Calculus 1B - Lecture 5

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Calculus 1B - Lecture 5

Complex numbers (Thomas' Calculus, Appendix A.7)

Themes:

1. Introduction to complex numbers ▸ Jump to Theme 1
   - Why do we need them?
   - What are they?
   - Arithmetic of complex numbers
2. Representation of complex numbers ▸ Jump to Theme 2
   - Cartesian coordinates
   - Polar coordinates
   - Complex powers of e
3. Solving equations with complex numbers ▸ Jump to Theme 3

Calculus 1B - Lecture 5 (part 1)

Complex numbers (Thomas' Calculus, Appendix A.7)

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Solving first-order DEs

Example 1. Consider the first-order differential equation

$$5y' + 4y = 0\,.$$

- This is a separable differential equation:

$$\begin{aligned} y' &= -\frac{4}{5}y \\ \frac{dy}{dx} &= -\frac{4}{5}y \\ \frac{1}{y}dy &= -\frac{4}{5}dx \end{aligned}$$

- We find the solution (see Lecture 4): $y(x) = Ce^{-\frac{4}{5}x}$.
- Now, we can apply "educated guessing" as a method for solving this type of DEs.

Solving first-order DEs

Example 1. Consider the first-order differential equation

$$5y' + 4y = 0\,.$$

- Try an exponential solution $y(x) = e^{rx}$. Then:

$$\begin{aligned} 5\,(e^{rx})' + 4e^{rx} &= 5re^{rx} + 4e^{rx} \\ &= \big(5r + 4\big)e^{rx}. \end{aligned}$$

- Since e^{rx} is never zero, we find

$$5\,(e^{rx})' + 4e^{rx} = 0$$

  if and only if $5r + 4 = 0$, hence

$$r = -4/5.$$

- We find a solution $\boxed{y(x) = e^{-\frac{4}{5}x}}$

Solving second-order DEs (sneak preview)

Example 2. Consider the **second**-order differential equation

$$y'' + 5y' + 4y = 0\,.$$

- Try an exponential solution $y(x) = e^{rx}$. Then

$$y' = (e^{rx})' = re^{rx} \quad \text{and} \quad y'' = (e^{rx})'' = r^2 e^{rx}$$

- Substituting this in the differential equation gives:

$$\begin{aligned}(e^{rx})'' + 5\,(e^{rx})' + 4e^{rx} &= r^2 e^{rx} + 5re^{rx} + 4e^{rx} \\ &= \left[r^2 + 5r + 4\right] e^{rx}\,.\end{aligned}$$

- Since e^{rx} is never zero, we find

$$(e^{rx})'' + 5\,(e^{rx})' + 4e^{rx} = 0$$

if and only if

$$r^2 + 5r + 4 = 0\,.$$

Solving second-order DEs (sneak preview)

- We found that $y(x) = e^{rx}$ is a solution of the differential equation

$$y'' + 5y' + 4y = 0$$

if and only if

$$r^2 + 5r + 4 = 0\,.$$

- By solving the quadratic equation, we find two roots:

$$r_1 = -1 \text{ and } r_2 = -4\,.$$

- Two solutions of $y'' + 5y' + 4y = 0$ are

$$\boxed{y(x) = e^{-x}} \quad \text{and} \quad \boxed{y(x) = e^{-4x}}$$

Solving second-order DEs (sneak preview)

Example 3. Consider the second-order differential equation

$$y'' + 2y' + 10y = 0.$$

- Substituting $y(x) = e^{rx}$ gives

$$(e^{rx})'' + 2\,(e^{rx})' + 10e^{rx} = \left[r^2 + 2r + 10\right] e^{rx}\,.$$

- Since e^{rx} is never zero, we find that $y(x) = e^{rx}$ is a solution if and only if

$$r^2 + 2r + 10 = 0\,.$$

- But: $b^2 - 4ac = 2^2 - 4 \cdot 10 < 0$, so this equation does not have a real solution! The "educated guessing" method does not seem to work directly in this case.
- We need... complex numbers !

