FEM analysis:
Linear Elastic Object

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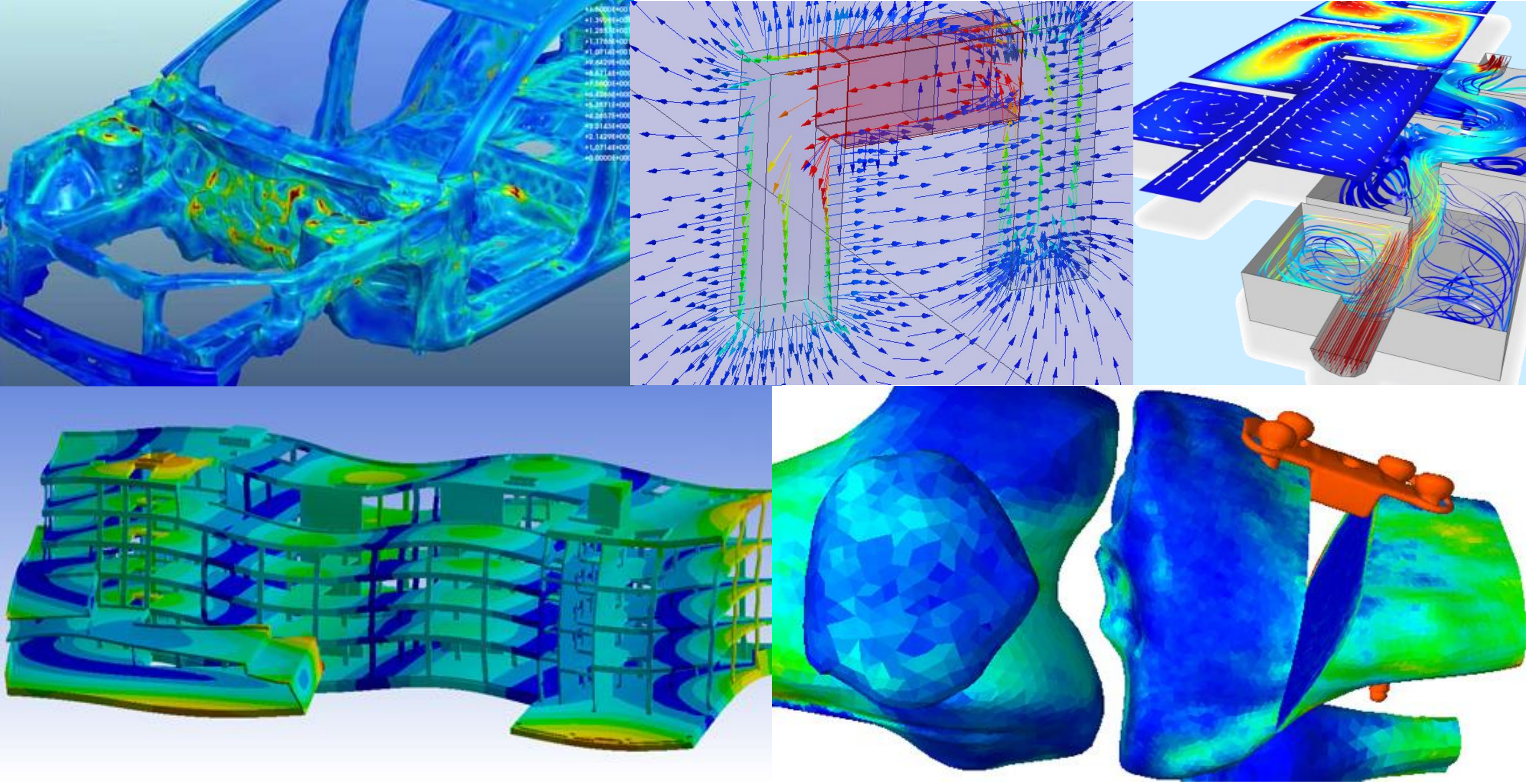
Physical simulation

Fundamentals

Physical simulation

Demonstration



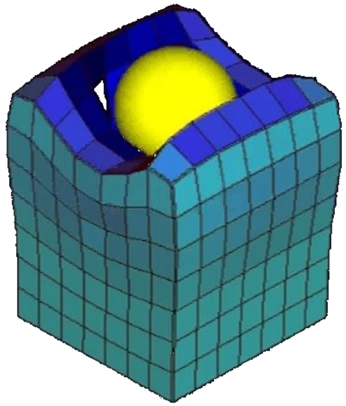
Physical simulation is an effective technology for various fields.

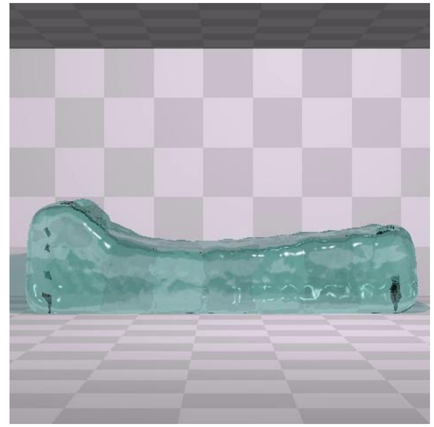
What is simulation?

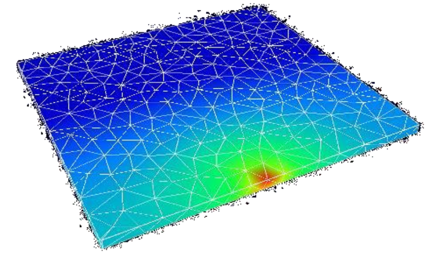
Meaning of simulation is imitation of real-world phenomenon.

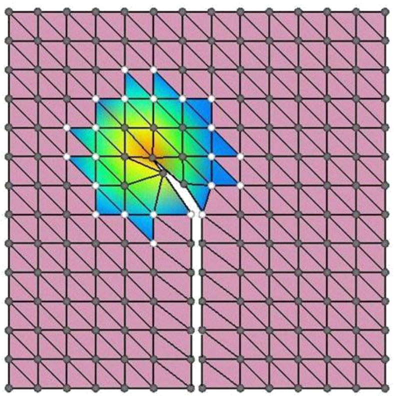
> “Investigation of objective signal and profile by imitating real-world phenomenon under controllable condition to reveal the mechanism” (from WIKI)

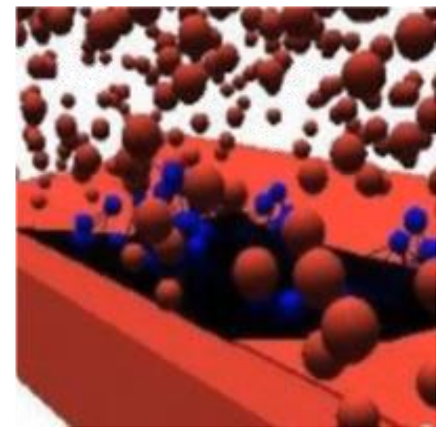
- Prediction of phenomenon for preparation
- Costless and convenient approach requiring no real resource
- Quantitative result is available.
- It is possible to Investigate the phenomenon of difficult to monitor.

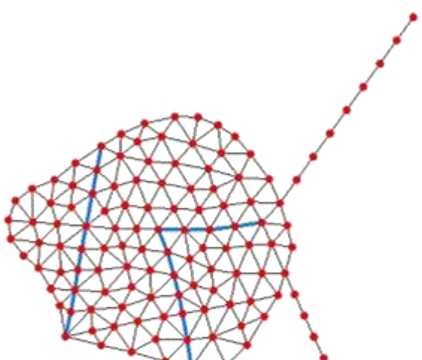


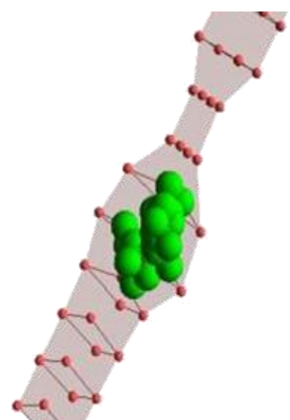


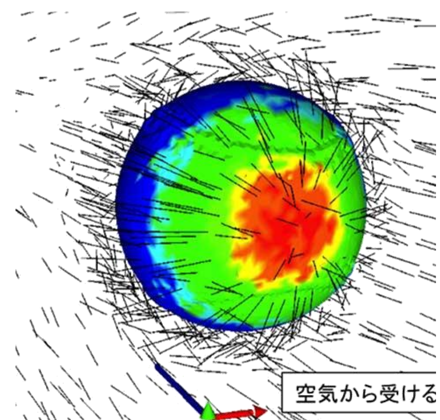











空気から受ける

Physical simulation:
Obtaining numerical solution for governing equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{F} - \frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v}$$

