Calculus 1B - Lecture 6

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January 2021

Calculus 1B - Lecture 6

Second-order differential equations
(Thomas' Calculus, Sections 17.1 & 17.2)

Themes:

1. Homogeneous equations ▸ Jump to Theme 1
   - Characteristic equation
   - General solution
2. Inhomogeneous equations ▸ Jump to Theme 2
   - Solution approach
   - Particular solutions
3. Initial value & boundary value problems ▸ Jump to Theme 3

Calculus 1B - Lecture 6 (part 1)

Second-order differential equations
(Thomas' Calculus, Sections 17.1 & 17.2)

Themes:

1. Homogeneous equations
   - Characteristic equation
   - General solution
2. Inhomogeneous equations
   - Solution approach
   - Particular solutions
3. Initial value & boundary value problems

Jump to Theme 2
Jump to Theme 3

First-order differential equations

Differential equations we have seen so far (Lecture 4):

- (Ordinary) **first-order** differential equations:

$$y' = f(x, y)$$

  - Separable first-order DEs:

Type	**Solution method**
$y' = f(x) \cdot g(y)$	Separation of variables
$y' + P(x)\, y = 0$	Separation of variables

  - Linear first-order DEs:

Type	**Solution method**
$y' + P(x)\, y = Q(x)$	Integrating factor

Second-order differential equations

Differential equations we will see today (Lecture 6):

- (Ordinary) **second-order** differential equations of the form[1]:

$$a\,y'' + b\,y' + c\,y = f(x)$$

  where a, b and c are real numbers and f is a real function.
- Sometimes, we use the following notation:

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + c\,y = f(x)$$

[1] Such equations are called second-order linear DEs with constant coefficients

Second-order differential equations

Differential equations we will see today (Lecture 6):

- (Ordinary) **second-order** differential equations of the form[2]:

  $$a\,y'' + b\,y' + c\,y = f(x)$$

  where a, b and c are real numbers and f is a real function.
- We distinguish two types:
  - Homogeneous DEs:

    $$a\,y'' + b\,y' + c\,y = 0$$

  - Inhomogeneous DEs:

    $$a\,y'' + b\,y' + c\,y = f(x)$$

[2] Such equations are called second-order linear DEs with constant coefficients

Second-order homogeneous DEs

Example 1. Solve the differential equation

$$y'' + 5y' + 4y = 0\,. \tag{1}$$

- We found (by educated guessing, see Lecture 5) that $y(x) = e^{rx}$ is a solution of equation (1) if and only if

$$r^2 + 5r + 4 = 0\,.$$

- By solving this quadratic equation, we find two roots:

$$r_1 = -1 \text{ and } r_2 = -4\,.$$

- Two solutions of $y'' + 5y' + 4y = 0$ are

$$\boxed{y(x) = e^{-x}} \quad \text{and} \quad \boxed{y(x) = e^{-4x}}$$

Second-order homogeneous DEs

- We have found two solutions of $y'' + 5y' + 4y = 0$, namely

$$\boxed{y(x) = e^{-x}} \quad \text{and} \quad \boxed{y(x) = e^{-4x}}$$

- Note that the following holds (check!):
  - for any $c_1, c_2 \in \mathbb{R}$,

    $$y(x) = c_1 e^{-x} \quad \text{and} \quad y(x) = c_2 e^{-4x}$$

    are solutions as well;
  - for any $c_1, c_2 \in \mathbb{R}$,

    $$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-4x}}$$

    is a solution as well.

Second-order homogeneous DEs

- All solutions of $y'' + 5y' + 4y = 0$ are given by

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-4x}, \qquad c_1, c_2 \in \mathbb{R}}$$

- The solution above is called the general solution of DE (1).
- The general solution is the solution containing **all** solutions.

The characteristic equation

Definition

Consider the differential equation

$$a\,y'' + b\,y' + c\,y = 0\,. \qquad (2)$$

The equation

$$a\,r^2 + b\,r + c = 0 \qquad (3)$$

is called the characteristic equation[3] of (2).