Complex numbers

- In $\mathbb{R}$, there is no number x which satisfies the equation

$$x^2 = -1\,.$$

- We extend our set of numbers by defining i as the solution of the above equation:

$$\boxed{i^2 = -1}$$

- or

$$\text{“}\, i = \sqrt{-1}\, \text{”}$$

Complex numbers

Definition

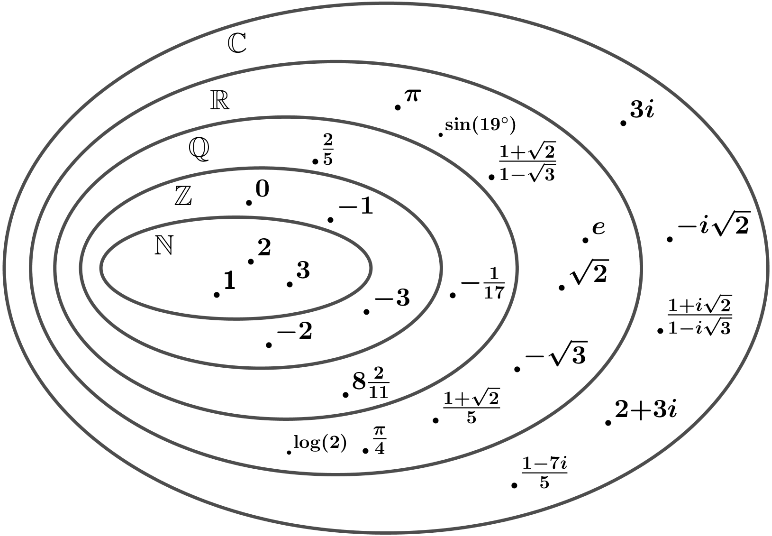
A complex number z is an expression of the form

$$z = a + bi,$$

where a and b are real numbers ($a, b \in \mathbb{R}$).

- Sometimes we write $z = a + ib$ instead of $z = a + bi$.
- As $i \notin \mathbb{R}$, another set of numbers is needed: $\mathbb{C}$, the set of complex numbers.

Complex numbers



$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Arithmetic of complex numbers

We can do calculations with complex numbers in the usual way. Whenever i^2 occurs we substitute -1.

- Addition:
$$(3+4i)+(7-9i)=10-5i.$$
- Subtraction:
$$(3+4i)-(7-9i)=-4+13i.$$

Arithmetic of complex numbers

We can do calculations with complex numbers in the usual way. Whenever i^2 appears, we substitute -1.

- Multiplication:

$$\begin{aligned}(3+4i)(7-2i) &= 3\cdot 7 - 3\cdot 2i + 7\cdot 4i + (4i)(-2i)\\ &= 21 - 6i + 28i - 8i^2\\ &= 21 + 22i - 8i^2\\ &= 21 + 22i - 8\cdot -1\\ &= 21 + 22i + 8\\ &= 29 + 22i.\end{aligned}$$

Arithmetic of complex numbers

We can do calculations with complex numbers in the usual way. Whenever i^2 occurs we substitute -1.

- Division:
  We do not want a complex number in the denominator of a fraction, so we multiply (in a clever way) by... 1:

$$\frac{3+4i}{7-2i} = \frac{3+4i}{7-2i} \cdot \frac{7+2i}{7+2i}$$
$$= \frac{(3+4i)(7+2i)}{(7-2i)(7+2i)}$$
$$= \frac{13+34i}{53}$$
$$= \frac{13}{53} + \frac{34}{53}i$$

Arithmetic of complex numbers

Quiz

If $z = i$ then $\dfrac{1}{z}$ is equal to

(a) i

(b) $-i$

(c) -1

(d) $1 - i$

- Answer (b) is correct

Complex numbers (Thomas' Calculus, Appendix A.7)

Themes:

