MCMT Homework 7

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Exercise 7.1

For any state x, $\sum_{y\in\Omega}\hat{P}(x,y) = \dfrac{\sum_{y\in\Omega}\pi(y)P(y,x)}{\pi(x)} = \dfrac{\pi(x)}{\pi(x)} = 1$. We know π, P are all non-negative. Multiplicaiton is closed for non-negative numbers, so $\hat{P}$ is non-negative.

So $\hat{P}$ is a stochastic matrix.

Exercise 7.2

Show $\hat{P}^n(n,\cdot) = \pi$ by induction on the number of states in the Markov Chain, which is $n+1$ (from 0 to n).

Base case: when $n = 0$, the Markov Chain has 1 state. $\hat{P}^0(0,\cdot) = \delta_0$. This is the stationary distribution.

Inductive step: assume that $\hat{P}^n(n,\cdot) = \pi$ for some $n \geq 0$. Consider a transition matrix Q on a Markov chain with $n+2$ states.

$$Q^n(n+1,m) = \begin{cases} \pi(m-1) & m > 0 \\ 0 & m = 0 \end{cases}$$

by induction hypothesis.

Consider $Q^{n+1}(n+1,\cdot)$, which is $Q^n(n+1,\cdot)Q$.

$Q^{n+1}(n+1,0) = Q^n(n+1,1)Q(1,0) = \dfrac{1}{2}1 = \dfrac{1}{2}$.

$Q^{n+1}(n+1,1) = Q^n(n+1,2)Q(2,1) = \dfrac{1}{2^2}1 = \dfrac{1}{2^2}$.

...

$Q^{n+1}(n+1,n-1) = Q^n(n+1,n)Q(n,n-1) = \dfrac{1}{2^n}1 = \dfrac{1}{2^n}$.

$Q^{n+1}(n+1,n) = Q^n(n+1,n+1)Q(n+1,n) = \dfrac{1}{2^n}\dfrac{1}{2} = \dfrac{1}{2^{n+1}}$.

$Q^{n+1}(n+1,n+1) = Q^n(n+1,n+1)Q(n+1,n+1) = \dfrac{1}{2^n}\dfrac{1}{2} = \dfrac{1}{2^{n+1}}$.

This is the stationary distribution for the Markov chain with $n+2$ states.