Generalized symmetry Day3

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1 (Temporal) Plan of this informal seminar

In this week, I will try to show you the topics below:

- The remaining part of $d = 3 + 1$ $U(1)$ Maxwell theory:
  - Gauging 1-form symmetry and 1-form anomaly
  - Duality between electric and magnetic 1-form symmetry
  - (If time allowed) the screening of string charge under the existence of matter field
- The introduction to discrete higher-form symmetry:
  - General property of discrete gauge theory, in brief
  - 3D toric code, as the manifestation of $\mathbb{Z}_2^e \times \mathbb{Z}_2^m$ 1-form symmetry

Also, there are other topics I want to cover (or, be covered by someone else):

- Spontaneous symmetry breaking of higher-form symmetry (There is a nice lecture note by McGreevy [4])
  - General property (such as topological order, Goldstone boson), from the perspective of higher-form symmetry
  - Examples (e.g. Z_N gauge theory, or BF theory)
- More about gauging higher-form symmetry and higher-form anomaly
- The classification of SPT phases with higher-form symmetry
- Lattice realizaton of higher-form symmetry
- More about higher group theory

I just intended to list up those topics as a suggestion for future seminar contents. Especially, the lattice one is totally unfamiliar to me and I want other people to cover that (, if necessary).

We can also go into non-invertible way of generalization, like

- Algebraic property of non-invertible symmetry: fusion category (or modular tensor category)
- Examples from rational CFT:
  - Ising CFT $\Leftrightarrow$ $\mathrm{TY}_+(\mathbb{Z}_2)$, its gauging and relationship to Majorana CFT (see Shu-Heng's lecture note [5])
  - $c = 1$ free compact boson, including Wess-Zumino-Witten model ($R = \sqrt{2}$) and orbifold CFT ($R = 1/\sqrt{2}$)
- Relationship to anyon braiding
- Lattice construction of non-invertible symmetry, its gauging, and relationship to LSM-type anomaly (e.g. Seifnashri's this work [6])
- Are there any others? (I don't know)

I think it's still enough if we cannot cover non-invertible side as long as we focus on higher-form side, but my interest may be a little bit more close to non-invertible stuff.

As applications of those two generalizations, we may go further into

- Higher gauging, and half-space gauging (though these two concept are essentially different)
- Symmetry TFT construction
- Subsystem symmetry: ”foliated” gauge field
- Lattice realization of various generalized symmetry (any kind of topics is appreciated)

although it will be very hard to cover the whole things (depending on the choice of next speakers).

2 Short comment on last week's discussion

2.1 About the derivation of operator action on charged objects

Last week, I tried to introduce a pure $3+1$-d $U(1)$ Maxwell theory and ”derived” the action of an electric 1-form operator on Wilson line from the definition of correlation function. I wrote the modified equation of motion as the form $d \star F = qg^2 \delta^3(\gamma)$. However, since both sides are not just numbers but operators acting on charged object (note that our 2-form electric symmetry), we should actually write it instead as

$$d \star F W(q, \gamma) = qg^2 \delta^3(\gamma) W(q, \gamma)$$

which makes much sense. This corresponds to considering Ward-Takahashi identity instead of just the classical equation of motion.

3 Example: continued

What is written below is totally the same as the one of last week's material (except the last subsubsection), so you may skip all of this.

3.1 $D = 3 + 1$ $U(1)$ Maxwell gauge theory

3.1.1 Electric and Magnetic 1-form symmetry

As an example of higher-form symmetry apprаring in physics, let's consider $U(1)$ pure Maxwell theory in $(3+1)$d spacetime. The action is given as

$$S = -\frac{1}{2g^2}\int F \wedge \star F = -\frac{1}{4g^2}\int F_{\mu\nu}F^{\mu\nu}, \tag{3.1}$$

where $F = dA$ is the 2-form field strength of gauge field A [*1]. Obviously, this action is invariant under gauge transformation $A \to A + d\lambda$ (since $F = dA$ transforms as $F \to F' = d(A + d\lambda) = dA + d(d\lambda) = dA + 0$). The variation of S under infinitesimal transformation $A \to A + \delta A$ is

$$\delta S = -\frac{1}{2g^2}\int (d(\delta A) \wedge \star F + F \wedge \star d(\delta A)) = -\frac{1}{g^2}\int d(\delta A) \wedge \star F = -\frac{1}{g^2}\int (\delta A \wedge d \star F + d(\delta A \wedge \star F)),$$

where we used $\omega \wedge \star\eta = \eta \wedge \star\omega$ and $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{k_\omega} \wedge \eta$ for any k_ω-form ω and k_η-form η. Therefore, the equation of motion for A is $d \star F = 0$. This means that we have a conserved 2-form current F, implying that this pure $U(1)$ gauge theory has a $U(1)$ 1-form symmetry (a.k.a *electric 1-form symmetry*).

Once we find a conserved current, we can write down the corresponding symmetry operator (which is topological) as an integral over closed 2-dim surface (λ is a parameter specifying a certain $U(1)$ element)

$$U^e_\lambda(\Sigma_2) = \exp\left\{ i\lambda \oint_\Sigma \frac{\star F}{g} \right\} =: \exp\left\{ i\lambda \oint_\Sigma J^e_2 \right\} \quad \text{(just a choice of normalization)}.$$

[*1] The representation of F in a certain local trivialization is $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. Recall that, for any k-forms α, β defined on n-dimensional manifold (mfd, in short), Hodge star operator is defined so that it satisfies

$$\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle \mathrm{vol},$$

where $\langle\ ,\ \rangle$ is the Gram determinant and vol is the n-dim volume form. Using this, we obtain

$$\begin{aligned} F \wedge \star F &= \langle F, F \rangle d^4x \quad \text{(Note that we consider Wick} - \text{rotated spacetime)} \\ &= frac14 \langle F_{\mu\nu}dx^\mu \wedge dx^\nu, F_{\rho\sigma}dx^\rho \wedge dx^\sigma \rangle d^4x \\ &= \frac{1}{4}F_{\mu\nu}F_{\rho\sigma} \det \begin{vmatrix} \langle dx^\mu, dx^\rho \rangle & \langle dx^\nu, dx^\rho \rangle \\ \langle dx^\mu, dx^\sigma \rangle & \langle dx^\nu, dx^\sigma \rangle \end{vmatrix} \\ &= \frac{1}{4}F_{\mu\nu}F_{\rho\sigma}(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}) \\ &= \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \left(= \frac{1}{4}(F_{\mu\nu}F^{\mu\nu} - F_{\mu\nu}F^{\nu\mu}) \right). \end{aligned}$$

The integral $\lambda \oint_\Sigma J_2^e$ is the "charge" of objects inside Σ. The charge quantization $\int_\Sigma \star F = 2\pi\mathbb{Z}$ [*2] tells us that the obtained symmetry is $U(1)$.