1. Introduction to complex numbers Jump to Theme 1
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   - What are they?
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2. Representation of complex numbers
   - Cartesian coordinates
   - Polar coordinates
   - Complex powers of e
3. Solving equations with complex numbers Jump to Theme 3

Complex numbers

Recall (see Lecture 5 - part 1):

Definition

A complex number z is an expression of the form

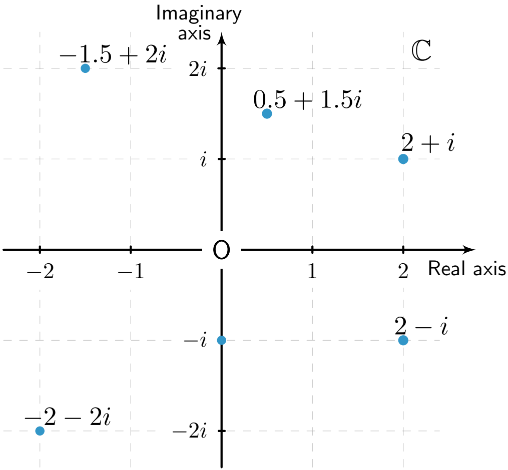
$$z = a + bi,$$

where a and b are real numbers ($a, b \in \mathbb{R}$).

- We sometimes write $z = a + ib$ instead of $z = a + bi$.
- The set of complex numbers is denoted by $\mathbb{C}$.

Cartesian coordinates

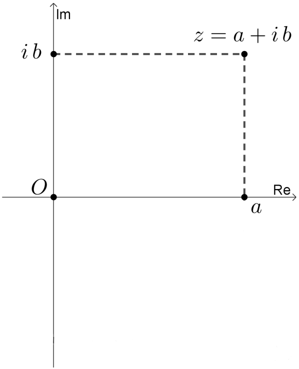
We can regard complex numbers as points in the complex plane.



Cartesian coordinates

If $z = a + i\,b$, then

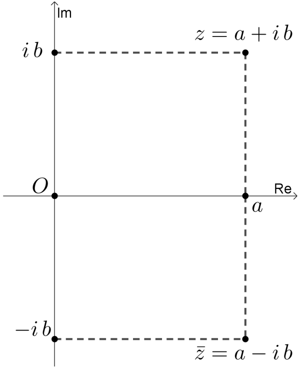
- a is called the real part of z.
- b is called the imaginary part of z.
- We write $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$.



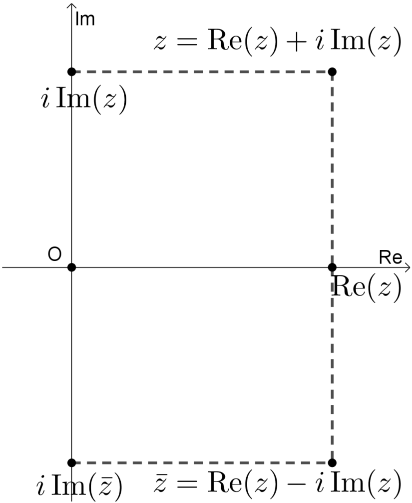
Cartesian coordinates

If $z = a + i\,b$, then

- $\bar{z} = a - i\,b$ is called the complex conjugate $\bar{z}$ of z.
- This can be written as $\bar{z} = \mathrm{Re}(z) - i\ \mathrm{Im}(z)$



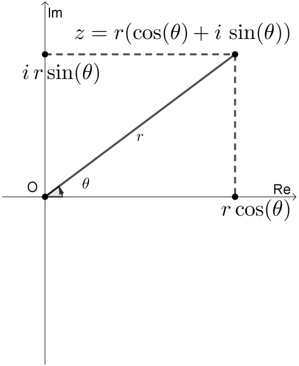
Cartesian coordinates



Polar coordinates

Given $z \in \mathbb{C}$.

