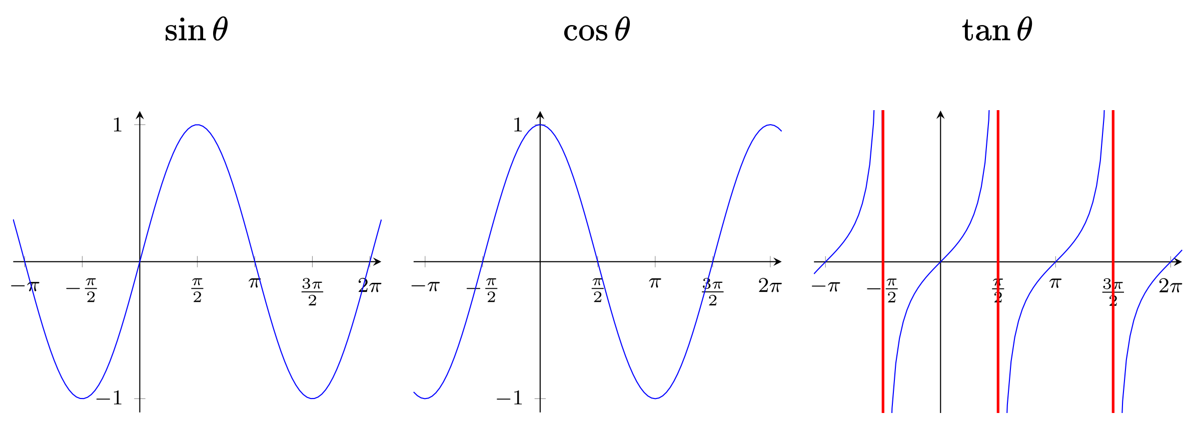
CLP-3 Multivariable Calculus

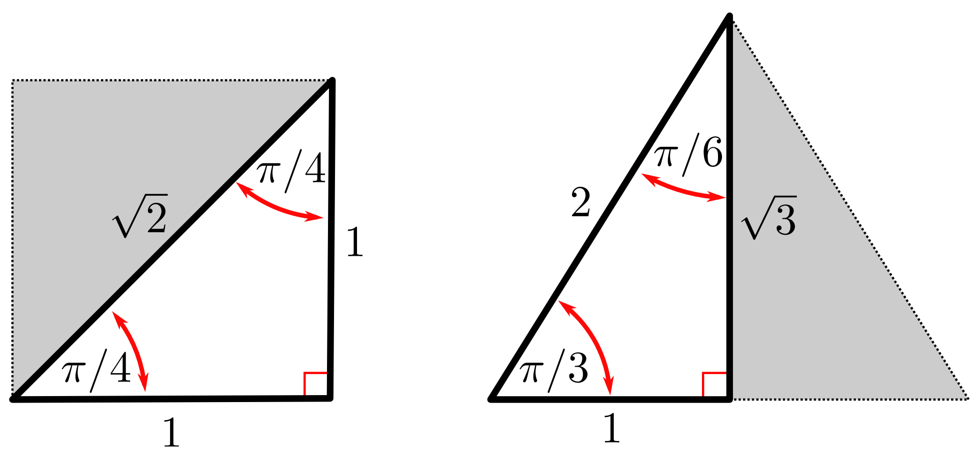
Joel Feldman, Andrew Rechnitzer, Elyse Yeager

A.1 Trigonometry

A.1.1 Trigonometry — Graphs



A.1.2 Trigonometry — Special Triangles



From the above pair of special triangles we have

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

A.1.3 Trigonometry — Simple Identities

- Periodicity

$$\sin(\theta + 2\pi) = \sin(\theta) \qquad \cos(\theta + 2\pi) = \cos(\theta)$$

- Reflection

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$

- Reflection around $\pi/4$

$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$

- Reflection around $\pi/2$

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$

- Rotation by π

$$\sin(\theta + \pi) = -\sin\theta \qquad \cos(\theta + \pi) = -\cos\theta$$

- Pythagoras

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

- $\sin$ and $\cos$ building blocks

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

A.1.4 Trigonometry — Add and Subtract Angles

- Sine

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

- Cosine

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

- Tangent

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ \tan(\alpha - \beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \end{aligned}$$

- Double angle

$$\begin{aligned} \sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \\ &= 1 - 2\sin^2(\theta) \\ \tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2\theta} \\ \cos^2\theta &= \frac{1 + \cos(2\theta)}{2} \\ \sin^2\theta &= \frac{1 - \cos(2\theta)}{2} \\ \tan^2\theta &= \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} \end{aligned}$$

- Products to sums

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$$
$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$$
$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$$

