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DIFFERENTIALS

DEFINITION

The differential at f at p is $df_p : T_p\Sigma \to \mathbb{R}$, which is linear

EXAMPLE

Say $X : \Omega \longrightarrow X(\Omega) \subseteq \Sigma$ is a local parametrization, s.t. $X(u_0) = p$

We define the C^k map $u^i : \underset{\text{open}}{X(\Omega)} \subseteq \Sigma \longrightarrow \mathbb{R}$ as local coord param

Then, $du^i_p(v) = \left(\frac{\partial}{\partial u^1} u^i \circ X|_{u_0}, \frac{\partial}{\partial u^2} u^i \circ X|_{u_0}\right) \cdot \underbrace{\left(\frac{du^1}{dt}(0), \frac{du^2}{dt}(0)\right)}_{} = v^i$ $\overset{(v^1, v^2)}{}$ ($\because$ Say $i = 1$, then $\frac{\partial}{\partial u^1} u^1 \circ X|_{u_0} = 1$, $\frac{\partial}{\partial u^2} u^1 \circ X|_{u_0} = 0$)

$e^*_i(v^1 e_1 + \dots + v^n e_n) = v^i$ by def

Now, as $du^1_p(v) = v^1$, $du^2_p(v) = v^2$, thus the dual basis $= \{v^1, v^2\}$

↳ Note, this is a basis. For $\omega : V \to \mathbb{R}$, $\omega(v) = \omega(\sum_i v^i e_i) = \sum_i v^i \omega(e_i) = \sum_i v^i \omega_i = (\sum_i \omega_i e^*_i)(v)$

EINSTEIN SUMMATION NOTATION

Writing $\sum_i$ is annoying!

So, we can write $\sum_{i=1}^{2} \frac{\partial}{\partial u^i} f \circ X|_{u_0} \frac{du^i}{dt}(0)$ as simply $\frac{\partial}{\partial u^i} f \circ X|_{u_0} \frac{du^i}{dt}(0)$. Here, i is automatically assumed to cycle all i's then sum

In fact, here is some rewritten notation:

$df_p(v) = D_v f(p) = \frac{d}{dt} f(\alpha(t))|_{t=0} = df_{\alpha(0)}(\alpha'(0))$ when eval at $t = 0$.

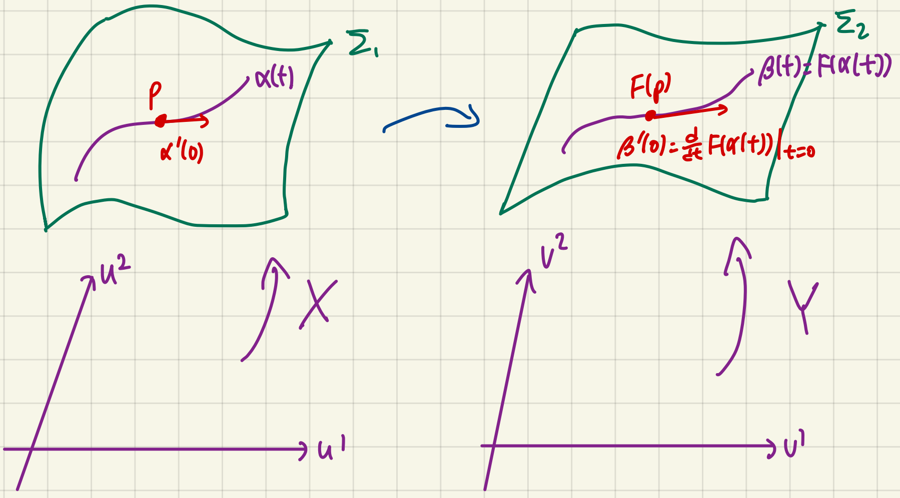
This makes it look more like chain rule.

