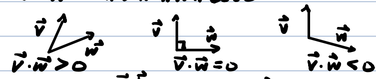
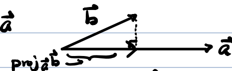
Chapter 1

dot product

- $\langle a,b,c\rangle \cdot \langle x,y,z\rangle = ax+by+cz$
- $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$
- $\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$

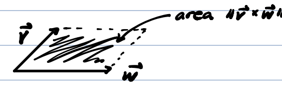


- $\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\,\vec{a}$



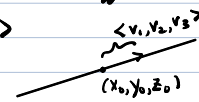
cross product

- $\langle a,b,c\rangle \times \langle x,y,z\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz-cy)\hat{i} - (az-cx)\hat{j} + (ay-bx)\hat{k}$
- $\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\,\|\vec{w}\|\sin\theta$



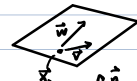
equations of lines

- $\langle x,y,z\rangle = \langle x_0,y_0,z_0\rangle + t\langle v_1,v_2,v_3\rangle$
  "parametric form"

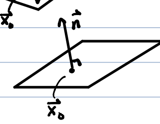


equations of planes

- $\langle x,y,z\rangle = \langle x_0,y_0,z_0\rangle + t\langle v_1,v_2,v_3\rangle + s\langle w_1,w_2,w_3\rangle$
  "parametric form"



- $\vec{n}\cdot(\vec{x}-\vec{x}_0) = 0$
  "vector equation"



$n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0$ "scalar eqn"

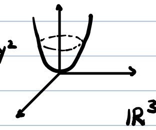
can read off normal vector; parallel planes have same normal vectors

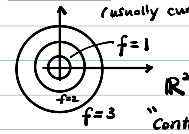
Chapter 2

graphs and level sets

- $y=f(x)$ 1-var. function, graph in $\mathbb{R}^2$ (curve), level sets $\overbrace{\text{level sets}}^{f=c}$ subsets of $\mathbb{R}^1$ (usually points)
- $z=f(x,y)$ 2-var function, graph in $\mathbb{R}^3$ (surface), level sets subsets of $\mathbb{R}^2$ (usually curves)

$z=f(x,y)=x^2+y^2$



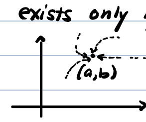


"contour diagram"

- $w=f(x,y,z)$ 3-var function, graph in $\mathbb{R}^4$ (not drawn), level sets subsets of $\mathbb{R}^3$ (usually surfaces)

limits

- $\lim_{(x,y)\to(a,b)} f(x,y)$ exists only if limit along all approach paths exist and agree



partial derivatives

- treat all other variables as constant ... $\frac{\partial}{\partial x}(x^2+y^2) = 2x+0$

equation of tangent plane

- to a graph $z=f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$:
  $z = f(x_0,y_0) + (\partial_x f)(x_0,y_0)(x-x_0) + (\partial_y f)(x_0,y_0)(y-y_0)$
- to a level surface $f(x,y,z)=c$ at (x_0,y_0,z_0):
  $\nabla f(x_0,y_0,z_0)\cdot\big((x,y,z)-(x_0,y_0,z_0)\big) = 0$
  see "vector eqn" of plane

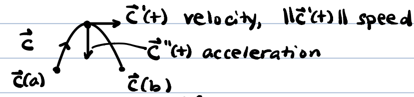
derivative

- $F:\mathbb{R}^m\to\mathbb{R}^n$, $DF = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{bmatrix}$; $n=1$, $Df=\nabla f$

chain rule

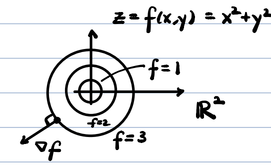
- $F:\mathbb{R}^m\to\mathbb{R}^n$, $G:\mathbb{R}^n\to\mathbb{R}^p$, $(D(G\circ F))(\vec{x}) = (DG)(F(\vec{x}))\cdot DF(\vec{x})$
- special case $p=1$, $G=G(x_1,\dots,x_n)$, $x_i = x_i(y_1,\dots,y_m)$
  $\dfrac{\partial G}{\partial y_k} = \dfrac{\partial G}{\partial x_1}\dfrac{\partial x_1}{\partial y_k} + \cdots + \dfrac{\partial G}{\partial x_n}\dfrac{\partial x_n}{\partial y_k}$ "cancel" the x_i's

curves · $\vec{c}: [a,b] \to \mathbb{R}^n$ (usually $n=2,3$)



gradient · $f: \mathbb{R}^n \to \mathbb{R}^1$, $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ points in direction of greatest increase, and perpendicular to level curves