- If r is a root of the characteristic equation (3), then $y(x) = e^{rx}$ is **a** solution of (2).
- To find the **general** solution of (2), we distinguish 3 cases:
  - $D > 0$
  - $D = 0$
  - $D < 0$

  where $D = b^2 - 4ac$ is the discriminant of equation (3).

[3] Also called the 'auxiliary equation' of (2)

The general solution (in case $D > 0$)

Theorem (The $D > 0$ case)

Consider the second-order homogeneous differential equation

$$a\,y'' + b\,y' + c\,y = 0$$

with $a, b, c \in \mathbb{R}$.

If $D = b^2 - 4ac > 0$, *then the general solution is*

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \qquad c_1, c_2 \in \mathbb{R}\,,$$

with

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The general solution (in case $D = 0$)

Example 2. Find the general solution of the differential equation

$$y'' + 2y' + y = 0\,. \tag{4}$$

- The characteristic equation of DE (4) is

$$r^2 + 2r + 1 = 0\,.$$

- Since $D = b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot 1 = 0$, the characteristic equation has only one (repeated) real root, namely $r = -1$.
- The general solution to (4) is **not** (as in the case $D > 0$)

$$y(x) = c_1\, e^{-x} + c_2\, e^{-x},$$

as **not** all solutions to (4) are of the form $c\, e^{-x}$.
- The general solution to (4) is

$$y(x) = c_1\, e^{-x} + c_2\, x\, e^{-x}.$$

The general solution (in case $D = 0$)

Theorem (The $D = 0$ case)

Consider the second-order homogeneous differential equation

$$a\,y'' + b\,y' + c\,y = 0$$

with $a, b, c \in \mathbb{R}$.

If $D = b^2 - 4ac = 0$, *then the general solution is*

$$y(x) = c_1 e^{rx} + c_2\, x\, e^{rx}, \qquad c_1, c_2 \in \mathbb{R}\,,$$

with

$$r = \frac{-b}{2a}.$$

The general solution (in case $D < 0$)

Example 3. Find the general solution of the differential equation

$$y'' + 2y' + 10y = 0\,. \tag{5}$$

- The characteristic equation of DE (5) is

$$r^2 + 2r + 10 = 0\,.$$

- Since $D = b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot 10 = -36 < 0$, the characteristic equation does not have any real roots.
- The two **complex** (non-real) roots of the characteristic equation are:

$$r_{1,2} = -1 \pm 3i\,.$$

The general solution (in case $D < 0$)

- Two **complex** solutions of the differential equation

$$y'' + 2y' + 10y = 0 \tag{5}$$

  are

$$y(x) = e^{(-1+3i)x} \quad \text{and} \quad y(x) = e^{(-1-3i)x}\,.$$

- Thus the **complex** general solution of DE (5) is

$$y(x) = \alpha e^{(-1+3i)x} + \beta e^{(-1-3i)x}\,,$$

  with arbitrary $\alpha, \beta \in \mathbb{C}$.

- But... this is a complex function, and we are only interested in **real** solutions when solving differential equations.
- Fortunately, we can rewrite the above complex general solution as a real function using... Euler's formula!

The general solution (in case $D < 0$)

- The solution of

$$y'' + 2y' + 10y = 0 \tag{5}$$

is given by

$$\begin{aligned}
y(x) &= \alpha e^{(-1+3i)x} + \beta e^{(-1-3i)x} \\
&= e^{-x}\big(\alpha e^{3ix} + \beta e^{-3ix}\big) \\
&= e^{-x}\big(\alpha\,[\cos(3x) + i\sin(3x)] + \beta\,[\cos(-3x) + i\sin(-3x)]\,\big) \\
&= e^{-x}\big(\alpha\,[\cos(3x) + i\sin(3x)] + \beta\,[\cos(3x) - i\sin(3x)]\,\big) \\
&= e^{-x}\big(\,(\alpha+\beta)\cos(3x) + (\alpha i - \beta i)\sin(3x)\big) \\
&= e^{-x}\big(c_1\cos(3x) + c_2\sin(3x)\big)
\end{aligned}$$

- By restricting ourselves to real values for c_1 and c_2, we get all **real** solutions of (5).

