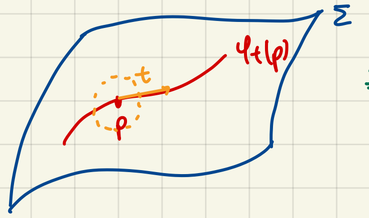
11-14-25 (WEEK 11) Shun/翔海(@shun4midx)

RECALL

Let V be a tangent vector field on Σ.

Given $p \in \Sigma$, let $\varphi_t(p)$ be the pt which starts at p and slides along an integral curve of V for time t.



$\frac{\partial}{\partial t}\varphi_t(p) = V(\varphi_t(p))$

PROPERTIES

① $\varphi_t \circ \varphi_s(p) = \varphi_{t+s}(p)$

$\because \frac{\partial}{\partial t}\varphi_{t+s}(p) = V(\varphi_{t+s}(p))$, $\varphi_{t+s}(p)|_{t=0} = \varphi_s(p)$, $t \mapsto \varphi_{t+s}(p)$

$\therefore \varphi_{t+s}(p) = \varphi_t(\varphi_s(p))$

② $\varphi_t(p)$ is defined for $t \in [0, T] \Rightarrow \exists$ nbd U of p on Σ, s.t. $\varphi_t(q)$ is defined for $t \in [0, T]$ $\forall q \in U$

Proof

$$\begin{cases} \frac{du^i}{dt} = V^i(u(t)), \quad t \in [0, t_1] \\ u(0) = u_0 \Rightarrow \exists \delta > 0 \text{ s.t. } \tilde{u}_0 \in B_\delta(u_0), \sup_{t \in [0,t_1]} |\varphi(u_0, t) - \varphi(\tilde{u}_0, t)| < \varepsilon \text{ if } |\tilde{u}_0 - u_0| < \delta \end{cases}$$

Notice, we can apply ① and ② repeatedly, use ② to extend to new pts, then ① to attach the new section

THEOREM

$\varphi(p,t) = \varphi_t(p)$ is smooth whenever it is defined — Proof

THEOREM

$\varphi_t : U \xrightarrow{\subseteq \Sigma} \Sigma$ is a local diffeomorphism, i.e. φ_t is defined $\Rightarrow \exists$ nbd U of p s.t. $\varphi_t : U \to \varphi_t(U) \subseteq \Sigma$ is a diffeo

Proof

As $\varphi_0 = id$, $q = \varphi_t(p) \Rightarrow p = \varphi_{-t}(q)$ ($\because \varphi_{-t} \circ \varphi_t = id$ near p)

In other words,

φ_t is defined on $U \ni p$, φ_{-t} is defined on $V \ni q \Rightarrow \varphi_t(U) \subseteq V$

$\Rightarrow \varphi_{-t} \circ \varphi_t = id$ on U

$\therefore D\varphi_{-t}(q) \circ D\varphi_t(p) = id$

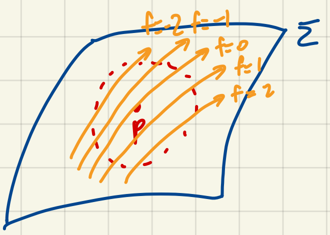
$\llcorner$ invertible by IFT, $\exists$ open set $\tilde{U} \subseteq U$, s.t. $\varphi_t(\tilde{U}) = \tilde{V}$ in an open nbd of q.

THEOREM

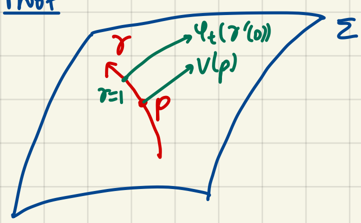
If $V(p) \neq 0$, then $\exists$ open nbd U of p on Σ and a smooth function $f: U \to \mathbb{R}$, s.t.

① $df_x \neq 0$ $\forall x \in U$

② Flow lines of V in U = level curve of f



Proof



$\gamma(0)=p$, $\{\gamma'(0), V(p)\}$: lin indep

Let $X=\varphi_t(\gamma(s))=X(s,t)$, then $X:(-\delta,\delta)\times(-\delta,\delta)\longrightarrow\Sigma$ is smooth

We can rewrite X: $X(s,t)=\varphi(\gamma(s),t)$

↳ $\partial_s X(0,0)=\frac{d}{ds}\gamma(s)\big|_{s=0}=\gamma'(0)$

↳ $\partial_t X(0,0)=\frac{d}{dt}\varphi_t(p)\big|_{t=0}=V(p)$

$\therefore$ By IFT, $X:(-\delta,\delta)\times(-\delta,\delta)\longrightarrow \overbrace{X((-\delta,\delta)^2)}^{U\text{ (open)}}\subseteq\Sigma$ is a diffeomorphism

As s,t: smooth, thus $\{ds, dt\}$: lin indep $\square$

Graphically,