- The distance r of z to the origin is the absolute value of z.
- The angle θ with the positive real axis is called the argument of z.
- We write $r = |z|$ and $\theta = \arg(z)$.

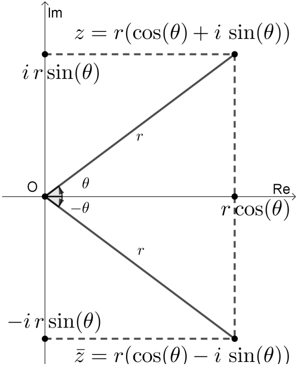


Remark: Adding an integer multiple of 2π to θ gives the same complex number. Usually, we choose θ such that $\theta \in (-\pi, \pi]$ or $\theta \in (0, 2\pi]$.

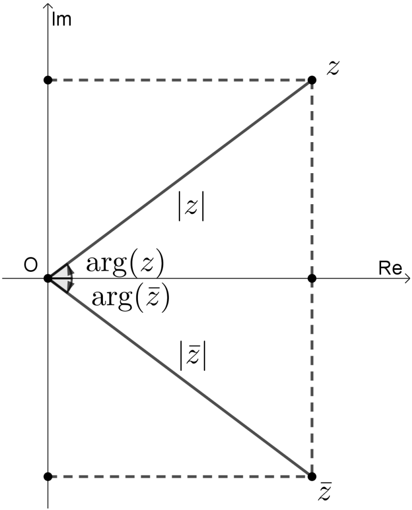
Polar coordinates

For the complex conjugate $\bar{z}$ we have:

- $|\bar{z}| = r$
- $\arg(\bar{z}) = -\theta$



Polar coordinates

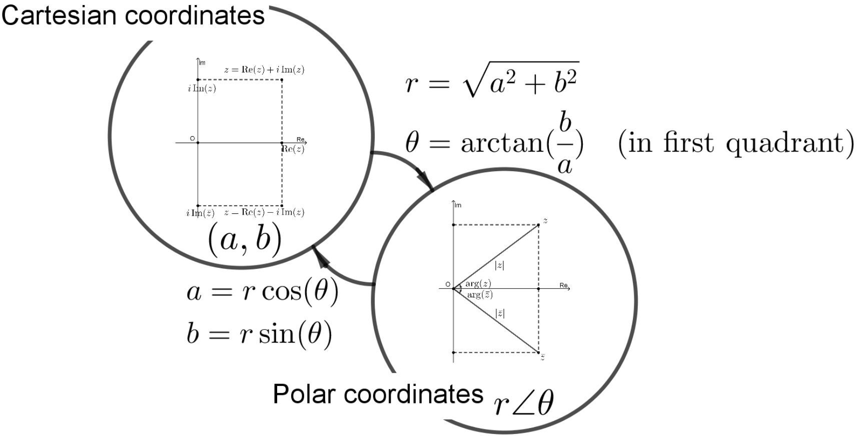


Cartesian vs polar coordinates

Exercise 1. Plot the following complex numbers in the complex plane and determine their polar coordinates:

- $1 + \sqrt{3}\, i$
- $1 - \sqrt{3}\, i$
- $-1 + \sqrt{3}\, i$
- $-1 - \sqrt{3}\, i$

Cartesian vs polar coordinates



Polar coordinates

- Observe that for any complex number $z = a + bi$, we have:

$$\begin{aligned} z \cdot \bar{z} &= (a+bi)(a-bi) \\ &= (a^2+b^2) \\ &= |z|^2 \end{aligned}$$

so

$$\boxed{z \cdot \bar{z} = |z|^2}$$

- Division:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1 \cdot \overline{z_2}}{|z_2|^2}$$

No i in the denominator!