$$C_v\frac{\partial T}{\partial t} = -\lambda\frac{\partial^2 T}{\partial x^2}$$

$$\frac{1}{s^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

$$\boldsymbol{F} = m\frac{d^2\boldsymbol{x}}{dt^2} + c\frac{d\boldsymbol{x}}{dt} + k\boldsymbol{x}$$

$$\frac{\partial \phi}{\partial t} = D\nabla^2\phi$$

$$\nabla \cdot \boldsymbol{B} = 0$$

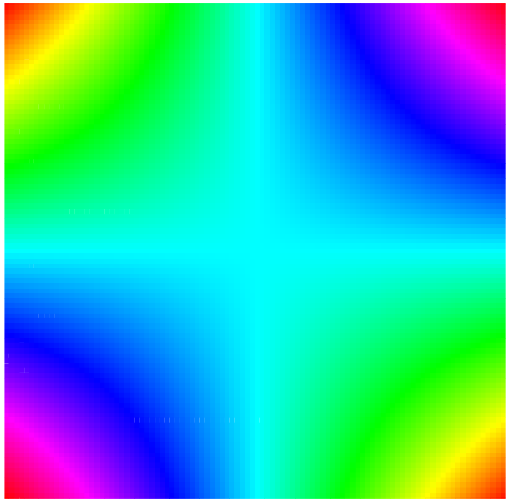
$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

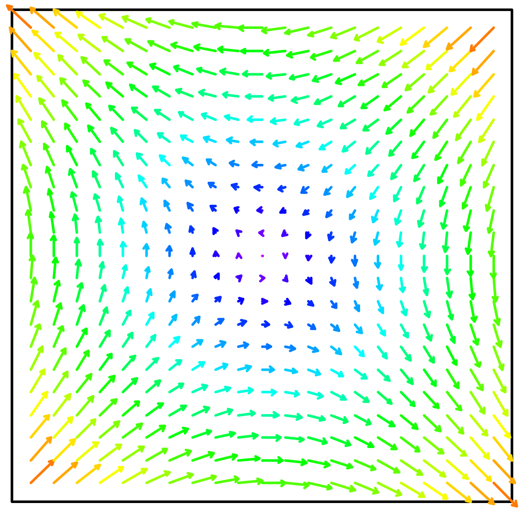
$$\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = j$$

$$\int_v \{\epsilon\}^{\mathrm{T}}\{\sigma\}dv - \int_v \{U\}^{\mathrm{T}}\{\bar{G}\}dv - \int_{S_\sigma}\{U\}^{\mathrm{T}}\{\bar{T}\}ds = 0$$

Scalar field, vector field and derivative operator



Scalar field



Vector field

$$\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$$

Spatial derivative

$$\nabla f = \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\} = \mathrm{grad} f$$

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \mathrm{rot} \boldsymbol{v}$$

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \mathrm{div} \boldsymbol{v}$$

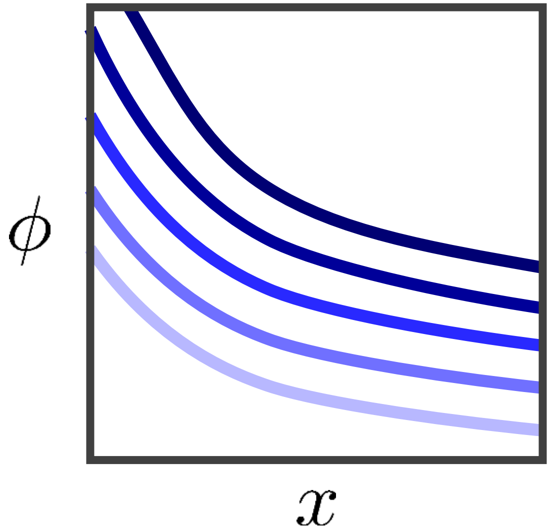
$$\Delta f = \nabla^2 f$$

Solution with initial and boundary conditions

Without boundary condition

$$\frac{d\phi(x)}{dx} = -\phi(x)$$

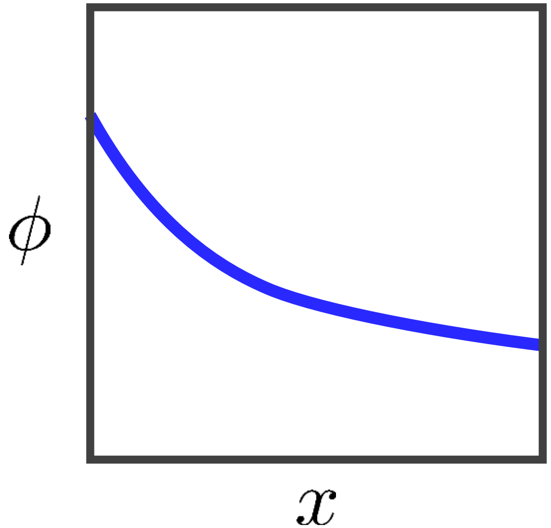
$$\phi(x) = A\exp(-x)$$



With boundary condition

$$\frac{d\phi(x)}{dx} = -\phi(x) \quad \boxed{\phi(0) = 1}$$

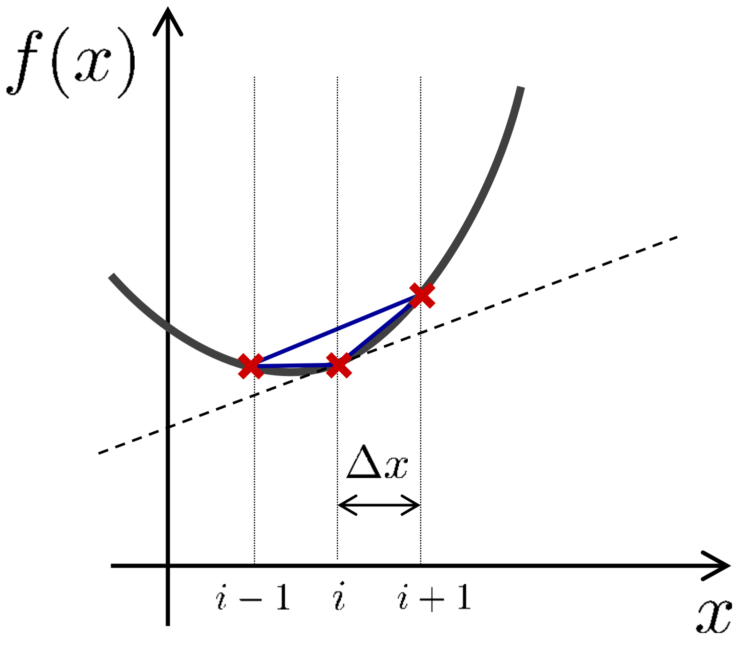
$$\phi(x) = \exp(-x)$$



Discretization for calculation in a computer

$$f(x) \xrightarrow{\text{discretization}} f[i]$$

$$\frac{df(x)}{dx} \xrightarrow{\text{discretization}}$$



Forward difference $$\frac{f[i+1]-f[i]}{\Delta x}$$

Central difference $$\frac{f[i+1]-f[i-1]}{2\Delta x}$$

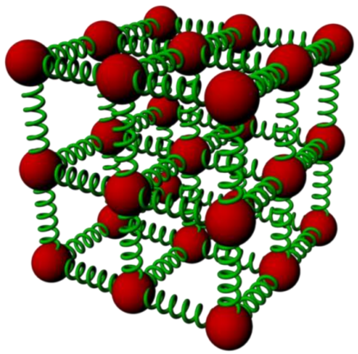
Backward difference $$\frac{f[i]-f[i-1]}{\Delta x}$$

Keywords in numerical analysis

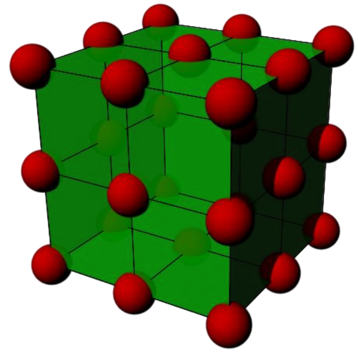
- Error, Accuracy, Stability
- Newton method, Bisection method
- Gaussian elimination, iterative method
- Polynomial interpolation, Least-squares method
- Quadrature by parts, Trapezoidal rule, Simpson’s rule
- Euler method, Crank-Nicholson scheme, Runge-Kutta method
- Method of Lagrange multiplier

