Exercises on 28 Sep. 2021

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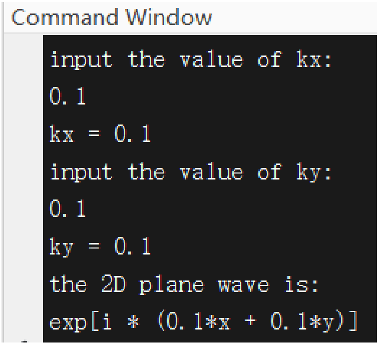
1. Plane waves

A plane wave in two dimensions is described by $f(x,y) = e^{i(k_x x + k_y y)}$ with k_x, $k_y \in \mathbb{R}$

− Write a program that calculates 2D plane waves for any given k_x, k_y.

After inputting the values of k_x and k_y, the command window will output the expression of the 2D plane wave.

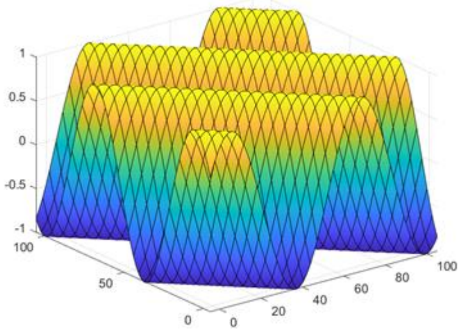

Command Window
```
input the value of kx:
0.1
kx = 0.1
input the value of ky:
0.1
ky = 0.1
the 2D plane wave is:
exp[i * (0.1*x + 0.1*y)]
```

− Display and examine Re(f), Im(f), Abs(f), Phase(f) for a selection of kx, ky.

$$Set:\ k_x = 0.1, \qquad k_y = 0.1$$

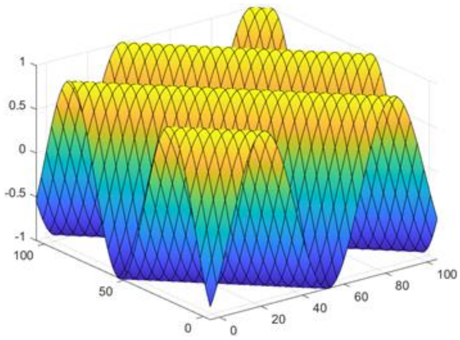
Real Part:

$$\cos(k_x x + k_y y)$$



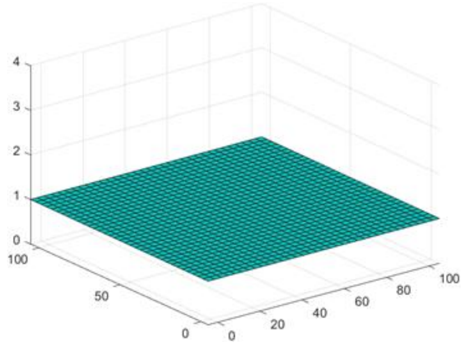
Imaginary Part:

$$\sin(k_x x + k_y y)$$



Compared with the real part, the imaginary part has a phase delay of $\frac{\pi}{2}$

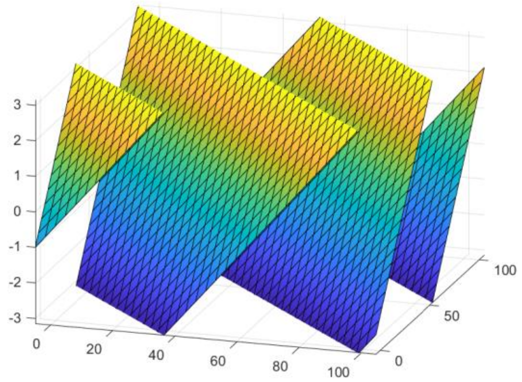
Absolute Value:



$$|f(x,y)| = \sqrt{Re^2[f(x,y)] + Im^2[f(x,y)]} = \sqrt{\cos^2(k_x x + k_y y) + \sin^2(k_x x + k_y y)}$$

Apparently, the absolute value is a constant.

Phase



The phase shows a periodicity of 2π

- What determines the direction of each wave?

