Calculus 1B lecture 4

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Definition

A first-order differential equation (DE) is an equation involving an unknown function together with its derivative[1].

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- The function $y(x) = e^x - x - 1$ satisfies (1):

$$y'(x) = e^x - 1,$$

consequently

$$y'(x) - y(x) = (e^x - \ 1\ ) - (e^x - x - \ 1\ )$$

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$$\begin{aligned} y'(x) - y(x) &= (\cancel{e^x} - \cancel{1}) - (\cancel{e^x} - x - \cancel{1}) \\ &= x. \end{aligned}$$

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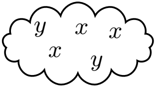
Definition

An ordinary first-order differential equation (ODE) is a differential equation where y' can be expressed as a function of y and x.

$$y' =$$



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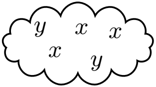
$y' =$ 

- The formal way to denote a normal equation is

$$y' = f(x, y).$$

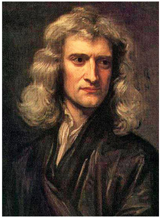
Example

The equation $y' - y = x$ can be rewritten as a normal equation:

$$y' = y + x. \tag{2}$$

Hence we have: $f(x, y) = y + x$.

Differential equations first appeared in the work of Newton and Leibniz, ± 1672.

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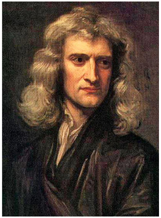

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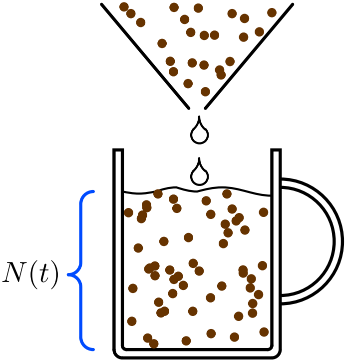
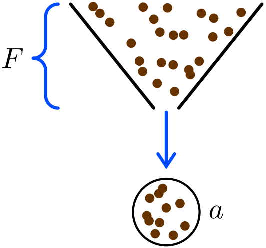
Next we derive a differential equation modelling the making of a mug of coffee via filtration.

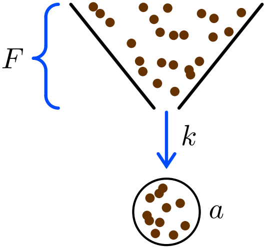


Const	**description**	**units**
F_0	initial number of coffee particles in the filter	–
k	coffee filtration constant	s^{-1}
Var	**description**	**units**
t	time	s
$F(t)$	nr of coffee particles in filter	–
$N(t)$	nr of coffee particles in mug	–



- The number a of coffee particles that leaves the filter during a short period of time Δt is proportional to the number of coffee particles F in the filter and to the length of the period Δt:

$$a \propto F\Delta t.$$



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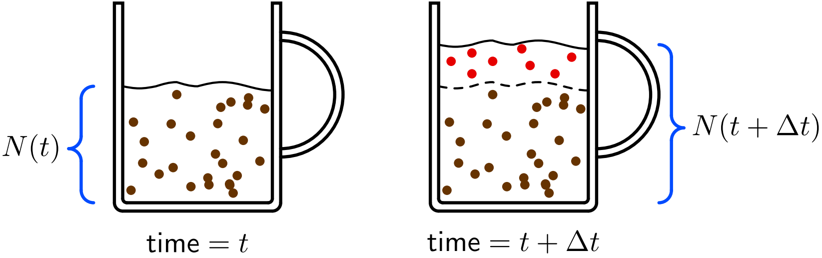
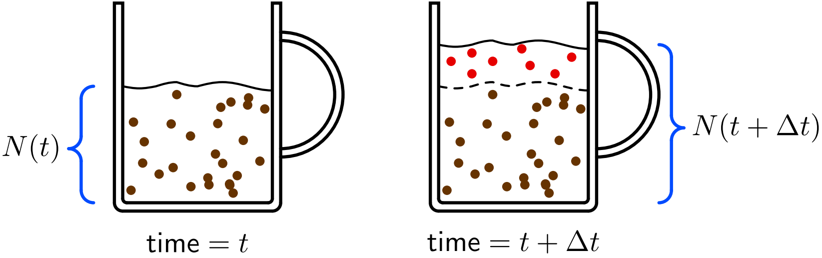
- The coffee filtration constant k is defined as the fraction particles that trickle through the filter per second, hence