TANGENT MAPS

DEFINITION

Say $F : \Sigma_1 \to \Sigma_2$ is a C^k map, $p \in \Sigma_1$, $v \in \Sigma_2$, $\alpha : (-\varepsilon, \varepsilon) \to \Sigma$, $\alpha(0) = p$, $\alpha'(0) = v$

We say $D_v F(p) = \frac{d}{dt} F(\alpha(t))|_{t=0}$



We say $dF_p(v) = D_v f(p)$ where $dF_p : T_p\Sigma_1 \longrightarrow T_{F(p)}\Sigma_2$ is also known as the tangent map

In fact, when evaluating, $dF_p(v) = \frac{d}{dt} F \circ \alpha(t)|_{t=0} = \frac{d}{dt}\left(\overset{(v_1, v_2)}{Y} \circ \overset{u \mapsto v}{(Y^{-1} \circ F \circ X)} \circ \overset{(u^1, u^2)}{(X^{-1} \circ \alpha)}\right)(t)|_{t=0}$

$= \frac{\partial Y}{\partial v^j}(v_0) \frac{\partial}{\partial u^i}(Y^{-1} \circ F \circ X)^j|_{u_0} \cdot \frac{du^i}{dt}(0)$

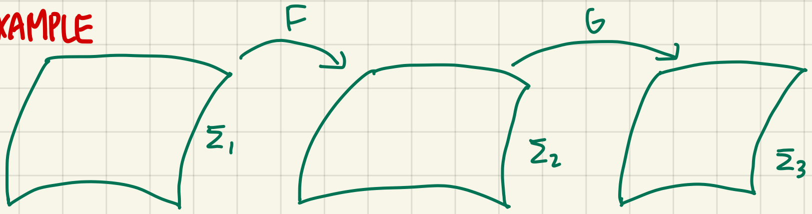
Matrix representation of tangent map

$$= \underbrace{\begin{pmatrix} \frac{\partial Y}{\partial v^1}(v_0) & \frac{\partial Y}{\partial v^2}(v_0) \end{pmatrix}}_{3 \times 2} \boxed{\underbrace{\begin{pmatrix} \frac{\partial}{\partial u^1}(Y^{-1} \circ F \circ X)^1 & \frac{\partial}{\partial u^1}(Y^{-1} \circ F \circ X)^2 \\ \frac{\partial}{\partial u^2}(Y^{-1} \circ F \circ X)^1 & \frac{\partial}{\partial u^2}(Y^{-1} \circ F \circ X)^1 \end{pmatrix}}_{2 \times 2} \underbrace{\begin{pmatrix} \frac{du^1}{dt}(0) \\ \frac{du^2}{dt}(0) \end{pmatrix}}_{2 \times 1}}$$

In fact, define $\hat{F} := Y^{-1} \circ F \circ X : \Omega_u \subseteq \mathbb{R}^2 \longrightarrow \Omega_v \subseteq \mathbb{R}^2$

Then, we can rewrite the middle matrix as the Jacobian matrix: $(\partial_i \hat{F}^j)$

EXAMPLE



Then, $d(G \circ F)_p(v) = \frac{d}{dt} G(F \circ \alpha(t))|_{t=0} = dG_{F(p)}\left(\frac{d}{dt} F(\alpha(t))|_{t=0}\right) = dG_{F(p)} \circ dF_p(v)$

DEFINITION

$F : \Sigma_1 \to \Sigma_2$ is a C^k diffeomorphism if F, F^{-1} are C^k maps and F is a homomorphism

EXAMPLE

$X : \Omega \to \underset{\text{open}}{X(\Omega)} \subseteq \Sigma$ is a diffeomorphism

REMARK

If $F : \Sigma_1 \to F_2$ is a diffeomorphism, then $dF_p : T_p\Sigma_1 \longrightarrow T_{F(p)}\Sigma_2$ is a linear isomorphism

Proof

$F^{-1} \circ F = id \Rightarrow dF^{-1} \circ dF = id$

REMARK

Moreover, if $F : \Sigma_1 \to \Sigma_2$, $p \in \Sigma_1$, $dF_p : T_p\Sigma_1 \to T_{F(p)}\Sigma_2$ is a linear isomorphism, then $F : \Sigma_1 \to \Sigma_2$ is a diffeomorphism (Proof by IFT)