Calculus 1B - Lecture 3

Pim van 't Hof
(✉ p.vanthof@utwente.nl)

November 2020

Themes:

1. Sequences and Series Jump to Theme 1
   - Thomas' Calculus, 10.1 & 10.2
2. Power Series Jump to Theme 2
   - Thomas' Calculus, 10.7
3. Taylor and Maclaurin Series Jump to Theme 3
   - Thomas' Calculus, 10.8

Calculus 1B - Lecture 3 (part 1)

Themes:

1. Sequences and Series
   - Thomas' Calculus, 10.1 & 10.2
2. Power Series
   - Thomas' Calculus, 10.7

   Jump to Theme 2
3. Taylor and Maclaurin Series
   - Thomas' Calculus, 10.8

   Jump to Theme 3

Sequences

Sequences

A sequence is a list of numbers

$$a_1, a_2, \ldots, a_n, \ldots$$

in a given order.

Sequences can be either **finite** or **infinite**.

- Examples:
  - $1, 3, 5, 7, 9$ (first five odd natural numbers)
  - $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ (Fibonacci sequence)
  - $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ (geometric sequence)
  - $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$ (alternating harmonic sequence)

Sequences

Sequences can be described in several ways. For example, the sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots$$

can be described

- by a formula that specifies its terms:

$$a_n = (-1)^{n+1}\frac{1}{n} \qquad\qquad n \geq 1$$

- by listing terms between curly brackets:

$$\{a_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1}\frac{1}{n}, \dots\right\}$$

Sequences - convergence and divergence (informal)

Informally, we say a sequence $\{a_n\}$ converges if its terms approach a single value as n increases.

- Examples:
  - $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\}$ (terms approach 0)
  - $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\}$ (terms approach 0)
  - $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots\}$ (terms approach 1)

If a sequence $\{a_n\}$ does not converge, we say it diverges.

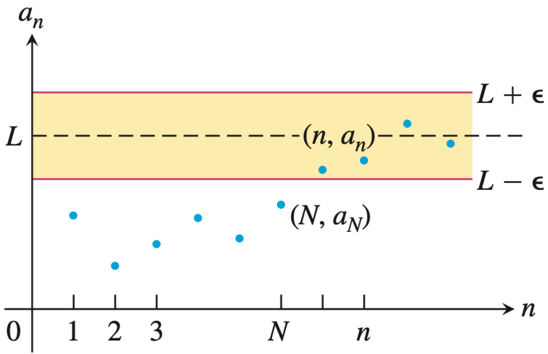
- Examples:
  - $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots, F_n, \ldots\}$ = Fibonacci sequence (terms get larger than any fixed number as n increases)
  - $\{1, -1, 1, -1, 1, \ldots, (-1)^{n+1}, \ldots\}$ (terms never converge to a single value)

Sequences - convergence and divergence (formal)

The sequence $\{a_n\}$ converges to the number L if for every positive number ϵ there is an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon$$

See Figure 10.2 in Thomas' Calculus (page 552):



Sequences - convergence and divergence (formal)

The sequence $\{a_n\}$ converges to the number L if for every positive number ϵ there is an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon$$

If $\{a_n\}$ converges to L, we write

$$\lim_{n\to\infty} a_n = L$$

or simply

$$a_n \to L\,.$$

and call L the limit of the sequence $\{a_n\}$.

If no such number L exists, we say that $\{a_n\}$ diverges.

Sequences - convergence and divergence

- Examples:
  - $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\}$
    - $\lim\limits_{n\to\infty} a_n = 0$, or equivalently, $a_n \to 0$.
    - The limit of the sequence $\{a_n\}$ is 0.
  - $\{b_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\}$
    - $\lim\limits_{n\to\infty} b_n = 0$, or equivalently, $b_n \to 0$.
    - The limit of the sequence $\{b_n\}$ is 0.
  - $\{c_n\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots\}$
    - $\lim\limits_{n\to\infty} c_n = 1$, or equivalently, $c_n \to 1$.
    - The limit of the sequence $\{c_n\}$ is 1.