$$a = kF\Delta t.$$



- The number of coffee particles that enters the mug between time t and $t + \Delta t$ is $N(t + \Delta t) - N(t)$.



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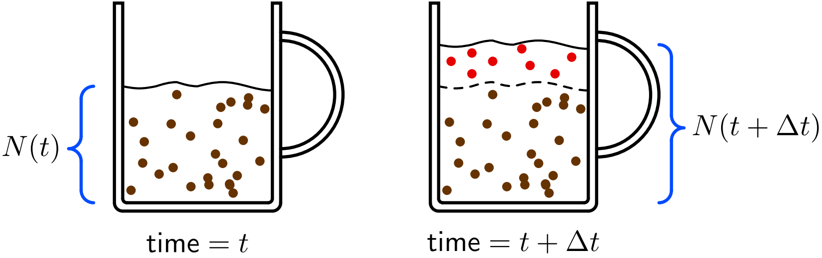

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- The number of coffeeparticles in the filter at time t is $F(t) = F_0 - N(t)$, so
- The number a of coffee particles that leaves the filter during a period of Δt seconds is approximately $a = k\big(F_0 - N(t)\big)\Delta t$.

Differential equations - Introduction Example:

- No coffee is wasted or added, so

$$N(t + \Delta t) - N(t) \approx k\big(F_0 - N(t)\big)\Delta t.$$

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- The function N satisfies the differential equation

$$\boxed{N' = k\big(F_0 - N\big)}$$

Given a differential equation, a natural question is how to find a solution.

We will show three techniques:

- Separation of variables, a way of calculating the solution;
- Direction fields, which is graphical construction;
- Integrating factor, a general way for calculating the solution.

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Quiz A solution of equation (1) is given by

(a) $y(x) = k\,e^{x}$; (b) $y(x) = -5e^{kx}$;

(c) $y(x) = 7e^{-kx}$; (d) $y(x) = \ln(kx)$;

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Answer (b) is correct. Why? Let us solve (1):

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- $y' = k\,y \Leftrightarrow \dfrac{dy}{dx} = k\,y$

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- $y = \pm e^{C_1}\,e^{k\,x}$
- $y = C\,e^{k\,x}$ with $C \in \mathbb{R}$

Indeed exponential change!

Examples of exponential growth are:

- the human population
  (annual growth percentage is approximately 1.14%);
- duckweed growth in a pond;
- carbon dating
  (the half-life of ^{14}C is approximately 5730 years);
- compound interest;
- Moore's law: the number of transistors on integrated circuits doubles approximately every two years.

The way of solving the differential equation for exponential change is an example of a more general method, applicable to differential equations that are separable:

Definition

A DE $y' = f(x, y)$ is separable if it can be written in the form

$$y' = g(x)h_1(y)$$

in which g is a function of x and h_1 is a function of y.

The solution of a separable differential equation can be found by the following steps:

- Let $y' = g(x)h_1(y)$ be written as $\dfrac{dy}{dx} = g(x)\dfrac{1}{h(y)}$.

Separable DEs (Thomas 7.2) General form

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- In which y is implicitly defined as a function of x. So
- In case H is invertible $y = H^{-1}\left(G(x) + C\right)$.

Solved!

Consider the differential equation[2]

$$\frac{dy}{dx} + P(x)\,y = 0$$

It can be solved applying the technique of the previous slide:

- Write as $\dfrac{dy}{dx} = -P(x)\,y$

[2] Thomas, end of page. 525.

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Let's have a break!