· directional derivative in direction of $\vec{v}$: $\nabla f \cdot \left(\frac{\vec{v}}{\|\vec{v}\|}\right)$



Chapter 3

length of a curve · $\vec{c}: [a,b] \to \mathbb{R}^n$, $\text{length}(\vec{c}) = \int_a^b \|\vec{c}\,'(t)\|\,dt$ (integrate the speed)

arc length function · for $a \le t \le b$, $s(t) = \int_a^t \|\vec{c}\,'(t)\|\,dt$ (length/distance up to time t)

· from $s(t)$ find inverse function $t(s)$, $\vec{c}(t(s))$ called "parametrization by arc length"

Chapter 4

Switching derivative order · Clairaut's thm: in general $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f$

critical points · critical points of $f: \mathbb{R}^n \to \mathbb{R}$ are where $\nabla f = 0$

· 2nd derivative test: $D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

at a critical point, $D > 0$, $f_{xx} < 0 \Rightarrow$ local max

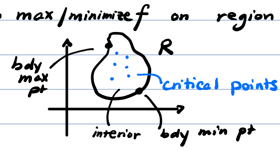
$D > 0$, $f_{xx} > 0 \Rightarrow$ local min

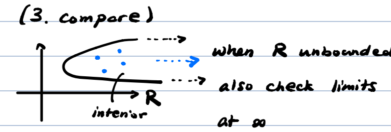
($D = 0$ inconclusive) $D < 0$ saddle point

finding extreme values · to max/minimize f on region R: 1. check interior critical points

2. check boundary max/min

(3. compare)

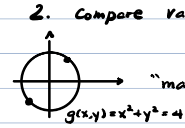




Lagrange multipliers · used to max/minimize f subject to constraints $g_1 = k_1, g_2 = k_2, \ldots, g_m = k_m$

1. Solve $\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_m \nabla g_m \\ g_1 = k_1, \ldots, g_m = k_m \end{cases}$

2. Compare values of f at solutions $(x_1, \ldots, x_n)$



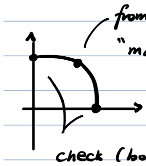
· with inequality constraints, need to check interior and boundary

by Lagrange or critical points, depending on situation

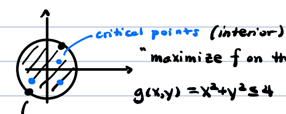
by Lagrange or manually

from Lagrange mult., $g(x,y) = x^2+y^2 = 4$ (interior)

"maximize f on the circle of radius 2, with $x, y \ge 0$"



check (boundary) points by hand



"maximize f on the disk bounded by circle of radius 2"

$g(x,y) = x^2+y^2 \le 4$

from Lagrange multipliers with constraint $g=4$ (bdy)

divergence · $F: \mathbb{R}^3 \to \mathbb{R}^3$, $\nabla \cdot F = (\partial_x, \partial_y, \partial_z) \cdot (F_1, F_2, F_3) = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$ (scalar)

measures volume distortion under flow ($\nabla \cdot F > 0$ expanding)

curl · $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i}(\partial_y F_3 - \partial_z F_2) - \hat{j}(\partial_x F_3 - \partial_z F_1) + \hat{k}(\partial_x F_2 - \partial_y F_1)$ (vector)

measures "swirling" of flow

identities · $\nabla \times (\nabla f) = \vec{0}$ ($f: \mathbb{R}^3 \to \mathbb{R}^1$)

· $\nabla \cdot (\nabla \times F) = 0$

Chapter 5

path integral of scalar functions · $\vec{c}: [a,b] \to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$; $\int_{\vec{c}} f\,ds = \int_a^b f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$

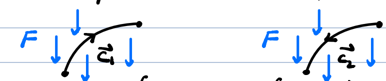
(integrate f, weighted by speed)

· $\int_{\vec{c}} f\,ds$ independent of parametrization and orientation

- $\int_{\vec c} 1\, ds = \text{length}(\vec c)$

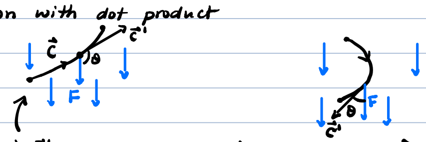
path integral of vector fields

- $\vec c : [a,b] \longrightarrow \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$; $\int_{\vec c} F \cdot d\vec s = \int_a^b F(\vec c(t)) \cdot \vec c\,'(t)\, dt$
  (integrate dot product of F and velocity)
  interpreted as work done by force F
- independent of parametrization except changes sign with orientation



$$\int_{\vec c_1} F \cdot d\vec s = -\int_{\vec c_2} F \cdot d\vec s$$

- relation with dot product



$F(\vec c(t)) \cdot \vec c\,'(t) < 0$ everywhere on $\vec c$

$\Rightarrow \int_{\vec c} F \cdot d\vec s = \int_a^b F(\vec c(t)) \cdot \vec c\,'(t)\, dt < 0$