Series

Series

Given a sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

we obtain a series when we add up the terms of the sequence:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

In other words, a series is the sum of (the terms of) a sequence.

Just like sequences, series can be either **finite** or **infinite**.

- Examples:

Sequence	Series
$1, 3, 5, 7, 9$	$1 + 3 + 5 + 7 + 9$
$1, 1, 2, 3, 5, 8, 13, 21, \ldots$	$1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + \cdots$
$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$	$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Sum of series

Given a series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots ,$$

we are interested in calculating the actual sum.

- For **finite** series, such as $1 + 3 + 5 + 7 + 9$, it is clear what that means; the sum always exists.
- For **infinite** series, this is less clear; for some series the sum exists, for others it doesn't.
  - $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + \cdots = ?$
  - $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = ?$
  - $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = ?$

Sum of infinite series

Given an infinite series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots ,$$

the sum of the first n terms

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the nth partial sum of the series.

Sum of infinite series

Given an infinite series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \ ,$$

the sum of the first n terms

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the nth partial sum of the series.

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \end{aligned}$$

Note that the partial sums form a sequence $\{s_n\}$.

Sum of infinite series

Example. Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

and a few of its partial sums:

$$\begin{aligned}
s_1 &= \tfrac{1}{2} && = 1 - \tfrac{1}{2} \\
s_2 &= \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4} && = 1 - \tfrac{1}{4} \\
s_3 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{7}{8} && = 1 - \tfrac{1}{8} \\
s_4 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{15}{16} && = 1 - \tfrac{1}{16} \\
&\vdots \\
s_n &= \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \tfrac{1}{2^n} = \tfrac{2^n - 1}{2^n} && = 1 - \tfrac{1}{2^n}
\end{aligned}$$

The sequence $\{s_n\}$ converges to 1 (i.e., $\lim_{n\to\infty} s_n = 1$).

Sum of infinite series

Example. Consider the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

We saw that the **sequence** $\{s_n\}$ **of partial sums** converges to 1.

Then we say that the original **series**

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges too, and has sum 1.

Sum of infinite series - convergence and divergence

Consider the infinite series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots .$$

- If the sequence $\{s_n\}$ of partial sums converges, that is, if the limit $\lim_{n\to\infty} s_n = s$ exists (as a real number, so not $\pm\infty$), we say that the series $a_1 + a_2 + \cdots + a_n + \cdots$ converges and we write:

  $$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s .$$

  The number s is called the sum of the series.

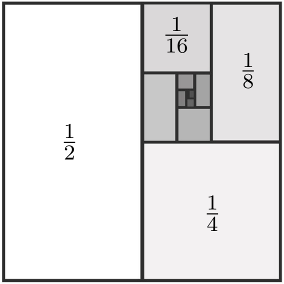
- If the sequence $\{s_n\}$ of partial sums does *not* converge, that is, if the limit $\lim_{n\to\infty} s_n$ does *not* exist, we say that the series $a_1 + a_2 + \cdots + a_n + \cdots$ diverges.

Example of a convergent series

Example. As we saw earlier, the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

is convergent. The sum of this series is 1 (see also figure below).

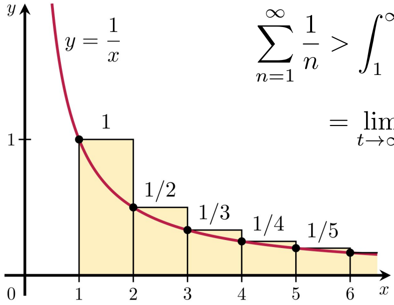


Example of a divergent series

Example. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

is divergent (see figure below).



$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \Big[\ln(x)\Big]_{x=1}^{t} = \lim_{t\to\infty} \Big(\ln(t) - \ln(1)\Big) = \infty$$

Sum of infinite series vs. improper integrals

Consider the following sum:

$$\sum_{n=1}^{\infty} a_n = \lim_{t\to\infty} \sum_{n=1}^{t} a_n$$

Compare it with the improper integral

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^{t} f(x)\,dx$$

- To find the improper integral, we first integrate from 1 to t and then let $t \to \infty$.
- To find the sum of the infinite series, we first sum from 1 to t and then let $t \to \infty$.

