Generalized symmetry Day2

Shuma NAKASHIBA

2025 年 5 月 16 日

目次

メモ: 前回資料の収集がつかなくなったので書き上げるのを (半ば) 諦めて, 今回の資料を改めて作成しました. preliminaries はもともと書き差しだったので日本語で書いています.

1 Historical background of generalized symmetry

1.1 higher-form symmetry

2 short comment on 0-form symmetry

As we saw last week, the discussion of extending (ordinary) 0-form symmetry to higher-form symmetry is essentially based on the current conservation: $d \star j = 0$. The existence of such a closed form enables us to identify the generators of symmetry as the set of d-dimensional topological operators, which allows us to extend the notion of symmetry as "action of topological operators acting on charged objects".
However, the current conservation in the context of classical field theory does not make sense in quantum field theory. Rather, we should consider Ward-Takahashi identity

$$\langle \delta_\epsilon(y)\mathcal{O}_R(x)\rangle = i\langle \epsilon(x)\partial_\mu j^\mu(y)\mathcal{O}_R(x)\rangle$$

as the property of quantum field theory with a certain symmetry. The WT identity is more natural when we consider QFT in a path-integral formulation.

3 Examples of higher-form symmetry

3.1 $d = 3+1$ $U(1)$ Maxwell gauge theory, with no matter field

As an example of higher-form symmetry appraring in physics, let's consider $U(1)$ pure Maxwell theory in $(3+1)$d spacetime. The action is given as

$$S = -\frac{1}{2g^2}\int F\wedge \star F = -\frac{1}{4g^2}\int F_{\mu\nu}F^{\mu\nu},$$

where $F = dA$ is the 2-form field strength of gauge field A [*1]. Obviously, this action is invariant under gauge transformation $A \to A + d\lambda$ (since $F = dA$ transforms as $F \to F' = d(A + d\lambda) = dA + d(d\lambda) =$

[*1] The representation of F in a certain local trivialization is $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. Recall that, for any k-forms α, β defined on n-dimensional manifold (mfd, in short), Hodge star operator is defined so that it satisfies

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta\rangle \mathrm{vol},$$

where $\langle\ ,\ \rangle$ is the Gram determinant and vol is the n-dim volume form. Using this, we obtain

$$\begin{aligned}
F\wedge\star F &= \langle F, F\rangle d^4x \quad (\text{Note that we consider Wick}-\text{rotated spacetime})\\
&= frac14\langle F_{\mu\nu}dx^\mu\wedge dx^\nu, F_{\rho\sigma}dx^\rho\wedge dx^\sigma\rangle d^4x\\
&= \frac{1}{4}F_{\mu\nu}F_{\rho\sigma}\det\begin{vmatrix}\langle dx^\mu, dx^\rho\rangle & \langle dx^\nu, dx^\rho\rangle\\ \langle dx^\mu, dx^\sigma\rangle & \langle dx^\nu, dx^\sigma\rangle\end{vmatrix}\\
&= \frac{1}{4}F_{\mu\nu}F_{\rho\sigma}(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma})\\
&= \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\left(= \frac{1}{4}(F_{\mu\nu}F^{\mu\nu} - F_{\mu\nu}F^{\nu\mu})\right).
\end{aligned}$$

$dA + 0$). The variation of S under infinitesimal transformation $A \to A + \delta A$ is

$$\delta S = -\frac{1}{2g^2}\int (d(\delta A) \wedge \star F + F \wedge \star d(\delta A)) = -\frac{1}{g^2}\int d(\delta A) \wedge \star F = -\frac{1}{g^2}\int (\delta A \wedge d \star F + d(\delta A \wedge \star F)),$$

where we used $\omega \wedge \star\eta = \eta \wedge \star\omega$ and $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{k_\omega} \wedge \eta$ for any k_ω-form ω and k_η-form η. Therefore, the equation of motion for A is $d \star F = 0$. This means that we have a conserved 2-form current F, implying that this pure $U(1)$ gauge theory has a $U(1)$ 1-form symmetry (a.k.a *electric 1-form symmetry*). Once we find a conserved current, we can write down the corresponding symmetry operator (which is topological) as an integral over closed 2-dim surface (λ is a parameter specifying a certain $U(1)$ element)

$$U^e_\lambda(\Sigma_2) = \exp\left\{ i\lambda \oint_\Sigma \frac{\star F}{g} \right\} =: \exp\left\{ i\lambda \oint_\Sigma J^e_2 \right\} \quad \text{(just a choice of normalization)}.$$

The integral $\lambda \oint_\Sigma J^e_2$ is the "charge" of objects inside Σ. The charge quantization $\int_\Sigma \star F = 2\pi\mathbb{Z}$ [*2] tells us that the obtained symmetry is $U(1)$.