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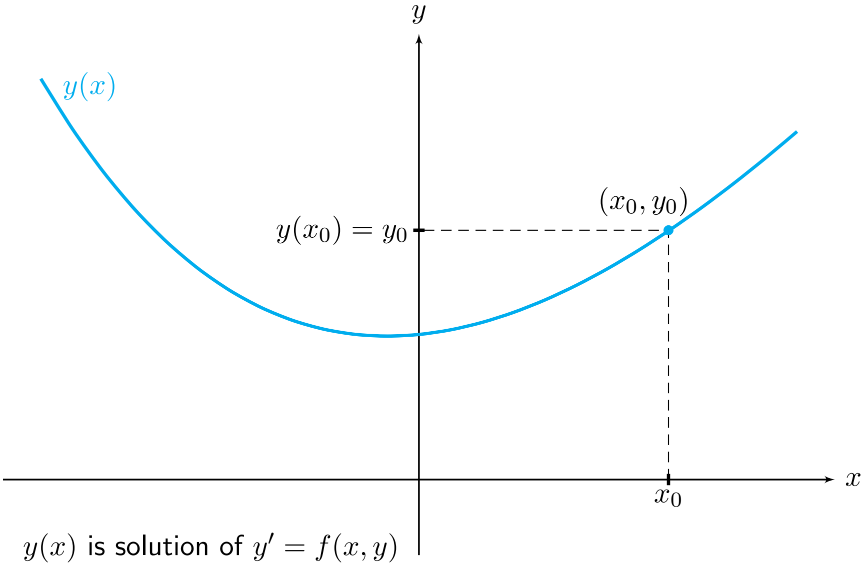

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TE.

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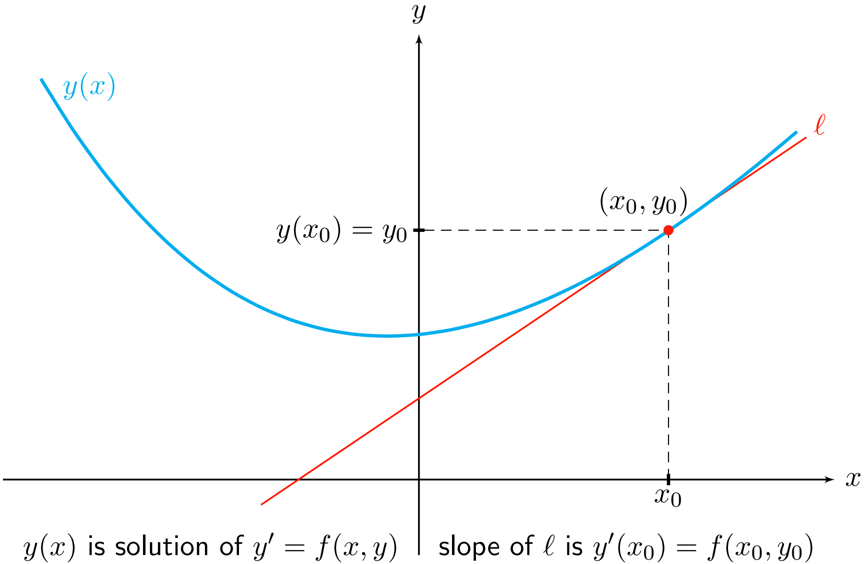

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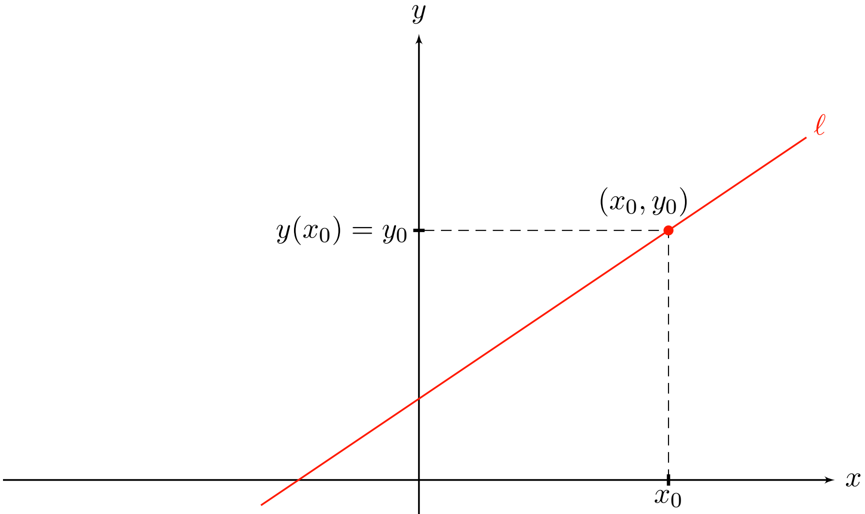