The corresponding charged operator $W(q,\gamma)$ is extended along 1-dim loop in spacetime, called **Wislon line**:

$$W(q,\gamma) = e^{iq\int_\gamma A} \ (q \in \mathbb{Z}).$$

The action of $U_\lambda^e(\Sigma)$ on this $W(q,\gamma)$ can be written as

$$\langle U_\lambda^e(\Sigma) W(q,\gamma)\rangle = e^{iq\lambda \mathrm{Link}(\Sigma,\gamma)} \langle W(q,\gamma) U_\lambda^e(\Sigma')\rangle,$$

where $\mathrm{Link}(\Sigma,\gamma)$ is the *linking number* of Σ and γ [*3] . This action can be derived by considering the

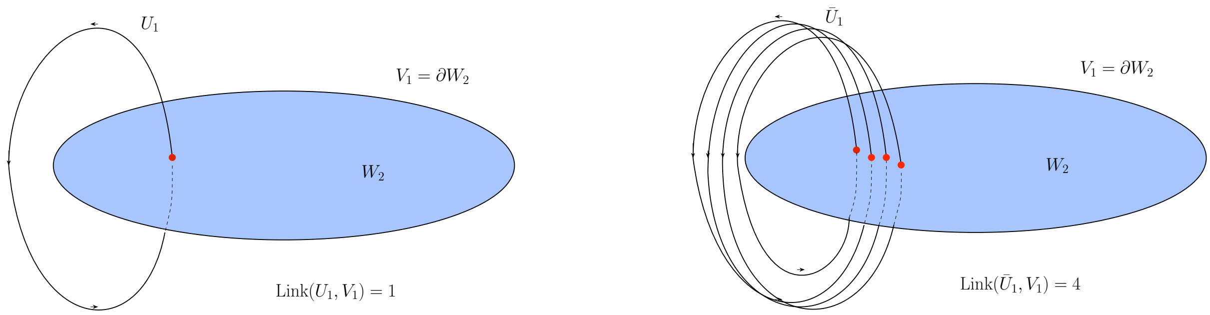


図 1: The illustration of the linking number in 3d.

insertion of a Wilson line in spacetime. Inserting a Wilson line (a source of electric flux) modifies the

[*2] This quantization is nothing but the consequence of Chern-Gauss-Bonnet theorem in $n = 2$ Riemannian mfd: The integral of the Pfaffian of curvature 2-form Ω over closed $2n$-dim mfd Σ is given by the Euler number $\chi(\Sigma)$ as

$$\int_\Sigma \mathrm{Pf}(\Omega) = (2\pi)^n \chi(\Sigma) \ .$$

[*3] Formally, this is defined as follows. First, we need to define "transverse intersection" of two submfds U_q (q-dim) and V_r (r-dim) on d-dim mfd M. At every intersection point $p \in U_q \cap V_r$, if their tangent spaces T_pU_q and T_pV_r together generate a $(q+r)$-dim subspace of T_pM, i.e., $T_pU_q \otimes T_pV_r \subseteq T_pM$ for $\forall p \in U_q \cap V_r$, then U_q and V_r are said to "intersect transversally".

The linking number is defined by two oriented submanifolds (U_q, V_r) in $d+1$-dim spacetime, satisfying $q + r = d$. We further assume that, U_q and V_r themself do not have the intersecting point, and each submanifold can be thought to form the boundary of a one-dim higher open manifold. Then, by denoting $\partial W_{r+1} = V_r$, U_q and W_{r+1} can intersect in a finite number of points p_i (the set $\{p_i\}$ must be finite as long as we consider compact spacetime M). Since $q + r + 1 = d + 1$, the two tangent spaces $T_{p_i}U_q$ and $T_{p_i}W_{r+1}$ at each p_i spans $T_{p_i}M$. Thus, the orientations on U_q and W_{r+1} are used to define a new orientation to the neighborfood of $p \in M$. If such an induced orientation is consistent with the original orientation of M, then we define $\mathrm{sign}(p_i) = +1$, and if not, $\mathrm{sign}(p_i) = -1$. The *linking number* is defined by summing up the signs over all the intersecting point of U_q and W_{r+1}:

$$\mathrm{Link}(U_q, V_r) = \delta_{r,d-q} \sum_i \mathrm{sign}(p_i).$$

path integral as

$$\int \mathcal{D}A\ e^{-S} \to \int \mathcal{D}A\ e^{iq\int_\gamma A} e^{-\frac{1}{2g^2}\int_{M_4 F\wedge\star F}} = \int \mathcal{D}A\ e^{iq\int_\gamma \delta^3(\gamma)\wedge A - \frac{1}{2g^2}\int_{M^4} F\wedge\star F},$$

where $\delta^3(\gamma)$ is a "3-form delta function" defined as $\int_{M_3^T} \delta^3(\gamma) = 1$ for a 3-dim mfd M_3^T which "transversely intersects" γ only once. Then, the equation of motion for A with the presence of Wilson line is

$$d\star F = qg^2\delta^3(x\in\gamma).$$

We now integrate both sides of the EoM over a 3-dim mfd Σ^3 whose boundary is $\partial\Sigma_3 = \Sigma_2$:

$$(LHS) = \int_{\Sigma_3} d\star F = \oint_{\Sigma_2} \star F \quad (\text{Stokes}'\ \text{theorem}),$$

$$(RHS) = qg^2\int_{\Sigma_3}\delta(\gamma) = qg^2\mathrm{Link}(\Sigma_2.\gamma).$$

