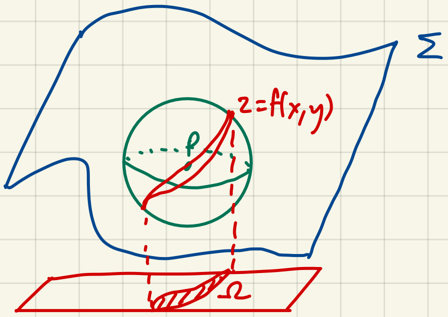
REGULAR SURFACES

DEFINITION

A set Σ in $\mathbb{R}^3$ is called a (C^k embedded) surface if near every point on Σ it is a graph, e.g. $z=f(x,y)$, of some C^k function.



$\Sigma \cap B_r(p) = \{(x,y,z) \mid z=f(x,y),\ (x,y)\in\Omega\subseteq\mathbb{R}^2,\ \Omega: \text{open}\}$

THEOREM

Given $\psi(x,y,z)=0$, if it satisfies the following, then Σ is a surface, for $\Sigma=\{(x,y,z) \mid \psi(x,y,z)=0\}$

① $\psi(x_0,y_0,z_0)=1$ for some (x_0,y_0,z_0)
② $\psi\in C^k$
③ $\nabla\psi(x,y,z)\neq 0$ on Σ

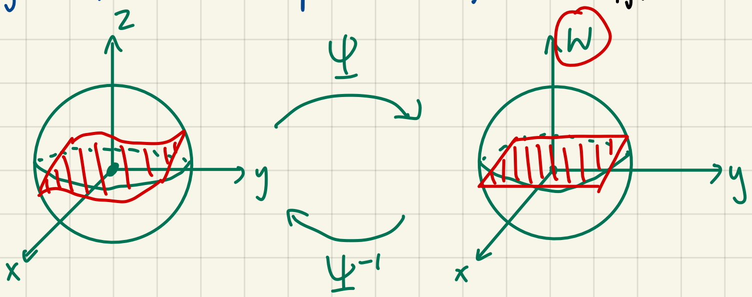
Proof

Fix $(x_0,y_0,z_0)\in\Sigma$, $\frac{\partial\psi}{\partial z}(x_0,y_0,z_0)\neq 0$

Consider $\Psi: (x,y,z) \longmapsto (x, y, \overset{=W}{\underline{\psi(x,y,z)}})$

Then, $D\Psi(x_0,y_0,z_0) = \left.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_1\psi & \partial_2\psi & \partial_3\psi \end{pmatrix}\right|_{(x_0,y_0,z_0)}$ is invertible

By InvFT, $\exists r>0$ and an open set $U\subseteq\mathbb{R}^3$, s.t. $\Psi|_{B_r(x_0,y_0,z_0)}: B_r(x_0,y_0,z_0)\longrightarrow U$ is a C^k diffeomorphism



Then, we obtain $\overbrace{\Psi^{-1}(x,y,w=0)}^{\psi(x,y,z)=0} = (x,y,\underline{z(x,y,w=0)})$ where $z(x,y,0)=f(x,y)$ □

DEFINITION

We notice $\nabla\psi$ is nonvanishing on $\Sigma=\{(x,y,z) \mid \psi(x,y,z)=\overset{(c)}{0}\}$ ⇔ every pt on Σ is not a critical pt of ψ.
In this case, $\overset{(c)}{0}$ is called a regular value of ψ.

REMARK

$\{\psi=c\}$ is a surface, provided c is a regular value of ψ.

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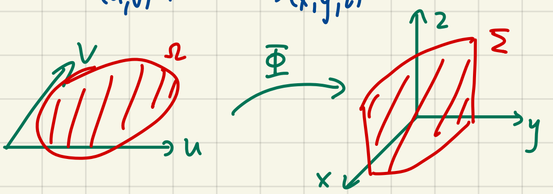
THEOREM

Say $\Phi: \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, and consider $X = \Phi(u,v)$
$(u,v) \longmapsto (x,y,z)$



If we have the following:
① $\Phi \in C^k$
② $\{\partial_u \Phi, \partial_v \Phi\}$ is linearly independent $\Leftrightarrow \partial_u \Phi \times \partial_v \Phi$ is nonvanishing

Then, $\forall (u_0, v_0) \in \Omega$, $\exists r > 0$, s.t. $\Phi(B_r(u_0, v_0))$ is a surface

Remark: It is a surface locally, may not be globally.