Representative model for deformation

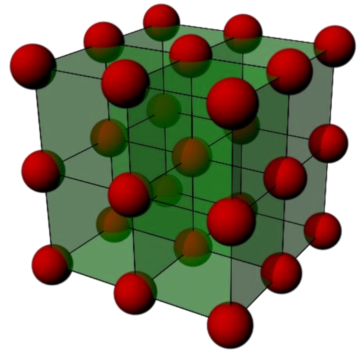
- Mass spring model (1D element)
  - Spring between nodes
  - Easy to implement, difficult to select parameter



- Boundary element model (2D element)
  - Element of boundary surface
  - homogeneous and isotropic object only



- Finite element model (3D element)
  - Simulation with measurable parameters
  - Large calculation cost



Explanation of simple problem

- Physical phenomenon
  - Deformation, potential distribution, heat conduction …
- Type of equation
  - Linear, nonlinear
- Analysis mode
  - Static (quasistatic), dynamic, frequency domain, buckling analysis

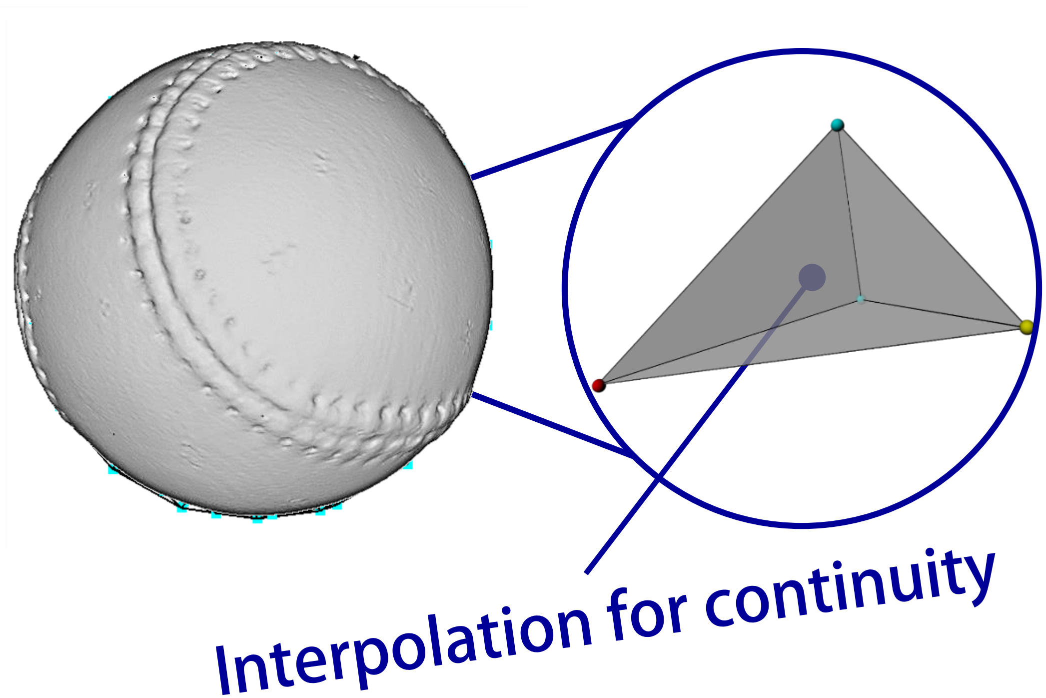
$$\{F\} = [k]\{\delta\}$$

Finite element method

Theory

Finite element method

- A method to solve partial differential equation
- Discretization of object shape to solve with a computer
- Adjustment of equation based on the above operation



$$\rho \frac{\partial \boldsymbol{v}}{\partial t}\bigg|_x = \rho \boldsymbol{g} + \nabla \cdot \boldsymbol{\sigma}$$

weak form

$$\int_v \{\epsilon\}^{\mathrm{T}}\{\sigma\}dv - \int_v \{U\}^{\mathrm{T}}\{\bar{G}\}dv - \int_{S_\sigma} \{U\}^{\mathrm{T}}\{\bar{T}\}ds = 0$$

$$[K^e]\{\delta^e\} = \{F^e\}$$

Steps for deformation analysis with FEM

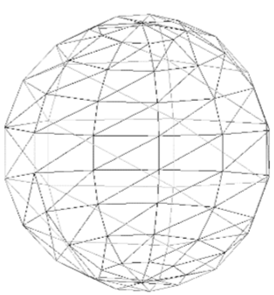
1. Mesh generation
2. Elastic parameter setting
3. Generation of stiffness equation
4. Solving equation with boundary condition
5. Visualization of physical quantitiy

Dirichlet condition: deformation is determined

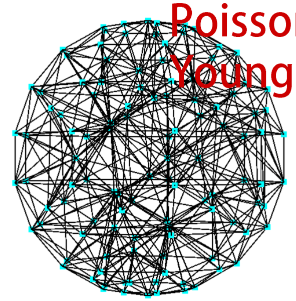
Neumann condition: force is zero

$$\{F\} = [k]\{\delta\}$$

Poisson ratio
Young's modulus

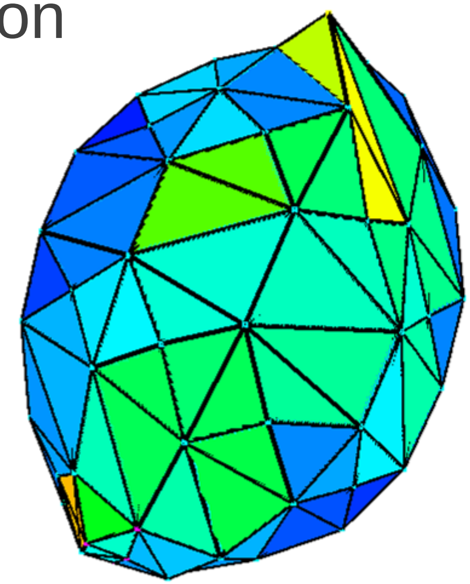


TetGen



Surface Mesh

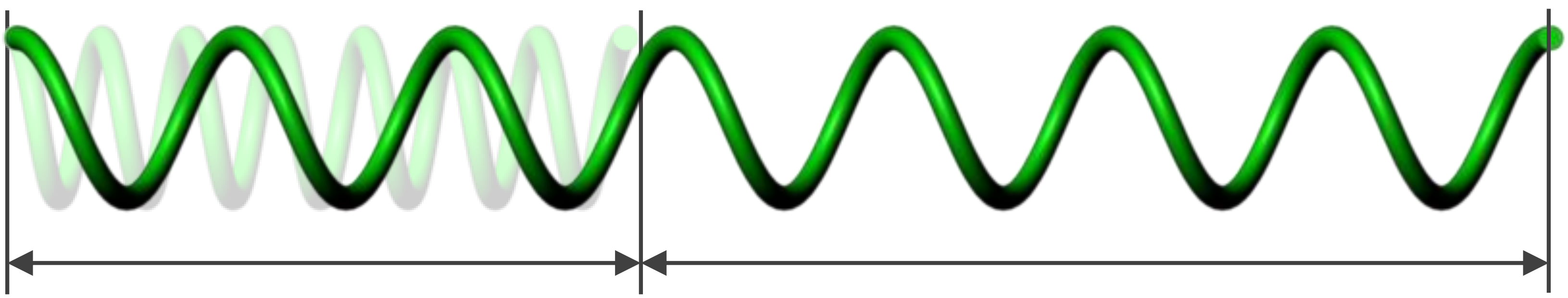
Volume Mesh



Generation of stiffness equation

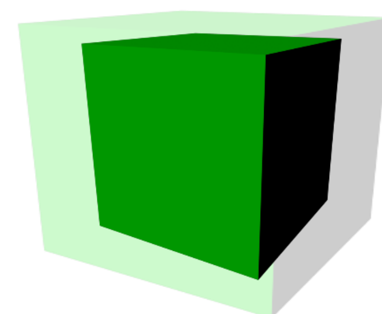
1. Mechanics of elasticity
2. Governing equation
3. Formularization
4. Element stiffness equation
5. Total stiffness equation

