11-19-25 (WEEK 12)

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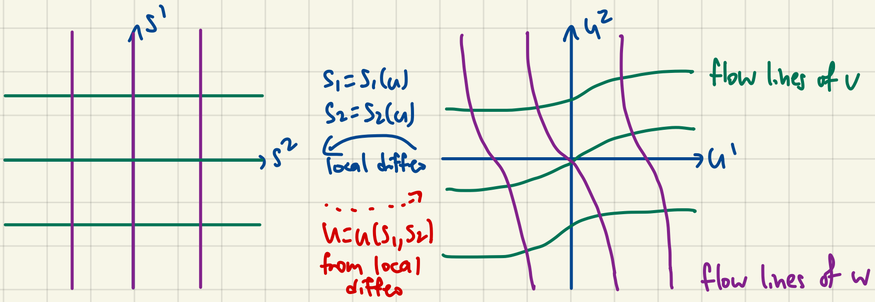
THEOREM

Let $\{v, w\}$ be linearly independent tangent vector fields on Σ near $p \in \Sigma$. Then, $\exists$ local param of Σ near p, s.t. the coord lines are the flow lines of v, w $\{v(p), w(p)\}$



Proof

level set val?

Let $S_1 =$ "index" of the flow lines of V

$S_2 =$ "index" of the flow lines of W

$dS_1(V)|_p = 0,\ dS_1(W)|_p = 0 \Rightarrow dS_1|_p = 0$ $\times$ ($\because$ s is a regular function)

So, $dS_1(V)|_p = 0 \quad dS_2(V)_p \neq 0$

$dS_1(W)|_p \neq 0 \quad dS_2(W)_p = 0$

$$\Rightarrow \begin{pmatrix} \frac{\partial S_1}{\partial u^1} & \frac{\partial S_1}{\partial u^2} \\ \frac{\partial S_2}{\partial u^1} & \frac{\partial S_2}{\partial u^2} \end{pmatrix} \begin{pmatrix} v^1 & w^1 \\ v^2 & w^2 \end{pmatrix} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \quad \square$$

REMARK

$\frac{\partial}{\partial s_2} u(s_1, s_2) = Du\left(\frac{\partial}{\partial s_2}\right) = \lambda V(u)$

Similarly, $\frac{\partial}{\partial s_1} u(s_1, s_2) = \mu W(u)$

COROLLARY

If $V \perp W$, then $X(u(s_1, s_2)) = \tilde{X}(s_1, s_2)$, $\frac{\partial \tilde{X}}{\partial s_1} \perp \frac{\partial \tilde{X}}{\partial s_2} \Rightarrow \tilde{g} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$

$$// \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{(0 \quad 1) \begin{pmatrix} g_{11}(u) & g_{21}(u) \\ g_{12}(u) & g_{22}(u) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(1 \quad 0) \begin{pmatrix} g_{11}(u) & g_{21}(u) \\ g_{12}(u) & g_{22}(u) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Let u be local coords.

If $\{\frac{\partial}{\partial u^1} \not\perp \frac{\partial}{\partial u^2}\}$, then we perform Gram-Schmidt, so $V = \frac{\partial}{\partial u^1} = (1, 0)$, $W = \frac{\partial}{\partial u^2} - \frac{\langle \partial/\partial u_2, \partial/\partial u_1 \rangle}{\langle \partial/\partial u_1, \partial/\partial u_1 \rangle} \frac{\partial}{\partial u_1}$.

EXAMPLE

Let P be a nonumbilical point $(k_1 < k_2) \Rightarrow V \perp W$ (principal directions)

ISOMETRIES

SETTING IN $\mathbb{R}^3$

Let $\phi: \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism

If it is length-preserving, then ϕ must be a rigid motion, where $\phi(x) = Ax + b$, $A \in O_3$ orthog matrix

PROPERTY IN GENERAL Σ

Let $\phi: \Sigma \to \tilde{\Sigma}$ be a diffeomorphism. If it's length-preserving, i.e. $L(\phi(\gamma)) = L(\gamma)\ \forall \gamma$