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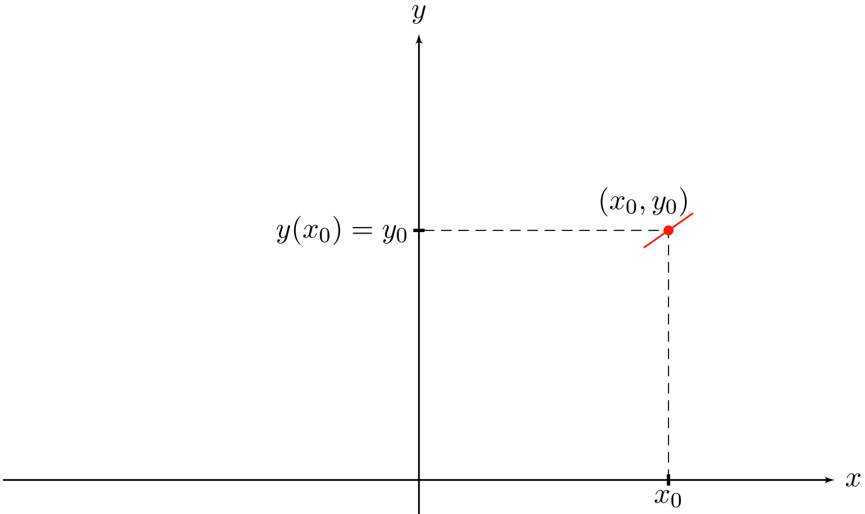

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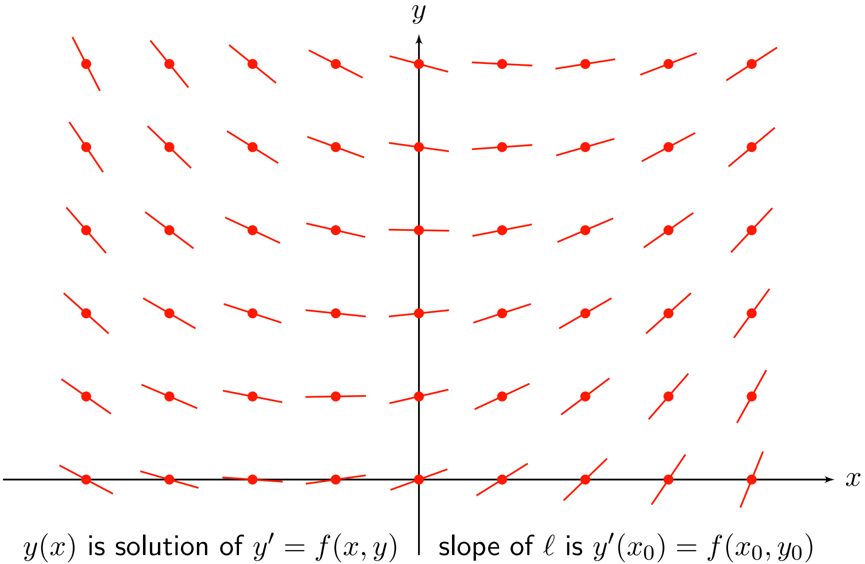