$F(\vec c(t)) \cdot \vec c\,'(t) > 0$ everywhere on $\vec c$

$\Rightarrow \int_{\vec c} F \cdot d\vec s = \int_a^b F(\vec c(t)) \cdot \vec c\,'(t)\, dt < 0$

conservative vector fields

- $F : \mathbb{R}^3 \to \mathbb{R}^3$, equivalent facts
  - $F = \nabla f$ for some $f : \mathbb{R}^3 \to \mathbb{R}$
  - $\nabla \times F = \vec 0$
  - $\int_{\vec c} F \cdot d\vec s$ depends only on endpoints of $\vec c$ (path indep)
  - $\int_{\vec c} F \cdot d\vec s = 0$ on any closed curve $\vec c$

Chapter 6

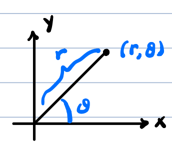
Fubini's theorem

- $\int_a^b \int_c^d f(x,y)\, dx\, dy = \int_c^d \int_a^b f(x,y)\, dy\, dx$
  (can switch integration order if bounds independent of variables)

change of variables

- if $T(u,v) = (x(u,v), y(u,v))$ changes variables from (u,v) in region S to (x,y) in region R
  then the Jacobian of T is
  $$|\det DT| = \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right|$$
- using the Jacobian an integral in (x,y) can be converted to an integral in (u,v):
  $$\iint_R f(x,y)\, dx\, dy = \iint_S f(x(u,v), y(u,v))\, |\det DT|\, du\, dv$$
  $$= \iint_S f(T(u,v))\, |\det DT|\, du\, dv$$

polar coordinates

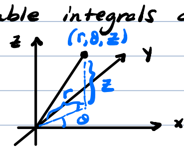
- 

  relation with Cartesian,
  $$\begin{cases} x = r\cos\theta & 0 \le r < \infty \\ y = r\sin\theta & 0 \le \theta < 2\pi \end{cases}$$
- relation between area forms of polar and Cartesian:
  $dx\,dy = r\,dr\,d\theta$

triple integrals

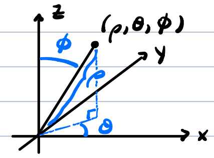
- analogues of Fubini and change of variables as stated for double integrals above also hold for triple integrals

cylindrical coordinates

- 

  relation with Cartesian,
  $$\begin{cases} x = r\cos\theta & 0 \le r < \infty \\ y = r\sin\theta & 0 \le \theta < 2\pi \\ z = z & -\infty < z < \infty \end{cases}$$
- relation between volume forms of cylindrical and Cartesian:
  $dx\,dy\,dz = r\,dr\,d\theta\,dz$

spherical coordinates



- relation with Cartesian,

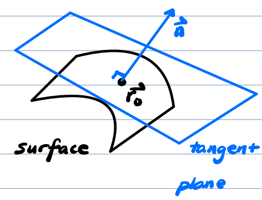
$$\begin{cases} x = \rho\cos\theta\sin\phi & 0 \le \rho < \infty \\ y = \rho\sin\theta\sin\phi & 0 \le \theta < 2\pi \\ z = \rho\cos\phi & 0 \le \phi \le \pi \end{cases}$$

- relation between volume forms of spherical and Cartesian:

$$dx\,dy\,dz = \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

Chapter 7

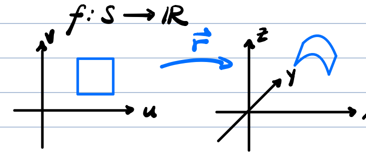
tangent plane to a parametrized surface



- if $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ then the tangent plane to the surface at
  $\vec{r}_0 = (x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))$
  is given by the equation
  $\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0$,
  where
  $\vec{n} = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$
  is the normal vector to the surface at $\vec{r}_0$, and
  $\vec{r} = (x,y,z)$ are the positions of points on the plane

surface integral of scalar functions

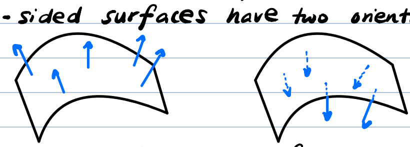
- surface S parametrized by $\vec{r}(u,v)$, $\vec{r}: D \to S$, and scalar function $f: S \to \mathbb{R}$



