応用幾何 ma・pa　　演習 11　　解答例. (2023.12.15)

空間曲線 $C : \boldsymbol{x}(t) = (t, t, t^2)\ \ (0 \le t \le a)\ \ (a > 0)$ を考える．次の 線積分 を求めよ．

(1) ベクトル場：　$\displaystyle\int_C \langle \boldsymbol{v}, d\boldsymbol{x} \rangle \qquad \boldsymbol{v}(x, y, z) = (y, x, z)$

(2) 微分 1 形式：　(i) $\displaystyle\int_C \alpha \qquad \alpha = xdx + zdy + ydz$ 　　(ii) $\displaystyle\int_C df \qquad f(x, y, z) = xy + z$

(解答例)　$\boldsymbol{x}'(t) = (1, 1, 2t)$

(1) $\langle \boldsymbol{v}(\boldsymbol{x}(t)), \boldsymbol{x}'(t) \rangle = \langle (t, t, t^2), (1, 1, 2t) \rangle = 2t + 2t^3$

$$\therefore \int_C \langle \boldsymbol{v}, d\boldsymbol{x} \rangle = \int_0^a \langle \boldsymbol{v}(\boldsymbol{x}(t)), \boldsymbol{x}'(t) \rangle \, dt = \int_0^a 2t + 2t^3 \, dt = \left[t^2 + \frac{1}{2} t^4 \right]_0^a = a^2 + \frac{a^4}{2}$$

(2) (i)
$$\int_C \alpha = \int_C (xdx + zdy + ydz) = \int_0^a \left(x(t) \frac{dx(t)}{dt} + z(t) \frac{dy(t)}{dt} + y(t) \frac{dz(t)}{dt} \right) dt$$
$$= \int_0^a \left(t \cdot 1 + t^2 \cdot 1 + t \cdot 2t \right) dt = \int_0^a t + 3t^2 \, dt = \left[\frac{1}{2} t^2 + t^3 \right]_0^a = \frac{1}{2} a^2 + a^3$$

(ii) $\displaystyle\int_C df = f(\boldsymbol{x}(a)) - f(\boldsymbol{x}(0)) = f(a, a, a^2) - f(0, 0, 0) = 2a^2 - 0 = 2a^2$