Chandrasekhar Limit for Pure Carbon White Dwarfs

Introduction

White dwarfs mark the end of the lives of main sequence stars with low (~ $0.6M_{\odot}$) and intermediate masses (~$10M_{\odot}$) [1] and are progenitors to Type Ia supernovae. They are objects in quasi-equilibrium, finely balanced between gravity and electron degeneracy pressure. The maximum mass at which this balance is exact is known as the Chandrasekhar mass limit.

The composition of a white dwarf varies depending on the mass of the progenitor star and may harbour a Helium, Carbon-Oxygen or Oxygen-Neon Core for those with lowest to highest masses [1]. For a (statistically) typical white dwarf, 99% of its mass is attributable to its Carbon-Oxygen core [1], despite detailed theories of their atmosphere is most important to explaining their observed spectra. Luckily for us, any given radius, their composition is most likely mono-elemental [1], making the modelling of these objects approachable.

Methods

Simplifying the Problem

Several simplifications are made:

1) The electron gas is completely degenerate - i.e. the electrons' kinetic energy exceed that needed to escape the bound states of nuclei; pressure due to Pauli Exclusion Principle, rather than temperature, overwhelmingly dictates electron number density; thermal and radiation pressure are negligible.
2) The white dwarf is composed of an element where the number of nucleons per electron is exactly 2, eg. Helium-4 or Carbon-12.
3) Coulomb interactions between nucleons and the electron soup don't contribute to the energy density of the white dwarf.
4) A somewhat arbitrary factor of $a = 10$ is chosen as the boundary of $p_F = a\, m_e c$, where p_F is Fermi momentum, m_e is electron rest mass and c is the speed of light in vacuum, which separates the relativistic and non-relativistic polytropic equations of state based solely on central density ρ_0.
5) General relativistic corrections are neglected (they are significant to the integrand only at the third-decimal place and become significant at densities higher than that of interest to this simulation).

Equations

Recall from Fermi-Dirac statistics applied to an electron soup at $T = 0$ K that electron number density is $n_e = \frac{2}{(2\pi)^3}\left(\frac{4}{3}\pi k_F^3\right)$ [2] where k_F is the Fermi wave vector. Applying the relation $p_F = \hbar k_F$:

$$n_e = \frac{p_F^3}{3\pi^2\hbar^3} \tag{1}$$

Observing that the white dwarf requires local charge neutrality to be stable in its configuration, taking the number of protons to balance the electrons for a species where $A/Z = 2$, thus the nucleon number density is $n_n = 2n_e \approx \rho/2m_n = \epsilon_n/2m_n c^2$ where $\rho = \rho(r)$ is the mass density at a given radius, m_n is the average mass of proton and neutron and ϵ is the energy density of nucleons. Free electrons contribute both rest mass energy and kinetic energy, leading to electron energy density of

$$\epsilon_e = \int_0^{p_F} \sqrt{p^2c^2 + m_e^2c^4}\,\frac{dn_e}{dp_F}\,dp \tag{2}$$

Then, the total energy density at a given radius is given by $\epsilon = \epsilon_n + \epsilon_e$. Treating pressure as a momentum flux which is isotropic [3], the pressure P from electron degeneracy is

$$P = \frac{1}{3}\int_0^{p_F} p\,v\,\frac{dn_e}{dp_F}\,dp \tag{3}$$

where $v = pc^2/E$, with p and E being the relativistic momentum and energy respectively.

This expression for pressure may be simplified to polytropic forms at the non-relativistic limit where $p_F \ll m_e c$ and at the extremely relativistic limit $p_F \gg m_e c$ to

$$P_{nr} = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2}{2m_n c^2}\right)^2 \epsilon^{4/3} \tag{4}$$

and

$$P_{rel} = \frac{\hbar^2}{15\pi^2 m_e}\left(\frac{3\pi^2}{2m_n c^2}\right)^2 \epsilon^{5/3} \tag{5}$$

respectively [3]. With pressure related to energy density, the rest is to integrate the coupled differential equations describing the condition of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{G\epsilon(r)M(r)}{c^2r^2} \tag{6}$$

and

$$\frac{dM}{dr} = \frac{4\pi r^2\epsilon(r)}{c^2} \tag{7}$$

from $r = 0$ to R where $M(0) = 0$ and $P(R) = 0$.