Strain: Normalized measure of displacement



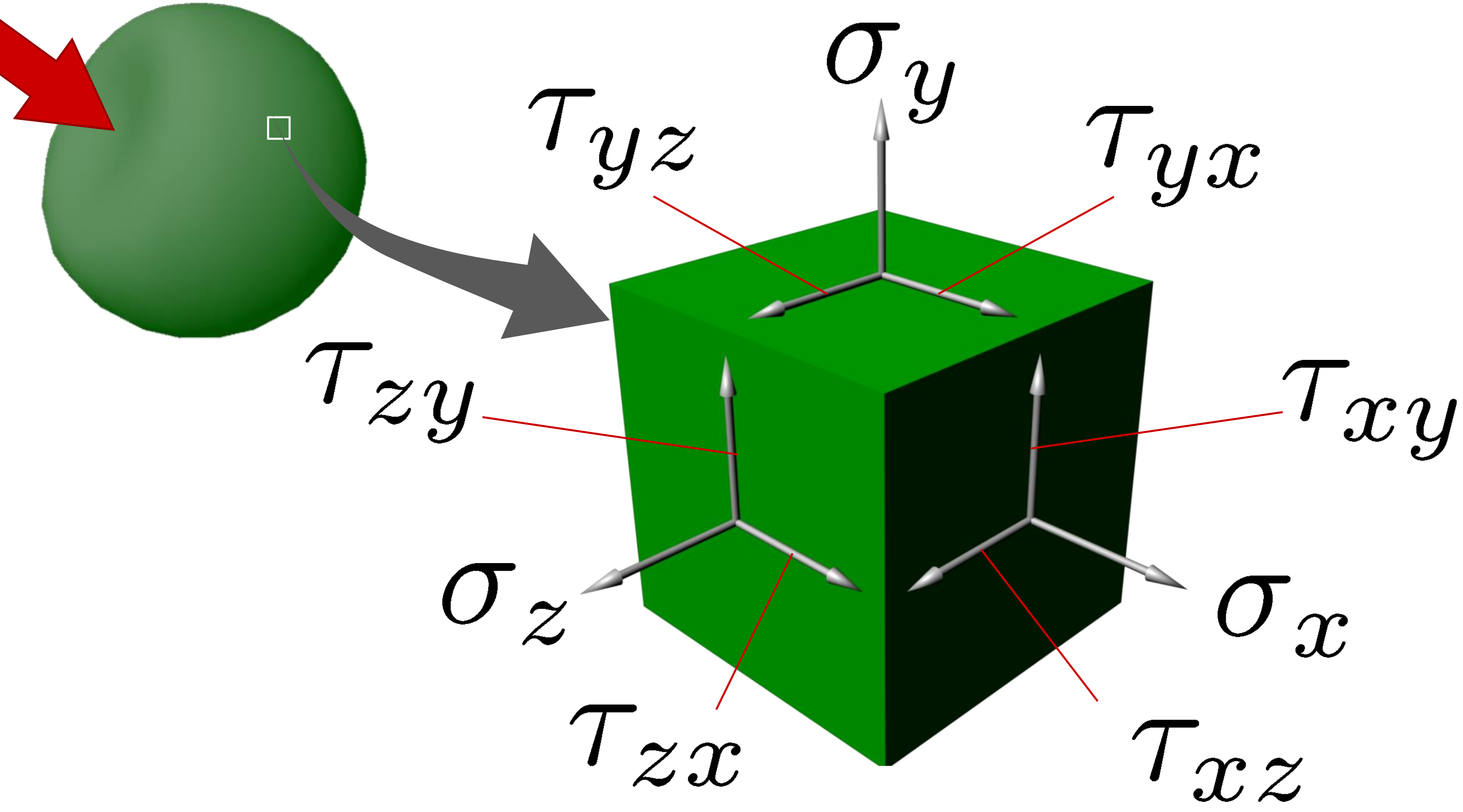
dx $d\delta$

$$\epsilon = \frac{\partial \delta}{\partial x}$$



9 components (3D)
(Normal and shear direction)

Stress: Force per unit of area



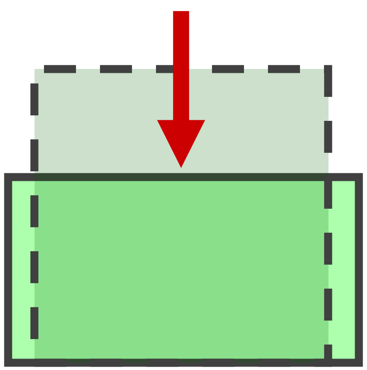
6 components for asymmetric object

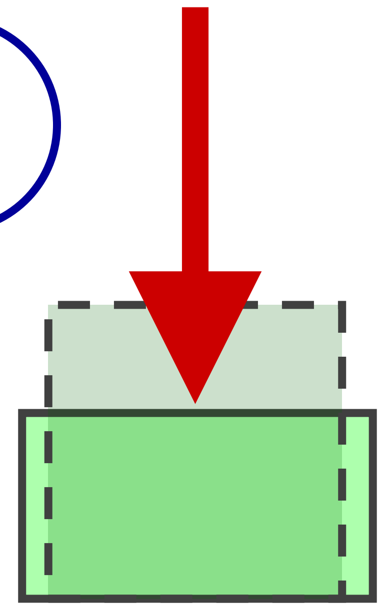
Physical parameter I: Young's modulus

Stress

$$E = \frac{\sigma}{\epsilon}$$

Strain





Object of large young's modulus is difficult to be deformed.

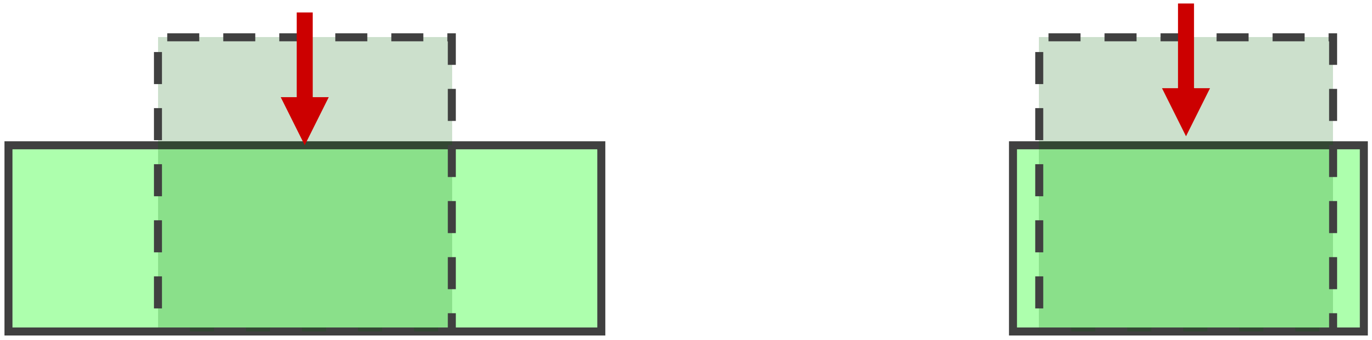
0.01 ~ 1200 GPa

Physical parameter II: Poisson ratio

$$\nu = -\frac{\epsilon}{\gamma}$$

Shear strain

Normal strain



Object of negative Poisson ratio extends transversally according to pressing force

-1 ~ 0.5

Governing equation: Cauchy's first law

Motion

Body force

Surface force

$$\rho \frac{\partial \boldsymbol{v}}{\partial t}\bigg|_{\boldsymbol{x}} = \rho \boldsymbol{g} + \nabla \cdot \boldsymbol{\sigma}$$

Zero for static analysis

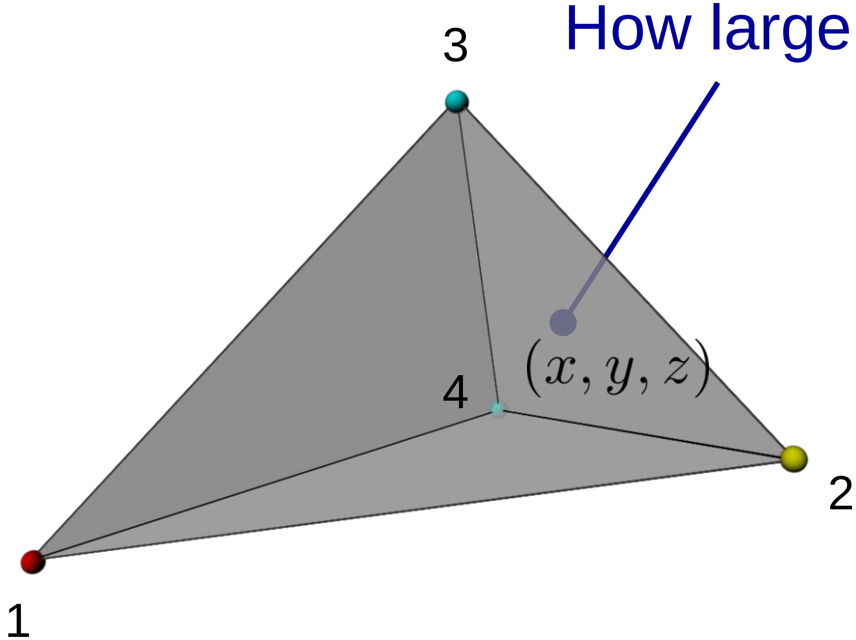
Volume integral

$$\int_v \{\epsilon\}^{\mathrm{T}}\{\sigma\}dv - \int_v \{U\}^{\mathrm{T}}\{\bar{G}\}dv - \int_{S_\sigma} \{U\}^{\mathrm{T}}\{\bar{T}\}ds = 0$$

Principle of virtual work (The law of the conservation of energy)

Formularization: Deformation field in element

How large is the deformation at this point?