The corresponding charged operator $W(q,\gamma)$ is extended along 1-dim loop in spacetime, called **Wislon line**:

$$W(q,\gamma) = e^{iq\int_\gamma A} \ (q \in \mathbb{Z}).$$

The action of $U^e_\lambda(\Sigma)$ on this $W(q,\gamma)$ can be written as

$$\langle U^e_\lambda(\Sigma) W(q,\gamma)\rangle = e^{iq\lambda \mathrm{Link}(\Sigma,\gamma)} \langle W(q,\gamma) U^e_\lambda(\Sigma')\rangle,$$

where $\mathrm{Link}(\Sigma,\gamma)$ is the *linking number* of Σ and γ [*3] . This action can be derived by considering the insertion of a Wilson line in spacetime. Inserting a Wilson line (a source of electric flux) modifies the path integral as

$$\int \mathcal{D}A\ e^{-S} \to \int \mathcal{D}A\ e^{iq\int_\gamma A} e^{-\frac{1}{2g^2}\int_{M_4} F \wedge \star F} = \int \mathcal{D}A\ e^{iq\int_\gamma \delta^3(\gamma) \wedge A - \frac{1}{2g^2}\int_{M^4} F \wedge \star F},$$

where $\delta^3(\gamma)$ is a "3-form delta function" defined as $\int_{M^T_3} \delta^3(\gamma) = 1$ for a 3-dim mfd M^T_3 which "transversely intersects" γ only once. Then, the equation of motion for A with the presence of Wilson line is

$$d \star F = qg^2 \delta^3(x \in \gamma).$$

We now integrate both sides of the EoM over a 3-dim mfd Σ^3 whose boundary is $\partial\Sigma_3 = \Sigma_2$:

$$(LHS) = \int_{\Sigma_3} d \star F = \oint_{\Sigma_2} \star F \quad (\text{Stokes}' \text{ theorem}),$$

[*2] This quantization is nothing but the consequence of Chern-Gauss-Bonnet theorem in $n = 2$ Riemannian mfd: The integral of the Pfaffian of curvature 2-form Ω over closed $2n$-dim mfd Σ is given by the Euler number $\chi(\Sigma)$ as

$$\int_\Sigma \mathrm{Pf}(\Omega) = (2\pi)^n \chi(\Sigma)\ .$$

[*3] Formally, this is defined as follows. First, we need to define "transverse intersection" of two submfds U_q (q-dim) and V_r (r-dim) on d-dim mfd M. At every intersection point $p \in U_q \cap V_r$, if their tangent spaces T_pU_q and T_pV_r together generate a $(q + r)$-dim subspace of T_pM, i.e., $T_pU_q \otimes T_pV_r \subseteq T_pM$ for $\forall p \in U_q \cap V_r$, then U_q and V_r are said to "intersect transversally".
Then, we define the linking number.

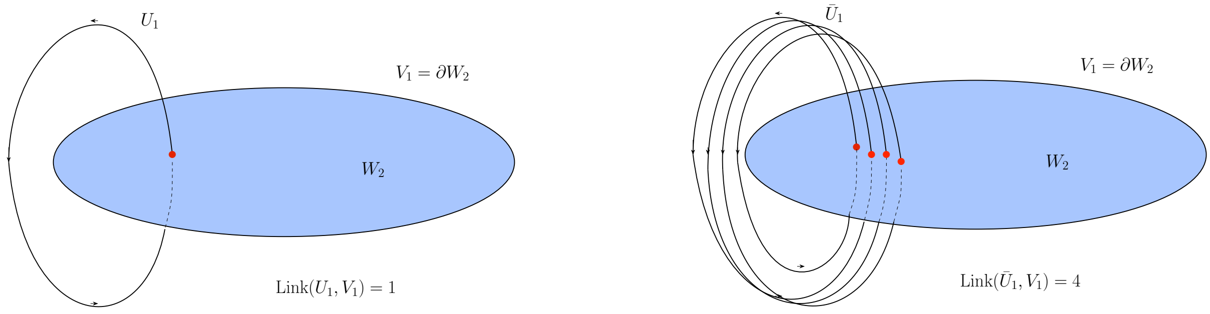

図 1: The illustration of the linking number in 3d.

$$(RHS) = qg^2 \int_{\Sigma_3} \delta(\gamma) = qg^2 \mathrm{Link}(\Sigma_2.\gamma).$$

Thus, by considering the "topological deformation" of Σ_2:

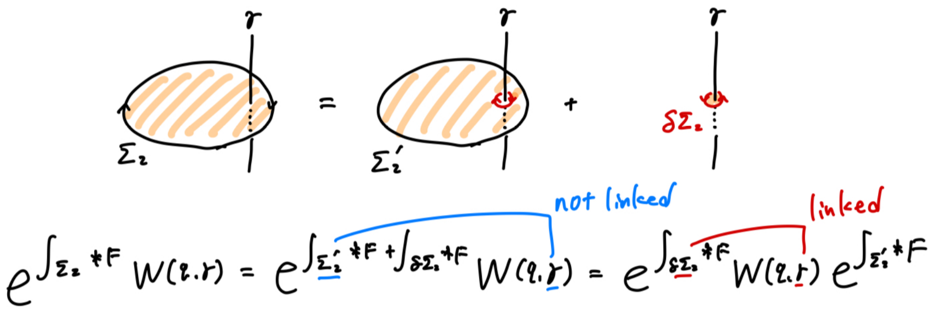

図 2: $\Sigma_2 = \Sigma_2' \cup \delta\Sigma_2$, where Σ_2 is a boundary of infinitesimal 3-dim mfd which links to γ.