Thus, by considering the "topological deformation" of Σ_2:

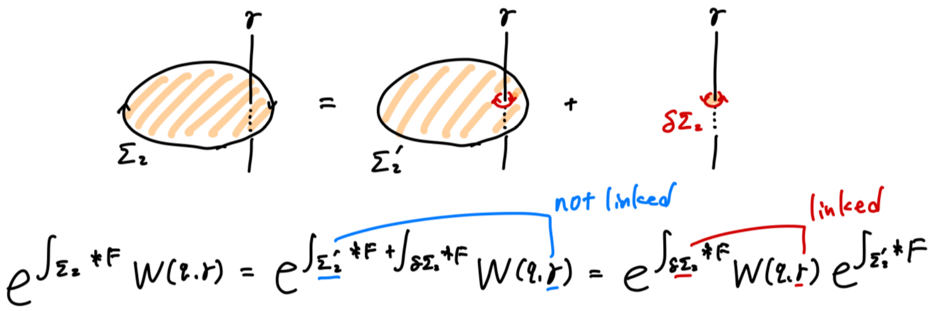


図 2: $\Sigma_2 = \Sigma_2' \cup \delta\Sigma_2$, where Σ_2 is a boundary of infinitesimal 3-dim mfd which links to γ.

Note that this deformation should be interpreted in path-integral formulation [*4] . Thus, we have

$$\langle \exp\left\{ i\frac{\lambda}{g^2}\oint_{\Sigma_2}\star F\right\} W(q,\gamma)\rangle = \langle e^{iq\lambda \mathrm{Link}(\Sigma_2,\gamma)} W(q,\gamma)\exp\left\{ i\frac{\lambda}{g^2}\oint_{\Sigma_2'}\star F\right\} W(q,\gamma)\rangle,$$

which is the same as (**??**).

By the way, the action of U_λ^e on $W(q,\gamma)$ can be seen as the shift of 1-form gauge field A by $A\mapsto A+a$, where $\oint_\gamma a = \lambda\mathrm{Link}(\Sigma_2,\gamma)$. In fact, the associate current $\star J^e$ is the conjugate momentum of A, generating the shift of A.

We can also consider the "dual" 1-form symmetry: due to the Bianchi identity, we have $dF = 0$

[*4] This means that we consider Ward-Takahashi identity as the current conservation, rather than Noether's theorem. In other words, we consider QFT rather than classical field theory.

(that is, the pure Maxwell field is flat). By regarding this identity as the conservation of 2-form current $J_2^m := \frac{1}{2\pi} \star F$, we can write down the correponding symmetry operator as

$$U_\lambda^m(\Sigma_2) = \exp\left\{i\lambda \oint_{\Sigma_2} \star J_2^m\right\} = \exp\left\{i\lambda \oint_{\Sigma_2} \frac{F}{2\pi}\right\}.$$

This 1-form symmetry is called *magnetic 1-form symmetry*. Its charged operator is the so-called **'t Hooft line**

$$T_1(m,\gamma) := e^{im\int_\gamma \tilde{A}},$$

where $\tilde{A}$ is the dual gauge field defined by $\star F = d\tilde{A}$. The action of the 1-form magnetic symmetry on a 't Hooft line is completely analogous to the case of electric 1-form symmetry:

$$\langle U_\lambda^m(\Sigma_2) T_1(m,\gamma)\rangle = e^{im\lambda \mathrm{Link}(\Sigma_2,\gamma)} \langle T_1(m,\gamma) U_\lambda^m(\Sigma_2')\rangle.$$

3.1.2 Gauging 1-form $U(1)$ symmetry

It is insightful to consider coupling "background gauge field" to our conserved 2-form currents, as we have done in the case of ordinary $U(1)$ gauge theory. In this case, the dynamic field A is 1-form, the conserved current is 2-form, and the background gauge field to be coupled is 2-form. This is done by adding to the action integral [*5] the term $\int B_2^e \wedge \star J_2^e$ and $\int B_2^m \wedge \star J_2^m$ respectively, where B_2^e and B_2^m are the 2-form background gauge field of 1-form electric/magnetic symmetry. The possible form of the modified action integral would be

$$S_1 = \frac{1}{2g^2}\int (F - B_2^e)\wedge\star(F - B_2^e) + \frac{i}{2\pi}\int B_2^m \wedge F = \frac{1}{2g^2}\int F\wedge\star F - \frac{1}{g^2}B^2\wedge F + \frac{1}{2g^2}\int B_2^e \wedge B_2^e + \frac{i}{2\pi}\int B_2^m \wedge F.$$

The new term $\int B_2^e \wedge B_2^e$ is to ensure the invariance of the kinetic term $\int F \wedge \star F$ under the gauge transformation $A \to A + \lambda_1^e,\ B_2^e \to B_2^e + d\lambda_1^e$ (λ_1^e : some (non-closed) 1-form). Note that, this term does not affect the dynamics of the theory unless we treat B_2^e and B_2^m just as "background" fields (so, the choice of such an additional term is not unique: we will point out that again below). This action is invariant under magnetic 1-form transformation $\tilde{A} \to \tilde{A} + \lambda_1^m,\ B_2^m \to B_2^m + d\lambda_1^m$, but not invariant under electric 1-form transformation $A \to A + \lambda_1^e,\ B_2^e \to B_2^e + d\lambda_1^e$:

$$\delta S_1 = \frac{i}{2\pi}\int B_2^m \wedge d\lambda_1^e.$$

Thus, the magnetic 1-form symmetry can be "gauged" consistently under this S_1 while the electric 1-form cannot.

