Indirect Inference for Nonlinear Panel Models with Fixed Effects*

Shuowen Chen†

October 8, 2021

Abstract

Fixed effects estimation for nonlinear panel data models suffers from incidental parameter problem. This leads to two undesirable consequences in applied research: point estimates are subject to large bias, and confidence intervals have incorrect coverage. This paper proposes a simulation–based method for bias reduction. The method simulates data using estimated individual effects, and finds values of parameters by equating the fixed effects estimators obtained from observed and simulated data. The asymptotic framework provides consistency, bias correction and asymptotic normality results. An application to labor force participation illustrates the finite–sample performance of the method.

*Conversations with Chuqing Jin inspired and improved this project. I am indebted to Iván Fernández-Val, Jean–Jacques Forneron and Hiroaki Kaido for patience, guidance and encouragement. For helpful discussions and suggestions, I thank Eric Hardy, Dennis Kristensen, Pierre Perron, Zhongjun Qu, Marc Rysman, Xiaoxia Shi, Martin Shi Yuen Shu, Shuang Wang, Guang Zhang and participants in BU econometrics seminars and the econometrics reading group. All errors are mine.

†Boston University. Email: swchen@bu.edu.

1 Introduction

Panel data refers to data for multiple entities (e.g., individuals, firms, etc.) observed at two or more time periods. Unobserved heterogeneity across entities often accounts for a large fraction of variations in the data. When unobserved heterogeneity is correlated with some explanatory variables included in the regression specifications, then the resultant omitted variable bias renders inconsistency of point estimates.

Adding individual fixed effects α_i's is a popular way to control for time–invariant heterogeneity. Compared to other approaches like random effects and correlated random effects, the fixed effects method does not impose a distributional assumption on α_i's or restrict their relationships with other explanatory variables. Instead, each α_i is treated as a parameter for estimation. However, because the number of α_i's increases with sample size and each α_i is estimated using only entity i's observations, adding fixed effects introduces the incidental parameter problem in estimating the vector of parameters of interest θ_0. This problem leads to two consequences for applied research: point estimates are subject to large bias and confidence intervals have incorrect coverage.

This paper proposes a simulation–based method to bias correct fixed effects estimators in a class of nonlinear panel models. The method is named *indirect fixed effects estimator* and features two main steps: the first one is to simulate data by using estimated individual effects from the observed data $\widehat{\alpha}_i$'s. The other step is to find the value of parameters that matches the fixed effects estimators using observed and simulated data.

The method has two advantages. First, it does not require an explicit characterization of the bias term, which can be hard to derive for complex models. Instead, the method finds the solution by automatically correcting the bias: the vector of parameter values that is the closest to θ_0 is the one that renders the same bias in fixed effects estimation. Second, standard errors can be derived using delta method, so there is no need to use bootstrap, which can be computationally intensive.

The two advantages are inherited from a precedent simulation–based estimation approach called indirect inference, which is first developed by Gouriéroux et al. (1993) and Smith (1993). In a nutshell, indirect inference uses an auxiliary model to summarize the statistical properties of the observed data and simulated data, and finds values of structural parameters that match the parameters of the auxiliary model, estimated using the observed and simulated data, in terms of a minimum–distance criterion function. Because the same regression is run on observed and simulated data, matched estimators have the same bias structure and thus get cancelled.

Gouriéroux et al. (2010) apply indirect inference to dynamic panel linear models, whose fixed effects estimators are known to be biased (Nickell, 1981). The linear structure allows Gouriéroux et al. (2010) to eliminate individual fixed effects by first–difference. Therefore, α_i's do not show up in the bias term, and data can be simulated without information on them.

However, indirect inference is not directly applicable to nonlinear panel models, which are widely used in various fields of economics like industrial organization and labor. Because α_i's cannot be cancelled out by differencing, data simulation seem infeasible without a functional form specification on them, and the bias term is a complicated function of θ_0 and α_i's.

To simulate data, this paper proposes using $\widehat{\alpha}_i$'s. These are informative proxies for the unknown individual effects α_i's because they become more accurate estimates when T grows large. Intuitively speaking, although data simulated using $\widehat{\alpha}_i$'s do not perfectly mimic the observed data, such a difference vanishes when the sample size increases.

The estimator then corrects bias of complicated forms by matching fixed effects estimates using observed and simulated data. This brings two advantages for implementation and theoretical analysis of the new estimator. First, the minimum–distance criterion function for matching is just–identified because the dimension of fixed effects estimators are identical. Therefore, there is no need to consider an estimation of the optimal weighting matrix. It further implies that the matching can be made as exact as machine precision permits.

The second advantage is with respect to the relationship between the vector of parameters of interest θ_0 and the unique maximizer of the limiting log likelihood function for fixed effects estimation. To back out point estimates of θ_0 from fixed effects estimators using simulated data, this relationship should be invertible. Because the unique maximizer is essentially θ_0, the relation turns out to be an identity function. Therefore, invertibility is satisfied trivially.

This paper presents consistency, bias correction and asymptotic normality results for the indirect fixed effects estimator. As usual in the indirect inference literature, consistency requires that the fixed effects estimators using observed and simulated data converge to the unique maximizer of the limiting log likelihood. Although the pointwise convergence of $\widehat{\theta}$ to θ_0 is a standard result in the large–T panel literature, three important differences arise in the analysis of fixed effects estimators using simulated data and pose technical challenges for establishing uniform convergence.

First, the simulated data are generated using $\widehat{\alpha}_i$'s instead of the true ones. To justify this practice, the corresponding log likelihood function should uniformly well approximates the one rendered by data simulated using the true individual effects. Otherwise, simulated fixed effects estimator is not even pointwise convergent.

The proof of this statement, however, is complicated by the fact that the log likelihood function using simulated data can be nonsmooth for important types of nonlinear panel models, with binary choice models as leading examples. Intuitively speaking, when the dependent variable is discrete, a small change in the parameter values can lead to discrete changes in the simulated data. As a result, the sample log likelihood function using simulated data is discontinuous.

Simulations often generate discontinuous objective functions (e.g., McFadden, 1989; Pakes and Pollard, 1989), but this paper confronts a second difference: the fixed effects estimator using simulated data is nonsmooth. Therefore, standard proof strategies in the panel literature (e.g., Hahn and Newey, 2004; Hahn and Kuersteiner, 2011) cannot be directly applied to characterize its limiting behavior.

Empirical process theory provides ample tools to handle nonsmoothness functions and moments in econometrics (Andrews, 1994), but the analysis of a nonsmooth fixed effects estimator is further complicated by the third difference: the presence of incidental parameters, whose number increases with sample size. Without restrictions on the structures of panel data and log likelihood function, general analysis of nonsmoothness and incidental parameters is very difficult.

To prove uniform convergence with nonsmoothness, this paper follows Newey (1991) by establishing pointwise convergence and stochastic equicontinuity of fixed effects estimator in the simulation world. Intuitively speaking, pointwise convergence is equivalent to uniform convergence for any finite number of points, but without smoothness, the gap between any two points can behave rather erratically. The stochastic equicontinuity condition is hence required to restrict such behaviors in probability.

Under the assumption that panel data are independent along the cross section dimension, this paper first justifies data simulation with $\widehat{\alpha}_i$'s by proving that the corresponding log likelihood function uniformly approximates the one from simulated data generated by true individual effects. As such, a uniform law of large number can be established and pointwise convergence in the simulation world follows from the standard consistency argument (Newey and McFadden, 1994).

To verify the stochastic equicontinuity condition, this paper profiles out incidental parameters and uses the concavity property of profiled log likelihood to show that simulated fixed effects estimators satisfy the L^2−smoothness property, which corresponds to the type−IV condition in Andrews (1994). The proof strategy might be of independent interest for future research.

Because fixed effects estimators using simulated data are not smooth in θ and $\widehat{\alpha}_i$'s, the conventional proof strategy in indirect inference (e.g., Gouriéroux et al., 1993) is not directly applicable for the asymptotic normality of the new estimator. A smoothness condition is thus imposed, which, combined with consistency, allows to explore bias correction and central limit theorem (CLT) through the lens of fixed effects estimators, whose asymptotic relationship with θ_0 has been established in the panel literature.

Proving bias correction involves two main steps. First, because $\widehat{\alpha}_i$ is uniformly consistent to α_{i0}, it is shown that the bias term from simulated data uniformly approximates an infeasible one from data simulated via α_0. As such, the bias term cancels out based on the classic statement in indirect inference.

Applying this strategy to proving asymptotic normality faces an extra technical challenges: the approximation error of the CLT term for each i has to vanish at a faster rate so that the accumulated errors are still negligible. The proof involves the following main steps: for each i, construct i.i.d sequences that approximate its simulated time series with a controlled remainder (Dedecker and Louhichi, 2002, Lemma 4.1), and find an envelope for its simulated CLT term. As such, the expectation of each simulated CLT term can be bounded a finite uniform entropy integral (van der Vaart, 2000, Corollary 19.35). These individual bounds are then aggregated to form a measure of approximation error that is asymptotically negligible.