The general solution (in case $D < 0$)

Theorem (The $D < 0$ case)

Consider the second-order homogeneous differential equation

$$a\,y'' + b\,y' + c\,y = 0$$

with $a, b, c \in \mathbb{R}$.

If $D = b^2 - 4ac < 0$ *and the roots of the characteristic equation* $ar^2 + br + c = 0$ *are*

$$r_{1,2} = \sigma \pm i\omega\,,$$

then the (real) general solution to the differential equation is

$$y(x) = e^{\sigma x}\left[c_1 \cos(\omega x) + c_2 \sin(\omega x)\right], \qquad c_1, c_2 \in \mathbb{R}\,.$$

The general solution (in case $D < 0$)

- Remark: Whenever we talk about 'the general solution' of a differential equation, we always mean 'the **real** general solution' of the equation, unless explicitly stated otherwise.

Calculus 1B - Lecture 6 (part 2)

Second-order differential equations
(Thomas' Calculus, Sections 17.1 & 17.2)

Themes:

Second-order differential equations

Differential equations we will see today (Lecture 6):

- (Ordinary) **second-order** differential equations of the form[4]:

$$a\,y'' + b\,y' + c\,y = f(x)$$

  where a, b and c are real numbers and f is a real function.

  - Homogeneous DEs:

$$a\,y'' + b\,y' + c\,y = 0$$

  - Inhomogeneous DEs:

$$a\,y'' + b\,y' + c\,y = f(x)$$

[4] Such equations are called second-order linear DEs with constant coefficients

Second-order (in)homogeneous DEs

Recall the following example:

Example 3. Solve the differential equation

$$y'' + 2y' + 10y = 0\,.$$

- This is a homogeneous differential equation.
- We found that the general solution of this DE is:

$$y(x) = e^{-x}\big(c_1 \cos(3x) + c_2 \sin(3x)\big)\,, \quad c_1, c_2 \in \mathbb{R}\,.$$

Now consider the following example:

Example 4. Solve the inhomogeneous differential equation

$$y'' + 2y' + 10y = 20\,.$$

Second-order inhomogeneous DEs

Example 4. Solve the inhomogeneous differential equation

$$y'' + 2y' + 10y = 20\,.$$

Quiz

Which of the following functions is a solution to the above differential equation? (Guess & try!)

(a) $e^{-x}\cos(3x)$

(b) e^{-x}

(c) 2

(d) $10x$

Answer (c) is correct

Second-order inhomogeneous DEs: Solution approach

Example 4. Solve the inhomogeneous differential equation

$$y'' + 2y' + 10y = 20\,. \tag{6}$$

- We need to find the general solution $y = y(x)$ to DE (6).
- We first find a particular solution $y_p = y_p(x)$ to DE (6).
- Since the right hand-side of (6) is a constant, we try a constant as a particular solution:

$$y_p(x) = A\,.$$

- Then $y_p'(x) = y_p''(x) = 0$, and substituting $y = y_p$ in (6) gives:

$$0 + 2 \cdot 0 + 10A = 20 \quad \Leftrightarrow \quad A = 2\,.$$

- So $y_p(x) = 2$ is a particular solution to DE (6).

