Shun/翔海 (@shun4midx)

10-17-25 (WEEK 7)

AREA WITH DIFFERENT PARAMETRIZATION

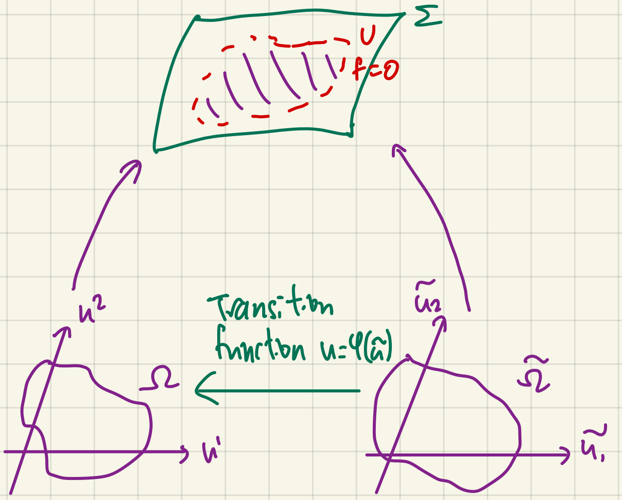
For a surface Σ, $X:\Omega \longrightarrow X(\Omega)=U\subseteq\Sigma$

Then, $\text{Area}(U)=\int_\Omega \underbrace{|\partial_1 X\times\partial_2 X|}_{=\sqrt{\det g}=\sqrt{EG-F^2}} du^1du^2$

$\text{spt } f=\overline{\{f\neq 0\}}$

We usually can write $X=\phi(u)$

For $f:X\to\mathbb{R}$, f is supported in U, $\int_\Sigma f(x)dS=\int_U f(x)dS=\int_\Omega f(\phi(u))\sqrt{\det g(u)}\,du^1du^2$



By change of variables, $\text{Area}=\int_{\tilde\Omega} f(\phi(\varphi(\tilde u))\sqrt{\det g(\varphi(\tilde u))}\left|\det\frac{\partial(u^1,u^2)}{\partial(\tilde u^1,\tilde u^2)}\right| d\tilde u^1 d\tilde u^2$
$=\int_{\tilde\Omega} f(\tilde\phi(\tilde u))\sqrt{\det\tilde g(\tilde u)}\,d\tilde u^1d\tilde u^2$

$\therefore$ Area is indep of param.

In fact, $\tilde g_{kl}(\tilde u)=g_{ij}(\varphi(u))\frac{\partial u^i}{\partial\tilde u^k}\frac{\partial u^j}{\partial\tilde u^l}$

Hence, $\tilde g=\left(\frac{\partial u}{\partial\tilde u}\right)^T g\left(\frac{\partial u}{\partial\tilde u}\right)\Rightarrow\sqrt{\det\tilde g}=\left|\det\frac{\partial u}{\partial\tilde u}\right|\det g$

(Notice this is analogous to $\int_\gamma f(x)ds$ as line integral)

Now, if we define $\phi:\Omega\longrightarrow U\subseteq\Sigma$, another way to look at it is:
$\int_U f(x)dS=\int_\Omega\phi^*(f(x)dS)$
$\phi^*dS=\phi^*dS(\partial_1,\partial_2)\,du^1\wedge du^2=dS(D\phi(\partial_1)\,D\phi(\partial_2))=dS(\partial_1X\,\partial_2X)\,du^1\wedge du^2$

DEFINITION

$f:\Sigma\to\mathbb{R}$ is compactly supported if $\overline{\{f\neq0\}}\subseteq\Sigma$ is a compact set

THEOREM

Given an open cover of Σ, say $\{U_i\}$, $\exists$ a partition of unity $\{\zeta_j:\Sigma\to[0,1]\}$ s.t.
1. $\text{spt }\zeta_j\subseteq U_{i_j}$ for some i_j
2. $\sum_j\zeta_j=1$
3. $\forall$ cpt set K of Σ, $\exists\, m_K\in\mathbb{N}$, s.t. $\zeta_j|_K=0$ $\forall j\geq m_K$

APPLICATION TO AREA

$\text{spt }\zeta_i\subseteq U_i$, $\phi_i:\Omega_i\to U_i\subseteq\Sigma$

Then, $\int_U f\,dS=\int_\Omega\phi^*(f\,dS)=\int_\Omega\phi^*(\sum\zeta_i f\,dS)=\sum_i\int_\Omega\phi^*(\boxed{\zeta_i f}\,dS)=\sum_i\int_{\Omega_i}\phi^*(\zeta_i f\,dS)$