Proof

Fix $(u_0, v_0) \in \Omega$, $\Phi(u,v) := (x(u,v), y(u,v), z(u,v))$, where $\partial_u \phi(u_0,v_0) \times \partial_v \phi(u_0, v_0) \neq 0$

Then, $\exists\ \varphi: (u,v) \longmapsto (x(u,v), y(u,v))$
$B_r(u_0, v_0) \longrightarrow \Omega$, φ^{-1}

Define $\Phi_3: (u,v) \longmapsto z$ as given from $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$
Then, $f(x,y) = \varphi_3 \circ \varphi^{-1}(x,y)$ provides an equation for z locally. □

PROPOSITION

Σ is a surface iff $\forall P \in \Sigma$, $\exists$ open nbd U of P in $\mathbb{R}^3$ and a $\boxed{\text{homeomorphism}}$ Φ: open $\Omega \subseteq \mathbb{R}^2 \longrightarrow \Sigma \cap U$ such that $\phi \in C^k$ with $\partial_1 \Phi(u,v) \times \partial_2 \Phi(u,v) \neq 0$ $\forall (u,v) \in \Omega$.

Proof

"$\Rightarrow$": Fix $P \in \Sigma$. $\exists$ open nbd U of P in $\mathbb{R}^3$ s.t. $\Sigma \cap U = \{(x,y,z) \mid z = f(x,y),\ (x,y) \in \Omega \subseteq \mathbb{R}^2 : \text{open}\}$

We define $\Phi: (x,y) \longmapsto (x, y, f(x,y))$, as a continuous bijection
$\Omega \in \mathbb{R}^2 \longrightarrow \Sigma \cap U$
projection

Notice, $\partial_1 \Phi = (1, 0, \partial_1 f)$, $\partial_2 \Phi = (0, 1, \partial_2 f) \Rightarrow \partial_1 \Phi \times \partial_2 \Phi = (-\partial_1 f, -\partial_2 f, 1) \neq 0$ ✓

"$\Leftarrow$": Fix $P \in \Sigma$.
As we know: $\partial_1 \Phi \times \partial_2 \Phi \neq 0$, say its k component is nonzero.

By IFT, $\exists$ open $\tilde{\Omega} \subseteq \Omega$ and open $D \subseteq \mathbb{R}^2$ s.t. $\varphi: \tilde{\Omega} \longrightarrow D$ is a diffeomorphism
$(u,v) \longmapsto (x,y)$

We know $\Phi(\tilde{\Omega}) = (\Sigma \cap U) \cap \overset{\text{open}}{\tilde{U}} = \Sigma \cap \underbrace{(\overset{\text{open}}{U \cap \tilde{U}})}_{\text{new nbd of } P \text{ in } \mathbb{R}^3 : \text{open}}$

$\therefore \overset{\Omega \to \Sigma}{\Phi}(\varphi^{-1}(x,y)) = f(x,y)$ gives Σ near P: $(x, y, f(x,y)) = \overset{\text{mapped via } \Omega \to \Sigma}{\Phi}(\underbrace{\varphi^{-1}(x,y)}_{\text{new } \Omega \text{ coordinate}})$ ✓

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PROPOSITION

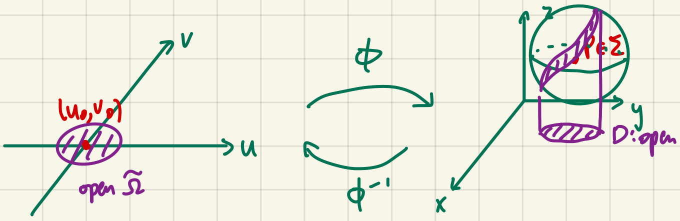
Let Σ be a surface in $\mathbb{R}^3$.
Suppose that $\phi: \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ satisfies:

- $\phi(\Omega) \subseteq \Sigma$
- $\phi \in C^k$
- $\partial_1\phi \times \partial_2\phi \neq 0$ everywhere on Ω
- ϕ is injective

Then, ϕ is a local parametrization of Σ

Proof

We only need to show ϕ is a homeomorphism



$\boxed{\text{continuous ?}}$

We know $\phi(u,v) = (x(u,v), y(u,v), z(u,v))$, where by IFT, $z(u,v) = f(x(u,v), y(u,v))$

Notice, $\partial_u\phi = (x_u, y_u, f_x x_u + f_y y_u) = x_u(1,0,f_x) + y_u(0,1,f_y)$
Similarly, $\partial_v\phi = x_v(1,0,f_x) + y_v(0,1,f_y)$

Jaccobian

Note, $\partial_u\phi \times \partial_v\phi = (x_u y_v - x_v y_u)(1,0,f_x)\times(1,0,f_y) = \det\frac{\partial(x,y)}{\partial(u,v)}(-f_x,-f_y,1) \neq 0$
$\therefore \det\frac{\partial(x,y)}{\partial(u,v)} \neq 0$
$\therefore \phi$ is a diffeo $\square$

REMARK

transition function

Then, as ϕ is diffeo, any parametrization differs from another by a reparametrization with change of parameters
↳ $\phi = F \circ \psi$