \Rightarrow |g'(\frac{1}{z})| > |f'(\frac{1}{z})| \quad (\because |f'(\frac{1}{z})| > 0. \text{ Otherwise, then } |g'(\frac{1}{z})| \text{ can be larger, take } f = g \star)$$

$|f'(\frac{1}{z})|$ is max for $B_{\frac{1}{z}}(z)$.

PROPOSITION

Say f is entire. If $|f(z)| < \frac{1}{|\operatorname{Im} z|}$ $\forall z$, then $f \equiv 0$

Proof

Define $g(z) = (z^2 - R^2)f(z)$, for some $R \in \mathbb{R}_{>0}$ (sufficiently large, e.g. $R \geq 1, R \rightarrow \infty$)

When $z \in C_R(0)$, $|z-R||z+R| \leq 2R|\operatorname{Im} z|$

$$\therefore |g(z)| \leq \frac{2R}{|\operatorname{Im} z|^2} \leq 2R \text{ when } z \in C_R(0)$$

By max modulus thm, $|g(z)| \leq 2R \forall z \in D(0, R)$

$$\Rightarrow |f(z)| \leq \frac{2R}{R^2 - R^2} = 1 \forall z \in D(0, R)$$

As $R \rightarrow \infty$, $|f(z)| \rightarrow 0$

$$\therefore f(z) = 0$$

MORERA'S THEOREM

THEOREM (MORERA'S THEOREM: (CONVERSE OF RECTANGLE THEOREM))

Let f be continuous on an open set $D \subseteq \mathbb{C}$, and Γ be the boundary of a closed rectangle $R \subseteq D$.

If $\int_{\Gamma} f(z) dz = 0 \forall \Gamma$ in $R \subseteq D$, then f is analytic in D .

Proof

Say $z_0 \in D$, D : open.

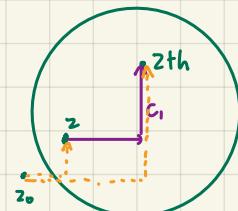
Then, $\exists \epsilon > 0$, s.t. $D(z_0, \epsilon) \subseteq D$.

Define $F(z) := \int_C f(z) dz \quad \forall z \in D(z_0, \epsilon)$, where $C: z_0 \rightarrow z_0 + Re(z-z_0) \rightarrow z$

For $z \in D(z_0, \epsilon)$ and h small enough s.t. $z+h \in D(z_0, \epsilon)$

Then,

$$\lim_{h \rightarrow 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \rightarrow 0} \int_{C_1} f(w) dw \quad \text{f: cont.}$$



EXAMPLE

Using $f(z) = \int_0^{\infty} \frac{e^{zt}}{t+1} dt$,

Claim: f is analytic $\forall z \in \mathbb{C} \setminus \{w \in \mathbb{C} \mid \operatorname{Re}(w) < 0\}$

Proof

We know for $z = x + iy$, $x < 0$, $|e^{zt}| = e^{xt}$

Here, $\int_{\Gamma} \int_0^{\infty} \frac{|e^{zt}|}{|t+1|} dt dz < \int_{\Gamma} \frac{1}{|t+1|} dz \text{ (ok for Fubini)}$

≈ 0 (\because By rectangle thm since $\frac{e^{zt}}{t+1}$: ana)

By Fubini's Thm, $\int_{\Gamma} \int_0^{\infty} \frac{e^{zt}}{t+1} dt dz = \int_0^{\infty} \int_{\Gamma} \frac{e^{zt}}{t+1} dz dt = 0$

\therefore By Morera's Thm, $f(z)$ is analytic on $\{w \in \mathbb{C} \mid \operatorname{Re}(w) < 0\}$. \square

DEFINITION

Let $\{f_n\}$ and f be defined on an open set D . We say that f_n converges uniformly on compacta if $f_n \rightarrow f$ uniformly on every compact subset $K \subseteq D$.

THEOREM

Let D be an open set in \mathbb{C} and $\{f_n\}$ be a sequence of ana functions s.t. $f_n \rightarrow f$ unif on cpt. Then, f is also ana in D .

Proof

$\because f_n$ is conti, $\forall K \subseteq D$: cpt set we have $f_n \rightarrow f$ unif on K

$\therefore f$ is conti on $K \forall K$, i.e. f is conti on D

We hope " $\int_{\Gamma} f dz = 0$ ", for Γ : boundary of a closed rectangle $R \subseteq D$

Hence, $\int_{\Gamma} f dz = \lim_{n \rightarrow \infty} \int_{\Gamma} f_n dz$

|| (f_n conti, $f_n \rightarrow f$ unif on R)

$\lim_{n \rightarrow \infty} (\int_{\Gamma} f_n dz)$

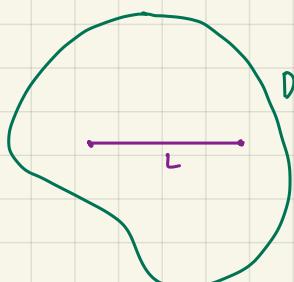
|| (Rectangle thm $\because f_n$: ana)

0

\therefore By Morera's Thm, f is conti. \square

THEOREM

f is continuous on an open set $D \subseteq \mathbb{C}$ and analytic except on a line segment in D . Then, f is analytic throughout D .

Proof

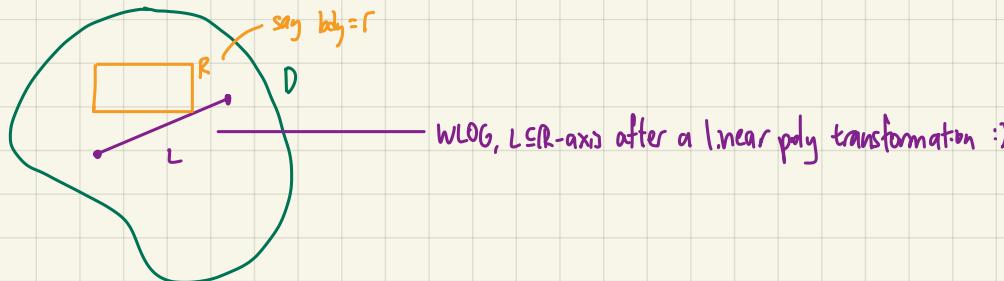
Fixed in next PDF cuz I was def high when I wrote the proof for the wrong thm \therefore (yes, hence the reupload)

REMARK

$f(z) = \frac{1}{z}$ is a counterexample as to why we cannot say " $\forall f: \text{ana ex} \text{e on finitely many pts in a region} \Rightarrow f: \text{ana}$ "

THEOREM

f is continuous on an open set $D \subseteq \mathbb{C}$ and analytic except on a line segment in D . Then, f is analytic throughout D .



Proof

We know $f|_{D \setminus L}$ is ana. Consider the following cases.

- ① $R \cap L = \emptyset \Rightarrow \int_L f(z) dz = 0$ as f is ana on $D \setminus L$
- ② $R \cap L \neq \emptyset$: Lift one side, we get a rectangle $R' \subseteq R$, $R' \cap L = \emptyset$



By case ①, as f is cont., $\lim_{\epsilon \rightarrow 0} \int_{R'} f(z) dz = 0 \Rightarrow \int_R f(z) dz = 0$

- ③ $R \cap L \neq \emptyset$



Then, $R = R_1 \cup R_2$, $\int_R f = \int_{R_1} f + \int_{R_2} f$

By cases ① and ②, hence $\int_R f(z) dz = 0$

∴ By Morera's Thm, f is analytic on D \square

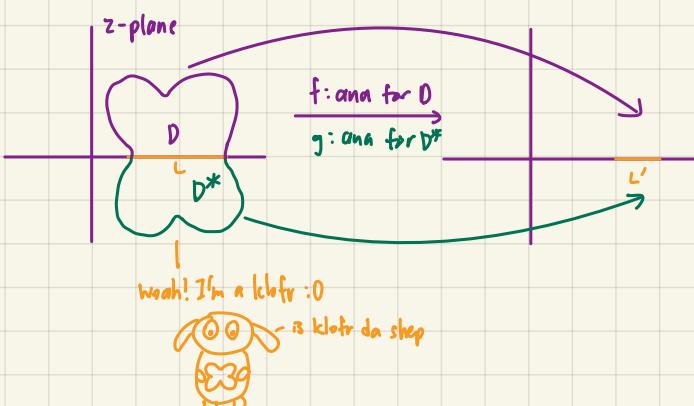
THEOREM (SCHWARZ REFLECTION PRINCIPLE)

Suppose f is C -analytic in a region D that is contained in either the upper or lower half plane and whose boundary contains a segment L on the real axis, and suppose f is real for real z .

Then, we can define an analytic "extension" g of f to the region $D \cup L \cup D^*$ that is symmetric with respect to the real axis by

$$g(z) = \begin{cases} f(z), & z \in D \cup L \\ f(\bar{z}), & z \in D^* \end{cases} \text{ where } D^* = \{z | \bar{z} \in D\}$$

Graphically, we represent it as follows:



Proof

① $z \in D$, then $f(z) = g(z)$, f : ana $\Rightarrow g$: ana

② $z \in D^*$ and $z \notin f(D)$, then:

$$\lim_{h \rightarrow 0} \frac{g(z+ih) - g(z)}{h} = \lim_{h \rightarrow 0} \left(\frac{f(z+ih) - f(z)}{h} \right) = f'(z)$$

$\therefore g$: ana

Since f : cont: on R -axis, so is g

\therefore We can apply the thm above so g : ana throughout $D \cup L \cup D^* \cup U$ \square

DEFINITION

A curve $\gamma: [a, b] \rightarrow C$ is called a regular analytic arc if γ is an analytic map, 1-1 on $[a, b]$ with $\gamma' \neq 0$.

FACT (WILL PROVE IN THE FUTURE)

$\forall a \in [a, b], \exists D(a, \epsilon), \text{ s.t. } \gamma: \text{ana on } D(a, \epsilon), \gamma'(z) \neq 0 \quad \forall z \in D(a, \epsilon)$

In fact, $\exists \gamma^{-1}: \text{ana}, \gamma: D(a, \epsilon) \xrightarrow{\sim} \gamma(D(a, \epsilon))$

Proof Sketch

Map the boundary via γ to a real segment $[a, b]$, apply the thm, then apply γ^{-1} . Then, reflect similarly in image via λ .

Mathematically, $\lambda(\gamma^{-1}(f(\gamma(\gamma^{-1}(z)))) \dots \times D$ (kill me)

SIMPLY CONNECTED DOMAIN

DEFINITIONS

Say $f: (S, \{U\}) \rightarrow (S, \{V\})$ for two topological spaces
 f is continuous if \forall open $V \subseteq S$, $f^{-1}(V)$ is open in S

Path-connected: $\forall \alpha, \beta \in S$, $\exists \gamma: [0, 1] \rightarrow S$ continuously s.t. $\gamma(0) = \alpha$, $\gamma(1) = \beta$ (E.g. $\{x=0\} \cup \{x=\frac{1}{n}\}$: connected but not path connected)
 Any open set cannot separate

IN C TOPOLOGICAL SPACE

We consider the topo space $C \cong \mathbb{R}^2$

We say S is simply connected if it is:

- ① Path-connected (\Rightarrow connected)
- ② For any continuous maps $f_i: [0, 1] \rightarrow S$ with $f_i(0) = f_i(1)$ and $f_i'(t) \neq 0$, \exists continuous $F: [0, 1] \times [0, 1] \rightarrow S$ s.t. $F(t, 0) = f_i(t)$, $F(t, 1) = f_i(t)$ (intuitively: they are connected via shrinking a rubber band for any two pts)

Actually, $S = C \cong \mathbb{R}^2$ is simply connected and a torus is not

\hookrightarrow can be seen as a disc with a hole taken out (disc裡挖出一個洞)

RECALL

$f: \text{ana on closed disc } D(0, 1)$, then $\int_{C(0, 1)} f dz = 0$ \iff Torus

However, remember $\int_{C(0, 1)} \frac{1}{z} dz = 2\pi i$, note that $\frac{1}{z}$ is well-defined on $D(0, 1) \setminus \{0\}$

Motivation: Simply via integration, we can determine the topo nature of subsets of C , equal zero or nonzero?

\hookrightarrow "Holomorphic simply connected" (= "topo simply connected")

DEFINITION

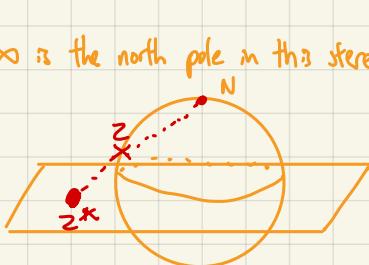
D is holomorphic simply connected (hsc) if $\forall f: \text{ana on } D$, $\int_C f dz = 0 \forall$ simple closed curve $C \subseteq D$

FACT: In C , hsc \Leftrightarrow sc

DEFINITION

We say the extended C -plane is $(C \cup \{\infty\}) \cong S^2$ (sphere) \hookrightarrow basically treat $\{\infty\}$ as one point, no more too as opposites, they are the same concept

∞ is the north pole in this stereographic projection



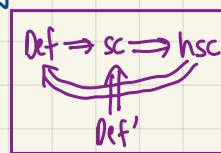
DEFINITION

If D is open connected $\subseteq C$, then D is sc if $(C \cup \{\infty\}) \setminus D$ is path connected

DEFINITION'

For a region $D \subseteq C$, we say D is sc' if $\forall z_0 \in C \setminus D$, $\forall \epsilon > 0$, \exists path $\gamma: (0, \infty) \rightarrow C \setminus D = \tilde{D}$, s.t.

- ① $d(\gamma, \tilde{D}) \leq \epsilon$
- ② $\gamma(0) = z_0$
- ③ $\lim_{t \rightarrow \infty} \gamma(t) = \infty$



REMARK

For $\text{open } D \subseteq \mathbb{C}$, D is open connected $\Leftrightarrow D$ is connected

Proof

" \Rightarrow ": Locally path connected (i.e. $\forall x \in D$, \exists open nbd $U(x) \subseteq D$ s.t. $U(x)$ is path-connected)

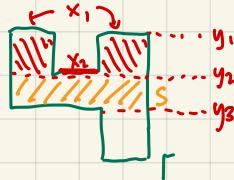
DEFINITION

Given a polygonal path Γ (e.g. level 3: ) , we define the level of $\Gamma := \#\text{diff values } y_0$ where the line $\text{Im}z = y_0$ contains a horizontal segment of Γ

LEMMA (no intersections w/ itself)

Γ : simple closed curve that is also a polygonal curve, say $\Gamma \subseteq D$, where D : sc region

Suppose the top level of Γ consists of points $y=y_1, x \in X$ and $y=y_2, x \in X_2$. Then the set $R := \{z + iy | y_2 \leq y \leq y_1, x \in X\}$ is contained in D

**Proof**

Consider induction on the level of Γ .

For $\text{lev}(\Gamma) \geq 2$,

- $\text{lev}(\Gamma) = 2$: $R = \bigcup_i R_i$, R_i : closed rectangle and $U(R_i) = \Gamma$

Let $z_0 \in R$.

Suppose $z_0 \notin R$. As D is simply connected, $\exists \gamma: [0, \infty] \rightarrow (\mathbb{C} \cup \{\infty\}) \setminus D$ s.t. $\gamma(0) = z_0$ and $\lim_{t \rightarrow \infty} \gamma(t) = \infty$, $t_0 := \sup \{t | \gamma(t) \in R\}$

(claim: $\gamma(t_0) \in \Gamma \subseteq D$ ($\Rightarrow \star$))

Proof

(i) $\gamma(t_0) \notin R \setminus \Gamma$: open (If no, then $\gamma(t_0) \in \Gamma$ or $\gamma(t_0) \in \mathbb{C} \setminus R$)

If yes, as $R \setminus \Gamma$: open, $\exists D(\gamma(t_0), \varepsilon) \subseteq R \setminus \Gamma$, $t_0 \in \gamma^{-1}(D(\gamma(t_0), \varepsilon)) \subseteq [0, \infty)$ (as $t_0 = \sup \{t | \gamma(t) \in R\}$) $\xrightarrow{\text{contradiction}}$

(ii) $\gamma(t_0) \notin \mathbb{C} \setminus R$: similarly, \star

$\therefore \gamma(t_0) \in \Gamma \subseteq D \xrightarrow{\text{contradiction}}$

- $\text{lev}(\Gamma) > 2$: Note: $U(\partial R_i)$ not necessarily equal to Γ

By the same argument as base case, $t_0 = \sup \{t | \gamma(t) \notin R\}$, $t_0 \in \partial R_i$ for some R_i :

If $t_0 \in \Gamma$, as in the base case, \star

Def $\tilde{R} := \mathbb{C} \setminus R$, $\Gamma' := (\Gamma \cap \tilde{R}) \cup L$, where $L := \{x + iy | y=y_2, x \in X_1 \setminus X_2\}$

For small h , $\gamma(t_0+h)$ is between the top two levels of Γ'

As $\gamma(t_0+h) \notin \tilde{R}$ and $\gamma(t_0+h) \in S$, $\text{level}(\Gamma') < \text{level}(\Gamma)$

\therefore By induction, $\gamma: [t_0+h, \infty) \rightarrow \mathbb{C} \setminus \Gamma'$ \square
 (as $\forall t > t_0$, $\gamma(t) \in R$)

10-9-25 (WEEK 6)

USED DEFINITION

For a region $D \subseteq \mathbb{C}$, D is simply connected if $(\mathbb{C} \setminus \{\infty\}) \setminus D$ is path connected

(We want to consider domain D s.t. $\int_C f(z) dz = 0$, f : analytic over D , $C \subseteq D$: simple closed curve)

THEOREM

f : ana in a s.c. region D and $C \subseteq D$ simple closed polygonal path. Then, $\int_C f(z) dz = 0$

Proof

Lemma from prev note $\Rightarrow R \subseteq D \Rightarrow \partial R \subseteq D$

As $C = \partial R + C'$, $\int_C f(z) dz = \int_{\partial R} f(z) dz + \int_{C'} f(z) dz$



By rectangle thm, $R \subseteq D$, f : ana in $D \Rightarrow f$: ana on $R \Rightarrow \int_{\partial R} f(z) dz = 0$

\therefore By induction on $\text{lev}(C')$, we get $\int_{C'} f(z) dz = 0 \square$

THEOREM

f : ana on a s.c. region $D \Rightarrow \exists$ primitive F , $F' = f$

Proof

Fix $z_0 \in D$, define $F(z) = \int_C f(s) ds$, where C = any polygonal path from z to $z_0 \subseteq D$

- F is well-defined: Suppose Γ_1, Γ_2 satisfy the polygonal path condition
Then, $\Gamma_1 - \Gamma_2 = \cup_i C_i$, C_i : simple closed polygonal curve $\subseteq D$
 $\Rightarrow \int_{\Gamma_1} f(s) ds - \int_{\Gamma_2} f(s) ds = \sum_i \int_{C_i} f(s) ds \stackrel{\text{by thm above}}{=} 0 \checkmark$

Now, let h be small enough s.t. $z + th \in D$

$$\Rightarrow \frac{F(z+th) - F(z)}{h} = \frac{1}{h} \left[\int_{\Gamma_1} f(s) ds - \int_{\Gamma_2} f(s) ds \right] = \frac{1}{h} \int_{\Gamma_3} f(s) ds, \text{ where}$$

Γ_1 : any poly path $z_0 \rightarrow z + th \subseteq D$

Γ_2 : any poly path $z_0 \rightarrow z \subseteq D$

Choose Γ_2 first, then $\Gamma_1 = \Gamma_2 + \Gamma_3$, where Γ_3 : any poly path $z \rightarrow z + th \subseteq D$

$$\text{Then, } \lim_{h \rightarrow 0} \left| \frac{F(z+th) - F(z)}{h} - f(z) \right| = \lim_{h \rightarrow 0} \frac{1}{h} \left| \int_{\Gamma_3} [f(s) - f(z)] ds \right| \stackrel{\substack{\text{by thm above} \\ z \rightarrow z+th \subseteq D}}{=} 0 \square$$

THEOREM (CLOSED CURVE THEOREM)

Let f : ana on a s.c. region D . Then, \forall simple closed curve $C \subseteq D$, $\int_C f(z) dz = 0$

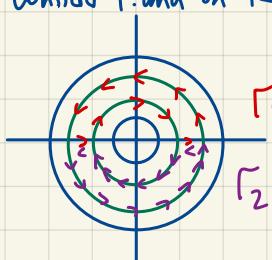
Proof

By Thm 3, $f = F'$ for some ana F

\therefore For any $C: \gamma(t): [0, 1] \rightarrow D$, $\int_C f(z) dz = F(\gamma(1)) - F(\gamma(0)) = 0 \quad \because \text{closed} \Rightarrow \gamma(1) = \gamma(0) \square$

EXAMPLE

Consider f : ana on $1 < |z| < 4$



Claim: $\int_{C_2(0)} f(z) dz = \int_{C_3(0)} f(z) dz$

Proof

We have $\int_{C_3(0)} f(z) dz - \int_{C_2(0)} f(z) dz = \boxed{\int_{r_1} f(z) dz + \int_{r_2} f(z) dz} = 0$ by closed curve thm \square

THE PROBLEM WITH DEFINING LOG

$\log z := u(z) + i v(z) \Rightarrow z = e^{u(z)} e^{i v(z)}$, but $\theta = v(z) + 2\pi k$, $k \in \mathbb{Z}$ all are fine, so how do we fix a value so $\log z$ well-def?