Geometric series

Geometric series

An important example of an infinite series is the geometric series:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}\,,$$

where a and r are fixed real numbers and $a \neq 0$. The number r is called the (common) ratio of the geometric series.

- Examples:
  - $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots \qquad (a = 1, r = \frac{1}{2})$
  - $2 - 6 + 18 - 54 + \cdots + 2 \cdot (-3)^{n-1} + \cdots \qquad (a = 2, r = -3)$

Geometric series - convergence and divergence ($|r| = 1$)

Whether or not a geometric series converges depends on r.

- If $r = 1$, then the nth partial sum of the geometric series is

$$s_n = a + a \cdot 1 + a \cdot 1^2 + a \cdot 1^3 + \cdots + a \cdot 1^{n-1} = an$$

  and $\lim_{n\to\infty} s_n = \pm\infty$ (depending on the sign of a). Hence, the geometric series **diverges** in this case.

- If $r = -1$, then the partial sum s_n of the geometric series is

$$\begin{aligned} s_n &= a + a \cdot (-1) + a \cdot (-1)^2 + a \cdot (-1)^3 + \cdots + a \cdot (-1)^{n-1} \\ &= a - a + a - a + \cdots + a \cdot (-1)^{n-1} \end{aligned}$$

  so the partial sums alternate between a and 0. Hence, $\lim_{n\to\infty} s_n$ does not exist and the geometric series **diverges**.

Geometric series - convergence and divergence ($|r| \neq 1$)

Whether or not a geometric series converges depends on r.

- If $|r| \neq 1$, then we have

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

and

$$rs_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n$$

If we subtract the second equation from the first, we get

$$s_n - rs_n = a - ar^n$$

$$s_n(1-r) = a(1-r^n)$$

so

$$\boxed{s_n = \frac{a(1-r^n)}{1-r}}$$

Geometric series - convergence and divergence ($|r| \neq 1$)

Whether or not a geometric series converges depends on r.

- If $|r| \neq 1$, then we have

$$\boxed{s_n = \frac{a(1-r^n)}{1-r}}$$

  - If $|r| < 1$, we know that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}\,.$$

    Hence, the geometric series **converges** in this case.

  - If $|r| > 1$, then $r^n \to \pm\infty$ and therefore $\lim_{n\to\infty} s_n = \pm\infty$. Hence, the geometric series **diverges** in this case.

Geometric series - convergence and divergence

Summarizing:

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1-r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}\,, \qquad |r| < 1\,.$$

If $|r| \geq 1$, the series diverges. (Recall that $a \neq 0$ by definition.)

In other words, the sum s of a geometric series with first term a and common ratio r is given by

$$s = \frac{a}{1-r}\,, \quad \text{or in words:} \quad \frac{\text{first term}}{1-\text{ratio}}$$

provided that $-1 < r < 1$.

Exercise 1. Find the sum s of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Solution. Note that the series can written as follows:

$$5 - 5 \cdot \frac{2}{3} + 5 \cdot \frac{4}{9} - 5 \cdot \frac{8}{27} + \cdots$$

or

$$5 + 5 \cdot \left(-\frac{2}{3}\right) + 5 \cdot \left(-\frac{2}{3}\right)^2 + 5 \cdot \left(-\frac{2}{3}\right)^3 + \cdots$$

So we find first term $a = 5$ and common ratio $r = -\frac{2}{3}$. Hence

$$s = \frac{a}{1-r} = \frac{5}{1-(-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3\,.$$

Exercise 2. Is the series $\displaystyle\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

Solution. We rewrite the nth term of the series in the form ar^{n-1}:

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{2n}3^{1-n} &= \sum_{n=1}^{\infty}(2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} 4^n \frac{1}{3^{n-1}} = \sum_{n=1}^{\infty} 4 \cdot 4^{n-1} \frac{1}{3^{n-1}} \\ &= \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{3}\right)^{n-1} = 4 + 4 \cdot \left(\frac{4}{3}\right) + 4 \cdot \left(\frac{4}{3}\right)^2 + \cdots\end{aligned}$$

This is a geometric series with $a = 4$ and $r = \frac{4}{3}$.

