Homework 1

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Exercise 1.1

Let $\mu(x) = \mathbb{P}(X_0 = x)$.

Base Case: The distribution of X_0 is characterized as μ as definition.

Inductive step: Assume that for some $n \geq 0$, $\mathbb{P}(X_n = x)$ can be represented with P and μ. $\mathbb{P}(X_{n+1} = y) = \mathbb{P}(X_n = x)\mathbb{P}(X_{n+1} = y|X_n = x) = \mathbb{P}(X_n = x)P(x, y)$. We already know that $\mathbb{P}(X_n = x)$ can be represented with P and μ. So $\mathbb{P}(X_{n+1} = y)$ can also be represented using P and μ.

Exercise 1.2

$\mathbb{P}(X_1 = y) = \sum_{x\in\Omega} \mathbb{P}(X_1 = y|X_0 = x)P(X_0 = x) = \sum_{x\in\Omega} P(x, y)P(X_0 = x) = \sum_{x\in\Omega} P_{xy}\mu_x = \begin{pmatrix}\mu_1 & \mu_2 & \cdots & \mu_k\end{pmatrix}\begin{pmatrix}P_{1y}\\ P_{2y}\\ \cdots\\ P_{ky}\end{pmatrix}$.

The distribution of X_1 is $(\mathbb{P}(X_1 = 1), \cdots, \mathbb{P}(X_1 = k)) = \begin{pmatrix}\mu_1 & \mu_2 & \cdots & \mu_k\end{pmatrix}\begin{pmatrix}P_{11} & \cdots & P_{1k}\\ P_{21} & \cdots & P_{2k}\\ \cdots & & \\ P_{ky} & \cdots & P_{kk}\end{pmatrix} = \mu P$.

To show the distribution of X_t is μP^t by way of induction. the base case is shown above. Assume for some $n > 0$, the distribution of X_n is μP^n, consider the distribution of X_{n+1}.

$\mathbb{P}(X_{n+1} = y) = \sum_{x\in\Omega} \mathbb{P}(X_{n+1} = y|X_n = x)P(X_n = x)$. Similar to the reasoning in the base case, this is $\mu P^n \begin{pmatrix}P_{1y}\\ P_{2y}\\ \cdots\\ P_{ky}\end{pmatrix}$. So the distribution of X_{n+1} is $\mu P^n P = \mu P^{n+1}$.