10-1-25 (WEEK 5)

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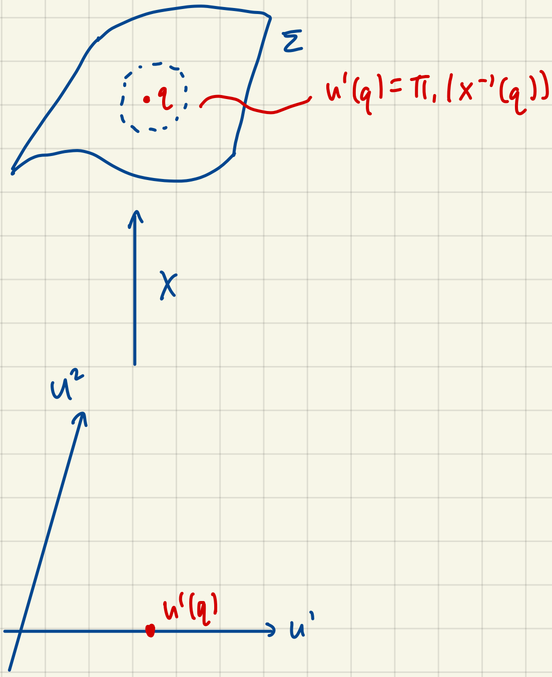
RECALL

Let Σ be a surface in $\mathbb{R}^3$. $f:\Sigma\to\mathbb{R}$ is C^k near $p\in\Sigma$ if $\exists$ local parametrization $X:\Omega\to\Sigma$, $X(u_1,v_1)=p$, s.t. $f\circ X$ is C^k near (u_1,v_1)

EXAMPLE

Local function $u^1=\pi_1\circ X^{-1}$ is a C^k function on the coordinate field:



EXAMPLE

If $f:\mathbb{R}^3\to\mathbb{R}$ is C^k, then $f|_\Sigma:\Sigma\to\mathbb{R}$ is a C^k function on Σ

Moreover, $f|_\Sigma\circ X := f\circ X(u^1,u^2)$ is C^k

For example: $f(x)=|x|^2$.

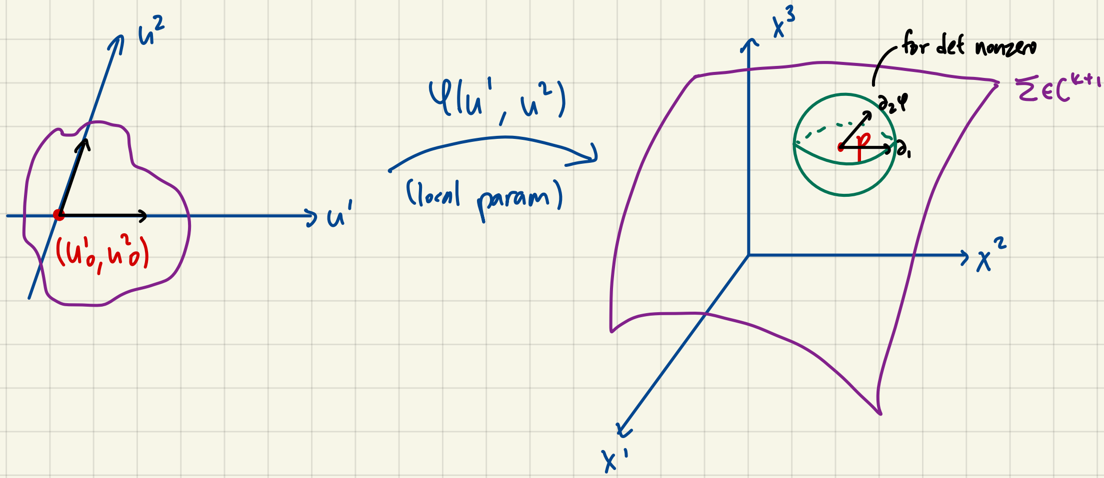
PROPOSITION

$f:\Sigma\to\mathbb{R}$ is C^k near $p \Leftrightarrow \exists r>0$ and $\tilde{f}:B_r(p)\to\mathbb{R}\in C^k$ s.t. $f=\tilde{f}|_\Sigma$ on $\Sigma\cap B_r(p)$

Proof

"$\Leftarrow$": Chain rule from above ✓

"$\Rightarrow$":



Here, $f\circ\varphi$ is C^k.

