11-12-25 (WEEK 11)

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VECTOR FIELDS

Let Σ be a surface, V: smooth tangent vector field on Σ ($V^i(u) \in C^\infty\ \forall i$)

$\Rightarrow V(p) \in T_p\Sigma\ \forall p \in \Sigma$

$$V \simeq \underbrace{V^i(u)}_{\text{local coordinate}} \partial_i \overset{\text{parametrization}}{X(u)}$$

Say $\exists$ another local parametrization $\tilde{X}(\tilde{u})$

We know $\exists$ reparametrization $X(\varphi(\tilde{u}))$, i.e. $u = \varphi(\tilde{u})$, φ: transition function

$$\text{Then, } V \simeq V^i(\varphi(\tilde{u})) \underbrace{\frac{\partial X}{\partial u^i}(\varphi(\tilde{u}))}_{= \sum_j \frac{\partial \tilde{X}}{\partial \tilde{u}^j}(\tilde{u}) \frac{\partial \tilde{u}^j}{\partial u^i}(\varphi(\tilde{u}))} = \underbrace{\left(V^i(\varphi(\tilde{u})) \frac{\partial \tilde{u}}{\partial u^i}(\varphi(\tilde{u}))\right)}_{\text{new coefficients}} \frac{\partial \tilde{X}}{\partial \tilde{u}^j}(\tilde{u}) = \tilde{V}^j(u) \frac{\partial \tilde{X}}{\partial \tilde{u}^j}(\tilde{u})$$

DEFINITION

A C^1 curve $\gamma(t)$ is called an **integral curve** of the vector field V if $\gamma'(t) = V(\gamma(t))$

CHANGE OF COORDINATES

For an integral curve $X(t)$,

$$\begin{array}{ccc} \frac{d}{dt}X(t) & \overset{?}{=\!=\!=} & V(X(t)) \\ \Vert & \text{local parametrization} & \Vert \\ \frac{d}{dt}X(u(t)) & \overset{?}{=\!=\!=} & V^i(u(t))\,\partial_i X(u(t)) \\ \Vert & & \\ \partial_i X(u(t)) \frac{du^i}{dt} & & \end{array}$$

$$\Leftrightarrow \frac{du^i}{dt} = V^i(u(t)),\ i \in \{1,2\};\ u(0) = u_0$$

THEOREM

By existence and uniqueness of ODE, $\exists!\ u(t) \in C^k(-\delta,\delta)$, s.t. $\frac{du^i}{dt} = V^i(u(t))$, $i \in \{1,2\}$; $u(0) = u_0$

COROLLARY

$\forall p \in \Sigma$, $\exists!$ integral curve γ of V starting at p

INTEGRAL CURVE VS FLOW LINE

Integral curve is parametrized with direction, a flow line is only the trajectory without info about direction/time

DEFINITION

Two vector fields V and $\tilde{V}$ have the same direction at every point if $\tilde{V}(x) = \lambda(x)V(x)$ for some smooth $\lambda(x) > 0$.

THEOREM

If $V, \tilde{V}$ have the same direction at every point, then they have the same flow lines (need not same integral curves)

Proof

Consider the integral curve of V starting at p,

$$\begin{cases} \frac{d}{dt}(x(t)) = V(x(t)) \\ x(0) = p \end{cases}$$

$\frac{d}{d\tau}(x(t(\tau)) = \frac{d}{dt}(x(t))\big|_{t=t(\tau)} \frac{dt}{d\tau} = V(x(t))\big|_{t=t(\tau)} \cdot \lambda(x(t)) = \tilde{V}(x(t(\tau)))$; $x(t(0)) = x(0) = p$ ✓

$(t = t(\tau) \Rightarrow x(t(\tau))$ is an integral curve of $\tilde{V})$

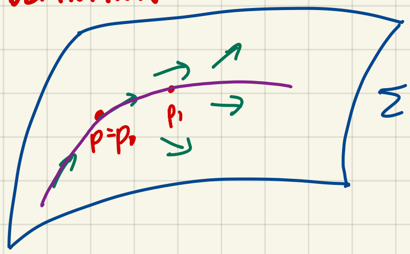
$\Updownarrow$

$$\begin{cases} \frac{dt}{d\tau} = \lambda(x(t)) \\ t(0) = 0 \end{cases}$$

∴ Flow line of $\tilde{V} \subseteq$ flow line of V

With the similar logic for $\tilde{V}(x)=\lambda(x)V(x) \Rightarrow V(x)=\frac{1}{\lambda(x)}\tilde{V}(x)$, we get the "$\supseteq$" direction
∴ Flow line of $\tilde{V}$ = flow line of V □

DEFINITION



we can connect mini integral curve segments together to form smth larger

$\gamma(t)$ is a maximal integral curve of V starting at $p\in\Sigma$ if $\beta'(t)=V(\beta(t)),\ \beta(0)=p,\ t\in(c,d) \Rightarrow (c,d)\subseteq(a,b),\ \gamma|_{(c,d)}=\beta$