Since $r > 1$, the series diverges.

Geometric series

$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ (iff $-1 < r < 1$)

Exercise 3. Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of two integers, i.e., as a rational number.

Solution. Note that $2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$.
The series

$$\frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

can be written as

$$\frac{17}{10^3} + \frac{17}{10^3} \cdot \left(\frac{1}{100}\right) + \frac{17}{10^3} \cdot \left(\frac{1}{100}\right)^2 + \cdots,$$

which is a geometric series with $a = \frac{17}{1000}$ and $r = \frac{1}{100}$. The sum of this series is

$$\frac{a}{1-r} = \frac{\frac{17}{1000}}{1 - \frac{1}{100}} = \frac{\frac{17}{1000}}{\frac{99}{100}} = \frac{17}{990}.$$

Therefore, $2.3\overline{17} = 2.3 + \frac{17}{990} = \ldots = \frac{1147}{495}$.

Geometric series

$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ (iff $-1 < r < 1$)

Exercise 4. Find the sum of the series $\sum_{n=0}^{\infty} x^n$ where $|x| < 1$.

Solution. Note that this series starts with $n = 0$ instead of $n = 1$, so the first term of the series is $x^0 = 1$. (We adopt the convention that $x^0 = 1$ even when $x = 0$.) So:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

is a geometric series with $a = 1$ and $r = x$.

Since $|r| = |x| < 1$, the series converges, and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,.$$

Calculus 1B - Lecture 3 (part 2)

Themes:

1. Sequences and Series
   - Thomas' Calculus, 10.1 & 10.2

Jump to Theme 1

2. Power Series
   - Thomas' Calculus, 10.7
3. Taylor and Maclaurin Series
   - Thomas' Calculus, 10.8

Jump to Theme 3

Short recap

Short recap

Recall that a geometric series is a series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1},$$

where a and r are fixed real numbers and $a \neq 0$. The number r is called the (common) ratio of the geometric series.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1-r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Short recap

In Exercise 4 of part 1, we observed that

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

is a geometric series with $a = 1$ and $r = x$.

Suppose we consider x in the above series to be a variable instead of a constant. Then the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

can be interpreted as an "infinite polynomial". This is an example of a power series.

Power series

Power series

A power series (about $x = 0$) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots ,$$

where

- x is a variable
- the c_is are constants called the coefficients of the series

Example: If we take $c_n = 1$ for all n, we get

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots ,$$

the geometric series [1] with first term 1 and common ratio x.

[1] see Lecture 3 - part 1

Power series

For general c_n, a power series is *not* a geometric series.

Some more examples of power series:

- $$\sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$
- $$\sum_{n=1}^{\infty}\frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$
- $$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1} = -x + \frac{x^3}{3} - \frac{x^5}{5} - \cdots$$

Power series

A power series (about $x = 0$) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots ,$$

where

- x is a variable
- the c_is are constants called the coefficients of the series

Power series

More generally, a power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots ,$$

where

- x is a variable
- the c_is are constants called the coefficients of the series
- a is a constant called the center of the series

A power series about $x = a$ can also be called:

- a power series around $x = a$
- a power series in $(x - a)$
- a power series centered at a

Power series - convergence

Given a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0+c_1(x-a)+c_2(x-a)^2+\cdots+c_n(x-a)^n+\cdots,$$

for what values of x does the series **converge**?

Example: If the center $a = 0$ and $c_n = 1$ for all n, we get

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and common ratio x, which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

In general, the set of values of x for which a power series is convergent is *always* an interval.