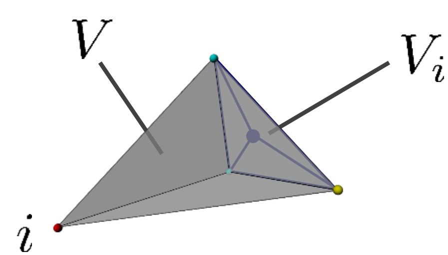
$$\delta_x = \sum_{i=1}^{4} N_i \delta_x^i = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$

$$\delta_y = \sum_{i=1}^{4} N_i \delta_y^i = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 z$$

$$\delta_z = \sum_{i=1}^{4} N_i \delta_z^i = \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} z$$

Deformation at an arbitrary point can be represented by using shape function N and node deformations

Linear interpolation

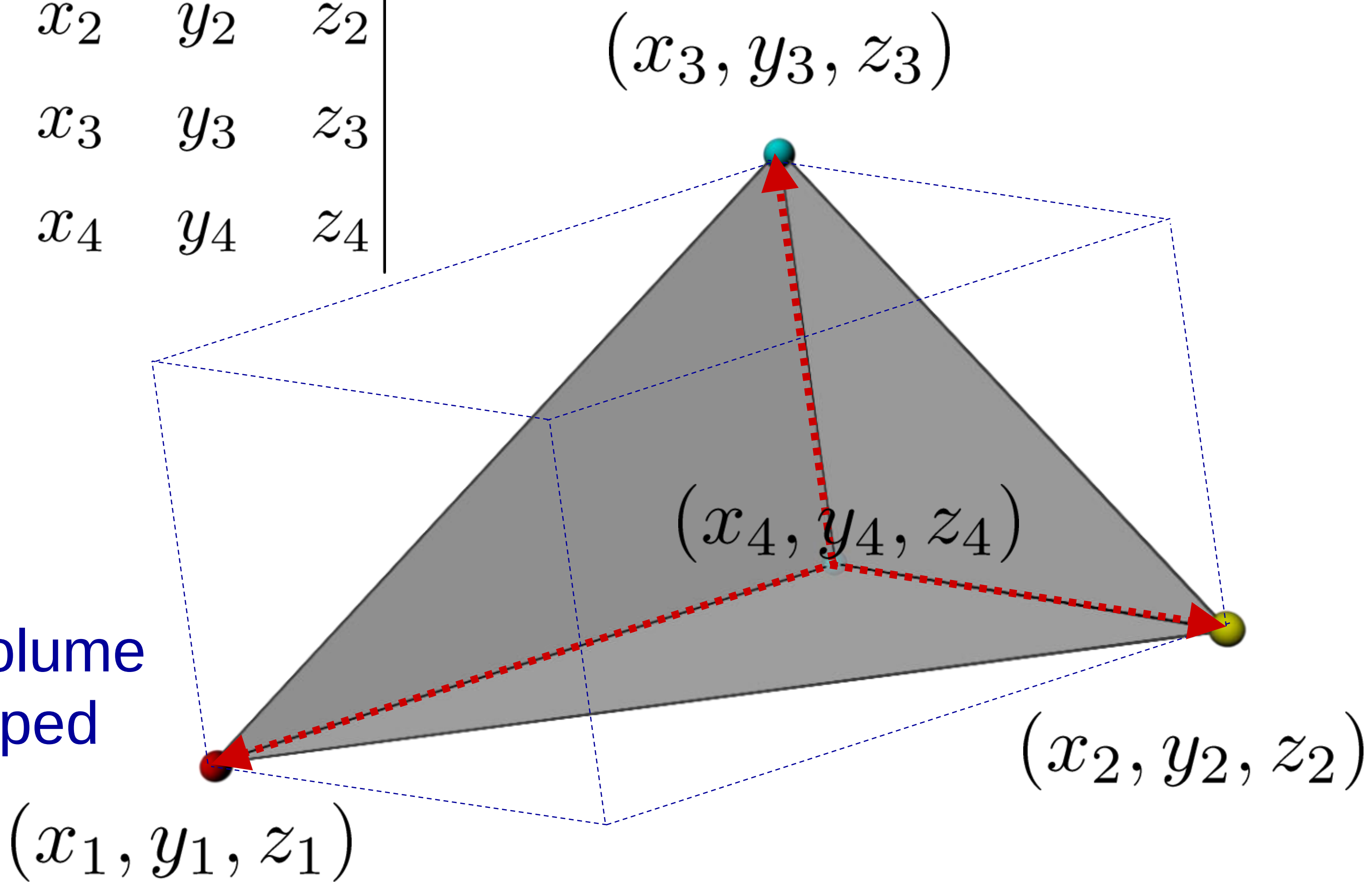


$$N_i = \frac{V_i}{V}$$

Function of x,y,z

Determinant and volume of tetrahedron

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$$



Six part of Volume of parallelepiped

How to calculate coefficients α

$$\delta_x = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$

Calculate coefficients by using 4 vertex coordinates

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}^{-1} \begin{bmatrix} \delta_x^1 \\ \delta_x^2 \\ \delta_x^3 \\ \delta_x^4 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \begin{bmatrix} \delta_x^1 \\ \delta_x^2 \\ \delta_x^3 \\ \delta_x^4 \end{bmatrix}$$

Representation of shape function

Assignment

$$\delta_x = \sum_{i=1}^{4} N_i \delta_x^i$$
$$= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$

Coefficient comparison

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$$

Element strain

$$\epsilon_x = \frac{\partial \delta_x}{\partial x} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial x} \cdot \delta_x^i$$

Deformation from original shape

$$\gamma_{xy} = \frac{\partial \delta_x}{\partial y} + \frac{\partial \delta_y}{\partial x} = \sum_{i=1}^{4} \left(\frac{\partial N_i}{\partial y} \cdot \delta_x^i + \frac{\partial N_i}{\partial x} \cdot \delta_y^i \right)$$

How to calculate spatial derivative of shape function

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial z} \\ \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial z} \\ \frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial z} \\ \frac{\partial N_4}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial z} \end{bmatrix} = \frac{1}{6V} \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{bmatrix}$$

Coefficients can be calculated by spatial derivative

Strain-deformation and stress-strain relations

$$[\epsilon] = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & & \frac{\partial N_4}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \dots & 0 & \frac{\partial N_4}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial z} & & 0 & 0 & \frac{\partial N_4}{\partial z} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & \dots & 0 & \frac{\partial N_4}{\partial z} & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & & \frac{\partial N_4}{\partial z} & 0 & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{bmatrix} \delta_x^1 \\ \delta_y^1 \\ \delta_z^1 \\ \vdots \\ \delta_x^4 \\ \delta_y^4 \\ \delta_z^4 \end{bmatrix} = [B][\delta]$$

$$[\sigma] = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & sym. & & & \frac{1-2\nu}{2} & 0 \\ & & & & & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = [D][\epsilon]$$

Element stiffness equation

$$[K^e]\{\delta^e\} = \{F^e\}$$

Hooke's law

How to derive this equation?

$$\rho \frac{\partial \boldsymbol{v}}{\partial t}\bigg|_x = \rho \boldsymbol{g} + \nabla \cdot \boldsymbol{\sigma}$$

$$\underbrace{\int_v \{\epsilon\}^{\mathrm{T}}\{\sigma\}dv}_{\text{Internal energy}} - \underbrace{\int_v \{U\}^{\mathrm{T}}\{\bar{G}\}dv - \int_{S_\sigma} \{U\}^{\mathrm{T}}\{\bar{T}\}ds}_{\text{External energy}} = 0$$

$$[\epsilon] = [B][\delta] \qquad [\sigma] = [D][\epsilon] \qquad [U] = [N][\delta]$$

$$[K^e] = \int_v [B]^{\mathrm{T}}[D][B]dv \qquad \{F^e\} = \int_v [N]^{\mathrm{T}}\{\bar{G}^e\}dv + \int_{S_\sigma} [N']^{\mathrm{T}}\{\bar{T}^e\}dv$$