Second-order inhomogeneous DEs: Solution approach

Example 4. Solve the inhomogeneous differential equation

$$y'' + 2y' + 10y = 20\,. \tag{6}$$

- We know that $y_p(x) = 2$ is a particular solution.
- This is only one solution. How do we get the general solution?
- Define $\tilde{y} = y - y_p$, so in our example:

$$\tilde{y} = y - 2\,.$$

- Then $\tilde{y}$ satisfies (check!):

$$\tilde{y}'' + 2\,\tilde{y}' + 10\,\tilde{y} = 0\,.$$

- The general solution of this homogeneous equation is[5]:

$$\tilde{y}(x) = e^{-x}\big(c_1 \cos(3x) + c_2 \sin(3x)\big)\,.$$

- Hence the general solution $y = y(x)$ of (6) is given by

$$y(x) = e^{-x}\big(c_1 \cos(3x) + c_2 \sin(3x)\big) + 2\,.$$

[5] As we saw in Example 3

Second-order inhomogeneous DEs: Solution approach

Theorem

Consider the second-order differential equation

$$a\,y'' + b\,y' + c\,y = f(x), \tag{7}$$

with $a, b, c \in \mathbb{R}$ *and* f *a real function.*
The general solution $y = y(x)$ *to equation (7) is of the form*

$$y = y_h + y_p\,,$$

where y_h *is the general solution to the homogeneous equation*

$$a\,y'' + b\,y' + c\,y = 0$$

and y_p *is a (one) particular solution to equation (7).*

Second-order inhomogeneous DEs: Solution approach

- So to find the general solution to an inhomogeneous differential equation of the form

$$a\,y'' + b\,y' + c\,y = f(x)\,,$$

we need to perform three steps:

Step 1. find the general solution y_h to the homogeneous differential equation $a\,y'' + b\,y' + c\,y = 0$;

Step 2. find a particular solution y_p to the (original) inhomogeneous differential equation $a\,y'' + b\,y' + c\,y = f(x)$.

Step 3. add the solutions together to find the general solution $y = y_h + y_p$ to the (original) inhomogeneous differential equation $a\,y'' + b\,y' + c\,y = f(x)$.

- We already know how to perform Step 1, as we know how to solve homogeneous differential equations.

Second-order inhomogeneous DEs: Solution approach

- So to find the general solution to an inhomogeneous differential equation of the form

$$a\,y'' + b\,y' + c\,y = f(x)\,,$$

  we need to perform three steps:

Step 1. find the general solution y_h to the homogeneous differential equation $a\,y'' + b\,y' + c\,y = 0$;

Step 2. find a particular solution y_p to the (original) inhomogeneous differential equation $a\,y'' + b\,y' + c\,y = f(x)$.

Step 3. add the solutions together to find the general solution $y = y_h + y_p$ to the (original) inhomogeneous differential equation $a\,y'' + b\,y' + c\,y = f(x)$.

- How do we find a particular solution (Step 2)?
- We make an educated guess (see next slide).

Second-order inhomogeneous DEs: Particular solutions

The following table provides several candidates for y_p:

If $f(x)$ is a constant multiple of...	Then try as candidate for y_p...
p (constant)	A (constant)
$px + q$	$Ax + B$
$px^2 + qx + m$	$Ax^2 + Bx + C$
e^{rx}	Ae^{rx}
$\cos(kx)$ or $\sin(kx)$	$A\cos(kx) + B\sin(kx)$

(This table is incomplete! We will return to it later.)

Example 5. Solve the differential equation

$$y'' + 2y' + 10y = \cos(2x)\,.$$

Step 1. (Find y_h)

We already know[6] that the general solution y_h to the homogeneous differential equation

$$y'' + 2y' + 10y = 0$$

is given by

$$y_h(x) = e^{-x}\big(c_1\cos(3x) + c_2\sin(3x)\big)\,.$$

[6] See Example 3

Second-order inhomogeneous DEs: Particular solutions

Example 5. Solve the differential equation

$$y'' + 2y' + 10y = \cos(2x)\,.$$

Step 2. (Find y_p)

As a particular solution y_p, we try (see table):

$$y_p(x) = A\cos(2x) + B\sin(2x)\,.$$

Then we have

$$y_p'(x) = -2A\sin(2x) + 2B\cos(2x)$$
$$y_p''(x) = -4A\cos(2x) - 4B\sin(2x)$$

Second-order inhomogeneous DEs: Particular solutions

Example 5. Solve the differential equation

$$y'' + 2y' + 10y = \cos(2x)\,.$$

Step 2. (Find y_p)

