10-7-25 (WEEK 6)

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SIMPLY CONNECTED DOMAIN

DEFINITIONS

Say $f:(S,\{U\}) \longrightarrow (S_1,\{N\})$ for two topological spaces

f is continuous if $\forall$ open $V \subseteq S_1$, $f^{-1}(V)$ is open in S

Path-connected: $\forall \alpha,\beta \in S$, $\exists \gamma:[0,1] \longrightarrow S$ continuously s.t. $\gamma(0)=\alpha$, $\gamma(1)=\beta$ (E.g. $\{x=0\} \cup \{\sin\frac{1}{x}\}$: connected but not path connected)

any open set cannot separate

IN $\mathbb{C}$ TOPOLOGICAL SPACE

We consider the topo space $\mathbb{C} \cong \mathbb{R}^2$

We say S is simply connected if it is:

① Path-connected ($\Rightarrow$ connected)

② For any continuous maps $f_i:[0,1] \longrightarrow S$ with $f_0(0)=f_1(0)$ and $f_0(1)=f_1(1)$, $\exists$ continuous $F:[0,1]\times[0,1] \longrightarrow S$ s.t. $F(t,0)=f_0(t)$, $F(t,1)=f_1(t)$ (intuitively: they are connected via shrinking a rubber band for any two pts)

Actually, $S=\mathbb{C}\cong\mathbb{R}^2$ is simply connected and a torus is not

can be seen as a disc with a hole taken out (disc裡挖出一個洞)

RECALL

f: ana on closed disc $D(0,1)$, then $\int_{C_1(0)} f\,dz = 0$

Torus

However, remember $\int_{C_1(0)} \frac{1}{z}\,dz = 2\pi i$, note that $\frac{1}{z}$: well-defined on $D(0,1)\setminus\{0\}$

Motivation: Simply via integration, we can determine the topo nature of subsets of $\mathbb{C}$, equal zero or nonzero?

↳ "Holomorphic simply connected" (= "topo simply connected")

DEFINITION

D is holomorphic simply connected (hsc) if $\forall f$: ana on D, $\int_\Gamma f\,dz = 0$ $\forall$ simple closed curve $\Gamma \subseteq D$

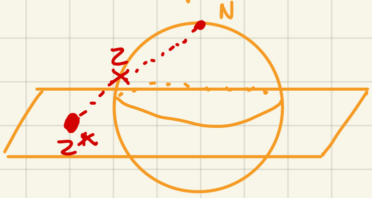
FACT: In $\mathbb{C}$, hsc $\Leftrightarrow$ sc

DEFINITION

basically treat $\{\infty\}$ as one point, no more $\pm\infty$ as opposites, they are the same concept

We say the extended $\mathbb{C}$-plane is $\mathbb{C}\cup\{\infty\} \cong S^2$ (sphere)

∞ is the north pole in this stereographic projection



DEFINITION

If D is open connected $\subseteq \mathbb{C}$, then D is sc if $(\mathbb{C}\cup\{\infty\})\setminus D$ is path connected

simply connected

DEFINITION'

For a region $D\subseteq\mathbb{C}$, we say D is sc' if $\forall z_0 \in \mathbb{C}\setminus D$, $\forall \varepsilon>0$, $\exists$ path $\gamma:[0,\infty) \longrightarrow \mathbb{C}\setminus D = \tilde{D}$, s.t.

① $d(\gamma,\tilde{D}) \le \varepsilon$

② $\gamma(0)=z_0$

③ $\lim_{t\to\infty} \gamma(t) = \infty$

$\boxed{\text{Def} \Rightarrow \text{sc} \Rightarrow \text{hsc}}$

Def'

REMARK

For open $D \subseteq \mathbb{C}$, D is open connected $\Leftrightarrow$ D is connected

Proof

"$\Rightarrow$": Locally path connected (i.e. $\forall x \in D$, $\exists$ open nbd $U(x) \subseteq D$ s.t. $U(x)$: path-connected)

DEFINITION

e.g. level 3: _‾‾

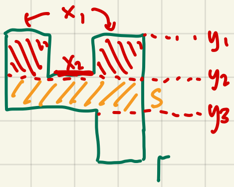
Given a polygonal path Γ ( ), we define the level of $\Gamma := \#$ diff values y_0 where the line $\operatorname{Im} z = y_0$ contains a horizontal segment of Γ

LEMMA

no intersections w/ itself

Γ: simple closed curve that is also a polygonal curve, say $\Gamma \subseteq D$, where D: SC region

Suppose the top level of Γ consists of points $y = y_1$, $x \in X_1$ and $y = y_2$, $x \in X_2$. Then the set $R := \{z = t + iy \mid y_2 \leq y \leq y_1, x \in X_1\}$ is contained in D



Proof

Consider induction on the level of Γ.

For $\operatorname{lev}(\Gamma) \geq 2$,

- $\operatorname{lev}(\Gamma) = 2$: $R = \bigcup_{i=1}^{n} R_i$, R_i: closed rectangle and $\cup \partial R_i := \Gamma$

  Let $z_0 \in R$.

  Suppose $z_0 \notin R$. As D is simply connected, $\exists \gamma : [0, \infty] \longrightarrow (\mathbb{C} \cup \{\infty\}) \setminus D$ s.t. $\gamma(0) = z_0$ and $\lim_{t \to \infty} \gamma(t) = \infty$, $t_0 := \sup\{t \mid \gamma(t) \in R\}$

  $\setminus D$ ※

  Claim: $\gamma(t_0) \in \Gamma \subseteq D$ ($\Rightarrow$ ※)

  Proof

  (i) $\gamma(t_0) \notin R \setminus \Gamma$: open (If no, then $\gamma(t_0) \in \Gamma$ or $\gamma(t_0) \in \mathbb{C} \setminus R$)

  If yes, as $R \setminus \Gamma$: open, $\exists D(\gamma(t_0), \varepsilon) \subseteq R \setminus \Gamma$, $t_0 \in \gamma^{-1}(D(\gamma(t_0), \varepsilon)) \subseteq [0, \infty)$ (as $t_0 = \sup\{t \mid \gamma(t) \in R\}$) ※ $\lim_{t \to \infty} \gamma(t) = \infty$

  (ii) $\gamma(t_0) \notin \mathbb{C} \setminus R$: Similarly, ※

  $\setminus D$ ※

  $\therefore$ $\gamma(t_0) \in \Gamma \subseteq D$ ※

- $\operatorname{lev}(\Gamma) > 2$: Note: $\cup(\partial R_i)$ not necessarily equal to Γ

  By the same argument as base case, $t_0 = \sup\{t \mid \gamma(t) \notin R\}$, $t_0 \in \partial R_i$ for some R_i

  If $t_0 \in \Gamma$, as in the base case, ※

  Def $\tilde{R} := \mathbb{C} \setminus R$, $\Gamma' := (\Gamma \cap \tilde{R}) \cup \overline{L}$, where $L := \{x + iy \mid y = y_2, x \in X_1 \setminus X_2\}$

  For small h, $\gamma(t_0 + h)$ is between the top two levels of Γ'

  As $\gamma(t_0 + h) \notin D$ and $\gamma(t_0 + h) \in S$, $\operatorname{level}(\Gamma') < \operatorname{level}(\Gamma)$

  $\therefore$ By induction, $\gamma : [t_0 + h, \infty] \to \infty$ intersects $\Gamma' \setminus \overline{L}$ $\square$

  as $\forall t > t_0$, $\gamma(t) \notin R$