How to calculate element stiffness matrix

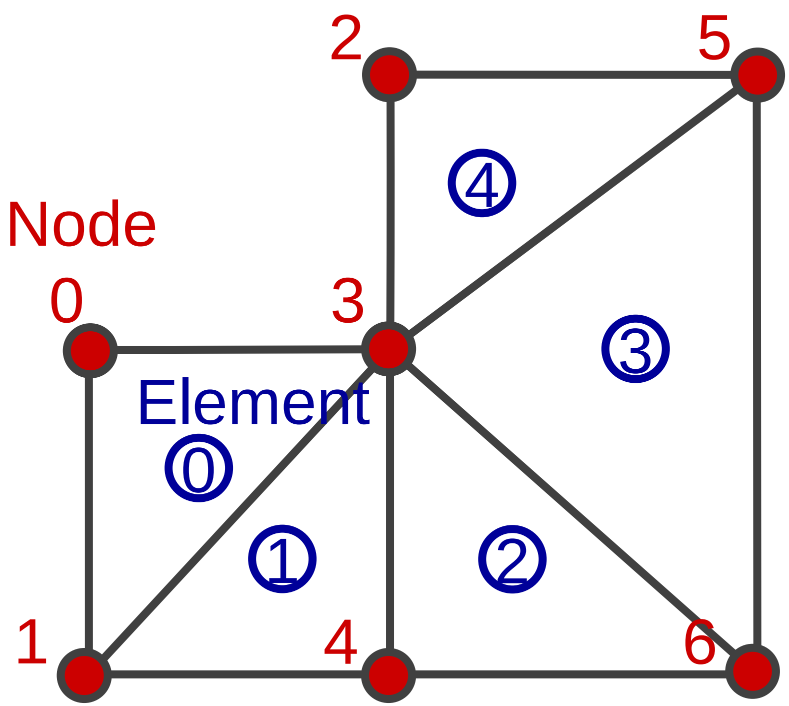
$$[K^e] = \int_v [B]^{\mathrm{T}}[D][B]dv = [B]^{\mathrm{T}}[D][B]\int_v dv$$

Incase of homogeneous object,
integral calculation can be replaced with volume calculation.

$$[K^e] = V[B]^{\mathrm{T}}[D][B]$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & & \frac{\partial N_4}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \dots & 0 & \frac{\partial N_4}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial z} & & 0 & 0 & \frac{\partial N_4}{\partial z} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & \dots & 0 & \frac{\partial N_4}{\partial z} & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & & \frac{\partial N_4}{\partial z} & 0 & \frac{\partial N_4}{\partial x} \end{bmatrix} \quad [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & sym. & & & \frac{1-2\nu}{2} & 0 \\ & & & & & \frac{1-2\nu}{2} \end{bmatrix}$$

Interaction between elements (nodes) can be represented by matrix



$$
\begin{array}{cc}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\
\begin{bmatrix}
k_{00} & k_{01} & 0 & k_{03} & 0 & 0 & 0 \\
k_{10} & k_{11} & 0 & k_{13} & k_{14} & 0 & 0 \\
0 & 0 & k_{22} & k_{23} & 0 & k_{25} & 0 \\
k_{30} & k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\
0 & k_{41} & 0 & k_{43} & k_{44} & 0 & k_{46} \\
0 & 0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\
0 & 0 & 0 & k_{63} & k_{64} & k_{65} & k_{66}
\end{bmatrix} &
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

List of node indexes belonging to each element is needed.

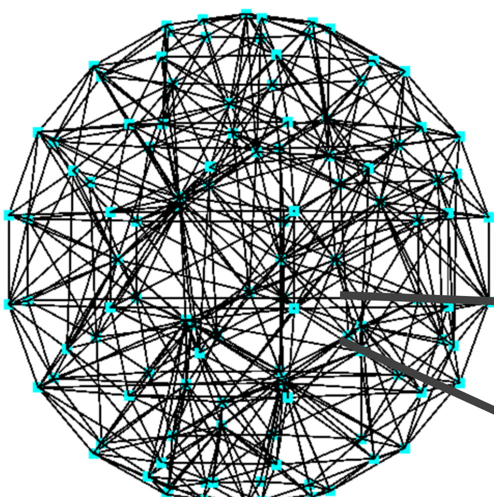
Shape variant (Tetrahedron in this explanation)

Extension of stiffness matrix for whole body

Superimpose all element stiffness matrix based on node index

$$[k]\{\delta\} = \{F\}$$

$$\begin{array}{ccc} & \text{Node 1} \qquad\qquad \text{Node 2} \qquad\qquad \text{Node N} & \\ [k] = & \left[\begin{array}{cccccc|c} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} & \\ \hline k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} & \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & \\ \hline & & & & & & \ddots \\ & & & & & & k_{3N3N} \end{array}\right] & \begin{array}{l} \text{Node 1} \\ \\ \text{Node 2} \\ \\ \text{Node N} \end{array} \end{array}$$



Element1: Node{1,2,3,4} $[K^{e_1}]\{\delta^{e_1}\} = \{F^{e_1}\}$

Element2: Node{2,5,6,7} $[K^{e_2}]\{\delta^{e_2}\} = \{F^{e_2}\}$

How to solve stiffness equation

1. Solving simultaneous equations
2. Boundary condition
3. Analysis for deformation input
4. Tips for calculation

Solve stiffness equation (simultaneous equations)