Related Literature

The indirect fixed effects estimator presented in this paper combines four strands of literature, and this section provides a non–exhaustive review. The incidental parameter problem is first discussed by Neyman and Scott (1948). When T is fixed, fixed effects estimation of nonlinear models is in general inconsistent because $\widehat{\theta}$ is not separable from $\widehat{\alpha}_i$'s and estimation errors do not vanish, even when number of cross–section n is very large (Chamberlain, 1984; Lancaster, 2000). Only some special models like static linear and conditional Logit specifications feature fixed–T consistent estimators (Andersen, 1970). A key insight of the large–T panel data literature is that the incidental parameter problem becomes an asymptotic bias problem when n and T grow at the same rate. When T is large, fixed effects estimators are consistent and asymptotically normal, but they still have a bias comparable to standard errors.

In the search for asymptotically unbiased estimators, there are two leading approaches. For certain types of models, the bias terms have been characterized analytically and corrected using a plugged–in approach (Hahn and Kuersteiner, 2002; Hahn and Newey, 2004; Fernández-Val, 2009; Hahn and Kuersteiner, 2011). However, such terms can be hard to derive for complicated models. Under further sampling and regularity conditions, bias terms can be automatically corrected using jackknife. For example, Hahn and Newey (2004) propose delete–one panel jackknife for data that do not have dependencies among observations of the same unit. Dhaene and Jochmans (2015) relax the assumption to stationarity along the time series, and propose a split–panel method. Under an unconditional homogeneity assumption, Fernández-Val and Weidner (2016) allow for two–way fixed effects and propose a jackknife method that corrects biases from both dimensions.

See Arellano and Hahn (2007) and Fernández-Val and Weidner (2018) for recent surveys. Standard errors are typically obtained by panel bootstrap, which can be computationally intensive[1].

Kim and Sun (2016) propose a parametric bootstrap bias correction method to nonlinear panel models considered in this paper. Similar to the indirect fixed effects estimator, parametric bootstrap corrects the bias term implicitly using simulations, but it is computationally intensive to obtain the standard errors, and the proof strategies are totally different.

Second, this paper extends the existing theory and practice of indirect inference. Since the introduction of the method, its asymptotic theory has mainly been focused on the times series data (Gouriéroux et al., 1993; Smith, 1993; Gallant and Tauchen, 1996). Some recent papers explore asymptotic properties in panel data with discrete dependent variables, but there are two key differences from this paper. First, their settings hold time series dimension fixed and study the different types of models. For example, Bruins et al. (2018) do not consider models with fixed effects, Frazier et al. (2019) impose standard normality distribution on α_i's, and Taber and Sauer (2021) assume bivarite normality distribution. Second, they deal with nonsmoothness by either smoothing discontinuous parts or constructing a differentiable criterion function that is asymptotically equivalent to the original one.

Gouriéroux et al. (2010) is the first paper that establishes theoretical properties of indirect inference for a class of large–T panel models. As discussed in Section 1, however, their analysis relies heavily on the linear structure of dynamic linear models, which does not apply to nonlinear models considered in this paper. This paper fills the gap by extending the theory to handle the presence of α_i's in data simulation and the bias term.

Indirect inference is popular in various fields of economics, including empirical industrial organizations (Collard-Wexler, 2013), labor economics (Altonji et al., 2013) and macroeconomics (Guvenen and Smith, 2014; Berger and Vavra, 2019). However, finding an informative auxiliary model is not a trivial task, and researchers often have to assume the invertibility of the limiting relationship between auxiliary parameters and parameters of interest. This paper provides an alternative choice, namely the log likelihood function from the nonlinear panel model, for researchers that employ panel data with fixed effects. The estimation procedures are simple to implement as fixed effects estimation schemes are available in free software like R and Julia.

Nonsmooth objective functions are common in econometrics, and empirical process methods are standard tools for asymptotic analysis (Andrews, 1994; Newey and McFadden, 1994; van der

[1]To obtain one debiased point estimate, fixed effects estimations are run three times: one for the whole sample, and twice for the two split samples. If the number of bootstraps is set to be 500, then the total number of fixed effects estimation becomes 1500. In addition, in practice it is often recommended to use multiple sample splits to improve the finite–sample performance.

Vaart and Wellner, 1996). The seminal work on simulation–based methods by Pakes and Pollard (1989) is predicated on the independence assumption of cross section data and therefore is not suitable for panel data, which feature dependence for each individual time series. Dedecker and Louhichi (2002) provide an overview of maximal inequalities for empirical central limit theorems for dependent data. Kato et al. (2012) provide new stochastic inequalities for mixing sequences and also establish stochastic equicontinuity in the presence of nuisance parameters, but their analysis focuses on a different class of nonlinear models, namely the panel quantile regression models.

Finally, this paper contributes to a burgeoning literature that considers simulations from semiparametric models. Simulation–based methods like simulated method of moments (McFadden, 1989; Pakes and Pollard, 1989; Lee and Ingram, 1991; Duffie and Singleton, 1993) and indirect inference are widely used to estimate models that do not render tractable moments or likelihood functions. See Gouriéroux and Monfort (1997) for an overview. These methods typically require models to be fully specified, but economic theory does not always provide guidance on functional forms, distributions of shocks or measurement error of observed data. Therefore, the resultant estimators can be subject to misspecification.

Dridi and Renault (2000) and Dridi et al. (2007) embed the semiparametric structural model into a full model for data simulation, and propose a partial encompassing principles where parameters of interest are consistently estimated even though nuisance parameters are inconsistently estimated due to misspecification of the full model. Schennach (2014) considers parameters estimation in moment conditions that contain unobservable variables, and proposes a simulation–based method that constructs equivalent moments involving only observable variables. Gospodinov et al. (2017) consider parameter estimation of autoregressive distributed lag models in which covariates are contaminated by serially correlated measurement errors. They propose a method such that simulated covariates preserve the dependence structure observed in the data even though the dynamics of latent covariates or measurement errors are not specified. Forneron (2020) approximates the distribution of shocks by sieves and proposes a sieve–SMM estimator that jointly estimates structural parameters and the distribution of shocks.

Structure of the Paper

The rest of the paper proceeds as follows: Section 2 introduces the model and describes the fixed effects estimator and incidental parameter problem. Section 3 provides an overview of the indirect fixed effects estimator and its implementation. Section 4 presents the theoretical properties of the estimator. Section 5 applies the method to dynamic labor force participation to illustrate the finite–sample properties of the estimator. Section 6 uses Monte Carlo simulations to compare the new estimator with other bias correction methods. Section 7 concludes and discusses open questions. Appendices A, B and C consist of proofs and computation details.

2 Nonlinear Panel Model and Fixed Effects Estimator

This section starts with a description of nonlinear panel models with fixed effects. Let the data observations be denoted by $\{z_{it} = (y_{it}, x_{it}) \colon i = 1, \ldots, n; t = 1, \ldots, T\}$, where y_{it} is dependent variable and x_{it} is a $p \times 1$ vector of explanatory variable. The observations are independent across individual i and weakly dependent across time t. The data generation process (DGP) of outcome y_{it} takes the following form:

$$y_{it} \mid x_i^T, \alpha \sim f(\cdot \mid x_{it}; \theta, \alpha_i), \tag{1}$$

where $x_i^T := (x_{i1}, \ldots, x_{iT})$, θ is a $p \times 1$ vector of parameters, α_i is a scalar individual effect and $\alpha := (\alpha_1, \ldots, \alpha_n)$. The explanatory variable x_{it} is strictly exogenous. The conditional density f denotes the parametric part of the model. The model is semiparametric in that neither the distribution of α_i's nor their relationships with explanatory variables x_{it} is specified. Depending on the assumptions on f, this type of models have been used to study many different questions of economic interest.

Example 1 (Panel Discrete Choice Models)**.** Let y_{it} denote a binary variable and F_u a cumulative distribution function (CDF), e.g., the standard normal or logistic distribution. Suppose the binary variable is generated by the following single index process with additive individual effects:

$$y_{it} = \mathbf{1}\{x_{it}'\theta + \alpha_i \geq u_{it}\}, \quad u_{it} \mid x_i^t, \alpha \sim F_u,$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function. Then the conditional distribution of y_{it} is expressed as

$$f(y_{it} \mid x_{it}, \alpha_i; \theta) = F_\varepsilon(x_{it}'\theta + \alpha_i)^{y_{it}}(1 - F_\varepsilon(x_{it}'\theta + \alpha_i))^{1-y_{it}}.$$

Helpman et al. (2008) model a country's export decision as Probit and estimate the gravity equation with country fixed effects. Fernández-Val (2009) uses a Probit specification to estimate determinants of females' labor force participation decisions in the presence of time–invariant heterogeneity such as willingness to work. Collard-Wexler (2013) uses binary Logit specification with market–fixed effects to study whether a ready–mix concrete plant decides to be active in a market.