Set θ as the direction angle, which is an angle between wave plane and direction-x. Write the wave vector as $\vec{k} = (k_x, k_y)$, then:

$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

- What is the wavelength as a function of k_x, k_y?

$$f(x,y) = e^{i(k_x x + k_y y)}$$

From the wave vector (k_x, k_y), we can get the propagation direction. In this direction, define vector $\vec{d}$ as a unit vector as following:

$$\vec{d} = \frac{k_x}{\sqrt{k_x^2 + k_y^2}}\vec{i} + \frac{k_y}{\sqrt{k_x^2 + k_y^2}}\vec{j}$$

At the origin $(0,0)$, the phase is 0. After a propagation of λ in the direction of $\vec{d}$, set the position as $(\Delta x, \Delta y)$, then we have:

$$(\Delta x, \Delta y) = \lambda\left(\frac{k_x}{\sqrt{k_x^2 + k_y^2}}, \frac{k_y}{\sqrt{k_x^2 + k_y^2}}\right)$$

According to the wave function, the phase at this point is:

$$\varphi = k_x \cdot \Delta x + k_y \cdot \Delta y = 2\pi$$

So the expression of λ:

$$\lambda = \frac{2\pi}{\sqrt{k_x^2 + k_y^2}}$$

– Think of some linear, shift-invariant (LSI) operation and apply it.

Consider an LSI Operation H, set:

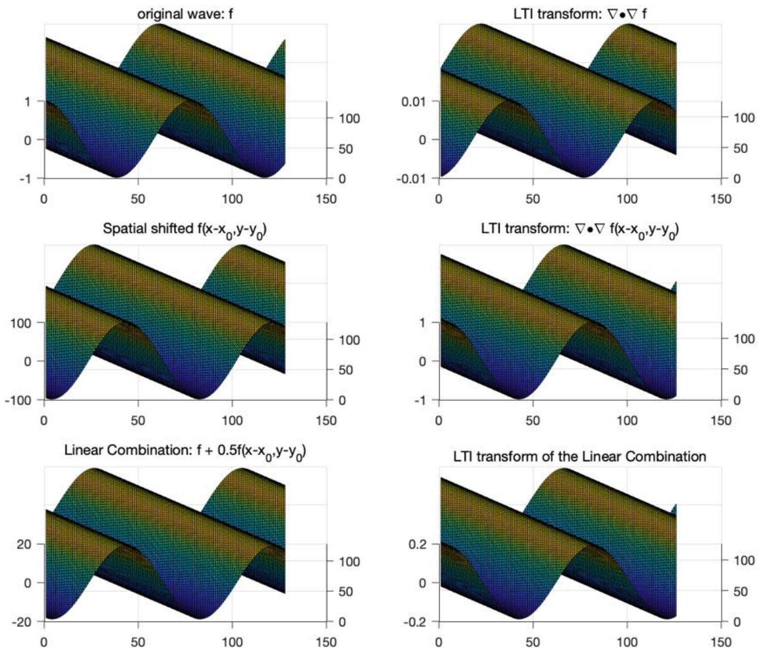
$$H[f(x,y)] = \nabla \cdot \nabla f(x,y) = F(x,y)$$

Then

$$H[f(x-x_0, y-y_0)] = F(x-x_0, y-y_0)$$

Also

$$H[a \cdot f(x,y) + b \cdot f(x-x_0, y-y_0)] = a \cdot H[f(x,y)] + b \cdot H[f(x-x_0, y-y_0)]$$
$$= a \cdot F(x,y) + b \cdot F(x-x_0, y-y_0)$$



– What happens to the waves?

Laplace operator ∇^2 multiplies the original function with its eigenvalue $(ik)^2$, where $k^2 = k_x^2 + k_y^2$. A multiplication of $(ik)^2$ is equivalent to a phase shift of π as well as an amplification of k^2.

2. Fast Fourier Transform (FFT)

This exercise studies the behavior of FFT, using a rectangle input as an example.

The prepared code works with a 1D input vector of length 256, with a rectangle of length 16 at the center. Run the code and answer the following questions. Along with the questions, review the related parts of the code.

