10-31-25 (WEEK 9)

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DEFINITIONS

$v = v^i \partial_i \xmapsto[\sim]{D_x} V = v^i \partial_i X$ (different based on upper/lowercase)

First fundamental form: $g(v,w) = V \cdot W$
Second fundamental form: $A(v,w) = -DN(V) \cdot W$
} bilinear form

PROPERTIES

$g(v,w) = v^i w^j g(\partial_i, \partial_j)$

$g_{ij} = g(\partial_i, \partial_j) = \partial_i X \cdot \partial_j X$

$A_{ij} = A(\partial_i, \partial_j) = \underbrace{-DN(\partial_i X)}_{= \partial_i N} \cdot \partial_j X$

DEFINITIONS

$S: v \xmapsto{D_x} -DN(V)$ (Shape operator)

$S(\partial_i) = S_i^j \partial_j \longmapsto -\partial_i N = S_i^j \partial_j X$

We may write $S_i^j = \underbrace{g^{jk} A_{ki}}_{g^{-1}A = S} = A_i^j$ (Superscript is inverse)

PROPERTIES

$\underbrace{A(v,w)}_{\text{symmetric}} = -DN(V) \cdot W = \underbrace{g(Sv, w)}_{\text{self adjoint w.r.t. } g}$

THEOREM

$\exists$ g-orthogonal $\{e_1, e_2\}$ s.t. $Se_1 = k_1 e_1$, $Se_2 = k_2 e_2$. Moreover, $k_1 = \min\{A(v,v) \mid |v| = 1\}$, $k_2 = \max\{A(v,v) \mid |v| = 1\}$

Proof

$v \in T_u\Omega \longmapsto A(\underbrace{v}_{v^i\partial_i}, \underbrace{v}_{v^j\partial_j}) = A_{ij} v^i v^j$

$|v|^2 = 1$

$\parallel$

$g(v,v) = g_{ij} v^i v^j$

Define $\underbrace{f}_{A}(v^1, v^2) = A_{11}(v^1)^2 + 2A_{12} v^1 v^2 + A_{22}(v^2)^2$ to consider extrema

Consider the Lagrange Multiplier,

$$\begin{cases} Av = \lambda g v \Leftrightarrow \underbrace{g^{-1}A}_{S} v = \lambda v \\ g(v,v) = 1 \end{cases}$$

Take the min as k_1, max as k_2: $ge_1 = \lambda e_1 \Rightarrow A(e_1, e_1) = k_1$ ($\because g(e_1,e_1) = 1$) $\therefore Se_1 = k_1 e_1$

Similarly, $Se_2 = k_2 e_2$

If $k_1 < k_2$, then $g(Se_1, e_2) = g(e_1, Se_2) \Rightarrow k_1 g(e_1, e_2) = k_2 g(e_1, e_2) \Rightarrow g(e_1, e_2) = 0$ ($\Rightarrow$ ortho)

If $\underbrace{k_1 = k_2}_{\text{"umbilical point"}}$, then $A(v,v) = k$ $\forall v \in T_u\Omega$, $|v| = 1$ $\square$

DEFINITION

If $|v| = 1$, $A(v,v) = 0$, then v is called an asymptotic direction

We also call $\{e_1, e_2\}$ principle directions/curvatures

CHANGE OF BASIS FROM $\{\partial_1, \partial_2\}$ TO $\{e_1, e_2\}$

We know $v = \tilde{v}^1 e_1 + \tilde{v}^2 e_2$, $|v| = 1 \Rightarrow \cos\theta e_1 + \sin\theta e_2$

Then, $K_n(v) = A(v,v) = g(Sv, v) = g(\cos\theta \underbrace{Se_1}_{k e_1} + \sin\theta \underbrace{Se_2}_{k e_2}, \cos\theta e_1 + \sin\theta e_2) = k_1 \cos^2\theta + k_2 \sin^2\theta$

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DEFINITIONS

Asymptotic curve: curve along the asymptotic direction
Line of curvature: curve along the principle direction

SHAPE OPERATOR (WEINGATEN OPERATOR) PROPERTIES

W.r.t. the basis $\{\partial_1, \partial_2\}$, S is self-adjoint

$$\Rightarrow S \sim \begin{pmatrix} S_1^1 & S_1^2 \\ S_2^1 & S_2^2 \end{pmatrix} = \begin{pmatrix} A_1^1 & A_1^2 \\ A_2^1 & A_2^2 \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix}$$

If we choose w.r.t. $\{e_1, e_2\}$,

$$S \sim \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

$\therefore$ $\det S = A_1^1 A_2^2 - A_1^2 A_2^1 = k_1 k_2 = K$ (Gaussian curvature)

$\operatorname{tr} S = A_1^1 + A_2^2 = k_1 + k_2 = H$ (Mean curvature)