Power series - convergence

The convergence of the power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is described by one of the following three cases:

1. There is a positive number R such that the series
   - converges for x with $|x-a| < R$, and
   - diverges for x with $|x-a| > R$.
2. The series converges for every x ($R = \infty$).
3. The series converges only for $x = a$ ($R = 0$).

R is called the radius of convergence of the series, and the interval $a - R < x < a + R$ is called the interval of convergence.

Power series - convergence

In other words, for a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots ,$$

the interval of convergence $a - R < x < a + R$ is the interval around a ($=$ the series' center) consisting of all values of x for which the series converges.

If x is an endpoint of the interval of convergence, that is, if $x = a - R$ or $x = a + R$, then the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ may or may not converge:

- the series may converge at one or both endpoints;
- the series may diverge at both endpoints.

Power series - convergence

Example (cont.): We already saw that the power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

converges when $-1 < x < 1$ and diverges when $|x| \geq 1$, so:

- the interval of convergence is $-1 < x < 1$, or $(-1, 1)$
- the radius of convergence is 1 $(R = 1)$

In this particular example, the power series diverges at both endpoints of the interval of convergence ($x = -1$ and $x = 1$).

Power series - sum

The sum of the power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

is the function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

whose domain is the set of all x for which the series **converges**.

Example (cont.): The sum of the power series $\sum_{n=0}^{\infty} x^n$ is the function

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

with domain $-1 < x < 1$.

Power series - the basic series

The previous example shows that, as long as $-1 < x < 1$, we can express the function $\frac{1}{1-x}$ as a power series:

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1\,.}$$

We will now see that we can express many more functions as power series. To do so, we will often use the above expression, which we will refer to as the **basic series**.

Exercise 1. Express $\frac{1}{1+x^2}$ as a power series and find the interval of convergence. Make use of the basic series.

Solution. If we replace x by $-x^2$ in the basic series, we get:

$$\begin{aligned}
\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} \\
&= \sum_{n=0}^{\infty} (-x^2)^n \\
&= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots
\end{aligned}$$

Since this is a geometric series (with first term $a = 1$ and common ratio $r = -x^2$), it converges when $|-x^2| < 1$, that is, if $x^2 < 1$ or $|x| < 1$. Hence, the interval of convergence is $(-1, 1)$.

Exercise 2. Express $\frac{1}{(1-x)^2}$ as a power series.

Note that we cannot use the same 'trick' as we used in Exercise 1, where we rewrote $\frac{1}{1+x^2}$ as $\frac{1}{1-(-x^2)}$.

However, since $\frac{1}{(1-x)^2}$ is the *derivative* of $\frac{1}{1-x}$, there's another 'trick' we can use to solve Exercise 2: we *differentiate* both sides of the basic series equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \,,$$

to get

$$\frac{1}{(1-x)^2} = \qquad 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} n x^{n-1} \,.$$

Differentiation & integration of power series

Differentiation & integration of power series

Theorem (Term-by-Term Theorem)

If the power series $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$, *then the function* f *defined by*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continous) and has an antiderivative on the interval of convergence $a - R < x < a + R$.

Differentiating or integrating the original series term-by-term yields

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

and

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C$$

The interval and radius of convergence of these new power series are (again) $a - R < x < a + R$ *and* R, *respectively.*

Differentiation & integration of power series

Exercise 2'. Express

$$\frac{1}{(1-x)^2}$$

as a power series by differentiating the basic series. What is the radius of convergence?

Solution. If we differentiate both sides of the basic series equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n\,,$$

we get

$$\frac{1}{(1-x)^2} = \qquad 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty} n x^{n-1}\,.$$

By the Term-by-term Theorem, the radius of convergence of this series is the same as that of the basic series: $R = 1$.

Differentiation & integration of power series

Exercise 3. Find a power series representation for $\ln(1-x)$, and determine its radius of convergence.