Another choice of modification might be

$$S_2 = -\frac{1}{2g^2}\int (F - B_2^e)\wedge\star(F - B_2^e) + \frac{i}{2\pi}\int B_2^m \wedge (F - B_2^e),$$

[*5] 註: 系の力学を記述する「作用積分」と, 演算子の場に対する「作用」を区別するために, 以降は (文脈から明らかな場合を除いて) 前者を"action integral" と呼ぶことにします.

The action is in turn invariant under electric 1-form symmetry, but not under the magnetic 1-form transforastion $A^m \to A^m + \lambda_1^m,\ B_2^m \to B_2^m + d\lambda_1^m$:

$$\delta S_2 = \frac{i}{2\pi} \int -d\lambda_1^m \wedge B_2^e. \tag{3.2}$$

Combining the two observation, we can see that we cannot make both electric and magnetic $U(1)$ 1-form symmetry simultaneous by any choice of additionallocal term. Such a situation is described as that the action of $3+1$-d Maxwell theory has a mixed 't Hooft anomaly of $U(1)_e^{(1)}$, $U(1)_m^{(1)}$.

Let us see the mixed anomaly (3.2) from the inflow pitcure briefly. We consider the following "anomaly inflow action" in five dimension:

$$S_{\text{inflow}} = -\frac{i}{2\pi} \int_{N_5} B_2^m \wedge dB_2^e,$$

where $\partial N_5 = M_4$ is our spacetime manifold. By the background gauge transformation $B_2^m \to B_2^m + d\lambda_1^m$, we have

$$\begin{aligned}
\delta S_{\text{inflow}} &= -\frac{i}{2\pi} \int_{N_5} d\lambda_1^m \wedge dB_2^e \\
&= -\frac{i}{2\pi} \int_{N_5} \left\{ d(d\lambda_1^m \wedge B_2^e) - d^2\lambda_1^m \wedge dB_2^e \right\} \\
&= -\frac{i}{2\pi} \int_{\partial N_5} d\lambda_1^m \wedge B_2^e + 0 \quad (\because \text{Stokes}'\text{thm}) \\
&= -\frac{i}{2\pi} \int_{M_4} d\lambda_1^m \wedge B_2^e,
\end{aligned}$$

which is just the same as δS_2 in (3.2). Therefore, we can add $-S_{\text{inflow}}$ in our action integral to cancel the mixed 't Hooft anomaly in the boundary M_4.

More generally, when a $U(1)$ p-form gauge field A_p and its $D-p$-form dual with action integral

$$S = \int_M \frac{1}{2g^2} F_{p+1} \wedge \star F_{p+1}, \quad F_{p+1} = dA_p$$

are simultaneously coupled to a background gauge field $B_{p+1}^{(e)}$ and $B_{D-p-1}^{(m)}$, then the action integral

$$S = \int \frac{1}{g^2} (F_{p+1} - B_{p+1}^{(e)}) \wedge \star (F_{p+1} - B_{p+1}^{(e)}) + \frac{i}{2\pi} F_{p+1} \wedge B_{D-p-1}^{(m)}$$

is not invariant under background gauge transformation $B_{p+1}^{(e)} \to B_{p+1}^{(e)} + \delta B_{p+1}^{(e)}$:

$$\delta S = i \int \Lambda_{p+1}^{(e)} \wedge \frac{B^{(m)_{D-p-1}}}{2\pi},$$

and is not canceled by any counter term. However, we can add the term of the integral of $D+1$-form

$$S_{\text{inflow}} = \frac{i}{2\pi} \int_N B_{p+1}^{(e)} \wedge dB_{D-p-1}^{(m)} \quad (\partial N = M)$$

to cancel the above δS.

3.1.3 Electric-magnetic duality

This part is mainly based on [8]. Please see his video for detail.

Let's now consider the following action integral, whose dynamical gauge field is $\tilde{A}$:

$$S[\tilde{A}, F] = -\frac{1}{2g^2}\int F \wedge \star F + \frac{i}{2\pi}\int d\tilde{A} \wedge F. \tag{3.3}$$

The action is the form of coupling magnetic 2-form current $\star J_2^m = \frac{1}{2\pi}F$ to dynamical gauge field $\tilde{A}$. "Integraling out" the $\tilde{A}$ field obviously reproduces the action (3.1), which means that gauging magnetic 1-form symmetry $U_m^{(1)}$ yields an electric 1-form symmetry $U_e^{(1)}$. Mathematically, gauging the symmetry $\mathcal{G}$ in a theory $\mathcal{T}$ gives another theory $\mathcal{T}/\mathcal{G}$, whose symmetry is given as the Pontryagin dual:

$$\hat{\mathcal{G}} := \mathrm{Hom}(\mathcal{G}, U(1)).$$

Hom is the set of (group) homomorphisms (群準同型) between two groups. For abelian group, the Pontryagin dual is known to be the same as the original one: $\hat{A} \simeq A$ (A: abelian). Therefore, by gauging (magnetic) 1-form $U(1)$ symmetry, we succeeded in obtaining another (electric) 1-form $U(1)$ symmetry as its dual.

Another way to study the relation of $U_e^{(1)}$ and $U_m^{(1)}$ is to consider the equation of motion for $\tilde{A}$ in the action (3.3), which is

$$\frac{i}{g^2}\star F = -\frac{1}{2\pi}d\tilde{A}\left(=: \frac{1}{2\pi}\tilde{F}\right).$$

By using the property of Hodge star operator for arbitrary p-form ω, $\star\star\omega = (-1)^{p(d+1-p)}\omega$ (for our case $p = 2$), we can also have

$$\star\tilde{F} = -\frac{2\pi i}{g^2}F.$$

Then we can rewrite the action (3.3) using $\tilde{A}$ only:

$$\begin{aligned}
\tilde{S}[\tilde{A}] &= -\frac{1}{2g^2}\int\left(\frac{g^2}{2\pi i}\star\tilde{F}\right)\wedge\left(\frac{g^2}{2\pi i}\tilde{F}\right) + \frac{i}{2\pi}\int\tilde{F}\wedge\left(\frac{g^2}{2\pi i}\star\tilde{F}\right)\\
&= -\frac{g^2}{8\pi^2}\int\tilde{F}\wedge\star\tilde{F}.
\end{aligned}$$

Then, this $\tilde{S}[\tilde{A}]$ becomes the same form of the original action under the EoM of dynamic field $\tilde{A}$. But the coupling constant is different: for the theory of $\tilde{S}$,

$$\frac{1}{2\tilde{g}^2} = \frac{g^2}{8\pi^2} \;\rightarrow\; \tilde{g}^2 = 4\pi^2/g^2.$$

This means that, when the original coupling constant of the original dynamical field A becomes smaller, the "dual" coupling constant of the dual field $\tilde{A}$ becomes larger.