Example 2 (Panel Poisson Model)**.** The Poisson distribution is useful in modeling arrivals of new events within a certain interval of time. Let y_{it} denote count data for firm i in year t. For Poisson parameter $\lambda_{it} = \exp(x_{it}'\theta + \alpha_i)$, the conditional density becomes

$$f(y_{it} \mid x_{it}, \alpha_i; \theta) = \frac{\lambda_{it}^{y_{it}} \exp(-\lambda_{it})}{y_{it}!} \mathbf{1}\{y_{it} \in \{0, 1, \dots\}\}.$$

Using the number of citation–weighted patents as a proxy for innovation, Aghion et al. (2005) employ this specification to study the relationship between innovation and competition with industry fixed effects.

Model (1) admits a log likelihood function that is concave in all parameters. The true values of the parameters, denoted by θ_0 and $\boldsymbol{\alpha}_0 := (\alpha_{10}, \dots, \alpha_{n0})$, are the unique solution to the population conditional maximum likelihood problem

$$(\theta_0, \alpha_{10}, \dots, \alpha_{n0}) = \underset{(\theta, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{\dim(\theta) + \dim(\alpha)}}{\arg\max} \mathbb{E}\Big(\frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \ln f(y_{it} \mid x_{it}; \theta, \alpha_i) \Big), \tag{2}$$

where the expectation is with respect to the distribution of the observed data, conditional on the unobserved effects and initial conditions. Section 4 discusses assumptions for the existence and uniqueness of the solution. The indirect fixed effects estimator relies on the uniqueness condition for consistency.

2.1 Fixed Effects Estimator

The fixed effects estimator of θ is obtained by doing maximum likelihood on sample analog of the population problem (2), treating each α_i as a parameter to be estimated.

$$(\widehat{\theta}, \widehat{\alpha}_1, \dots, \widehat{\alpha}_n) = \underset{(\theta, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{\dim(\theta) + \dim(\alpha)}}{\arg\max} \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \ln f(y_{it} \mid x_{it}; \theta, \alpha_i).$$

To facilitate theoretical analysis, this equation is rewritten such that the individual effects are profiled out. More specifically, given θ, the optimal $\widehat{\alpha}_i(\theta)$ for each i is defined as

$$\widehat{\alpha}_i(\theta) = \underset{\alpha \in \mathbb{R}}{\arg\max} \frac{1}{T} \sum_{t=1}^{T} \ln f(y_{it} \mid x_{it}; \theta, \alpha).$$

The estimators $\widehat{\theta}$ and $\widehat{\alpha}_i$ are then

$$\widehat{\theta} = \underset{\theta \in \mathbb{R}^{\dim(\theta)}}{\arg\max} \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \ln f(y_{it} \mid x_{it}; \theta, \widehat{\alpha}_i(\theta)), \quad \widehat{\alpha}_i = \widehat{\alpha}_i(\widehat{\theta}). \tag{3}$$

Section 4 discusses assumptions under which these estimators exist and are unique with probability approaching one as n and T become large.

2.2 Incidental Parameter Problem

In panel models, the individual effects are incidental parameters, i.e., nuisance parameters whose dimension grows with the number of cross sectional observations n. As equation (3) shows, the fixed effects estimator $\widehat{\theta}$ cannot generally be separated from the estimation of individual effects $\widehat{\alpha}_i$'s. Because each $\widehat{\alpha}_i$ is only estimated using the T observations for i, estimation errors do not vanish if T fixed, even as n approaches infinity. These estimation errors in turn contaminate $\widehat{\theta}$. This is the incidental parameter problem for fixed effects estimation. Mathematically,

$$\widehat{\theta} \xrightarrow{p} \theta_T := \underset{\theta \in \mathbb{R}^{\dim(\theta)}}{\arg\max} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Big(\frac{1}{T} \sum_{t=1}^{T} \ln f(y_{it} \mid x_{it}, \theta, \widehat{\alpha}_i(\theta)) \Big),$$

but $\theta_T \neq \theta_0$ in general because the population problem uses the true individual effects α_{i0}'s.

To illustrate the problem, suppose y_{it} has the normal distribution $\mathcal{N}(\alpha_{i0}, \theta_0)$, and the goal is to estimate the variance θ_0 in the presence of unknown individual–specific means α_{i0}'s. The fixed effects estimator is $\widehat{\theta} = \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} (X_{it} - \widehat{\alpha}_i)^2$, where $\widehat{\alpha}_i = \frac{1}{T} \sum_{t=1}^{T} X_{it}$. When T is fixed and n approaches infinity, Neyman and Scott (1948) show that $\widehat{\theta} \xrightarrow{p} \theta_0 - \frac{\theta_0}{T}$.

On the other hand, when T also grows to infinity, the bias term $-\frac{\theta}{T}$ approaches zero. The large–T panel literature generalizes this insight and shows that the incidental parameter problem becomes an asymptotic bias problem when n and T grow at the same rate.

3 An Indirect Fixed Effects Estimator

The key feature of the indirect fixed effects estimator is to match $\widehat{\theta}$ and fixed effects estimators from simulated data generated by $\widehat{\alpha}_i$'s and a given θ. To avoid confusion, it is necessary to distinguish the notations of parameters in the simulation world from those in Section 2. More specifically, this paper uses β and γ_i to denote the vector of parameters of interest and individual effects in the log likelihood function using simulated data.

To clarify the notations and introduce the implementation of indirect fixed effects estimator, this section first revisits the Neyman–Scott example. Using the panel Probit model as a concrete example, this section then illustrates the presence of nonsmoothness and discusses the general estimation procedures.

3.1 Neyman–Scott Example Revisited

The normality assumption on the structure of the panel implies the following data generating process (DGP):

$$y_{it}(\alpha_{i0}, \theta) = \alpha_{i0} + \sqrt{\theta} u_{it}, \quad u_{it} \sim \mathcal{N}(0,1).$$

This equation is not directly estimable without information on the distribution of α_{i0}'s. The indirect fixed effects estimator uses $\widehat{\alpha}_i$'s instead, and the simulated data have the following representation:

$$y_{it}^h(\widehat{\alpha}_i, \theta) = \widehat{\alpha}_i + \sqrt{\theta} u_{it}^h, \quad u_{it}^h \sim \mathcal{N}(0,1),$$

where the superscript h denotes a simulation path. The fixed effects estimator using $\{y_{it}^h(\widehat{\alpha}_i, \theta)\}$ is

$$\widehat{\beta}^h(\theta) := \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} (y_{it}^h(\widehat{\alpha}_i, \theta) - \widehat{\gamma}_i)^2,$$

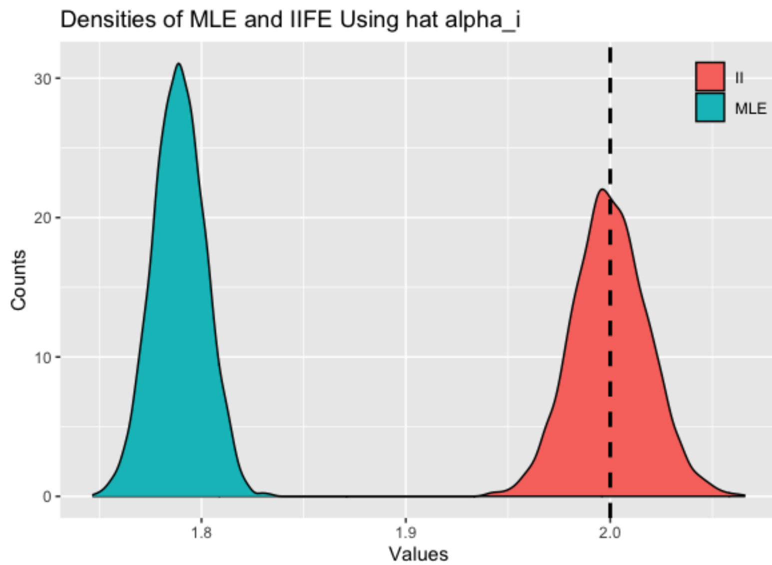
where $\widehat{\gamma}_i = \frac{1}{T} \sum_{t=1}^{T} y_{it}^h(\widehat{\alpha}_i, \theta)$. The interpretation of $\widehat{\beta}^h(\theta)$ is that the estimator changes if a different value of θ is used to simulate the data. Because $\widehat{\alpha}_i$'s are fixed throughout the simulation process, the notation is suppressed. The indirect fixed effects estimator $\widetilde{\theta}$ is the solution to

$$\widehat{\theta} = \widehat{\beta}^h(\widetilde{\theta}).$$

Figure (1) illustrates the issues of $\widehat{\theta}$ and the performance of $\widetilde{\theta}$ for this example. The green region plots the density distribution of the fixed effects estimator $\widehat{\theta}$ and it conveys two messages: the estimator is subject to a large bias, and the intervals around the mean do not have the correct coverage. The red region, on the other hand, plots the density distribution of $\widetilde{\theta}$. The new estimator corrects the bias significantly. The intervals are wider than those for $\widehat{\theta}$, and this is because simulations introduce an extra source of variation into the estimation scheme.