Complex powers of e

Definition

The identity

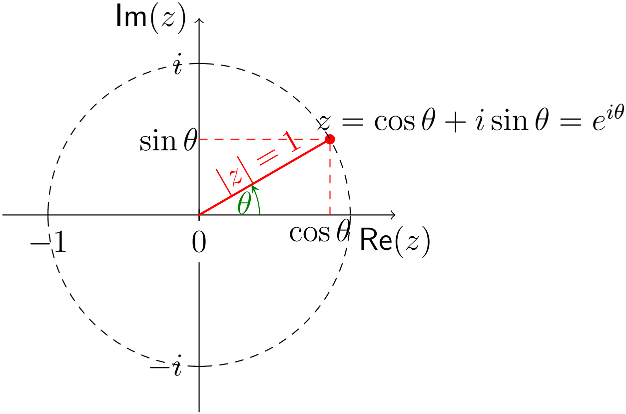
$$\boxed{e^{i\theta} = \cos\theta + i\,\sin\theta}$$

is called Euler's formula.

Complex powers of e

Question

Where is $e^{i\theta}$ located in the complex plane?

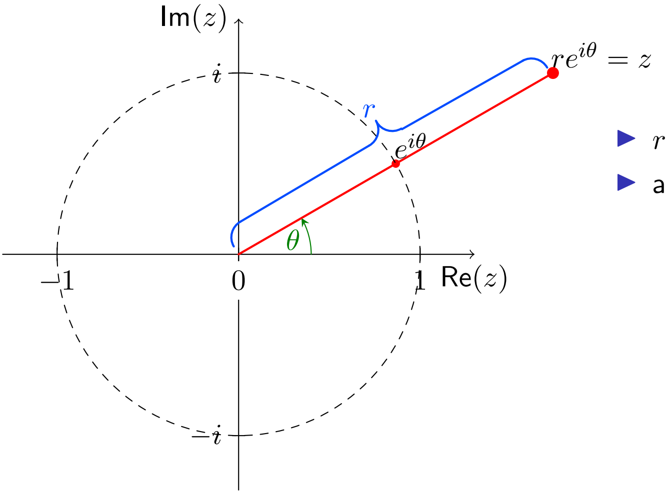


Choose $\theta = \pi$ and we have Euler's Identity: $\boxed{e^{i\pi} + 1 = 0}$

Complex powers of e

Question

Where is $z = re^{i\theta}$ located in the complex plane?



- $r = |z|$
- $\arg(z) = \theta$

Complex powers of e

- The following identity holds for every complex number z:

$$\boxed{z = |z|e^{i\arg(z)}}$$

- So: any $z \in \mathbb{C}$ can be written as a complex power of e.
- Why is this useful?

Complex powers of e: Multiplication

If we multiply two complex numbers z_1 and z_2, we get:

$$\begin{aligned} z_1 \cdot z_2 &= |z_1| e^{i \arg(z_1)} \cdot |z_2| e^{i \arg(z_2)} \\ &= |z_1||z_2| e^{i \arg(z_1) + i \arg(z_2)} \\ &= |z_1||z_2| e^{i\,(\arg(z_1) + \arg(z_2))} \,. \end{aligned}$$

Observe that

- absolute values are multiplied
- arguments are added

Example. If we multiply $z_1 = 4e^{i\pi/3}$ and $z_2 = 6e^{i\pi/4}$, we get:

$$z_1 z_2 = 4e^{i\pi/3} \cdot 6e^{i\pi/4} = 24e^{i(\pi/3+\pi/4)} = 24e^{7/12\, i\pi} \,.$$

Complex powers of e: Multiplication

Example. Let $z = 1 + \sqrt{3}i$. Write the complex number z^5 in the form $x + yi$.