- $\iint_S f\, dS = \iint_D f(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, du\, dv$
- in spherical coordinates if
  $\vec{r}(\theta,\phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \cos\phi)$
  for ρ = constant then
  $\|\vec{r}_\theta \times \vec{r}_\phi\| = \rho^2 \sin\phi$
- in cylindrical coordinates if
  $\vec{r}(\theta,z) = (r\cos\theta, r\sin\theta, z)$
  for r = constant then
  $\|\vec{r}_\theta \times \vec{r}_\phi\| = r$
- $\iint_S f\, dS$ independent of orientation and parametrization

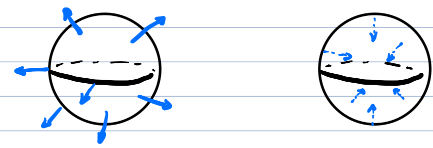
surface integral of vector fields

- two-sided surfaces have two orientations



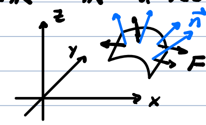
corresponding to a choice of direction of normal vectors

- for closed surfaces (no edges)
  outward normals $\longrightarrow$ "positive orientation"
  inward normals $\longrightarrow$ "negative orientation"



positive orientation   negative orientation

- $\vec{r}(u,v)$, $\vec{r}: D \to S$ parametrized surface, $F: S \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ a vector field



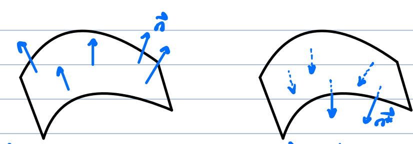
$$\iint_S F \cdot d\vec{S} = \iint_S F \cdot \vec{N}\, dS$$

unit normal

$$= \iint_D F(r(u,v)) \cdot (r_u \times r_v)\, du\, dv$$
$$= \iint_D F \cdot \vec{n}\, du\, dv$$

normal vector from parametrization $\vec{r}$ (may not be unit length)

- also called flux; sign depends on orientation from $\vec{r}$ parametrization but otherwise independent of parametrization

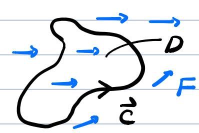


$\vec{r}: D \to S$ $\vec{r}^*: D^* \to S$

$$\iint_D F(u,v) \cdot \vec{n}(u,v)\, du\, dv = -\iint_{D^*} F(u,v) \cdot \vec{n}^*(u,v)\, du\, dv$$

Chapter 8

Green's theorem



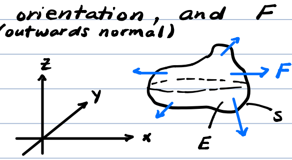
- $F(x,y) = (P(x,y), Q(x,y))$ vector field
  $\vec{c}: [a,b] \to \mathbb{R}^2$ counterclockwise closed curve

$$\underbrace{\int_{\vec{c}} P\, dx + Q\, dy}_{\substack{\text{path integral} \\ \int_{\vec{c}} F \cdot d\vec{s}}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy}_{\text{double integral}}$$

- area of D using Green's theorem,
  $$\iint_D 1\, dx\, dy = \frac{1}{2}\int_{\vec{c}} -y\, dx + x\, dy$$

Gauss's theorem

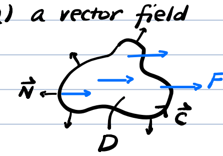
- E a solid region in $\mathbb{R}^3$ with boundary surface S and positive orientation (outwards normal), and F a vector field on E



$$\underbrace{\iint_S F \cdot d\vec{S}}_{\text{outward flux of } F \text{ across } S} = \underbrace{\iiint_E \operatorname{div} F\, dV}_{\text{triple integral of divergence}}$$

- version in the plane
  D a region in $\mathbb{R}^2$ with boundary curve $\vec{c}$

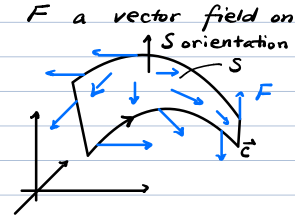
of positive orientation (counterclockwise), and
$F = (P, Q)$ a vector field



$$\int_{\vec c} \underbrace{F \cdot \vec N \, ds}_{\text{outward flux of } F \text{ across } \vec c} = \int_{\vec c} -Q\,dy + P\,dx = \iint_D \underbrace{\nabla \cdot F \, dA}_{\text{double integral of divergence}}$$

$\vec N$ is the unit outwards normal to $\vec c$

Stokes's theorem · S an oriented surface bounded by closed curve $\vec c$ of positive orientation (by right-hand rule), and F a vector field on S



$$\underbrace{\int_{\vec c} F \cdot d\vec s}_{\text{circulation of } F \text{ along } \vec c} = \underbrace{\iint_S (\nabla \times F) \cdot d\vec S}_{\text{surface integral of curl } F \text{ across } S}$$