応用幾何 ma・pa　　課題 #11　　解答例. (2023.12.15)

(1) 次の 線積分 を求めよ.

$$\int_C f\,ds \qquad f(x,y,z)=xz+y \qquad C:\ \boldsymbol{x}(t)=(e^t,\sqrt{2}\,t,e^{-t})\ \ (0\le t\le 1)$$

(解答例)

$$\boldsymbol{x}'(t)=(e^t,\sqrt{2},-e^{-t}) \qquad \|\boldsymbol{x}'(t)\|^2=e^{2t}+2+e^{-2t}=(e^t+e^{-t})^2 \qquad \therefore\ \|\boldsymbol{x}'(t)\|=e^t+e^{-t}$$

$$ds=\|\boldsymbol{x}'(t)\|\,dt=(e^t+e^{-t})\,dt \qquad f(\boldsymbol{x}(t))=f(e^t,\sqrt{2}\,t,e^{-t})=1+\sqrt{2}\,t$$

$$\begin{aligned}\int_C f\,ds &= \int_0^1 f(\boldsymbol{x}(t))\|\boldsymbol{x}'(t)\|\,dt=\int_0^1(1+\sqrt{2}\,t)(e^t+e^{-t})\,dt\\ &= \Big[(1+\sqrt{2}\,t)(e^t-e^{-t})\Big]_0^1-\sqrt{2}\int_0^1(e^t-e^{-t})\,dt=(1+\sqrt{2})(e-e^{-1})-\sqrt{2}\Big[e^t+e^{-t}\Big]_0^1\\ &= (1+\sqrt{2})(e-e^{-1})-\sqrt{2}\,(e+e^{-1}-2)=e-(1+2\sqrt{2})e^{-1}+2\sqrt{2}\end{aligned}$$

(2) 曲線　$C:\ \boldsymbol{x}(t)=(\cos t,\sin t,t^2)\ \ (0\le t\le 1)$　を考える. 次の 線積分 を求めよ.

(i) $\displaystyle\int_C\langle\boldsymbol{v},d\boldsymbol{x}\rangle \qquad \boldsymbol{v}(x,y,z)=(y,x,z)$　　　(ii) $\displaystyle\int_C\alpha \qquad \alpha=x^2dx+y^2dy+z^2dz$

(解答例)　　$\boldsymbol{x}'(t)=(-\sin t,\cos t,2t)$

(i) $$\begin{aligned}\langle\boldsymbol{v}(\boldsymbol{x}(t)),\boldsymbol{x}'(t)\rangle &=\langle\boldsymbol{v}(\cos t,\sin t,t^2),\boldsymbol{x}'(t)\rangle=\langle(\sin t,\cos t,t^2),(-\sin t,\cos t,2t)\rangle\\ &=-\sin^2 t+\cos^2 t+2t^3=\cos 2t+2t^3\end{aligned}$$

$$\therefore\ \int_C\langle\boldsymbol{v},d\boldsymbol{x}\rangle=\int_0^1\langle\boldsymbol{v}(\boldsymbol{x}(t)),\boldsymbol{x}'(t)\rangle\,dt=\int_0^1(\cos 2t+2t^3)\,dt=\Big[\frac{1}{2}\sin 2t+\frac{1}{2}\,t^4\Big]_0^1=\frac{1}{2}\sin 2+\frac{1}{2}$$

(iii) $$\begin{aligned}\int_C\alpha &= \int_C(x^2dx+y^2dy+z^2dz)=\int_0^1\Big(\cos^2 t\cdot(-\sin t)+\sin^2 t\cdot\cos t+t^4\cdot 2t\Big)\,dt\\ &= \Big[\frac{1}{3}\cos^3 t+\frac{1}{3}\sin^3 t+\frac{1}{3}t^6\Big]_0^1=\frac{1}{3}(\cos^3 1+\sin^3 1+1-1)=\frac{1}{3}(\cos^3 1+\sin^3 1)\end{aligned}$$

(別解) $\mathbb{R}^3$ の 関数　$h(x,y,z)=\dfrac{1}{3}(x^3+y^3+z^3)$　をとると

$dh=h_x\,dx+h_y\,dy+h_z\,dz=\alpha$　を満たす.

$$\begin{aligned}\therefore\ \int_C\alpha=\int_C dh &=h(\boldsymbol{x}(1))-h(\boldsymbol{x}(0))=h(\cos 1,\sin 1,1)-h(1,0,0)\\ &=\frac{1}{3}(\cos^3 1+\sin^3 1+1)-\frac{1}{3}=\frac{1}{3}(\cos^3 1+\sin^3 1)\end{aligned}$$

(3) (平面のグリーンの定理)　　平面 $\mathbb{R}^2$ で 微分 1 形式　$\alpha=(xy^2)\,dx+(x^2\cos y)\,dy$　を考える.

C を 閉領域　$D=\{(x,y)\in\mathbb{R}^2\mid 0\le x\le 1,\ 0\le y\le x\}$　の境界とする. 向きは 反時計回りとする.

線積分　$\displaystyle\int_C\alpha$　を求めよ.

(解答例)　平面のグリーンの定理 を用いる.

$$d\alpha = \big(-\partial_y(xy^2) + \partial_x(x^2\cos y)\big)\,dx\wedge dy = (-2xy + 2x\cos y)\,dx\wedge dy = 2x(\cos y - y)\,dx\wedge dy$$

$$\begin{aligned}\int_C \alpha \ &= \ \int_D d\alpha = \iint_D 2x(\cos y - y)\,dxdy = \int_0^1 dx\int_0^x 2x(\cos y - y)\,dy \\ &= \int_0^1 dx\ 2x\Big[\sin y - \frac{1}{2}y^2\Big]_0^x = \int_0^1 (2x\sin x - x^3)\,dx = 2(\sin 1 - \cos 1) - \frac{1}{4}\end{aligned}$$

$$\circ\ \int_0^1 x\sin x\,dx = \Big[-x\cos x\Big]_0^1 + \int_0^1 \cos x\,dx = -\cos 1 + \Big[\sin x\Big]_0^1 = -\cos 1 + \sin 1$$