Due to the simple structure of this example, $\widehat{\theta}$ and $\widehat{\beta}^h(\theta)$ have closed–form expressions, and the bias term does not contain α_i's. However, these are not the cases for model (1). In addition, the discrete nature of dependent variable leads to a nonsmooth log likelihood function in the simulation world.

Figure 1: Comparison of FE and IFE



Note: Density plots of fixed effects and indirect fixed effects estimator for θ_0. The DGP is $y_{it} = \alpha_{i0} + \sqrt{\theta_0} u_{it}$, where $u_{it} \sim \mathcal{N}(0, 1)$. The true value $\theta_0 = 2$ is depicted by the dashed line and $\alpha_{i0} = i$ for $i = 1, \ldots, n$. The sample size is $n = 2500, T = 5$ and number of simulation H is set to be 1. The simulations are conducted 5000 times.

3.2 Nonsmoothness

Consider the binary choice panel Probit model as a concrete example. Given θ, $\widehat{\alpha}_i$'s and x_{it}, the simulated dependent variable is

$$y_{it}^h(\theta, \widehat{\alpha}_i) = \mathbf{1}(x_{it}'\theta + \widehat{\alpha}_i > u_{it}^h), \quad u_{it}^h \sim \mathcal{N}(0, 1),$$

where u_{it}^h are simulation draws from the standard normal distribution. The corresponding log likelihood function is

$$\frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} y_{it}^h(\theta, \widehat{\alpha}_i) \log \Big(\Phi(x_{it}'\beta + \gamma_i) \Big) + (1 - y_{it}^h(\theta, \widehat{\alpha}_i)) \log \Big(1 - \Phi(x_{it}'\beta + \gamma_i) \Big). \tag{4}$$

This equation illustrates the three non–standard aspects of simulated fixed effects estimator $\widehat{\beta}^h(\theta)$. Because the simulated data are discontinuous in θ or $\widehat{\alpha}_i$'s, the log likelihood is discontinuous, and this nonsmoothness carries over to its maximizer $\widehat{\beta}^h(\theta)$. In addition, estimating $\widehat{\beta}^h(\theta)$ involves incidental parameters γ_i's. The deterministic limit is

$$\frac{1}{nT} \mathbb{E}\Big[\sum_{i=1}^{n} \sum_{t=1}^{T} y_{it}(\theta, \alpha_{i0}) \log \Big(\Phi(x_{it}'\beta + \gamma_i) \Big) + (1 - y_{it}(\theta, \alpha_{i0})) \log \Big(1 - \Phi(x_{it}'\beta + \gamma_i) \Big) \Big], \tag{5}$$

where the expectation is over u_{it}^h and x_{it}, and $\widehat{\alpha}_i$'s are replaced by α_{i0}'s.

Remark 1 (A comparison with panel quantile regression (QR) models)**.** One important type of nonlinear panel models that is not included in model (1) but also features nonsmoothness is panel QR models[2]. Kato et al. (2012) consider the following setup with individual effects:

$$Q_\tau(y_{it} \mid x_{it}, \gamma_{i0}(\tau)) = \gamma_{i0}(\tau) + x_{it}'\beta_0(\tau),$$

where $\tau \in (0,1)$ is a quantile index, and $Q_\tau(y_{it} \mid x_{it}, \gamma_{i0}(\tau))$ is the conditional τ−quantile of y_{it} given $(x_{it}, \gamma_{i0}(\tau))$. The fixed effects quantile regression (FE−QR) estimator for this model is

$$(\widehat{\gamma}_{\text{FE-QR}}, \widehat{\beta}_{\text{FE-QR}}) = \arg\min \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\rho_\tau(y_{it} - \gamma_i - x_{it}'\beta),$$

where $\gamma := (\gamma_1, \ldots, \gamma_n)'$ and $\rho_\tau(u) := \{\tau - \mathbf{1}\{u \le 0\}\}u$ is the check function. Because the check function is non−differentiable, the FE−QR estimator is also non−smooth. However, what makes it different from this paper is that the criterion function (4) is still smooth in β and γ_i's, and this is the key component for the asymptotic properties of the indirect fixed effects estimator[3].

3.3 General Estimation Procedures

From the known distribution F_u, the simulated unobservables $\{u_{it}^h\}$ are independently drawn for $h = 1, \ldots, H$, where H denotes the number of panel data simulated. For a given value of θ, let $y_{it}^h(\theta, \widehat{\alpha}_i)$ denote the simulated dependent variable for simulation path h, then the sample log likelihood function using the h−th simulated data is

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\ln f(y_{it}^h(\theta, \widehat{\alpha}_i) \mid x_{it}; \beta, \gamma_i), \tag{6}$$

where β and γ_i respectively denote the finite−dimensional parameter and incidental parameter in the simulation world. The fixed effects estimator to this problem is

$$\widehat{\beta}^h(\theta) = \arg\max_{\beta \in \mathcal{B}} \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\ln f(y_{it}^h(\theta, \widehat{\alpha}_i) \mid x_{it}; \beta, \widehat{\gamma}_i(\beta, \theta)),$$

[2] See Galvao and Kato (2018) for a recent survey.

[3] More specifically, the CLT and bias terms for smooth likelihood are combinations of high−order stochastic expansion terms of score functions, which is hard to derive for panel QR models.

where

$$\widehat{\gamma}_i(\beta, \theta) = \arg\max_{\gamma \in \Gamma_\gamma} \frac{1}{T} \sum_{t=1}^{T} \ln f(y_{it}^h(\theta, \widehat{\alpha}_i) \mid x_{it}; \beta, \gamma).$$

Repeating this estimation for all simulated panel, the following average can be computed:

$$\widehat{\beta}^H(\theta) := \frac{1}{H} \sum_{h=1}^{H} \widehat{\beta}^h(\theta),$$

The indirect fixed effects estimator $\widetilde{\theta}^H$ is the solution to

$$\widehat{\theta} = \widehat{\beta}^H(\theta), \tag{7}$$

where the superscript H stresses that the finite–sample performance depends on the number of simulations conducted. The box below summarizes the steps required to compute the estimator.

Algorithm: Computing the indirect fixed effects estimator

(i) Obtain fixed effects estimators $(\widehat{\theta}, \widehat{\alpha}_1, \ldots, \widehat{\alpha}_n)$ using the observed data.

(ii) Set a random seed. For each i and t, draw unobservables $\{u_{it}^h\}_{h=1}^H$ from F_u.

(iii) Given θ and $\widehat{\alpha}_i$'s, use model (1) and $\{u_{it}^h\}$ to simulate dependent variable. $\{y_{it}^h(\theta, \widehat{\alpha}_i)\}$ and create simulated data $\{y_{it}^h(\theta, \widehat{\alpha}_i), x_{it}\}$, where $i = 1, \ldots, n$ and $t = 1, \ldots, T$.

(iv) Obtain fixed effects estimator $\widehat{\beta}^h(\theta)$ using the simulated data in Step (iii).

(v) Do Steps (ii) and (iii) for all $h = 1, \ldots, H$. The indirect fixed effects estimator $\widetilde{\theta}_H$ is the solution to equation (7).

Remark 2 (Common random number)**.** Step (ii) follows the standard practice for simulations (Glasserman and Yao, 1992) by drawing unobserved shocks only once at the beginning of the algorithm. It implies that $\widehat{\beta}^h(\theta)$ and $\widehat{\beta}^{h'}(\theta)$ are independent for $h \neq h'$ conditional on x_{it}.

Remark 3 (The role of H)**.** The number of H affects the finite–sample performance of the estimator, and increasing H reduces the asymptotic variance. Just like SMM and indirect inference, there is a trade off between precision of the estimator and intensity of computation. The estimation method, however, is different from simulated maximum likelihood (Manski and Lerman, 1981), which is inconsistent for fixed H due to nonlinear transformation of simulated choice probabilities.

Remark 4 (Choices of optimization methods)**.** When computing fixed effects estimators in the simulation world, the discontinuity nature makes gradient–based optimization methods unsuitable, so simplex–based methods like Nelder–Mead are used instead.