Substituting $y = y_p$ in the above DE gives:

$$-4A\cos(2x) - 4B\sin(2x) \;+\; 2\big[-2A\sin(2x) + 2B\cos(2x)\big] + 10\big[A\cos(2x) + B\sin(2x)\big] = \cos(2x)\,.$$

Regrouping:

$$(6A + 4B)\cos(2x) + (6B - 4A)\sin(2x) = \cos(2x)\,.$$

This equation holds for all x if and only if

$$6A + 4B = 1 \quad \wedge \quad 6B - 4A = 0$$

Example 5. Solve the differential equation

$$y'' + 2y' + 10y = \cos(2x)\,.$$

Step 2. (Find y_p)

Solving the equations

$$6A + 4B = 1 \quad \wedge \quad 6B - 4A = 0$$

yields $A = \frac{3}{26}$ and $B = \frac{1}{13}$.

So the particular solution we find is:

$$y_p(x) = \tfrac{3}{26}\cos(2x) + \tfrac{1}{13}\sin(2x)\,.$$

Example 5. Solve the differential equation

$$y'' + 2y' + 10y = \cos(2x)\,.$$

Step 3. (Find $y = y_h + y_p$)

The general solution to the above differential equation is

$$\boxed{y(x) = e^{-x}\big(c_1\cos(3x) + c_2\sin(3x)\big) + \tfrac{3}{26}\cos(2x) + \tfrac{1}{13}\sin(2x)\,.}$$

Second-order inhomogeneous DEs: Particular solutions

The following table provides several candidates for y_p:

If $f(x)$ is a constant multiple of...	Then try as candidate for y_p...
p (constant)	A (constant)
$px + q$	$Ax + B$
$px^2 + qx + m$	$Ax^2 + Bx + C$
e^{rx}	Ae^{rx}
$\cos(kx)$ or $\sin(kx)$	$A\cos(kx) + B\sin(kx)$

The following examples show why this table is incomplete.

Example 6. Solve the differential equation

$$y'' + 5y' + 4y = 6e^{-4x}\,.$$

Step 1. (Find y_h)

We already know[7] that the general solution y_h to the homogeneous differential equation

$$y'' + 5y' + 4y = 0$$

is given by

$$y_h(x) = c_1 e^{-x} + c_2 e^{-4x}\,.$$

[7] See Example 1 (part 1)

Second-order inhomogeneous DEs: Particular solutions

Example 6. Solve the differential equation

$$y'' + 5y' + 4y = 6e^{-4x}\,.$$

Step 2. (Find y_p)

As a particular solution y_p, we would normally try (see table):

$$y_p(x) = Ae^{-4x}\,.$$

However, since $y_h(x) = c_1e^{-x} + c_2e^{-4x}$, we know that, for every constant A, the function Ae^{-4x} is a solution to the homogeneous equation

$$y'' + 5y' + 4y = 0$$

and thus can **not** be a solution to the inhomogeneous equation

$$y'' + 5y' + 4y = 6e^{-4x}\,.$$

Second-order inhomogeneous DEs: Particular solutions

Example 6. Solve the differential equation

$$y'' + 5y' + 4y = 6e^{-4x}\,.$$

Step 2. (Find y_p)

We try a different particular solution y_p, namely:

$$y_p(x) = A\,x\,e^{-4x}\,.$$

Then

$$\begin{aligned} y_p'(x) &= Ae^{-4x} - 4Axe^{-4x} \\ &= (A - 4Ax)e^{-4x} \\ y_p''(x) &= -4Ae^{-4x} - 4(A - 4Ax)e^{-4x} \\ &= (-8A + 16Ax)e^{-4x} \end{aligned}$$

