UNIVERSITY OF TWENTE.

Calculus 1B- Lecture 2

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16 November, 2020

Calculus 1B lecture 2

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Calculus 1 B lecture 2

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5.5

Indefinite Integrals and the substitution method

Introduction Substitution Method

Example:

Find: $$\int \frac{\cos x}{\sin x}\, dx$$

- We first show how the substitution $\sin(x) = u$ works here.
- Then we prove that it is mathematically correct
- Finally we explain how we found this substitution.

Introduction Substitution Method

$$\int \frac{\cos x}{\sin x}\, dx$$

If we substitute $\sin(x) = u$, what to do with $\cos(x)$ and dx??

We try to find out by “calculating” du.

We have:

$$u = \sin x \quad \text{so} \quad \frac{du}{dx} = \cos x \quad \text{and} \quad du = \cos x\, dx$$

This suggests to replace “cos x dx” by “du” !

$\sin(x) = u$ and $\cos x\, dx = du$ leads to

$$\int \frac{\cos x}{\sin x}\,dx = \int \frac{1}{\sin x}\cos x\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

$$(backwards - substitution) = \ln|\sin x| + C$$

Check by differentiation:

$$\frac{d}{dx}(\ln|\sin x| + C) = \frac{1}{\sin x}\cdot\cos x = \frac{\cos x}{\sin x}$$

Substitution Method

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

$$[= F(u) + C = F(g(x)) + C \;]$$

Check by differentiation:

$$\left(F(g(x)) + C\right)' = F'(g(x)) \cdot g'(x) = f(g(x)).g'(x)$$

Substitution Method

Strategy: replace the expression you don't like by "u":

- u = e-power
- u = (square) root
- u = sin or cos or tan
- $du = u'(x)dx$

Can lead to a simpler function in u… to integrate

Substitution Method

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example:

Find: $\displaystyle \int \frac{e^x}{e^x+1}\,dx$

Solution:

Substitution Method

Another example: Sometimes a useful substitution is more difficult …

Find: $\int x^3 \sqrt{x^2+1}\, dx$

Substitution Method

Another example: Sometimes a useful substitution is more difficult …

Find: $\int x^3 \sqrt{x^2+1}\, dx$

Solution: Just try the substitution: $u = \sqrt{x^2+1}$

Then: $\dfrac{du}{dx} = \dfrac{x}{\sqrt{x^2+1}}$ or: $dx = \dfrac{\sqrt{x^2+1}}{x}\, du$

Substituting gives:

$$\int x^3 \sqrt{x^2+1}\, dx = \int x^3 u \frac{\sqrt{x^2+1}}{x}\, du = \int x^2 u \sqrt{x^2+1}\, du$$

$$= \int x^2 u \cdot u\, du$$

$$= \int x^2 u^2\, du$$

Solution (continued):

In $\int x^2u^2\ du$ **we have to express *x* in *u* !!**

Now: $u=\sqrt{x^2+1}$ so $u^2=x^2+1$ so $x^2=u^2-1$

Hence: $\int x^3\sqrt{x^2+1}\,dx=\int x^2u^2\,du=\int (u^2-1)u^2\ du=\int\left(u^4-u^2\right)du$

$$=\frac{1}{5}u^5-\frac{1}{3}u^3+C$$

$$=\frac{1}{5}\left(\sqrt{x^2+1}\right)^5-\frac{1}{3}\left(\sqrt{x^2+1}\right)^3+C$$

You can check the result by differentiation.