DEFINITION

We say f is an analytic branch of $\log z$ in a domain D if:

(i) f is analytic

(ii) $e^{f(z)} = z$, candidate: $f(z) = \underbrace{\log|z|}_{\text{log over } \mathbb{R}} + i \operatorname{Arg} z \in [0, 2\pi)$

10-14-25 (WEEK 7)

RESOLVING THE WELL-DEFINEDNESS OF LOG OVER \mathbb{C}

We can consider approaching $\log z$ via integration.

We want $\log z$ s.t. ① $f: \text{ann}$

$$\textcircled{2} \quad \exp(f(z)) = z$$

\therefore If $f(z) = \log z$, we want it to satisfy $f'(z) = \frac{1}{z}$

We can fix $z_0 \in \mathbb{R}^+$, so $f(z) := \int_C \frac{1}{z} dz + \log(z_0)$ where $C: z_0 \rightarrow z$

However, we need $\int_C \frac{1}{z} dz$ to be well-defined indep of path...

\therefore Choose a s.c. region D , then $\forall C \subseteq D$, $\int_C \frac{1}{z} dz$ is well-defined (details in proof below)

Proof (Sketch)

$\forall C_1, C_2 \subseteq D$ with the same endpoints, $C_1 - C_2$ forms a closed path in D

$$\therefore \int_{C_1 - C_2} \frac{1}{z} dz = 0 \Rightarrow \int_{C_1} \frac{1}{z} dz = \int_{C_2} \frac{1}{z} dz$$

THEOREM

Set $f(z) := \int_{z_0}^z \frac{1}{z} dz + \log z_0$ on a s.c. region $D \subseteq \mathbb{C} \setminus \{0\}$, we fix a $z_0 \in D$ and choose $\log z_0$.

Then, f is an analytic branch of $\log z$ in D .

Proof

As D is s.c., $C_1 - C_2$: closed curve

$$\therefore \text{By closed curve thm, } \int_{C_1 - C_2} \frac{1}{z} dz = \int_{C_1} \frac{1}{z} dz - \int_{C_2} \frac{1}{z} dz = 0$$

$\therefore f$ is analytic

Example for D 

Moreover, we want " $\exp(f(z)) = z \Leftrightarrow z e^{-f(z)} = 1$ "

$$\text{Set } g(z) := z e^{-f(z)} \Rightarrow g'(z) = e^{-f(z)} - z f'(z) e^{-f(z)} = 0$$

$$\therefore g(z) = \text{const} = g(z_0) = z_0 e^{-\log z_0} = 1$$

APPLICATION

Instead of this directly only used for \log , we can use analytic branch to define \sqrt{z} .

Reason:

Say $z = r e^{i\theta}$, then $(\sqrt{r} e^{i(\frac{\theta}{2} + \pi k)})^2 = r e^{i\theta} = z \quad \forall k \in \mathbb{Z}$, so \sqrt{z} is not uniquely defined

\therefore For $f(z) = \sqrt{z}$, we can define an analytic branch for $\log z$ by $\sqrt{z} = e^{\frac{1}{2}\exp(\log z + 2\pi i n)}$

SINGULARITY

DEFINITION

A deleted neighborhood of z_0 is an open set of $\{z \mid 0 < |z - z_0| < \delta\}$

↑ the actual deleted thing lmao

DEFINITION

f is said to have an isolated singularity at z_0 if f is analytic in a deleted neighborhood D of z_0 but is not analytic in z_0

EXAMPLES (Intuition, formal names given later)

① "Artificial" singularity: $f(z) = \begin{cases} \sin^2 z, & z \neq 0 \\ 0, & z = 0 \end{cases}$

② "Fixable by multiplying a polynomial": $\frac{1}{z}$ at $z=0$

③ "Unfixable": $\exp(\frac{1}{z})$ at $z=0$

DEFINITION

Say z_0 is a singularity of f , we can classify it as follows:

- ① If $\exists g: \text{ana at } z_0$ and $f(z) = g(z)$ in some deleted nbd of z_0 , we say f has a removable singularity at z_0
- ② If for $z \neq z_0$, f can be written as $f(z) = \frac{A(z)}{B(z)}$ where A and B are analytic at z_0 , $A(z_0) \neq 0$, $B(z_0) = 0$, we say f has a pole at z_0 . In particular, if B has a zero of order k at z_0 , then we say z_0 is a pole of f of order k
- ③ f has neither a removable singularity nor a pole at z_0 , then we call z_0 an essential singularity of f (not the focus of this course)

THEOREM (RIEMANN'S PRINCIPLE OF REMOVABLE SINGULARITIES)

If f has an isolated singularity at z_0 and $\lim_{z \rightarrow z_0} (z - z_0) f(z) = 0$, then the singularity is removable

Proof

Define $D'(z_0, \delta) := D(z_0, \delta) \setminus \{z_0\}$, $\exists \delta$, s.t. $f: \text{ana on } D'(z_0, \delta)$

$$\text{Set } g(z) := \begin{cases} (z - z_0) f(z), & z \in D'(z_0, \delta) \\ 0, & z = z_0 \end{cases}$$

Since $\lim_{z \rightarrow z_0} (z - z_0) f(z) = 0 = g(z_0)$, hence g is conti at z_0 .

$\therefore f: \text{ana on } D'(z_0, \delta)$

$\therefore g: \text{ana on } D'(z_0, \delta)$

— Morera needs it to be conti on the whole domain.

$\therefore g: \text{conti on } D(z_0, \delta) + \text{ana on } D'(z_0, \delta)$

$\therefore g: \text{ana on } D(z_0, \delta)$ (apply the conti except on a line segment thing)

Now, set:

$$h(z) := \begin{cases} \frac{g(z) - g(z_0)}{z - z_0}, & z \in D'(z_0, \delta) \\ g'(z), & z = z_0 \end{cases}$$

h is ana because g is ana.

Moreover, as $f(z) = h(z)$ on $D'(z_0, \delta)$, thus z_0 is a removable singularity

COROLLARY

f has an isolated singularity at z_0 . If f is bounded on some deleted nbd of z_0 , then z_0 is a removable singularity

Proof

$\exists \delta$, s.t. $f: \text{ana and bounded on } D'(z_0, \delta)$

Given $\varepsilon > 0$, $\forall 0 < |z - z_0| < \frac{\varepsilon}{M}$, $|(z - z_0) f(z)| < \varepsilon \Rightarrow \lim_{z \rightarrow z_0} (z - z_0) f(z) = 0$

\therefore Conclude with them above. \square

THEOREM 3

Say f has an isolated singularity at z_0 .

If $\exists k \in \mathbb{Z}_{>0}$, s.t. $\lim_{z \rightarrow z_0} (z - z_0)^k f(z) \neq 0$ but $\lim_{z \rightarrow z_0} (z - z_0)^{k+1} f(z) = 0$, then f has a pole of order k at z_0 (rem. sing = pole of order 0)

Proof

$$\text{Set } g(z) := \begin{cases} (z - z_0)^{k+1} f(z), & z \in D'(z_0, \delta) \\ 0, & z = z_0 \end{cases}$$

$\therefore \lim_{z \rightarrow z_0} (z - z_0)^{k+1} f(z) = 0 \therefore g: \text{conti at } z_0$

$\therefore f: \text{ana on } D'(z_0, \delta)$

$\therefore g: \text{ana on } D'(z_0, \delta)$

$\therefore g: \text{conti on } D(z_0, \delta) + \text{ana on } D'(z_0, \delta)$

$\therefore g: \text{ana on } D(z_0, \delta)$

$$\text{Set } h(z) := \begin{cases} \frac{g(z)-g(z_0)}{z-z_0} = (z-z_0)^k f(z), & z \in D'(z_0, \delta) \\ g'(z), & z=z_0 \end{cases}$$

$\therefore h$ is ana on $D(z_0, \delta)$.

As we know, by assumption, $\lim_{z \rightarrow z_0} h(z) \neq 0 \Rightarrow h(z_0) \neq 0$ ($\because h$ is ana)
 $\therefore f(z) = \frac{h(z)}{(z-z_0)^k} \Rightarrow f$ has a pole at order k at z_0 \square

REMARK

$|f(z)| < \frac{1}{|z|}$ in a deleted nbd of 0 and f has an isolated singularity at 0 $\Rightarrow 0$ is a removable singularity
 \therefore There exists nonbounded removable singularity

Proof

Actually, $|zf(z)| < \frac{1}{|z|} \Rightarrow \lim_{z \rightarrow z_0} zf(z) = 0 \Rightarrow 0$ is a removable singularity \square

REMARK

Similarly, if we have $|f(z)| < \frac{1}{|z|^{\alpha}}$, then we know $|z^2 f(z)| < \sqrt{|z|} \Rightarrow \lim_{z \rightarrow z_0} z^2 f(z) = 0$
 $\therefore \begin{cases} \text{Case 1: } \lim_{z \rightarrow z_0} z^2 f(z) = 0, \text{ then removable singularity (pole of order 0)} \\ \text{Case 2: } \lim_{z \rightarrow z_0} z^2 f(z) \neq 0, \text{ then pole of order 1} \end{cases}$

\Rightarrow It has a pole of at most order 1 (higher the order \Rightarrow the worse the pole)

THEOREM (CASORATI - WEIERSTRASS THEOREM)

If f has an essential singularity at z_0 and D is a deleted neighborhood of z_0 , where f is analyt.2, then the range $R := \{f(z) | z \in D\}$ is dense in C

Proof

Suppose not, then $\exists w \in C$ and $\delta > 0$, s.t. open $D(w, \delta) \cap R = \emptyset$

In other words, $\forall z \in D$, $|f(z) - w| \geq \delta \Rightarrow \left| \frac{1}{f(z) - w} \right| \leq \frac{1}{\delta} \quad \forall z \in D \Rightarrow \frac{1}{f(z) - w}$ is bounded in the del nbd

By coro, $\frac{1}{f(z) - w}$ has a removable singularity at z_0

$\therefore \exists g$: ana on $D' \cup \{z_0\}$, s.t. $g(z) = \frac{1}{f(z) - w} \Rightarrow f(z) = w + \frac{1}{g(z)}$ $\forall z \in D'$

$\therefore z_0$ is a zero of $g(z)$ of finite order or $g(z_0) \neq 0$

!!
n

$\therefore f(z)$ has a pole of order $\leq n$ at z_0 , so not an essential singularity \star

10-16-25 (WEEK 7)

Shun/翔海 (@shun4midx)

DEFINITION

We say $\sum_{i=0}^{\infty} \mu_i = L$ if $\sum_{i=0}^{\infty} \mu_i$ and $\sum_{j=1}^{\infty} \mu_{-j}$ both converge and their sum is L.

THEOREM

$f(z) = \sum_{k=0}^{\infty} a_k z^k$ is convergent in the domain $D = \{z : R_1 < |z| < R_2\}$, where $R_2 = (\lim_{k \rightarrow \infty} \sup |a_k|^{\frac{1}{k}})^{-1}$, $R_1 = \lim_{k \rightarrow -\infty} \sup |a_{-k}|^{-\frac{1}{k}}$.

Proof

$\sum_{k=0}^{\infty} a_k z^k$ converges when $|z| < (\lim_{k \rightarrow \infty} \sup |a_k|^{\frac{1}{k}})^{-1}$

$\sum_{k=1}^{\infty} a_{-k} (z^{-1})^k$ converges when $|z^{-1}| < (\lim_{k \rightarrow -\infty} \sup |a_{-k}|^{-\frac{1}{k}})^{-1} \Rightarrow |z| > \lim_{k \rightarrow -\infty} \sup |a_{-k}|^{-\frac{1}{k}}$

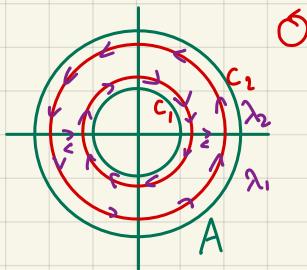
$\therefore f$ is conv $\forall z \neq 0$.

THEOREM

Let $A := \{z : R_1 < |z| < R_2\}$. If f is analytic in A, then f has a Laurent expansion $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ in A.

Proof

Say $C_1 := R_1(0)$, $R_1 < r_1 < r_2 < R_2$, $A := A_1 \cup A_2$, $A_1 := \{z : z \in A \text{ and } \operatorname{Im} z > -1\}$, $A_2 := \{z : z \in A \text{ and } \operatorname{Im} z < 1\}$.



$$\therefore C_2 - C_1 = \lambda_1 + \lambda_2$$

$$\therefore \int_{C_2 - C_1} \frac{f(w) - f(z)}{w - z} dw = \int_{\lambda_1 + \lambda_2} \frac{f(w) - f(z)}{w - z} dw \quad \forall w \in A$$

(*)

As f is ana. in A,

$$g(w) := \begin{cases} \frac{f(w) - f(z)}{w - z}, & w \in A \setminus \{z\} \\ f'(w), & w = z \end{cases} \quad \text{is ana.}$$

$$\Rightarrow (*) = \int_{\lambda_1} \frac{f(w) - f(z)}{w - z} dw + \int_{\lambda_2} \frac{f(w) - f(z)}{w - z} dw = 0 \quad \text{by closed curve thm.}$$

Moreover, by closed curve thm, $\int_{C_1} \frac{dw}{w-z} = 0$

$$\text{As } \int_{C_2} \frac{dw}{w-z} = 2\pi i, \quad f(z) \int_{C_1} \frac{dw}{w-z} = 2\pi i f(z)$$

[not A, why]

$$\begin{aligned} \therefore 2\pi i f(z) &= \int_{C_2 - C_1} \frac{f(w) - f(z)}{w - z} dw = \int_{C_2} \frac{f(w) - f(z)}{w - z} dw - \int_{C_1} \frac{f(w) - f(z)}{w - z} dw \\ &= \int_{C_2} \frac{f(w)}{w(1-\frac{z}{w})} dw + \int_{C_1} \frac{f(w)}{z(1-\frac{z}{w})} dw \\ &= \boxed{\int_{C_2} \left[\frac{f(w)}{w} \sum_{i=0}^{\infty} \left(\frac{z}{w}\right)^i \right] dw}^{(2)} + \boxed{\int_{C_1} \left[\frac{f(w)}{z} \sum_{i=0}^{\infty} \left(\frac{z}{w}\right)^i \right] dw}^{(1)} \end{aligned}$$

We know $\sum_{i=0}^{\infty} \left(\frac{z}{w}\right)^i$ conv abs. unif. on C_2 .

Hence, $(2) = \sum_{i=0}^{\infty} \left[\int_{C_2} \frac{f(w)}{w^{i+1}} dw \right] z^i$

$$(1) = \sum_{i=0}^{\infty} \left[\int_{C_1} f(w) w^i dw \right] z^{-(i+1)}$$

$$\therefore f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \quad \forall R_1 < |z| < R_2 \quad \square$$

Uniqueness

If $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ & $R_1 < |z| < R_2$, then $\int_C \frac{f(z)}{z^{k+1}} dz = \int_C \sum_{i=-\infty}^{\infty} c_i z^{i-(k+1)} dz = \sum_{i=-\infty}^{\infty} c_i \int_C z^{i-(k+1)} dz$ for $C = C_r(0) \subseteq A$.

$\therefore \int_C \frac{f(z)}{z^{k+1}} dz = c_k 2\pi i$ & $C = C_r(0) \subseteq A$

Thus, (i) $\int_C \frac{f(z)}{z^{k+1}} dz = c_k$ is indep of r for $C = C_r(0) \subseteq A$

(ii) $a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}} dz$ is indep of C_1 and C_2

\therefore The Laurent expansion of f in A is unique \square

REMARK

Taking $R_1=0$, we can take Laurent expansion at a pole or removable singularity.

10-23-25 (WEEK 8) (Sorry & the notes are shit, I'm very fucking upset at myself and attended CompAma today) Shun (翔) 海 (@shun4mix)

EXAMPLES OF LAURENT EXPANSIONS

Around $z=0$,

$$\frac{(2+z)^2}{z} = 2 + \frac{1}{z} + 2z^2 + \dots$$

$$\exp(\frac{1}{z}) = 1 + \frac{1}{z} + \frac{1}{2z^2} + \dots$$

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}(1/z + z^2 + \dots) = \frac{1}{z^2} + \frac{1}{z} + 1 + z + \dots$$

Why am I even fucking offering my notes as such a shit and lazy student anyway, it's not like my notes are beneficial

DEFINITION

If $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ is a Laurent expansion of f around an isolated singularity z_0 . Then, $\sum_{k=0}^{-1} a_k(z-z_0)^k$ is called the principal part of f at z_0 and $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ is called the analytic part of f .

