MCMT Homework 6

Shun Zhang

Exercise 6.1

$2||\mu P^t - \pi||_{TV} = \sum_{y\in\Omega} |(\mu P^t)(y) - \pi(y)|$
$= \sum_{y\in\Omega} |\sum_{x\in\Omega} \mu(x)P^t(x,y) - \pi(y)|$
$= \sum_{y\in\Omega} |\sum_{x\in\Omega} \mu(x)P^t(x,y) - \sum_{x\in\Omega} \mu(x)\pi(y)|$
$= \sum_{y\in\Omega} |\sum_{x\in\Omega} \mu(x)(P^t(x,y) - \pi(y))|$
$\leq \sum_{y\in\Omega} \sum_{x\in\Omega} \mu(x)|P^t(x,y) - \pi(y)|$
$= \sum_{x\in\Omega} \mu(x)2||P^t(x,\cdot) - \pi||_{TV}$
$\leq \sup_{x\in\Omega} 2||P^t(x,\cdot) - \pi||_{TV}$

In the last step, we know $\mu(x)$ sums to 1 for $x \in \Omega$. So to maximize it, let $\mu = \delta_x$ so that $2||P^t(x,\cdot)-\pi||_{TV}$ is maximized. Therefore, $\sup_\mu ||\mu P^t - \pi||_{TV} = d(t)$.

$2||\mu P^t - \nu P^t||_{TV} = \sum_{y\in\Omega} |(\mu P^t)(y) - (\nu P^t)(y)|$
$= \sum_{y\in\Omega} |\sum_{x\in\Omega} \mu(x)P^t(x,y) - \sum_{z\in\Omega} \nu(z)P^t(z,y)|$
$= \sum_{y\in\Omega} |\sum_{x\in\Omega} \mu(x)\sum_{z\in\Omega} \nu(z)(P^t(x,y) - P^t(z,y))|$
$\leq \sum_{y\in\Omega} \sum_{x\in\Omega} \mu(x)\sum_{z\in\Omega} \nu(z)|P^t(x,y) - P^t(z,y)|$
$= \sum_{x\in\Omega} \mu(x)\sum_{z\in\Omega} \nu(z)2||P^t(x,\cdot) - P^t(z,\cdot)||_{TV}$
$\leq \sup_{x,z\in\Omega} 2||P^t(x,\cdot) - P^t(z,\cdot)||_{TV}$

The last step is similar to the reasoning to the previous proof. We choose $\mu = \delta_x, \nu = \delta_z$ so that $2||P^t(x,\cdot) - P^t(z,\cdot)||_{TV}$ is maximized. Therefore, $\sup_{\mu,\nu} ||\mu P^t - \nu P^t||_{TV} = \bar{d}(t)$.

Exercise 6.2

$2||\mu P - \nu P||_{TV} = \sum_{y\in\Omega} |(\mu P)(y) - (\nu P)(y)|$
$= \sum_{y\in\Omega} |\sum_{x\in\Omega}(\mu(x)P(x,y) - \nu(x)P(x,y))|$
$\leq \sum_{y\in\Omega} \sum_{x\in\Omega} |\mu(x)P(x,y) - \nu(x)P(x,y)|$
$= \sum_{y\in\Omega} \sum_{x\in\Omega} P(x,y)|\mu(x) - \nu(x)|$
$= \sum_{x\in\Omega} |\mu(x) - \nu(x)| \qquad \sum_{y\in\Omega} P(x,y) = 1$
$= 2||\mu - \nu||_{TV}$

So $||\mu P - \nu P||_{TV} \leq ||\mu - \nu||_{TV}$. We have $||\mu P^{t+1} - \nu P^{t+1}||_{TV} \leq ||\mu P^t - \nu P^t||_{TV}$ by substituting μ, ν in the proof above by $\mu P^t, \nu P^t$, respectively. So we have $\bar{d}(t+1) \leq \bar{d}(t)$.

We have $||\mu P^{t+1} - \pi||_{TV} \leq ||\mu P^t - \pi||_{TV}$ by substituting μ, ν in the proof above by $\mu P^t, \pi$, respectively. Note that $\pi P = \pi$. So we have $\bar{d}(t+1) \leq \bar{d}(t)$.