5.6

Substitution Method
(Definite integrals)

Substitution Method

The substitution method for definite integrals is:

$$\int_a^b f(g(x))\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Proof:

According to the substitution method for indefinite integrals:

$$\int_a^b f(g(x))\cdot g'(x)dx = [F(g(x))]_{x=a}^{x=b} = F(g(b)) - F(g(a))$$

Also:

$$\int_{g(a)}^{g(b)} f(u)du = [F(u)]_{u=g(a)}^{u=g(b)} = F(g(b)) - F(g(a))$$

Substitution Method

$$\int_{a}^{b} f\big(g(x)\big)\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

So there are three changes:

1. The integrand changes (*via* $u = g(x)$)
2. The differential dx changes (dx = d$u/g'(x)$)
3. The limits of integration change

Example: Calculate $\int_0^{\frac{\pi}{4}} \tan x dx$

Solution: Because $\tan x = \dfrac{\sin x}{\cos x}$, we substitute $u = \cos(x)$

So du = –sin(x) dx, and hence sin(x) dx = –du

$$\int_0^{\frac{\pi}{4}} \tan x dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx = \int_{\cos 0}^{\cos(\frac{\pi}{4})} -\frac{du}{u} = \int_1^{\frac{\sqrt{2}}{2}} -\frac{1}{u} du =$$

$$= \left[-\ln|u|\right]_{u=1}^{u=\frac{\sqrt{2}}{2}} = -\ln\frac{\sqrt{2}}{2} + \ln 1 = -\ln\frac{\sqrt{2}}{2}$$

8.1

Integration by Parts

Integration by parts

By definition of integral and derivative

$$f(x)g(x) = \int \frac{d}{dx}\Big(f(x)g(x)\Big)dx$$

… and by the product rule

$$= \int \Big(f'(x)g(x) + f(x)g'(x)\Big)dx$$

Integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

Integration by parts

$$\int u \cdot dv = uv - \int v \cdot du$$

Example: Evaluate $\int x \sin x dx$

Solution:

You can check the result by differentiation. Work out!

Integration by parts twice

$$\int u \cdot dv = uv - \int v \cdot du$$

Example:

$$\int x^2 e^x \, dx.$$

Solution:

Integration by parts

$$\int u \cdot dv = uv - \int v \cdot du$$

Find: $$\int \ln x dx$$

Solution:

Find: $\int \ln x dx$

Solution:

Use partial integration with $u = \ln(x)$ and $dv = 1dx$.
Then $du = 1/x\, dx$ and $v = x$.

We conclude:

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx$$
$$= x \ln x - x + C$$

Because ln *x* is standard function, learn by head!

8.7

Improper Integrals

Improper integrals

In this section, we extend the concept $\int_a^b f(x)dx$ of a definite integral with continuous *f(x)* on finite bounded interval [a,b] to the cases where:

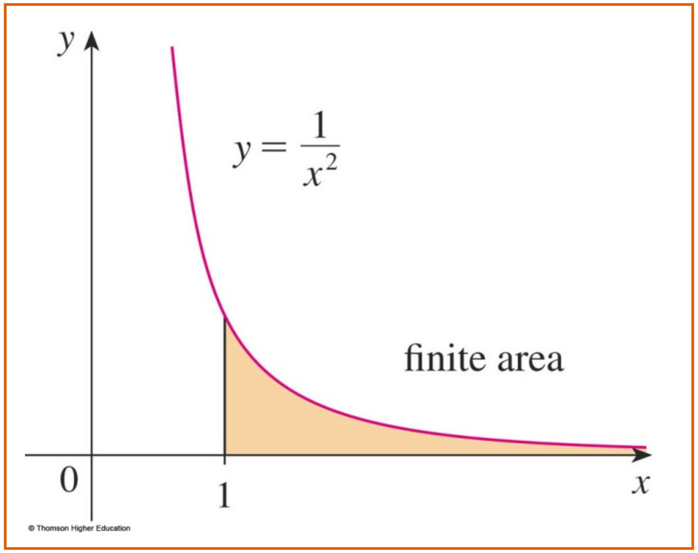
- the interval is infinite, thus $[a,\infty)$ or $(-\infty,b]$
- f has a vertical asymptote in $(a, b]$ or in $[a, b)$
- f has a discontinuity in $[a, b]$

In either case or combination of cases, the integral is called an ***improper integral***. For combinations the integral has to be split up in singular improperties.