$X(u^1,u^2,u^3)=\varphi(u^1,u^2)+u^3\partial_1\varphi(u^1,u^2)\times\partial_2\varphi(u^1,u^2)$ is C^k

Note that $X(u^1,u^2,0)=\varphi(u^1,u^2)$

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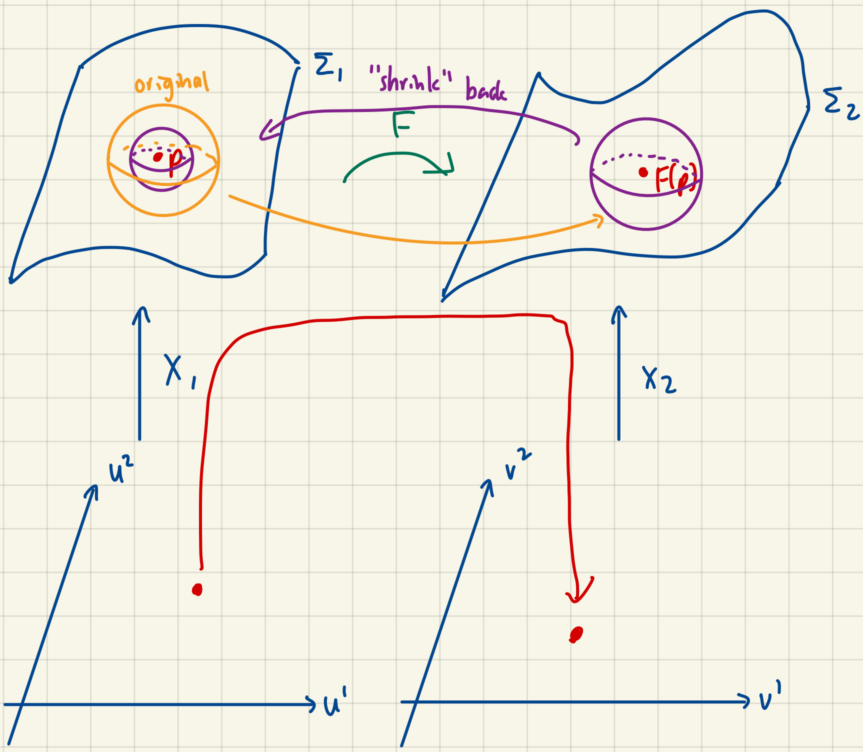
$$\det(\partial_1 X, \partial_2 X, \partial_3 X)\big|_{(u_0^1, u_0^2, 0)} = \det\big(\partial_1\varphi + \cancel{u^3(\partial_1\varphi\times\partial_2\varphi + \partial_1\varphi + \partial_2\varphi)}^{\to 0}, \partial_2\varphi + \cancel{u^2(\dots)}^{\to 0}, \partial_1\varphi \times \partial_2\varphi\big)$$
$$= |\partial_1\varphi(u_0^1, u_0^2) \times \partial_2\varphi(u_0^1, u_0^2)|^2 \underbrace{> 0}_{\neq 0}$$

$\therefore$ We can say $\underbrace{\tilde{f}\circ X(u^1, u^2, u^3) = f\circ\varphi(u^1, u^2)}_{\det\neq 0}$, so f is C^k □

DEFINITION

$F: \Sigma_1 \longrightarrow \Sigma_2$ is C^k near $p \in \Sigma$ if $X_2^{-1}\circ F\circ X_1$ is C^k near (u_0^1, u_0^2) (Note: f_c, X_1, X_2, Σ are cont.)



We consider $\tilde{X}_1: \tilde{\Omega}_1 \to \Sigma_1$, $\tilde{X}_2: \tilde{\Omega}_2 \to \Sigma_2$, then $\tilde{X}_2^{-1}\circ F\circ\tilde{X}_1 = (\tilde{X}_2^{-1}\circ X_2)\circ\underbrace{(X_2^{-1}\circ F\circ X_1)}_{C^k}\circ(X_1^{-1}\circ\tilde{X}_1)$, so it is well-defined across different params.

