9-3-25 (WEEK 1)

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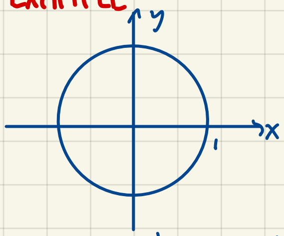
CURVES

DEFINITION

A set $\Gamma \subseteq \mathbb{R}^2$ (or $\mathbb{R}^3$) is called a C^k curve ($k \in \mathbb{N}$) if $\forall p \in \Gamma$, $\exists r > 0$, s.t. $\Gamma \cap B_r(p)$ (neighborhood of p) is a graph of some C^k function

curve is a set of pts

EXAMPLE



Each quadrant corresponds to a C^k graph (We say $\underbrace{\phi(x,y)=0}_{\text{level set}}$, $\phi(x,y) = x^2+y^2-1$, $\nabla\phi = (2x, 2y)$)

THEOREM (IMPLICIT FUNCTION THEOREM)

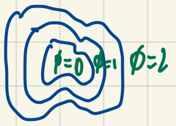
- $\phi \in C^k$ on Ω
- $\nabla \phi \neq 0$ on $\{(x,y) \mid \phi(x,y) = 0\} = \Gamma$
  ($\partial_x \phi$, $\partial_y \phi$)

$\Rightarrow \Gamma$ is a C^k curve

LEVELS AND REGULAR VALUES

Say $\phi(x,y)$:

level sets:



If $\nabla \phi \neq 0$ on $\{\phi = c\}$, then c is a regular valu

INTUITION FOR IFT FOR CURVES (PROOF)

With $\Gamma = \{\phi(x,y) = 0\}$, if we have

- $\phi(x_0, y_0) = 0$
- ϕ is C^k near (x_0, y_0)
- $\nabla\phi(x_0, y_0) \neq 0$

Then, $\exists r > 0$, s.t. $\Gamma \cap B_r(x_0, y_0) = \{y = f(x) \mid f \in C^k\}$ □

EXTEND TO $\mathbb{R}^3$

Say we have the curve as the intersection of two planes

- $\phi(x,y,z) = 0$
- $\psi(x,y,z) = 0$

And also:
- $\phi, \psi \in C^k$
- $\{\nabla\phi, \nabla\psi\}$ is linearly independent along Γ

Then, Γ is a C^k curve

Alternate writing: (doesn't emphasize components)

- $F(x,y,z) = 0$, where $F(x,y,z) = \begin{pmatrix} \phi(x,y,z) \\ \psi(x,y,z) \end{pmatrix}$
- $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
  $(x,y,z) \longmapsto (\phi, \psi)$
- $F \in C^k$
- $\operatorname{rank} DF = 2$, $DF = \begin{pmatrix} \nabla\phi \\ \nabla\psi \end{pmatrix} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is surjective

$\Rightarrow \{F(x,y,z) = \vec{0}\}$

INTUITION FOR $\mathbb{R}^3$ IFT (PROOF)

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Say $\begin{cases} \phi(x_0, y_0, z_0) = 0 \\ \psi(x_0, y_0, z_0) = 0 \end{cases}$

- $\phi, \psi \in C^k$ near (x_0, y_0, z_0)
- $\begin{pmatrix} \phi_x & \phi_y & \phi_z \\ \psi_x & \psi_y & \psi_z \end{pmatrix}\Big|_{(x_0,y_0,z_0)}$ is full rank

Then, as "full rank $\Rightarrow$ det(submatrix) $\neq 0$", we can choose $\begin{pmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{pmatrix}$

$\Rightarrow \exists r > 0$, s.t. $\Gamma \cap B_r(x_0, y_0, z_0) = \{x = f(z),\ y = g(z) \mid f, g \in C^k\}$ $\square$

CURVE PARAMETRIZATION

Say $\Sigma(t) = (x(t), y(t))$, $\Sigma \in C^k$, and $\Sigma'(t) = (x'(t), y'(t)) \neq 0\ \forall t$

Then, $\Gamma = \{\Sigma(t) \mid t \in (a,b)\}$ is a C^k curve

└ Alternatively called a "trace"

(Note, $\Sigma: (a,b) \longrightarrow \mathbb{R}^2$, $t \longmapsto (x,y)$ is an "immersed curve")

┌ interval notation, assume finite

└ may have self intersection

Proof

Fix $t_0 \in (a,b)$. We know $(x'(t_0), y'(t_0)) \neq (0,0)$. WLOG, say $x'(t_0) \neq 0$.

As $x(t) \in C^k$, $-\frac{dx}{dt}\big|_{t=t_0} \neq 0$, by IFT, $\exists\, t = T(x)$ for the neighborhood of $t = t_0$, s.t. $T \in C^k$

<Change of variables>

Now, locally, we can rewrite $\Sigma(T(x)) = (x(t), y(t))|_{t=T(x)} = (x, y(T(x)))$. As $y, T \in C^k$ locally, thus $y \circ T \in C^k$ locally

If we continue the logic for all t, we form the same representation of a C^k curve. $\square$ (Note: This only applies for graphs with no self intersections, i.e. not embedded curves)

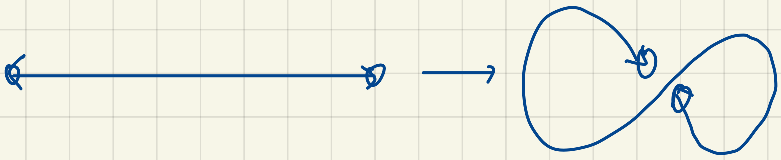
PROPOSITION

┌ counterexample for why not (a,b) later

Furthermore, if $\Sigma: [a,b] \longrightarrow \mathbb{R}^2$ is continuous and 1-1, then $\Gamma = \{\Sigma(t) \mid t \in [a,b]\}$ is an embedded curve

EXAMPLE FOR WHY WE NEED CLOSED INTERVAL AS DOMAIN

If $\Sigma: (a,b) \longrightarrow \mathbb{R}^2$, with



Then, we notice a curve that is 1-1 but still immersed

$\therefore$ We need to write $[a,b]$ not just (a,b)