The duality picture gives us an important viewpoint of the symmetry: symmetry operators act on defect operators, which generates the symmetry of the system. In other words, the algebra of symmetry operators is the representation of the symmetry, whose representation space is spanned by defect operators. Generally, this algebraic structure is not necessarily group-like (called "higher group"), so we will go on a journey to the representation theory of category (圏) in order to study the physics under generalized symmetry.

4 Screening of 1-form charge

Theories with higher-form symmetry have some extended operators [*6] in more than 1 dimension. For such extended operators, we can consider the **screening** of a certain operator $\mathcal{O}_p$ to another one $\mathcal{O}_p$. This will actually turn out to be a strong method to perform higher-form symmetries of gauge theories.

The screening is defined as follows:

定義 4.1.
Screening A p (≥ 1)-dimensional operator $\mathcal{O}_p$ can be "**screened** to another p-dimensional $\mathcal{O}'_p$ if we can insert a $(p-1)$-dimensional operator $\mathcal{O}_{p-1}$ between $\mathcal{O}_p$ and $\mathcal{O}'_p$, depicted as follows:

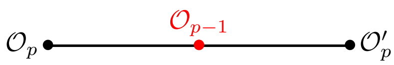
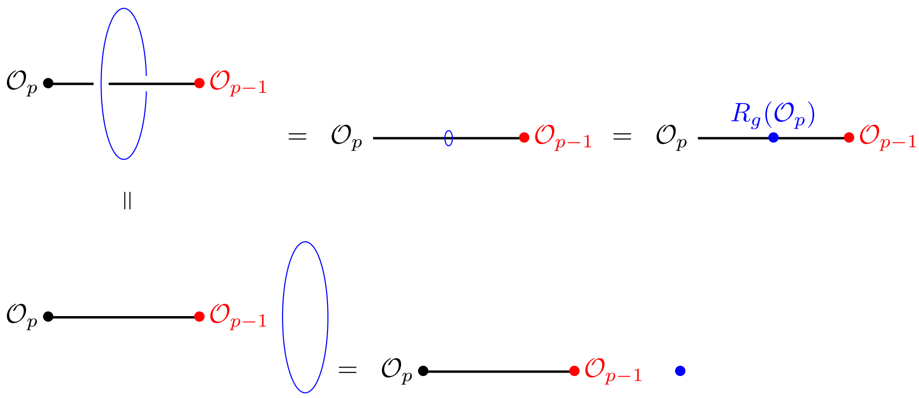


図 3: Screening of 1-form charged defect by 0-form defect.

Moreover, a p-dim $\mathcal{O}_p$ is **completely screened** if is can end on a $(p-1)$-dim operator.

Some literatures (such as [1]) uses the term *screened* to mean *completely screened* defined above.

The important point of this screening is that, if one operator $\mathcal{O}_p$ is screened to another $\mathcal{O}'_p$, then they have the same p-form charge. In particular, for a *completely screended* operator $\mathcal{O}_p$, the p-form charge becomes zero, which is pictorically apparent:



In other words, when some defects are completely screened by 1-dim lower (1-codim higher) objects, then the defects cease to be topological. This can be interpreted as the breaking of p-form symmetry induced by lower-dimensional objects of nontrivial vacuum expected value (vev). Here we stop by just mentioning the relationship between screening and SSB.

[*6] Note that, we frequently use the term "operators" and "defects" interchangeably.

4.1 studying p-form symmetry from (absense of) screening

We can say the converse of the above statement: if $\mathcal{O}_p$ and $\mathcal{O}'_p$ are not screened to each other, then we can construct a representation of p-form symmetry action on p-dim charged defects, so that $\mathcal{O}_p$ and $\mathcal{O}'_p$ have a different charge under it.

In practice, we define the "symmetry" under the following strategy. We first identify two p-dim operators if they are related by screenings. Then, we can define the equivalence classes of p-dim operators by the above identification (we denote the set of all eq. classes as D_p). We can induce a ring (環) strucure on the set D_p, but for many cases the D_p can be an abelian group.

In our setup, we must have a p-form symmetry whose charge distinguishes objects of different equivalence classes i.e. different elements of D_p. Such a p-form symmetry is the group $G^{(p)} = \hat{D}_p$ (Pontryagin dual). In contrast, the possible charges under this p-form symmetry group forms the group D_p itself. By definition, this corresponds to elements of the equivalence class of charged operators $[\mathcal{O}_p] \in D_p$ (the equivalence class of charged operators are labeled by their charges).

For the case of 1-form symmetries, line operators may be screened by local (0-form) operators. The global form of the gauge group determines what representation of local operators are allowed, which in turn specifies what set of line operators are screened (endable at local operators). To be more concrete, consider the Wilson lines of a gauge theory which transform in a representation R of the gauge group

$$W_R(q^a, \gamma) = \mathrm{Tr}_R \; P e^{iq^a \int_\gamma \; A^a}.$$

In other words, they are world-lines along the path γ of heavy probe particles with rep. R. If a Wilson line W_R is endable on a local operator $\mathcal{O}_R$ which transforms in the same representation R, then the $\mathcal{O}_R$ corresponds to an annihilation operator of the heavy probe particle. If a theory does not have such a particular operator representation, the Wilson line does not end, and carries a non-trivial charge under the 1-form symmetry.