PROPOSITION

(i) f has a removable singularity at z_0 . Then, $a_k = 0 \forall k < 0$

(ii) f has a pole of order k at z_0 . Then, $a_i = 0 \forall i < -k$ but $a_{-k} \neq 0$

(iii) f has an essential singularity at z_0 . Then, it must have inf many nonzero terms in its principal part

Proof

(i) $\exists D'(z_0, \delta)$ and \exists ana g on $D(z_0, \delta)$, s.t. $g=f$ on $D'(z_0, \delta)$

$\therefore g$ is ana at z_0

$$\therefore g = \sum_{i=0}^{\infty} b_i(z-z_0)^i, |z-z_0| < \delta$$

$\therefore f$ is ana on $D'(z_0, \delta)$

$$\therefore f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} b_k(z-z_0)^k \text{ on } D'(z_0, \delta)$$

By uniqueness of Laurent expansion, $a_k = 0 \forall k < 0$ \square

(ii) $f = \frac{A(z)}{B(z)}$, A, B ana on $D(z_0, \delta)$ with $A(z_0) \neq 0$ and $B(z)$ has a zero of order k at z_0 .

$$\therefore B(z) = c_k(z-z_0)^k + c_{k+1}(z-z_0)^{k+1} + \dots, c_k \neq 0$$

Thus, $B(z) = (z-z_0)^k(c_k + c_{k+1}(z-z_0)^1 + \dots)$

$\Rightarrow H(z)$

$\therefore H(z)$ is ana on $D(z_0, \delta)$ and $H(z_0) \neq 0$

$$\therefore f(z) = \frac{1}{(z-z_0)^k} \frac{A(z)}{H(z)}$$

As H : cont., $\exists D(z_0, \delta_1)$, $\delta_1 \leq \delta$, s.t. H has no zeros on $D(z_0, \delta_1)$.

$$\therefore f = \frac{1}{(z-z_0)^k} \frac{A(z)}{H(z)}$$
 on $D'(z_0, \delta_1)$ with $A(z)$ ana on $D(z_0, \delta_1)$

We consider Taylor expansion of $\frac{A}{H}$ around z_0 .

$$\text{Then, } \frac{A(z)}{H(z)} = \sum_{i=0}^{\infty} e_i(z-z_0)^i$$

$\therefore A(z_0) \neq 0, H(z_0) \neq 0$

$\therefore e_0 \neq 0 \Rightarrow a_i = 0 \forall i < -k, a_{-k} \neq 0 \square$

(iii) z_0 is an essential singularity, $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ on $D'(z_0, \delta)$

If only finitely many $a_k \neq 0$ for $k > 0$, then $\lim_{z \rightarrow z_0} f(z)(z-z_0)^N = 0$ for big enough $N \rightarrow \infty$

PROPOSITION 10

P, Q : poly with $\deg P < \deg Q$.

Say $Q(z) = \prod_{i=1}^n (z-z_i)^{e_i}$ with distinct z_i . Then, $R(z) = \frac{P(z)}{Q(z)}$ is a sum of polynomials in $\frac{1}{z-z_i}$ with $1 \leq i \leq n$

Proof

$R(z)$ is ana on $C \setminus \{z_1, \dots, z_n\}$.

2. z is an isolated singularity of R which is a pole of order at most e_i :

$$\text{Then, } R(z) = \sum_{i=0}^{\infty} c_i (z-z_1)^i = \underbrace{\sum_{i=0}^{-1} c_i}_{\sim P_1(\frac{1}{z-z_1})} (z-z_1)^{-1} + \underbrace{\sum_{i=0}^{\infty} c_i}_{\sim A_1(z)} (z-z_1)^i$$

Then, $A_1(z) = R(z) - P_1(\frac{1}{z-z_1})$ is ana on U .

$\lim_{z \rightarrow z_1} (z-z_1) A_1(z) = 0 \Rightarrow z_1$ is a removable singularity at $A_1(z) \Rightarrow A_1(z_1) := \lim_{z \rightarrow z_1} A_1(z)$

$\therefore A_1(z)$ ana on $C \setminus \{z_1, \dots, z_n\}$ \Rightarrow Inductively, we get $A_n(z) := A_{n-1}(z) - P_n(\frac{1}{z-z_n}) \Rightarrow$ principal part of $R(z)$

However, A_n is bounded since $R, P_n \xrightarrow{z \rightarrow \infty} 0$.

\therefore By Liouville's Thm, A_n is const., so $A_n \equiv 0$

$$\therefore R(z) = P_1(\frac{1}{z-z_1}) + \dots + P_n(\frac{1}{z-z_n}) \quad \square$$

THE RESIDUE THEOREM

KEY POINT

$$C_r(0) \Rightarrow \int_{C_r(0)} z^k dz = \begin{cases} 2\pi i, & k = -1 \\ 0, & \text{otherwise} \end{cases}$$

(★)

f is ana on $D'(z_0, \delta) \Rightarrow f = \sum_{i=-\infty}^{\infty} a_i (z-z_0)^i, 0 < |z-z_0| < \delta$.

As $C_r(z_0) \subseteq D'(z_0, \delta) \Rightarrow \int_{C_r(z_0)} f dz = a_{-1} (2\pi i)$

DEFINITION

(★), then we define $\text{Res}(f, z_0) := -1$

PROPOSITION

Given (★),

(i) z_0 is a simple pole (pole of order 1) then $a_{-1} = \lim_{z \rightarrow z_0} (z-z_0) f(z) = \frac{A(z_0)}{B'(z_0)}$

Proof

$$(z-z_0) f(z) = A(z) \div \frac{B(z)-B(z_0)}{z-z_0} \quad A, B: \text{ana} \Rightarrow \text{done}$$

(ii) If f has a pole of order k , then $a_{-1} = \frac{1}{(k-1)!} \left. \frac{d^{k-1}}{dz^{k-1}} ((z-z_0)^k f(z)) \right|_{z=z_0}$

EXAMPLE

$$\text{Res}(\csc z, 0) = \frac{1}{\cos z} \Big|_{z=0} = 1$$

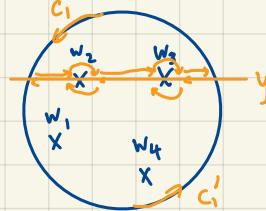
$\frac{1}{\sin z}$ has a simple pole at $z=0$.

PROPOSITION

For a region D , and f analytic on $D \setminus \{w_1, \dots, w_n\}$, with $w_i \in D(0, r)$, we have $\int_{C_r(0)} f(z) dz = 2\pi i \sum_{i=1}^n \text{Res}(f, w_i)$

Proof

Recall: $\int_{C_r(0)} \frac{1}{z} dz = 2\pi i$; (indep of r) $\Rightarrow \int_{C_r(0)} \frac{1}{z-a} dz = 2\pi i$



Let $Y := \{\text{Im}(w_i) \mid 1 \leq i \leq n\}$, where we order Y as $y_1 > y_2 > \dots > y_n$

Choose ε , s.t. $D(w_i, \varepsilon) \subseteq D(0, r)$ and $D(w_i, \varepsilon)$ are mutually disjoint

$$\Rightarrow \int_{C_ε + C_r} f(z) dz = \int_{C_r(0)} f(z) dz - \sum_{i: \text{Im}(w_i)=y_i} \int_{C_ε(w_i)} f(z) dz$$

By Closed Curve Thm on simply connected domain, $\int_{C_ε + C_r} f(z) dz = \int_{C_r(0)} f(z) dz - \sum_{i: \text{Im}(w_i)=y_i} \int_{C_ε(w_i)} f(z) dz$

As we know, $C_2 + C'_2 = C_1 - C_\varepsilon(w_i) \Rightarrow \int_{C_ε} f(z) dz = \int_{C_2} \square + \int_{C'_2} + \sum \int_{C_ε(w_i)}$

Inductively, we get: $\int_{C_r(0)} f(z) dz = \sum_{i=1}^n \int_{C_ε(w_i)} f(z) dz$

With the Laurent Expansion $f(z) = \sum_{k=-\infty}^{\infty} (z-w_i)^k$, hence $\int_{C_r(0)} f(z) dz = 2\pi i \sum_{i=1}^n \text{Res}(f, w_i)$ \square

OBSERVATION

For a simple closed curve Γ , if we can define the "interior" of Γ , then we can replace the above proposition by $\partial(\Gamma \cup \text{int}(\Gamma))$ with replacing $C_r(0)$ by Γ .

*e.g. not (e)***WINDING NUMBER — CAUCHY RESIDUE THEOREM****DEFINITION**

(i) Suppose γ is a closed curve and $a \notin \gamma$. We define the winding number of γ around a to be $\eta(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$

(E.g. $\gamma: C_r(a)$, $\eta(\gamma, b) = 1 \Leftrightarrow b \in D(a, r)$)

(ii) γ is called a regular closed curve if γ is a simple closed curve and $\eta(\gamma, a) \in \{0, 1\}$ $\forall a \notin \gamma$

THEOREM

For any closed curve γ and $a \notin \gamma$, $\eta(\gamma, a)$ is an integer

REMARK

$F: \times \gamma, \eta(\gamma, a)$ is cont: in $a \Rightarrow \eta(\gamma, a)$ is a locally constant function ($\eta(\gamma, a)$ is constant on each connected component of $\mathbb{C} \setminus \gamma$)

PROOF OF THEOREM

For $\gamma: z(t)$, $t \in [0, 1]$, define $F(s) = \int_0^s \frac{z'(t)}{z(t)-a} dt \quad \forall s \in [0, 1]$.

 $\mapsto \frac{d}{ds}$

Consider $g(z(s)) := (z(s)-a) \exp(-F(s))$

$\therefore g'(s) = z'(s) \exp(-F(s)) + (z(s)-a) \exp(-F(s)) (-F'(s))$, where $F'(s) = \frac{z''(s)}{z(s)-a} \Rightarrow g'(s) = 0$

Thus, $g(s) = g(0) = z(0)-a$

" $g(1) = (z(1)-a) \exp(-F(1))$

$$\therefore \exp(-F(1)) = \frac{z(1)-a}{z(0)-a}$$

As γ : closed curve, thus $z(1) = z(0) \Rightarrow \exp(-F(1)) = 1 \Rightarrow F(1) = 2\pi i k$ for some $k \in \mathbb{Z}$.

$$\therefore \eta(r, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz \in \mathbb{Z} \quad \square$$

REMARK

Fix $r, a \rightarrow \infty$. Then, $\left| \int_{\gamma} \frac{f(z)}{z-a} dz \right| \rightarrow 0$

$\therefore \eta(r, a) = 0$ when $a \rightarrow \infty \Rightarrow \eta(r, a) = 0$ on unbounded connected component of $\mathbb{C} \setminus \gamma$.

FACT (JORDAN CURVE THEOREM)

$\eta(r, a) = 0 \Rightarrow a \in$ unbounded connected component of $\mathbb{C} \setminus \gamma$.

THEOREM (CAUCHY RESIDUE THEOREM)

For $f: \text{ana}$ on a s.c. domain D except at w_1, \dots, w_n , γ : closed curve $\subseteq D \setminus \{w_1, \dots, w_n\}$, then $\int_{\gamma} f(z) dz = 2\pi i \sum_{j=1}^n \text{Res}(f, w_j) \eta(\gamma, a)$

Proof

Considering the Laurent series around w_i ,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-w_i)^k \quad \text{on } 0 < |z-w_i| < \epsilon$$

$$P_i\left(\frac{1}{z-w_i}\right) = \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{z-w_i}\right)^k$$

Notice, $P_i\left(\frac{1}{z-w_i}\right) \rightarrow \text{ana}$ on $D \setminus \{w_i\}$.

$\therefore g(z) = f(z) - \sum_{i=1}^n P_i\left(\frac{1}{z-w_i}\right) \rightarrow \text{ana}$ on $D \setminus \{w_1, \dots, w_n\}$.

Around w_i with $0 < |z-w_i| < \epsilon$, $g(z) = \sum_{k=0}^{\infty} a_k (z-w_i)^k - \sum_{j \neq i} P_j\left(\frac{1}{z-w_j}\right) \Rightarrow g$ is ana at $z=w_i$

$\therefore w_1, \dots, w_n$ are removable singularities at w_i .

By (closed) Curve Thm, $\int_{\gamma} g(z) dz = 0 \Rightarrow \int_{\gamma} f(z) dz = \sum_{i=1}^n \int_{\gamma} P_i\left(\frac{1}{z-w_i}\right) dz = \sum_{i=1}^n 2\pi i \text{Res}(f, w_i) \int_{\gamma} \frac{1}{z-w_i} dz = 2\pi i \sum_{i=1}^n \text{Res}(f, w_i) \eta(\gamma, a) \quad \square$

DEFINITION

f is meromorphic on a domain D if $f \rightarrow \text{ana}$ on D except at isolated poles

with winding it, we don't need to define inside or outside anymore

10-30-2S (WEEK 9)

THEOREM

Let γ be a regular closed curve and f be a meromorphic function on and inside γ .
 Suppose f has no poles nor zeros of f inside γ .

Set $n_z = \# \text{zeros inside } \gamma$, $n_p = \# \text{poles inside } \gamma$ Then, $\int_{\gamma} \frac{f'}{f} dz = (n_z - n_p) 2\pi i$; \star S rly fits winding numberProof

① f has a zero of order k at w , around w , we have $f(z) = \frac{f^{(k)}(w)}{k!} (z-w)^k + \dots = (z-w)^k g(z)$ and $g(z) \neq 0$ ana around w

$$\Rightarrow \frac{f'}{f} = \frac{k(z-w)^{k-1} g(z) + (z-w)^k g'(z)}{(z-w)^k g(z)} = \frac{k}{z-w} + \frac{g'(z)}{g(z)} \text{ around } w$$

$\therefore \frac{f'}{f}$ has a simple pole at w with radius k

② f has a pole at w of order k around w .

$$\therefore f = (z-w)^{-k} \sum_{i=0}^{\infty} a_i; (z-w)^{k+i} = (z-w)^{-k} g(z) \Rightarrow \frac{f'}{f} = -k \frac{1}{z-w} + \frac{g'(z)}{g(z)} \text{ around } w, \frac{f'}{f}$$
 has a simple pole with residue $-k$

③ w is not a pole nor a zero, then $\text{Res}(\frac{f'}{f}, w) = 0$ as $\frac{f'}{f}$ is ana at w

By Cauchy Residue Thm, $\int_{\gamma} \frac{f'}{f} dz = 2\pi i \sum_{i=1}^n n(r, w_i) \text{Res}(\frac{f'}{f}, w_i)$

Now, we know:

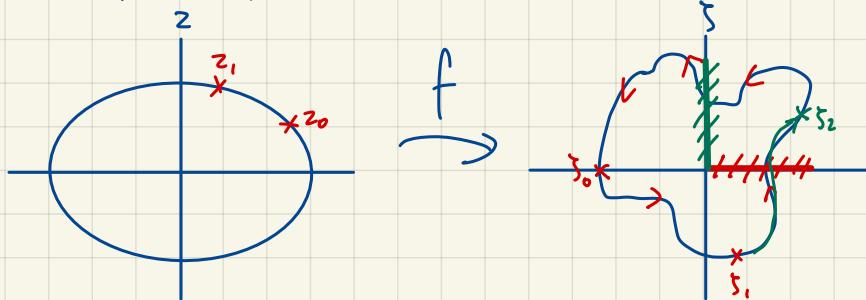
 $n(r, w_i) = 1$ for w_i : inside γ

$$\text{Res}(\frac{f'}{f}, w_i) = \begin{cases} k, & w_i: \text{zero of order } k \\ -k, & w_i: \text{pole of order } k \end{cases}$$

$$\therefore \int_{\gamma} \frac{f'}{f} dz = (n_z - n_p) 2\pi i \quad \square$$

COROLLARY (ARGUMENT PRINCIPLE) \star IMPORTANT TO KNOW THE INTUITION

If f is ana inside and on a closed curve γ and f has no zeros on γ , then $n_z = \frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} dz$

REMARK ON COROLLARYFor $\gamma: z(t)$, $\gamma' = f(z(t))$,

We can integrate over these curves:

$$\log(s) = \int_{s_0}^s \frac{ds}{s} + \log(s_0)$$

$$\log(s_1) = \int_{s_0}^{s_1} \frac{ds}{s} + \log(s_0) \leftarrow \arg \in [\pi, 2\pi]$$

$$\log(s_2) = \int_{s_1}^{s_2} \frac{ds}{s} + \log(s_1) \leftarrow \arg \in [\pi, 2\pi]$$

} Different analytic branches (need to choose one where the entire path is cont.)

(If we want to find a whole loop's worth of integration, we can do so by finding enough ana branches)

$$\Rightarrow \int_{\gamma'} \frac{dz}{z} = \log(s_n) - \log(s_0) \text{ where } s_n \text{ is the "overlapping" } s_0 \text{ point on a new ana branch}$$

APPLICATIONS OF RESIDUE THEOREM

THEOREM (ROUCHE'S THEOREM)

Suppose f, g are analytic inside and on a regular closed curve γ , and suppose $|f(z)| > |g(z)| \forall z \in \gamma$, then $n_z(f+g) = n_z(f)$ inside γ

Proof

As $|f(z)| > |g(z)| \geq 0 \forall z \in \gamma$, $f+g$ has no zeros on γ (and f has no zeros on γ)

$f+g$ has no poles since f, g : ana. $\boxed{f, 1+\frac{g}{f} \text{ have no zeros or poles on } \gamma}$

$$\therefore n_z(f+g) + 0 = \int_{\gamma} \frac{(f+g)}{f+g} dz = \int_{\gamma} \frac{[f(1+\frac{g}{f})]'}{f(1+\frac{g}{f})} dz = \int_{\gamma} \frac{f'}{f} dz + \int_{\gamma} \frac{(1+\frac{g}{f})'}{1+\frac{g}{f}} dz = n_z(f) + n_z(1+\frac{g}{f})$$

Notice, $1+\frac{g}{f}$ will never equal zero, since that will need $|\frac{g(z)}{f(z)}|=1$, but by max mod thm, $\forall z \in \text{inside}, |\frac{g}{f}| < 1$

$$\therefore n_z(1+\frac{g}{f}) = 0$$

$$\therefore n_z(f+g) = n_z(f) \quad \square$$

EXAMPLE

For $f=2z^{10}+4z^2+1$, how many zeros of f are inside $D(0, 1)$?

Take $g=4z^2, h=2z^{10}+1 \Rightarrow \forall |z|=1, |g(z)| > |h(z)|$

By Rouché's Thm, $n_z(g+h) = n_z(g)$ in $D(0, 1) \Rightarrow f$ has two zeros in $D(0, 1)$.

THEOREM (GENERALIZED CAUCHY INTEGRAL FORMULA)

Let f be analytic on a s.c. domain D , and γ be a regular closed curve contained in D . Then, for each z inside γ , $k \in \mathbb{Z}_{\geq 0}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{k+1}} dw$$

Proof

Around z , we have $f(w) = f(z) + f'(z)(w-z) + \dots + \frac{f^{(k)}(z)}{k!} (w-z)^k + \dots$

Now, we know $\text{Res}(\frac{f(w)}{(w-z)^{k+1}}, z) = \frac{f^{(k)}(z)}{k!}$

$$\therefore \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{k+1}} dw = \frac{f^{(k)}(z)}{k!} \quad \square$$

RECALL

For ana $\{f_n\}_n$, if $f_n \rightarrow f$ uniformly on compacta of D , it means " $\forall V \subseteq D: \text{cpt}, f_n \rightarrow f$ uniformly on V "

Moreover, by Morera's Thm, we had $f_n: \text{ana} \Rightarrow f: \text{ana}$

THEOREM

Let $\{f_n\}$ be a sequence of analytic functions that converges uniformly on compacta of region D

Then, f is analytic and $f_n \rightarrow f$ uniformly on compacta of D

Proof

$\forall z_0 \in D$, choose $r_0 > 1$, s.t. $\overline{D(z_0, r_0)} \subseteq D$, $R := C_{r_0}(z_0)$

Then, by Generalized Cauchy integral formula,

$$f'_n(z) - f'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(w) - f(w)}{(w-z)^2} dw$$

Recall, $f_n \rightarrow f$ uniformly on $\overline{D(z_0, r_0)}$. By def, given $\epsilon > 0$, $\exists N$, s.t. $|f_n(z) - f(z)| < \frac{\epsilon r_0^2}{4} \quad \forall z \in D(z_0, r_0)$

In particular, $\forall z \in D(z_0, \frac{1}{2}r_0)$, $|f_n(z) - f(z)| < \frac{\epsilon r_0^2}{4}$

By ML-formula, $|f'_n(z) - f'(z)| \ll \frac{1}{2\pi} \frac{\frac{\epsilon r_0^2}{4}}{(\frac{r_0}{2})^2} \cdot 2\pi r_0 = \epsilon$

Now, \forall compact $V \subseteq D$, $\exists r < 1$, s.t. $f_n' \rightarrow f'$ uniformly on $\overline{D(z, \frac{r}{2})}$

By def, $V \subseteq \bigcup_{z \in V} (D, \frac{r}{2}) \Rightarrow \exists z_i, r_i$, s.t. $V = \bigcup_i D(z_i, \frac{r_i}{2})$

$\therefore f_n' \rightarrow f$ uniformly on V . \square

THEOREM (HURWITZ'S THEOREM)

Let $\{f_n\}$ be a sequence of non-vanishing analytic functions in a region D .