PROPOSITION

the flow exists for all time

If the tangent vector field V on Σ has a compact support, then the flow is complete
(i.e. $K=\overline{\{V(x)\neq 0\}}$ is a cpt subset in Σ) ← e.g. Σ is cpt without bdry (closed surface)

Proof

Let $\gamma:(a,b)\to\Sigma$ be a maximal integral curve of V
└ Goal: Extend to $(a,b]$ via limits

Claim: $a=-\infty$, $b=\infty$

Proof

Suppose the contrary that $b<\infty$.
Note: $V(\gamma(t))\neq 0\ \forall t\in(a,b)$

As $\{\gamma(t)\}_{t\in(a,b)} \underset{\text{cpt}}{\subseteq} K$, thus $\exists\, t_i\to b$ as $i\to\infty$, s.t. $\gamma(t_i)\to p\in K$

Claim: $\gamma(t)\to p$ as $t\to b$

Proof

$|\gamma(t')-\gamma(t)| = \left|\int_t^{t'}\gamma'(\tau)\,d\tau\right| = \left|\int_t^{t'} V(\gamma(\tau))\,d\tau\right| \le \int_t^{t'}|V(\gamma(\tau))|\,d\tau \le C|t'-t|$ where $\sup_\Sigma |V|\le C$

∵ $|\gamma(t)-p| \le |\gamma(t)-\gamma(t_i)| + |\gamma(t_i)-p|$
∴ $\gamma(t)\to p$ as $t\to b$ ✓

Define $\tilde{\gamma}(t)=\begin{cases}\gamma(t), & t\in(a,b)\\ p, & t=b\end{cases}$
Then, $\tilde{\gamma}\in C(a,b]$

In fact, $\tilde{\gamma}$ is diffable at b with $\tilde{\gamma}'(b)=V(p) \Rightarrow \tilde{\gamma}'\in C(a,b]$
Proof: $\frac{\tilde{\gamma}^i(t)-\tilde{\gamma}^i(b)}{t-b} = (\gamma^i)'(t_*^i) = V^i(\gamma(t_*^i)) \xrightarrow{t\to b} V^i(p) \Rightarrow \lim_{t\to b}\frac{\tilde{\gamma}(t)-\tilde{\gamma}(b)}{t-b}=V(p)$ ✓

∴ $\tilde{\gamma}'(t)=\begin{cases}\gamma'(t)=V(\tilde{\gamma}(t)), & t\in(a,b)\\ V(p)=V(\tilde{\gamma}(b)), & t=b\end{cases}$ is s.t. $\tilde{\gamma}\in C^1(a,b],\ \tilde{\gamma}'(t)=V(\tilde{\gamma}(t))$, contradicting that γ is max $\times$

DEFINITION

Let $\varphi_t(p) = \gamma(t)$, $p \in \Sigma$, where $\gamma(t)$ is the integral curve of V starting at p

We also write:

$$\begin{aligned} \varphi_t : \Sigma &\longrightarrow \Sigma \\ p &\longmapsto \varphi_t(p) = \varphi(p,t), \end{aligned}$$ where φ is defined as below

$$\begin{aligned} \varphi : \Sigma \times \mathbb{R} &\longrightarrow \Sigma \\ (p,t) &\longmapsto \varphi_t(p) \end{aligned}$$

is called the flow generated by V

REMARK

① $\varphi_0 := \varphi(\cdot, 0) = id$

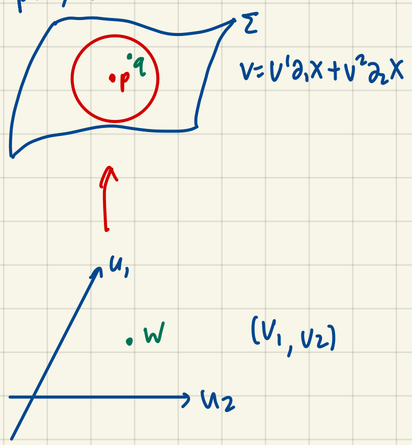
② $\frac{\partial}{\partial t}\varphi(p,t) \overset{?}{=} V(\varphi(p,t))$

$$\begin{array}{ccc} \frac{\partial}{\partial t}\varphi(p,t) & \overset{?}{=} & V(\varphi(p,t)) \\ \| & & \| \\ \frac{\partial}{\partial t}\gamma(t) & = & V(\gamma(t)) \end{array}$$

③ φ is smooth (i.e. locally at each point, the coords are smooth)

Proof

$p \in \Sigma$, $t=0$



$$\begin{cases} \frac{du^i}{dt} = V^i(u(t)) \\ u(0) = w \in \Omega \end{cases} \Rightarrow \begin{aligned} u &= u_w(t),\ t \in (-\delta, \delta) \\ &:= \varphi(w,t) \in C^\infty,\ w \in \Omega \end{aligned}$$

by ODE Thm: since the equations were C^∞, thus ans is C^∞.