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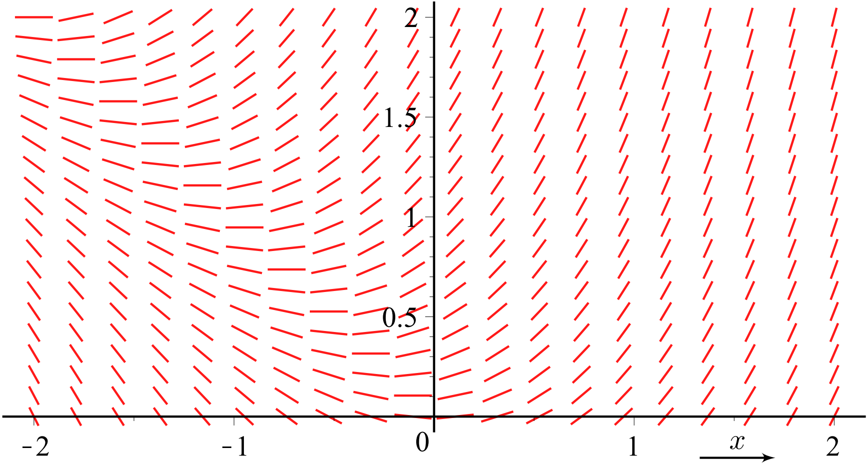

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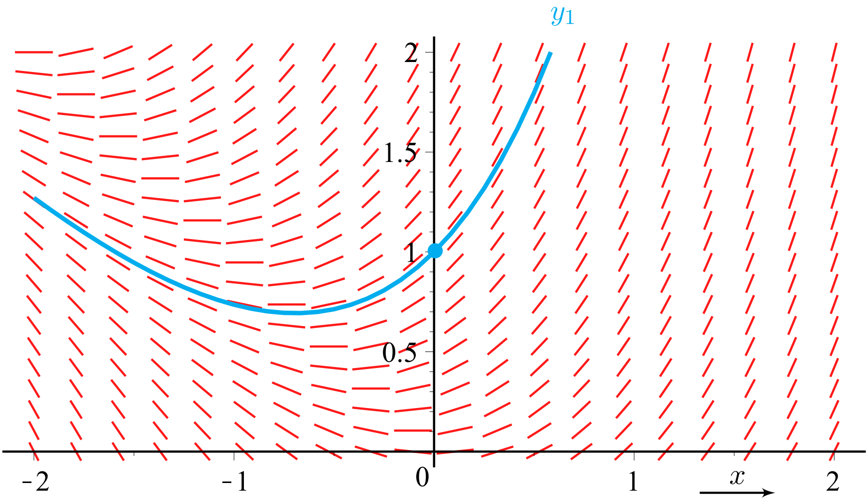
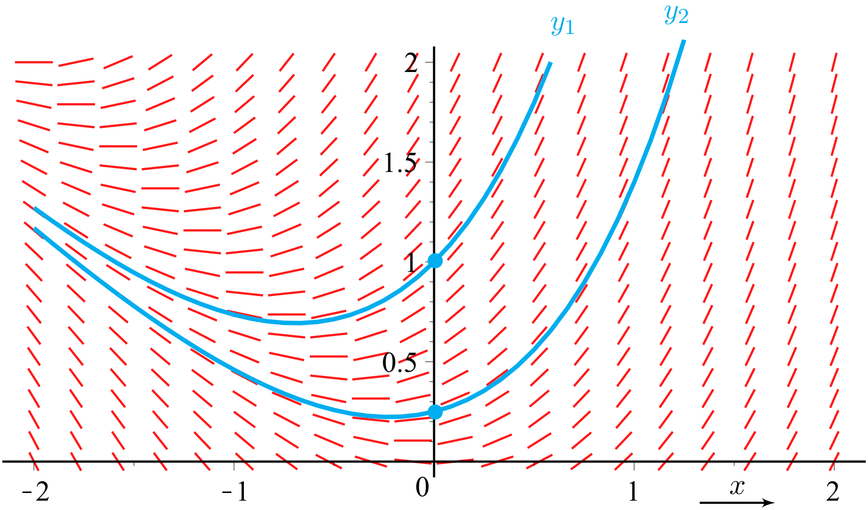


$y' = y + x$ $\qquad$ $y_1(0) = 1$

Direction Fields (Thomas 9.1) General idea



$y' = y + x$ $\qquad$ $y_1(0) = 1$ $\qquad$ $y_2(0) = .25$

TE.