Suppose that $f_n \rightarrow f$ uniformly on compacta of D . Then, either $f \equiv 0$ on D or $f(z) \neq 0 \forall z \in D$

Proof

Suppose $\exists z \in D$, s.t. $f(z) = 0$

Claim: $f \equiv 0$ on D

Proof

As f_n is analytic and has no zero on D , choose r , s.t. $\overline{D(z, r)} \subseteq D$ and f has no zero on $C_r(z)$ (otherwise, f has an acc point of zero + D : region $\Rightarrow f \equiv 0$ on D)

$$\text{Now, } \frac{1}{2\pi i} \int_{C_r(z)} \frac{f'}{f} dz = n_z(f) \text{ in } D(z, r) \geq 1$$

We know $f_n' \rightarrow f'$ uniformly on $D(z, r)$, so $\frac{f_n'}{f_n} \rightarrow \frac{f'}{f}$ uniformly on $C_r(z)$, since f_n, f have no zeros on $C_r(z)$

$$\text{However, } \int_{C_r(z)} \frac{f'}{f} dz = \lim_{n \rightarrow \infty} \int_{C_r(z)} \frac{f_n'}{f_n} dz = 0 \text{ (since } f_n \text{ has no zeros in } D) \rightarrow \times$$

COROLLARY

Replace "0" with "a" in Hurwitz's Thm

Proof

Consider $g_n(z) = f_n(z) - a$. \square

EXAMPLE

We know $\sin z = \sum_{i=0}^{\infty} (-1)^i \frac{z^{2i+1}}{(2i+1)!}$, take $f_n(z) = \sum_{i=0}^n (-1)^i \frac{z^{2i+1}}{(2i+1)!}$

$f_n \rightarrow \sin z$ uniformly on any compact set $V \subseteq \mathbb{C}$.

\therefore By Hurwitz's Thm, $f_n(z)$ has a zero in $D(0, 2\pi)$ $\forall n \geq N$ for some $N > 0$. !!

THEOREM

Let $\{f_n\}$ be a seq of ana functions, and $f_n \rightarrow f$ uniformly on compacta of a region D . If f_n is 1-1 $\forall n$, then either $f: \text{const}$ or 1-1

Proof

Suppose $\exists z_1, z_2 \in D$, $z_1 \neq z_2$ s.t. $f(z_1) = f(z_2) = a$

$\exists r$, s.t. $D(z_1, r) \subseteq D$ and $D(z_1, r) \cap D(z_2, r) = \emptyset$

On $D(z_1, r)$, $f_n \rightarrow f$ on compacta and $f(z_1) = a$. Hence, if $f_n(z) \neq a \quad \forall z \in D(z_1, r)$, then $f \equiv a$ on $D(z_1, r)$.

If $f \equiv a$ on $D(z_1, r)$, as D is a region, $f \equiv a$ on D .

If $f \neq a$ on D , then by coro, $\exists N$, s.t. $f_n(z) = a$ has a zero on $D(z_1, r)$ for $n \geq N$.

f_n are 1-1 $\Rightarrow \forall n \geq N$, $f_n - a$ has no zeros on $D(z_1, r)$

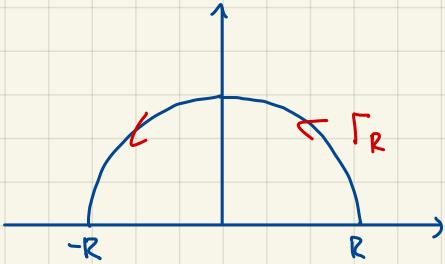
$\therefore f \equiv a$ on D $\rightarrow \times$

RESIDUE THEOREM AND EVALUATION OF INTEGRALS AND SUMS

TYPE I INTEGRALS

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{P(x)}{Q(x)} dx$$

If P, Q : poly and $\deg P \leq \deg Q + 2$, $\gcd(P, Q) = 1$, and $Q(x)$ has no real roots



Define $\gamma: \Gamma_R + (-R \rightarrow R)$

Then, $\int_{\gamma} \frac{P(z)}{Q(z)} dz = \underbrace{\int_{\Gamma_R} \frac{P(z)}{Q(z)} dz}_{\text{Residue Thm}} + \boxed{\int_{-R}^R \frac{P(x)}{Q(x)} dx} = 2\pi i \left(\sum_{\substack{w: \text{root of } Q(z) \\ \text{in upper-half plane}}} \text{Res}\left(\frac{P(z)}{Q(z)}, w_i\right) \right)$

When R is big enough,

Claim: $\lim_{R \rightarrow \infty} \left| \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz \right| = 0$

Proof

Notice, for $R > 0$, $\left| \frac{P(z)}{Q(z)} \right| \leq \frac{A}{|z|^2} \leq \frac{A}{R^2} \quad \forall |z| \geq R$

By ML-Formula,

$$\int_{\Gamma_R} \frac{P(z)}{Q(z)} dz \ll \frac{A}{R^2} \cdot \pi R = \frac{\pi A}{R} \xrightarrow{R \rightarrow \infty} 0$$

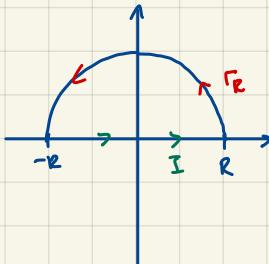
$$\therefore \int_{-R}^R \frac{P(x)}{Q(x)} dx = 2\pi i \left(\sum_{\substack{w: \text{root of } Q(x) \\ \text{in upper-half plane}}} \text{Res}\left(\frac{P(x)}{Q(x)}, w_i\right) \right)$$

EXAMPLE

$$\int_{-\infty}^{\infty} \frac{1}{x^{1/4} + 1} dx = 2\pi i \left(\text{Res}\left(\frac{1}{x^{1/4} + 1}, e^{\pi i/4}\right) + \text{Res}\left(\frac{1}{x^{1/4} + 1}, e^{3\pi i/4}\right) \right) = \frac{\sqrt{2}}{2} \pi$$

11-6-25 (WEEK 10) Ok, elephant in the room: I think I'll still wanna cry every CompAnal lecture so I'll stop Shun/淳海(@shun4midx) venting about it in my comments but yeah feel free to not use my notes anymore due to the lower quality. Don't ask why I wanna cry, and before you wonder, no, I didn't fail any exam; yes, I still love CompAnal.

If we want to apply the route in (I), we must check $\lim_{R \rightarrow \infty} \int_{\Gamma_R} f(x) \cos x dx = 0$ (or the equivalent for $\sin x$)



The key is in $e^{iz} = \cos z + i \sin z$ (However, note $|e^{iz}| = e^{Re z}$ may not vanish!)

$$N(m) := \int_{\Gamma_{M+1}}^{\infty} f(z) e^{iz} dz \Rightarrow \text{We have } \int_{-\infty}^{\infty} f(x) \sin x dx = \lim_{M \rightarrow \infty} \operatorname{Im}(N(m)), \quad \int_{-\infty}^{\infty} f(x) \cos x dx = \lim_{M \rightarrow \infty} \operatorname{Re}(N(m))$$

* Caution: $f(z)e^{iz}$ may have singularities on the real axis, we may need to consider $e^{iz}-c, c \in \mathbb{C}^*$

EXAMPLE

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = ?$$

We CANNOT consider $\int_{\Gamma} \frac{e^{iz}}{z} dz \Rightarrow$ consider $\int_{\Gamma} \frac{e^{iz}-1}{z} dz$ ($z=0$: removable singularity)

$$\text{Then, } \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \operatorname{Im} \left(\lim_{M \rightarrow \infty} \int_{\Gamma_{M+1}} \frac{e^{iz}-1}{z} dz \right)$$

LEMMA 1 * Key lemma for cool complex analysis integral types

Let f be an analytic function defined on a sector of the upper-half plane

Suppose $|f(z)| \rightarrow 0$ as $z \rightarrow \infty$

Then, we have

$$\left| \int_{\Gamma_R} f(z) e^{iz} dz \right| \rightarrow 0 \text{ as } R \rightarrow \infty, \text{ where } \Gamma_R: Re^{i\theta}, 0 \leq \theta_1 \leq \theta \leq \theta_2 \leq \pi$$

Proof

Define $M := \max_{z \in \Gamma_R} |f(z)|$, notice $M \rightarrow 0$ as $R \rightarrow \infty$

$$\text{Then, } \left| \int_{\Gamma_R} f(z) e^{iz} dz \right| \leq M \int_{\Gamma_R} |e^{iz}| dz = M \int_{\theta_1}^{\theta_2} e^{-R \sin \theta} R d\theta \leq M \int_0^\pi e^{-R \sin \theta} R d\theta = 2\pi \int_0^{\frac{\pi}{2}} e^{-R \sin \theta} R d\theta$$

$$\text{Notice, } \frac{2}{\pi} \leq \frac{\sin \theta}{\theta} \leq 1 \quad \text{concave down: } \frac{\sin \theta}{\theta} \geq \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

$$\text{Thus, } \left| \int_{\Gamma_R} f(z) e^{iz} dz \right| \leq 2M \int_0^{\frac{\pi}{2}} R e^{-R \sin \theta} d\theta \leq M\pi$$

$$\therefore \text{As } R \rightarrow \infty, \left| \int_{\Gamma_R} f(z) e^{iz} dz \right| \rightarrow 0 \quad \square$$

APPLICATION TO CASE (II) INTEGRALS

For $f(x) = \frac{P(x)}{Q(x)}$, for big enough R , $f(x)e^{iz}$ has no poles on Γ_R , hence: $\lim_{R \rightarrow \infty} \int_{\Gamma_R+I} f(z) e^{iz} dz = \lim_{R \rightarrow \infty} \int_{-R}^R f(z) dz$

CALCULATED EXAMPLE

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \text{Im} \left[\lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{iz}-1}{z} dz \right] = \text{Im} \left(2\pi i; \text{Res} \left(\frac{e^{iz}-1}{z}, 0 \right) \right) = \pi$$

Shun/翔海 (@shun4midx)

Do not separate this into $\int \frac{e^{iz}}{z}$ and $\int \frac{1}{z}$, they both are divergent.

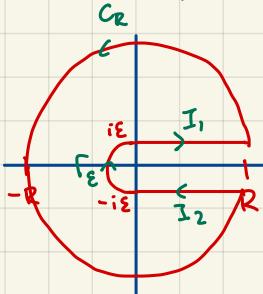
TYPE (III) INTEGRALS

$$\int_0^\infty \frac{P(x)}{Q(x)} dx \text{ for } \gcd(P, Q)=1, \deg Q \geq \deg P+2, Q(x) \neq 0 \quad \forall x \geq 0$$

If $\frac{P(x)}{Q(x)}$ is even, we should just do $\frac{1}{2} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$

Let $f(x) := \frac{P(x)}{Q(x)}$. Consider if $f(x)$ is NOT even.

Consider the path:



Abuse of notation to write C_R, Γ_ε . Consider $R \rightarrow \infty, \varepsilon \rightarrow 0$.

Consider $\log z$ on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}, \arg z \in (0, 2\pi)$

For $f(z)\log z$,

$$\begin{aligned} I_1: \log(t+i\varepsilon) &= \log|t+i\varepsilon| + \theta(t)i \\ I_2: \log(t-i\varepsilon) &= \log|t-i\varepsilon| + [2\pi - \theta(t)]i \end{aligned} \quad \left. \begin{array}{l} \text{difference in arg due to symmetry} \\ \text{d.f.} \end{array} \right\}$$

Notice, $\theta(t) \xrightarrow{\varepsilon \rightarrow 0} 0$ rapidly

Rmk

If $f(0) \neq 0$, then $f(z)\log z$ is not defined at $z=0$, so don't write $\lim_{R \rightarrow \infty} \int_{-\varepsilon}^{\varepsilon} f(x)\log x dx = \int_0^\infty f(x)\log x dx$

We consider the paths I_1 and I_2 ,

$$I_1: \int_0^R f(t+i\varepsilon) \log(t+i\varepsilon) dt$$

$$I_2: \int_0^R f(t-i\varepsilon) \log(t-i\varepsilon) dt$$

$$\begin{aligned} \Rightarrow \text{Total: } & \int_0^R [f(t+i\varepsilon) \log(t+i\varepsilon) - f(t-i\varepsilon) \log(t-i\varepsilon)] dt \\ &= \int_0^R [f(t+i\varepsilon) - f(t-i\varepsilon)] \log(t+i\varepsilon) dt + \int_0^R f(t+i\varepsilon) [\log(t+i\varepsilon) - \log(t-i\varepsilon)] dt \end{aligned}$$

For $(x, \varepsilon) \in [0, \varepsilon_0] \times [0, \varepsilon_0]$,

$$[f(x+i\varepsilon) - f(x-i\varepsilon)] \log(x+i\varepsilon) =: g(x, \varepsilon). \text{ Let } \alpha := \max_{x, \varepsilon} |f(x+i\varepsilon) - f(x-i\varepsilon)| \log(x+i\varepsilon)$$

Given $R, \delta_1, d > 0, \exists \varepsilon, s.t. \quad ① \theta(x+i\varepsilon) < \delta, \forall x \geq d$

$$② |f(x+i\varepsilon) - f(x-i\varepsilon)| < \frac{\delta_1}{m}$$

Then, as $\varepsilon \rightarrow 0$,

$$\int_0^R f(t+i\varepsilon) \log(t+i\varepsilon) dt - \int_0^R f(t-i\varepsilon) \log(t-i\varepsilon) dt = \int_0^R f(t+i\varepsilon) 2\pi i dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^R f(t) 2\pi i dt \quad ①$$

Now, we know $\left| \frac{P(x)}{Q(x)} \right| \leq \frac{A}{x^m}$

$$\text{So, as } \varepsilon \rightarrow 0, \left| \int_{-\varepsilon}^{\varepsilon} f(z) \log z dz \right| \leq \frac{A}{R} (\log R + 2\pi) \cdot 2\pi R \xrightarrow{R \rightarrow \infty} 0 \quad ②$$

Choose a small ε , s.t. $\Omega(z) \neq 0 \forall z \in D(0, \varepsilon)$.

Let $m := \max_{z \in D(0, \varepsilon)} |f(z)|$.

$$\therefore \int_{\Gamma_\varepsilon} f(z) \log z dz \ll m(\log \varepsilon + c) \cdot \pi \varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \quad (3)$$

$$\therefore -2\pi i \int_0^\infty f(t) dt = 2\pi i \sum_{\substack{w_k: \text{roots} \\ \text{of } Q(z)}} \operatorname{Res}(f(z) \log z, w_k)$$

$$\Rightarrow \int_0^\infty f(t) dt = -\sum_{\substack{w_k: \text{roots} \\ \text{of } Q(z)}} \operatorname{Res}(f(z) \log z, w_k)$$

EXAMPLE

$$\int_0^\infty \frac{1}{1+x^3} dx = -\sum_{k=1,3,5} \operatorname{Res}\left(\frac{1}{1+z^3} \log z, w_k\right)$$

ALTERNATE FORM

$\int_a^\infty \frac{P(x)}{Q(x)} dx$ can be evaluated by considering $\int_{C_R} \log(z-a) \frac{P(z)}{Q(z)} dz$. In fact, $\int_a^\infty = \int_0^\infty - \int_a^\infty$

ALTERNATE FORM

$$\int_0^\infty \frac{x^{\alpha-1}}{Q(x)} dx, \quad 0 < \alpha < 1 \text{ and } Q: \text{poly w/ deg 1}$$

Notice, $z^{\alpha-1} := \exp((\alpha-1) \log z)$

Then,

$$\int_{C_R} \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$\int_{\Gamma_\varepsilon} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

$$I_1, \varepsilon \rightarrow 0, \int_0^\infty \frac{x^{\alpha-1}}{Q(x)} dx$$

$$I_2, \varepsilon \rightarrow 0, \int_2 \frac{z^{\alpha-1}}{Q(z)} dz = \int_0^\infty \frac{z^{\alpha-1} e^{2\pi i(\alpha-1)}}{Q(z)} dz$$

$$\therefore [1 - e^{2\pi i(\alpha-1)}] \int_0^\infty \frac{x^{\alpha-1}}{Q(x)} dx = 2\pi i \sum_{w_k: \text{roots of } Q} \operatorname{Res}\left(\frac{z^{\alpha-1}}{Q(z)}, w_k\right)$$

TYPE (IV) INTEGRALS

$$\text{For } R(x,y) = \frac{P(x,y)}{Q(x,y)}, \quad P, Q \in \mathbb{C}[x,y], \quad \int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta$$

We take $z = \sin \theta + i \cos \theta, \quad d\theta = \frac{dz}{iz}$.

$$\cos \theta = \frac{z+z^{-1}}{2}, \quad \sin \theta = \frac{z-z^{-1}}{2i}$$

$$\Rightarrow \text{change of variables gives us } \int_{|z|=1} \frac{P\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)}{Q\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)} \frac{dz}{iz} = 2\pi i \sum_{\substack{w_i: \text{polys} \\ \text{of } f(z)}} \operatorname{Res}(f(z), w_i)$$

EXAMPLE

$$\int_0^{2\pi} \frac{d\theta}{2 \cos \theta} = \int_{|z|=1} \frac{-2i}{z^2 + 4z + 1} dz = 4\pi \operatorname{Res}\left(\frac{1}{z^2 + 4z + 1}, \sqrt{3} - i\right)$$

ESTIMATING SUMS

TYPE (I)

$$\sum_{n=-\infty}^{\infty} f(n) \stackrel{(1)}{=} \sum_{n=-\infty}^{\infty} (-1)^n f(n) \stackrel{(2)}{=}$$

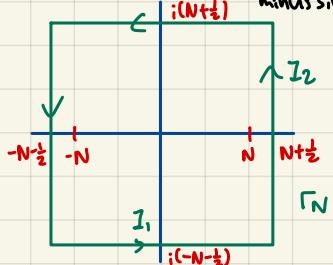
Want: Find an analytic $g(z)$ on $(-\infty, \infty) \setminus \{z \in \mathbb{Z}\}$, s.t. $\operatorname{Res}(g(z), n) = c f(n)$, $c \in \mathbb{C}^*$

$\hookrightarrow \frac{1}{\sin \pi z}$ has simple roots at $z \in \mathbb{Z}$ and ana elsewhere.