- Option 1: 'Direct' calculation:

$$\begin{aligned} z^5 &= (1 + \sqrt{3}i)^5 \\ &= \dots \\ &= 16 - 16\sqrt{3}\, i \end{aligned}$$

- Option 2: Using the fact that $(1 + \sqrt{3}i) = 2e^{\frac{\pi}{3}i}$ (check!)[1]:

$$\begin{aligned} z^5 &= \left(2e^{\frac{\pi}{3}i}\right)^5 = 2^5 e^{\frac{5\pi}{3}i} = 32e^{\frac{5\pi}{3}i} \\ &= 32(\cos(\tfrac{5\pi}{3}) + i\sin(\tfrac{5\pi}{3})) = 32(\tfrac{1}{2} + i(-\tfrac{\sqrt{3}}{2})) \\ &= 16 - 16\sqrt{3}\, i \end{aligned}$$

[1] see Exercise 1 on a previous slide

Complex powers of e: Division

If we divide two complex numbers z_1 and z_2, we get:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{|z_1|e^{i\arg(z_1)}}{|z_2|e^{i\arg(z_2)}} \\ &= \frac{|z_1|}{|z_2|}e^{i\arg(z_1)-i\arg(z_2)} \\ &= \frac{|z_1|}{|z_2|}e^{i\,(\arg(z_1)-\arg(z_2))}\,.\end{aligned}$$

Observe that

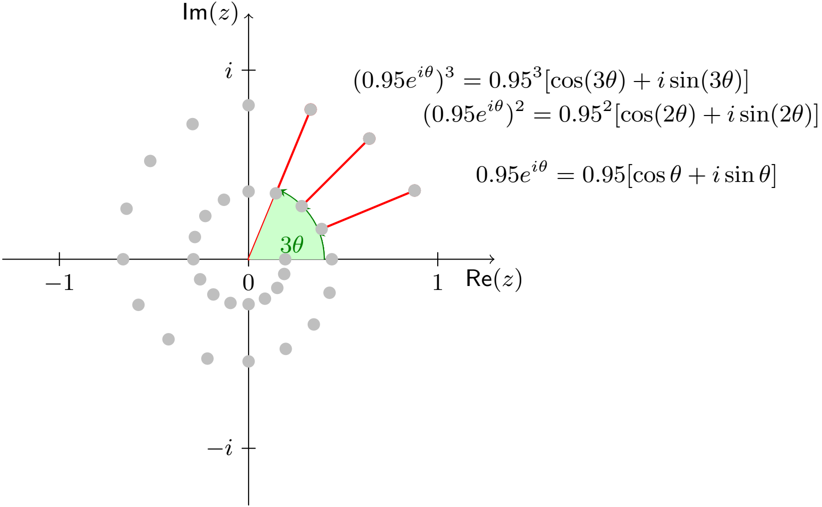
- absolute values are divided
- arguments are subtracted

Example. If we divide $z_1 = 4e^{i\pi/3}$ and $z_2 = 6e^{i\pi/4}$, we get:

$$\frac{z_1}{z_2} = \frac{4e^{i\pi/3}}{6e^{i\pi/4}} = \tfrac{4}{6}e^{i(\pi/3-\pi/4)} = \tfrac{2}{3}e^{i\pi/12}\,.$$

Complex powers of e: De Moivre's Theorem

Example. Consider powers of a complex number $z = 0.95e^{i\theta}$:



Complex powers of e: De Moivre's Theorem

- We see that

$$(\cos\theta + i\sin\theta)^2 = e^{i\theta}e^{i\theta} = e^{i2\theta} = \cos(2\theta) + i\sin(2\theta)$$

- This leads to De Moivre's Theorem :

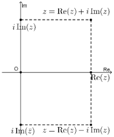
Theorem

For $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$ we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Overview

Cartesian coordinates



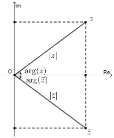
(a, b)

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right) \quad \text{(in first quadrant)}$$

$$a = r\cos(\theta)$$

$$b = r\sin(\theta)$$



Polar coordinates $r\angle\theta$

Complex power of e

$$z = r\,e^{i\,\theta}$$

$$\bar{z} = r\,e^{-i\,\theta}$$

Euler

$$e^{i\,\theta} = \cos\theta + i\,\sin\theta$$

$(\cdot)^n$

De Moivre

$$\cos(n\,\theta) + i\,\sin(n\,\theta) = (\cos\theta + i\,\sin\theta)^n$$

Complex numbers (Thomas' Calculus, Appendix A.7)

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Jump to Theme 1

Jump to Theme 2

The quadratic formula

Theorem

For any real numbers a, b, c with $a \neq 0$, the solutions $z \in \mathbb{C}$ to the equation $az^2 + bz + c = 0$ are given by

$$\boxed{z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

- This formula is called the quadratic formula.

