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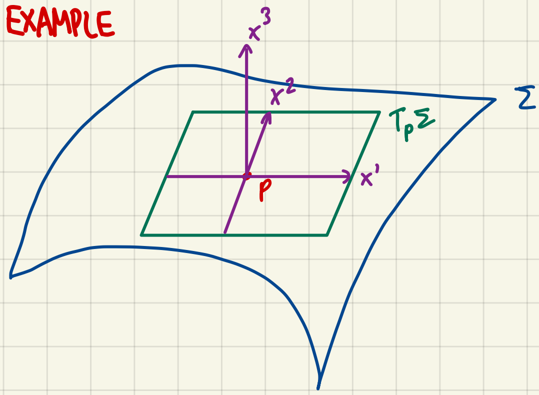
SECOND FUNDAMENTAL FORM CALCULATIONS

EXAMPLE



$\exists$ local param s.t. $x^3 = f(x^1, x^2)$, so $X = (x^1, x^2, f(x^1, x^2))$

$\Rightarrow \partial_1 X = (1, 0, \partial_1 f)$, $\partial_2 X = (0, 1, \partial_2 f)$

$$\therefore\ g = (\partial_i X \cdot \partial_j X)_{ij} = \begin{pmatrix} 1 + (\partial_1 f)^2 & \partial_1 f \partial_2 f \\ \partial_1 f \partial_2 f & 1 + (\partial_2 f)^2 \end{pmatrix}$$

For second fundamental form,

$\partial_{11} X = (0, 0, \partial_{11} f)$, $\partial_{12} X = (0, 0, \partial_{12} f)$, $\partial_{22} X = (0, 0, \partial_{22} f)$

Recall: $N = \dfrac{\partial_1 X \times \partial_2 X}{|\partial_1 X \times \partial_2 X|} = \dfrac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}$

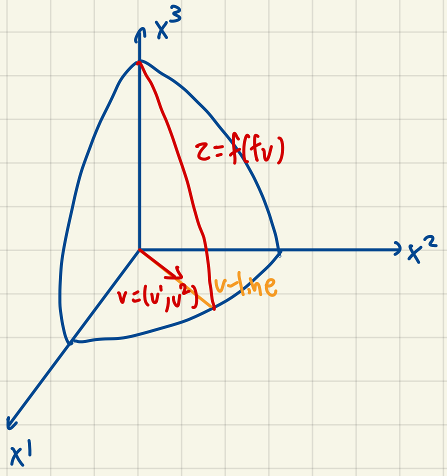
$$\therefore\ A = (\partial_{ij} X \cdot N)_{ij} = \frac{1}{\sqrt{1 + |\nabla f|^2}} \begin{pmatrix} \partial_{11} f & \partial_{12} f \\ \partial_{12} f & \partial_{22} f \end{pmatrix}$$

If we consider a transformed graph s.t. $f(0,0) = 0$, $\nabla f(0,0) = 0$, then:

$$g|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad A|_{(0,0)} = \begin{pmatrix} \partial_{11} f(0) & \partial_{12} f(0) \\ \partial_{12} f(0) & \partial_{22} f(0) \end{pmatrix} = \text{Hessian matrix}$$

EXAMPLE

On the other hand, if we consider $z = f(tv)$



Then, the second directional derivative at $v = (1,0)$, $z = f(t,0)$ is:

$$\frac{d^2}{dt^2} f(tv)\Big|_{t=0} = \frac{d}{dt} \partial_i f(tv) v^i \Big|_{t=0} = \partial_{ji} f(tv) v^j v^i \Big|_{t=0} = A_{ij}(0) v^i v^j$$

$\therefore\ S|_0 = g^{-1} A|_0 = \nabla^2 f(0) \Rightarrow \partial_{11}(0) = k_1,\ \partial_{12}(0) = 0,\ \partial_{22}(0) = k_2$ (rmk: $\underbrace{f(x_1, x_2)}_{f(x)} = \overset{0}{\cancel{f(0)}} + \frac{1}{2} \partial_{ij}(0) x_i x_j + O(|x|^3)$ at max)