We have: $\operatorname{Res}\left(\frac{1}{\sin \pi z}, n\right) = \frac{1}{\pi} (-1)^n$

Let $g(z) = f(z) \frac{\pi}{\sin \pi z} \Rightarrow \operatorname{Res}(g(z), n) = (-1)^n f(n)$

$\therefore \sum_{n=-\infty}^{\infty} f(n) (-1)^n = \sum_{n=-\infty}^{\infty} \operatorname{Res}(g(z), n) + \sum_{\substack{\text{missing} \\ \omega \in \text{sing of } f}} \operatorname{Res}(g(z), \omega_k) = \frac{1}{2\pi i} \int_{\Gamma_N} g(z) dz$, where Γ_N is as follows:



Ensure $\sum f(n), \sum (-1)^n f(n)$ conv, we assume $|f(z)| \leq \frac{A}{|z|^2}$

Here, $\frac{1}{\sin \pi z} = \frac{2i}{e^{i\pi z} - e^{-i\pi z}} = \frac{2i e^{i\pi z}}{e^{2i\pi z} - 1}$ ⚡ key term w/ poles

$$I_1: \left| \frac{1}{\sin \pi z} \right| = \left| \frac{2e^{\pi(N+\frac{1}{2})}}{e^{2\pi(N+\frac{1}{2})} - 1} \right| < 1$$

$$I_2: \left| \frac{1}{\sin \pi z} \right| = \left| \frac{2e^{-y\pi}}{e^{-2\pi y} e^{(2N+1)\pi} - 1} \right| = \frac{e^{-y\pi}}{e^{-2\pi y} + 1}$$

$$\therefore \lim_{N \rightarrow \infty} \left| \int_{\Gamma_N} f(z) \frac{\pi}{\sin \pi z} dz \right| \leq \lim_{N \rightarrow \infty} 4(2N+1) c \cdot \frac{A}{|N+\frac{1}{2}|^2} = 0 \Rightarrow \sum_{n=-\infty}^{\infty} f(n) = - \sum_{\substack{\text{missing} \\ \omega \in \text{sing}}} \operatorname{Res}(g(z), \omega_k)$$

- cos πz gives the $(-1)^n$
- (A): $g(z) = f(z) \cot \pi z$
 - (B): $g(z) = f(z) \csc \pi z$

WARNING

When evaluating $\sum_{n=-\infty}^{\infty} \frac{1}{n^2}$, notice $\frac{1}{0^2}$ is undefined.

$$\Rightarrow \int_{\Gamma_N} \frac{\pi}{z^2} \frac{z \cos \pi z}{\sin \pi z} dz = 2\pi i \sum_{n=0}^{\infty} \operatorname{Res}(g(z), n) + \operatorname{Res}(g(z), 0)$$

Then we evaluate as usual.

SUMMARY

For (A), consider $g(z) := f(z) \frac{\cos \pi z}{\sin \pi z} \pi$

For (B), consider $g(z) := f(z) \frac{1}{\sin \pi z} \pi$

TYPE (II) — BINOMIAL COEFFICIENTS

We know $\binom{n}{k} \sim \text{coef of } z^k \text{ in } (1+z)^n$, so $\binom{n}{k} = \frac{1}{2\pi i} \int \frac{(1+z)^n}{z^{k+1}} dz$

EXAMPLE

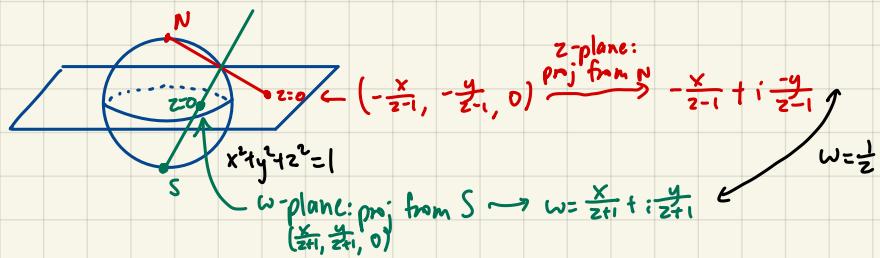
$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C_R(0)} \frac{(1+z)^{2n}}{(5z)^n} \frac{dz}{z} = \frac{1}{2\pi i} \int_{C_R(0)} \sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{(5z)^n} \frac{dz}{z} = \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{3z-1-z^2} dz = 5 \operatorname{Res}\left(\frac{1}{3z-1-z^2}, \frac{5\sqrt{5}}{2}\right)$$

RESIDUE AT INFINITY

DEFINITION

For f : ana on $\mathbb{C} \setminus \{w_1, \dots, w_k\}$, $\text{Res}(f, \infty) := - \int_{C_R} f(z) dz$

For $R > 0$, s.t. $|w_i| < R$, we have " ∞ " as follows:



→ We can switch to the w-plane, to only have one residue (N) remaining
[change of coords lol]

RESIDUE AT INFINITY

DEFINITION from the w -plane.

$$\text{Res}(f, \infty) := -\frac{1}{2\pi i} \int_C f(z) dz, \quad C := C_R(0)$$

INTUITION/PROOF OF THEOREM

Say $f(z)$ is a poly of deg n , hence it is entire

Then, $z \rightarrow \infty \Rightarrow |f(z)| \rightarrow \infty$, i.e. $w=0$ is a pole ($w \rightarrow 0 \Rightarrow |f(w)| \rightarrow 0$) at ∞

Using $z = \frac{1}{w}$,

$$\int_{C_R(0)} f(z) dz = \int_{-\frac{1}{C_R(0)}}^{\frac{1}{C_R(0)}} f\left(\frac{1}{w}\right) \cdot -\frac{1}{w^2} dw$$

$\vdots g(w)$
[change of direction from change of variables]

Notice, $g(w)$ is ana on $C \setminus \{\frac{1}{z_i}, 0\}$ $z_i \neq 0$ on the w -plane. Define $\{w_j\} := \{\frac{1}{z_i}, 0\}$ $z_i \neq 0$

Then, $\int_{C_{\frac{1}{R}(0)}} f\left(\frac{1}{w}\right) \cdot \frac{1}{w^2} dw = -2\pi i \sum_{|w_j| < \frac{1}{R}} \text{Res}\left(f\left(\frac{1}{w}\right) \frac{1}{w^2}, w_j\right)$
 $= -2\pi i \text{Res}\left(f\left(\frac{1}{w}\right) \frac{1}{w^2}, 0\right)$

THEOREM

$$\text{Res}(f, \infty) = -\text{Res}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right)$$

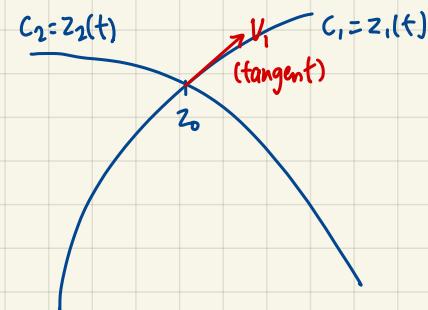
EXAMPLE

Say $f(z) = \frac{z_2 - 2}{z(z-1)}$, then $\int_{C_R(0)} f(z) dz = -2\pi i \text{Res}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right) = 10\pi i$
[encompass all residues ($\because \frac{1}{0} = \infty$)]

CONFORMAL MAPPING (ANGLE-PRESERVING)

NOTES CONVENTION

All curves $z(t)$ here are assumed to have $\dot{z}(t) \neq 0 \forall t \in [a, b]$



Say $z_i(t) = x_i(t) + iy_i(t)$, v_i = tangent of C_i at $z=z_i$, $\angle C_1, C_2$ = counterclockwise angle from v_1 to v_2

DEFINITION

For f : ana at z_0 ,

(i) f is locally 1-1 at z_0 if $\exists \delta > 0$, s.t. $f|_{B(z_0, \delta)} \rightarrow 1-1$

(ii) For a region D , f is 1-1 throughout D if f is locally 1-1 at $z \forall z \in D$

(iii) f is 1-1 on D , if $f(z_1) \neq f(z_2) \forall z_1, z_2 \in D$

EXAMPLE

$$f(z) = \exp(z) \Rightarrow$$

- Locally 1-1 on \mathbb{C} (\because we can always take a small enough ball s.t. it doesn't reach $2\pi i$ more)
- NOT 1-1 on \mathbb{C}

EXAMPLE

$$f(z) = z^2 \text{ is locally 1-1 } \forall z \neq 0 \quad (\because f(z) = f(-z))$$

DEFINITION

For f ana at $z=z_0$,

- (i) f is conformal at z_0 if $\forall C_1, C_2$ passing through z_0 , $\angle C_1, C_2 = \angle f(C_1), f(C_2)$
- (ii) For a region D , f is conformal in D if f is conformal at $z \forall z \in D$

THEOREM

Say f is ana at z_0 and $f'(z_0) \neq 0$. Then, f is conformal and locally 1-1

REMARK/KEY EXAMPLE

Reflections aren't conformal mappings. In particular, $f(z) = \bar{z}$ is not conformal

PROOF OF THEOREM

For $C_i : z_i(t) = r_i(t) + iy_i(t)$, $z_i(t_0) = z_0$,

Define $v_i(t_0) = \dot{z}_i(t_0)$

We know $v_i(t) = |v_i| e^{i\theta_i(t)}$, $\theta_i(t) \in [0, 2\pi)$

$$\Rightarrow \angle C_1, C_2 : \theta_2(t) - \theta_1(t)$$

Then, for $f(C_i)$,

$$w_i(t) = f(z_i(t)) \Rightarrow w_i(t) = f'(z_i(t)) \dot{z}_i(t) \stackrel{f'(z_0) \neq 0}{=} |f'(z_i(t))| e^{i\varphi_i(t)} |\dot{z}_i(t)| e^{i\theta_i(t)} = |e^{i(\varphi_i(t) + \theta_i(t))}|$$

$$\therefore \angle f(C_1), f(C_2) \Big|_{t=t_0} = (\varphi_2(t_0) + \theta_2(t_0)) - (\varphi_1(t_0) + \theta_1(t_0)) = \angle C_1, C_2$$

(Continue next time to have not just at $t=t_0$)

THEOREM

If f is ana at z_0 and $f'(z_0) \neq 0$, f is conformal and locally 1-1 at z_0

Proof

" f is locally 1-1 at z_0 ": Set $h(z) = f(z) - f(z_0) = \alpha$

[conformal proved last] Then, $h(z_0) = 0$, $h'(z_0) \neq 0 \Rightarrow h(z)$ is not const in a nbhd of z_0 .
time $\Rightarrow \exists D'(z_0, \delta) \text{ s.t. } \forall z \in D'(z_0, \delta), h(z) \neq \alpha$
 $D(z_0, \delta) \setminus \{z_0\}$

Prove loc const

$$\forall \delta_1 < \delta, C_1 := C(z_0, \delta_1) \subseteq D(z_0, \delta),$$

$$n_z(h) = \frac{1}{2\pi i} \int_{C_1} \frac{h'}{h} dz, \quad h: \text{ana in } D(z_0, \delta)$$

Turn into integral

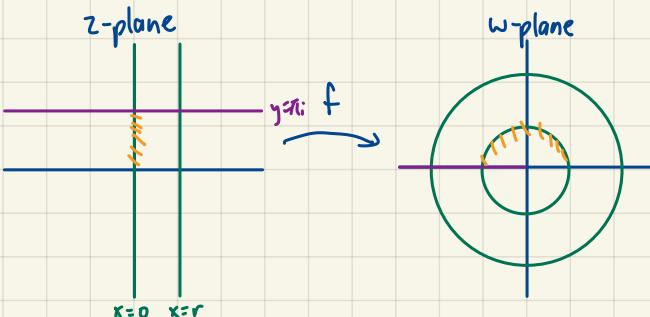
Define $w := h(z)$, $\int_{C_1} \frac{h'}{h} dz = \int_{h(C_1)} \frac{dw}{w} = \eta(h(C_1), 0)$ [winding number]

$$w := h(z)$$

winding # \Rightarrow shrink to As $\eta(h(C_1), 0)$ is locally const in α , $\exists \varepsilon, \text{s.t. } \eta(h(C_1), 0) = 1 \quad \forall \alpha \in D(0, \varepsilon)$
Show $\eta = \text{loc const}$ means Consider $\delta_2 < \delta$, s.t. $D(z_0, \delta_2) \subseteq h^{-1}(D(0, \varepsilon))$, i.e. $\forall z \in D(z_0, \delta_2), 1 = \frac{1}{2\pi i} \int_{C_1} \frac{dh(z)-h(z_1)}{h(z)-h(z_1)} dz = n_z(h(z) - h(z_1))$
(locally 1-1) \Rightarrow only $z_1 \in D(z_0, \delta_2)$, s.t. $h(z_1) = h(z_0) \leftarrow$ as in only z_1 is s.t. $h(\cdot) = h(z_1)$ \square

EXAMPLE

$f(z) = e^z$: entire, $f'(z) \neq 0 \quad \forall z \in \mathbb{C} \Rightarrow$ everywhere conformal + locally 1-1

**DEFINITION**

Let $k \in \mathbb{Z}_{>0}$, f is a k-to-1 mapping of D_1 onto D_2 if $\forall x \in D_2$, $f(z) = x$ has k roots in D_1 counting by multiplicity

LEMMA

Let $f(z) = z^k$, $k \in \mathbb{Z}_{>0}$. Then, f magnifies angles at 0 by a factor of k and f maps $D(0, \delta)$ onto $D(0, \delta^k)$ in a k-to-1 manner.

Proof

For $z \neq 0$, we have $z = |z|e^{i\theta}, \theta \in (0, 2\pi)$, $z^k = |z|^k e^{ik\theta}$

For $x = |x|e^{i\theta} \neq 0$, (Case 1)

$f(z) = x \Rightarrow z = |x|^{\frac{1}{k}} e^{i(\theta+2\pi n)/k}$, $0 \leq n \leq k-1$ are the roots of $f(z) = x$

For $x = 0$, (Case 2)

$f(z) = 0$ has a zero at $z=0$ with multiplicity k \square

THEOREM

$$\begin{cases} f: \text{ana at } z_0 & (\Delta) \\ f'(z_0) \neq 0 \end{cases}$$

Suppose f is not const. Then, \exists nbhd U of z_0 s.t. $f|_U$ is a k-to-1 mapping and magnifies angles at z_0 by a factor of k , where k is the least positive integer s.t. $f^{(k)}(z_0) \neq 0$ (\star)

$$\Delta \Rightarrow k \geq 2$$

Proof

After replacing f by $f(z) - f(z_0)$, we can assume $f(z_0) = 0$

$$\begin{aligned} \text{As } f \text{ is ana at } z=z_0 + (\Delta), \exists D(z_0, \delta_0), \text{ s.t. } f(z) = \sum_{i=k}^{\infty} a_i (z-z_0)^i \text{ with } a_k \neq 0 \\ \Rightarrow f(z) = (z-z_0)^k (a_k + a_{k+1}(z-z_0)^1 + \dots), \text{ with } g(z_0) \neq 0 \quad (\because a_k \neq 0) \end{aligned}$$

↑ Taylor expansion

Taylor expansion + factor

$\exists \varepsilon, \text{ s.t. } D(a_k, \varepsilon) \neq 0 \Rightarrow \text{we can choose a branch of } \log \text{ defined on } D(a_k, \varepsilon)$
 $\therefore \forall z \in g^{-1}(D(a_k, \varepsilon)), \text{ we can define } \exp(\frac{1}{k} \log g(z)) := h \quad (\text{k}^{\text{th}} \text{ root})$

choose a branch to k^{th} root

$\Downarrow D,$

$$\therefore f(z) = [(z-z_0)h]^k, \text{ define } H := (z-z_0)h \Rightarrow f = m_k \circ H \text{ on } D_1, \text{ where } m_k(z) := z^k$$

Apply root + unroot

As H is ana on D_1 , and $H'(z_0) = h(z_0) \neq 0$ ($\because g(z_0) \neq 0$), thus H is locally 1-1 + conformal. Prove root: conformal

\therefore By lemma, we have f 's angles magnified by k and $f: k\text{-to-1} \square$

↑ H preserves angles, then m_k magnifies

THEOREM

Say $f: 1\text{-1 ana on a region } D$.

Then, (i) f^{-1} exists and is ana on $f(D)$

(ii) f and f^{-1} are conformal in D and $f(D)$ respectively

Proof

As $f: 1\text{-1}$, on $f(D)$, $\forall y \in f(D)$, $g(y) = x$ s.t. $f(x) = y$ is an inverse of f

Claim: g is conti: (" $g^{-1}(\text{open}) = \text{open}$ ")

Proof

$\forall \text{ open } U \subseteq D$, notice: $g^{-1}(U) := \{y \in f(D) \mid g(y) \in U\} = \{f(x) \mid x \in U\} = f(U)$

By open mapping Thm, $g^{-1}(U) = f(U) \rightarrow \text{open } \checkmark$

Define g as f^{-1} , prove it's conti w/ " $g^{-1}(\text{open}) = \text{open}$ "

As $f \circ g = \text{id}_{f(D)}$, $g \circ f = \text{id}_D$, g : conti, $f'(z) \neq 0 \quad \forall z \in D$, thus $f: \text{ana} \Rightarrow g: \text{ana}$, $g'(ff(z)) = \frac{1}{f'(z)}$

We get $g: 1\text{-1}$ and $g'(z) \neq 0 \forall z$

As $f'(z) \neq 0$, thus $g'(ff(z)) \neq 0$ and is well-def $\Rightarrow g$: locally 1-1 and conformal on $f(D)$

As $f: 1\text{-1}$, thus $g: 1\text{-1} \square$

DEFINITION

① A 1-1 analytic mapping is called a conformal mapping

② Two regions D_1 and D_2 are conformally equivalent if \exists conformal mapping from D_1 onto D_2

FACT

Conformal equivalence is an equivalence relation

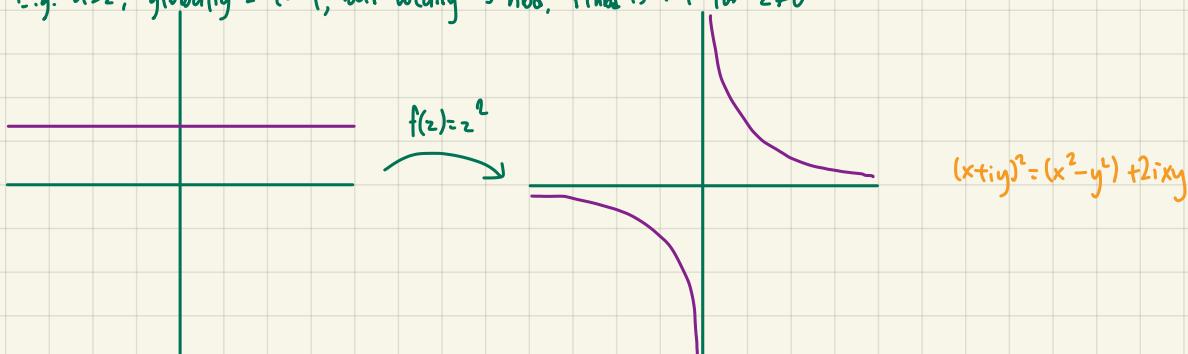
SPECIAL MAPPINGS (EXAMPLES OF CONFORMAL MAPPINGS)