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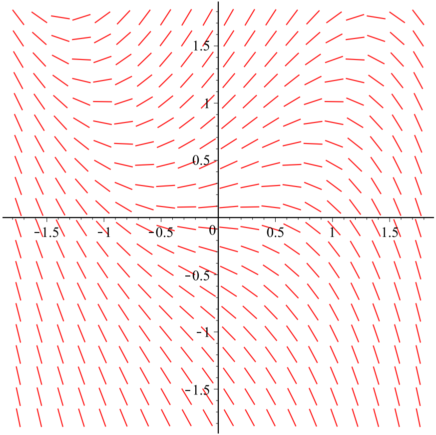

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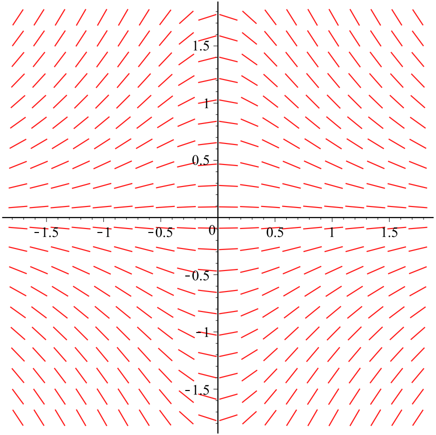
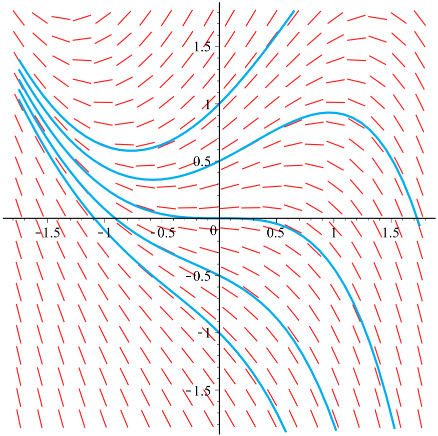
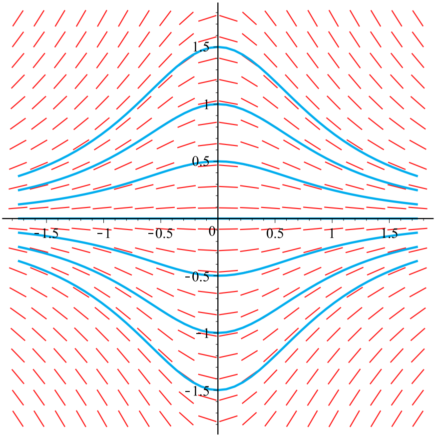


$$y' = -\frac{2xy}{x^2+1}$$





$$y' = y - x^2$$

$$y(x) = x^2 + 2x + 2 + Ce^x$$

$$y' = -\frac{2xy}{x^2+1} \qquad y(x) = \frac{C}{x^2+1}$$

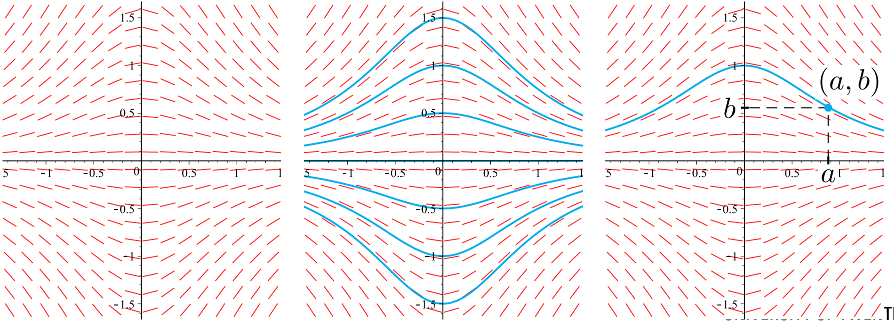
Direction Fields (Thomas 9.1) Initial value problem