Solution. We notice that, except for a factor of -1, the derivative of $\ln(1-x)$ is $\frac{1}{1-x}$ (= the sum of the basic series). Consequently:

$$\begin{aligned}
\ln(1-x) = \int \frac{-1}{1-x}\,dx &= -\int \frac{1}{1-x}\,dx \\
&= -\int (1 + x + x^2 + x^3 + \cdots)\,dx \\
&= -\Big(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots\Big) + C \\
&= -\sum_{n=1}^{\infty} \frac{x^n}{n} + C
\end{aligned}$$

Differentiation & integration of power series

Exercise 3. Find a power series representation for $\ln(1-x)$, and determine its radius of convergence.

Solution (cont.). So we have found:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} + C\,.$$

To determine the value of C, we put $x = 0$ in this equation and obtain $\ln(1-0) = C$. So $C = 0$, and we conclude that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}\,.$$

By the Term-by-term Theorem, the radius of convergence of this series is the same as that of the basic series: $R = 1$.

Exercise 3. Find a power series representation for $\ln(1-x)$, and determine its radius of convergence.

Remark: If we put $x = \frac{1}{2}$ in the equation

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}\,,$$

we get:

$$\ln(\tfrac{1}{2}) = -\sum_{n=1}^{\infty} \frac{(\frac{1}{2})^n}{n} = -\frac{1}{2} - \frac{1}{8} - \frac{1}{24} - \frac{1}{64} - \cdots\,.$$

Since $\ln(\frac{1}{2}) = -\ln(2)$, we find that

$$\ln(2) = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \cdots\,.$$

Calculus 1B - Lecture 3 (part 3)

Themes:

1. Sequences and Series Jump to Theme 1
   - Thomas' Calculus, 10.1 & 10.2
3. Power Series Jump to Theme 2
   - Thomas' Calculus, 10.7
3. Taylor and Maclaurin Series
   - Thomas' Calculus, 10.8

Taylor and Maclaurin series

Taylor and Maclaurin series are useful for:

- approximating functions by polynomials
- integrating functions without elementary antiderivatives
- solving differential equations

Taylor polynomials (sneak preview)

Problem 1. Calculate $\sin(1)$.

Solution. Get out your pocket calculator/phone. It will tell you that

$$\sin(1) \approx 0.84147098\ldots$$

But... a pocket calculator/phone can only add and multiply finitely many numbers!

- How does it 'do it'?
- How does it 'know' the value of $\sin(1)$?
- How does it approximate the value of $\sin(1)$?

Taylor polynomials (sneak preview)

Problem 2. Calculate $\sin(1)$. Only use pen and paper.

Solution. Taylor polynomials!

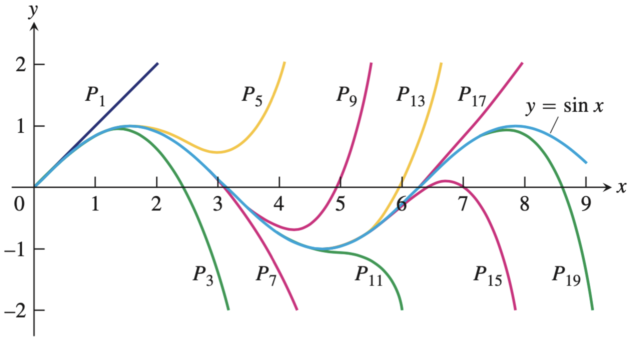


FIGURE 10.20

Taylor and Maclaurin series

For a function f, the Taylor series generated by f at $x = a$ is

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots}$$

Here, $f^{(n)}(a)$ denotes the nth derivative of f evaluated at $x = a$.
In particular:

$$\begin{aligned} f^{(0)}(a) &= f(a) \\ f^{(1)}(a) &= f'(a) \\ f^{(2)}(a) &= f''(a) \end{aligned}$$

We assume that $f^{(n)}(a)$ of f exists for every $n = 0, 1, 2, \ldots$

Taylor and Maclaurin series

For a function f, the Taylor series generated by f at $x = a$ is

$$\boxed{\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots\end{aligned}}$$

Alternative names for this series:

- Taylor series generated by f around $x = a$
- Taylor series of f at/around $x = a$
- Taylor series of f at/around a
- Taylor series of f centered at a

Taylor and Maclaurin series

For a function f, the Taylor series generated by f at $x = a$ is

$$\boxed{\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots\end{aligned}}$$

The Maclaurin series generated by f is

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots ,}$$

that is, the Taylor series generated by f at $x = 0$ (often just called 'the Taylor series of f').