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(i) $f(z) = az + b$, $a \neq 0$ (rotation + shrinkage)

(ii) $f(z) = z^\alpha$, locally at $z \neq 0$, for $\alpha > 0$

E.g. $\alpha = 2$, globally $z \mapsto -1$, but locally \exists nbd, $f|_{\text{nbd}}$ is 1-1 for $z \neq 0$

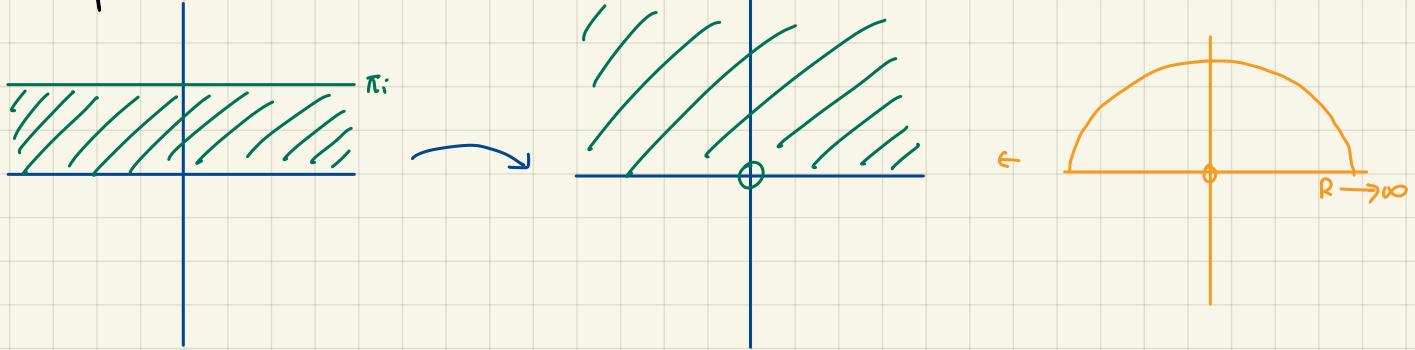


$$\text{For } z \neq 0, z = r e^{i\theta} \Rightarrow f(z) = r^\alpha e^{i\alpha\theta}$$

$$\{z \mid \theta_1 \leq \arg z \leq \theta_2\} \xrightarrow{f(z)} \{z \mid \alpha\theta_1 \leq \arg z \leq \alpha\theta_2\}$$

If $\alpha\theta_2 - \alpha\theta_1 < 2\pi$, then f : 1-1 in a nbd of z and conformal

(iii) $f(z) = \exp z$



(iv) Bilinear transformation

$$f(z) = \frac{az+b}{cz+d}, \quad \underline{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0}, \text{ ana on } \mathbb{C} \setminus \{-\frac{d}{c}\}$$

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

Then, f is locally 1-1 + conformal in $\mathbb{C} \setminus \{-\frac{d}{c}\}$

As $f(z_1) = f(z_2) \Rightarrow (ad-bc)(z_1 - z_2) = 0$, thus f is globally 1-1 ($\infty \notin \text{Im } f$)

If we extend f as a function on the Riemann sphere $(\mathbb{C} \cup \{\infty\})$, then f : conformal + globally 1-1

LEMMA

If S is a circle or a line, and $f(z) = \frac{1}{z}$, then $f(S)$ is a circle or a line

Proof

Wasn't this a HW problem lmfao

Rmk: This extends to $f(z) = \frac{az+b}{cz+d}$

11-20-25 (WEEK 12)

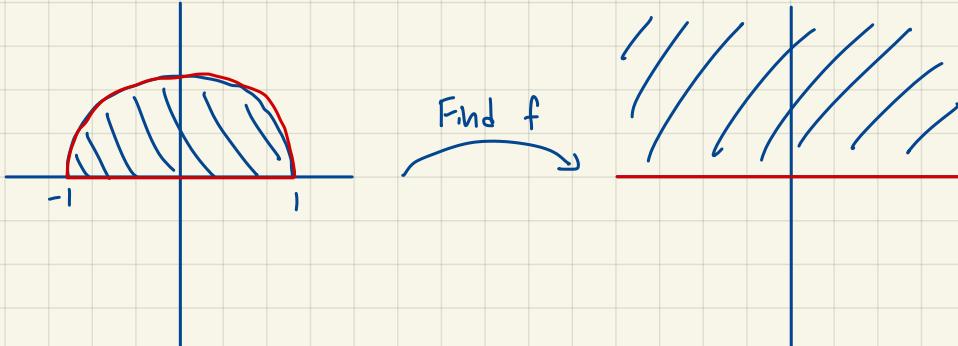
THEOREM (RECALL)

 $f(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ maps circles and lines to circles and lines.

Proof

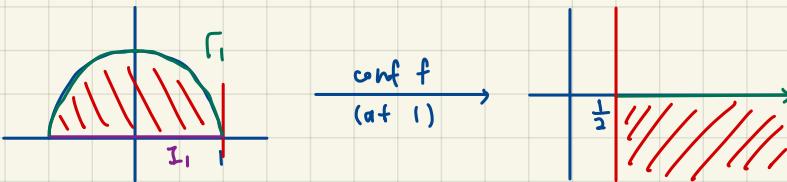
If $c=0$, then trivial.If $c \neq 0$, $f(z) = \frac{1}{c}(a - \frac{ad-bc}{cz+d})$ key transformation to only have $\frac{1}{z}$ term
 \Rightarrow Consider $z \rightarrow cz+d \rightarrow \frac{1}{cz+d} \rightarrow \frac{ad-bc}{cz+d}$
from lemma

EXAMPLE

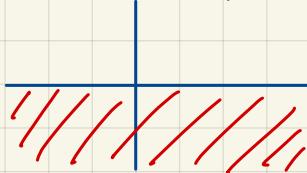


$$S = \{z \mid |z| < 1, \operatorname{Im} z > 0\}$$

$$H = \{z \mid \operatorname{Im} z > 0\}$$

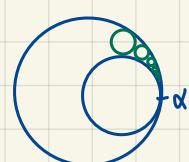
Intuition: Cut the Δ , then map to straight lineSo we set $-1 \rightarrow \infty$ (i.e. a pole)Consider $f_1(z) = \frac{1}{z+1}$, then $f_1(1) = \frac{1}{2}$, $[-1, 1] \xrightarrow{f_1} [\frac{1}{2}, \infty)$. Notice, f_1 maps circles and lines to circles and lines.

$$\therefore f_1(\Gamma_1) = \{z \mid \operatorname{Im} z \leq 0, \operatorname{Re} z = \frac{1}{2}\}, f_1(I_1) = [\frac{1}{2}, \infty)$$

After $f_2(z) = (z - \frac{1}{2})^2$, then it will become

$$\therefore f(z) = -\left(\frac{1}{z+1} - \frac{1}{2}\right)^2$$

EXAMPLE

 C_1, C_2 : circles tangent at α Find a chain of circles tangent to C_1 and C_2 and each other.

Then, the points of tangency lie on a circle or a line.

$$\hookrightarrow \text{Choose } f(z) = \frac{1}{z-\alpha}$$

DEFINITION $f: D_1 \rightarrow D_2$, if f is analytic and bijective

A conformal mapping of a region D onto itself is called an automorphism of D , which we denote as $\text{Aut}(D)$

LEMMA

If $f: D_1 \rightarrow D_2$, D_1 and D_2 are regions, and f is a conformal mapping onto D_2 ,

Then, (i) For any other $h: D_1 \rightarrow D_2$, conformal onto D_2 , then $\exists g \in \text{Aut}(D_2)$, s.t. $h = g \circ f$

(ii) $\forall h: D_1 \rightarrow D_1$, $\exists g \in \text{Aut}(D_1)$, s.t. $h = f^{-1} \circ g \circ f$

Proof

(i) For f : conformal and $f(D_1) = D_2$, we know $\exists f^{-1}: D_2 \rightarrow D_1$, conformally onto D_1

Then, $g = h \circ f^{-1} \in \text{Aut}(D_2) \Rightarrow h = g \circ f$

(ii) $f \circ h \circ f^{-1} \in \text{Aut}(D_2) \Rightarrow h = f^{-1} \circ g \circ f \quad \square$

$\therefore g$

LEMMA

The only automorphism of a unit disc with $f(0)=0$ are given by $f(z) = e^{i\theta}z$

Proof

Let $D := D(0, 1)$.

As $f \in \text{Aut}(D)$ and $f(0)=0$, by Schwarz's Lemma, $|f(z)| \leq |z|$

Schwarz's Lemma on f

By Thm, $\exists f^{-1}$ and $f^{-1} \in \text{Aut}(D) \Rightarrow$ By Schwarz's Lemma, $|f^{-1}(z)| \leq |z|$
 $\Rightarrow |z| = |f^{-1}(f(z))| \leq |f(z)| \leq |z|$
 $\Rightarrow |f(z)| = |z|$

Schwarz's Lemma on f^{-1}
 $|f(z)| = |z|$

By Schwarz's Lemma, $f(z) = e^{i\theta}z \quad \square$ (recall "equality $\Rightarrow f(z) = e^{i\theta}z$ ")

LEMMA

Let h be a bilinear transformation. If h maps D to D , where $D := D(0, 1)$, and $h(\alpha) = 0$ for some $|\alpha| < 1$, then $h = e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha}z}$

Proof def of bilinear transformation

$h(z) = \frac{az+b}{cz+d} \rightarrow$ globally 1-1, $h(\alpha) = a \left(\frac{\alpha-z}{1-\bar{\alpha}z} \right)$, $a \neq 0$

$\vdash B\alpha(z)$

Then, $h(D) \subseteq D \Rightarrow h$: ana on D

$h(D) \subseteq D \Rightarrow h$: ana on D

By Schwarz Reflection Principle, $h(\bar{\alpha}) = \overline{h(\alpha)} = \overline{h(\alpha)}^{-1} = \infty$

$\lim_{z \rightarrow \infty} \rightarrow \infty$, so we can define a pole via Schwarz Reflection Principle

$\therefore h(z) = A \left(\frac{z-\alpha}{z-\bar{\alpha}} \right)$, $A \neq 0$

However, by open mapping Thm, bdry \rightarrow bdry

$\therefore |h(1)| = 1$

$\therefore |A - \frac{1}{\bar{\alpha}}| \cdot |1 - \frac{\alpha}{\bar{\alpha}}| = 1 \Rightarrow A = \bar{\alpha}e^{i\theta} \quad \square$

THEOREM

$D := D(0, 1)$.

Then, $\text{Aut}(D) = \{e^{i\theta} \left(\frac{z-\alpha}{1-\bar{\alpha}z} \right) \mid |\alpha| < 1, 0 \leq \theta \leq 2\pi\}$

Proof

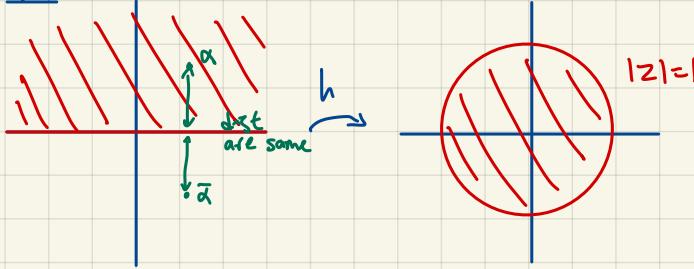
$g \in \text{Aut}(D) \Rightarrow \alpha := g^{-1}(0) \in D \Rightarrow |\alpha| < 1$

Set $h = \frac{z-\alpha}{1-\bar{\alpha}z}$, then $g \circ h^{-1}(0) = 0$

$\therefore g \circ h^{-1} \in \text{Aut}(D)$, so by lemma, $g(h^{-1}(z)) = e^{i\theta}z \Rightarrow g(z) = e^{i\theta}h(z) \quad \square$

THEOREM

The conformal mapping $h: H := \{z \mid Im z > 0\} \xrightarrow{\text{onto}} D(0, 1)$ are of the form $h(z) = e^{i\theta} \frac{z-a}{z-\bar{a}}$ for some $a \in H$

Proofsame dist if \bar{a}

As we want $|z|=1$, consider $f(z) = \frac{z-a}{z-\bar{a}}$, $a \in H$, $z \in \mathbb{R} \Rightarrow |f(z)|=1$

check $f(z) = \frac{z-a}{z-\bar{a}}$ on real plane

As f is globally 1-1 and ana on $C \setminus \{\bar{a}\} \supseteq H$, and e.g. $\left| \frac{i-a}{i-\bar{a}} \right| < 1$, $i \in H$
 $\therefore \forall z \in H, |f(z)| \leq 1$, i.e. $f(H) \subseteq D(0, 1)$

find the inside, so we need:

1. $|f(z)| \leq 1 \quad \forall z \in H$
2. $f^{-1}(D(0, 1)) \subseteq H$

Then, check f^{-1} to see $f^{-1}(D(0, 1)) \subseteq H$ (trivial)

By lemma, $\forall h$ satisfying $H \rightarrow D(0, 1)$ that are onto, $h = g \circ f$, $g \in \text{Aut}(D(0, 1)) = e^{i\theta} B_\theta \Rightarrow h = (e^{i\theta} B_\theta) \circ f$ \square

THEOREM

$$h \in \text{Aut}(H) \Rightarrow h = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, \quad ad-bc > 0$$

Proof Sketch

$\forall h \in \text{Aut}(H), \exists g \in \text{Aut}(D(0, 1))$, s.t. $h = f^{-1} \circ g \circ f = f^{-1} \circ (e^{i\theta} B_\theta) \circ f$, then apply Thm above for f

DEFINITION

For f : func on C , z_0 is a fixed point of f if $f(z_0) = z_0$.

PROPOSITION

A bilinear transformation f other than identity has ≤ 2 fixed points. If we regard f as a function on $(C \cup \{\infty\})$, then it has 2 fixed pts counted by multiplicity

Proof

$$\text{For } f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0, \quad f(z_0) = z_0 \Rightarrow az_0+b = z_0(cz_0+d)$$

Consider two cases,

- ① $c \neq 0$, then z_0 is a root of $cx^2 + (d-a)x + b = 0 \Rightarrow f$ has ≤ 2 fixed pts ($f(\infty) = \frac{a}{c}$)
- ② $c = 0$, then $f(z) \ni$ a linear function $\Rightarrow f$ has one fixed pt ($f(\infty) = \infty$)

LEMMA

Let z_1, z_2, z_3 be distinct pts in C .

(★)

The unique bilinear transformation sending z_1, z_2, z_3 to $\infty, 0, 1$ respectively is given by $T(z_1, z_2, z_3)(z) = \frac{z-z_3}{z-z_1} \cdot \frac{z_2-z_1}{z_3-z_2}$

Proof• Uniqueness

Let S be a bilinear transformation satisfying ★

Then, $S^{-1} \circ T(z_i) = z_i, i=1, 2, 3$, so by prop above, $S^{-1} \circ T = id$ ($\because 3$ fixed pts)

Consider another S , then $S^{-1} \circ T$

We say it's $S^{-1} \circ T = id$.

• Existence: Trivial. \square **REMARK**

For the lemma above, $z_1 = \infty \Rightarrow \frac{z-z_3}{z-z_1}, z_2 = \infty \Rightarrow \frac{z_3-z_1}{z_3-z_2}, z_3 = \infty \Rightarrow \frac{z-z_2}{z-z_3}$ (just consider dominating terms, don't memorize)

DEFINITION

For distinct $z_1, z_2, z_3, z_4 \in \mathbb{C}$, the cross-ratio is defined as $(z_1, z_2, z_3, z_4) := T(z_1, z_2, z_3, z_4)$

PROPOSITION

The cross-ratio is invariant under bilinear transformation S , i.e. $(z_1, z_2, z_3, z_4) = (Sz_1, Sz_2, Sz_3, Sz_4)$

Proof

$$z_1, z_2, z_3 \xrightarrow[T(z_1, z_2, z_3)]{\downarrow S} \infty, 0, 1$$

$$Sz_1, Sz_2, Sz_3 \xrightarrow[T(Sz_1, Sz_2, Sz_3)]{} \infty, 0, 1$$

Then, $T'(Sz_1, Sz_2, Sz_3) = T(z_1, z_2, z_3)(z_i) = S(z_i)$ for $i=1, 2, 3$

By prop., $T'(Sz_1, Sz_2, Sz_3) \circ T(z_1, z_2, z_3) = S$ (3 fixed pts)

$$\Rightarrow T'(Sz_1, Sz_2, Sz_3) \circ T(z_1, z_2, z_3)(z_4) = S(z_4) \Rightarrow T(z_1, z_2, z_3)(z_4) = T(Sz_1, Sz_2, Sz_3)(S(z_4)) \quad \square$$

RIEMANN MAPPING THEOREM**OVERVIEW**

"holomorphically simply connected D " (D : open + connected)

$\forall f: \text{ana on } D$, $C: \text{simple closed curve} \subseteq D$, and $\int_C f dz = 0$ (\star)

We define "s.c." in \mathbb{C} as $(C \cup \text{foo}) \setminus D$: path-connected, then we say $D \ni$ s.c.

\Rightarrow (Already Proved) D satisfies (\star)

With Riemann Mapping Thm, we get $D \xrightarrow[\text{conformal map}]{\varphi} D(0, 1) \Rightarrow D \ni$ s.c. in topological sense

RIEMANN MAPPING THEOREM

Let $R :=$ s.c. region $\subseteq \mathbb{C}$, $U := D(0, 1)$.

(Δ)

Intuitively, use Schwarz Lemma, $\because 0 \rightarrow 0$

Then, given $z_0 \in R$, \exists a unique conformal mapping φ of R onto U s.t. $\varphi'(z_0) > 0$ and $\varphi(z_0) = 0$

ProofUniqueness

Let φ_1, φ_2 satisfy (Δ).

Uniqueness

$\varphi := \varphi_2 \circ \varphi_1^{-1} \in \text{Aut}(U)$ with $\varphi(0) = 0$

Then, $\varphi := \varphi_2 \circ \varphi_1^{-1} \in \text{Aut}(U)$ with $\varphi(0) = 0$

$\therefore \varphi = e^{i\theta} \varphi$

\therefore By Lemma, $\varphi(z) = e^{i\theta} z \Rightarrow \varphi'(z) = e^{i\theta}$

However, $\varphi'(0) = \varphi_2'(z_0)(\varphi_1'(z_0))^{-1} \Rightarrow \theta = 0$

Existence

Let $F_{z_0} := \{f \mid f: R \rightarrow U, \text{ conformal}, f'(z_0) > 0\}$

Uniqueness

$\varphi := \varphi_2 \circ \varphi_1^{-1}, \varphi(0) = 0, \varphi \in \text{Aut}(U)$

CLAIM A: $F_{z_0} \neq \emptyset$

Consider $p_0 \in \mathbb{C} \setminus R$. Then, $\frac{1}{z-p_0}: 1 \rightarrow 1 + \text{analytic}$

Case ①: $\exists r, s.t. D(p_0, r) \cap R = \emptyset \Rightarrow \frac{1}{z-p_0} \leq \frac{1}{r}$ (bounded \Rightarrow can normalize to $\leq 1 \mapsto U$)

Case ②: $\exists \{z_n\} \rightarrow p_0, z_n \in R, R: \text{s.c.}$

We can define a branch of \log on R

\Rightarrow We can choose a branch of \log of $f(z) := \sqrt{\frac{z-z_0}{z-p_0}}$ s.t. $f(z_0) = 1$ (\Rightarrow We chose s.t. -1 would not appear!)

universal bound

Claim: $|f(z) - (-1)| > \delta$ for some $\delta > 0 \quad \forall z \in R$

Proof

Suppose not, $\exists \{z_n\}$ with $\lim_{n \rightarrow \infty} f(z_n) = -1 \Rightarrow \frac{z_n - p_0}{z_n - z_0} \rightarrow 1$ as $n \rightarrow \infty \Rightarrow z_n \rightarrow z_0$

However, $f: \text{cont} \Rightarrow \lim_{n \rightarrow \infty} f(z_n) = f(z_0) = 1 \quad \times$

Set $g(z) = \frac{1}{f(z)+1}$, then $|g(z)| \leq \frac{1}{\delta}$ $\forall z \in \mathbb{R}$ (bounded \Rightarrow can normalize to $\leq 1 \rightarrow 0$)

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For Cases ① and ②, (let $g(z) := \frac{1}{z-p_0}$ for ①), $g''(z_0) \neq 0 \Rightarrow \exists \theta$, s.t. $e^{i\theta} g'(z_0) = |g'(z_0)| > 0$
 $\Rightarrow F_{z_0} \neq \emptyset \square$

Next time: Claim B+C.

