11-11-25 (WEEK 11)

Shun/翔海(@shun4midx)

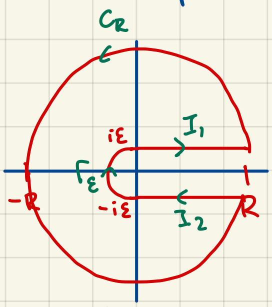
TYPE (III) INTEGRALS

$\boxed{\int_0^\infty \frac{P(x)}{Q(x)}dx}$ for $\gcd(P,Q)=1$, $\deg Q \geq \deg P+2$, $Q(x)\neq 0\ \forall x\geq 0$

If $\frac{P(x)}{Q(x)}$ is even, we should just do $\frac{1}{2}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx$

Let $f(x):=\frac{P(x)}{Q(x)}$. Consider if $f(x)$ is NOT even.

Consider this path:



Abuse of notation to write C_R, Γ_ε. Consider $R\to\infty$, $\varepsilon\to 0$.

Consider $\log z$ on $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$, $\arg\in(0,2\pi)$

For $f(z)\log z$,

I_1: $\log(t+i\varepsilon)=\log|t+i\varepsilon|+\theta(t)i$
I_2: $\log(t-i\varepsilon)=\log|t-i\varepsilon|+[2\pi-\theta(t)]i$
} difference in arg due to symmetry

Notice, $\theta(t)\xrightarrow{\varepsilon\to 0} 0$ rapidly

Rmk

If $f(0)\neq 0$, then $f(z)\log z$ is not defined at $z=0$, so don't write $\lim_{\substack{M\to\infty\\ \varepsilon\to 0}}\int_\varepsilon^M f(x)\log x\,dx=\int_0^\infty f(x)\log x\,dx$

We consider the paths I_1 and I_2,

I_1: $\int_0^R f(t+i\varepsilon)\log(t+i\varepsilon)dt$

I_2: $\int_R^0 f(t-i\varepsilon)\log(t-i\varepsilon)dt$

$\Rightarrow$ Total: $\int_0^R [f(t+i\varepsilon)\log(t+i\varepsilon)-f(t-i\varepsilon)\log(t-i\varepsilon)]dt$

$$=\int_0^R \underbrace{[f(t+i\varepsilon)-f(t-i\varepsilon)]}_{c_1\varepsilon+c_2\varepsilon^2+\dots\text{ (begins at } \varepsilon \text{ term)}}\underbrace{\log(t+i\varepsilon)}_{\log|t+i\varepsilon|+\theta(t)i}dt+\int_0^R f(t+i\varepsilon)\underbrace{[\log(t+i\varepsilon)-\log(t-i\varepsilon)]}_{[-2\pi+2\theta(t)]i}dt$$

For $(x,\varepsilon)\in[0,\varepsilon_0]\times[0,\varepsilon_0]$,

$[f(x+i\varepsilon)-f(x-i\varepsilon)]\log(x+i\varepsilon)=:g(x,\varepsilon)$. Let $\alpha:=\max_{x,\varepsilon}(f(x+i\varepsilon)-f(x-i\varepsilon))\log(x+i\varepsilon)$

Given $R,\delta_1,d>0$, $\exists\,\varepsilon$, s.t. ① $\theta(x+i\varepsilon)<\delta_1\ \forall x\geq d$

② $|f(x+i\varepsilon)-f(x-i\varepsilon)|<\frac{\delta_1}{M}$

Then, as $\varepsilon\to 0$,

$$\int_0^R f(t+i\varepsilon)\log(t+i\varepsilon)dt-\int_0^R f(t-i\varepsilon)\log(t-i\varepsilon)dt=\int_0^R f(t+i\varepsilon)2\pi i\,dt\xrightarrow{\varepsilon\to 0}\int_0^R f(t)2\pi i\,dt \quad ①$$

Now, we know $\left|\frac{P(x)}{Q(x)}\right|\leq\frac{A}{|R|^2}$

So, as $\varepsilon\to 0$, $\left|\int_{C_R} f(z)\log z\,dz\right|\leq\frac{A}{R^2}(\log R+2\pi)\cdot 2\pi R\xrightarrow{R\to\infty} 0 \quad ②$

Choose a small ε, s.t. $Q(z)\neq 0\ \forall z\in D(0,\varepsilon)$.
Let $m := \max_{z\in \overline{D(0,\varepsilon)}} |f(z)|$.
$\therefore \int_{\gamma_\varepsilon} f(z)\log z\, dz \ll m(|\log\varepsilon|+c)\cdot \pi\varepsilon \xrightarrow{\varepsilon\to 0} 0$ ③