Improper integrals – infinite intervals

Consider the infinite region S that lies:

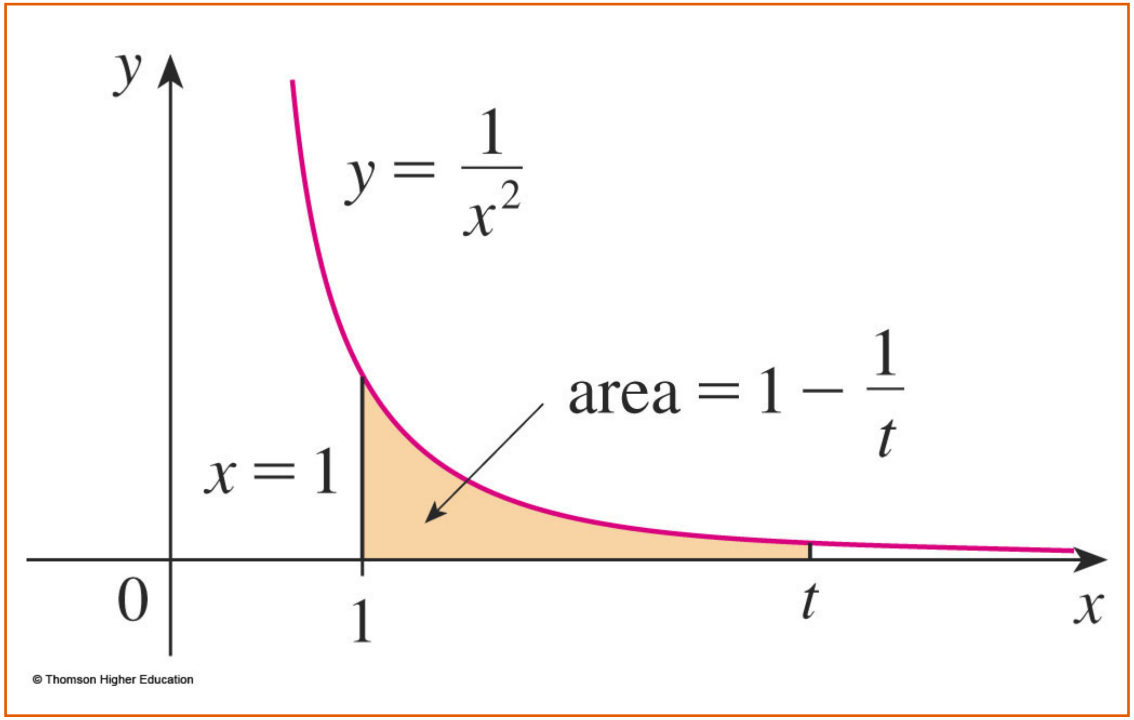
- Under the curve $y = 1/x^2$
- Above the x-axis
- To the right of the line $x = 1$



The area of the part of *S* that lies to the left of the line *x* = 1 (shaded) is:

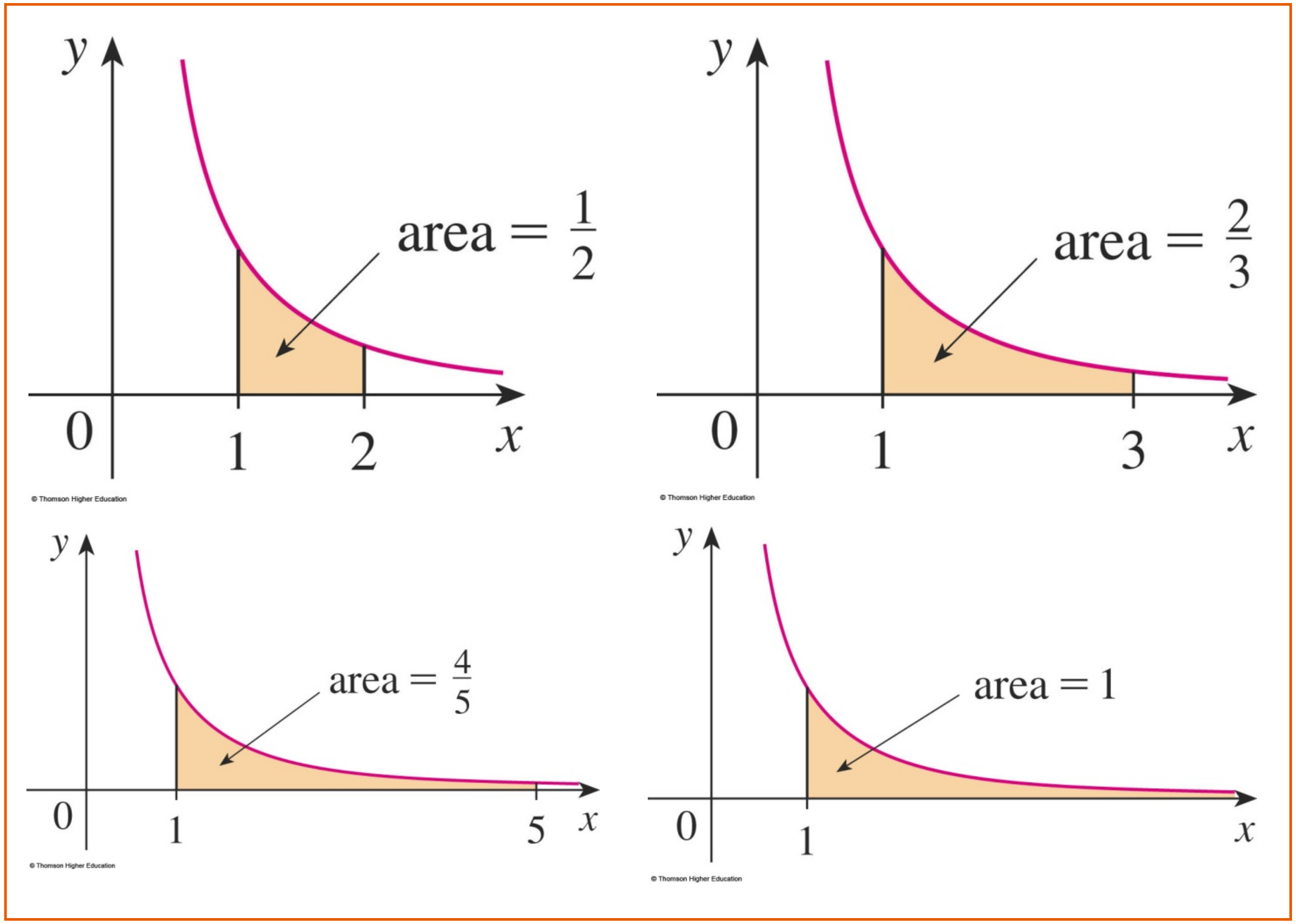
$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$

- Notice that $A(t) < 1$ no matter how large *t* is chosen.



INFINITE INTERVALS

The area of the shaded region approaches 1 as $t \to \infty$.



INFINITE INTERVALS

We observe that:

$$\lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left(1 - \frac{1}{t} \right) = 1$$

So, we say that the area of the infinite region S is equal to 1 and we write:

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x^2}\, dx = 1$$

Definition improper integral

If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_{t}^{b} f(x)dx$$

provided this limit exists (as a finite number).

Convergent and divergent

The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called:

- *Convergent* if the corresponding limit exists.
- *Divergent* if the limit does <u>not</u> exist.

