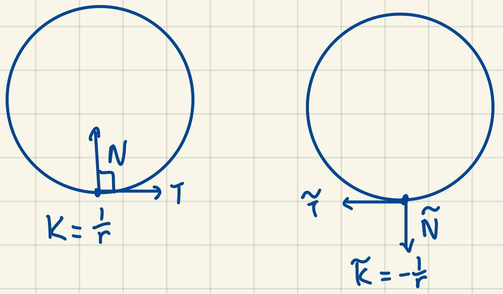
9-17-25 (WEEK 3) (From here on, I will write κ as K for convenience ∵ Blame me for not using LaTeX) Shun/翔海 (@shun4midx)

REMARK



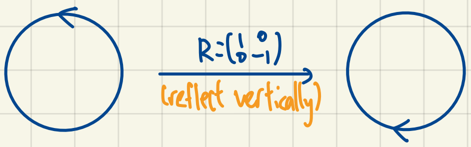
⇒ Curvature differs by a negative sign if it is reflected
↳ $\det(R) = 1 \Rightarrow$ No change
↳ $\det(R) = -1 \Rightarrow$ Multiply by -1

Proof

For $\tilde{x}(s) = R(x(s) - x(0)) + \tilde{x}(0)$,
$\tilde{T} = \frac{d\tilde{x}}{ds} = R\frac{dx}{ds} = RT$, $\frac{d}{ds}(RT) = R\frac{dT}{ds} = K(RN)$

Define $\tilde{N}$ as in $\tilde{T} \times \tilde{N} = (0, 0, 1)$
$\therefore RN = \begin{cases} \tilde{N}, & \det R = 1 \\ -\tilde{N}, & \det R = -1 \end{cases}$
$\Rightarrow \tilde{K} = \begin{cases} \tilde{K}, & \det R = 1 \\ -\tilde{K}, & \det R = -1 \end{cases}$ □

EXAMPLE

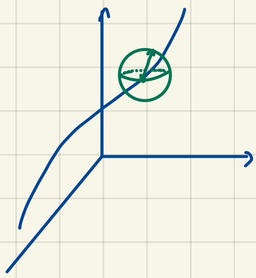


⇒ Swapped direction

∴ For simplicity, we assume R is pure rotation

FROM PLANE CURVE TO SPACE CURVE

Let $\Gamma \subseteq \mathbb{R}^3$ be a curve, $x(s)$ be a unit-speed parametrization of Γ.



Then, we define: Avoid degeneracy
- Unit tangent: $T = \frac{dx}{ds}$
- $\frac{d^2x}{ds^2} = \frac{dT}{ds} \underset{K>0}{=} K\left(\frac{dT}{ds} / \left|\frac{dT}{ds}\right|\right) = KN$
  ↳ Define $K = \left|\frac{dT}{ds}\right| \geq 0$

In the case of $\Gamma \subseteq \mathbb{R}^3$, we can consider $\frac{dN}{ds}$
As $|N|^2 \equiv 1 \Rightarrow N \cdot N \equiv 1$, thus $\frac{dN}{ds} \cdot N = 0$. In 3D, $\frac{dN}{ds}$ doesn't have to be parallel to T.

DEFINITION

We say the binormal is given by $T \times N = B$

DERIVATION OF TORSION

We can express $\frac{dN}{ds} = \alpha T + \beta B$

We know:

$\hookrightarrow \alpha = \frac{dN}{ds} \cdot T = \frac{d}{ds}(\cancel{N \cdot T})^{0} - N \cdot \underbrace{\frac{dT}{ds}}_{\kappa N} = -\kappa$

$\hookrightarrow \beta = \frac{dN}{ds} \cdot B = -\tau(s)$ "Torsion"

$\therefore \boxed{\frac{dN}{ds} = -\kappa T - \tau B}$

We try to find more properties of τ.

