Lecture 1

Concepts

1, $\boldsymbol{\Pi - class}$ closed under finite intersections. i.e. $A,\ B \in \mathcal{A},\ \implies\ A \cap B \in \mathcal{A}$

2, $\boldsymbol{\sigma - \cap - closed}$ closed under countable intersections. i.e. $A_1,\ A_2,\ \cdots \in \mathcal{A},\ \implies\ \cap_{i=1}^{\infty} A_i \in \mathcal{A}$

3, $\boldsymbol{\cup - closed}$ closed under finite unions. i.e. $A,\ B \in \mathcal{A},\ \implies\ A \cup B \in \mathcal{A}$

4, $\boldsymbol{\sigma - \cup - closed}$ closed under countable unions. i.e. $A_1,\ A_2,\ \cdots \in \mathcal{A},\ \implies\ \cup_{i=1}^{\infty} A_i \in \mathcal{A}$

5, $\boldsymbol{\setminus - closed}$ closed under set differences i.e. $A,\ B \in \mathcal{A},\ \implies\ A \setminus B \in \mathcal{A}$

6, **closed under complements** i.e. $A \in \mathcal{A},\ \implies\ A^c \in \mathcal{A}$

Important Concepts

$\boldsymbol{\sigma - field}$ / $\boldsymbol{\sigma - algebra}$ A collection of sets $\mathcal{F} \subseteq 2^{\Omega}$ is called a σ-field if the following 3 conditions are satisfied:
(i) $\Omega \in \mathcal{F}$;
(ii) $\mathcal{F}$ is closed under complements;
(iii) $\mathcal{F}$ is closed under countable union.

field / algebra A class of subsets $\mathcal{A} \subseteq 2^{\Omega}$ is called a field or algebra if the following 3 conditions are satisfied:
(i) $\Omega \in \mathcal{A}$;
(ii) $\mathcal{A}$ is closed under complements;
(iii) $\mathcal{A}$ is closed under finite unions.

Theorem

In the definition of σ-field we can replace (iii) with (iii)' $\mathcal{F}$ is closed under countable intersections.
Similarly, in the definition of field, we can replace (iii) with (iii)' $\mathcal{A}$ is closed under finite intersections.

Concepts

$\mathcal{A}$ is called a **ring** if :
(i) $\emptyset \in \mathcal{A}$
(ii) $\mathcal{A}$ is closed under set difference
(iii) $\mathcal{A}$ is closed under finite unions

Remarks:
If we replace “finite union” in definition of ring to “countable union” we get a σ-ring.
$\mathcal{A}$ is a field $\implies$ $\mathcal{A}$ is a ring.

$\mathcal{A}$ is called a **semi-ring** if:
(i) $\emptyset \in \mathcal{A}$
(ii) $A,\ B \in \mathcal{A},\ \implies\ B \setminus A$ is a finite union of disjoint sets from $\mathcal{A}$
(iii) $\mathcal{A}$ is closed under finite intersections

$\mathcal{A}$ is called a $\boldsymbol{\lambda - class}$ if :
(i) $\Omega \in \mathcal{A}$
(ii) For any 2 sets $A, B \in \mathcal{A}$ with $A \subseteq B\ \implies\ B \setminus A \in \mathcal{A}$ is closed under set difference
(iii) Suppose $A_1, A_2, \cdots \in \mathcal{A}$ and further for all $i \neq j,\ A_i \cap A_j = \emptyset\ \implies\ \cup_{i=1}^{\infty} A_i \in \mathcal{A}$

Theorem

1, $\mathcal{A}$ is $\sigma - field \implies \mathcal{A}$ is a $\lambda - class$, algebra, $\sigma - ring$, ring, semi-ring.
2, $\sigma - ring \implies$ ring $\implies$ semi-ring.
3, Field $\implies$ ring; field on a finite set $\implies \sigma - field.$

Theorem

Let I be an arbitrary index set and suppose $\mathcal{A}_i$ is a $\sigma - field$ for all $i \in I$. Then $\mathcal{A}_I := \{A \subset \Omega : A \in \mathcal{A}_i \; for \; every \; i \in I\} = \cap_{i\in I}\mathcal{A}_i$ is also a $\sigma - field$. Similar statement is true for rings, sigma-rings, fields, lambda-systems but NOT semi-rings.

Let $\varepsilon \subset 2^\Omega$ be an arbitrary collection of sets. Then there exists a smallest sigma-field containing ε. Writing $\sigma(\varepsilon)$ for this sigma-field, this can be obtained via $\sigma(\varepsilon) = \cap_{\mathcal{A}\subset 2^\Omega, \; \mathcal{A} \; is \; a \; \sigma - field, \; \mathcal{A}\supseteq\varepsilon}\mathcal{A}$. This will be called the sigma-field generated by ε. Similarly for lambda-system generated by ε.