MCMT Homework 16

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Exercise 16.1

Assume G is transient. For a vertex a, there exists a flow from a to inifinity with finite energy. Consider its k-fuzz graph, we just set the flow on the additional edges to be 0. Then the new flow is a finite flow for the k-fuzz graph.

If the k-fuzz of a graph is transient. Consider deriving the original graph G by removing edges. Consider an edge e' in k-fuzz that connects two vertices with distance of k. Let the path in G that connects these two vertices be $e_1, e_2, \cdots, e_k$. By removing e', let $\theta(e')$ go through $e_1, e_2, \cdots, e_k$ instead.
All the edges have unit conductances. So the energy before removing e' is $\sum_{i=1}^k \theta(e_i)^2 + \theta(e')^2$. The energy after removing e' is $\sum_{i=1}^k (\theta(e_i) + \theta(e'))^2$
$= \sum_{i=1}^k \theta(e_i)^2 + 2\sum_{i=1}^k \theta(e_i)\theta(e') + k\theta(e')^2$
$\leq \sum_{i=1}^k \theta(e_i)^2 + \sum_{i=1}^k (\theta(e_i)^2 + \theta(e')^2) + k\theta(e')^2$
$\leq 2k(\sum_{i=1}^k \theta(e_i)^2 + \theta(e')^2)$.
So the energy increment by removing one edge in k-fuzz graph can be locally finitelly bounded. For any original edge in G, its energy can be augemented at most $(\Delta^{k+1})^2$ times, where Δ^{k+1} upper bounds the number of the nodes in a tree with depth of k and branching factor of Δ. So the energy of the flow can be increased at most $(2k)^{(\Delta^{k+1})^2}$ times (which is a loose upper bound), which is still finite.