10-15-25 (WEEK 7) Shun/翔海(@shun4midx)

Exam Material:
↳ 4 Questions
↳ All HW related subchapter + remaining of ch 2

THE GRADIENT VECTOR

METRIC TENSOR

Say $T_p\Sigma = \boxed{(T_{u_0}\Omega \cdot g_{u_0})}$, g_{u_0} is a metric tensor for $X: \Omega \longrightarrow X(\Omega) \subseteq \Sigma$
Inner product space; notation; change from Σ to Ω; $u_0 = (u_0^1, u_0^2) \mapsto p$

Then, $v = v^1\partial_1 + v^2\partial_2 \longmapsto v^1\partial_1 X + v^2\partial_2 X$ ($\partial_1 = e_1$, $\partial_2 = e_2$)
$\Rightarrow$ We define $g(v,w) := \langle v, w\rangle = \sum_{i,j}(v^i\partial_i X)(w^j\partial_j X)$

PROPERTIES

$g \cong (g_{ij})_{i,j\in\{1,2\}}$ has a matrix representation $\begin{pmatrix} E & F \\ F & G\end{pmatrix} \Rightarrow$ symmetric positive definite (hence invertible cuz $\begin{pmatrix} E & F \\ F & G\end{pmatrix} \sim \begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$, $\lambda_1, \lambda_2 > 0$)

DEFINITION

Fix $v = v^1\partial_1 + v^2\partial_2 \cong (v^1, v^2)$
$w \mapsto \langle v, w\rangle = v^\flat(w)$
$v \mapsto v^\flat$

$\therefore T_{u_0}\Omega \xrightarrow{\sim} T^*_{u_0}\Omega$ is a linear isomorphism ($v + cv' \mapsto v^\flat + c(v')^\flat$)
$T_{u_0}\Omega = \text{span}\{u^1, u^2\}$, $T^*_{u_0}\Omega = \text{span}\{du^1, du^2\}$

In other words, $\forall \nu \in T^*_{u_0}\Omega$, $\exists!\, v \in T_{u_0}\Omega$, s.t. $\nu(w) = \langle v, w\rangle\ \forall w \in T_{u_0}\Omega$, we say $v^\flat := \nu$

THE GRADIENT VECTOR (DEFINITION)

Hence, we can look at $f: \Sigma \to \mathbb{R}$ as $f: \Omega \to \mathbb{R}$

For $v = v^1\partial_1 + v^2\partial_2$, thus $\boxed{d}f_{u_0}(v) = D_v f = v^1\partial_1 f + v^2\partial_2 f = \boxed{\partial_1 f\, du^1 + \partial_2 f\, du^2(v)}$, so $du^1(v) = v^1$, $du^2(v) = v^2$ (from comparing like terms)
$\therefore$ The gradient vector of f at v_0 is $df_{u_0}(v) = \langle \nabla f_{u_0}, v\rangle$
$\nabla f_{u_0} = a\partial_1 + b\partial_2 \cong (a,b)$

REPRESENTATION VECTOR

Say $\nu(v) = \nu(v^1\partial_1 + v^2\partial_2) = \underbrace{du^1(v)}_{v_1}\nu(\partial_1) + \underbrace{du^2(v)}_{v_2}\nu(\partial_2) = (\nu(\partial_1)du^1 + \nu(\partial_2)du^2)(v)$, where we can define $\nu_i = \nu(\partial_i)$ so $\nu(v) = (\nu_1 du_1 + \nu_2 du_2)(v)$

To solve for ν_1, ν_2,
Notice, $\nu(w) = \langle v, w\rangle\ \forall w \Rightarrow \forall w^j$, $w^j\nu_j = v^i w^j\langle \partial_i, \partial_j\rangle = v^i w^j g_{ij}$ ($v = v^i\partial_i$, $w = w^j\partial_j$)

★ $g_{ij} = g_{ji}$, $g^{ij} = g^{ji}$

$\therefore \nu_j = g_{ij}v^i$, i.e. $\boxed{(\nu_1\ \nu_2) = (v_1\ v_2)\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}}$ (lower indices) $\Rightarrow$ flat / $\flat$ → $\nu = v^\flat$

Moreover, $\boxed{(v_1\ v_2) = (\nu_1\ \nu_2)\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{pmatrix}}$ (raised indices) $\Rightarrow$ sharp / #, where we say $v^i = g^{ij}\nu_j$ → $\nu = v^\#$
Not the natural first fundamental form

Now, $\nabla^i f = g^{ij}\partial_j f \Rightarrow (\nabla^1 f\ \ \nabla^2 f) = (\partial_1 f\ \ \partial_2 f)\begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1}$

IMPLICATION

Thus, $D_v f_{u_0} = D_{\alpha'(0)} f(\alpha(0)) = \boxed{\frac{d}{dt} f(\alpha(t))\big|_{t=0}}$
$\quad \parallel$
$df(v) = df_{u_0}(\alpha'(0)) = \boxed{\langle \nabla f_{u_0}, \alpha'(0)\rangle}$

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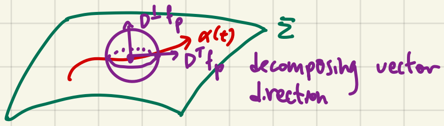
OBSERVATION / REMARK

① $D_v f = \frac{d}{dt} f(\alpha(t))\big|_{t=0} = \underbrace{Df_p}_{(\partial_1 f,\, \partial_2 f,\, \partial_3 f)} \cdot (\alpha'(0)) = Df_p \cdot v$