Second-order inhomogeneous DEs: Particular solutions

Example 6. Solve the differential equation

$$y'' + 5y' + 4y = 6e^{-4x}\,.$$

Step 2. (Find y_p)

Substituting $y = y_p$ in the above DE gives:

$$(-8A + 16Ax)e^{-4x} \; + \; 5\big[(A - 4Ax)e^{-4x}\big] + 4\big[Axe^{-4x}\big] = 6e^{-4x}\,.$$

Regrouping:

$$(-3A)e^{-4x} = 6e^{-4x}\,.$$

This equation holds for all x if and only if

$$-3A = 6 \quad \Leftrightarrow \quad A = -2\,.$$

Second-order inhomogeneous DEs: Particular solutions

Example 6. Solve the differential equation

$$y'' + 5y' + 4y = 6e^{-4x}\,.$$

Step 2. (Find y_p)

So the particular solution we find is:

$$y_p(x) = -2xe^{-4x}\,.$$

Step 3. (Find $y = y_h + y_p$)

The general solution to the above differential equation is

$$\boxed{y(x) = c_1e^{-x} + c_2e^{-4x} - 2xe^{-4x}\,.}$$

Example 7. Solve the differential equation

$$y'' + 2y' + y = e^{-x}\,. \tag{8}$$

Step 1. (Find y_h)

We already know[8] that the general solution y_h to the homogeneous differential equation

$$y'' + 2y' + y = 0$$

is given by

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x}\,.$$

Consequently, neither Ae^{-x} nor Axe^{-x} is a solution to DE (8).

[8] See Example 2 (part 1)

Second-order inhomogeneous DEs: Particular solutions

Example 7. Solve the differential equation

$$y'' + 2y' + y = e^{-x}\,. \tag{8}$$

Step 2. (Find y_p)

We try a different particular solution y_p, namely:

$$y_p(x) = A\,x^2\,e^{-x}\,.$$

Computing y_p' and y_p'' and substituting $y = y_p$ in DE (8) gives:

$$(2A)e^{-x} = e^{-x}\,.$$

This equation holds for all x if and only if

$$2A = 1 \quad \Leftrightarrow \quad A = \tfrac{1}{2}\,.$$

Second-order inhomogeneous DEs: Particular solutions

Example 7. Solve the differential equation

$$y'' + 2y' + y = e^{-x}\,. \tag{8}$$

Step 2. (Find y_p)

So the particular solution we find is:

$$y_p(x) = \tfrac{1}{2}x^2e^{-x}\,.$$

Step 3. (Find $y = y_h + y_p$)

The general solution to differential equation (8) is

$$\boxed{y(x) = c_1e^{-x} + c_2xe^{-x} + \tfrac{1}{2}x^2e^{-x}\,.}$$

Second-order inhomogeneous DEs: Particular solutions

The following table provides several candidates for y_p:

If $f(x)$ is a constant multiple of...	Then try as candidate for y_p...
p (constant)	A (constant) (see Remark 2)
$px+q$	$Ax+B$ (see Remark 2)
px^2+qx+m	Ax^2+Bx+C (see Remark 2)
e^{rx}	Ae^{rx} (see Remark 1)
$\cos(kx)$ or $\sin(kx)$	$A\cos(kx)+B\sin(kx)$

Remarks:

1. If r is a single or double root of the characteristic equation, then we try Axe^{rx} or Ax^2e^{rx}, respectively, instead of Ae^{rx}.
2. If 0 is a single or double root of the characteristic equation, then we increase the degree of the candidate polynomial by 1 or 2, respectively.

