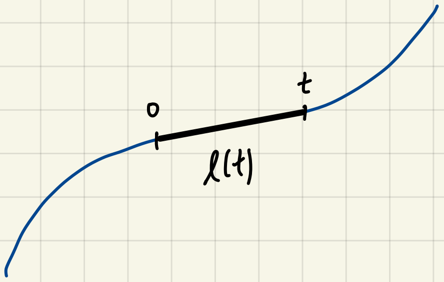
9-10-25 (WEEK 2)

Shun/翔海 (@shun4midx)

UNIT-SPEED PARAMETRIZATION (TANGENT VECTOR)

DERIVING THE UNIT TANGENT VECTOR

For $X(t): [a,b] \longrightarrow \Gamma \subseteq \mathbb{R}^2$, we can define the arclength function $l(t) = \int_0^t |x'(t)|\,dt$. We may also denote $s = l(t)$



In this case, we get $l'(t) = \frac{d}{dt}\int_a^t |x'(\tau)|\,d\tau \overset{FTC}{=} |x'(t)| > 0$ (i.e. it will never stop)

$\therefore$ By IFT, we get $t = l^{-1}(s)$ (many things you just need nonzero derivative $\Rightarrow$ IFT)

Reparametrization: We can rewrite as $X(t)|_{t=l^{-1}(s)} = X(l^{-1}(s))$

$\therefore$ We get $\frac{dX}{ds} = \frac{dX}{dt}\cdot\frac{dt}{ds} = X'(t)/\frac{ds}{dt} = \frac{X'(t)}{|X'(t)|}\Big|_{t=l^{-1}(s)}$, which is a unit vector

This is the unit tangent vector: $T(s) = \frac{dX}{ds} = \frac{X'(t)}{|X'(t)|}\Big|_{t=l^{-1}(s)}$

It is in fact, unit-speed: Notice, $l(t_0) = \int_0^{t_0} \cancel{|x'(t)|}^{\,1}\,dt = t_0$

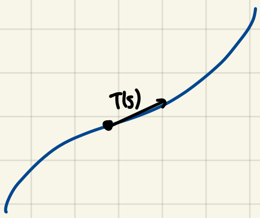
REMARK

If s, c are both unit-speed parametrization, then so is $\tilde{s} = s + c$, i.e. $\frac{d\tilde{s}}{ds} = 1$

OTHER DERIVED QUANTITIES (CURVATURE, NORMAL)

Given $X(s)$ = unit-speed parametrization of Γ, we know the unit tangent vector is $\frac{dX}{ds} = T(s)$

In fact, we can consider $\frac{d^2X}{ds^2} = \frac{dT}{ds}$. Intuitively, the rate of change corresponds to how "curved" the curve is.



DEFINITION

Curvature is defined as $\kappa = \left|\frac{dT}{ds}\right|$ (In space curves, to maintain N dir w.r.t. RH rule, κ may be negative $\Rightarrow$ κ is called the signed curvature)

└ ok definition in plane curves

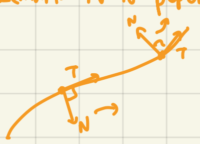
DERIVING NORMAL VECTOR (WLOG, $\kappa \neq 0$)

We get $\frac{dT}{ds} = \left|\frac{dT}{ds}\right| \cdot \left(\frac{dT}{ds} \div \left|\frac{dT}{ds}\right|\right) = \kappa N$, where N here is the unit normal vector

Intuition

We know $|T(s)| = 1 \Rightarrow T(s)\cdot T(s) = 1 \Rightarrow 2T\cdot\frac{dT}{ds} = 0 \Rightarrow \frac{dT}{ds} \perp T$

Remark: N is perpendicular in the direction the curve is curving. It



REPARAMETRIZATION OF T(s)

We know $|T(s)|=1 \ \forall s\in[0,\ell]$
To be honest, we can create the mapping $T:[0,\ell]\longrightarrow S^1\subseteq\mathbb{R}^2$

When $\frac{dT}{ds}\neq 0$, by IFT, $T(s)$ is a local parametrization of S^1

However, we know S^1 is parametrized as $(\cos\theta, \sin\theta)$, so T is only a reparametrization away.
Hence, $T(s)=(\cos\theta(s), \sin\theta(s))$ as a reparametrization

In this case, $\frac{dT}{ds}=\underbrace{(-\sin\theta(s), \cos\theta(s))}_{=N}\underbrace{\frac{d\theta}{ds}}_{=\kappa \text{ (signed curvature)}}$

$\therefore$ We get another representation for signed curvature: $\kappa=\frac{d\theta}{ds}$

CALCULATIONS IN PLANE CURVES

Notice, $\frac{dN}{ds}\perp N$ ($N\cdot N=1\Rightarrow\frac{dN}{ds}\cdot N=0$), so under a plane curve, we know $\frac{dN}{ds}=cT$

Notice, $T\cdot T=1\Rightarrow c=\frac{dN}{ds}\cdot T=\frac{d}{ds}(\cancelto{0}{N\cdot T})-N\cdot\underbrace{\frac{dT}{ds}}_{\kappa N}=-\kappa \quad \therefore \frac{dN}{ds}=-\kappa T$