$\therefore D_v f = Df_p \cdot v = D^T f_p \cdot v$

As we have isometry $T_{u_0}\Omega \xrightarrow{\sim} (T_p\Sigma, \cdot)$,
Hence, $D_v f = D^T f_p \cdot v = \langle \nabla f, v \rangle$, i.e. $D^T f_p = \nabla^1 f \partial_1 x + \nabla^2 f \partial_2 x$

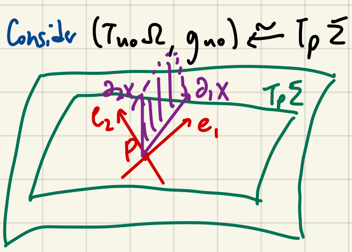
② As $D_v f = \langle \nabla f, v \rangle$, v: unit vector, thus $|\nabla f_{u_0}| = \max\{D_v f_{u_0} \mid |v| = 1\} > 0$
Thus, $\frac{\nabla f_{u_0}}{|\nabla f_{u_0}|}$ is the direction along which the direction derivative is max ($\because$ Cauchy-Schwartz: $|\langle \nabla f, v \rangle| \leq |\nabla f||v|$)

REMARK

By IFT, if $\{f = c\}$ is a curve for regular value c, $df \neq 0 \Rightarrow \nabla f \neq 0$ ($\because \frac{d}{dt} f(\alpha(t))\big|_{t=0} = \langle \nabla_{f(0)} f, \alpha'(0) \rangle \neq 0$)

INTERPRETING AREA

Consider $(T_{u_0}\Omega, g_{u_0}) \xrightarrow{\sim} T_p\Sigma$



We can view the determinant here as our signed area:

$$\begin{vmatrix} \partial_1 x \cdot e_1 & \partial_2 x \cdot e_1 \\ \partial_1 x \cdot e_2 & \partial_2 x \cdot e_2 \end{vmatrix} = \begin{vmatrix} \langle \partial_1 x, e_1 \rangle & \langle \partial_2 x, e_1 \rangle \\ \langle \partial_1 x, e_2 \rangle & \langle \partial_2 x, e_2 \rangle \end{vmatrix} = \begin{vmatrix} e_1^\flat(v) & e_1^\flat(w) \\ e_2^\flat(v) & e_2^\flat(w) \end{vmatrix}$$

DEFINITION

Given $\nu, \omega \in T^*_{u_0}\Omega$, we define the wedge product $\boxed{\nu \wedge \omega := \nu \otimes \omega - \omega \otimes \nu}$
$\Rightarrow \underline{(\nu \wedge \omega)(v, w) = \nu(v)\omega(w) - \omega(v)\nu(w)}$

Note: $\nu \wedge \omega$ is a skew-symmetric bilinear form (2-form)

REWRITING AREA

Hence, area of parallelogram spanned by $v, w = (e_1^\flat \wedge e_2^\flat)(v, w)$
$\Rightarrow$ We call the area form $:= dS = dA = e_1^\flat \wedge e_2^\flat$

REMARK

All 2-forms η can be written as $\eta(\partial_1, \partial_2)\, du^1 \wedge du^2$ (Trivial proof too lazy to write lol)

REWRITING THE AREA FORM

Hence, $dS = dS(\partial_1, \partial_2)\, du^1 \wedge du^2$

Define $A := \begin{bmatrix} \langle e_1, \partial_1 \rangle & \langle e_2, \partial_1 \rangle \\ \langle e_1, \partial_2 \rangle & \langle e_2, \partial_2 \rangle \end{bmatrix} \Rightarrow dS(\partial_1, \partial_2) = \det(A)$

By doing $\partial_i = \langle \partial_i, e_1 \rangle e_1 + \langle \partial_i, e_2 \rangle e_2$, we can do normal dot product:

$$A^T A = \begin{bmatrix} \langle e_1, \partial_1 \rangle & \langle e_1, \partial_2 \rangle \\ \langle e_2, \partial_1 \rangle & \langle e_2, \partial_2 \rangle \end{bmatrix} \begin{bmatrix} \langle e_1, \partial_1 \rangle & \langle e_2, \partial_1 \rangle \\ \langle e_1, \partial_2 \rangle & \langle e_2, \partial_2 \rangle \end{bmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = g$$

$\Rightarrow \det(A^TA) = \det(g)$

$\therefore \det(A) = \pm\sqrt{\det(g)}$

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ORIENTING AREA

Choose: $dS = \sqrt{\det g}\, du^1 \wedge du^2 = e_1^b \wedge e_2^b$

$\hookrightarrow$ dS is oriented by $\{e_1, e_2\}$

AREA FORMULA

For $X: \Omega \longrightarrow X(\Omega) \subseteq \Sigma$, $\mathrm{Area}(X(\Omega)) = \int_\Omega |\partial_1 X \times \partial_2 X|\, du^1 du^2$

Proof Sketch

$|\partial_1 X \times \partial_2 X|^2 = |\partial_1 X|^2 |\partial_2 X|^2 \sin^2\theta = |\partial_1 X|^2 |\partial_2 X|^2 - |\partial_1 X|^2 |\partial_2 X|^2 \cos^2\theta = (\partial_1 X \cdot \partial_1 X)(\partial_2 X \cdot \partial_2 X) - (\partial_1 X \cdot \partial_2 X)^2 = g_{11} g_{22} - g_{12}^2 = \det(g)$

Thus, we have $dS(\partial_1, \partial_2)\, du^1 du^2 = |\partial_1 X \times \partial_2 X|\, du^1 du^2 \Rightarrow$ Obviously, $A = \int_\Omega dS$ $\square$

NEXT TIME'S FOCUS: ORIENTATION