4 Asymptotic Properties

This section starts with a discussion of the main assumptions that lead to theoretical properties of $\widehat{\theta}$ and $\widehat{\alpha}_i$'s. These assumptions are standard in large$-T$ panel data models (Hahn and Kuersteiner, 2011), and they also impose certain structures that help establish the asymptotic properties of the indirect fixed effects estimator. Additional assumptions are imposed to ensure simulations do not affect the data structure.

Assumption 1 (Large T asymptotics)**.** $n, T \to \infty$ such that $\frac{n}{T} \to \kappa \in (0, \infty)$.

Assumption 1 requires that time series dimension grows at the same rate as the cross section dimension. The assumption defines the large$-T$ asymptotics framework and is a necessary condition for consistency of fixed effects estimators using observed data.

Assumption 2 (Sampling of Observed Data)**.** (*i*) For each i, $\{z_{it}\}_{t=1}^{\infty}$ is a stationary mixing sequence; (*ii*) $\{z_{it}\}_{t=1}^{\infty}$ are independent across i; (*iii*) $\sup_i |\alpha_i(m)| \leq C a^m$ for some a such that $0 < a < 1$ and some $C > 0$, where $\mathcal{A}_t^i \equiv \sigma(z_{it}, z_{i,t-1}, z_{i,t-2}, \ldots)$, $\mathcal{B}_t^i \equiv \sigma(z_{it}, z_{i,t+1}, z_{i,t+2}, \ldots)$, and

$$\alpha_i(m) \equiv \sup_t \sup_{A \in \mathcal{A}_t^i, B \in \mathcal{B}_{t+m}^i} |P(A \cap B) - P(A)P(B)|.$$

Assumption 2(*i*) imposes temporal dependence on each individual time series. Assumption 2(*ii*) imposes independence along the cross–section dimension, and it rules out factors. This assumption is crucial for theoretical analysis because it allows to decompose the aggregate log likelihood to individual contributions, each of which then only contains a fixed number of parameters. Assumption 2(*iii*) specifies that the each individual time series satisfies $\alpha-$mixing, and the quantity $\alpha_i(m)$ measures for each i how much dependence exists between data separated by at least m time periods. This allows to bound covariances and moments when using law of large numbers (LLN) and central limit theorem (CLT). Note that Assumption 2 rules out time effects and linear trend.

Assumption 3 (Identification)**.** Denote $G_{(i)}(\theta, \alpha_i) \equiv \mathbb{E}[\frac{1}{T}\sum_{t=1}^{T} \ln f(y_{it} \mid x_{it}, \theta, \alpha_i)]$. For each $\eta > 0$,

$$\inf_i \left[G_{(i)}(\theta_0, \alpha_{i0}) - \sup_{(\theta,\alpha): \|(\theta,\alpha)-(\theta,\alpha_{i0})\| \geq \eta} G_{(i)}(\theta, \alpha) \right] > 0.$$

Assumption 3 imposes a sufficient condition such that the log likelihood function admits a unique maximizer based on time series variation. This is a standard assumption in the large$-T$ panel literature to prove the consistency of fixed effects estimators. The indirect fixed effects estimator also requires this assumption for consistency.

Assumption 4 (Bounding Condition). (*i*) The parameter $\varphi_i := (\theta, \alpha_i) \in \text{int}\, \Theta \times \Gamma_\alpha$, where Θ and Γ_α are compact, convex subset of $\mathbb{R}^p$ and $\mathbb{R}$ respectively. (*ii*) There exists an envelope function $M(z_{it})$ such that

$$\|D^\nu G_{(i)}(\varphi_1; \theta) - D^\nu G_{(i)}(\varphi_2; \theta)\| \leq M(z_{it})\|\varphi_1 - \varphi_2\|,$$

where $D^\nu G_{(i)}(\varphi_1; \theta) := \partial^{|\nu|} G_{(i)}(\varphi) / (\partial \varphi_1^{\nu_1} \ldots \partial \varphi_p^{\nu_p})$ and $|\nu| \leq 5$.

Assumption 4(*i*) imposes compactness of parameter space, which is standard for establishing asymptotic properties of extremum estimators. Compactness is indispensable for proving uniform convergence with nonsmooth criterion functions (Newey, 1991). Assumption 4(*ii*) imposes a Lipschitz condition on the log likelihood function and a moment condition on the envelope function. This allows to establish uniform law of large number (ULLN) of sample log likelihood function and hence the pointwise consistency of $\widehat{\theta}$.

Under these assumptions and some regularity conditions, Hahn and Kuersteiner (2011) establish the following two results:

$$\max_{1 \leq i \leq n} |\widehat{\alpha}_i - \alpha_{i0}| = o_p(1), \tag{8}$$

$$\widehat{\theta} = \theta_0 + \frac{A(\theta_0, \alpha_0)}{\sqrt{nT}} + \frac{B(\theta_0, \alpha_0)}{T} + o_p\left(\frac{1}{T}\right); \tag{9}$$

Equation (8) states that the maximal deviation of $\widehat{\alpha}_i$ from α_{i0} converges to zero. This uniform consistency result is employed to justify the usage of $\widehat{\alpha}_i$'s for data simulations. Equation (9) characterizes the asymptotic relationship between $\widehat{\theta}$ and θ. The term $A(\theta_0, \alpha_0)$ satisfies the central limit theorem (CLT) with zero mean, and the term $B(\theta_0, \alpha_0)$ converges to its expected value. Therefore, $\widehat{\theta}$ is consistent, asymptotically normal, but biased. Both terms are complicated functions of θ_0 and α_0. Because the same regression is run on simulated data, the same structure arises. The indirect fixed effects estimator finds solution by matching the bias terms, but the asymptotic variance is inflated.

Assumption 5 (Simulation). (*i*) Assumption 2 holds for the simulated process for all $\theta \in \Theta$. (*ii*) The parameter spaces for β and γ_i are Θ_β and Γ_γ and are compact.

Assumption 5(*i*) requires that simulation does not affect the mixing properties of the observed data. Assumption 5(*ii*) requires the parameter space in the simulation world to be compact. Since (β, γ) is just a change of notation from (θ, α), this is a natural assumption.

4.1 Backing Out Point Estimates

The non–stochastic limiting function for equation (6) is

$$\mathbb{E}\Big[\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\ln f(y_{it}(\theta,\alpha_{i0})\mid x_{it};\beta,\gamma_i))\Big],$$

where the expectation is taken over simulation draws and sampling of observed data, and $\widehat{\alpha}_i$ is replaced with α_{i0}. This limiting function is essentially the population function for the fixed effects estimation problem, except that β and γ_i are used to denote parameters for estimation in the simulation world. By Assumption 3, this limiting function has a unique maximizer, which is denoted as $\beta(\theta)$.

To back out $\widetilde{\theta}^H$ from $\widehat{\beta}^H(\widetilde{\theta}^H)$, an invertible relationship between θ and $\beta(\theta)$ is required. However, $\beta(\theta)$ is essentially θ, so invertibility is satisfied trivially[4]. Although $\beta(\theta)$ is an identity function, for the rest of the paper this notation is kept to avoid the confusion between maximum of the limit and a parameter for data generation.

4.2 Consistency

In order for the indirect inference–type estimator to be consistent, three conditions should be satisfied (Gouriéroux et al., 1993): an invertible relationship between θ and $\beta(\theta)$, pointwise convergence of $\widehat{\theta}$ to $\beta(\theta_0)$, and uniform convergence of $\widehat{\beta}^h(\theta)$ to $\beta(\theta)$ over the compact parameter space Θ. The previous subsection establishes the invertibility condition, and equation (9) gives the second condition. The following proposition formally states the uniform convergence condition.

Proposition 1 (Uniform convergence of fixed effects estimator using simulated data)**.** *Under Assumptions 1–5,*

$$\sup_{\theta\in\Theta}\|\widehat{\beta}^h(\theta)-\beta(\theta)\|\xrightarrow{p}0.$$

The current proof specializes in panel Probit models, but it is generalizable to other models that features concavity and smoothness in (β,γ_i). Details are available in Appendix B, and here the main ideas are discussed.

Proving the uniform convergence condition with nonsmoothness requires two steps: pointwise

[4]For readers who are familiar with indirect inference, the relationship essentially means that the binding function is an identity. This is because the auxiliary model is identical to the structural model, and thus the parameters in the two models coincide. Many papers that employ indirect inference often have to assume invertibility of the binding function (Collard-Wexler, 2013; Gospodinov et al., 2017), but this assumption is guaranteed in this paper.

convergence of $\widehat{\beta}^h(\theta)$ to $\beta(\theta)$, and a stochastic equicontinuity condition as follows:

$$\mathbb{E}\Big(\sup_{\|\theta_1-\theta_2\|\leq\delta}\|\widehat{\beta}^h(\theta_1)-\widehat{\beta}^h(\theta_2)\|\Big)\leq C\delta, \tag{10}$$

where C is a constant and δ is a positive scalar.