Notice, $\frac{dB}{ds} = \frac{d}{ds}(T \times N) = \frac{dT}{ds} \times N + T \times \frac{dN}{ds} = (\kappa N) \times N + T \times (-\kappa T - \tau B) = -\tau(T \times B)$

As $T \times B = -N$, hence $\frac{dB}{ds} = \tau N$

FERRET-SFRET EQUATIONS FOR PLANE CURVES AND PROPERTIES OF TORSION

EQUATIONS

$$\boxed{\begin{cases} \frac{dT}{ds} = \kappa N \\ \frac{dN}{ds} = -\kappa T - \tau B \\ \frac{dB}{ds} = \tau N \end{cases}}$$

PROPERTIES

Up to non-reflective rigid motion, these equations still mostly hold

PROOF

Define $\tilde{X}(s) = R(X(s) - X(0)) + \tilde{X}(0)$ where $R \in O(3)$

Then, $\frac{d\tilde{X}}{ds} = R\frac{dX}{ds} = \tilde{T}$, $\frac{d^2\tilde{N}}{ds^2} = \frac{d}{ds}(RT) = \kappa RN \Rightarrow \tilde{\kappa} = \kappa$, $\tilde{N} = RN$ as per the 2D-case

We know $\{RT, RN, RB\}$ is an orthonormal basis

Here, $RT = \tilde{T}$, $RN = \tilde{N}$.

Then, $\tilde{B} = \tilde{T} \times \tilde{N} = \begin{cases} RB, & \det R = 1 \\ -RB, & \det R = -1 \end{cases}$

Hence, $\frac{d\tilde{B}}{ds} = \pm\frac{d}{ds}(RB) = \pm\tau\tilde{N} = \pm\tau RN$

RELATING TORSION TO PARAMETRIZATION W.R.T. TIME

Let $X(t)$ be a regular parametrization of Γ.

So, $\dot{X} = \underbrace{\frac{dX}{dt}}_{\text{velocity}} = \frac{dX}{ds} \cdot \frac{ds}{dt} = T\underbrace{v}_{\text{speed}}$

Hence, $\ddot{X} = \frac{dv}{dt}T + v\frac{dT}{dt} = \frac{dv}{dt}T + \kappa v^2 N$

$\Rightarrow \dot{X} \times \ddot{X} = \kappa v^3 T \times N$

$\therefore$ We still have $\kappa = \frac{|\dot{X} \times \ddot{X}|}{|v|^3}$

Now, $\dddot{X} = \frac{d}{dt}(\dot{v}T + \kappa v^2 N) = \ddot{v}T + \dot{v}\kappa N v + \frac{d\kappa}{ds}v^3 N + 2\kappa v\dot{v}N + \kappa v^3(-\kappa T - \tau B)$

$= (\ddot{v} - \kappa^2 v^3)T + (\kappa v\dot{v} + \frac{d\kappa}{ds}v^3 + 2v\dot{v}\kappa)N - \kappa\tau v^3 B$

$\therefore \det(\dot{X}, \ddot{X}, \dddot{X}) = \det(vT, \kappa v^2 N, -\kappa\tau v^3 B) = -v^6\kappa^2\tau \underbrace{\det(T, N, B)}_{=1} = -v^6\kappa^2\tau$

$\therefore \tau = -\frac{(\dot{X} \times \ddot{X}) \cdot \dddot{X}}{|\dot{x}|^6 \kappa^2} = -\frac{(\dot{X} \times \ddot{X}) \cdot \dddot{X}}{|\dot{X} \times \ddot{X}|^2}$

Shun/翔海(@shun4midx)

APPLICATIONS

Consider the reparametrization $X(x) = (x, f(x), g(x))$ by IFT

Then, $X' = (1, f'(x), g'(x))$

$X'' = (0, f''(x), g''(x))$

$X''' = (0, f'''(x), g'''(x))$

From the graph, for the case of a plane curve,

$$(\dot{X} \times \ddot{X}) \cdot \dddot{X} = \begin{vmatrix} 1 & f'(x) & g'(x) \\ 0 & f''(x) & g''(x) \\ 0 & f'''(x) & g'''(x) \end{vmatrix} = |f''(x)\cancelto{0}{g'''(x)} - \cancelto{0}{g''(x)} f'''(x)| = 0 \Rightarrow \boxed{\tau = 0}$$

If not a plane curve,

$$\dot{X} \times \ddot{X} = \begin{vmatrix} i & j & k \\ 1 & f'(x) & g'(x) \\ 0 & f''(x) & g''(x) \end{vmatrix} \Rightarrow |\dot{X} \times \ddot{X}|^2 = (f''(x))^2 + (g''(x))^2$$

$$\therefore \boxed{\tau = -\frac{|f''(x)g'''(x) - g''(x)f'''(x)|}{(f''(x))^2 + (g''(x))^2}}$$