10-30-25 (WEEK 9)

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THEOREM

Let γ be a regular closed curve and f be a meromorphic function on and inside γ.
Suppose f has no poles nor zeros of f inside γ.

Set n_z = #zeros inside γ, n_p = #poles inside γ
Then, $\int_\gamma \frac{f'}{f}dz = (n_z-n_p)2\pi i$ ☆ $\int$ rly fits winding number

Proof

① f has a zero of order k at ω, around ω, we have $f(z)=\frac{f^{(k)}(\omega)}{k!}(z-\omega)^k+\dots=(z-\omega)^k g(z)$ and $g(z)\neq 0$ ana around ω

$\Rightarrow \frac{f'}{f}=\frac{k(z-\omega)^{k-1}g(z)+(z-\omega)^k g'(z)}{(z-\omega)^k g(z)}=\boxed{\frac{k}{z-\omega}+\frac{g'(z)}{g(z)}}$ around ω

$\therefore \frac{f'}{f}$ has a simple pole at ω with radius k

② f has a pole at ω of order k around ω.

$\therefore f=(z-\omega)^{-k}\sum_{i=0}^{\infty} a_i(z-\omega)^{k+i}=(z-\omega)^{-k}g(z) \Rightarrow \boxed{\frac{f'}{f}=-k\frac{1}{z-\omega}+\frac{g'(z)}{g(z)}}$ around ω, $\frac{f'}{f}$ has a simple pole with residue $-k$

③ ω is not a pole nor a zero, then $\text{Res}(\frac{f'}{f},\omega)=0$ as $\frac{f'}{f}$ is ana at ω

By Cauchy Residue Thm, $\boxed{\int_\gamma \frac{f'}{f}dz=2\pi i\sum_{i=1}^{n} n(\gamma,\omega_i)\,\text{Res}(\frac{f'}{f},\omega_i)}$

Now, we know:

$n(\gamma,\omega_i)=1$ for ω_i inside γ

$$\text{Res}\left(\frac{f'}{f},\omega_i\right)=\begin{cases} k, & \omega_i\text{: zero of order } k \\ -k, & \omega_i\text{: pole of order } k\end{cases}$$

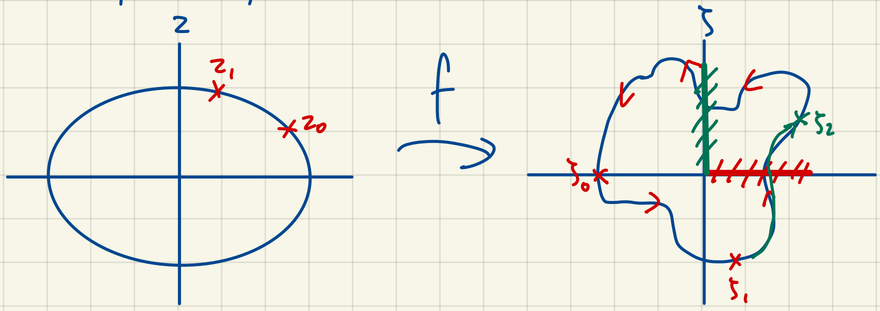
$\therefore \int_\gamma \frac{f'}{f}dz=(n_z-n_p)2\pi i$ □

COROLLARY (ARGUMENT PRINCIPLE) ☆ IMPORTANT TO KNOW THE INTUITION

If f is ana inside and on a closed curve γ and f has no zeros on γ, then $n_z=\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz$

REMARK ON COROLLARY

For $\gamma: z(t)$, $\gamma'=f(z(t))$,



We can integrate over these curves:

$\log(\zeta)=\int_{\zeta_0}^{\zeta}\frac{d\zeta}{\zeta}+\log(\zeta_0)$

$\log(\zeta_1)=\int_{\zeta_0}^{\zeta_1}\frac{d\zeta}{\zeta}+\log(\zeta_0) \leftarrow \arg\in[\pi,2\pi]$ } Different analytic branches (need to choose one where the entire path is cont:)

$\log(\zeta_2)=\int_{\zeta_1}^{\zeta_2}\frac{d\zeta}{\zeta}+\log(\zeta_1) \leftarrow \arg\in[\pi,2\pi]$

└ $\arg\in(\frac{\pi}{2},\frac{5\pi}{2})$

(If we want to find a whole loop's worth of integration, we can do so by finding enough ana branches)

$\Rightarrow \int_{\gamma'}\frac{d\zeta}{\zeta}=\log(\zeta_n)-\log(\zeta_0)$ where ζ_n is the "overlapping" ζ_0 point on a new ana branch