CS 391L Machine Learning Assignment 3

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Problem 1

(a)

$$
\begin{aligned}
Cov(y) &= E((y - E(y)(y - E(y)^T) && (1)\\
&= E((Ax + b - E(Ax + b))(Ax + b - E(Ax + b))^T) && (2)\\
&= E((Ax - E(Ax))(Ax - E(Ax))^T) && \text{Linearity of } E \quad (3)\\
&= AE((x - E(x))(x - E(x))^T)A^T && (4)\\
&= A\Sigma A^T && \text{Def. of Cov.} \quad (5)
\end{aligned}
$$

(b) Base case: when $k = 0$, $Ix = \lambda^0 x$. When $k = 1$, by the definition of eigenvalue and eigenvector, $Ax = \lambda x$.

Inductive hypothesis: assume $A^k x = \lambda^k x$ for some $k \in N$. Want to show $A^{k+1}x = \lambda^{k+1}x$.

$$
\begin{aligned}
A^k x &= \lambda^k x && \text{I.H.} \quad (6)\\
A^{k+1}x &= A\lambda^k x && (7)\\
A^{k+1}x &= \lambda^k Ax && (8)\\
A^{k+1}x &= \lambda^{k+1}x && Ax = \lambda x \quad (9)
\end{aligned}
$$

If $k < 0$, we already know $A^{-k}x = \lambda^{-k}x$. By algebra, $\lambda^k A^{-k}x = x$, $\lambda^k x = A^k x$.

Problem 2

(a)

$$
\begin{aligned}
r &= 1 - \frac{H(Y|X)}{H(X)} && (10)\\
&= \frac{H(X) - H(Y|X)}{H(X)} && (11)\\
&= \frac{I(Y|X)}{H(X)} \qquad \text{Def. of Mutual Information} && (12)
\end{aligned}
$$

(b) $H(Y|X) \geq 0$, $H(X) > 0$. So $\frac{H(Y|X)}{H(X)} \geq 0$, $1 - \frac{H(Y|X)}{H(X)} \leq 1$.

$H(Y|X) < H(X)$, so $\frac{H(Y|X)}{H(X)} <= 1$, $1 - \frac{H(Y|X)}{H(X)} \geq 0$.

Therefore, $0 \leq r \leq 1$.

(c) $r = 0$ when two variables are independent. $r = 1$ when two variables are perfectly correlated.

Problem 3

(a) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$. Compared to $\frac{1}{1 + e^{-x}}$, tanh has a steeper slope.

To be an appropriate sigmoid function, tanh needs to be scaled to range of [0, 1].

(b) We know W^k is 2×2.

$$
\begin{aligned}
\frac{\partial H}{\partial w_{ij}^k} &= \frac{\partial}{\partial w_{ij}^k} (\lambda^{k+1})^T g(W^k x^k) && (13)\\
&= \frac{\partial}{\partial w_{ij}^k} (\lambda_1^{k+1} g(w_{11} x_1^k + w_{12} x_2^k) + \lambda_2^{k+1} g(w_{21} x_1^k + w_{22} x_2^k)) && (14)\\
&= \lambda_1^{k+1} \frac{\partial}{\partial w_{ij}^k} g(w_{11} x_1^k + w_{12} x_2^k) + \lambda_2^{k+1} \frac{\partial}{\partial w_{ij}^k} g(w_{21} x_1^k + w_{22} x_2^k) && (15)
\end{aligned}
$$

Therefore, for $w_{11}, w_{12}, w_{21}, w_{22}$, there is $\frac{\partial H}{\partial w_{ij}^k} = \lambda^{k+1} x_j^k g'(w_i^k x^k)$.

Problem 4

(a) The code for computing these results is attached separately.

IG(Color) = 0.1043
IG(Size) = 0.4086
IG(Noise) = 0.0207

For small size,

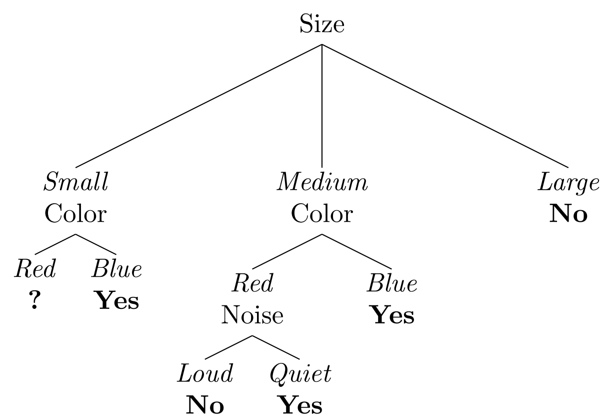
IG(Color) = 0.3219
IG(Noise) = 0.0729

For medium size,

IG(Color) = 0.1226
IG(Noise) = 0.1226

For large size, IG is clearly 0.

The decision tree is



(b) If the event of missing a datum is uniformly random over all the attributes, then it doesn't harm if we simply delete that line.

Problem 5

(a) $\min \dfrac{1}{3}\pi r^2 h$ such that $A = \pi r s + \pi r^2$, $s^2 = r^2 + h^2$.

Use Lagrange multiplier, $H = \frac{1}{3}\pi r^2 h + \lambda(\pi r s + \pi r^2 - A) + \gamma(r^2 + h^2 - s^2)$.

$\frac{\partial H}{\partial r} = \frac{2}{3}\pi r h + \lambda(\pi s + 2\pi r) + 2\gamma r = 0$

$\frac{\partial H}{\partial h} = \frac{1}{3}\pi r^2 + 2\gamma h = 0$

Problem 6

Each node has m possible attributes. So there are no more than m^n configuration of the decision tree. The number of different classifiers is also no more than m^n.

We know that for data set with size of k, a decision tree should have 2^k decisions. So the VC dimension is $\log_2 m^n = n \log_2 m = O(n \log(m))$.

Problem 7

The smallest positive integer p is 2.

When $p = 1$, $k(x, x_i) = 1 + x^T x_i = 1 + x_1 x_{i1} + x_2 x_{i2}$. So $\Phi(x) = (1, x_1, x_2)^T$. Clearly, as (x_1, x_2) cannot be separated by SVM, adding a bias of 1 doesn't help either.

When $p = 2$ — same case as the class note — $k(x, x_i) = (1 + x^T x_i)^2 = (1 + x_1 x_{i1} + x_2 x_{i2})^2 = 1 + x_1^2 x_{i1}^2 + x_2^2 x_{i2}^2 + 2x_1 x_{i1} + 2x_2 x_{i2} + 2x_1 x_{i1} x_2 x_{i2}$. Then $\Phi(x) = (1, x_1^2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2, \sqrt{2}x_1 x_2)^T$.

Let $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}$. $Q(\lambda) = I\lambda - \frac{1}{2}\lambda^T A \lambda$, where $A = \begin{pmatrix} 9 & -1 & -1 & 1 \\ -1 & 9 & 1 & -1 \\ -1 & 1 & 9 & -1 \\ 1 & -1 & -1 & 9 \end{pmatrix}$, the elements of which are computed by the kernel function. The solution is $\lambda_i = \frac{1}{8}$ for $i = 1, 2, 3, 4$. They are all support vectors, and they are linearly separable.

Using a value of p larger than minimum would unnecessarily map the data to higher dimension. The classifier becomes more flexible in lower dimension, and thus may overfit the data.

However, if it has a regularization term, it doesn't have such downside. For example, if the regularization term is $||w||_1$, then it becomes sparse coding.