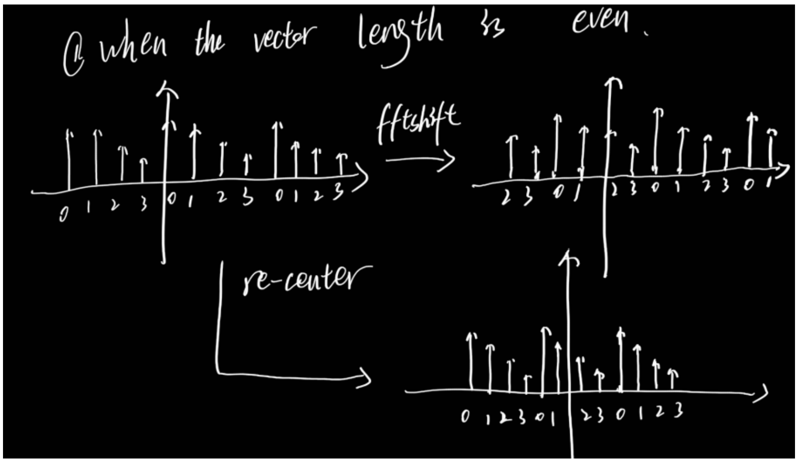
a) **The code first calculates and displays the FFT of the input vector straightforwardly (Figure 1). Why is the sinc-shaped Fourier transform split in two halves?**
Because within FFT, the component of 0 Hz is located at k=0. When constructing the array, we can only get the components of positive frequencies. And because of the periodicity of FFT, the right half is corresponding to negative components. In order to get a sinc-shaped Fourier transform, we should recenter the graph.

b) **The code then performs 'fftshift' on the transform, which swaps its first and second halves (Figure 2). Why does the phase of the transform oscillate rapidly? Isn't the rectangle symmetric so that the transform should be purely real?**
Because the center of the input rectangle is not located at the origin. Spatial shift of half a period corresponds to phase shift of half a period, which is π. Then such a phase shift is represented as an imaginary part.

c) **The code now includes phase correction, shifting the origin in the original domain to the center of the rectangle (Figure 3). Review the related lecture notes.**
Spatial shift corresponds to phase shift, following:

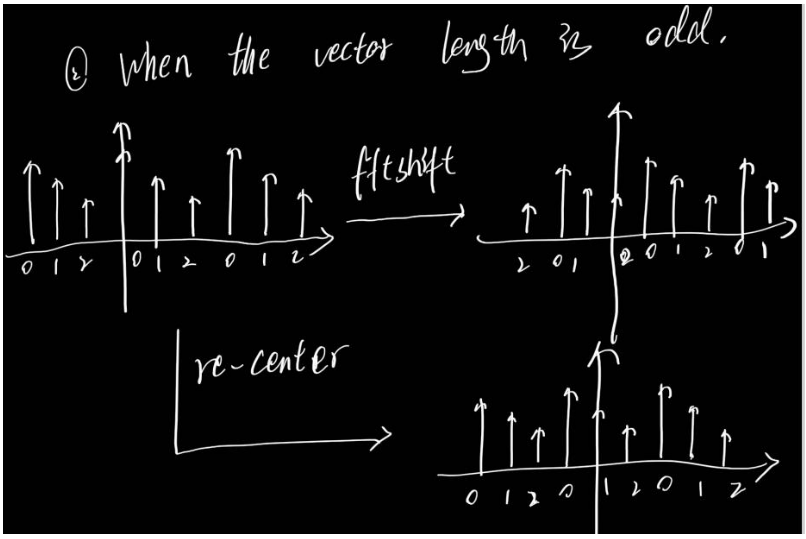
$$\mathcal{F}[f(x-x_0)] = e^{(-i\omega x_0)}F(\omega)$$

d) **Next the code attempts to shift the origin simply by fftshift instead, similar to what it does in the Fourier domain (Figure 4). As you see, this works only almost. Why doesn't it work fully?**



According to the graphs above, as for an array with even elements, the center still has a displacement of $\frac{1}{2}\Delta x$ (which cannot be eliminated in a discrete system!), therefore, there is still a phase shift of $e^{-ik\frac{1}{2}\Delta x}$.

To solve this problem, we move $\frac{1}{2}\Delta x$ to the right in the spatial domain. Corresponding to the frequency domain, it is equivalent as a factor $e^{i\frac{1}{2}k\Delta x}$.