Param γ w.r.t. t, $L(\phi(\gamma)) = L(\gamma) \Rightarrow \int_a^b \left|\frac{d}{dt}(\phi(\gamma(t)))\right| dt = \int_a^b |\gamma'(t)| dt \Rightarrow \left|\frac{d}{dt} \phi(\gamma(t))\right| = |\gamma'(t)|$

In other words, $|D\phi_p(v)| = |v|\ \forall v \in T_p\Sigma$

DEFINITION

If $\forall v \in T_p\Sigma$, $|D\phi_p(v)| = |v|$, then $D\phi: T_p\Sigma \to T_{\phi(p)}\tilde{\Sigma}$ is a linear isometry

REMARK

$D\phi(v)\cdot D\phi(w) = v\cdot w$ $\forall v,w\in T_p\Sigma$ if $D\phi$ is an isometry

Proof

$|D\phi(v+w)|^2 = |v+w|^2 \Rightarrow |D\phi(v)|^2 + |D\phi(w)|^2 + 2D\phi(v)\cdot D\phi(w) = |v|^2+|w|^2+2v\cdot w$ □

PROPERTY

$\phi:\Sigma\to\tilde\Sigma$ is an isometry

$\Leftrightarrow D\phi: T_p\Sigma \to T_{\phi(p)}\tilde\Sigma$ is a linear isometry (extrinsic POV)

$\Leftrightarrow \varphi^*\tilde g = g$ ← focus here below (recall: $\varphi^* g$ is the pull-back) (intrinsic POV)

$$\begin{tikzcd}
\Sigma \arrow[r, "\phi"] & \tilde\Sigma \\
(\Omega, g) \arrow[u, "X"] \arrow[r, dashed] & (\tilde\Omega, \tilde g) \arrow[u, "\tilde X"']
\end{tikzcd}$$

$\|$

pull-back of dot product by X

$g_{ij} = g(\partial_i,\partial_j) = \partial_i X\cdot\partial_j X$

OBSERVE

$D\phi(\partial_i X)\cdot D\phi(\partial_j X) = \partial_i X\cdot \partial_j X = g_{ij}$

$\|$

$\partial_i(\phi(X))\cdot\partial_j(\phi(X))$ (if we rewrite $\phi(X)=\tilde X(\varphi)$, so $X(u^1,u^2) = \tilde X(\tilde u^1,\tilde u^2)$, $(\tilde u^1,\tilde u^2)=\varphi(u^1,u^2)$)

$\|$

$\tilde\partial_k \tilde X(\varphi)\frac{\partial\tilde u^k}{\partial u^i}\cdot\tilde\partial_l\tilde X(\varphi)\frac{\partial \tilde u^l}{\partial u^j} = \tilde g_{kl}(\varphi)\frac{\partial\tilde u^k}{\partial u^i}\frac{\partial\tilde u^l}{\partial u^j} = \varphi^*\tilde g(\partial_i,\partial_j)$

REMARK (COMPARISON WITH AREA-PRESERVATION)

Recall, $\phi:\Sigma\to\tilde\Sigma$ is area-preserving $\Leftrightarrow \int_{\tilde\Omega}\sqrt{\det\tilde g}\,d\tilde u^1 d\tilde u^2 = \int_\Omega\sqrt{\det g}\,du^1du^2$

$\|$ ($\tilde u = \varphi(u)$)

$\int_\Omega \sqrt{\det\tilde g(\varphi)}\,\left|\det\frac{\partial\tilde u}{\partial u}\right|\,du$

$\Leftrightarrow \sqrt{\det\tilde g(\varphi)}\,\left|\det\frac{\partial\tilde u}{\partial u}\right| = \sqrt{\det g}$

$\|$

$\sqrt{\det\varphi^*(\tilde g)}$ ← similar to the intrinsic POV above

Now, from the extrinsic POV, what we needed was:

$\int_\Omega |\partial_1(\phi(X))\times\partial_2(\phi(X))|\,du^1du^2 = A(\tilde\Sigma)$

$\|$

$\int_\Omega |\partial_1X\times\partial_2X|\,du^1du^2 = A(\Sigma)$

$\Leftrightarrow |\partial_1(\phi(X))\times\partial_2(\phi(X))| = |\partial_1X\times\partial_2X|$