Exercise†. Find the Taylor series generated by the function $f(x) = \frac{1}{x^2}$ at $x = 1$.

Solution. We first compute:

$$\begin{aligned} f(x) &= \frac{1}{x^2} & f(1) &= 1 \\ f'(x) &= \frac{-2}{x^3} & f'(1) &= -2 \\ f''(x) &= \frac{6}{x^4} & f''(1) &= 6 \\ f^{(3)}(x) &= \frac{-24}{x^5} & f^{(3)}(1) &= -24 \\ &\vdots \end{aligned}$$

So for every $n = 0, 1, 2, \ldots$ we have:

$$f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}} \qquad f^{(n)}(1) = (-1)^n (n+1)!$$

† Thomas' Calculus, Section 10.8, Exercise 27.

Exercise*. Find the Taylor series generated by the function $f(x) = \frac{1}{x^2}$ at $x = 1$.

Solution (cont.). We found that for every $n = 0, 1, 2, \ldots$, we have

$$f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}} \qquad f^{(n)}(1) = (-1)^n (n+1)!$$

Hence, the Taylor series generated by f at $x = 1$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!}(x-1)^n \\
&= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n \\
&= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots
\end{aligned}$$

*Thomas' Calculus, Section 10.8, Exercise 27.

Exercise 1. Find the Maclaurin series of the function $f(x) = e^x$.

Solution. First note that, for every $n = 0, 1, 2, \ldots$, we have

$$f^{(n)}(x) = e^x \quad \text{and} \quad f^{(n)}(0) = 1$$

Hence, the Maclaurin series of f (or, equivalently, the Taylor series generated by f at $x = 0$) is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

It can be shown that:

- this series converges for every x (that is, $R = \infty$)
- this series converges to e^x for every x

So the function e^x equals the above power series.

Exercise 2. Find the Maclaurin series of the function $f(x) = \sin x$.

Solution. We first compute:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

As the derivatives repeat in a cycle of four, the Maclaurin series for $\sin x$ is given by:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$

Exercise 2. Find the Maclaurin series of the function $f(x) = \sin x$.

Solution (cont.). It can be shown that the series we found

- converges for every x (that is, $R = \infty$)
- converges to $\sin x$ for every x

So:

$$\boxed{\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for every } x \end{aligned}}$$

Exercise 3. Find the Maclaurin series of the function $f(x) = \cos x$.

Solution. We can **differentiate** the Maclaurin series for $\sin x$:

$$\begin{aligned}\cos x = \frac{d}{dx}(\sin x) &= \frac{d}{dx}\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\Big)\\ &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots\\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\end{aligned}$$

So:

$$\boxed{\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for every } x\end{aligned}}$$

Exercise 4. Find the Maclaurin series of the function $f(x) = x \cos x$.

Solution. Instead of computing the derivatives $f^{(n)}(x)$ and substituting $x = 0$, we **multiply** the Maclaurin series for $\cos x$ by x:

$$
\begin{aligned}
x \cos x &= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \\
&= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots
\end{aligned}
$$

Important Maclaurin series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

Remark. Not every function equals its Taylor/Maclaurin series:

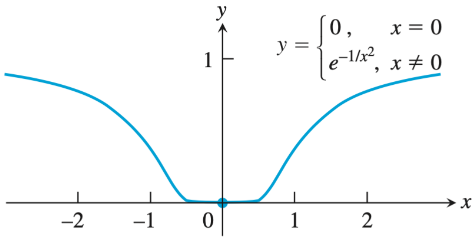


FIGURE 10.19 The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4). Therefore its Taylor series is not the function itself.