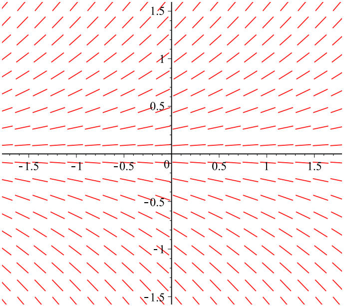
- In general a differential equation has infinitely many solutions.

Definition

An initial value problem consists of a differential equation and an equation of the form $y(a) = b$.

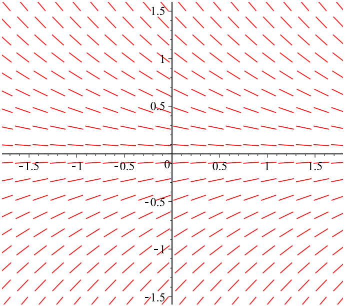
- An initial value problem in general has one solution.





$k > 0$

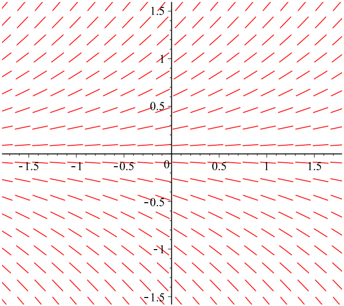
$$\boxed{y' = ky}$$



$k < 0$

- Let $y(t)$ be a solution. Define $Y(t) = y(t)e^{-kt}$, then

$$Y' = y'e^{-kt} - kye^{-kt}$$



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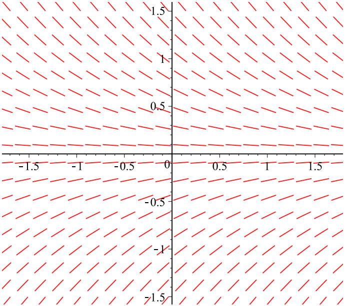

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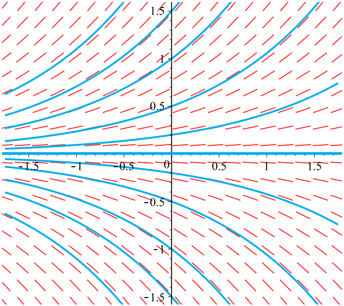
$$\begin{aligned} Y' &= y'e^{-kt} - kye^{-kt} \\ &= \big(y' - ky\big)e^{-kt} = 0, \end{aligned}$$

hence $Y(t)$ is constant. So $Y(t) = y(t)e^{-kt} = C$, and ..

Direction Fields (Thomas 9.1) Exponential change



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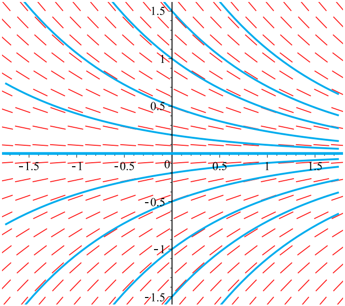

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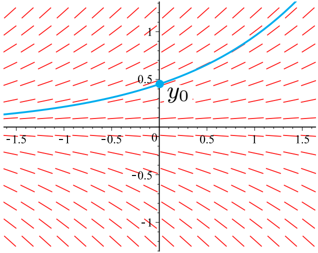
  hence $Y(t)$ is constant. So $Y(t) = y(t)e^{-kt} = C$, and ..
- The general solution of $y' = ky$ is $y(t) = Ce^{kt}$ with $C \in \mathbb{R}$.



Exponential growth with an initial condition

The (unique) solution of the initial value problem

$$\begin{cases} y' = ky, \\ y(0) = y_0 \end{cases}$$

is $y(t) = y_0 e^{kt}$.

- We often write $y(t) = y(0)e^{kt}$.

