10-8-25 (WEEK 6) Shun/翔海 (@shun4midx)

PULL-BACK AND PUSH-FORWARD

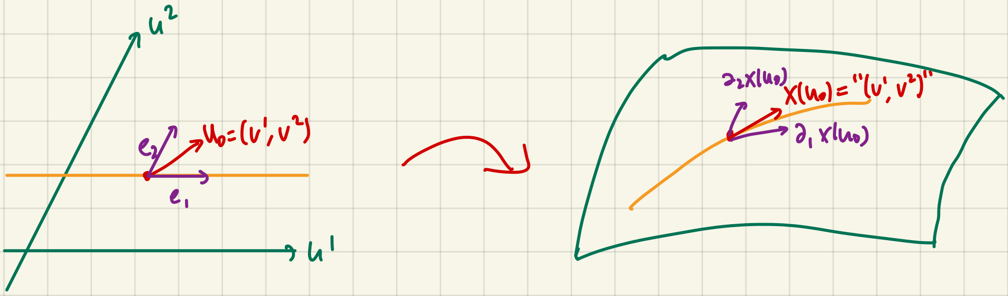
COORDINATE TRANSFORMATION

For a surface Σ, $X:\Omega \longrightarrow X(\Omega)$ open $\subseteq \Sigma$ as a local parametrization is a diffeomorphism $\left(\Rightarrow d(X^{-1})_{X(p)} = (dX)_p^{-1}\right)$

$u_0 \longmapsto p$

$\Rightarrow dX: \underset{\mathbb{R}^2}{\underset{\wr\mid}{T_{u_0}\Omega}} \longrightarrow T_p\Sigma$ is a linear isomorphism

Actually, $v^1e_1+v^2e_2 \longmapsto v^1\partial_1X(u_0)+v^2\partial_2X(u_0)$ as visible below (both are $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$)



Hence, we can simply use the same coordinates

DEFINITION OF PULL-BACK/PUSH-FORWARD

Notation: We write $F^*f=f\circ F$ (dual space), then a $\boxed{\text{pull back is } f^*: T^*_{F(p)}\Sigma_2 \longrightarrow T^*_p\Sigma_1,\ \text{push forward: } df_p: T_p\Sigma_1 \to T_p\Sigma_2}$

REMARK

Consider
$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{F} & \Sigma_2 \\ X\,\cup\!\uparrow & & Y\,\cup\!\uparrow \\ \Omega_1 & \cdots\to & \Omega_2 \end{array}$$

$dF_p: T_p\Sigma_1 \to T_{F(p)}\Sigma_2$, $f:\Sigma_2\to\mathbb{R}$, $F^*f=f\circ F:\Sigma_1\to\mathbb{R}$

Then for $\omega\in T^*_{F(p)}\Sigma_2$ (i.e. $\omega: T_{F(p)}\Sigma_2\to\mathbb{R}$ is a linear function), $F^*\omega_p(\underset{T_p\Sigma_1}{\underset{\cap}{v}}) := \omega_{F(p)}(\underset{T_{F(p)}\Sigma_2}{\underset{\cap}{dF_p(v)}}) \in \mathbb{R}$

We have: $F^*\omega_p(\alpha v_1+\beta v_2)=\alpha F^*\omega_p(v_1)+\beta F^*\omega_p(v_2)$, so $F^*\omega_p\in T^*_p\Sigma_1$ ($F^*: T^*_{F(p)}\Sigma_2\to T^*_p\Sigma_1$ is linear)

PROPERTIES OF PULL-BACK

For $f:\Sigma_2\to\mathbb{R}\Rightarrow df\in T^*_{F(p)}\Sigma_2$, $p\in\Sigma_1$, $F(p)\in\Sigma_2$, $(F^*df)_p(\underset{T_p\Sigma_1}{\underset{\cap}{v}})=(df_{F(p)}\circ dF_p)(v)=d(f\circ F)_p(v)$

Hence, we have this pull-back (*) relationship: $F^*df=d(f\circ F)=dF^*f$

Say $\Sigma_1\xrightarrow{F}\Sigma_2\xrightarrow{G}\Sigma_3$, $f:\Sigma_3\to\mathbb{R}$

Then, $(G\circ F)^*f=f\circ G\circ F=F^*\circ(f\circ G)=F^*\circ G^*\circ f \Rightarrow (G\circ F)^*=F^*\circ G^*$

Notice, this implies $\forall\omega\in T^*_{G\circ F(p)}\Sigma_3$, $G^*\omega\in T^*_{F(p)}\Sigma_2$, $(G\circ F)^*\omega_p(v)=F^*(G^*\omega)_p(v)$ (Similar derivation tbh: $d(F\circ G)_p=dG_{F(p)}\circ dF_p$)

EXAMPLE

Consider
$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{F} & \Sigma_2 \\ X\uparrow s & F & Y\uparrow s \\ \Omega_1 & \xrightarrow{\cong} & \Omega_2 \end{array}$$

Here, for $f:\Sigma_2\to\mathbb{R}$, $F^*df=d(f\circ F)=\frac{\partial}{\partial u^i}(f\circ F)du^i=\partial_jf\,\partial_iF^j du^i$ ($h(u^1,u^2):\Sigma_1\to\mathbb{R}$, then $dh=\frac{\partial h}{\partial u^1}du^1+\frac{\partial h}{\partial u^2}du^2$)

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In particular, take $f = v^k$.
Then, $F^* dv^k = \delta^k_j \partial_i F^j du^i = \partial_i F^* du^i \Rightarrow F^*(\omega) = F^*(\omega_j dv^j) = \omega_j F^*(dv^j) = \omega_j(\partial_i F^j du^i)$

Hence, $\omega_j \longmapsto \partial_i F^j \omega_j$
$\omega \longmapsto \partial_i F^j \omega_j du^i$ $\Rightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \mapsto (dF)^T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$

☆ The pull-back corresponds to the transpose of tangent map!

