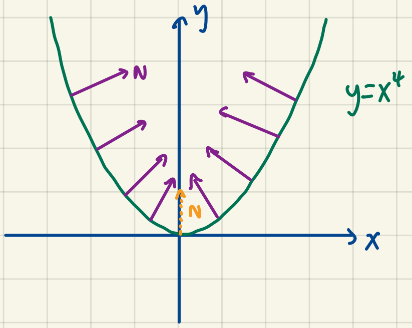
9-19-25 (WEEK 3)

Shun/翔海 (@shun4midx)

WHAT IS UNIT NORMAL IF $\kappa=0$? (SO THAT IT IS SMOOTH AND SATISFIES FS-EQ)

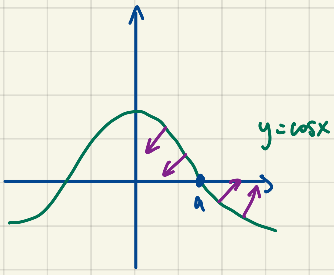
EXAMPLE



Then, $X=(x,x^4)$, $\kappa(x)=\frac{f''(x)}{(1+f'^2(x))^{1.5}} \Rightarrow \kappa(0)=f''(0)=0$

However, notice the $N(x)$ at other points.
Naturally, we can define $N(x)=\frac{(-f'(x),1)}{\sqrt{1+f'(x)^2}}$

EXAMPLE



In this case, we cannot define $N(a)$ so that N is smooth.

EXAMPLE

Say $X(z)=(0,0,z)$, then $\frac{dX}{dz}=(0,0,1)=T$

We can choose $N(z)=(1,0,0)$. Then, $\frac{dT}{dz}=\kappa N(z) \Rightarrow \kappa=0$.
$B(z)=T(z)\times N(z)=(0,1,0) \Rightarrow \tau=0$ (satisfy $\frac{dN}{dz}=-\kappa T-\tau B$, $\frac{dB}{dz}=\tau N$)

We can also choose $N(z)=(\cos z,\sin z,0)$. Then, $\kappa=0$.
$B(z)=T(z)\times N(z)=(-\sin z,\cos z,0) \Rightarrow \frac{dB}{dz}=-N(z) \Rightarrow \tau=-1$

$\therefore$ Torsion depends on how you choose N.

FUNDAMENTAL THEOREM OF SPACE CURVES

Two space curves with the same $\kappa(0)$ and $\tau(0)$ are congruent

Proof

UNIQUENESS

We know we can write $X(s)=\underbrace{R}_{\in SO(3)}(Y(s)-Y(0))+X(0)$ (We denote curve $X \rightsquigarrow$ curve Y)

Then, $R(T_y(0),N_y(0),B_y(0))=(T_x(0),N_x(0),B_x(0))$

$$\therefore R=(T_x(0),N_x(0),B_x(0))\begin{pmatrix} T_y(0)^T \\ N_y(0)^T \\ B_y(0)^T \end{pmatrix}$$

If we denote curve $\tilde{\gamma}$ as any curve satisfying $(T_{\tilde{\gamma}}(0), N_{\tilde{\gamma}}(0), B_{\tilde{\gamma}}(0)) = (T_x(0), N_x(0), B_x(0))$ and $X(0) = \tilde{\gamma}(0)$ Shun/翔海(@shun4midx)

We get
$$\begin{cases} T_{\tilde{y}}(0) = RT_y(0) = T_x(0) \\ N_{\tilde{y}}(0) = RN_y(0) = N_x(0) \\ B_{\tilde{y}}(0) = RB_y(0) = B_x(0) \end{cases}$$

Notice, $\begin{cases} \frac{dT_{\tilde{y}}}{ds} = \kappa N_{\tilde{y}} \\ \frac{dN_{\tilde{y}}}{ds} = -\kappa T_{\tilde{y}} - \tau B_{\tilde{y}} \\ \frac{dB_{\tilde{y}}}{ds} = \tau N_{\tilde{y}} \end{cases}$, which matches the FS-eq for curve X.

$\therefore$ By uniqueness of ODE solution, $(T_x, N_x, B_x) \equiv (T_{\tilde{y}}, N_{\tilde{y}}, B_{\tilde{y}})$

Finally, by FTC, $X(s) = X(0) + \int_0^s T_x(\sigma)d\sigma = \tilde{\gamma}(0) + \int_0^s T_{\tilde{y}}(\sigma)d\sigma = \tilde{\gamma}(s) = R(Y(s) - Y(0)) + X(0)$ □

Given $\kappa(s)$ and $\tau(s)$, there exists a curve with the prescribed curvature and torsion

Proof (EXISTENCE)

We know the curve must satisfy FS-eq,
$$\begin{cases} \frac{dT}{ds} = \kappa N \\ \frac{dN}{ds} = -\kappa T - \tau B \\ \frac{dB}{ds} = \tau N \end{cases}$$

Suppose $T(0) = e_1$, $N(0) = e_2$, $B(0) = e_3$ (initial condition)

Want: $\{T(s), N(s), B(s)\}$ is orthonormal, i.e. $\det(T(s), N(s), B(s)) = 1$

Clearly, $\frac{d}{ds}|T|^2 = 2\frac{dT}{ds}\cdot T = 2\kappa TN$

$\frac{d}{ds}|N|^2 = -2\kappa TN - 2\tau BN$

$\frac{d}{ds}|B|^2 = 2\tau NB$

Note: $\frac{d}{ds}(T\cdot N) = -\kappa|T|^2 + \kappa|N|^2 - \tau TB$

$\frac{d}{ds}(T\cdot B) = \kappa NB + \tau TN$

$\frac{d}{ds}(N\cdot B) = -\kappa TB + \tau|N|^2 - \tau|B|^2$

Closed ODE system

So, we know $\exists$ unique solution for the closed ODE system

However, a trivial solution exists: $|T|^2 = |N|^2 = |B|^2 = 1$, $T\cdot N = T\cdot B = N\cdot B = 0$

$\Rightarrow \det(T(s), N(s), B(s)) = 1$ □

Finally, we know $X(s) := \int_0^s T(\sigma)d\sigma \Rightarrow \frac{dX}{ds} = T(s)$, $\frac{d^2X}{ds^2} = \kappa N$, $B = T\times N$, $\frac{dB}{ds} = \tau N$, so $\exists X(s)$, s.t. T, N, B are as defined. □