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◎ベクトル球面調和関数

文献：D.A. Varshalovich, A.N. Moskalev, and V.K. Khersonskii, "Quantum Theory of Angular Momentum", World Scientific (1988).

☑定義

$$\mathbb{Y}^{l}_{JM}(\hat{\mathbf{r}}) \equiv \sum_{m=-l}^{l}\sum_{\mu=-1}^{1} C^{JM}_{lm1\mu} Y_{lm}(\hat{\mathbf{r}})\, \mathbf{e}_{\mu}$$

$C^{JM}_{lm1\mu}$ ：Clebsch-Gordan係数

$Y_{lm}(\hat{\mathbf{r}})$ ：(普通の)球面調和関数

$\mathbf{e}_{\mu}$ ：(共変)球面基底ベクトル

$$\mathbf{e}_1 = -\frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y), \quad \mathbf{e}_0 = \mathbf{e}_z, \quad \mathbf{e}_{-1} = \frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y)$$

☑例

- $l=0$

$$\mathbb{Y}^{0}_{1M} = Y_{00}\mathbf{e}_M = \frac{1}{\sqrt{4\pi}}\mathbf{e}_M \quad (M=-1,0,1)$$

- $l=1$

$$\mathbb{Y}^{1}_{00} = \frac{1}{\sqrt{3}}(Y_{11}\mathbf{e}_{-1} - Y_{10}\mathbf{e}_0 + Y_{1-1}\mathbf{e}_1) = -\frac{1}{\sqrt{4\pi}}(\hat{x}\mathbf{e}_x + \hat{y}\mathbf{e}_y + \hat{z}\mathbf{e}_z)$$

$$\mathbb{Y}^{1}_{1-1} = \frac{1}{\sqrt{2}}(Y_{10}\mathbf{e}_{-1} - Y_{1-1}\mathbf{e}_0) = \frac{1}{\sqrt{4\pi}}\cdot\frac{\sqrt{3}}{2}\left\{\hat{z}(\mathbf{e}_x - i\mathbf{e}_y) - (\hat{x} - i\hat{y})\mathbf{e}_z\right\}$$

$$\mathbb{Y}^{1}_{10} = \frac{1}{\sqrt{2}}(Y_{11}\mathbf{e}_{-1} - Y_{1-1}\mathbf{e}_1) = \frac{i}{\sqrt{4\pi}}\sqrt{\frac{3}{2}}(\hat{x}\mathbf{e}_y - \hat{y}\mathbf{e}_x)$$

$$\mathbb{Y}^{1}_{11} = \frac{1}{\sqrt{2}}(Y_{11}\mathbf{e}_0 - Y_{10}\mathbf{e}_1) = \frac{1}{\sqrt{4\pi}}\frac{\sqrt{3}}{2}\left\{\hat{z}(\mathbf{e}_x + i\mathbf{e}_y) - (\hat{x} + i\hat{y})\mathbf{e}_z\right\}$$

$$\mathbf{Y}^1_{2-2} = Y_{1-1}\mathbf{e}_{-1} = \frac{1}{\sqrt{4\pi}}\frac{\sqrt{3}}{2}(\hat{x} - i\hat{y})(\mathbf{e}_x - i\mathbf{e}_y)$$

$$\mathbf{Y}^1_{2-1} = \frac{1}{\sqrt{2}}(Y_{1-1}\mathbf{e}_0 + Y_{10}\mathbf{e}_{-1}) = \frac{1}{\sqrt{4\pi}}\frac{\sqrt{3}}{2}\{\hat{z}(\mathbf{e}_x - i\mathbf{e}_y) + (\hat{x} - i\hat{y})\mathbf{e}_z\}$$

$$\mathbf{Y}^1_{20} = \frac{1}{\sqrt{6}}(Y_{1-1}\mathbf{e}_1 + 2Y_{10}\mathbf{e}_0 + Y_{11}\mathbf{e}_{-1}) = \frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{2}}(2\hat{z}\mathbf{e}_z - \hat{x}\mathbf{e}_x - \hat{y}\mathbf{e}_y)$$

$$\mathbf{Y}^1_{21} = \frac{1}{\sqrt{2}}(Y_{10}\mathbf{e}_1 + Y_{11}\mathbf{e}_0) = -\frac{1}{\sqrt{4\pi}}\frac{\sqrt{3}}{2}\{\hat{z}(\mathbf{e}_x + i\mathbf{e}_y) + (\hat{x} + i\hat{y})\mathbf{e}_z\}$$

$$\mathbf{Y}^1_{22} = Y_{11}\mathbf{e}_1 = \frac{1}{\sqrt{4\pi}}\frac{\sqrt{3}}{2}(\hat{x} + i\hat{y})(\mathbf{e}_x + i\mathbf{e}_y)$$

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☑ "直交性"

$$\int d\Omega_{\hat{r}} \sum_{\alpha\beta} \left(\mathbb{Y}^{\ell_1}_{J_1M_1}(\hat{r}) \cdot \sigma i\sigma^y \right)_{\alpha\beta} \left(\mathbb{Y}^{\ell_2}_{J_2M_2}(\hat{r}) \cdot \sigma i\sigma^y \right)^{\dagger}_{\alpha\beta}$$

$$= \sum_{m_1\mu_1} \sum_{m_2\mu_2} C^{J_1M_1}_{\ell_1m_1 1\mu_1} C^{J_2M_2}_{\ell_2m_2 1\mu_2} \underbrace{\int d\Omega_{\hat{r}} Y_{\ell_1m_1}(\hat{r}) Y^{*}_{\ell_2m_2}(\hat{r})}_{\delta_{\ell_1\ell_2}\delta_{m_1m_2}} \underbrace{\sum_{\alpha\beta} (\sigma_{\mu_1} i\sigma^y)_{\alpha\beta} (\sigma_{\mu_2} i\sigma^y)^{\dagger}_{\alpha\beta}}_{2\delta_{\mu_1\mu_2}}$$

$$= 2 \sum_{m_1\mu_1} C^{J_1M_1}_{\ell_1m_1 1\mu_1} C^{J_2M_2}_{\ell_1m_1 1\mu_1} \delta_{\ell_1\ell_2}$$

$$= 2\, \delta_{J_1J_2} \delta_{M_1M_2} \delta_{\ell_1\ell_2}$$