$$\{F\} = [k]\{\delta\}$$

$$\{\delta\} = [k]^{-1}\{F\}$$

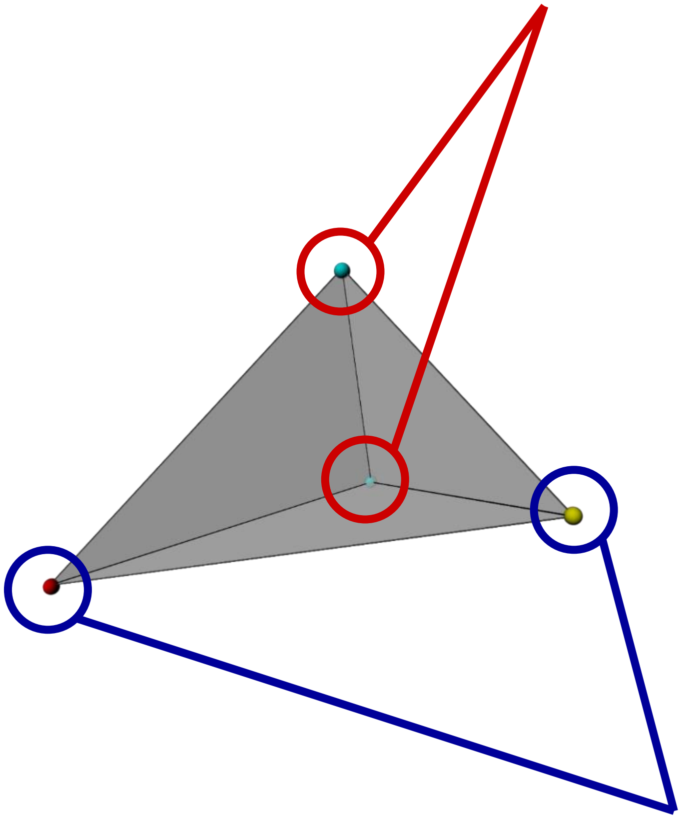
Underspecified equation

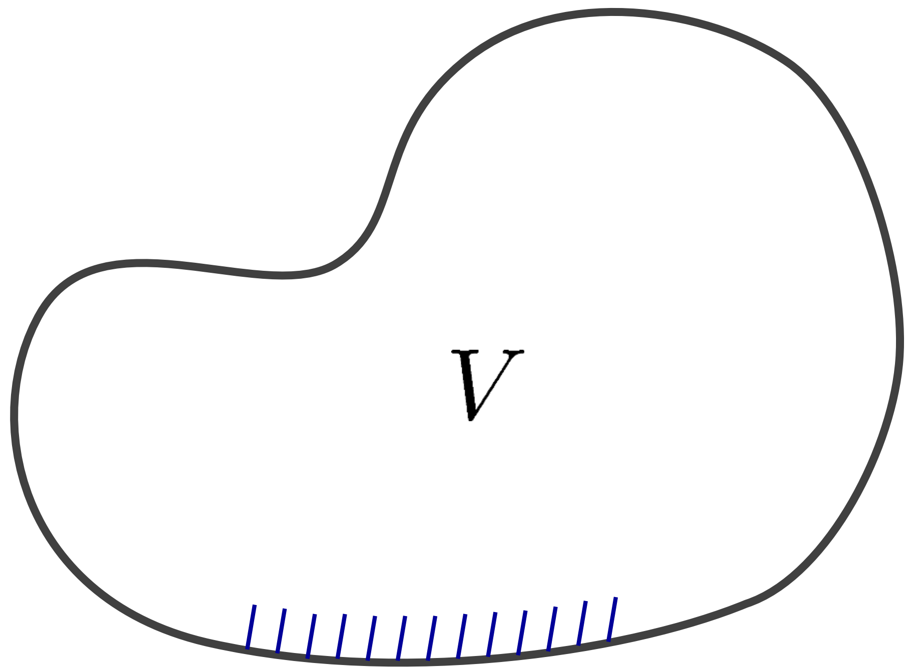
- Matrix calculation instead of solving partial difference equation
- Simultaneous equations of number of node $\times$ degree of freedom
- Number of equation should be more than number of unknown parameter.
- What are unknown and known parameter?

Boundary condition: Dirichlet and Neumann

Behaviors are determined

$S_n \qquad \frac{d\boldsymbol{x}}{dn} = \boldsymbol{x}_1$



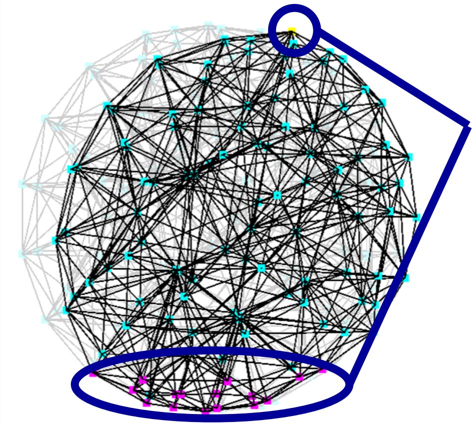


Values are determined

$S_d \qquad \boldsymbol{x} = \boldsymbol{x}_0$

Deformation analysis under forced displacement

$$\begin{Bmatrix} \boxed{\{F_d\}} \\ \boxed{\{F_n\}} \end{Bmatrix} = \begin{bmatrix} [K_{dd}] & [K_{dn}] \\ [K_{nd}] & [K_{nn}] \end{bmatrix} \begin{Bmatrix} \boxed{\{\delta_d\}} \\ \boxed{\{\delta_n\}} \end{Bmatrix}$$

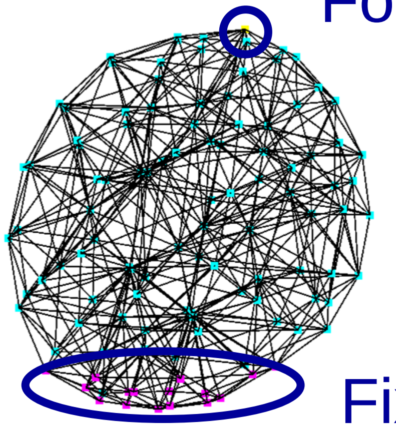


Dirichlet condition: deformation is determined

Neumann condition: force is zero

$$\boxed{\begin{aligned} \{\delta_n\} &= -[K_{nn}]^{-1}[K_{nd}]\{\delta_d\} \\ \{F_d\} &= [K_{dd}]\{\delta_d\} - [K_{dn}][K_{nn}]^{-1}[K_{nd}]\{\delta_d\} \end{aligned}}$$

Reducing calculation cost for real time simulation



Forced displacement S_d

Fixed S_f

$$\begin{Bmatrix} \{F_f\} \\ \{F_s\} \end{Bmatrix} = \begin{bmatrix} [K_{ff}] & [K_{fs}] \\ [K_{sf}] & [K_{ss}] \end{bmatrix} \begin{Bmatrix} \{0\} \\ \{\delta_s\} \end{Bmatrix}$$

$$\{F_s\} = [K_{ss}]\{\delta_s\} \qquad \{\delta_s\} = [L_{ss}]\{F_s\}$$

$$\begin{Bmatrix} \{\delta_d\} \\ \{\delta_n\} \end{Bmatrix} = \begin{bmatrix} [L_{dd}] & [L_{dn}] \\ [L_{nd}] & [L_{nn}] \end{bmatrix} \begin{Bmatrix} \{F_d\} \\ \{0\} \end{Bmatrix}$$

$$\{\delta_n\} = -[L_{nd}][L_{dd}]^{-1}\{\delta_d\}$$

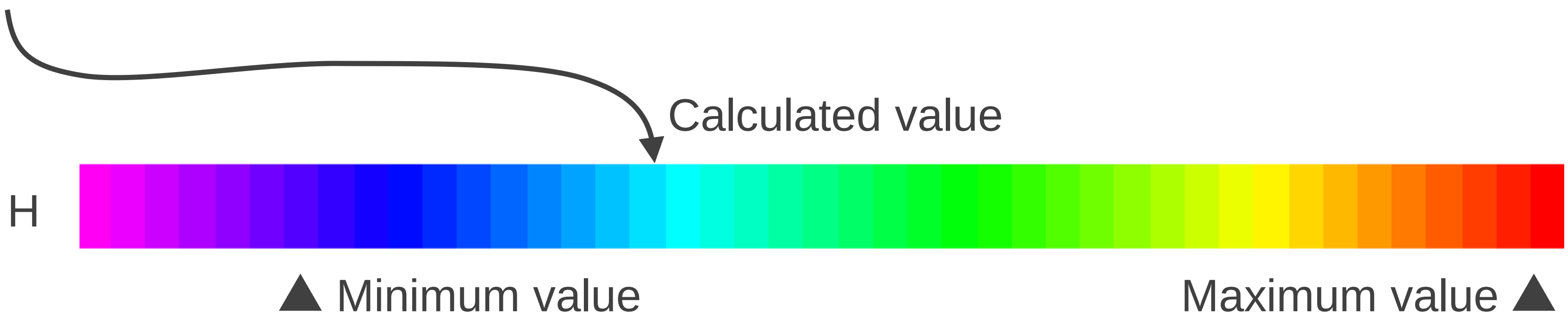
Effective when

- S_f is constant
- $dimS_f < dimS$

Visualization of physical quantity

Example: Von Mises stress

$$\sigma_{VM} = \sqrt{\frac{1}{2}\left\{(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2+3(\sigma_{xy}^2+\sigma_{xz}^2+\sigma_{yx}^2+\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{zy}^2)\right\}}$$



- Each element has value.
- Color map is useful for visualizing the spatial distribution of the stress.
- H value of HSV color space is a general representation of the color map.

Finite element method

Implementation

List of variables

- Element
  - Node index
  - Node coordinate
  - Displacement vector
  - Stiffness matrix
  - Stress strain matrix
  - Strain displacement matrix
  - Shape function
  - Poisson ratio
  - Young’s modulus
  - Volume
  - von Mises stress
  - Strain vector
  - Stress vector
- Entire model
  - Number of node
  - Number of element
  - Node coordinate
  - Element
  - Node set of not Dirichlet condition
  - Node set of Dirichlet condition
  - Node set of Neumann condition
  - Flag for boundary condition
  - Force vector
  - Displacement vector
  - Stiffness matrix

List of functions

- Matrix multiplication, matrix-vector multiplication, transposed matrix, inverse matrix and determinant
- Loading a model
- Physical parameter setting for each element
- Shape function, strain, stress, von mises stress and volume calculations for each element
- Setting of strain-displacement relation matrix [Be] for each element
- Setting of stress-strain relation matrix [De] for each element
- Stiffness matrix [Ke] setting for each element
- Total stiffness matrix [K]
- Boundary condition setting and pre-calculation of inverse matrix
- Setting of load condition
- Solving stiffness equation (displacements for all node will be calculated)
- Calculation of von mises stress for each element
- Releasing displacement

.fem file format (ASCII)

nNodes number of nodes

nTetrahedra number of elements

@1 Node coordinate

x value y value z value

・・・

$\}$ Number of nodes

@2 Node index for each element

Node index Node index Node index Node index

・・・

$\}$ Number of elements

@3 Material index for each element

Material index

・・・

$\}$ Number of elements

Implementation of stress-strain matrix setting

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & sym. & & & \frac{1-2\nu}{2} & 0 \\ & & & & & \frac{1-2\nu}{2} \end{bmatrix}$$