RIEMANN-MAPPING THEOREM PROOF CONTINUED

CLAIM C: $\exists f \in \mathcal{F}_{z_0}$, s.t. $f: \text{onto to } U$ To do this, we find $f \in \mathcal{F}_{z_0}$, s.t. $f'(z_0) = \max$ Idea: Suppose $f \in \mathcal{F}_{z_0}$, s.t. $f'(z)$ is maximum(i) $f(z_0) = 0$

Proof

If $f(z_0) = \alpha \neq 0 \in U$, then $\boxed{\text{B}(\alpha, f(z_0)) = 0}$ up to $e^{i\theta}$ and $\boxed{\text{B}(0, f(z_0)) \in \mathcal{F}_{z_0}}$
 $\Rightarrow g'(z_0) = \frac{f'(z_0)}{1 - |\alpha|^2} > f'(z_0) \rightarrow \times$

shift center to equal 0
then derivative results in
a larger one *

(ii) f is onto

Proof

If not, $\alpha \neq 0$, $\alpha \in U \setminus \text{Im}(f)$, $|f'(z_0)|$ still maxReplace f by some $e^{i\theta}$, then we can assume $\alpha \in \mathbb{R}_{<0} \Rightarrow$ set $t \in \mathbb{R}_{>0}$, $\alpha = -t^2$ shift so $f(z_0) \in \mathbb{R}_{>0}$

$$\therefore f_1(z) = \frac{f(z) - \alpha}{1 - \alpha f(z)} = \frac{f(z) + t^2}{1 + t^2 f(z)}$$

 $0 \notin \text{Im}(f_1)$, $f_1: \mathbb{R} \rightarrow U$, 1-1, analytic $\therefore f_1(z), f_1'(z)$ anal on \mathbb{R} must check $\frac{f_1'}{f_1}$ analAs R : s.c., by closed curve thm, \forall closed curve $C \subseteq R$, $\int_C \frac{f_1'(z)}{f_1(z)} dz = 0$ \therefore We can define $\log f_1(z)$ on R want to define $\log^2 \int_C \frac{f_1'(z)}{f_1(z)} dz$ In particular, we can choose a branch of $f_1(z)$ on R , s.t. $\sqrt{f_1(z_0)} = t$ so we can have $\sqrt{f_1(z_0)} = t$

$$\text{Let } f_2(z) := \sqrt{f_1(z)}, \quad f_3(z) := \frac{f_1(z) - t}{1 - t f_2(z)}, \quad f_3: 1-1.$$

$$\begin{aligned} \text{Notice, } f_1'(z_0) &= f'(z_0)(1-t^4) \\ f_2'(z_0) &= \frac{f_1'(z_0)}{2t} \\ f_3'(z_0) &= \frac{f_1'(z_0)}{1-t^2} \end{aligned} \quad \Rightarrow f_3'(z_0) = \frac{f'(z_0)(1+t^4)}{2t} >> f'(z_0)$$

If we set $g(z) := e^{i\theta} f_3(z)$, then $g \in \mathcal{F}_{z_0}$, $g'(z_0) > f'(z_0) \rightarrow \times$ $\therefore f$ must be onto ✓Conclusion: $|f'(z_0)| \max \Rightarrow f'(z_0) = 0$ and $f: \text{onto} \square$ CLAIM B: $\exists \Psi \in \mathcal{F}_{z_0}$, s.t. $\Psi'(z_0) = M$, where $M = \sup_{f \in \mathcal{F}_{z_0}} |f'(z_0)| < \infty$ ① $M < \infty$ Choose r , s.t. $\overline{D(z_0, r)} \subseteq R$ \therefore By Cauchy Integral Formula, $f'(z_0) = \frac{-1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{(z-z_0)^2} dz$

$$\therefore |f'(z_0)| \leq \frac{1}{2\pi} \cdot \frac{1}{r^2} \cdot 2\pi r = \frac{1}{r}$$

$$f^n(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

② $\{f_n\}_{n=1}^\infty$ is a countable dense subset in R , where we choose $\{f_n\}_{n=1}^\infty \subseteq \mathcal{F}_{z_0}$, s.t. $\lim_{n \rightarrow \infty} f_n'(z_0) = M$ Find a subseq $\{f_{n_k}\}$ st. $\{f_{n_k}(z_1)\}$ conv, find $\{f_{n_k}\}$ subseq of $\{f_{n_l}\}$ s.t. $\{f_{n_l}(z_2)\}$ conv $\Rightarrow \{f_{n_k}(z_2)\}$ and $\{f_{n_l}(z_2)\}$ conv
Continuing this process, $\{f_{n_l}\}_{l=1}^\infty$ conv at $z_1, z_2, \dots, z_i \Rightarrow \{f_{n_l}\}$ conv at all z_i Find all f_{n_l} via recursion:

Claim: $\{\varphi_n\}$ conv to an analytic function

Shun/翔海 (@shun4midx)

Proof

(i) $\forall z \in \mathbb{C}_0 \Rightarrow |\varphi_i(z)| < 1 \quad \forall z \in \mathbb{R} \Rightarrow \{\varphi_i\}$: uniformly bounded (indep of i, z) ✓

(ii) $\{\varphi_n\}$ conv unif on $\overline{D(w_0, r)} \subseteq \mathbb{R}$

$\mathbb{C} \setminus R$: closed, so $d := d(\tilde{R}, \overline{D(w_0, r)}) = \inf \overline{d(z_1, z_2)}, z_1 \in \tilde{R}, z_2 \in \overline{D(w_0, r)}$

$\forall z \in \overline{D(w_0, r)} \Rightarrow \overline{D(z, d)} \subseteq R$, so $|\varphi_n(z)| = \frac{1}{2\pi} \left| \int_{C_d(z)} \frac{\varphi_n(w)}{(w-z)^2} dw \right| \leq \frac{1}{d}$

$\Rightarrow \forall z_1, z_2 \in \overline{D(w_0, r)}, |\varphi_n(z_1) - \varphi_n(z_2)| = \left| \int_{z_1}^{z_2} \varphi_n'(z) dz \right| = \frac{|z_2 - z_1|}{d}$

Given $\epsilon, \forall n, \forall |z_1 - z_2| < \epsilon d$, we have $|\varphi_n(z_1) - \varphi_n(z_2)| < \epsilon$

$\therefore \varphi$: uniformly equicontinuous (indep of n, z_1, z_2) ✓

CONTINUED RIEMANN MAPPING THEOREM PROOF

As S is dense in D , $\exists z_i \in S$, s.t. $|z - z_i| < \epsilon_d$

$\therefore \lim_{n \rightarrow \infty} \varphi_n(z_i)$ exists

$$\therefore \exists N > 0, \text{ s.t. } |\varphi_n(z_i) - \varphi_m(z_i)| < \frac{\epsilon}{3} \quad \forall n, m > N$$

$$\text{Hence, } |\varphi_n(z) - \varphi_m(z)| \leq |\varphi_n(z) - \varphi_n(z_i)| + |\varphi_n(z_i) - \varphi_m(z_i)| + |\varphi_m(z_i) - \varphi_m(z)| < \epsilon$$

$\therefore \lim_{n \rightarrow \infty} \varphi_n(z)$ exists $\forall z \in D$

(iii) Define $\varphi(z) := \lim_{n \rightarrow \infty} \varphi_n(z) \quad \forall z \in D$.

We want " $\varphi_n \rightarrow \varphi$ on cpts of D "

$\forall K \subseteq D$: cpt, given $\epsilon > 0$, $V_j := \{z \in K \mid |\varphi_n(z) - \varphi(z)| < \epsilon \}$

Then, $K = \bigcup_{j=1}^{\infty} V_j \stackrel{K \text{ cpt}}{\implies} K = \bigcup_{j=1}^{\infty} V_j \Rightarrow \varphi_n \rightarrow \varphi \text{ unif on } K$

As $\varphi_n \text{ ana} \Rightarrow \varphi \text{ ana}$ and $\varphi'_n \rightarrow \varphi'$, thus $\varphi \text{ ana}$ and $\varphi'(z_0) = \lim_{z \rightarrow z_0} \varphi'(z) = M$ \square

RIEMANN-ZETA FUNCTION

DEFINITION

notice, $n \mapsto$ the base now, not z !!!

A Dirichlet series is in the form of $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$, where $n^z = \exp(z \log n)$ and we choose the branch s.t. $\log n \in R$

In particular, the Riemann zeta function is $\sum_{n=1}^{\infty} \frac{1}{n^z}$

THEOREM

If $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges at $z=z_0$, then it converges at all $z \in \{ \operatorname{Re}(z) > \operatorname{Re}(z_0) \} := H_0$. Moreover, it conv unif on cpts $\forall K \subseteq H_0$

Proof

Fix z_0 , we want " $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ conv"

Claim: Given $\epsilon > 0$, $\exists N_0$, s.t. $N, M > N_0 \Rightarrow \left| \sum_{n=M}^N \frac{a_n}{n^z} \right| < \epsilon$

Proof

Define $A_i := \sum_{n=1}^i \frac{a_n}{n^{z_0}}$, $b_i := \frac{1}{n^{z-z_0}}$. Then, $\sum_{n=M}^{\infty} \frac{a_n}{n^z} = A_{M-1} b_M + A_M (b_M - b_{M+1}) + \dots + A_{N-1} (b_{N-1} - b_N) + A_N b_N$ $\quad (\star)$

Since $\sum_{n=0}^{\infty} \frac{a_n}{n^{z_0}}$ conv, $\exists A > 0$, s.t. $|A_i| < A$, $b_n - b_{n+1} = \frac{1}{n^{z_0}} - \frac{1}{(n+1)^z} = (-t^z)^{-\frac{1}{n+1}} = \int_n^{n+1} w t^{-z-1} dt \Rightarrow |b_n - b_{n+1}| < \frac{|z|}{n^{z+1}}$, $\delta = \operatorname{Re}(z - z_0) > 0$

Moreover, $\star \leq A \cdot \sum_{n=1}^{N-1} \frac{|z|}{n^{z+1}}$, where $\sum_{n=1}^{\infty} \frac{|z|}{n^{z+1}}$ conv $\forall \delta > 0$

$\therefore \exists C > 0$, s.t. $\forall n, n_1, n_2 > C$, $\left| \sum_{n=n_1}^{n_2} \frac{a_n}{n^{z_0}} \right| < \frac{\epsilon}{A}$, so $\star < \epsilon$
 $\Rightarrow \forall N, M > C$, $\left| \sum_{n=M}^N \frac{a_n}{n^{z_0}} \right| < 3\epsilon \quad \square$

REMARK / COUNTEREXAMPLE

$A_i = \sum_{n=1}^i \frac{a_n}{n^{z_0}}$, indep of $z \in H_0$. $\quad (\Delta)$

In particular, $\exists A$, s.t. $|A_i| < A \quad \forall i$. Then, $\sum_{n=1}^{\infty} \frac{a_n}{n^{z_0}}$ converges? (No!)

For example, let $a_n := (-1)^n$, then for $z=0$, $\sum_{n=1}^{\infty} (-1)^n$ does not converge, but it satisfies (Δ) . Hence, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{z_0}}$ conv $\forall \operatorname{Re}(z) > 0$

THEOREM

If $\sum_{n=0}^{\infty} \frac{a_n}{n^2}$ converges for some z but not all $z_0 \in \mathbb{C}$, then $\exists x_0 \in \mathbb{R}$ (called abscissa of convergence) s.t. $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges if $\operatorname{Re}(z) > x_0$
 and $\sum_{n=0}^{\infty} \frac{a_n}{n^2}$ diverges if $\operatorname{Re}(z) < x_0$

Proof

Consider the Thm, we know $[\inf \{\operatorname{Re}(w) | \sum \frac{a_n}{n^2} \text{ conv}\}] > -\infty$ as $f(z) := \sum \frac{a_n}{n^2}$ not conv $\forall z \in \mathbb{C}$.

 $\geq c$

↑ def for inf

$\forall z'$ with $\operatorname{Re}(z') > c$, $\exists z_0$ with $c < \operatorname{Re}(z_0) < \operatorname{Re}(z')$, s.t. $f(z) : \text{conv at } z_0 \Rightarrow f(z') : \text{conv}$

By Thm, $x_0 = c$

↑ conv at z_0

$$\text{Now, } \sum \left| \frac{a_n}{n^2} \right| = \sum \frac{|a_n|}{n^{2c+1}}$$

$$\forall z \text{ with } \operatorname{Re}(z) > \operatorname{Re}(z_0), \sum \frac{|a_n|}{n^{2c+1}} < \sum \frac{|a_n|}{n^{2\operatorname{Re}(z_0)}} \quad \square$$

EXAMPLE

$$\zeta(z) := \sum \frac{1}{n^2} \Rightarrow \text{abscissa of } \zeta \text{ is } z=1$$

↑ square, i.e. $\zeta \cdot \zeta$.

$$\zeta^2(z) = \sum \frac{c_n}{n^2}, c_n = \sum_{d|n} 1$$

$$\text{In general, } f(z) := \sum \frac{a_n}{n^2}, g(z) := \sum \frac{b_n}{n^2} : \text{conv } \forall \operatorname{Re}(z) > c$$

$$\text{Then, we can define } f(z) + g(z), f(z)g(z) = \sum \frac{c_n}{n^2}, c_n = \sum_{d|n} a_d b_{n/d} \text{ for } \operatorname{Re}(z) > c$$

ANALYTIC CONTINUATION**MAIN QUESTION** any arbitrary $g(z)$: ana on $D(0, R)$

Given $\sum_{n=0}^{\infty} a_n z^n$ conv abs for $|z| < R$, \exists ana f s.t. $f|_{D(0, R)} = \sum a_n z^n$ but f defined on $D \neq D(0, R)$?

↪ We have seen examples such as Schwarz Lemma and removable singularities: $\frac{1}{1-z}$: ana cont: of $\sum_{n=1}^{\infty} z^n$, $|z| < 1$

(I) POWER SERIES**THEOREM**

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a positive radius of conv R , then f has a singularity at $|z|=R$

EXAMPLE

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} \text{ conv at } z=1$$

If f : ana in a nbhd of $z=1$ and $f(z) = \sum \frac{z^n}{n^2}$ at $z=1$, $f(z)|_{D(1, \varepsilon)} = \sum \frac{z^n}{n^2}$,

Then, $f''(z)$: ana on $D(1, \varepsilon)$

$$\text{However, } f''(z) = \sum_{n=1}^{\infty} \frac{n(n-1)}{n^2} z^{n-2} \text{ div at } z=1 \times$$

$\Rightarrow z=1$ is a singular point of $\sum \frac{z^n}{n^2}$

PROOF OF THEOREM

If not, $\forall |z|=R, \exists \max \varepsilon_z > 0$, s.t. f can be continued analytically on $D(z, \varepsilon_z)$ to \tilde{f} , where ε_z varies cont: in z .

$$|z|=R: \text{cpt} \Rightarrow \varepsilon = \min_{|z|=R} \varepsilon_z, \exists |z_0|=R, \varepsilon_{z_0} = \varepsilon$$

Then, \tilde{f} can be defined on $D(0, R+\varepsilon)$ analytically $\Rightarrow \tilde{f} = \sum_{n=0}^{\infty} b_n z^n, \tilde{f}|_{D(0, R)} = f \xrightarrow{\text{Uniqueness}} a_n = b_n$, so radius of conv = $R \times$

12-4-25 (WEEK 14) (I'm so fucking stressed but we don't talk about that... hope I'm gonna be okay...)

Shun/翔海(@shun4midx)

DEFINITION

Suppose that f is analytic in a disc D and $z_0 \in \partial D$. Then, f is said to be regular at z_0 if f can be continued analytically to a region D_1 with $z_0 \in D_1$. Otherwise, f is said to be a singularity at z_0 .

CAUTION

$z=0$ is a singularity does NOT depend on if f is continuous at $z=0$

Example 1: $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $|z| < 1$, but $f(z)$ is cont. on $|z|=1$

Example 2: $f(z) = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$, $\lim_{z \rightarrow 0} f(z)$ DNE but $\frac{1}{z-0}$ is defined $\forall z \in \mathbb{C} \setminus \{0\} \Rightarrow z=0$: regular pt. of f

THEOREM

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a radius of conv $R < \infty$ and $a_n \neq 0 \forall n$. Then, $f(z)$ becomes a singularity at some $z=R$

Proof

By Thm last time, f has a singularity at some $|z|=R$

Let $z = Re^{i\theta}$ be such a singularity.

Consider f at $z = pe^{i\theta}$, $0 < p < R$

By Taylor expansion at $z = pe^{i\theta}$, $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(pe^{i\theta})}{n!} (z - pe^{i\theta})^n$ in $|z - pe^{i\theta}| < R - p$ as $Re^{i\theta} \Rightarrow$ a singularity of f

Consider $\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(pe^{i\theta})}{n!} \right| |z - pe^{i\theta}|^n$, where $f^{(n)}(pe^{i\theta}) = \sum_{i=n}^{\infty} n(n-1)\dots(n-i)(a_i)(pe^{i\theta})^{i-n}$

$\therefore |f^{(n)}(pe^{i\theta})| \leq f^{(n)}(p)$

\therefore At $z=p$, $f(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(p)}{n!} (z-p)^n$, moreover, radius of conv at $z=pe^{i\theta} \geq$ radius of conv at $z=p$

$\therefore z=R$ is a singular point of f .

DEFINITION

If $\sum_{n=0}^{\infty} a_n z^n$ has a singularity at every point on its circle of convergence, then the circle is called a natural boundary (e.g. $\sum_{n=0}^{\infty} z^n$, $|z|=1$: natural boundary)

THEOREM If auto, $\exists k \in \mathbb{N} \Rightarrow \lim_{k \rightarrow \infty} \frac{|c_k|}{k} = 1$, so " $>$ " means the spacing between nonzero terms is large.