4.2 Maxwell theory, with matter field

This example is based on [7]. Consider a $d+1$-dimensional gauge theory with $U(1)$ gauge group, along with a matter field ϕ with charge of a certain value $q \in \mathbb{Z}$ under the $U(1)$ gauge group. We will see that the introduction of this ϕ breaks the $U(1)$ 1-form symmetry to its $\mathbb{Z}_q$ subgroup (then, this theory turns out to have a $\mathbb{Z}_q$ 1-form symmetry).

The key point is that, the matter field ϕ gives rise to a *non-genuine* local operator $\phi(x)$ that screens a Wilson line of charge q as shown:

$$W(q,\gamma) \;\frac{\qquad\qquad\qquad\qquad}{}\!\bullet\; \phi(x)$$

Here, a *non-genuine* q-dimensional operator is defined to be an operator that is attached to a collection of $p(>q)$-dimensional operators. An insertion of a genuine field $\phi(x)$ cannot be done because it is not gauge invariant. Still, we can consider the gauge-invariant way to insert ϕ, by attaching it to a Wilson line on a (semi-)finite Line L at $x=\partial L$. In fact, the gauge transformation $A(x)\mapsto A(x)-\frac{d\theta(x)}{2\pi}$ gives

$$\begin{aligned}&\phi(x)\mapsto e^{iq\theta(x)}\phi(x),\\ W(q,\gamma)=\exp\left(2\pi iq\int_L A\right)\mapsto \exp\left(-iq\int_L d\theta\right)&W(q,\gamma)=\exp\left(-iq\int_{\partial L}\theta\right)W(q,\gamma)=e^{-iq\theta(x)}W(q,\gamma),\end{aligned}$$

thus the two factors cancel each other to make $\phi(x)W(q,\gamma)$ gauge invariant.

Similarly, by inserting a power ϕ^n, we can screen a Wilson line of charge q_0+nq to another line of charge q_0. Therefore, when we have a matter field ϕ of charge q, the charge of 1-form symmetry (charge carried by Wilson lines) is meaningful only modulo q, and the group of equivalence classes of unscreened line operators becomes the quotient by $q\mathbb{Z}$: $D_1=\frac{\mathbb{Z}}{q\mathbb{Z}}=\mathbb{Z}_q$. The group $\mathbb{Z}$ in numerator is the group of equivalence classes of Wilson lines before screening (note that the action of 1-form symmetry on Wilson lines is defined by linking number), and $\mathbb{Z}_q$ in the denominator is the group of completely screened Wilson lines by ϕ (with 0-form charge q). This is how a 0-form charged operator can screen the charge of 1-form operators.

As is seen before, the 1-form symmetry distinguishing the $\mathbb{Z}_q$-valued charge of (screened) Wlison lines is the Pontryagin dual $G^{(1)}=\hat{\mathbb{Z}}_q\sim\mathbb{Z}_q$ [*7]. The corresponding codim-2 topological operators generating this 1-form symmetry are a subset of the operators in the case of pure $U(1)$ gauge theory, given by

$$U^e_n(\Sigma_{d-2})=\exp\left(\frac{i}{g^2}\frac{2\pi n}{q}\int_{\Sigma_{d-2}}\star F\right),\quad n\in\{0,1,\cdots,q-1\}.$$

These operators acts on Wilson lines of charge q trivially, as the phase factor obtained by action becomes

$$\exp\left(i\frac{2\pi n}{q}\cdot q\cdot \mathrm{Link}(\Sigma_{d-2},\gamma)\right)=\exp(2\pi in\cdot\mathrm{Link}(\Sigma_{d-2},\gamma))=1,$$

since the linking number is always an integer.

[*7] The Pontryagin dual of $\mathbb{Z}_q$ is $\mathrm{Hom}(\mathbb{Z}_q,U(1))\simeq\mathbb{Z}_q$. In fact, if we label the element of $\mathbb{Z}_q$ as $g_n=e^{\frac{2\pi in}{q}}$ ($n=0,1,\cdots,q-1$), then the possible homomorphism takes the form of

$$\varphi_N:\mathbb{Z}_q\to U(1),\quad \varphi_N(g_n)=(e^{\frac{2\pi in}{q}})^N\quad (N=0,1,\cdots,q-1).$$

Thus, the set of such φs is equivalent to $\mathbb{Z}_q$.

More generally, if G is a finite abelian group, then we can have

$$\hat{G}=\mathrm{Hom}(G,U(1))\simeq G.$$

Note that G must be finite. In fact, we can see $\hat{\mathbb{Z}}=U(1)$ by considering the map

$$\varphi_\alpha:\mathbb{Z}\to U(1),\quad n\mapsto e^{in\alpha}.$$

The label α of $\mathrm{Hom}(\mathbb{Z},U(1))$ takes value on $[0,2\pi)$, then the set of φ_αs form a $U(1)$ group.

5 More About Higher-Form Symmetry

Maybe this will be skipped or given talks by another person.

5.1 Anomaly of Higher-Form Global Symmerty

5.1.1 Anomaly in general

5.2 Spontaneous Symmetry Breaking of Higher-form Symmetry

6 Discrete Gloal Symmetry

So far, we have only considered systems with continuous symmetry. In that case, we can find conserved $(d-p)$-form current, or closed d-form explicitly, by considering infinitesimal transformations of group elements parametrized by continuous numbers.

However, we cannot do in the same manner for the case of discrete symmetry, such as $\mathbb{Z}_2$ or $\mathbb{Z}_N$. Indeed, the concept of "infinitesimal transformation" is totally nonsense for discrete symmetry since all the elements are "finitely separated" [*8] from identity. Then, how can we characterize the symmetry of discrete group? Remember that the theory with (generalized) symmetry has symmetry operators which are topological, and symmetry action on charged objects known as defect operators is defined. For continuous case, we can construct such topological operators explicitly from the conserved current. How can we find such topological operators in the case of discrete symmetry?

Note that, even in the case of discrete symmetry, we can still have the notion of "defect operator". There are several characterization of defect operators. One way is the following.

Another way is from the perspective of gauging: the insertion of symmetry defect operators corresponds to turning on a background gauge field.

