MCMT Homework 9

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Exercise 9.1

Let Y_t denote the number of coordinates that have been selected after t rounds. Define the distinguishing statistic $f : \{0,1\}^n \to R$ by $f(x) = \sum_{i=1}^n x_i$.

1. The probability that one coordinate is not selected in t steps is $p = (1-\frac{1}{n})^t$. The expectation of the number of coordinates that are not selected in t steps is $\mathrm{E}(n - Y_t) = np = n(1-\frac{1}{n})^t$. So $\mathrm{E}(Y_t) = n - n(1-\frac{1}{n})^t$.

2. Let $I_i(t)$ be the indicator that i-th coordinate has been selected at t-th step.

   $\mathrm{E}(I_i(t)I_j(t)) = 2(1-(1-\frac{1}{n})^t) - (1-(1-\frac{2}{n})^t) = 1 - 2(1-\frac{1}{n})^t + (1-\frac{2}{n})^t$.

   $\mathrm{E}(I_i(t))E(I_j(t)) = (1-(1-\frac{1}{n})^t)^2$.

   $\mathrm{Cov}(I_i(t), I_j(t)) = \mathrm{E}(I_i(t)I_j(t)) - \mathrm{E}(I_i(t))E(I_j(t)) = (1-\frac{2}{n})^t - (1-\frac{1}{n})^t \le 0$.

   $\mathrm{Var}(Y_t) = \mathrm{Var}(\sum_i I_i(t)) = \sum_i(\mathrm{Var}(I_i(t))) + \sum_{i \neq j} \mathrm{Cov}(I_i(t), I_j(t)) \le \sum_i(\mathrm{Var}(I_i(t))) \le np(1-p) \le \frac{n}{4}$.

3. $E_0(X_{ti}|Y_t) = \frac{Y_t}{n}\frac{1}{2}$. This is the probability that x_i is chosen times the probability that it is set to be 1.

   $E_0(f(X_t)|Y_t) = \sum_i E_0(X_{ti}|Y_t) = \frac{Y_t}{2}$.