$\|$

$|D\phi(\partial_1X)\times D\phi(\partial_2X)|$ (similar to extrinsic POV of isometry)

DEFINITION

Σ and $\tilde\Sigma$ are locally isometric if $\forall p\in\Sigma$, $\exists \tilde p\in\tilde\Sigma$ and local param $X:\Omega\to\Sigma$ and $\tilde X:\Omega\to\tilde\Sigma$, s.t. $\tilde g = g$

(Translation for stupid ppl like Shun himself: "Locally isometric" = w.r.t. the same coord space, they have the same first fund form)

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PROPOSITION

Σ and $\tilde{\Sigma}$ are locally isometric $\Leftrightarrow$ $\exists$ diffeo $\phi : \underset{\subset \Sigma}{X(\Omega)} \longrightarrow \underset{\subset \tilde{\Sigma}}{\tilde{X}(\Omega)}$ that is an isometry

Proof

"$\Leftarrow$": $\tilde{X} = \phi(X)$

$$\tilde{g}_{ij} = \partial_i \tilde{X} \cdot \partial_j \tilde{X} = \partial_i(\phi(X)) \cdot \partial_j(\phi(X)) = D\phi(\partial_i X) \cdot D\phi(\partial_j X) = \partial_i X \cdot \partial_j X = g_{ij}$$

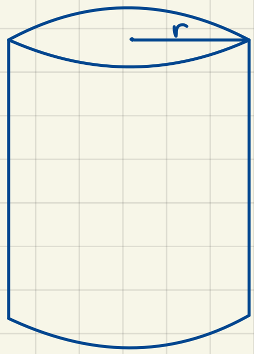
"$\Rightarrow$": Let $\phi = \tilde{X} \circ X^{-1}$, which is a diffeo.

Then,

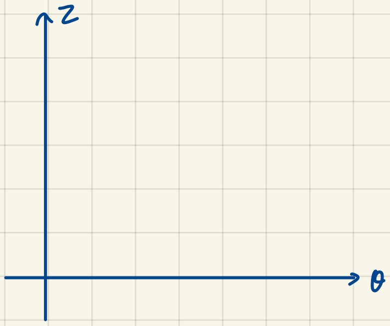
$$\begin{tikzcd} \Sigma \arrow[rr, "\phi"] & & \tilde{\Sigma} \\ & \Omega \arrow[ul, "X"] \arrow[ur, "\tilde{X}"'] & \end{tikzcd}$$

Then, $\underbrace{D\phi(\partial_i X)}_{\partial_i \phi(X) = \partial_i \tilde{X}} \cdot D\phi(\partial_j X) = \partial_i \tilde{X} \cdot \partial_j \tilde{X} = \tilde{g}_{ij} = g_{ij} = \partial_i X \cdot \partial_j X$ $\square$

EXAMPLE



$\cong$



Consider the cylinder: $X = (r\cos\frac{\theta}{r}, r\sin\frac{\theta}{r}, z) = X(\theta, z)$

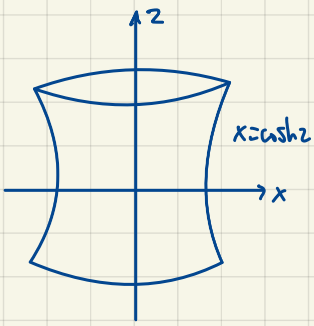
Then, $\partial_\theta X = (-\sin\frac{\theta}{r}, \cos\frac{\theta}{r}, 0)$

$\partial_z X = (0, 0, 1)$

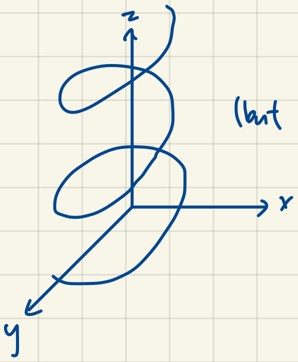
$\Rightarrow g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_{(\theta, z)} \Rightarrow$ isometry!

EXAMPLE

Catenoid $\cong$ Helicoid!! (Aaaaaa my fav example for years QuQ!!)



$\cong$



(but not a line, it's a plane)