Convergent and divergent

Example:

Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

According to the definition,
we have:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty}(\ln t - \ln 1)$$

$$= \lim_{t\to\infty} \ln t = \infty$$

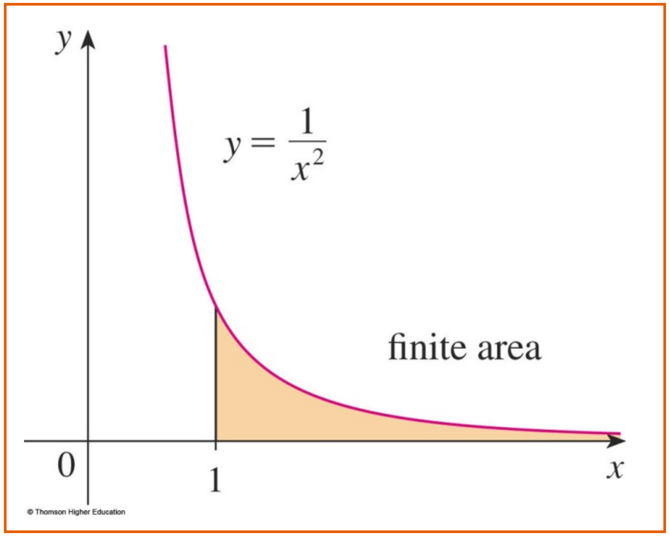
- The limit does not exist as a finite number.
- So, the integral is divergent.

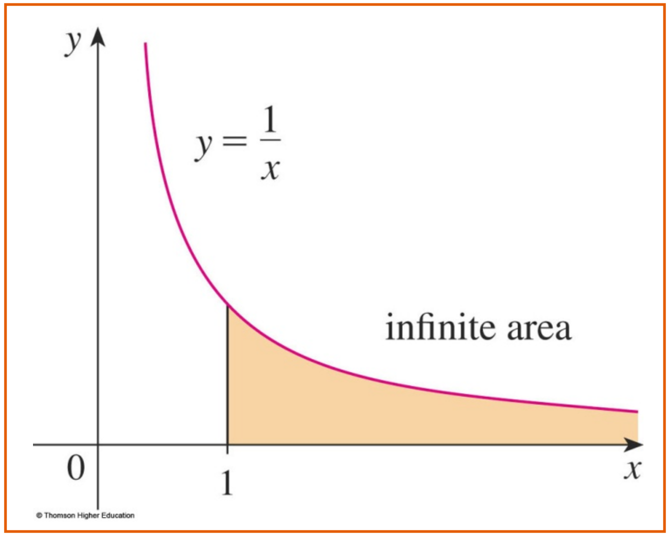
Convergent and divergent

Let's compare this result to
the example at the beginning of the section:

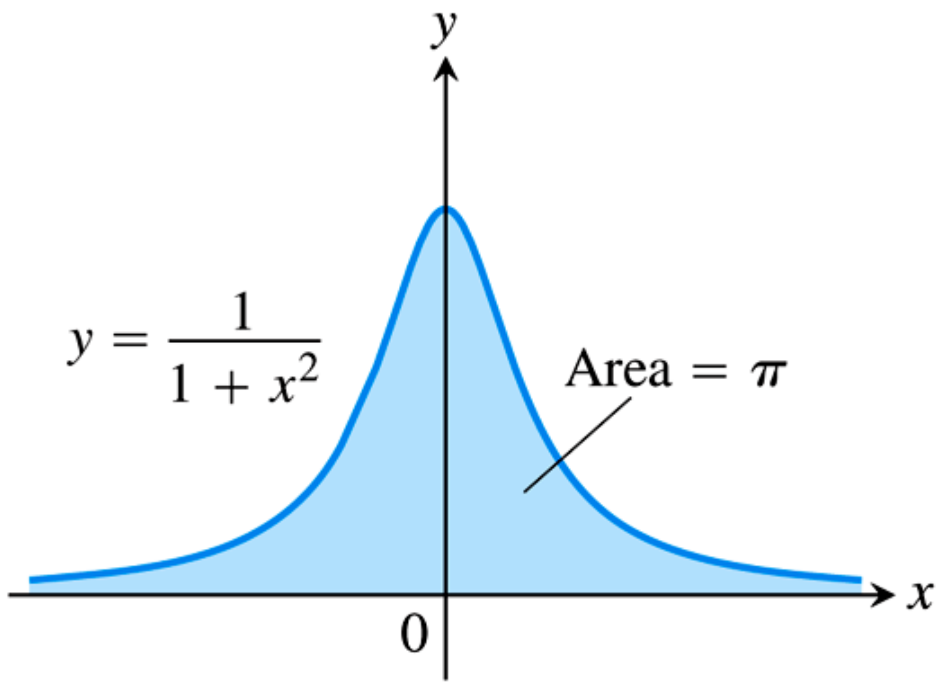
$$\int_1^{\infty} \frac{1}{x^2}\,dx \text{ converges} \qquad\qquad \int_1^{\infty} \frac{1}{x}\,dx \text{ diverges}$$