CURVATURE WITHOUT UNIT-SPEED PARAMETRIZATION

We know reparametrization can be done so that $X(t)=\overline{X}(s(t))$ as a diffeomorphism

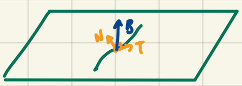
Hence, $\underbrace{\frac{dX}{dt}}_{\text{velocity}}=\frac{dX}{ds}\cdot\frac{ds}{dt}$, assume $\frac{ds}{dt}>0$ (same orientation / orientation-preserving)
We can rewrite it as: $\frac{dX}{dt}=T\cdot v$, $v=\frac{ds}{dt}$ (speed, no direction)

Taking another derivative, we get $\frac{d^2X}{dt^2}=\frac{dv}{dt}T+v\underbrace{\frac{dT}{ds}}_{\kappa N}\underbrace{\frac{ds}{dt}}_{v}=\frac{dv}{dt}T+\kappa v^2N$ (acceleration $=a_{\parallel}+a_{\perp}$)

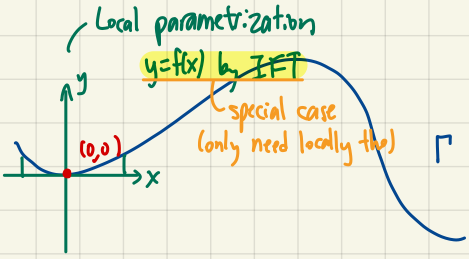
Now, to solve for κ,
Consider $\frac{dX}{dt}\times\frac{d^2X}{dt^2}$, $\underbrace{\frac{dX}{dt}\times\frac{d^2X}{dt^2}}_{\text{rewritten: } \dot{X}\times\ddot{X}}=vT\times(\frac{dv}{dt}T+\kappa v^2N)=vT\times\kappa v^2N=\kappa v^3\, T\times N$

$\therefore |\dot{X}\times\ddot{X}|=|\kappa|v^3\Rightarrow|\kappa|=\frac{|\dot{X}\times\ddot{X}|}{v^3}$ (if we need signed curvature, then use $\dot{X}\times\ddot{X}=\kappa v^3\,T\times N$)

Remark: $T\times N=B$ (binormal unit vector)



SPECIAL CASE OF CURVATURE



Here, $X=(x,y)=(x,f(x))\Rightarrow X'=(1,f'(x))$, $X''=(0,f''(x))$
Then $X'\times X''=(1,f'(x),0)\times(0,f''(x),0)=(0,0,f''(x))$. As in this case, $B=T\times N=(0,0,1)$, thus: $X'\times X''=|\kappa|v^3(0,0,1)$

As $v = |x'| = \sqrt{1+(f')^2}$, thus we have: $\kappa(x) = \frac{f''(x)}{(1+f'(x))^{3/2}}$ (general form for curvature)

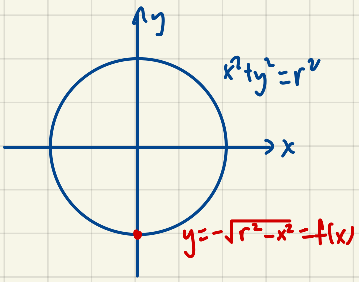
Shun/翔海(@shun4midx)

Now, for the (0,0) on the graph (i.e. any turning point in general),

$$\kappa(0) = \frac{f''(0)}{(1+\cancel{f'(0)})^{3/2}} = f''(0)$$

CALCULATED EXAMPLES

EXAMPLE (CIRCLE)



We see that $f'(x) = \frac{x}{\sqrt{r^2-x^2}}$ and $|\kappa| = |f''(0)| = \left[\frac{1}{\sqrt{r^2-x^2}} - \frac{1}{2}(r^2-x^2)^{-\frac{3}{2}}(2x^2)\right]\Big|_{x=0} = \frac{1}{r}$

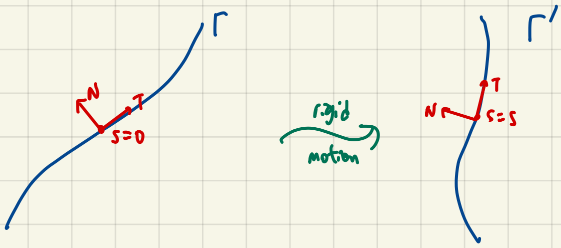
RIGID MOTION PRESERVING CURVATURE

INFORMAL DEFINITION

Rigid motion refers to any series of translations and rotations, i.e. it preserves the shape of the curve.

INFORMAL PROOF

Consider the following rigid motion:



Then, we can derive $\tilde{X}(s) = R(X(s) - X(0)) + \tilde{X}(0)$, where R is an orthogonal matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, which preserves length

Evidently, $\frac{d\tilde{X}}{ds} = R\underbrace{\frac{dX}{ds}}_{T} = RT$ is still a unit vector, where $\tilde{T} = RT$

Moreover, $\frac{d^2\tilde{X}}{ds^2} = R\underbrace{\frac{dT}{ds}}_{\kappa N} = \kappa(RN) = \kappa\tilde{N}$ (since RN is still a unit vector), where $\tilde{N} = RN$

$\therefore$ Rigid motion preserves signed curvature. □