Let $f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$ with $c_k \neq 0 \forall k$. Suppose $\lim_{k \rightarrow \infty} \frac{n_{k+1}}{n_k} > 1$. Then, the circle of convergence of f is a natural boundary

Proof

As the assumption is indep of c_k ($c_k \neq 0$), we can replace z by Rz and assume the radius of convergence = 1

If $f(z)$ is singular at some $z=e^{i\theta}$, replace z with $ze^{i\theta}$ so $f(z)$ is singular at $z=1$

\therefore It suffices to prove, $\forall \theta$, $f(z)$ has a singularity at $z=1$

Consider the map $w \xrightarrow{n} w$ ($n > 1$), it fixes $|w|=1$, $|w| < 1$, $|w| > 1$ to their corresponding (in)equality with 1

For $h: w \xrightarrow{n} \frac{w^n + w^{n+1}}{2}$

Define $g(z) := f \circ h$

As $|h(w)| < 1 \forall |w|=1$ but $w \neq 1$, thus g is regular at $|w|=1$ but $w \neq 1$.

If we can show that the radius of conv of g is 1, by Thm, we know that $w=1$ is a singular point of $g \Rightarrow f$ is singular at $h(1)=1$

Claim: Conv radius of $g=1$ for some n as n , exponent of w .

Proof

$\lim_{k \rightarrow \infty} \frac{n_{k+1}}{n_k} = 1+\delta, \delta > 0 \Rightarrow$ choose m s.t. $\frac{m+1}{m} < 1+\delta$, as $k \gg 0$, set $n=m$. We can even assume $n_0 \gg 0$.

$$\text{Then, } g(w) = f\left(\frac{w^m + w^{m+1}}{2}\right) = C_0 \left(\frac{w^n + w^{n+1}}{2}\right)^{n_0} + C_1 \left(\frac{w^n + w^{n+1}}{2}\right)^{n_1} + \dots$$

$$= \frac{C_0 w^{mn_0}}{2^{n_0}} + \frac{C_0 h_0 w^{mn_0+1}}{2^{n_0}} + \dots + \frac{C_0}{2^{n_0}} w^{\text{mn}_0+n_0} + \frac{C_1}{2^{n_1}} w^{\text{mn}_1} + \frac{C_1 n_1}{2^{n_1}} w^{mn_1+1} + \dots + \frac{C_1}{2^{n_1}} w^{mn_1+n_1} + \dots$$

If conv radius of $g=r>1$, then

$$\left| \frac{C_0 w^{mn_0}}{2^{n_0}} \right| + \left| \frac{C_0 h_0 w^{mn_0+1}}{2^{n_0}} \right| + \dots + \left| \frac{C_0}{2^{n_0}} w^{\text{mn}_0+n_0} \right| + \left| \frac{C_1}{2^{n_1}} w^{\text{mn}_1} \right| + \left| \frac{C_1 n_1}{2^{n_1}} w^{\text{mn}_1+1} \right| + \dots + \left| \frac{C_1}{2^{n_1}} w^{\text{mn}_1+n_1} \right| + \dots \quad (\star)$$

conv & $|w|<1$. In particular, $\exists 1 < r < r$, s.t. $|w|=r$, s.t. (\star) conv and $(\star) = f\left(\frac{w^m + w^{m+1}}{2}\right)$ ~~\neq~~ $\therefore f$ has radius of conv 1

\therefore Radius of conv of $g=1$ \square

THE METHOD OF MOMENT

Say $\sum_{n=0}^{\infty} c_n z^n = f(z)$, how do we find an ana contn?

CASE 1

$$c_n = \int_a^b g(t) t^n dt \Rightarrow f(z) = \sum_{n=0}^{\infty} (\int_a^b g(t) t^n dt) z^n \quad \begin{matrix} |t_z| < 1 : \text{abs conv} \\ |t_z| < 1 \end{matrix}$$

Say $h(z) := \int_a^b \frac{g(t)}{t-z} dt$, then $z \notin [a, b] \Rightarrow 1-tz \neq 0$

Is $h(z)$ ana on $\mathbb{C} \setminus [a, b]$? (Morera. Prove cont. then Morera)

LEMMA

Suppose $\varphi(z, t)$ is a continuous function of $t \forall t \in [a, b]$ for fixed z and is analytic for $z \in D$ for fixed t .

Then, $f(t) = \int_a^b \varphi(z, t) dt$ is analytic $\forall z \in D$.

Proof

Notice, $\varphi(z, t)$: ana in $z \Rightarrow \varphi(z, t)$: cont. in $z \Rightarrow f(z)$: cont. in z

\forall closed rectangle $R \subseteq D$,

$$\int_{\partial R} f(z) dz = \int_{\partial R} \int_a^b \varphi(z, t) dt dz = \int_a^b \int_{\partial R} \varphi(z, t) dz dt = \int_a^b 0 dt = 0$$

$\stackrel{!}{=} 0 \quad (\because \varphi: \text{ana})$

\therefore By Morera's Thm, $f(z)$: ana in z . \square

COROLLARY

$\int_a^b \frac{g(t)}{t-z} dt$ is an analytic continuation of f

REMARK

We can replace $[a, b]$ with $[a, \infty]$: We check if we can $\int_{\partial R} \leftrightarrow \int_a^b$!

EXAMPLE

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1}, |z| < 1$$

$$\star \frac{1}{n+1} = \int_0^1 t^n dt$$

Then, $f(z) = \sum_{n=0}^{\infty} (\int_0^1 t^n dt) z^n = \int_0^1 \sum_{n=0}^{\infty} (tz)^n dt = \int_0^1 \frac{1}{1-tz} dt$: ana on $\mathbb{C} \setminus [1, \infty)$

CASE 2

$$\int_0^{\infty} e^{-zt^2} dt = \frac{\sqrt{\pi}}{2\sqrt{z}}$$

$$\text{Then, } \sum_{n=0}^{\infty} \frac{z^n}{n+1} = \frac{1}{\sqrt{z}} \left[\sum_{n=1}^{\infty} \left(\int_0^{\infty} e^{-zt^2} dt \right) z^n \right] = \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \int_0^{\infty} (ze^{-tz^2})^n dt = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{z}{e^{tz^2}} dt$$

\hookrightarrow Ana when $ze \mathbb{C} \setminus [1, \infty) \Rightarrow e^{tz^2} - z \neq 0$

If $\int_0^{\infty} \frac{z}{te^{tz^2}-z} dt$ exists

Here, $\int_{\partial R} \int_0^{\infty} \frac{z}{te^{tz^2}-z} dt = \int_0^{\infty} \int_{\partial R} \frac{z}{te^{tz^2}-z} dt$: ana cont.

$$\star \frac{1}{n+1} = \int_0^{\infty} e^{-nt} e^{-at} dt$$

ANALYTIC CONTINUATION FOR DIRICHLET SERIES (I.E. $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$)

Shun/翔海 (@shun4mido)

THEOREM (LANDAU'S THEOREM)

Suppose that $a_n \geq 0 \forall n$, $b \in \mathbb{R}$ is a boundary point (i.e. f conv $\forall \operatorname{Re}(z) > b$). Then, b is a singular point of f .

Proof Idea: If b : regular, then $\exists c < b$ s.t. $f(z)$ conv at $c \Rightarrow f(z)$ conv $\forall \operatorname{Re}(z) > c \Rightarrow b$: not bdry pt

Proof \curvearrowright contn ana with b : ana

If b : regular, then $\exists \tilde{f}$: ana at b , s.t. $\tilde{f}|_{\operatorname{Re}(z) > b} = f$

Recall Riemann-zeta function is a Dirichlet series, conv at $\operatorname{Re} z > 2$ ("abscissa")
will test in exam

We can consider the Taylor expansion of \tilde{f} at some $a > b$.

$$\Rightarrow \tilde{f} = \sum_{k=0}^{\infty} C_k (z-a)^k, \quad C_k = \frac{\tilde{f}^{(k)}(a)}{k!} = \frac{f^{(k)}(a)}{k!} \quad \therefore \tilde{f} = \sum_{n=1}^{\infty} \frac{a_n (\log n)^k}{n^a k!}$$

As \tilde{f} : ana at b , radius of conv of $(*)$ $> a-b$

In particular, $\exists \varepsilon > 0$, s.t.

$$\begin{aligned} \sum_{k=0}^{\infty} |C_k| |a-b+\varepsilon|^k &\stackrel{\text{conv}}{\leq} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_n (\log n)^k}{n^a k!} \right) (a-b+\varepsilon)^k \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{a_n}{n^a} \left[\frac{(\log n)^k}{k!} (a-b+\varepsilon)^k \right] = \sum_{n=1}^{\infty} \frac{a_n}{n^{b-\varepsilon}} \end{aligned}$$

$\therefore f$: conv at $b-\varepsilon \square$

$$\exp(\log n \cdot a - b + \varepsilon) = n^{a-b+\varepsilon}$$

COROLLARY

If $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ with $a_n \geq 0$ and can be analytically continued to the entire complex plane, then it converges throughout the complex plane

$\exists \tilde{f}$: entire, $\tilde{f}|_{\operatorname{Re} z > b} = f$

f : conv throughout

GAMMA FUNCTION

CONSTRUCTION

$$f(z) = \int_0^\infty e^{-t} t^z dt \quad \stackrel{\text{IBP}}{=} z f(z-1)$$

Notice, $|f(z)| \leq \int_0^\infty |e^{-t} t^z| dt = \int_0^\infty e^{-t} t^{\operatorname{Re}(z)} \Rightarrow f(z)$: well-def $\forall \operatorname{Re}(z) > -1$

$\therefore \Gamma(z) := f(z-1)$ is well-def $\forall \operatorname{Re}(z) > 0$

ANALYTIC CONTINUATION OF Γ

As $\Gamma(z+1) = z\Gamma(z)$, "ana" can be checked via diffability.

① Note, $\Gamma(z+1) = z\Gamma(z) \Rightarrow \Gamma(z) = \frac{\Gamma(z+1)}{z} \Rightarrow$ simple pole at $z=0$

② $\Gamma(z+1)$ is defined $\forall \operatorname{Re}(z) > -1$

$$\therefore \text{Define } \tilde{\Gamma}_1(z) := \begin{cases} \frac{\Gamma(z+1)}{z}, & -1 < \operatorname{Re}(z) < 0 \\ \Gamma(z), & \operatorname{Re}(z) \geq 0 \end{cases} \quad \text{analytic}$$

Check: " $\tilde{\Gamma}_1(z)$: ana except for $z=0$ "

By Morera, it suffices to check $\tilde{\Gamma}_1(z)$ is conti:

$$\text{For } y \neq 0, \lim_{z \rightarrow iy} \tilde{\Gamma}_1(z) = \lim_{z \rightarrow iy} \frac{\Gamma(z+1)}{z} = \frac{\Gamma(iy+1)}{iy} = \Gamma(iy)$$

$\therefore \tilde{\Gamma}_1(z)$ is conti at $z=iy, \forall y \neq 0 \checkmark$

Continue this process, e.g.

$$\tilde{\Gamma}_2(z) = \begin{cases} \tilde{\Gamma}_1(z), & \operatorname{Re}(z) > -1 \\ \frac{\tilde{\Gamma}_1(z+1)}{z}, & -2 < \operatorname{Re}(z) < -1 \end{cases} \rightarrow \text{simple pole at } z=-1$$

etc...

In this way, Γ can be extended analytically to $\mathbb{C} \setminus \{n \ln \mathbb{Z}_{\leq 0}\}$

Shun/翔海 (@shun4midx)

RESIDUE OF Γ

EXAMPLE

$$\text{Res}(\Gamma(z), -1) = \text{Res}(\tilde{\Gamma}_2(z), -1) = \lim_{z \rightarrow -1} (z+1)\tilde{\Gamma}_2(z) = \lim_{z \rightarrow -1} (z+1) \frac{\Gamma(z+2)}{z(z+1)} = -1$$

$$\text{Moreover, } \text{Res}(\Gamma(z), -n) = \frac{(-1)^k}{k!}$$

SPECIAL EQUALITIES

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad (\text{i.e. } \Gamma \text{ has no zeros})$$

ZETA FUNCTION (VERY HANDWAHY)

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \Rightarrow \frac{1}{2^z} \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{(2n)^z} \Rightarrow \left(1 - \frac{1}{2^z}\right) \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^z} \Rightarrow \text{etc...}$$

$$\zeta(z) = \prod_{p: \text{prime}} \left(1 - \frac{1}{p^z}\right)^{-1} = \prod_{p: \text{prime}} \left(1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \dots\right)$$

EQUATIONS

$$\exists g(z): \text{entire, s.t. } \zeta(z) = \frac{1}{g(z)} [g(z) + \frac{1}{z-1} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \dots]$$

80%~90% OF QUESTIONS IN THE FINAL WILL BE FROM THE HWs THIS SEM

NOTICE

In \mathbb{C} , $\log a + \log b \neq \log(ab)$ in general

ZETA FUNCTION

AIM

Can we extend ζ to a meromorphic function on \mathbb{C} ?

LINK TO GAMMA FUNCTION

$$\text{Recall } \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

$$\text{Notice, } \int_0^\infty e^{-nt} t^{z-1} dt \stackrel{s=nt}{=} \int_0^\infty e^{-s} \left(\frac{s}{n}\right)^{z-1} \frac{ds}{n} = \underbrace{\frac{1}{n^z}}_{\text{for } \zeta(z)} \int_0^\infty e^{-s} s^{z-1} ds \approx \Gamma(z)$$

$$\therefore \text{For } \operatorname{Re}(z) > 1, \quad \zeta(z) = \frac{1}{\Gamma(z)} \sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^{z-1} dt$$

$$\begin{aligned} \Gamma(z) \quad &\text{(1) has no zeros} \\ &\text{(2) has only poles as singularity} \end{aligned} \quad \left. \begin{aligned} &\Rightarrow \frac{1}{\Gamma(z)} : \text{entire} \\ &\text{depends on } z \quad \text{holomorphic } \forall z \in \mathbb{C} \end{aligned} \right\}$$

$$\text{Thus, } \operatorname{Re}(z) > 1 \Rightarrow \sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^{z-1} dt = \int_0^\infty \left(\sum_{n=1}^{\infty} (e^{-nt}) \right) t^{z-1} dt = \int_0^\infty (\sum_{n=1}^{\infty} e^{-nt}) t^{z-1} dt + \int_0^\infty (\sum_{n=1}^{\infty} e^{-nt}) t^{z-1} dt$$

$$\text{Notice, } \int_0^1 (\sum_{n=1}^{\infty} e^{-nt}) t^{z-1} dt = \int_0^1 \frac{t^{z-1}}{e^{t-1}} dt \quad e^{t-1} = x + tx^2 + \dots$$

$$\text{We know } \frac{1}{e^{t-1}} \text{ has a simple pole at } t=0, \quad \operatorname{Res}\left(\frac{1}{e^{t-1}}, 0\right) = 1$$

Consider the Laurent Expansion around $t=0$,

$$\frac{1}{e^{t-1}} = \frac{1}{t} + \sum_{i=0}^{\infty} A_i t^i. \quad \text{As } \frac{1}{e^{t-1}} \text{ defined } \forall t \neq 0, \text{ thus } \sum_{i=0}^{\infty} A_i t^i \text{ conv } \forall t \in \mathbb{C}.$$

$$\therefore \text{For } \operatorname{Re}(z) > 1, \quad \int_0^1 \frac{t^{z-1}}{e^{t-1}} dt = \int_0^1 t^{z-2} + \sum_{i=0}^{\infty} A_i t^{i-2} dt = \frac{1}{z-1} + \sum_{i=0}^{\infty} \frac{A_i}{z+i} \quad \text{conv } \forall z \in \mathbb{C} \setminus \{z \leq 1\} \text{ because } \begin{aligned} &\text{(1) } \sum_{i=0}^{\infty} A_i t^i \text{ conv } \forall t \in \mathbb{C} \\ &\text{(2) } \frac{1}{z+i} \in \mathbb{C} \text{ for } i \text{ large enough} \end{aligned}$$

$$\text{Thus, } \operatorname{Re}(z) > 1 \Rightarrow \zeta(z) = \frac{1}{\Gamma(z)} \left(\frac{1}{z-1} + \sum_{i=0}^{\infty} \frac{A_i}{z+i} + g(z) \right) \quad \begin{aligned} &\text{def on } \mathbb{C} \setminus \{z \leq 1\} \\ &\text{holo in } \mathbb{C} \\ &\text{entire w/ zero of order 1 at } z=0 \end{aligned}$$

THEOREM

$\zeta(z)$ can be analytically continued to $\mathbb{C} \setminus \{z=1\}$

Proof

$\frac{1}{z-1}$ has zero of order 1 at $z=0$ and (a) has simple poles at $z \leq 1$

SET-UP

$$\text{Recall, } \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p: \text{prime}} (1 - p^{-z})^{-1}, \quad \operatorname{Re}(z) > 1$$

$\therefore \zeta(z)$ has no zeros for $\operatorname{Re}(z) > 1$

THEOREM

$\zeta(z)$ has no zeros for $\operatorname{Re}(z) \geq 1$ (i.e. by set-up, $\zeta(z)$ has no zeros at $x=1$)

Proof Sketch

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{p: \text{prime}} \frac{[(1-p^{-z})^{-1}]}{(1-p^{-z})^{-1}} = \sum_p \frac{\log p}{p^{z-1}} = \phi(z) + \underbrace{\sum_p \frac{\log p}{p^z(p^z-1)}}_{\text{holomorphic for } \operatorname{Re}(z) > \frac{1}{2}}, \text{ where } \phi(z) = \sum_{p: \text{prime}} \frac{\log p}{p^z}$$

Use prime not log

Suppose \exists zero for ζ at $1+ia_0$ with $a_0 \neq 0$, of order $\mu > 0$

We know $z \in \mathbb{R}_{>1} \Rightarrow \zeta(z) \in \mathbb{R}$

\therefore By Reflection Principle, $\zeta(\bar{z}) = \overline{\zeta(z)}$

$\therefore 1-ia_0$ is a zero of order μ of ζ

Say ζ has a zero at $1 \pm 2ia_0$ of order ν ($\nu = 0$ if $1 \pm 2ia_0$ is a "zero")

Thus, $\phi(z)$ has poles of the same order as $-\frac{\zeta'(z)}{\zeta(z)}$ = zeros and poles of $\zeta(z)$

↓

$$\lim_{\varepsilon \rightarrow 0} \sum \varepsilon \phi(1+\varepsilon) = 1 \quad (\zeta \text{ has simple pole at } z=1 \text{ with residue 1})$$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \phi(1+ia_0 + \varepsilon) = \mu$$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \phi(1+2ia_0 + \varepsilon) = \nu$$

Finally,

$$\begin{aligned} \sum_{-2 \leq k \leq 2} \binom{4}{2+k} \phi(1+\varepsilon+i\lambda a_0) &= \sum_p \frac{\log p}{p^{1+\varepsilon}} \left(\sum_{-2 \leq k \leq 2} \binom{4}{2+k} \left(\frac{1}{p^{1+\frac{i\varepsilon}{2}}} \right)^{k+2} \left(\frac{1}{p^{-\frac{i\lambda a_0}{2}}} \right)^{-k-2} \right) \\ &= \sum_p \frac{\log p}{p^{1+\varepsilon}} \underbrace{\left(p^{\frac{i\varepsilon}{2}} + p^{-\frac{i\lambda a_0}{2}} \right)^4}_{\geq 0} \end{aligned}$$

$$\text{As } \varepsilon \rightarrow 0, -2\nu - 8\mu + 6 \geq 0 \Rightarrow \mu = 0 \rightarrow$$