Following the standard argument in Newey and McFadden (1994), pointwise convergence requires a ULLN result of log likelihood function using simulated data (4) to the limiting log likelihood (5). The log likelihood (4) has two sources of randomness: the first source comes from sampling variation of observed data, and the other is from simulations of unobservables. The non–standard part, however, is that data are simulated using $\widehat{\alpha}_i$'s. Therefore, it is necessary to first show that (4) uniformly well approximates the log likelihood using data generated by α_{i0}'s:

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\int_U\Big[y_{it}^h(\theta,\alpha_{i0})\log\Big(\Phi(x_{it}'\beta+\gamma_i)\Big)+(1-y_{it}^h(\theta,\alpha_{i0}))\log\Big(1-\Phi(x_{it}'\beta+\gamma_i)\Big)\Big]dF_u, \tag{11}$$

where the integration is with respect to the distribution of simulation draws to eliminate randomness from simulations. The details are available in Lemma 1, and intuition is provided here. Because panel data are independent along cross section, it suffices to show that each individual's log likelihood

$$\frac{1}{T}\sum_{t=1}^{T}y_{it}^h(\theta,\widehat{\alpha}_i)\log\Big(\Phi(x_{it}'\beta+\gamma_i)\Big)+(1-y_{it}^h(\theta,\widehat{\alpha}_i))\log\Big(1-\Phi(x_{it}'\beta+\gamma_i)\Big)$$

satisfies this property. Given θ, this individual log likelihood is an additive and multiplicative combinations of indicator functions of scalar $\widehat{\alpha}_i$ and smooth functions of (β,γ_i), who belong to classes of functions that satisfy stochastic equicontinuity (van der Vaart and Wellner, 1996). Therefore, its empirical process

$$\begin{aligned}\nu_T(\tau)=\frac{1}{T}\sum_{t=1}^{T}\Big[&y_{it}^h(\theta,\tau)\log\Big(\Phi(x_{it}'\beta+\gamma_i)\Big)+(1-y_{it}^h(\theta,\tau))\log\Big(1-\Phi(x_{it}'\beta+\gamma_i)\Big)\\&-\int_U y_{it}^h(\theta,\tau)\log\Big(\Phi(x_{it}'\beta+\gamma_i)\Big)+(1-y_{it}^h(\theta,\tau))\log\Big(1-\Phi(x_{it}'\beta+\gamma_i)\Big)dF_u\Big]\end{aligned}$$

is stochastic equicontinuous. Combined with uniform consistency result of $\widehat{\alpha}_i$'s and LLN of $\nu_T(\alpha_{i0})$, an application of the triangular inequality leads to the uniform approximation result. Now that (11) only has randomness from observed data, its uniform convergence to the limiting log likelihood (5) follows the argument as in Hahn and Kuersteiner (2011). As such, the pointwise convergence of $\widehat{\beta}^h(\theta)$ follows through[5].

To verify the stochastic equicontinuity condition (10), note that the profiled log likelihood

$$\widehat{Q}(\beta;\theta) = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} y_{it}^h(\theta,\widehat{\alpha}_i)\log\left(\Phi(x_{it}'\beta+\widehat{\gamma}_i)\right) + (1-y_{it}^h(\theta,\widehat{\alpha}_i))\log\left(1-\Phi(x_{it}'\beta+\widehat{\gamma}_i)\right)$$

is concave in β. By definition, $\widehat{\beta}^h(\theta_1)$ satisfies the first–order condition $\partial\widehat{Q}(\widehat{\beta}^h(\theta_1);\theta_1)/\partial\theta = 0$. A first–order Taylor expansion with respect to $\widehat{\beta}^h(\theta_1)$ around $\widehat{\beta}^h(\theta_2)$ shows that $\widehat{\beta}^h(\theta_1) - \widehat{\beta}^h(\theta_2)$ is bounded by $\left|\frac{\widehat{Q}(\widehat{\beta}^h(\theta_2);\theta_2)}{\partial\beta} - \frac{\partial\widehat{Q}(\widehat{\beta}(\theta_2);\theta_1)}{\partial\beta}\right|$, which, by Cauchy–Schwarz is bounded by the product of two terms: a smooth function of (β,γ_i) and

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}(y_{it}^h(\theta_1,\widehat{\alpha}_i) - y_{it}^h(\theta_2,\widehat{\alpha}_i)). \tag{12}$$

Therefore, it suffices to bound the two terms in expectation. The technical challenge mainly comes from proving this for equation (12). Although indicator functions are well–known to have controlled complexities (Andrews, 1994), and a similar result on the difference of indicator functions with univariate variable is given in Chen et al. (2003), the non–standard part here is that θ's are multi–dimensional. It turns out, however, that it satisfies the L^2–smoothness regularity condition. Therefore, the stochastic equicontinuity condition for $\widehat{\beta}^h(\theta)$ is verified.

Armed with Proposition 1, the consistency of $\widetilde{\theta}^H$ follows the arguments as in Gouriéroux et al. (1993). The proof is straightforward because there is no need to consider the weighting matrix.

Theorem 1 (Consistency of indirect fixed effects estimator)**.** *Under Assumptions 1–5,*

$$\widetilde{\theta}^H \xrightarrow{p} \theta_0.$$

4.3 Bias Correction and Asymptotic Normality

Recall that the indirect fixed effects estimator using H simulations $\widetilde{\theta}^H$ is the solution to $\widehat{\theta} = \widehat{\beta}^H(\widetilde{\theta}^H)$. Non–differentiability of $\widehat{\beta}^H(\theta)$ means that the techniques in the indirect inference literature (e.g., Gouriéroux et al., 1993) is not applicable. The following smoothness assumption is imposed.

Assumption 6. For all positive random sequences $\delta_{nT} \to 0$,

$$\sup_{\|\theta_1-\theta_2\|\le\delta_{nT}} \sqrt{nT}\|\widehat{\beta}^H(\theta_1) - \widehat{\beta}^H(\theta_2) - \mathbb{E}(\widehat{\beta}^H(\theta_1) - \widehat{\beta}^H(\theta_2))\| \xrightarrow{p} 0.$$

[5] Details are available in Lemma 2.

Assumption 6 imposes that the difference between $\widehat{\beta}^H(\theta_1)$ and $\widehat{\beta}^H(\theta_2)$ can be approximated by its expectation at a $\sqrt{nT}$ rate. Combined with consistency of $\widetilde{\theta}^H$, it allows to analyze the asymptotic normality of $\widetilde{\theta}^H$ through the lens of fixed effects estimators as follows:

$$\sqrt{nT}(\widetilde{\theta}^H - \theta_0) = \sqrt{nT}(\widehat{\theta} - \widetilde{\beta}^H(\theta_0)) + o_p(1). \tag{13}$$

Equation (9) characterizes the representation of $\widehat{\theta} - \theta_0$. Because the same regression is run on simulated data h and the likelihood is smooth in (β, γ_i), the same structure of representation arises, namely that

$$\widehat{\beta}^h(\theta_0) - \theta_0 = \frac{A^h(\theta_0, \widehat{\alpha}; \theta_0, \widehat{\alpha})}{\sqrt{nT}} + \frac{B^h(\theta_0, \widehat{\alpha}; \theta_0, \widehat{\alpha})}{T} + o_p\Big(\frac{1}{T}\Big). \tag{14}$$

The terms $A^h(\theta_0, \widehat{\alpha}; \theta_0, \widehat{\alpha})$ and $B^h(\theta_0, \widehat{\alpha}; \theta_0, \widehat{\alpha})$ reflects that the data are generated using θ_0, $\widehat{\alpha}$ and simulated unobservables $\{u_{it}^h\}$, and evaluated at θ_0 and $\widehat{\alpha}$. A combination of (9), (13) and (14) therefore leads to

$$\begin{aligned}\sqrt{nT}(\widetilde{\theta}^H - \theta_0) = &\Big(A(\theta_0, \alpha_0) - \frac{1}{H}\sum_{h=1}^{H} A^h(\theta_0, \widehat{\alpha}; \theta_0, \widehat{\alpha})\Big) \\ &+ \sqrt{\frac{n}{T}}\Big(B(\theta_0, \alpha_0) - \frac{1}{H}\sum_{h=1}^{H} B^h(\theta_0, \widehat{\alpha}; \theta_0, \widehat{\alpha})\Big) + o_p\Big(\sqrt{\frac{n}{T^3}}\Big).\end{aligned} \tag{15}$$

Bias Correction

For intuition of the proof, consider an infeasible fixed effects estimator $\widehat{\beta}^h(\theta_0, \alpha_0)$ obtained from data simulated by (θ_0, α_0). Then the representation of $\widehat{\beta}^h(\theta_0, \alpha_0) - \theta_0$ takes the form

$$\sqrt{nT}(\widehat{\beta}^h(\theta_0, \alpha_0) - \theta_0) = A^h(\theta_0, \alpha_0) + \sqrt{\frac{n}{T}} B^h(\theta_0, \alpha_0) + o_p\Big(\sqrt{\frac{n}{T^3}}\Big).$$

The bias correction property of this infeasible estimator then follows from the classic indirect inference. The terms $B(\theta_0, \alpha_0)$ and $B^h(\theta_0, \alpha_0)$ have the same probability limit and converge to the asymptotic bias $\mathbb{E}(B(\theta_0, \alpha_0))$.