↑ diag matrix $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$

UMBILICAL POINT PROPERTIES

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DEFINITION

P is called an umbilical point of Σ if $k_1=k_2 \Leftrightarrow Sv=kv\ \forall v\in T_p\Sigma$, i.e. $S=kI$

PROPOSITION

Σ is (connected) everywhere umbilical ⇔ Σ must be either contained in a plane or a sphere

Proof

We notice $\underset{\text{shape operator}}{Sv=kv} \Rightarrow -\partial_i N = S_i^j \partial_j X \overset{\text{by def expanding } X}{=} k\delta_i^j \partial_j X = k\partial_i X$

∴ $\partial_j(-\partial_i N)=\partial_j(k\partial_i X)$

↳ $\partial_{ji} N = \partial_j k \partial_i X + k\partial_{ij} X$

↳ $\partial_{ij} N = \partial_i k \partial_j X + k\partial_{ji} X$

∴ $\partial_2 k \partial_1 X = \partial_1 k \partial_2 X \Rightarrow \partial_2 k \partial_1 X + \partial_1 k \partial_2 X = \underbrace{0\partial_1 X + 0\partial_2 X}_{\{\partial_1 X, \partial_2 X\}\text{ is a basis}}$

∴ $k=\text{const}$

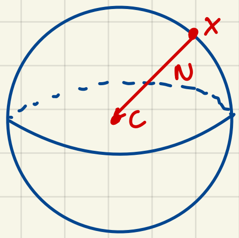
Case 1: $k=0$

$\partial_1 N = \partial_2 N = 0 \Rightarrow N$ is locally const ($\partial_i N = -k\partial_i X$)

∴ Σ is locally contained in a plane

Case 2: WLOG, $k\neq 0$

We want C to be constant here:



$C := X+\frac{1}{k}N \Rightarrow \partial_i C = \partial_i X + \frac{1}{k}(\underbrace{-k\partial_i X}_{-\partial_i N = k\partial_i X}) = 0$

∴ C: const ✓

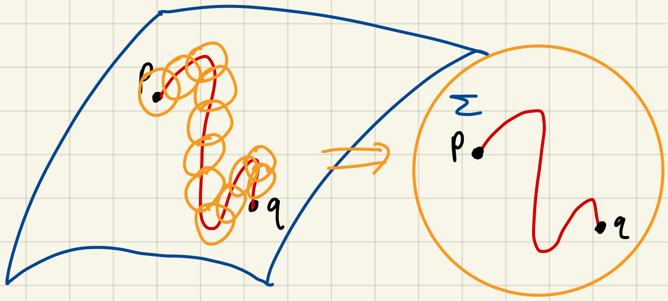
$|X-C| = |\frac{1}{k}N| = \frac{1}{k} = \text{const}$ ✓

∴ Σ is locally contained in a sphere

"Locally contained $\xrightarrow{?}$ Globally contained"

Assume Σ is connected + locally path-connected ⇒ Σ is path-connected

Sketch of final part of the proof:



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THE NON-UMBILICAL CASE

Say P is a non-umbilical point of Σ, i.e. $k_1 < k_2$, k_1, k_2: eigenvalues of S

Notice, k_1, k_2 must satisfy: $\det(\lambda I - S) = 0$

$$\det(\lambda I - S) = (\lambda - k_1)(\lambda - k_2) = \lambda^2 - (k_1 + k_2)\lambda + k_1 k_2 = \lambda^2 - H\lambda + K = 0, \quad H := \operatorname{tr} S,\ K := \det S$$

REMARK

① k_1, k_2 are cont.

② If p is a non-umbilical pt, i.e. $k_1(p) < k_2(p) \Rightarrow k_1, k_2 \in C^\infty$ (locally smooth)

DEFINITION

γ is called a line of curvature (principal curve) if $S(\gamma') = \lambda \gamma'$