BILINEAR FORMS AND TENSOR PRODUCTS

DEFINITION

Say we have $f, g \in T_p^*\Sigma$. Then, we define $(f \otimes g)(v, w) := (f(v), g(w))$ for any $v, w \in T_p\Sigma$. This is a bilinear form by def.

In fact, from Linear Algebra (I), we know $f \otimes g$ can be generated by a basis: $\{du^i \otimes du^j\}$
$\Rightarrow T_p^*\Sigma \otimes T_p^*\Sigma = \text{span}\{f \otimes g \mid f, g \in T_p^*\Sigma\}$

PROPERTIES

Say $F: \Sigma_1 \to \Sigma_2$, $\eta \in T_p^*\Sigma_2 \otimes T_p^*\Sigma_2$
Define $F^*\eta(v, w) := \eta(dF(v), dF(w))$, then $F^*: T^*_{F(p)}\Sigma_2 \otimes T^*_{F(p)}\Sigma_2 \longrightarrow T^*_{F(p)}\Sigma_1 \otimes T^*_{F(p)}\Sigma_1 \Rightarrow T^*\eta \in T_p\Sigma_1 \otimes T_p\Sigma_1$

KEY EXAMPLE (INCLUSION MAP)

$$\begin{tikzcd} \Sigma \arrow[r, "i"] & \mathbb{R}^3 \\ \Omega \arrow[u, "X"] \arrow[ur, dotted, "X(u^1,u^2)"'] & \end{tikzcd}$$
(inclusion map)

(u^1, u^2)

Then, $di: T_p\Sigma \longrightarrow T_p\Sigma \subseteq \mathbb{R}^3$
$v^i \partial_i \longmapsto v^i \partial_i X$

FIRST FUNDAMENTAL FORM

Define $h(V, W) := V \cdot W$, $V, W \in \mathbb{R}^3$

We say for $g := i^*h$, $g(V, W) = h(\underbrace{dX(V), dX(W)}_{\text{Supposed to be } di(V),\ di(W)}) = dX(V) \cdot dX(W)$

DEFINITION

$g(V, W) := dX(V) \cdot dX(W)$

Hence, $g(v, w) = v^i w^j g(\partial_i, \partial_j)$, $\because dX(\partial_i) = \partial_i X$
Actually, g is an inner product, i.e. it is a symmetric, positive definite. Also, $(T_{u_0}\Omega, g_{u_0}) \cong (T_p\Sigma, \cdot)$

DEFINITION

We can define $\langle v, w \rangle = (v^1\ v^2) \underbrace{\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}}_{\text{Symmetric positive definite}} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$

In fact, the matrix can be rewritten as: $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$, $E = \partial_1 X \cdot \partial_1 X$, $F = \partial_1 X \cdot \partial_2 X$, $G = \partial_2 X \cdot \partial_2 X$

Moreover, the first fundamental form is defined as $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \mapsto \left\langle \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \right\rangle = (v^1\ v^2) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$

ANOTHER PERSPECTIVE OF ARCLENGTH

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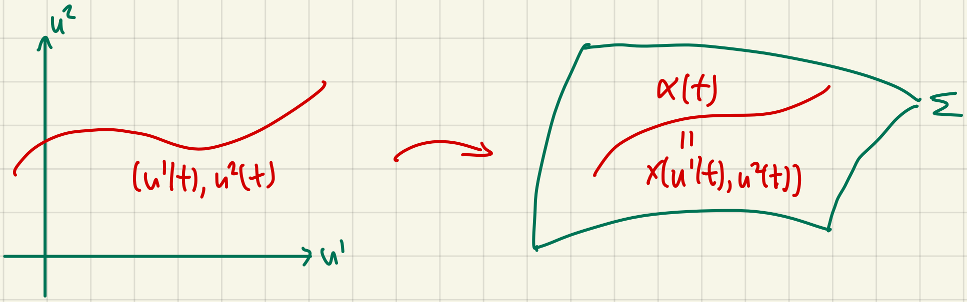

Notice, $\alpha'(t) = \frac{d}{dt}X(u(t)) = \sum \partial_i X(u(t))\,\dot{u}^i(t)$

$$\therefore \int_a^b |\alpha'(t)|\,dt = \int_a^b \sqrt{\sum (\partial_i X \dot{u}^i)(\partial_j X \dot{u}^j)}\,dt = \int_a^b \sqrt{g_{ij}\dot{u}^i\dot{u}^j}\,dt$$

In terms of inner product,

$$\int_a^b |\alpha'(t)|\,dt = \int_a^b \sqrt{\langle \dot{u}, \dot{u}\rangle}\,dt = \int_a^b \overbrace{|\dot{u}|_g}^{\sqrt{(\dot{u}^1)^2+(\dot{u}^2)^2}}\,dt$$

(In fact, the determinant can be used to determine area similarly)