- Geometrically, this means the following.





Splitting integrals



If $f(x)$ is continuous on $(-\infty, \infty)$, then

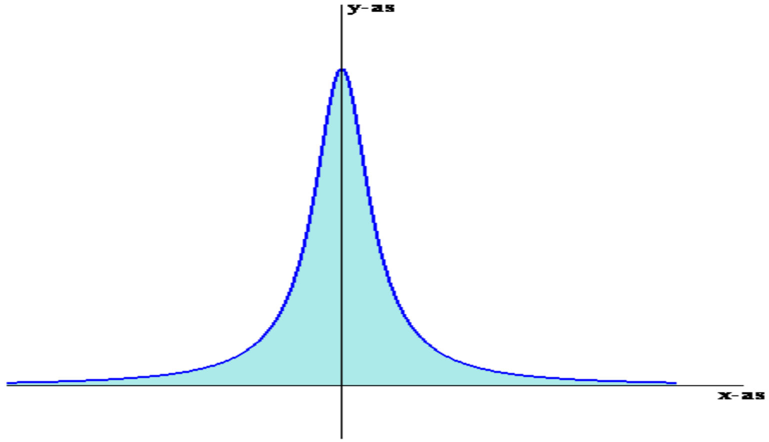
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx,$$

independent of the choice of c

Splitting integrals

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$



Solution: (Split)

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx \quad = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

Splitting integrals

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

$$\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{x^2+1}\,dx = \lim_{t\to-\infty} \left[\tan^{-1} x\right]_{x=t}^{x=0}$$

$$= \lim_{t\to-\infty} \left(\tan^{-1} 0 - \tan^{-1} t\right) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

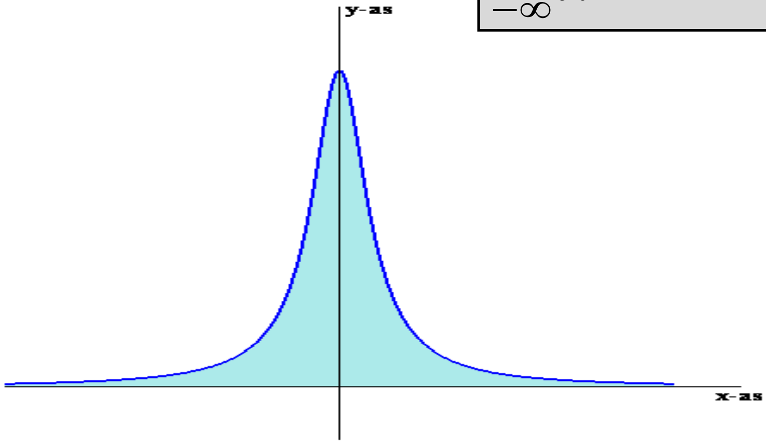
$$\int_{0}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left[\tan^{-1} x\right]_{x=0}^{x=t}$$

$$= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Splitting integrals

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

Conclusion:



$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Functions with ‘bad’ behaviour