6.1 BF theory

Let's consider $\mathbb{Z}_N$ p-form gauge theory. One way to construct such a theory is called **BF theory**, which is a constrained $U(1)$ gauge theory so that only $\mathbb{Z}_N$ gauge fields contribute to the path integral (note that $\mathbb{Z}_N \subset U(1)$ as a subgroup). In this case, we only encounter Wilson lines whose test charges only have a meaning modulo N.

We will consider a $D = d+1$-dimensional theory of a $U(1)$ p-form gauge field A_p with a $U(1)$ $(d-p-1)$-form background gauge field. The action is of the form

$$S = \frac{iN}{2\pi}\int B_{d-p} \wedge dA_p,$$

which is invariant under the transformation $A_p \mapsto A_p + d\Lambda_{p-1}$ and $B_{d-p} \mapsto B_{d-p} + d\Lambda_{d-p-1}$. We normalize the q-form ($q = p$ or $d-p$) so that $\oint_{\Sigma_{q+1}} \frac{d\Lambda_q}{2\pi} \in \mathbb{Z}$. The equation of motion tells us that the gauge fields A_p, B_{d-p} are flat:

$$N\frac{dA_p}{2\pi} = 0, \quad N\frac{dB_{d-p}}{2\pi} = 0.$$

The equation of motion restricts the path integral to the sum only over $\mathbb{Z}_N$ gauge fields.

6.1.1 Discrete Global Symmetry of BF Theory

The EoM of BF theory seems to be trivial, but we still have the field configurations where the holomomies are $\mathbb{Z}_N$-valued:

$$N$$

6.2 Example From Lattice: 3D Toric Code

Now we will see the example of lattice models which have some higher-form symmetries. 3D toric code [*9] , whose target space is 3-dimensioanl [*10] , is known as the example of 1-form discrete global symmetry: $\mathbb{Z}_2^e \times \mathbb{Z}_2^m$.

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*8 More precisely, discrete group is equipped with discrete topology (離散位相) while continuous group can have the structure of metric space (距離空間). Especially, Lie group is equipped with differential structure (微分構造).

*9 As is mentioned last week, when the target space dimension is D, the possible (higher-form) symmetries are of $p = 0, 1, \cdots, D-2$. Therefore, 2D toric code cannot have any higher-form symmetry, but $3D$ toric code can.

*10 Note that toric code model is not necessary defined on a torus. In fact, the Hamiltonian can be defined on any square lattice, as long as it has vertices and plaquettes.

6.2.1 Brief review of toric code model

The **toric code** model is constructed on an arbitrary dimensioanl lattice (typically square lattice), with periodic boundary condition on both directions (i.e. the Hilbert space is on torus T^2). We put a qubit on every link. The Hamiltonian is

$$H = -\sum_j A_j - \sum_p B_p, \tag{6.1}$$

where j and p is the index of site and plaquette respectively, $A_j \equiv \prod_{l:\text{link ending at j}} Z_l$ is defined for every site j, and $B_p \equiv \prod_{l:\text{bdy of p}} X_l$ is for every plaquette p. A_j and B_p are sometimes called "Gauss law operator" and "flux operator" respectively.

All of those operators A_j and B_p commute with each other since they all share an even number (0, 2, or 4) of Xs and Zs (so we always have $(-1)^{\text{even}}$ factor). Then, we can diagonalize all terms in simultaneously.

Let's consider states whose eigenvalue of A_j is 1 for every vertex j. For such states, all vertex should be associated with an even number of X_ls ending on it. The states satisfying this condition are "closed-string states", where all lines of electric flux are closed and have no charges to end on. Therefore, the groundstate of the toric code model is degenerated as many as the possible closed loop configuration, which is the order of the exponential of the system size.

The excitation of toric code is created by open strings of Zs or Xs. When we have a Z on a link (j_1, j_2), then the eigenvalues of A_{j_1} and A_{j_2} are flipped $1 \to -1$, and the total energy is increased by 4. In this case, we say that the Z creates a pair of *e-excitations* at vertex j_1 and j_2. Similarly, a X on a link (j_1, j_2) creates a pair of *m-excitations* at plaquette p_1 and p_2, where p_1 and p_2 are the plaquettes sharing the link (j_1, j_2).

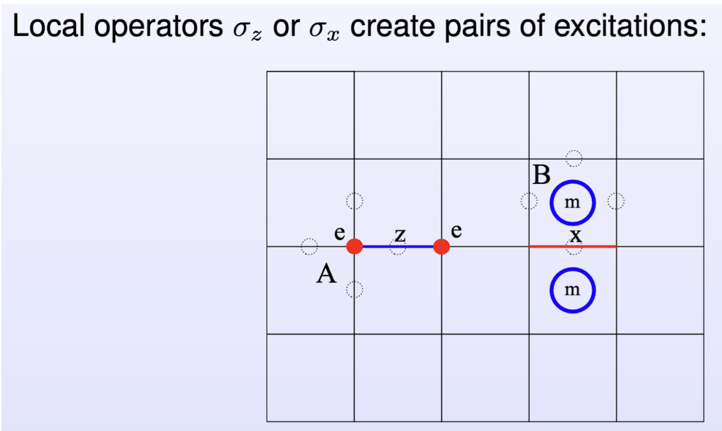


図 4: Pari creation of e-excitations and m-excitations. Image from [3].

The e-excitations can be moved by other Zs, and the m-excitations can be moved by other Xs. Note that string operators of Xs are defined on the dual lattice, whereas those of Zs are on the original lattice. We can annihilate the pair of excitations by closing a string of Z or X to form a loop. Therefore, the electric and magnetic charge of such excitations are said to be $\mathbb{Z}_2$-valued. The symmetry of the toric code is generated by closed strings of σ_z and σ_x. There are two types of closed strings: trivial ones and non-trivial ones. Non-trivial ones correspond to the non-trivial homology of the model.