DEFINITION

$f:\Sigma\to\mathbb{R}$ is compactly supported if $\int_\Sigma f\,dS=\sum_i\int_{\Omega_i}\phi_i^*(\zeta_i f\,dS)$

Check for well-defined:

Say $\sum_j\tilde\zeta_j=1$, $\text{spt }\tilde\zeta_j\subseteq\tilde U_j$, then $\sum_j\int_{\tilde\Omega_j}\tilde\phi_i^*(\underbrace{\tilde\zeta_j f\,dS}_{=\sum_i\zeta_i\tilde\zeta_j f\,dS})=\sum_i\sum_j\int_{\tilde\Omega_j}\tilde\phi_j^*(\underbrace{\zeta_i\tilde\zeta_j f\,dS}_{\subseteq U_i})=\sum_i\sum_j\int_{\Omega_i}\phi_i^*(\zeta_i\tilde\zeta_j f\,dS)=\sum_i\int_{\Omega_i}\phi_i^*((\sum_j\tilde\zeta_j)\zeta_i f\,dS)$

Shun/翔海(@shun4midx)

2-FORM

Say η is a 2-form field on S.
Consider the coord nbd $x:\Omega \longrightarrow U \subseteq \Sigma$.

$\phi^*(\eta)(\partial_1, \partial_2)du^1\wedge du^2 = \eta(D\phi(\partial_1)\ D\phi(\partial_2))du^1\wedge du^2 = \eta(\partial_1 X\ \partial_2 X)\,du^1\wedge du^2$

We say $\int_U \eta = \int_\Omega \underline{\phi^*\eta}$

Here, $\int_\Omega \eta(\partial_1 X\ \partial_2 X)\,du^1\wedge du^2 := \int_\Omega \eta(\partial_1 X\ \partial_2 X)\,du^1 \wedge du^2$

$\frac{\partial \tilde{u}^1}{\partial u^2}\tilde{\partial}_1\tilde{X}(\tilde{u}) + \frac{\partial \tilde{u}^2}{\partial u^2}\tilde{\partial}_2\tilde{X}(\tilde{u})$

Change of variables with $u=\varphi(\tilde{u})$, $= \int_{\tilde{\Omega}} \eta(\partial_1 X(\varphi(\tilde{u}))\ \ \partial_2 X(\varphi(\tilde{u})))\left|\det \frac{\partial(u^1,u^2)}{\partial(\tilde{u}^1,\tilde{u}^2)}\right| d\tilde{u}_1 d\tilde{u}_2$

"$\frac{\partial \tilde{u}^1}{\partial u^1}\tilde{\partial}_1\tilde{X}(\tilde{u}) + \frac{\partial \tilde{u}^2}{\partial u^1}\tilde{\partial}_2\tilde{X}(\tilde{u})$

$= \int_{\tilde{\Omega}} \eta(\tilde{\partial}_1\tilde{X}(\tilde{u})\ \ \tilde{\partial}_2(\tilde{X}(\tilde{u}))\ \left|\frac{\partial \tilde{u}^1}{\partial u^1}\frac{\partial \tilde{u}^2}{\partial u^2} - \frac{\partial \tilde{u}^2}{\partial u^1}\frac{\partial \tilde{u}^1}{\partial u^2}\right| \left|\det \frac{\partial u}{\partial \tilde{u}}\right| \partial\tilde{u}$

MIDTERM EXAM

Updated info: Sections tested until Ch2.5 only, and only chapters that appeared in HW
$\Rightarrow$ Ch 1.2, 1.3, 1.4, 1.5, 2.2, 2.3, 2.4, 2.5

REVIEW (RMB THE PROOFS OF THMS TOO)

FRENET FORMULA

$\alpha: I \to \mathbb{R}^3$ curve param by arc length s. (i.e. unit-speed param)
Curvature $k = |\alpha''|$.
Assume $k \neq 0 \Rightarrow t = \alpha'$, $n = \frac{\alpha''}{|\alpha''|}$, $b = t \times n$

$$\begin{pmatrix} t \\ n \\ b \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}}_{\text{skew-symmetric}} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

Non-unit-speed version:
Consider $v(t) = |\alpha'(t)| \Rightarrow t = \frac{\alpha'}{|\alpha'|}$

$$\begin{pmatrix} t \\ n \\ b \end{pmatrix}' = \begin{pmatrix} 0 & kv & 0 \\ -kv & 0 & -\tau v \\ 0 & \tau v & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}$$

Alternatively, for $X(u,v) = ?$, $N = \frac{X_u \times X_v}{|X_u \times X_v|}$.