PROPOSITION

$F: \Sigma_1 \to \Sigma_2$ is C^k near $p \in \Sigma$ iff $\exists$ C^k extension $\tilde{F}: U_1 \to U_2$, where U_1 and U_2 are open sets in $\mathbb{R}^3$ containing Σ_1 and Σ_2 near p and $F(p)$ respectively.

Proof

"$\Leftarrow$": We can view $F: U_1 \supseteq \Sigma_1 \longrightarrow U_2 \supseteq \Sigma_2$ as a C^k map.
We want to show that $\underbrace{Y^{-1}\circ F\circ X}_{\text{"}X\text{"} = X_1,\ \text{"}Y\text{"} = X_2} = \phi$ is a C^k map.

Notice, $F\circ X(u^1, u^2): \Omega_1 \longrightarrow \mathbb{R}^3$ is a C^k map s.t. $F\circ X_1(\Omega_1) \subseteq \Sigma_2$
$\quad\ \ \|$
$(Y_1(u^1, u^2), Y^2(u^1, u^2), \underbrace{Y^3(u^1, u^2)}_{=\, h(Y^1(u^1,u^2),\, Y^2(u^1,u^2))}) \in C^k$

Additionally, $\hat{Y}_1^{-1}\circ F\circ X = (Y^1(u^1, u^2), Y^2(u^1, u^2)) \in C^k$
Thus, $Y^{-1}\circ F\circ X = \underbrace{(Y^{-1}\circ\hat{Y}_1)}_{C^k}\circ(\hat{Y}_1^{-1}\circ F\circ X)$ is C^k. □

"$\Rightarrow$": $F\circ X: \Omega \to \Sigma_2 \subseteq \mathbb{R}^3$ is C^k
$\quad \| $
$Y\circ\underbrace{(Y^{-1}\circ F\circ X)}_{C^k}$

Denote $\phi := Y^{-1}\circ F\circ X$, then we can write $(Y^1\circ\underbrace{\phi(u^1, u^2)}_{=(v^1, v^2)}, Y^2\circ\phi(u^1, u^2), Y^3\circ\phi(u^1, u^2))$ is C^k

We can apply extension on each component, so we finally got it. □

THE TANGENT PLANE

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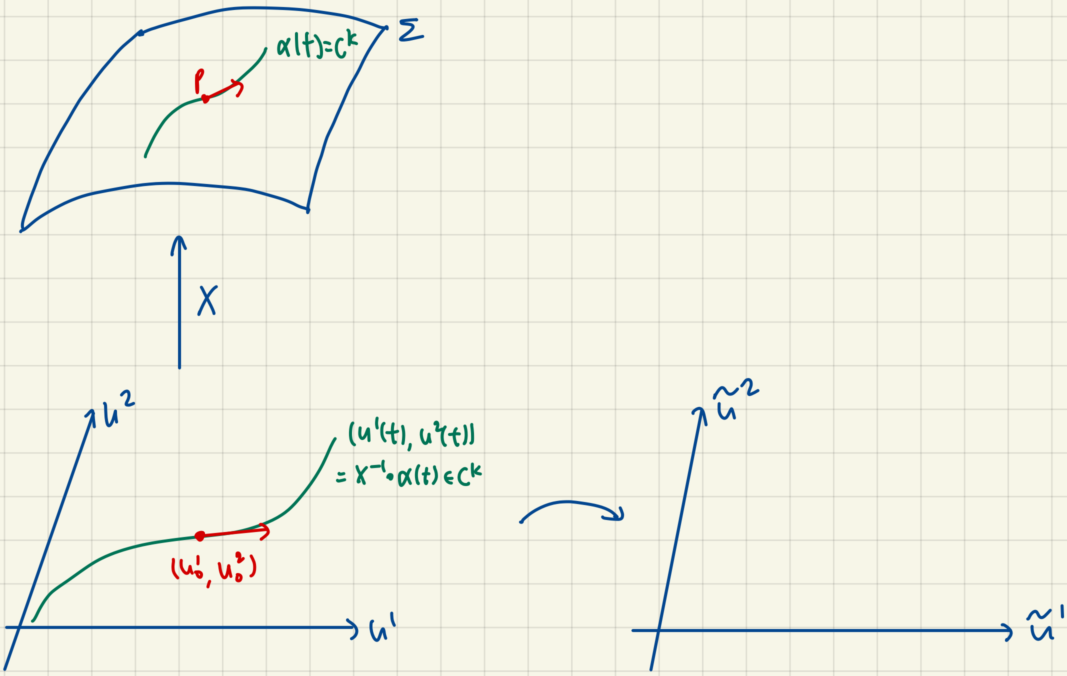
DEFINITIONS/PROPERTIES



