9-12-25 (WEEK 2)

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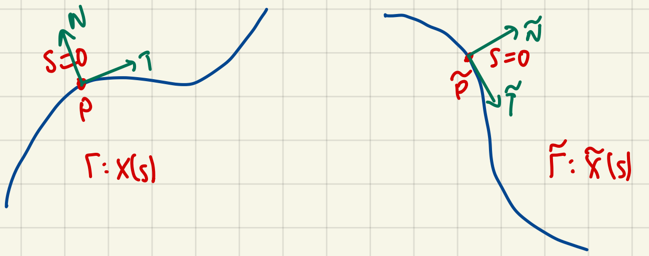
THEOREM

normally we focus on embedded

Two plane immersed curves having the same $\kappa(s)$ are **congruent**



(Γ, $\tilde{\Gamma}$ are unit-speed parametrization)

Proof

We define the rigid motion $\tilde{X}=R(X-P)+\tilde{P}$, where R is the orthogonal matrix s.t. $R\tilde{T}(0)=T(0)$, $R\tilde{N}(0)=N(0)$

WLOG, we may assume that $\tilde{X}(0)=X(0)$, $\tilde{T}(0)=T(0)$, $\tilde{N}(0)=N(0)$ (We want $X(s)=\tilde{X}(s)\ \forall s$)

Recall the **Fernet-Serret equations** for unit-parametrized curves: $\frac{dT}{ds}=\kappa N$, $\frac{dN}{ds}=-\kappa T$. Hence, here we also have $\frac{d\tilde{T}}{ds}=\kappa N$, $\frac{d\tilde{N}}{ds}=-\kappa T$
(Note κ really stands for $\kappa(s)$)

We now have this ODE system:

$$\begin{cases} \frac{dT}{ds}=\kappa(s)N \\ \frac{dN}{ds}=-\kappa(s)T \\ \frac{d\tilde{T}}{ds}=\kappa(s)\tilde{N} \\ \frac{d\tilde{N}}{ds}=-\kappa(s)\tilde{T} \\ \tilde{T}(0)=T(0),\ \tilde{N}(0)=N(0) \end{cases}$$

By the uniqueness thm of ODE, as both sets of T,N equations have the same initial conditions, hence $\tilde{T}(s)=T(s)$, $\tilde{N}(s)=N(s)$

Now, we see that
$$X(s)=\int_0^s X'(\sigma)\,d\sigma+X(0)=\int_0^s T(\sigma)\,d\sigma+X(0)$$
$$\tilde{X}(s)=\int_0^s \tilde{X}'(\sigma)\,d\sigma+\tilde{X}(0)=\int_0^s T(\sigma)\,d\sigma+X(0)$$

$\therefore$ $X(s)=\tilde{X}(s)\ \forall s$ $\square$

COROLLARY

$\kappa\equiv 0 \Leftrightarrow$ straight line, $\kappa\equiv 1 \Leftrightarrow$ unit circle

THEOREM

Given a smooth $\kappa(s)$, there exists an immersed curve **$X(s)$ with curvature $\kappa(s)$**

Proof

By the Fernet-Serret equations, we have $\frac{dT}{ds}=\kappa(s)N$, $\frac{dN}{ds}=-\kappa(s)T$
Assume the initial conditions $T(0)=e_1$, $N(0)=e_2$ ($e_2=(0,1)$, $e_1=(1,0)$)

Then, by solving ODE, we have a solution for $T=T(s)$ and $N=N(s)$. However, we still don't know if T and N are tangent and normal.

- $T\cdot N=?$: Notice, $\frac{d}{ds}(T\cdot N)=\frac{dT}{ds}\cdot N+\frac{dN}{ds}\cdot T=\kappa N\cdot N-\kappa T\cdot T=-\kappa(|T|^2-|N|^2)$
  $\frac{d}{ds}(|T|^2-|N|^2)=2\frac{dT}{ds}T-2\frac{dN}{ds}N=2\kappa NT-2(-\kappa T)N=4\kappa(T\cdot N)$

  $\therefore$ Denote $A=T\cdot N$, $B=|T|^2-|N|^2$, we have the system $\frac{dA}{ds}=-\kappa B$, $\frac{dB}{ds}=4\kappa A$, $A(0)=0$, $B(0)=0$
  As $A\equiv 0$ and $B\equiv 0$ is a solution, by the uniqueness of ODE sol, we have $T\cdot N=0$, $|T|^2-|N|^2=0 \Rightarrow |T|=|N|$
- $|T|=?$: Notice, $\frac{d}{ds}|T|^2=2\frac{dT}{ds}T=2\kappa\cancel{NT}^{0}$ $\therefore$ We know $|T|$ is constant
  As $|T(0)|=1$, hence $|T|=1$. Moreover, by $|T|^2-|N|^2=0$, $|N|=1$

$\therefore$ $\{T,N\}$ is an orthonormal basis of $\mathbb{R}^3$.

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Finally, consider $T \times N \overset{?}{=} (0,0,1)$

↳ We consider $(T \times N) \cdot (0,0,1)$, with the only possible sol we know as $\pm 1\ \forall s$.

We know $[T(0) \times N(0)] \cdot (0,0,1) = 1$. However, as $(T(s) \times N(s)) \cdot (0,0,1)$ is continuous, it cannot equal -1 suddenly.

$\therefore$ $T(s) \times N(s) \equiv (0,0,1)\ \forall s$

Hence, we obtain a unique solution for $T(s)$ and $N(s)$ satisfying tangent and normal properties.

Now, $X(s) := \int_0^s T(\sigma)\, d\sigma \Rightarrow \frac{dX}{ds} = T(s)$: unit-vector

$\therefore$ $X(s)$ is a unit-speed parametrization

unit vector

As $\frac{d^2X}{ds^2} = \frac{dT}{ds} = \kappa N$, hence we deduce $X(s)$ has curvature $\kappa(s)$ □

(Note: $N = \frac{dT}{ds} \div \left|\frac{dT}{ds}\right|$)