MCMT Homework 9

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Exercise 9.1

Let Y_t denote the number of coordinates that have been selected after t rounds. Define the distinguishing statistic $f : \{0,1\}^n \to R$ by $f(x) = \sum_{i=1}^{n} x_i$.

1. The probability that one coordinate is not selected in t steps is $p = (1-\frac{1}{n})^t$. The expectation of the number of coordinates that are not selected in t steps is $\mathrm{E}(n - Y_t) = np = n(1-\frac{1}{n})^t$. So $\mathrm{E}(Y_t) = n - n(1-\frac{1}{n})^t$.

2. Let $I_i(t)$ be the indicator that i-th coordinate has been selected at t-th step.
   $\mathrm{E}(I_i(t)I_j(t)) = 2(1-(1-\frac{1}{n})^t) - (1-(1-\frac{2}{n})^t) = 1 - 2(1-\frac{1}{n})^t + (1-\frac{2}{n})^t$.
   $\mathrm{E}(I_i(t))E(I_j(t)) = (1-(1-\frac{1}{n})^t)^2$.
   $\mathrm{Cov}(I_i(t), I_j(t)) = \mathrm{E}(I_i(t)I_j(t)) - \mathrm{E}(I_i(t))E(I_j(t)) = (1-\frac{2}{n})^t - (1-\frac{1}{n})^t \le 0$.