The quadratic formula

$$\boxed{z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

- If $b^2 - 4ac > 0$, then the quadratic formula yields two *real* solutions.
- If $b^2 - 4ac = 0$, then the quadratic formula yields one *real* solution.
- If $b^2 - 4ac < 0$, then the quadratic formula yields two *non-real* complex solutions:
  - in that case, we have $b^2 - 4ac = -|b^2 - 4ac| = i^2|b^2 - 4ac|$, so

$$z_{1,2} = \frac{-b \pm i\sqrt{|b^2 - 4ac|}}{2a}\,.$$

The quadratic formula

Example 1.

Find the solutions in $\mathbb{C}$ of $z^2 + 2z + 10 = 0$.

- By the quadratic formula, the solutions are:

$$\begin{aligned} z_{1,2} &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} \\ &= \frac{-2 \pm \sqrt{-36}}{2} \\ &= \frac{-2 \pm i\sqrt{36}}{2} \\ &= -1 \pm 3i, \end{aligned}$$

  in other words: $z_1 = -1 + 3i$ and $z_2 = -1 - 3i$.

- Note that $\overline{z_1} = z_2$.

Equality of complex numbers

Question

When are two complex numbers z_1 and z_2 equal to one another?

- This seems a trivial question:

$$z_1 = z_2 \Leftrightarrow \mathrm{Re}(z_1) = \mathrm{Re}(z_2) \wedge \mathrm{Im}(z_1) = \mathrm{Im}(z_2)\,.$$

- However, recall that:

$$z_1 = z_2 \Leftrightarrow |z_1| = |z_2| \wedge \arg(z_1) = \arg(z_2) + k \cdot 2\pi \text{ and } k \in \mathbb{Z}\,.$$

- The argument of a complex number is unique except for an integer multiple of 2π.

Roots of complex numbers

Example 2.

Find all complex numbers z such that $z^3 = 8$.

- Write $z = re^{i\theta}$, then
$$z^3 = r^3 e^{i3\theta} = 8.$$
- This holds if and only if
$$|r^3 e^{i3\theta}| = |8| \quad \text{and} \quad \arg(r^3 e^{i3\theta}) = \arg(8) + k \cdot 2\pi \quad (k \in \mathbb{Z})$$
-
$$r^3 = 8 \quad \text{and} \quad 3\theta = 0 + k \cdot 2\pi \quad (k \in \mathbb{Z})$$
-
$$r = \sqrt[3]{8} = 2 \quad \text{and} \quad \theta = k \cdot \frac{2\pi}{3} \quad (k \in \mathbb{Z})$$
- If we would require that $\theta \in [0, 2\pi)$, we have:
$$\theta = 0, \quad \theta = \tfrac{2\pi}{3}, \quad \text{or} \quad \theta = \tfrac{4\pi}{3}.$$

Roots of complex numbers

Example 2.

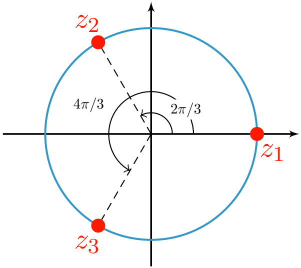
Find all complex numbers z such that $z^3 = 8$.

