

MCMT Homework 7

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Exercise 7.1

For any state x , $\sum_{y \in \Omega} \hat{P}(x, y) = \frac{\sum_{y \in \Omega} \pi(y)P(y, x)}{\pi(x)} = \frac{\pi(x)}{\pi(x)} = 1$. We know π, P are all non-negative. Multiplication is closed for non-negative numbers, so \hat{P} is non-negative.

So \hat{P} is a stochastic matrix.

Exercise 7.2

Show $\hat{P}^n(n, \cdot) = \pi$ by induction on the number of states in the time reversal of winning streak chain, which is $n + 1$ (from 0 to n).

Base case: when $n = 0$, the chain has 1 state. $\hat{P}^0(0, \cdot) = \delta_0$. This is the stationary distribution.

Inductive step: assume that $\hat{P}^n(n, \cdot) = \pi$ for some $n \geq 0$. Consider a transition matrix Q on a chain with $n + 2$ states.

$$Q^n(n + 1, m) = \begin{cases} \pi(m - 1) & m > 0 \\ 0 & m = 0 \end{cases}$$

by induction hypothesis. Here π is for \hat{P} .

Consider $Q^{n+1}(n + 1, \cdot)$, which is $Q^n(n + 1, \cdot)Q$.

$$Q^{n+1}(n + 1, 0) = Q^n(n + 1, 1)Q(1, 0) = \frac{1}{2}1 = \frac{1}{2}.$$

$$Q^{n+1}(n + 1, 1) = Q^n(n + 1, 2)Q(2, 1) = \frac{1}{2^2}1 = \frac{1}{2^2}.$$

...

$$Q^{n+1}(n + 1, n - 1) = Q^n(n + 1, n)Q(n, n - 1) = \frac{1}{2^n}1 = \frac{1}{2^n}.$$

$$Q^{n+1}(n + 1, n) = Q^n(n + 1, n + 1)Q(n + 1, n) = \frac{1}{2^n} \frac{1}{2} = \frac{1}{2^{n+1}}.$$

$$Q^{n+1}(n + 1, n + 1) = Q^n(n + 1, n + 1)Q(n + 1, n + 1) = \frac{1}{2^n} \frac{1}{2} = \frac{1}{2^{n+1}}.$$

This is the stationary distribution for the Markov chain with $n + 2$ states.