Second-order inhomogeneous DEs: Particular solutions

Example 8. Solve the differential equation

$$y'' - 2y' = -4x - 2\,. \tag{9}$$

Step 1. (Find y_h)

The characteristic equation of the associated homogeneous DE

$$y'' - 2y' = 0$$

is

$$r^2 - 2r = 0\,.$$

This characteristic equation has roots $r_1 = 0$ and $r_2 = 2$, so we find

$$y_h(x) = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}\,.$$

Second-order inhomogeneous DEs: Particular solutions

Example 8. Solve the differential equation

$$y'' - 2y' = -4x - 2\,. \qquad (9)$$

Step 2. (Find y_p)

Since the right-hand side of (9) is $-4x-2$, our first guess would be (see table, ignoring Remark 2):

$$y_p(x) = Ax + B\,.$$

Then $y_p'(x) = A$ and $y_p''(x) = 0$. Substituting $y = y_p$ in DE (9) then gives:

$$0 - 2A = -4x - 2\,.$$

There is no constant A for which this equation holds for all x.

Second-order inhomogeneous DEs: Particular solutions

Example 8. Solve the differential equation

$$y'' - 2y' = -4x - 2\,. \tag{9}$$

Step 2. (Find y_p)

We try a different particular solution y_p by increasing the degree of the polynomial by 1. In other words, we try:

$$y_p(x) = Ax^2 + Bx + C\,.$$

Then $y_p'(x) = 2Ax + B$ and $y_p''(x) = 2A$. Substituting $y = y_p$ in DE (9) then gives:

$$2A - 2(2Ax + B) = -4x - 2\,.$$

This equation holds for all x if and only if $A = 1$ and $B = 2$.

Example 8. Solve the differential equation

$$y'' - 2y' = -4x - 2\,. \tag{9}$$

Step 2. (Find y_p)

So the particular solution we find is:

$$y_p(x) = x^2 + 2x\,.$$

Step 3. (Find $y = y_h + y_p$)

The general solution to differential equation (9) is

$$\boxed{y(x) = c_1 + c_2 e^{2x} + x^2 + 2x\,.}$$

Second-order inhomogeneous DEs: Particular solutions

The following table[9] provides several candidates for y_p:

If $f(x)$ is a constant multiple of...	Then try as candidate for y_p...
p (constant)	A (constant) (see Remark 2)
$px + q$	$Ax + B$ (see Remark 2)
$px^2 + qx + m$	$Ax^2 + Bx + C$ (see Remark 2)
e^{rx}	Ae^{rx} (see Remark 1)
$\cos(kx)$ or $\sin(kx)$	$A\cos(kx) + B\sin(kx)$

Remarks:

1. If r is a single or double root of the characteristic equation, then we try Axe^{rx} or Ax^2e^{rx}, respectively, instead of Ae^{rx}.
2. If 0 is a single or double root of the characteristic equation, then we increase the degree of the candidate polynomial by 1 or 2, respectively.

[9] See also Table 17.1 in Thomas' Calculus

Second-order inhomogeneous DEs: Particular solutions

- If $f(x)$ is the sum of several functions listed in the left column of the table, then include an appropriate candidate from the right column for each of those functions in y_p.
  - For example, as a particular solution to the inhomogeneous differential equation

    $$y'' - y' - 6y = e^{-x} - 7\cos x\,,$$

    try the function

    $$y_p(x) = Ae^{-x} + B\cos x + C\sin x\,.$$

    As an exercise[10], show that the general solution to this differential equation is

    $$y(x) = c_1 e^{3x} + c_2 e^{-2x} - \tfrac{1}{4}e^{-x} + \tfrac{49}{50}\cos x + \tfrac{7}{50}\sin x\,.$$

  - See also Example 5 in Section 17.2.

[10] This is Exercise 11 in Section 17.2

Final remarks

- Notation and terminology in Thomas' Calculus:
  - 'auxiliary equation' instead of 'characteristic equation'
  - 'nonhomogeneous' instead of 'inhomogeneous'
  - '$G(x)$' instead of '$f(x)$' for the right-hand side of an inhomogeneous differential equation
  - 'y_c' instead of 'y_h' for the general solution to the associated homogeneous differential equation[11]

$$a\,y'' + b\,y' + c\,y = 0$$

when solving an inhomogeneous differential equation

$$a\,y'' + b\,y' + c\,y = f(x)\,.$$

[11] Called the 'complementary equation' in Thomas' Calculus

Second-order differential equations
(Thomas' Calculus, Sections 17.1 & 17.2)

Themes:

1. Homogeneous equations
   - Characteristic equation
   - General solution

   Jump to Theme 1

3. Inhomogeneous equations
   - Solution approach
   - Particular solutions

   Jump to Theme 2

3. Initial value & boundary value problems

Second-order DEs: General solution

Recall the following example.