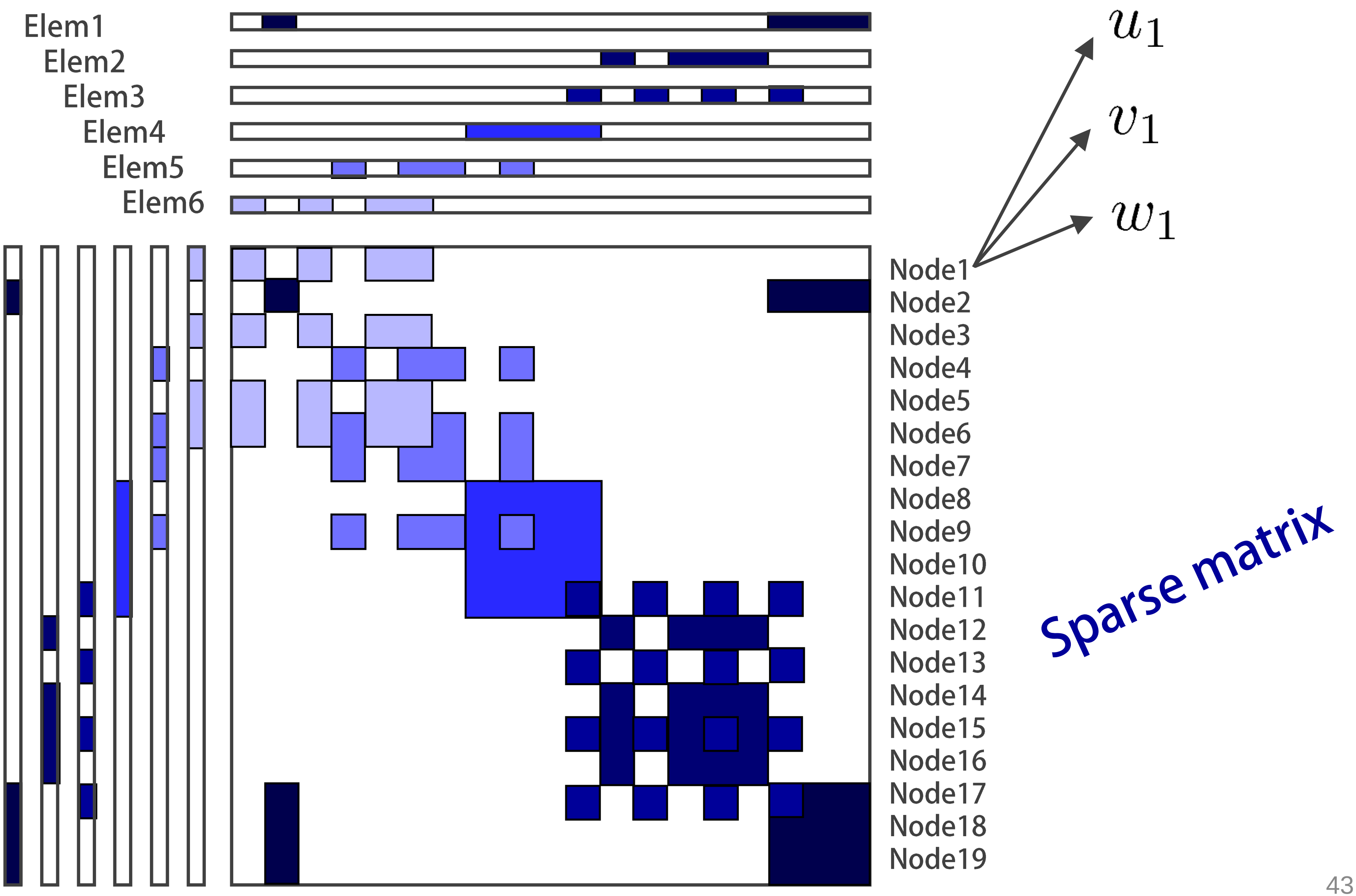
```
unsigned int i;

double Dscale;

Dscale = young_modulus / ( ( 1.0 + poisson_ratio )
                          * ( 1.0 - 2.0 * poisson_ratio ) );

for( i = 0; i < 3 ; i ++ ){
    D[ 6 * i + i ] = Dscale * ( 1.0 - poisson_ratio );
    D[ 6 * i + ( i + 1 ) % 3 ] = Dscale * poisson_ratio;
    D[ 6 * i + ( i + 2 ) % 3 ] = Dscale * poisson_ratio;
    D[ 6 * ( i + 3 ) + i + 3 ] = Dscale * ( 1.0 - 2 * poisson_ratio ) / 2;
}
```

Overview of total stiffness matrix generation



Decomposition of vector and matrix for boundary condition

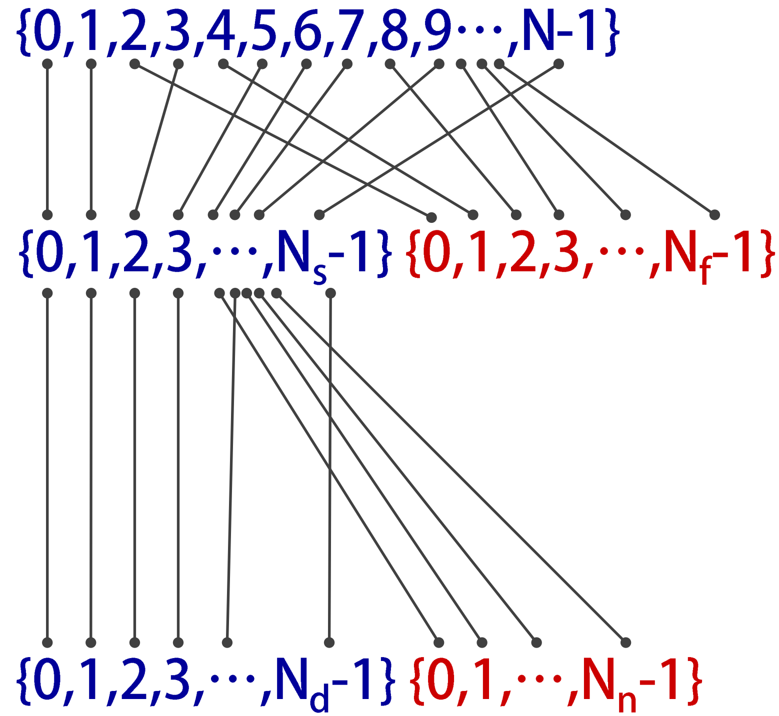
$$\{F\} = [k]\{\delta\}$$

$$\begin{Bmatrix}\{F_f\}\\ \{F_s\}\end{Bmatrix} = \begin{bmatrix}[K_{ff}] & [K_{fs}]\\ [K_{sf}] & [K_{ss}]\end{bmatrix}\begin{Bmatrix}\{0\}\\ \{\delta_s\}\end{Bmatrix}$$

$$\{F_s\} = [K_{ss}]\{\delta_s\} \qquad \{\delta_s\} = [L_{ss}]\{F_s\}$$

$$\begin{Bmatrix}\{\delta_d\}\\ \{\delta_n\}\end{Bmatrix} = \begin{bmatrix}[L_{dd}] & [L_{dn}]\\ [L_{nd}] & [L_{nn}]\end{bmatrix}\begin{Bmatrix}\{F_d\}\\ \{0\}\end{Bmatrix}$$

$$\{\delta_n\} = -[L_{nd}][L_{dd}]^{-1}\{\delta_d\}$$



List of corresponding node indexs

Physical simulation

Discussion

This is the only overview.

- Learn continuum mechanics,
  if you don’t understand governing equation.
- Learn calculusinfinitesimal calculus and algebra,
  if you don’t understand formulization.
- Learn numerical analysis,
  if you don’t understand programming of simultaneous equations.
- Learn C language,
  if you don’t understand implementation.
- The shortest way to learn is to ask professionals what you don’t understand.

※ Difficult thing is that the description is different between the textbooks.

What you have to consider in simulation

- Explain your analysis design logically.
  - How to construct a model for the simulation.
  - Which is your analysis mode, static, dynamic, frequency, or bulking analysis?
  - What is boundary condition, input and output?
- It is possible to show the property but difficult to explain the essence.
- The investigation of validity is difficult.
- No appropriate imitation, no valid result.
- You cannot find your mistake if your model is too complicated.

References

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