6.2.2 anyon braiding

In toric code model, we have two types of excitations: e-excitations created by z-strings, and m-excitations by x-strings. These string excitations can be moved by other zs and xs freely, then they themselves obey a bosonic

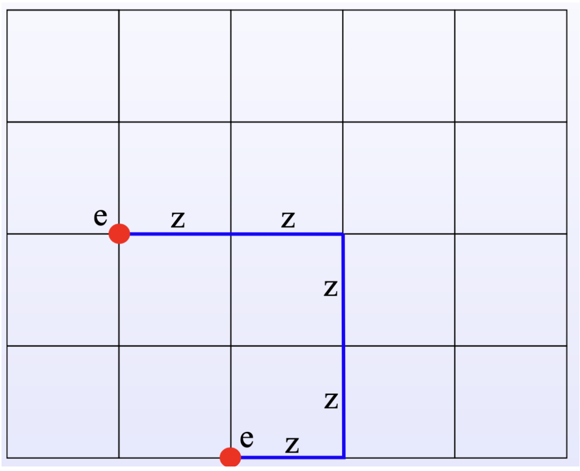


図 5: e-excitations can be moved by Z. Image from [3].

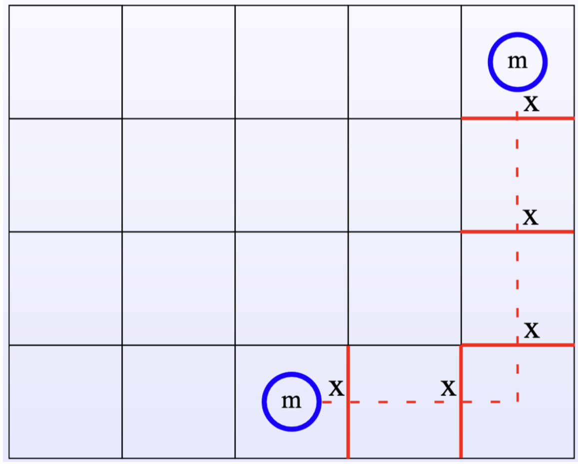


図 6: m-excitations can be moved by X. Image from [3].

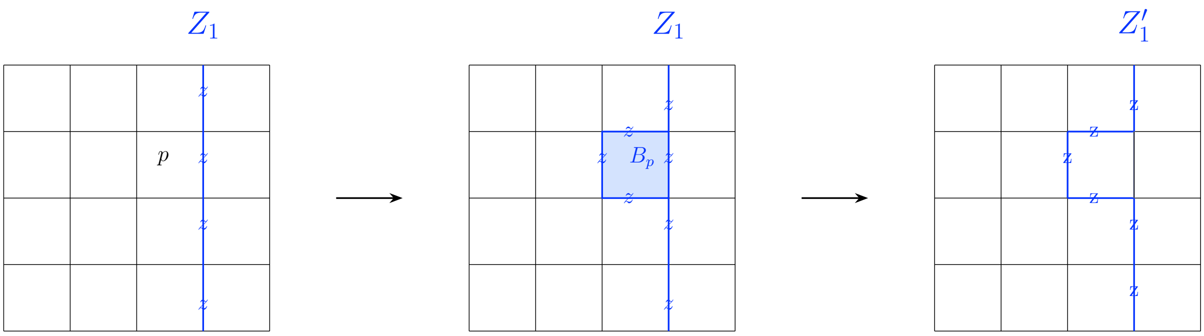
statistics.

In contrast, when we move the e-excitations around m-excitations, then we will have a non-trivial phase.

6.2.3 1-form symmetry of toric code

There are two types of 1-dimensional topological operators defined on the toric code, commuting with the Hamiltonian (6.2.1). They generate 1-form symmetry $\mathbb{Z}_2^e \times \mathbb{Z}_2^m$.

The symmetry operator of $\mathbb{Z}_2^e$ is the Wilson loop $W[C] = \prod_{i \in C} \sigma_i^z$ (corresponding to **magnetic string operator** along the non-contractible closed loop), and that of $\mathbb{Z}_2^m$ is the 't Hooft loop $T[\tilde{C}] = \prod_{i \in \tilde{C}} \sigma_i^x$ (corresponding to **electric string operator** along the non-contractible closed loop). Such kinds of string operators are topological in that string operators in the same homology equivalence class can be transposed to each other through the multiplication by stabilizers [*11].



The corresponding charged operator of $\mathbb{Z}_2^m$ is the non-contractable closed electric loop ('t Hooft loop), and that of $\mathbb{Z}_2^e$ is the non-contractable closed magnetic loop (Wlison loop). The action is written as

$$W[C]T[\tilde{C}]W[C]^{-1} = e^{i\pi \mathrm{Link}(C,\tilde{C})} T[\tilde{C}], \tag{6.2}$$

[*11] The operators A_j (defined on each vertex) and B_p (on each plaquette) are called "stabilizers", since they commute with all the terms in the Hamiltonian (6.2.1).

where $W[C]^{-1}$ is the Wlison loop on the inverse direction of C. Intuitively, this action is interpreted as follows: when $W[C]$ and $T[\tilde{C}]$ are linked, we have to intersect the twoo loops as many as the linking number to "delink" the two. Intersecting the loop of Z and that of X one time gives a phase factor (-1), then we totally get the phase factor above.

6.2.4 1-form symmetry breaking of GS

INCOMPLETE

6.2.5 Mixed 't Hooft anomaly of $\mathbb{Z}_2^e \times \mathbb{Z}_2^m$

INCOMPLETE

As we have seen before, the electric and magnetic 1-form $U(1)$ symmetries causes mixed 't Hooft anomaly when they are gauged together. The similar thing happens in toric code, but the symmetry here is 1-form $\mathbb{Z}_2$ instead of $U(1)$.

Consider the action (6.3)

$$W[C]T[\tilde{C}]W[C]^{-1} = e^{i\pi \mathrm{Link}(C,\tilde{C})}T[\tilde{C}]. \tag{6.3}$$

If the Linking number is odd, then we have

$$W[C]T[\tilde{C}] = -T[\tilde{C}]W[C].$$

This means that the two symmetry operators do not commute. It is problematic when we consider the "gauging" of the two 1-form symmetry simultaneously.

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