$\alpha(t) = X(u^1(t), u^2(t))$

$\alpha'(0) = \partial_1 X(u_0^1, u_0^2)\underbrace{\dot{u}^1(0)} + \partial_2 X(u_0^1, u_0^2)\underbrace{\dot{u}^2(0)}$ components of $\alpha'(0)$ w.r.t. basis $\{\partial_1 X, \partial_2 X\}$

We define $T_p\Sigma = \text{span}\{\partial_1 X(u_0^1, u_0^2), \partial_2 X(u_0^1, u_0^2)\}$ $\simeq \mathbb{R}^2$ as the tangent plane Σ at p.

Moreover, $v = v^1\partial_1 X(u_0^1, u_0^2) + v^2\partial_2 X(u_0^1, u_0^2) \longmapsto \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

CHANGE OF BASIS

$\partial_1 X(u_0) = \frac{\partial X}{\partial u^1}(u_0) = \frac{\partial}{\partial u^1}\left(\tilde{X}\circ(\tilde{X}^{-1}\circ X)\right)\Big|_{u_0} = \boxed{\frac{\partial \tilde{X}}{\partial \tilde{u}^1}(\tilde{u}_0)}\frac{\partial \tilde{u}^1}{\partial u^1}(u_0) + \boxed{\frac{\partial \tilde{X}}{\partial \tilde{u}^2}(\tilde{u}_0)}\frac{\partial \tilde{u}^2}{\partial u^1}(u_0)$

$\Rightarrow (\partial_1 X(u_0), \partial_2 X(u_0)) = (\tilde{\partial}_1 \tilde{X}(\tilde{u}_0), \tilde{\partial}_2 \tilde{X}(\tilde{u}_0))\underbrace{\frac{\partial(\tilde{u}^1, \tilde{u}^2)}{\partial(u^1, u^2)}}$

Jacobian

Moreover, $v = \partial_1 X v^1 + \partial_2 X v^2 = (\partial_1 X, \partial_2 X)\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = (\tilde{\partial}_1 \tilde{X}, \tilde{\partial}_2 \tilde{X})\boxed{\frac{\partial(\tilde{u}^1, \tilde{u}^2)}{\partial(u^1, u^2)}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}} = \begin{pmatrix} \tilde{v}^1 \\ \tilde{v}^2 \end{pmatrix}$

Remark: No matter what coordinates are used to express, they differ only by a matrix multiplication

DIRECTIONAL DERIVATIVE ⇝ DIFFERENTIALS

Say we have $f: \Sigma \to \mathbb{R}$.

Then, $D_v f(p) = \frac{d}{dt}(f(\alpha(t))\Big|_{t=0} = \frac{d}{dt}((f\circ X)(X^{-1}\circ\alpha)(t))\Big|_{t=0}$

$= \frac{\partial}{\partial u^1} f\circ X\Big|_{u_0}\dot{u}^1(0) + \frac{\partial}{\partial u^2} f\circ X\Big|_{u_0}\dot{u}^2(0)$

$= \left(\frac{\partial}{\partial u^1} f\circ X, \frac{\partial}{\partial u^2} f\circ X\right)\Big|_{u_0}\cdot\underbrace{(\dot{u}^1(0), \dot{u}^2(0))}$

directional derivative

components of $\alpha'(0)$ w.r.t. $\{\partial_1 X, \partial_2 X\}$

Here, $v \mapsto D_v f(p) = df_p(v)$ is linear, i.e. $df_p : T_p\Sigma \to \mathbb{R}$ is a linear function

differential

Moreover, consider the dual space: $\{\omega: T_p\Sigma \to \mathbb{R} \text{ is linear}\} = T_p^*\Sigma$ is a 2D lin space.