- The coffee problem is described by the initial value problem

$$\begin{cases} N' = k(F_0 - N) \\ N(0) = 0. \end{cases}$$

Direction Fields (Thomas 9.1) Making coffee

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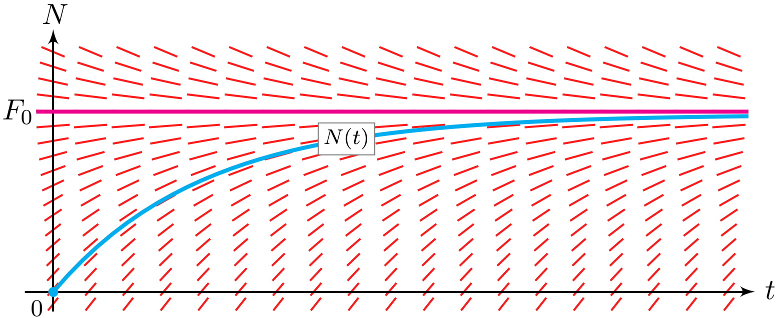
- $$\boxed{N(t) = F_0\bigl(1 - e^{-kt}\bigr)}$$



- The number of coffee particles in the mug at time t is

$$N(t) = F_0\big(1 - e^{-kt}\big), \qquad t \geq 0.$$

- For large t almost all coffee particles will be extracted from the filter into the brew.

Definition

A linear first-order equation is a differential equation of the form

$$y' + P(x)y = Q(x). \qquad (*)$$

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- Product rule: let $v(x)$ be an arbitrary function, then

$$\boxed{(v\,y)'} = vy' + \boxed{v'}\,y$$

- If $v' = vP$ then $(vy)' = vQ$.
  Consequently vy (and therefore y) can be obtained by integrating vQ.

- Rewrite $v' = vP$ to

$$\frac{v'}{v} = P(x).$$

[3] The constant of integration was taken 0, the anti-derivative is as simple as possible.

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$$\ln v(x) = \int P(x)\ \mathrm{d}x$$
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-
$$v(x) = e^{\int P(x)\,\mathrm{d}x}.$$
The function $v(x)$ is called the integrating factor [3].

[3] The constant of integration was taken 0, the anti-derivative is as simple as possible.

Solve the differential equation $xy' - y = x^2 e^x$.

write in standard form	$y' + Py = Q$	
find the integrating factor $v(x)$	$v = e^{\int P(x)\, \mathrm{d}x}$	
multiply with v	$vy' + vPy = vQ$	
use $v' = vP$	$vy' + v'y = vQ$	
product rule	$(vy)' = vQ$	
integrate	$vy = \displaystyle\int v(x)Q(x)\, \mathrm{d}x$	
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Integrating factor (Thomas 9.2) Example

Solve the differential equation $xy' - y = x^2\, e^x$.

write in standard form	$y' + Py = Q$	$y' - \dfrac{1}{x} y = x\, e^x$ $\boxed{\begin{matrix} \mathbf{P(x) = -\frac{1}{x}} \\ \mathbf{Q(x) = x\, e^x} \end{matrix}}$
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divide by v	$y = \frac{1}{v} \int v(x) Q(x)\, \mathrm{d}x$	$y(x) = x\left(e^x + C\right)$

Summarizing Exercises - 1

See slide #18: Given the DEs

$$y' = -\frac{2xy}{x^2+1} \qquad\qquad y' = y - x^2$$

Are they

- first order?
- Separable?
- Of the form dy/dx+P(x)y=0 ?
- Of the form dy/dx+P(x)y=Q(x) ?

Summarizing Exercises - 2

Solve

$$xy' + \frac{y}{x} = 0 \qquad ; \qquad y(1) = 1$$

Solve

$$xy' + \frac{y}{x} = e^{1/x}$$

-Contents-

- ☑ Integrals
- ☑ Calculation techniques for integrals
- ☑ Power and Taylor series

- ☑ First order ODEs
- ❑ Complex numbers
- ❑ Second order ODEs