MCMT Project Report
Mixing Time in Reinforcement Learning

Shun Zhang

1 Introduction

Reinforcement Learning is a learning method for the problem of learning with feedback from the environment. In the literature of Reinforcement Learning, Markov Decision Process is generally used to model the environment. There are many learning algorithms defined on the Markov Decision Process. However, even though most of the algorithms are proved to eventually converge to the optimal solution, the time required to achieve a near-optimal solution is not theoretically bounded. In this report, I use mixing time to analyze the convergence time.

2 Markov Decision Process

Markov Decision Process (MDP) can be regarded as an extension to Markov chain. If an agent starts in a state in a Markov chain, it can only drift according to the transition probability. It cannot indicate its preference on where to go. In MDP, the states have rewards defined on them. The agent obtains the reward when reaching a state. It wants to maximize the rewards it gets over time, so we introduce *actions* in MDP.

MDP is defined below compared to the definition of Markov Chain using the same notations.

Definition 1. *Markov Chain is a two-element tuple* (Ω, P)*, where*

- Ω *is the state space.*
- P *is the transition probability.* $P : \Omega \times \Omega \to \mathbb{R}$.

Definition 2. *Markov Decision Process is a four-element tuple* (Ω, A, P, R)*, where*

- Ω *is the state space.*
- A *is an action set.*
- P *is the transition probability.* $P : \Omega \times A \times \Omega \to \mathbb{R}$.

- *R is the reward upon reaching a state. $R : \Omega \to \mathbb{R}$. [3]*

There is a difference in the transition probability compared to Markov chain — it measures the probability reaching another state, given a state and an action.

We use the term utility to describe the accumulated rewards after starting from a state.

Definition 3. *A policy in a MDP is a mapping $\pi : \Omega \to A$.*

Definition 4. *The utility of a state s following policy π is: $U(s) = \mathrm{E}(R(s) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots)$, where $\gamma \in (0, 1]$, is a discounting factor.*

3 Learning Algorithms

The learning process is represented by updating *utility*, which evaluates the expected accumulated payoff starting from a state.

The goal of the agent is to maximize the accumulated rewards it gets. Usually, the agent should take the action that leads to the maximum expected utilities, that is $\mathrm{argmax}_a \sum_{s'} P(s, a, s')(R(s') + \gamma \hat{U}(s'))$. However, the U may be inaccurate, it may also want to explore the actions randomly with some probability. The is called the Explore or Exploit Dilemma — exploration may takes the agent to the subset of the states with higher rewards that are previously not realized. One popular way to resolve this dilemma. is called ϵ-greedy, which the agent takes the action that maximizes the estimated utility with probability $1 - \epsilon$, and takes a random action with probability ϵ.

We introduce two learning algorithms. One is model-based, which assumes that we know R and P. The other is model-free, which assume that we don't know R or P.

3.1 Value Iteration

Theorem 1. *The utility for the optimal policy satisfies $U(s) = \max_a \sum_{s'} P(s, a, s')(R(s') + \gamma U(s'))$ for all s. [3]*

To update, we can use the following rule. This is a bootstrapping way to solve the nonlinear equations above.

$$U(s) \leftarrow \max_a \sum_{s'} P(s, a, s')(R(s') + \gamma U(s')) \tag{1}$$

This assumes that we know the transition and the reward functions.

3.2 Q-learning

When the transition and reward functions are not known, computing U won't be helpful for decision making. We define a new function that evaluates a state, action pair.

Definition 5. $Q(s,a) = \sum_{s'} P(s,a,s')(R(s') + \gamma U(s'))$.

Note that by definition, $U(s) = \max_a Q(s,a)$. The Q function can be computed by the following update rule.

$$Q(s,a) \leftarrow (1-\alpha)Q(s,a) + \alpha(R(s') + \gamma \max_{a'} Q(s',a')) \tag{2}$$

where $\alpha \in (0,1)$.

Theorem 2. *Following the update rule Equation 2, Q eventually converges to Q^*. [3]*

Note that Q-learning doesn't give any bound on convergence. Actually, this is affected by the transition model.

4 Convergence Rates of Utilities

One measure of the learning performance is to compare the current utility with that of the optimal solution, that is $\sum_s ||U(s) - U^*(s)||$ or $\sum_{s,a} ||Q(s,a) - Q^*(s,a)||$ for $s \in \Omega$.

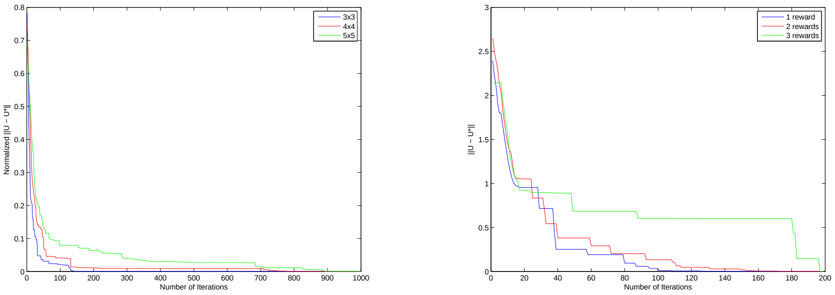


Figure 1: (Left) Convergence rates of different domains, with sizes of 3x3, 4x4, 5x5. (Right) Convergence rates of domains with size 4x3 of different number of rewards.