Because the actual simulated data are generated by $\widehat{\alpha}$, it suffices to show that $B^h(\theta_0, \widehat{\alpha})$ uniformly well approximates $B^h(\theta_0, \alpha_0)$ at an appropriate rate. As such, the approximation error is negligible, and the indirect fixed effects estimator corrects the bias.

Under the independence assumption along the cross section,

$$B^h(\theta_0, \widehat{\alpha}) = -\Big[\frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_i(\theta_0, \widehat{\alpha}_i)\Big]^{-1} \frac{1}{n}\sum_{i=1}^{n} B_i^h(\theta_0, \widehat{\alpha}_i),$$

where $\mathcal{I}_i(\theta_0, \widehat{\alpha}_i)$ is individual i's information matrix, and it is a smooth function of all its arguments. Therefore, by continuous mapping theorem $\mathcal{I}_i(\theta_0, \widehat{\alpha}_i) \xrightarrow{p} \mathcal{I}_i(\theta_0, \alpha_{i0})$ for each i. Although each $B_i^h(\theta_0, \widehat{\alpha}_i)$ is nonsmooth in $\widehat{\alpha}_i$, uniform consistency of $\widehat{\alpha}_i$'s allows to establish another stochastic equicontinuity condition such that $B^h(\theta_0; \widehat{\alpha})$ replaces $B^h(\theta_0, \alpha_0)$ with negligible errors (Proposition 2), and the bias terms get cancelled out based on the classic treatment in indirect inference.

Proposition 2 (Bias correction of $\widetilde{\theta}^H$)**.** *Under Assumptions 1–6,*

$$|B^h(\theta_0, \widehat{\alpha}) - B^h(\theta_0, \alpha_0)| \xrightarrow{p} 0.$$

Asymptotic Normality

The same intuition could be applied to establishing the asymptotic normality: if $A^h(\theta_0, \widehat{\alpha}_i)$ can uniformly well approximate $A^h(\theta_0, \alpha_0)$, then the asymptotic normality result in indirect inference literature follows through. The technical challenge, however, is to ensure that the approximation error is decreasing at a sufficiently fast rate. More specifically, the independence assumption along the cross section implies that

$$A^h(\theta_0, \widehat{\alpha}) = \Big[\frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_i^h(\theta_0, \widehat{\alpha}_i)\Big]^{-1} \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T} A_{it}^h(\theta_0, \widehat{\alpha}_i),$$

where $A_{it}^h(\theta_0, \widehat{\alpha}_i)$ is a combination of high–order derivatives of the log likelihood. For panel probit models,

Theorem 2. *Under Assumptions 1–6,*

$$\sqrt{nT}\Big(\widetilde{\theta}^H - \theta_0\Big) \xrightarrow{d} \mathcal{N}\Big(0, \Omega + \frac{1}{H}\Omega\Big)$$

5 Application to Labor Force Participation

Research on the causal effect of fertility on female labor force participation is complicated by the presence of unobserved factors that affect both decisions. Following Hyslop (1999), this paper addresses this omitted variable issue by including individual fixed effects into the binary response panel probit model.

The data is a nine–year longitudinal sample from the Panel Study of Income Dynamics (PSID), spanning from 1979 to 1988. The sample includes women aged 18—60 in 1985 who were continuously married with husbands in the labor force in each of the sample periods. Of the 1461 women in the data, 664 changed their labor force participation statuses. Consider the following static specification:

$$y_{it} = \mathbf{1}\{X_{it}'\theta + \alpha_i > u_{it}\}, \quad u_{it} \sim \mathcal{N}(0,1),$$

where y_{it} denotes the labor force participation indicator for individual i at time t, and X_{it} denotes a vector of time–varying covariates. These covariates include numbers of children of at most 2 years of age, between 3 and 5 years of age, between 6 and 17 years of age; log of the husband's income[6], age and age squared. The individual effects α_i's are included to control for time–invariant unobserved heterogeneity such as unwillingness to work.

6 Monte Carlo Simulations

This section considers Monte Carlo simulations calibrated to the same PSID data. The details of calibration procedures are available in Appendix C.1. The indirect inference fixed effects estimator is compared with MLE, analytical bias correction and jackknife bias correction.

All simulations are done 500 times and H is set to 10. All statistics are relative to the true parameters and multiplied by 100. Tables (1) and (2) tabulate the results for fixed effects and indirect fixed effects estimators for the probit coefficients. Fixed effects estimators are subject to a bias that is of the same order of magnitude as the standard deviation. This leads to severe under–coverage of the confidence intervals. The indirect fixed effects estimators, on the other hand, reduce bias by a wide margin without too much inflation in the standard deviation.

Table 1: FE Estimators for Static LFP

	kids0_2	kids3_5	kids6_17	loghusinc	age	age2
Bias	14.93	14.67	12.44	13.86	13.35	13.45
Std Dev	10.02	14.11	36.82	25.63	19.08	20.68
RMSE	17.98	20.35	38.83	29.12	23.27	24.65

[6]This variable serves as a proxy for permanent nonlabor income (Hyslop, 1999).

Table 2: Indirect–FE Estimators for Static LFP

	kids0_2	kids3_5	kids6_17	loghusinc	age	age2
Bias	-4.92	-7.71	-19.67	-5.35	0.03	-1.08
Std Dev	8.62	13.26	42.67	29.03	8.16	22.19
RMSE	9.92	15.32	46.95	29.49	8.15	22.19

7 Conclusion

Fixed effects estimation for nonlinear panel models is subject to large bias of point estimates and incorrect coverage of confidence intervals. This paper proposes an indirect fixed effects estimator that reduces the bias and obtains standard errors without bootstrap. The current theory is restricted to strictly exogenous explanatory variables, but Monte Carlo simulations in Appendix C.2 shows that the method can accommodate lagged dependent variables as well. Naturally, the next step is to extend the current theory to allow for dynamics in the DGP.

There are at least four other questions for further explorations. First, average partial effects are often the quantities of interest in nonlinear models. This paper establishes theoretical properties of finite dimensional parameters, and it could be interesting to explore if they can be extended to handle average partial effects, which is a function of explanatory variables, parameters of interest and incidental parameters.

Second, this paper directly works with non–smooth log likelihood function and establishes the asymptotic properties of the estimator. However, a practical concern of non–smoothness is that gradient–based optimization schemes cannot be used for estimation, and gradient–free schemes like Nelder–Mead face computational difficulty in high–dimensional problems. The indirect fixed effects estimator might benefit from smoothing approaches like kernel smoothing, but the theoretical justification can be nontrivial as smoothing can introduce an extra bias.

Third, measurement error is common in panel data (Meijer et al., 2015). Fixed effect estimation in linear panel models suffers from attenuation bias when explanatory variables are mismeasured (Griliches and Hausman, 1986), and a recent paper by Evdokimov and Zeleneev (2020) extends this result to nonlinear panel models considered in this paper. Indirect inference holds out the promise of automatic bias correction when the same regression is run on observed and simulated data. However, current practices fully specify individual effects and measurement error (e.g., Guvenen and Smith, 2014), which might lead to a concern of misspecification. It would be interesting to investigate if the indirect fixed effects estimator can be extended to handle measurement error without having to impose a particular structure.

Finally, incorporating unobserved heterogeneity into dynamic discrete choice (DDC) models is an active area of research. One popular approach treats unobserved heterogeneity as an unobserved state variable and assumes individuals can be categorized into a finite number of types (Kasahara and Shimotsu, 2009; Arcidiacono and Miller, 2011). Introducing fixed effects circumvents the need to take a stand on the number of types, but can potentially complicate identification and estimation: the individual effects show up in both the current payoff and the continuation value, the latter of which has to be solved using a fixed–point algorithm. It would be exciting to investigate whether some of the ideas in this paper can be applied to incorporate fixed effects into DDC models.

References

Aghion, P., N. Bloom, R. Blundell, R. Griffith, and P. Howitt (2005): "Competition and Innovation: an Inverted-U Relationship," *The Quarterly Journal of Economics*, 120, 701–728.

Altonji, J. G., A. A. Smith, and I. Vidangos (2013): "Modeling Earnings Dynamics," *Econometrica*, 81, 1395–1454.