Note that this deformation should be interpreted in path-integral formulation. Thus, we have

$$\langle \exp\left\{ i\frac{\lambda}{g^2} \oint_{\Sigma_2} \star F \right\} W(q,\gamma) \rangle = \langle e^{iq\lambda \mathrm{Link}(\Sigma_2,\gamma)} W(q,\gamma) \exp\left\{ i\frac{\lambda}{g^2} \oint_{\Sigma_2'} \star F \right\} W(q,\gamma) \rangle,$$

which is the same as (**??**).
By the way, the action of U^e_λ on $W(q,\gamma)$ can be seen as the shift of 1-form gauge field A by $A \mapsto A + a$, where $\oint_\gamma a = \lambda \mathrm{Link}(\Sigma_2, \gamma)$. In fact, the associate current $\star J^e$ is the conjugate momentum of A, generating the shift of A.

We can also consider the "dual" 1-form symmetry: due to the Bianchi identity, we have $dF = 0$ (that is, the pure Maxwell field is flat). By regarding this identity as the conservation of 2-form current

$J_2^m := \frac{1}{2\pi} \star F$, we can write down the correponding symmetry operator as

$$U_\lambda^m(\Sigma_2) = \exp\left\{ i\lambda \oint_{\Sigma_2} \star J_2^m \right\} = \exp\left\{ i\lambda \oint_{\Sigma_2} \frac{F}{2\pi} \right\}.$$

This 1-form symmetry is called *magnetic* 1-*form symmetry.* Its charged operator is the so-called **'t Hooft line**

$$T_1(m,\gamma) := e^{im\int_\gamma \tilde{A}},$$

where $\tilde{A}$ is the dual gauge field defined by $\star F = d\tilde{A}$. The action of the 1-form magnetic symmetry on a 't Hooft line is completely analogous to the case of electric 1-form symmetry:

$$\langle U_\lambda^m(\Sigma_2) T_1(m,\gamma)\rangle = e^{im\lambda \mathrm{Link}(\Sigma_2,\gamma)} \langle T_1(m,\gamma) U_\lambda^m(\Sigma_2')\rangle.$$

It is insightful to consider coupling "background 1-form $U(1)$ gauge field" to our conserved 2-form currents, as we have done in the case of ordinary $U(1)$ gauge theory. This is done by modifying the action integral [*4]

$$S = \frac{1}{2g^2}$$

3.2

4 Non-invertible symmetry

Let us now turn our attention to the other of the "two generalization directions" mentioned earlier. In the previous discussions, symmetry operators have always had their inverses. Now, let us consider what kind of "symmetry" will appear when such restrictions are relaxed. This is no longer the usual group-like symmetry, but is what we call "non-invertible symmetry" or "non-invertible symmetry".

4.0.1 Algebraic Structure of Non-invertible Symmetry: Fusion Category

The algebraic structure of non-invertible symmetry is not a group, but it in general has a categorical structure called a fusion category. To say a little more, in a theory with irreversible symmetries, there is a "fusion rule" relation between symmetry defect operators that appear in the theory. This is regarded as a projection between defect and defect, in that defects "fuse" with each other to create another defect. defects in the sense that defects "fuse" with each other to produce another defect. Then, we can consider a category with (simple) defects as "objects" and the fusion rule between defects as "morphisms". This sphere has a "richer structure" algebraically than the general sphere in that it has a tensor product and

[*4] 註: 系の力学を記述する「作用積分」と, 演算子の場に対する「作用」を区別するために, 以降は前者を"action integral" と呼ぶことにします.

a natural transformation called ”F-symbol”. (=fusion sphere) in that this sphere has tensor products and “F-symbol” natural transformations.
The fusion category consists of the following structures:

- tensor functor
- natural transformation

But why we should care about such a categorical structure? Note that the algebraic structure of non-invertible symmetry cannot be totally arbitrary: there should be some rules or orders. More precisely, we need some ”consistency condition” for valid physical systems.

4.0.2 Example from lattice

The (probably) most typical example of field theory with non-invertible symmetry is of the critical transverse-field Ising model (TFIM). Its Hamiltonian is given by

$$H = -\sum_{j=1}^{L} (Z_{j-1}Z_j + X_j),$$

with periodic boundary condition $X_j = X_{j+L}, Z_j = Z_{j+L}$. This model has $\mathbb{Z}_2$ symmetry

4.0.3 Example from continuous

5 What comes next?

In this section, we will see what comes as following topics after studying gerenalized symmetry in this text. I appologize that I do not understand the following topics and cannot explain any details.

5.1 Generalized charges

5.2 Symmetry TFT

参考文献

[1] T. Daniel Brennan, Sungwoo Hong. Introduction to Generalized Global Symmetries in QFT and Particle Physics. arXiv: 2306.00912