I evaluated the Q-learning algorithm in gridworld domains, with the sizes of 3x3, 4x4 and 5x5. Apparently, the mixing time of the transition models for these domains vary. The convergence rates of U differ. This is shown in the left figure in Figure 1.

In the second experiments, I evaluated Q-learning in 4x3 gridworld domains but with different number of rewards, 1, 2, 3, respectively. The result is shown in right figure in Figure 1. In the figure, the U converges to its optimum slowly when there are more rewards to consider, even though the transition functions are same for all these three settings.

5 Mixing Time in Literature

Kearns et al. proposed the idea of using mixing time on utility as a metric for learning performance [2]. Their main contribution of the paper is an innovative algorithm that takes mixing time of the transition function into consideration. Their new algorithm, Explicit Explore or Exploit (E^3), shows that the mixing time of the learning algorithm can be polynomially bounded by the mixing time of the transition function. The idea is that the agent needs to visit a state a certain number of times to mark it as *known*. The mixing time of the transition matrix is an important factor for such confidence.

This encourages the agent to cover the state space efficiently, and expedite the convergence of U. Even though, this doesn't outperform traditional algorithms in other metrics, for example, minimizing the regrets. Regrets are defined as the difference between the rewards got in the simulation and the rewards the agent could have got. In Q-learning, for example, the agent may stick to any reward it observes while exploring. In E^3, the agent explicitly drops the chance to collect small rewards, but tries to explore the state space.

The main theorem of their work is as follows.

Theorem 3. *Let $U(i)$ denote the value function for the policy with the optimal expected discounted return in M. Then there exists an algorithm A, taking inputs ϵ, δ, N and $U(i)$,such that the total number of actions and computation time taken by A is polynomial in $1/\epsilon, 1/\sigma, N$,the mixing time of the transition function [1], and the maximum reward. With probability at least $1 - \delta$, A will halt in a state i, and output a policy such that following such policy, $U(i) \geq U^*(i) - \epsilon$.*

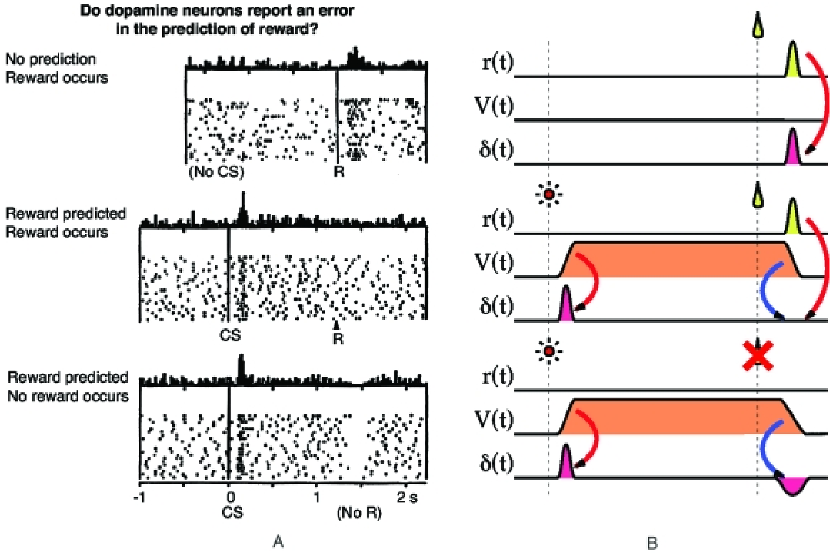
In Theorem 3, the authors claim that the utility of the states can be polynomially bounded by some factors of the domain, including the mixing time of the optimal policy. Another related work is a more popular algorithm R-max [1], following Kearns et al., that makes the distinction of exploration and exploitation phases implicit.

6 Discussion and Conclusion

In this project, I introduced the concept of reinforcement learning, and how it is related to Markov Chain. I generated empirical data of the convergence time of

[1] It is horizon time in the original paper for the discounted case. They use the term mixing time for the undiscounted case, that is, when $\gamma = 1$.

utilities of a grid world domain. An application of mixing time is also surveyed in the literature.



To conclude, there is a journal article in neuroscience that shows a correlation between the dopemine that a creature fires and the temporal difference error [4]. The temporal difference error is defined as $R(s') - (\gamma U(s') - U(s))$. This means that, if the agent uses U to make decisions, how unexpected he feels after a step. The temporal difference error can occur when an unseen reward time is observed for the first time, or a reward that is believed to exist is missing. One experiment of this paper is shown in Figure 6. In the top subfigure, the temporal difference error is positive when a monkey observes a banana. In the bottom figure, the temporal difference error is negative when the banana is missing, while it is supposed to be there. So it is possible that animals use a reinforcement-learning-like approach to learn the world.

References

[1] Ronen I Brafman and Moshe Tennenholtz. R-max-a general polynomial time algorithm for near-optimal reinforcement learning. *The Journal of Machine Learning Research*, 3:213–231, 2003.

[2] Michael Kearns and Satinder Singh. Near-optimal reinforcement learning in polynomial time. *Machine Learning*, 49(2-3):209–232, 2002.

[3] Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*, volume 1. Cambridge Univ Press, 1998.

[4] Saori C Tanaka, Kenji Doya, Go Okada, Kazutaka Ueda, Yasumasa Okamoto, and Shigeto Yamawaki. Prediction of immediate and future rewards differentially recruits cortico-basal ganglia loops. *Nature neuroscience*, 7(8):887–893, 2004.