The solutions to $z^3 = 8$ are

$$z_1 = 2e^{i0} = 2$$

$$\begin{aligned} z_2 &= 2e^{i\frac{2\pi}{3}} \\ &= 2(\cos(\tfrac{2\pi}{3}) + i\,\sin(\tfrac{2\pi}{3})) \\ &= -1 + i\sqrt{3} \end{aligned}$$

$$\begin{aligned} z_3 &= 2e^{i\frac{4\pi}{3}} \\ &= 2(\cos(\tfrac{4\pi}{3}) + i\,\sin(\tfrac{4\pi}{3})) \\ &= -1 - i\sqrt{3} \end{aligned}$$



Roots of complex numbers

Problem

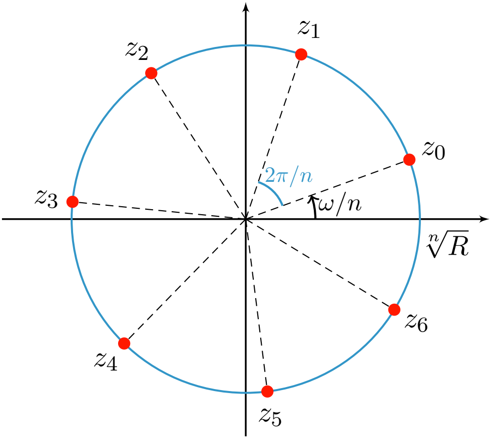
Let n be a positive integer and let c be an arbitrary complex number, unequal to 0. Find all z such that $z^n = c$.

- Write $z = r\,e^{i\theta}$ and $c = R\,e^{i\omega}$, then $z^n = c$ can be written as
$$r^n\,e^{in\theta} = R\,e^{i\omega}.$$
- This equation holds if and only if
$$|r^n\,e^{in\theta}| = |R\,e^{i\omega}| \quad \text{and} \quad \arg(r^n\,e^{in\theta}) = \arg(R\,e^{i\omega}) + k\cdot 2\pi$$
$$r^n = R \quad \text{and} \quad n\theta = \omega + k\cdot 2\pi$$
$$r = \sqrt[n]{R} \quad \text{and} \quad \theta = \frac{\omega}{n} + k\cdot\frac{2\pi}{n}$$
- The solutions of $z^n = c$ (i.e., the nth roots of c) are:
$$\boxed{z_k = \sqrt[n]{R}\,e^{i\left(\frac{\omega}{n}+\frac{2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1}$$

Roots of complex numbers

The solutions of $z^n = c = R\,e^{i\omega}$ are

$$\boxed{z_k = \sqrt[n]{R}\,e^{i\left(\frac{\omega}{n}+\frac{2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1}$$



Roots of complex numbers

Quiz

What are the solutions of $z^4 = 16$?

(a) $z_1 = 2$, $z_2 = -2$

(b) $z_1 = 2$, $z_2 = -2$, $z_3 = 2i$, $z_4 = -2i$

(c) $z_1 = 2$

(d) $z_1 = 2 + 2i$, $z_2 = 2 - 2i$, $z_3 = -2 + 2i$, $z_4 = -2 - 2i$

Answer (b) is correct

Why?

Roots of complex numbers

$z_k = \sqrt[n]{R}\, e^{i\left(\frac{\omega}{n}+\frac{2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1$

Example 3.

Find all complex numbers z such that $z^4 = 16$.

- Write $z = re^{i\theta}$ and $16 = 16e^{i0}$, then $z^4 = 16$ can be written as
$$r^4 e^{i4\theta} = 16e^{i0}\,.$$
- This equation holds if and only if $r^4 = 16$ and
$$4\theta = 0 + k \cdot 2\pi \quad \Leftrightarrow \quad \theta = 0 + k \cdot \tfrac{\pi}{2}\,.$$
- Therefore $r = \sqrt[4]{16} = 2$ and $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$ or $\theta = \frac{3\pi}{2}$.
- The solutions of $z^4 = 16$ are
$$z_1 = 2e^{i0} = 2, \qquad z_2 = 2e^{i\pi/2} = 2i,$$
$$z_3 = 2e^{i\pi} = -2, \quad z_4 = 2e^{i3\pi/2} = -2i.$$