MCMT Homework 4

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Exercise 4.1

$\mathbb{P}_\pi(X_0 = x_0, X_1 = x_1, \cdots, X_t = x_t)$
$= \mathbb{P}_\pi(X_0 = x_0)\mathbb{P}_\pi(X_1 = x_1|X_0 = x_0) \cdots \mathbb{P}_\pi(X_t = x_t|X_{t-1} = x_{t-1})$
$= \pi(x_0)P(x_0, x_1) \cdots P(x_{t-1}, x_t)$
$= P(x_1, x_0)\pi(x_1) \cdots P(x_{t-1}, x_t)$ Detailed balance on x_0, x_1
$= P(x_1, x_0)P(x_2, x_1)\pi(x_2) \cdots P(x_{t-1}, x_t)$ Detailed balance on x_1, x_2
$\cdots$
$= P(x_1, x_0)P(x_2, x_1) \cdots P(x_t, x_{t-1})\pi(x_t)$
$= \pi(x_t)P(x_t, x_{t-1}) \cdots P(x_2, x_1)P(x_1, x_0)$
$= \mathbb{P}_\pi(X_0 = x_t, X_1 = x_{t-1}, \cdots, X_t = x_0)$

Exercise 4.2

Consider the transition from x to y, for arbitrary $x, y \in \Omega$.

Define $V_1 = \{v \in V : y \in \Omega(x, v)\}$, and $V_2 = V \setminus V_1$. For $v \in V_1$, or equivalently $y \in \Omega(x, v)$, $x(u) = y(u)$ for all $u \neq v$. This is equivalent to say $x \in \Omega(y, v)$. So $\Omega(x, v) = \Omega(y, v)$.

$\pi(x)P(x, y)$
$= \pi(x)\dfrac{1}{|V|}\sum_v \mu^{x,v}(y)$ v is chosen uniformly random
$= \pi(x)\dfrac{1}{|V|}(\sum_{v\in V_1} \dfrac{\pi(y)}{\pi(\Omega(x, v))} + \sum_{v\in V_2} 0)$
$= \pi(x)\dfrac{1}{|V|}\sum_{v\in V_1} \dfrac{\pi(y)}{\pi(\Omega(y, v))}$ $\Omega(x, v) = \Omega(y, v)$
$= \pi(y)\dfrac{1}{|V|}\sum_{v\in V_1} \dfrac{\pi(x)}{\pi(\Omega(y, v))}$
$= \pi(y)\dfrac{1}{|V|}(\sum_{v\in V_1} \dfrac{\pi(x)}{\pi(\Omega(y, v))} + \sum_{v\in V_2} 0)$
$= \pi(y)\dfrac{1}{|V|}\sum_v \mu^{y,v}(x)$ $\{v \in V : y \in \Omega(x, v)\} = \{v \in V : x \in \Omega(y, v)\}$
$= \pi(y)P(y, x)$

So the detailed balance holds for arbirary x, y. It is reversible. From a previous theorem that if P is reversible, then P is stationary. We also know that P is stationary.