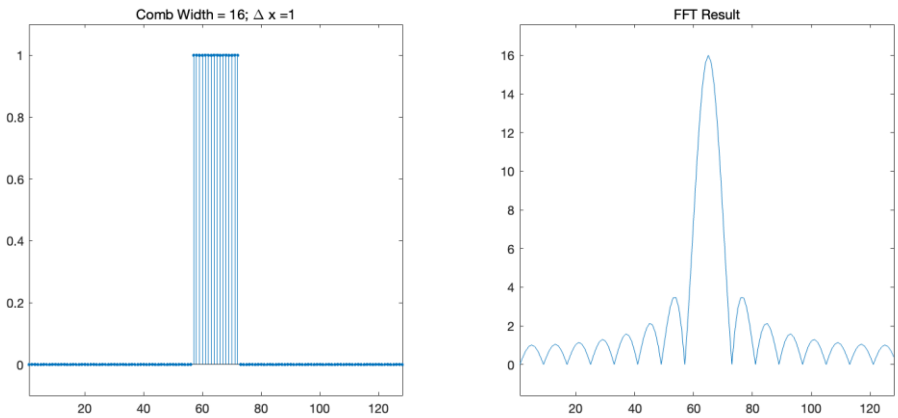


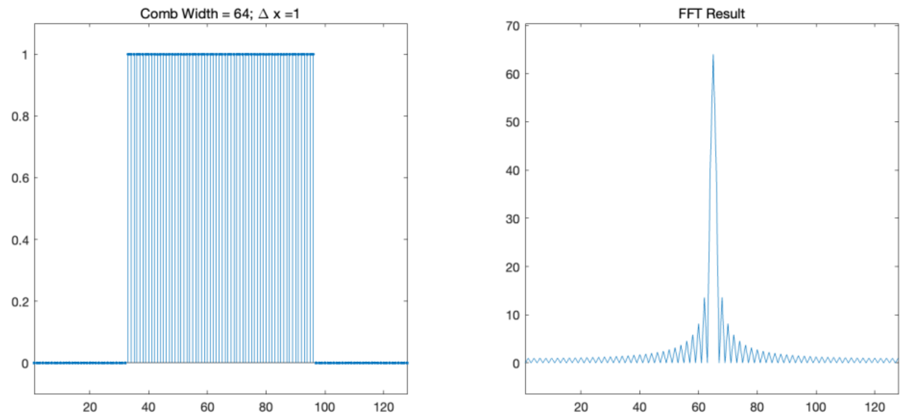
As for an array with odd elements, the displacement occurs again. This is due to the function: fftshift in MATLAB, which plays as $01234 \rightarrow 34012$. To solve this problem, we move the array by Δx to the right, in the frequency domain, it is equivalent to multiply a factor $e^{ik\Delta x}$

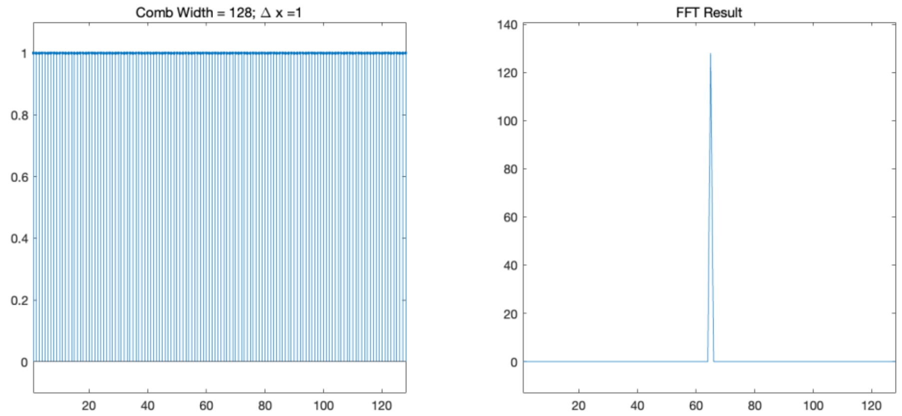
3. Building a Comb

Build a 1D comb function by starting with a single point impulse in the center and successively adding impulses on the left and right

-At each stage, calculate and examine the Fourier transform. What happens?

