応用幾何 ma・pa　　課題　#4　　解答例. (2023.10.20)

(1) 次の関数を考える．　$\boldsymbol{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : \boldsymbol{F}(x,y,z) = \begin{pmatrix} x+y+z \\ x^2+y^2+z^2 \\ xyz \end{pmatrix}$

(i) 微分行列 $\dfrac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}$ を求めよ．

(ii) ヤコビアン $J_F(\boldsymbol{x})$ を (x, y, z の 1次式 に 因数分解 した形で) 求めよ．

(解答例)

(i) $$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}} = \begin{pmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{pmatrix}$$

(ii) $$J_F(\boldsymbol{x}) = \det \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & xz & xy \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ yz & xz-yz & xy-yz \end{vmatrix}$$

$$= 2\begin{vmatrix} y-x & z-x \\ z(x-y) & y(x-z) \end{vmatrix} = 2(y-x)(z-x)\begin{vmatrix} 1 & 1 \\ -z & -y \end{vmatrix} = -2(y-x)(z-y)(x-z)$$

(2) 関数 $f : \mathbb{R}^3 \longrightarrow \mathbb{R} :\ f(x,y,z) = xy + \sin yz$ に対して 次を求めよ．

(i) ∇f　　(ii) Δf

(解答例)

(i) $$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (y, x + z\cos yz, y\cos yz)$$

(ii) $$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 - z^2 \sin yz - y^2 \sin yz = -(y^2+z^2)\sin yz$$

(3) 次の関数を考える．　$\boldsymbol{G} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 : \boldsymbol{G}(x,y) = \begin{pmatrix} e^{xy} \\ \cos xy \\ \sin xy \end{pmatrix}$

(i) 微分行列 $\dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{x}}$ を求めよ．

(ii) 偏微分ベクトル $\dfrac{\partial \boldsymbol{G}}{\partial x}$ と $\dfrac{\partial \boldsymbol{G}}{\partial y}$ の 内積 $\left\langle \dfrac{\partial \boldsymbol{G}}{\partial x}, \dfrac{\partial \boldsymbol{G}}{\partial y} \right\rangle$ 及び 外積 $\dfrac{\partial \boldsymbol{G}}{\partial x} \times \dfrac{\partial \boldsymbol{G}}{\partial y}$ を求めよ．

(解答例)

(i) $$\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{x}} = \left(\frac{\partial \boldsymbol{G}}{\partial x}, \ \frac{\partial \boldsymbol{G}}{\partial y}\right) = \begin{pmatrix} ye^{xy} & xe^{xy} \\ -y\sin xy & -x\sin xy \\ y\cos xy & x\cos xy \end{pmatrix}$$

(ii) $$\left\langle \frac{\partial \boldsymbol{G}}{\partial x}, \frac{\partial \boldsymbol{G}}{\partial y} \right\rangle = \begin{pmatrix} ye^{xy} \\ -y\sin xy \\ y\cos xy \end{pmatrix} \cdot \begin{pmatrix} xe^{xy} \\ -x\sin xy \\ x\cos xy \end{pmatrix} = xy\begin{pmatrix} e^{xy} \\ -\sin xy \\ \cos xy \end{pmatrix} \cdot \begin{pmatrix} e^{xy} \\ -\sin xy \\ \cos xy \end{pmatrix}$$

$$= xy(e^{2xy} + \sin^2 xy + \cos^2 xy) = xy(e^{2xy}+1)$$

$$\frac{\partial \boldsymbol{G}}{\partial x} \times \frac{\partial \boldsymbol{G}}{\partial y} = \begin{pmatrix} ye^{xy} \\ -y\sin xy \\ y\cos xy \end{pmatrix} \times \begin{pmatrix} xe^{xy} \\ -x\sin xy \\ x\cos xy \end{pmatrix} = xy\begin{pmatrix} e^{xy} \\ -\sin xy \\ \cos xy \end{pmatrix} \times \begin{pmatrix} e^{xy} \\ -\sin xy \\ \cos xy \end{pmatrix} = \boldsymbol{0}$$