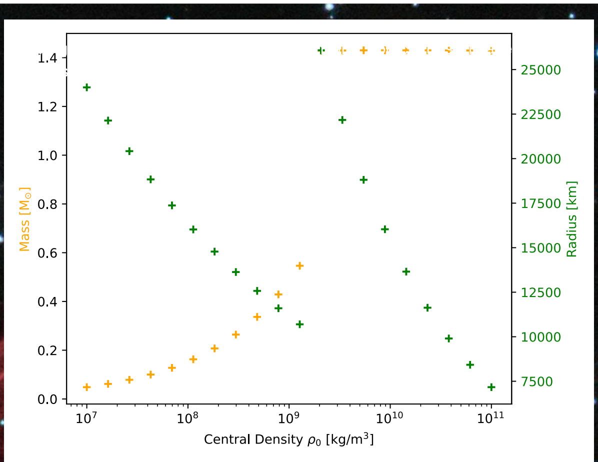

Fig. 1. Chandrasekhar masses and radii at those masses for a density range centred at the typical white dwarf density of 10^9 kg/m^3, neglecting electron energy density ϵ_e. The discontinuity in the masses and radii of white dwarfs is product of the choice of assuming either non-relativistic or extremely relativistic cases in the equation of state.

Discussion

As shown in Fig. 1, a maximum mass is obtainable in the extremely relativistic limit, while the radius of the star tends to zero as central density continues to increase. The effect of including electron energy density ϵ_e is roughly negligible. Using the expression for full energy density ϵ, the Chandrasekhar mass at $\rho_0 = 10^{11}$ kg/m^3 is 1.4274M$_{\odot}$. If electron energy density is neglected, as is done for Fig. 1, the Chandrasekhar mass at $\rho_0 = 10^{11}$ kg/m^3 is 1.4289 M$_{\odot}$. Seen in Fig. 2, the cut-off at the maximum mass is chosen to be 1.4 M$_{\odot}$, while white dwarfs of the smallest masses are rarely observed unless formed in a binary system, in part attributable to the longevity of stars with extremely low mass when left unperturbed [1] [4]. Further, the composition of a white dwarf depends on its progenitor's mass, thus at low central density in Fig. 1, the core is likely Helium-4, while at high density, the core corresponds more closely to Carbon-12. Fig. 2 additionally hints at the effect of the entropy of mixing and the effect of non-zero temperature at regions of densities much lower than the central density. In conclusion, the Chandrasekhar mass obtained by solving the simplified model agrees with upper bound of observational evidence, but more detailed modelling is desirable for testing theories of condensed matter.

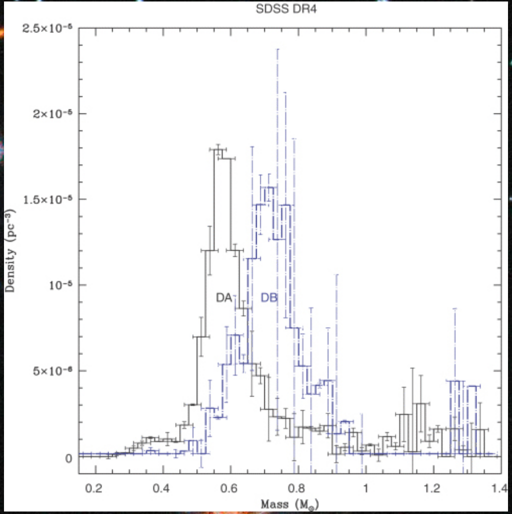

Fig. 2. The distribution of observed white dwarf masses in the Sloan Digital Sky Survey for those with Hydrogen-dominated atmospheres (DA) and those with Helium-dominated atmospheres (DB). [4]

References

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