$\therefore -2\pi i\int_0^\infty f(t)\,dt = 2\pi i \sum_{\omega_k:\,Q\text{'s roots}} \operatorname{Res}(f(z)\log z, \omega_k)$

$\Rightarrow \int_0^\infty f(t)\,dt = -\sum_{\omega_k:\,Q\text{'s roots}} \operatorname{Res}(f(z)\log z, \omega_k)$

EXAMPLE

$\int_0^\infty \frac{1}{1+x^3}\,dx = -\sum_{k=1,3,5} \operatorname{Res}\left(\frac{1}{1+z^3}\log z, \omega_k\right)$

ALTERNATE FORM

$\int_a^\infty \frac{P(x)}{Q(x)}\,dx$ can be evaluated by considering $\int_{C_M} \log(z-a)\frac{P(z)}{Q(z)}\,dz$. In fact, $\int_0^a = \int_0^\infty - \int_a^\infty$

ALTERNATE FORM

$\int_0^\infty \frac{x^{\alpha-1}}{Q(x)}\,dx$, $0<\alpha<1$ and Q: poly w/ deg 1

Notice, $z^{\alpha-1} := \exp[(\alpha-1)\log z]$
Then,
$\int_{C_R} \longrightarrow 0$ as $R\longrightarrow\infty$
$\int_{\gamma_\varepsilon} \longrightarrow 0$ as $\varepsilon\longrightarrow 0$
I_1, $\varepsilon\to 0$, $\int_0^\infty \frac{x^{\alpha-1}}{Q(x)}\,dx$
I_2, $\varepsilon\to 0$, $\int_{I_2} \frac{z^{\alpha-1}}{Q(z)}\,dz = \int_0^\infty \frac{x^{\alpha-1}e^{2\pi i(\alpha-1)}}{Q(x)}\,dx$
$\therefore [1-e^{2\pi i(\alpha-1)}]\int_0^\infty \frac{x^{\alpha-1}}{Q(x)}\,dx = 2\pi i \sum_{\omega_k:\,\text{roots of } Q} \operatorname{Res}\left(\frac{z^{\alpha-1}}{Q(z)}, \omega_k\right)$

TYPE (IV) INTEGRALS

$\boxed{\text{For } R(x,y) = \frac{P(x,y)}{Q(x,y)},\ P,Q\in\mathbb{C}[x,y],\ \int_0^{2\pi} R(\cos\theta,\sin\theta)\,d\theta}$

We take $z = \sin\theta + i\cos\theta$, $d\theta = \frac{dz}{iz}$.
$\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$

$\Rightarrow$ Change of variables gives us $\int_{|z|=1} \underbrace{\frac{P\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)}{Q\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)}}_{=:f(z)} \frac{dz}{iz} = 2\pi i \sum_{\omega_i:\,\text{poles of } f(z)} \operatorname{Res}(f(z), \omega_i)$

EXAMPLE

$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{|z|=1} \frac{-2i}{z^2+4z+1}\,dz = 4\pi \operatorname{Res}\left(\frac{1}{z^2+4z+1}, \sqrt{3}-i\right)$

ESTIMATING SUMS

Shun/翔海(@shun4midx)

TYPE (I)

$\boxed{\sum_{n=-\infty}^{\infty} f(n)^{Ⓐ}, \sum_{n=-\infty}^{\infty} (-1)^n f(n)}^{Ⓑ}$

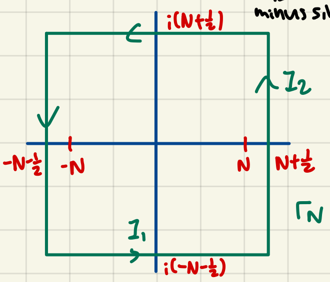
WANT: Find an analytic $g(z)$ on $\mathbb{C}\setminus\{n\in\mathbb{Z}\}$, s.t. $\text{Res}(g(z),n) = cf(n)$, $c\in\mathbb{C}^\times$

$\hookrightarrow \frac{1}{\sin\pi z}$ has simple roots at $z\in\mathbb{Z}$ and ana elsewhere

We have: $\text{Res}(\frac{1}{\sin\pi z}, n) = \frac{1}{\pi}(-1)^n$

Let $g(z) = f(z)\frac{\pi}{\sin\pi z} \Rightarrow \text{Res}(g(z),n) = (-1)^n f(n)$

$\therefore \sum_{n=-\infty}^{\infty} f(n)(-1)^n = \sum_{\substack{n=-\infty \\ \text{minus sing}}}^{\infty} \text{Res}(g(z),n) + \sum_{\substack{w_k:\text{sing} \\ \text{of } f}} \text{Res}(g(z),w_k) = \frac{1}{2\pi i}\int_{\Gamma_N} g(z)\,dz$, where Γ_N is as follows:



Ensure $\sum f(n)$, $\sum(-1)^n f(n)$ conv, we assume $|f(z)| \le \frac{A}{|z|^2}$

Here, $\frac{1}{\sin\pi z} = \frac{2i}{e^{i\pi z}-e^{-i\pi z}} = \frac{2ie^{i\pi z}}{e^{2i\pi z}-1}$ ☆ key term w/ poles

$I_1: \left|\frac{1}{\sin\pi z}\right| = \left|\frac{2e^{\pi(N+\frac{1}{2})}}{e^{2\pi(N+\frac{1}{2})}-1}\right| < 1$

$I_2: \left|\frac{1}{\sin\pi z}\right| = \left|\frac{2e^{-y\pi}}{e^{-2\pi y}e^{(2N+1)\pi}-1}\right| = \frac{e^{-y\pi}}{e^{-2\pi y}+1}$

$\therefore \lim_{N\to\infty}\left|\int_{\Gamma_N} f(z)\frac{\pi}{\sin\pi z}dz\right| \le \lim_{N\to\infty} 4(2N+1)c\cdot\frac{A}{|N+\frac{1}{2}|^2} = 0 \Rightarrow \sum_{\substack{n=-\infty \\ n\ne\text{sing}}}^{\infty} f(n) = -\sum_{w_k:\text{sing}}\text{Res}(g(z),w_k)$

$\cos\pi z$ gives the $(-1)^n$

Ⓐ: $g(z) = f(z)\cot\pi z$

Ⓑ: $g(z) = f(z)\csc\pi z$

WARNING

When evaluating $\sum_{n=1}^{\infty}\frac{1}{n^2}$, notice $\frac{1}{n^2}$ is undefined.

$\Rightarrow \int_{\Gamma_N}\frac{\pi}{z^2}\frac{2\cos\pi z}{\sin\pi z}dz = 2\pi i\sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty}\text{Res}(g(z),n) + \text{Res}(g(z),0)$

Then we evaluate as usual.

SUMMARY

For Ⓐ, consider $g(z) := f(z)\frac{\cos z\pi}{\sin z\pi}\pi$

For Ⓑ, consider $g(z) := f(z)\frac{1}{\sin z\pi}\pi$

TYPE (II) — BINOMIAL COEFFICIENTS

We know $\binom{n}{k} \leadsto$ coef of z^k in $(1+z)^n$, so $\binom{n}{k} = \frac{1}{2\pi i}\int\frac{(1+z)^n}{z^{k+1}}dz$

EXAMPLE

$\sum_{n=0}^{\infty} \binom{2n}{n}\frac{1}{5^n} = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{C_r(0)}\frac{(1+z)^{2n}}{(5z)^n}\frac{dz}{z} = \frac{1}{2\pi i}\int_{C_r(0)}\sum_{n=0}^{\infty}\underline{\frac{(1+z)^{2n}}{(5z)^n}}\frac{dz}{z} = \frac{5}{2\pi i}\int_{|z|=1}\frac{1}{3z-1-z^2}dz = 5\,\text{Res}\left(\frac{1}{3z-1-z^2}, \frac{5-\sqrt{5}}{2}\right)$

$\sum_{n=0}^{\infty}\frac{(1+z)^{2n}}{(5z)^n}$ conv abs

$\left[\frac{(1+z)^2}{5z}\right]^n$ ← geom series

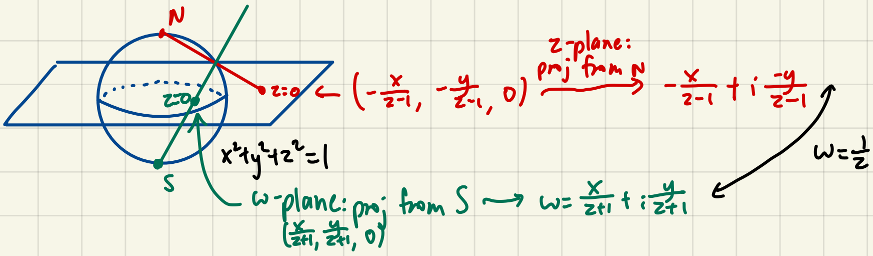
Shun/翔海 (@shun4midx)

RESIDUE AT INFINITY

DEFINITION

For f: ana on $\mathbb{C} \setminus \{w_1, \dots, w_k\}$, $\text{Res}(f, \infty) := -\int_{C_R} f(z)\,dz$

For $R >> 0$, s.t. $|w_i| < R$, we have "∞" as follows:



$\rightsquigarrow$ We can switch to the w-plane, to only have one residue (N) remaining

change of coords lol