FUNDAMENTAL THEOREM OF CURVES

Given $k(s) > 0$, $\tau(s)$, $\exists$ curve param by arc length with curvature k, torsion τ unique up to rigid motion
Local canonical form: $\alpha(s) = \alpha(0) + (s - \frac{k(0)^2}{3!}s^3)t(0) + (\frac{k(0)}{2} + \frac{k'(0)}{3!}s^3)n(0) + \frac{k(0)\tau(0)}{3!}s^3 b(0) + o(n^3)$

SURFACES

For $S \subseteq \mathbb{R}^3$,
$\forall p \in S$, $\exists$ param $x: U \longrightarrow V$ s.t. ($U \subseteq \mathbb{R}^2$ open, $V \subseteq \mathbb{R}^3$ open)
1. x is C^k
2. $x: U \to V$ is a homeomorphism
3. $\forall q \in U$, dx_q is 1-1, i.e. $x_u(q) \times x_v(q) \neq 0$

Prop
$x: U \to S$, $y: V \to S$, $W := x(U) \cap y(V) \neq \emptyset$
Then, $x^{-1} \circ y: y^{-1}(W) \to x^{-1}(W)$ is C^k diffeo

REGULAR SURFACES

Prop
$f: U \underset{\text{open}}{\subseteq} \mathbb{R}^3 \to \mathbb{R}$, $a \in \mathbb{R}$: regular value, $f^{-1}(\{a\}) \neq \emptyset$
Then, $f^{-1}(\{a\})$ is a regular surface, e.g. graph of $g: \mathbb{R}^2 \to \mathbb{R}$

Prop
S: regular surface, then $\forall p \in S$, $\exists$ nbd of p that is the graph $z = f(x,y)$, $y = g(x,z)$, or $x = h(y,z)$

Say S: C^k regular surface

Shun/翔海(@shun4midx)

$f: S \to \mathbb{R}$ is C^k if $\forall p \in S$, $\exists$ param $x: U \to S$ with $p \in x(U)$, s.t. $f \circ x: U \to \mathbb{R}$ is C^k.

Restriction of C^k: $f: V \underset{\text{open}}{\subseteq} \mathbb{R}^3 \to \mathbb{R}$ to S is C^k

S_1, S_2: C^k regular surfaces

$\varphi: S_1 \to S_2$ is C^k if φ is cont. and $\forall p \in S$, $\exists$ param $x: U \to S_1$, $y: V \to S_2$, with $p \in x(U)$, $\varphi(x(U)) \subseteq y(V)$, s.t. $y^{-1} \circ \varphi \circ x: U \to V$ is C^k

TANGENT PLANES

$T_pS := \{\alpha'(0) \mid \alpha: (-\varepsilon, \varepsilon) \to S \text{ curve with } \alpha(0) = p\}$

Prop

$x: U \to S$ param $p \in x(U)$, then $T_pS = dx_{x^{-1}(p)}(\mathbb{R}^2)$

$\varphi: S_1 \to S_2$ is C_k, $d\varphi_p: T_pS_1 \longrightarrow T_{\varphi(p)}S_2$

$\alpha'(0) \longmapsto (\varphi \circ \alpha)'(0)$

$x: U \to S_1$, $y: V \to S_2$

Param $\varphi(u,v) = (x(u,v), y(u,v))$

Bases $\{x_u, x_v\}$, $\{y_{\bar{u}}, y_{\bar{v}}\}$ $\quad y^{-1} \circ \varphi \circ x: U \to V$

Matrix of $d\varphi_p$:

$$d\varphi_p = \begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{u}}{\partial v} \\ \frac{\partial \bar{v}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix} (x^{-1}(p))$$

φ: change of variables

☆ Chain rule

Prop

φ: diffeo $\Rightarrow d\varphi_p$: isom

Prop:

$d\varphi_p$: isom $\Rightarrow \varphi$: local diffeo at p

FIRST FUNDAMENTAL FORM

$I_p: T_pS \longrightarrow \mathbb{R}$

$v \longmapsto \langle v, v \rangle$

$E = \langle x_u, x_u \rangle$, $F = \langle x_u, x_v \rangle$, $G = \langle x_v, x_v \rangle$

$I(ax_u + bx_v) = Ea^2 + 2Fab + Gb^2$

$(u,v) \longmapsto \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} = \text{arc-length}$

Applications: $\text{Area} = \int \sqrt{EG - F^2}\, du\, dv$

COMPUTATIONS TO REMEMBER

- Check if S is a regular surface
- Computing first fundamental form $\Rightarrow$ area
- Compute tangent plane at a point
- Calculate t, n, b, τ, κ, v using Frenet

Misc: Osculating plane $= \text{span}\{T(s), N(s)\}$