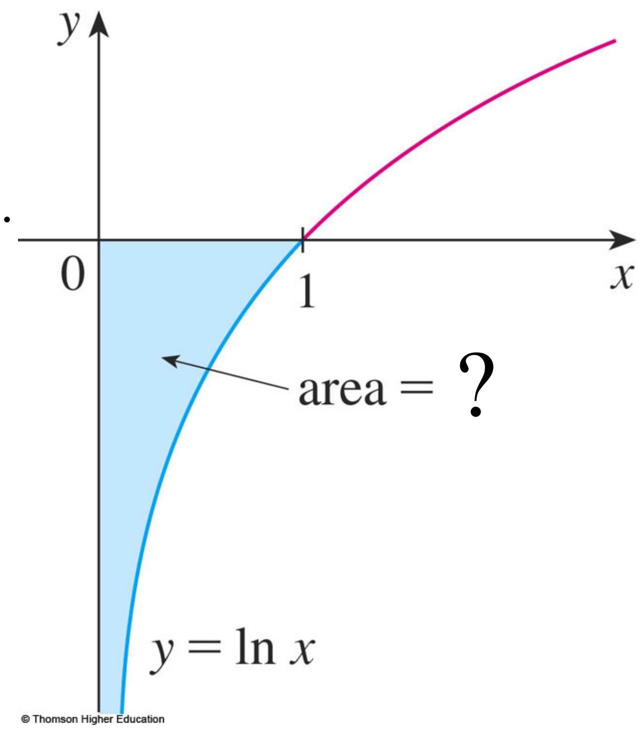
Example: Evaluate $\int_0^1 \ln x \, dx$

- We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since

$$\lim_{x \to 0^+} \ln x = -\infty$$

- Thus, the given integral is improper, and we have:

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$



Functions with 'bad' behaviour

Now, we integrate by parts with $u = \ln x$, and $dv = dx$,

$du = dx/x$, and $v = x$:

$$\int_t^1 \ln x\, dx = x \ln x\Big]_t^1 - \int_t^1 dx$$

$$= 1 \ln 1 - t \ln t - (1 - t)$$

$$= -t \ln t - 1 + t$$

Functions with ‘bad’ behaviour

To find the limit of the first term,
we use l’Hospital’s Rule:

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t}$$

$$= \lim_{t\to 0^+} \frac{1/t}{-1/t^2}$$

$$= \lim_{t\to 0^+} (-t)$$

$$= 0$$

Functions with ‘bad’ behaviour

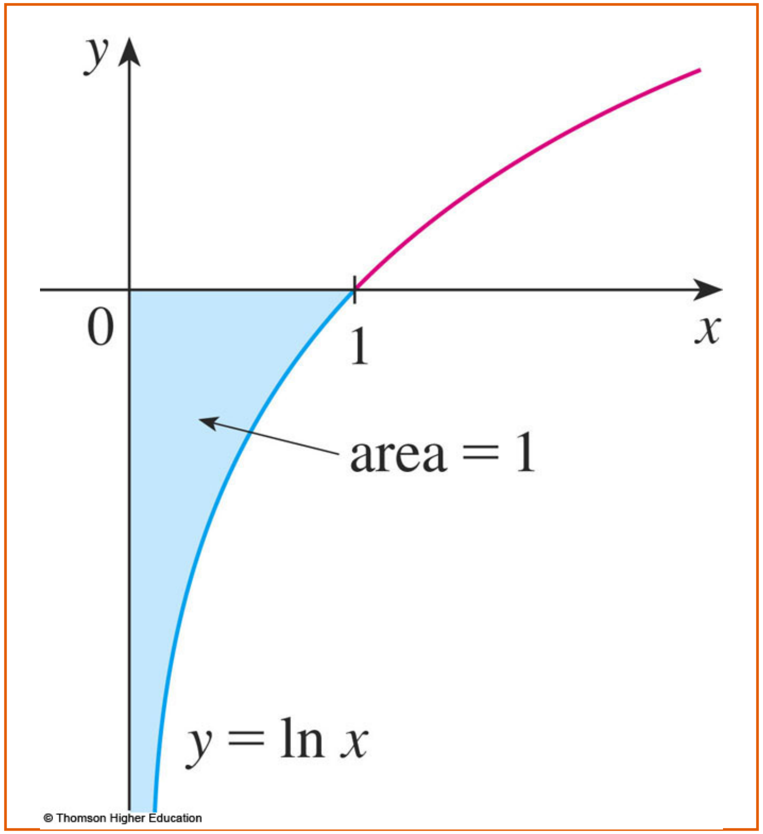
Therefore, $\displaystyle\int_0^1 \ln x\,dx = \lim_{t\to 0^+}(-t\ln t - 1 + t)$