Taylor polynomials

Taylor polynomials

Recall the Taylor series generated by a function f at $x = a$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \,.$$

The Taylor polynomial of order n generated by f at $x = a$ is defined as follows:

$$\boxed{\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \,. \end{aligned}}$$

In other words, the Taylor polynomial of order n generated by f at $x = a$ (also called 'the nth-order Taylor polynomial of f around a') is the nth partial sum of the Taylor series generated by f at $x = a$.

Taylor polynomials

$$\boxed{\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \,. \end{aligned}}$$

The first order Taylor polynomial of f around a is given by:

$$\boxed{P_1(x) = f(a) + f'(a)(x-a)\,.}$$

This is exactly the linearization[3] of f at $x = a$.

[3] See Calculus 1A, or Thomas' Calculus (page 202).

Exercise 5. Find the Taylor polynomial of order 4 generated by $\sin x$ at $x = 0$.

Solution. In Exercise 2, we found that

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

So the 4th-order Taylor polynomial generated by $\sin x$ at $x = 0$ is

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k = 0 + x + 0 - \frac{x^3}{3!} + 0 = x - \frac{x^3}{3!} \, .$$

Note that the *degree* of this polynomial is 3, one less than the *order* of the polynomial. In this case, $P_3(x) = P_4(x)$.

Exercise 5. Find the Taylor polynomial of order 4 generated by $\sin x$ at $x = 0$.

Remark. In the figure below, we see the graph of $\sin x$ and the graphs of the Taylor polynomials of orders 1 through 19 generated by $\sin x$ at $x = 0$. The higher the index, the better the approximation.

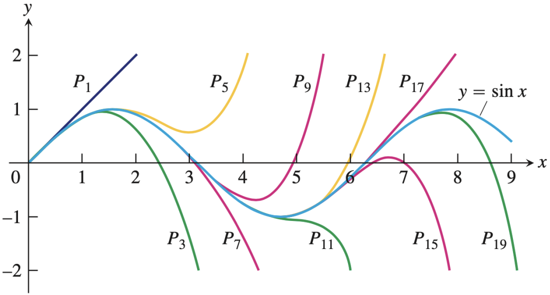


FIGURE 10.20

Taylor polynomials

$\sin(1) \approx 0.84147098\ldots$

Problem 2. Calculate $\sin(1)$. Only use pen and paper.

Solution. We have seen that the function $f(x) = \sin(x)$ around $x = 0$ can be approximated by its Taylor polynomials (the higher the index, the better the approximation):

- $P_1(x) = x$
  - $\sin(1) \approx P_1(1) = 1$
- $P_3(x) = x - \frac{x^3}{3!}$
  - $\sin(1) \approx P_3(1) = 1 - \frac{1}{6} = \frac{5}{6} \approx 0.83333333\ldots$
- $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$
  - $\sin(1) \approx P_5(1) = 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} \approx 0.84166667\ldots$
- $P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
  - $\sin(1) \approx P_7(1) = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} = \frac{4241}{5040} \approx 0.84146825\ldots$

Important ('basic') Taylor polynomials

The table below contains the Taylor polynomials of order n generated by some important functions at $x = 0$:

$$\frac{1}{1-x} : \quad P_n(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^n$$

$$e^x : \quad P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$$

$$\sin x : \quad P_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x : \quad P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$

We will sometimes refer to these as 'basic' Taylor polynomials.

Using basic Taylor polynomials

Exercise 6. Find the Taylor polynomial of order 10 generated by $\sin(x^2)$ at $x = 0$. (Hint: Use the basic Taylor polynomial for $\sin x$.)

Solution. Recall the basic Taylor polynomial for $\sin x$ (see previous slide). In particular, the Taylor polynomial of order 5 generated by $\sin x$ at $x = 0$ is given by

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \,.$$

If x is close to 0, then so is x^2. Replacing x by x^2 in the above equation for $P_5(x)$ yields the following polynomial:

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} \,,$$

which is the 10th-order Taylor polynomial of $\sin(x^2)$ around $x = 0$.