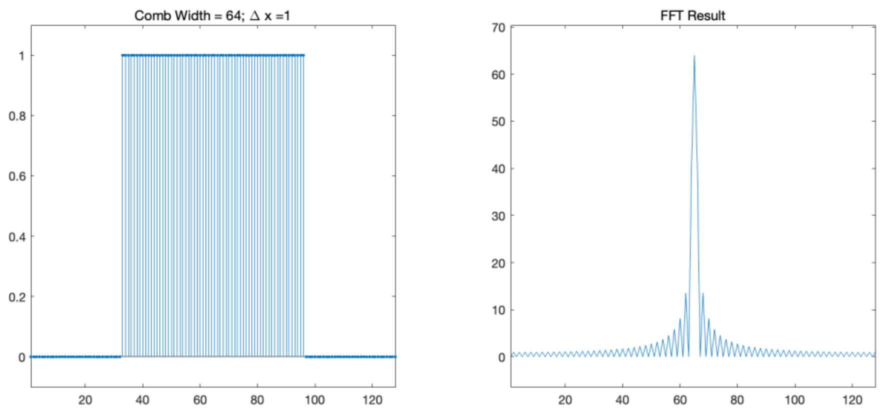


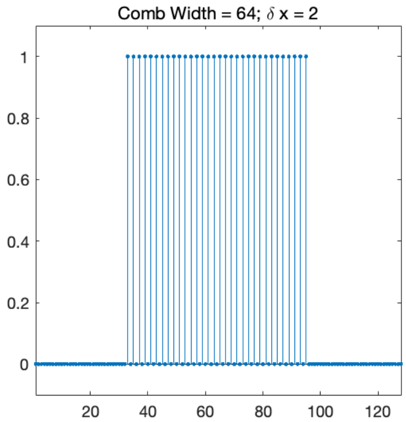


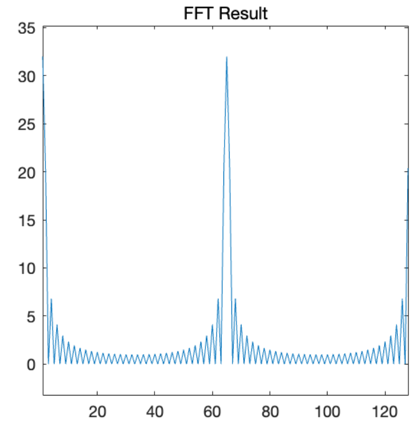


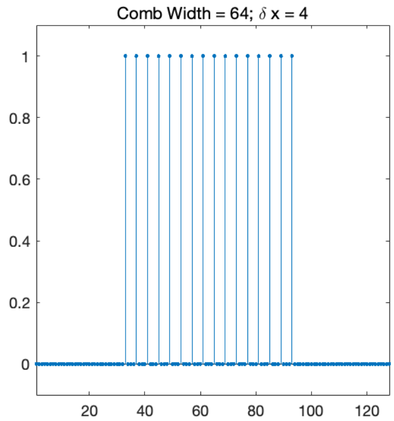
In this experiment, we use the comb function defined in DFT (the amplitude equals to 1). As the comb becomes wider (continuously adding impulses), the frequency band becomes narrower and the spectrum becomes closer to an impulse, which means the comb function is closer to a constant function, with only one frequency component: 0 Hz.

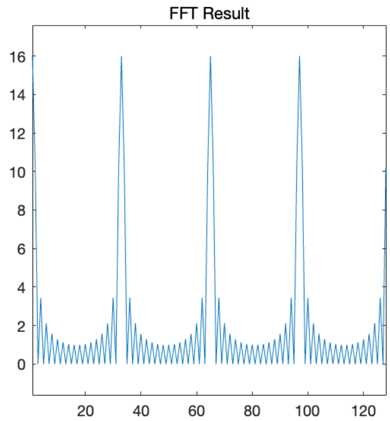
-What happens when you vary Δx of the original comb?











To increase Δx is equivalent to lower the sampling rate. Therefore, adding Δx in the image domain causes aliasing and overlapping in the frequency domain. According to the sampling theorem, the relationship between Δx and Δk is as following:

$$\Delta x \times \Delta k = 2\pi$$