- Sums to products

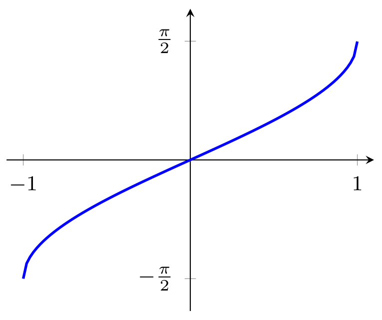
$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

A.1.5 Inverse Trigonometric Functions

$\arcsin x$

Domain: $-1 \le x \le 1$

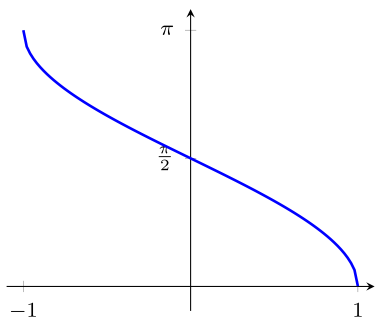
Range: $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$



$\arccos x$

Domain: $-1 \le x \le 1$

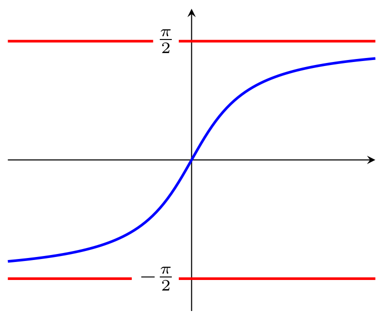
Range: $0 \le \arccos x \le \pi$



$\arctan x$

Domain: all real numbers

Range: $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$



Since these functions are inverses of each other we have

$$\begin{aligned} \arcsin(\sin\theta) &= \theta & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \arccos(\cos\theta) &= \theta & 0 \le \theta \le \pi \\ \arctan(\tan\theta) &= \theta & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{aligned}$$

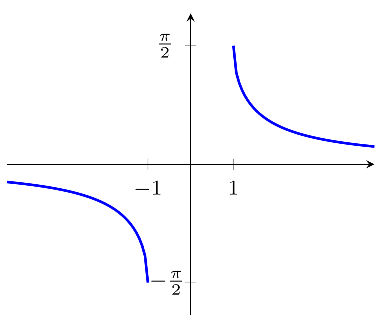
and also

$$\begin{aligned} \sin(\arcsin x) &= x & -1 \le x \le 1 \\ \cos(\arccos x) &= x & -1 \le x \le 1 \\ \tan(\arctan x) &= x & \text{any real } x \end{aligned}$$

$\operatorname{arccsc} x$

Domain: $|x| \ge 1$

Range: $-\frac{\pi}{2} \le \operatorname{arccsc} x \le \frac{\pi}{2}$

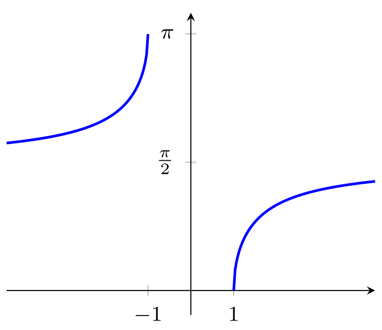


$\operatorname{arccsc} x \ne 0$

$\operatorname{arcsec} x$

Domain: $|x| \ge 1$

Range: $0 \le \operatorname{arcsec} x \le \pi$

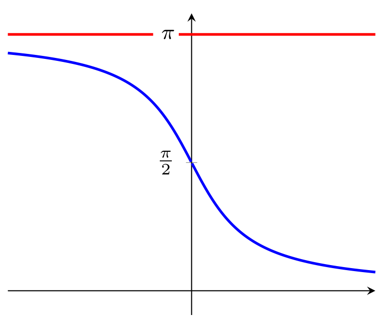


$\operatorname{arcsec} x \ne \frac{\pi}{2}$

$\operatorname{arccot} x$

Domain: all real numbers

Range: $0 < \operatorname{arccot} x < \pi$



Again

$$\begin{aligned} \operatorname{arccsc}(\csc\theta) &= \theta & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ \theta \ne 0 \\ \operatorname{arcsec}(\sec\theta) &= \theta & 0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2} \\ \operatorname{arccot}(\cot\theta) &= \theta & 0 < \theta < \pi \end{aligned}$$

and

$$\begin{aligned} \csc(\operatorname{arccsc} x) &= x & |x| \ge 1 \\ \sec(\operatorname{arcsec} x) &= x & |x| \ge 1 \\ \cot(\operatorname{arccot} x) &= x & \text{any real } x \end{aligned}$$