Example 4. Solve the inhomogeneous differential equation

$$y'' + 2y' + 10y = 20\,.$$

- We found that the general solution of this DE is given by

$$y(x) = e^{-x}\big(c_1\cos(3x) + c_2\sin(3x)\big) + 2\,.$$

- This general solution contains **all** solutions to the DE, one for each choice of c_1 and c_2.
- To find a **unique** solution, we need to specify two conditions[12] that determine the values of the arbitrary constants c_1 and c_2.

[12] Recall that for a **first**-order linear DE, only one condition is needed to determine a unique solution.

Second-order DEs: Unique solution

To find a **unique** solution to a second-order linear differential equation of the form

$$a\, y'' + b\, y' + c\, y = f(x)\,,$$

we need to specify two conditions that determine the values of the two arbitrary constants in the general solution.

Two options:

1. We specify the value of the solution function y and the value of its derivative y' at a single point (initial conditions):

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1\,.$$

2. We specify the value of the solution function y at two different points (boundary values):

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2\,.$$

Initial value problem & boundary value problem

This leads to the following two definitions:

1. An initial value problem consists of a differential equation together with specified initial conditions $y(x_0)$ and $y'(x_0)$.
   - An initial value problem always has a unique solution.
2. A boundary value problem consists of a differential equation together with specified boundary values $y(x_1)$ and $y(x_2)$.
   - A boundary value problem can have
     - a unique solution
     - more than one solution
     - no solution
   - See Section 17.1, Example 5 (unique solution) and Exercise 65 (no unique solution)[13].

[13] Exercise 65 is one of this week's tutorial exercises

Example 9. Solve the following differential equation subject to the given initial conditions:

$$y'' + 2y' + 10 = 20\,, \quad y(0) = 0\,, \quad y'(0) = 14\,.$$

- The general solution to the differential equation is given by[14]

$$y(x) = e^{-x}\big(c_1\cos(3x) + c_2\sin(3x)\big) + 2\,.$$

- We will use the initial conditions to find c_1 and c_2.
  First, we compute

$$\begin{aligned} y'(x) &= -e^{-x}\big(c_1\cos(3x) + c_2\sin(3x)\big) + \\ &\qquad e^{-x}\big(-3c_1\sin(3x) + 3c_2\cos(3x)\big) \\ &= e^{-x}\big((3c_2 - c_1)\cos(3x) - (3c_1 + c_2)\sin(3x)\big)\,. \end{aligned}$$

[14] See Example 4 (part 1)

Initial value problem: An example

- We have

$$\begin{aligned} y(x) &= e^{-x}\big(c_1\cos(3x) + c_2\sin(3x)\big) + 2 \\ y'(x) &= e^{-x}\big((3c_2 - c_1)\cos(3x) - (3c_1 + c_2)\sin(3x)\big) \end{aligned}$$

- Substituting $x = 0$ gives

$$\begin{aligned} y(0) &= e^{-0}\big(c_1\cos(0) + c_2\sin(0)\big) + 2 = c_1 + 2 \\ y'(0) &= e^{-0}\big((3c_2 - c_1)\cos(0) - (3c_1 + c_2)\sin(0)\big) = 3c_2 - c_1 \end{aligned}$$

- Together with the initial conditions $y(0) = 0$ and $y'(0) = 14$, we find that $c_1 = -2$ and $c_2 = 4$.
- The solution to the initial value problem in Example 9 is:

$$\boxed{y(x) = e^{-x}\big(-2\cos(3x) + 4\sin(3x)\big) + 2}$$