$$= -0 - 1 + 0$$

$$= -1$$

The geometric interpretation of the result is shown.

- The area of the shaded region above $y = \ln x$ and below the x-axis is 1.



DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

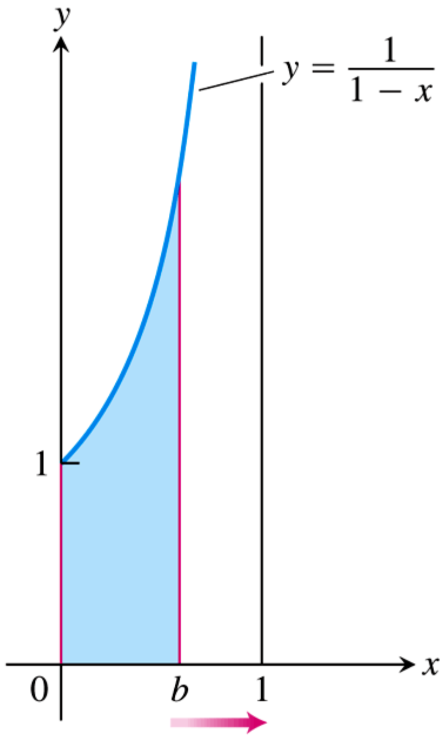
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Terminology

Definition for improper integrals of Type 1 or Type 2 :

If (one of) the limit(s) that define(s) the integral does not exist (or is infinite), then the integral is called ***divergent***.

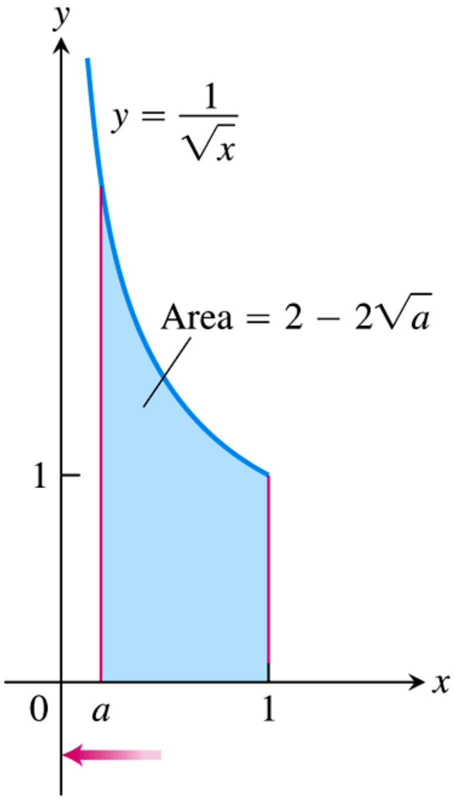


$\to -\ln(1-x)$

The integral is ***divergent***. (the area is infinite)

Terminology

If the limit(s) that define(s) the integral exist(s) (and are finite), then the integral is called ***convergent***.



$\rightarrow 2\sqrt{x}$

The integral is ***convergent*** (the area is finite)

Summarizing Exercises

a. Compute $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\, dx$

b. Compute $\displaystyle\int x \ln^2(x)\, dx$

Hint: apply partial integration twice

Summarizing Exercises

Compute $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$

Summarizing Exercises

Compute $\displaystyle\int x \ln^2(x)\ dx$

Hint: apply partial integration twice