Andersen, E. B. (1970): "Asymptotic Properties of Conditional Maximum–Likelihood Estimators," *Journal of the Royal Statistical Society. Series B (Methodological)*, 32, 283–301.

Andrews, D. W. (1994): "Empirical Process Methods in Econometrics," in *Handbook of Econometrics*, ed. by J. J. Heckman and E. Leamer, Elsevier, vol. 4, chap. 37, 2247–2294.

Arcidiacono, P. and R. A. Miller (2011): "Conditional Choice Probability Estimation of Dynamic Discrete Choice Models With Unobserved Heterogeneity," *Econometrica*, 79, 1823–1867.

Arellano, M. and J. Hahn (2007): "Understanding Bias in Nonlinear Panel Models: Some Recent Developments," in *Advances in Economics and Econometrics, Theory and Applications, Ninth World Congress*, ed. by R. Blundell, W. Newey, and T. Persson, Cambridge University Press, chap. 12, 381–409.

Berger, D. and J. Vavra (2019): "Shocks versus Responsiveness: What Drives Time-Varying Dispersion?" *Journal of Political Economy*, 127, 2104–2142.

Bruins, M., J. A. Duffy, M. P. Keane, and A. A. Smith (2018): "Generalized Indirect Inference for Discrete Choice Models," *Journal of Econometrics*, 205, 177–203.

Chamberlain, G. (1984): "Panel Data," in *Handbook of Econometrics*, ed. by J. Griliches and M. Intriligator, Elsevier, vol. 2, chap. 22, 1247–1318.

Chen, X., O. Linton, and I. Van Keilegom (2003): "Estimation of Semiparametric Models when the Criterion Function is not Smooth," *Econometrica*, 71, 1591–1608.

Collard-Wexler, A. (2013): "Demand Fluctuations in the Ready-Mix Concrete Industry," *Econometrica*, 81, 1003–1037.

Dedecker, J. and S. Louhichi (2002): "Maximal Inequalities and Empirical Central Limit Theorem," in *Empirical Process Techniques for Dependent Data*, ed. by H. Dehling, T. Mikosch, and M. Sørensen, Springer, vol. 7, chap. 6, 487–565.

Dhaene, G. and K. Jochmans (2015): "Split–Panel Jackknife Estimation of Fixed-effect Models," *The Review of Economic Studies*, 82, 991—-1030.

Dridi, R., A. Guay, and E. Renault (2007): "Indirect Inference and Calibration of Dynamic Stochastic General Equilibrium Models," *Journal of Econometrics*, 136, 397–430.

Dridi, R. and E. Renault (2000): "Semi–Parametric Indirect Inference," Unpublished manuscript.

Duffie, D. and K. J. Singleton (1993): "Simulated Moments Estimation of Markov Models of Asset Prices," *Econometrica*, 61, 929–952.

Evdokimov, K. S. and A. Zeleneev (2020): "Errors–In–Variables in Large Nonlinear Panel and Network Models," Working paper.

Fernández-Val, I. (2009): "Fixed Effects Estimation of Structural Parameters and Marginal Effects in Panel Probit Models," *Journal of Econometrics*, 150, 71–85.

Fernández-Val, I. and M. Weidner (2016): "Individual and time effects in nonlinear panel models with large N, T," *Journal of Economics*, 192, 291–312.

——— (2018): "Fixed Effects Estimation of Large–T Panel Data Models," *Annual Review of Economics*, 10, 109–138.

Forneron, J.-J. (2020): "A Sieve–SMM Estimator for Dynamic Models," Working paper.

Frazier, D. T., T. Oka, and D. Zhu (2019): "Indirect Inference with a Non–Smooth Criterion Function," *Journal of Econometrics*, 212, 623–645.

Gallant, R. A. and G. Tauchen (1996): "Which Moments to Match?" *Econometric Theory*, 12, 657–681.

Galvao, A. F. and K. Kato (2018): "Quantile Regression Methods for Longitudinal Data," in *Handbook of Quantile Regression*, ed. by R. Koenker, V. Chernozhukov, X. He, and L. Peng, Chapman and Hall/CRC, chap. 19, 363–380.

Glasserman, P. and D. D. Yao (1992): "Some Guidelines and Guarantees for Common Random Numbers," *Management Science*, 38, 884–908.

Gospodinov, N., I. Komunjer, and S. Ng (2017): "Simulated Minimum Distance Estimation of Dynamic Models with Errors-in-Variables," *Journal of Econometrics*, 200, 181–193.

Gouriéroux, C. and A. Monfort (1997): *Simulation-based Econometric Methods*, Oxford University Press.

Gouriéroux, C., A. Monfort, and E. Renault (1993): “Indirect Inference,” *Journal of Applied Econometrics*, 8, 85–118.

Gouriéroux, C., P. Phillips, and J. Yu (2010): “Indirect Inference for Dynamic Panel Models,” *Journal of Econometrics*, 157, 68–77.

Griliches, Z. and J. A. Hausman (1986): “Errors in Variables in Panel Data,” *Journal of Econometrics*, 31, 93–118.

Guvenen, F. and A. A. Smith (2014): “Inferring Labor Income Risk and Partial Insurance from Economic Choices,” *Econometrica*, 82, 2085–2129.

Hahn, J. and G. Kuersteiner (2002): “Asymptotically Unbiased Inference for a Dynamic Panel Model with Fixed Effects When Both n and T Are Large,” *Econometrica*, 70, 1639–1657.

——— (2011): “Bias Reduction for Dynamic Nonlinear Panel Models with Fixed Effects,” *Econometric Theory*, 27, 1152 – 1191.

Hahn, J. and W. Newey (2004): “Jackknife and Analytical Bias Reduction for Nonlinear Panel Models,” *Econometrica*, 72, 1295–1319.

Helpman, E., M. Melitz, and Y. Rubinstein (2008): “Estimating Trade Flows: Trading Partners and Trading Volumes,” *The Quarterly Journal of Economics*, 123, 441–487.

Hyslop, D. R. (1999): “State Dependence, Serial Correlation and Heterogeneity in Intertemporal Labor Force Participation of Married Women,” *Econometrica*, 67, 1255–1294.

Kasahara, H. and K. Shimotsu (2009): “Nonparametric Identification of Finite Mixture Models of Dynamic Discrete Choices,” *Econometrica*, 77, 135–175.

Kato, K., A. F. Galvao Jr., and G. V. Montes-Rojas (2012): “Asymptotics for Panel Quantile Regression Models with Individual Effects,” *Journal of Econometrics*, 170, 76–91.

Kim, M. S. and Y. Sun (2016): “Bootstrap and k-step Bootstrap Bias Corrections for the Fixed Effects Estimator in Nonlinear Panel Data Models,” *Econometric Theory*, 32, 1523–1568.

Lancaster, T. (2000): “The Incidental Parameter Problem Since 1948,” *Journal of Econometrics*, 95, 391–413.

Lee, B.-S. and B. F. Ingram (1991): “Simulation estimation of time-series models,” *Journal of Econometrics*, 47, 197–205.

Manski, C. and S. Lerman (1981): "On the Use of Simulated Frequencies to Approximate Choice Probabilities," in *Structural Analysis of Discrete Data with Econometric Applications*, ed. by C. Manski and D. McFadden, MIT Press, chap. 7, 2–50.

McFadden, D. (1989): "A Method of Simulated Moments for Estimation of Discrete Response Models Without Numerical Integration," *Econometrica*, 57, 995–1026.

Meijer, E., L. Spierdijk, and T. J. Wansbeek (2015): "Measurement Error in Panel Data," in *Oxford Handbook of Panel Data*, ed. by B. H. Baltagi, Oxford University Press, vol. 4, chap. 11, 325–362.

Newey, W. K. (1991): "Uniform Convergence in Probability and Stochastic Equicontinuity," *Econometrica*, 59, 1161–1167.

Newey, W. K. and D. McFadden (1994): "Large Sample Estimation and Hypothesis Testing," in *Handbook of Econometrics*, ed. by J. J. Heckman and E. Leamer, Elsevier, vol. 4, chap. 36, 2111–2245.

Neyman, J. and E. L. Scott (1948): "Consistent Estimates Based on Partially Consistent Observations," *Econometrica*, 5, 999–1033.

Nickell, S. (1981): "Biases in Dynamic Models with Fixed Effects," *Econometrica*, 49, 874–897.

Pakes, A. and D. Pollard (1989): "Simulation and the Asymptotics of Optimization Estimators," *Econometrica*, 57, 1027–1057.

Schennach, S. M. (2014): "Entropic Latent Variable Integration via Simulation," *Econometrica*, 82, 345–385.

Smith, A. A. (1993): "Estimating Nonlinear Time–Series Models Using Simulated Vector Autoregressions," *Journal of Applied Econometrics*, 8, 63–84.

Taber, C. and R. M. Sauer (2021): "Understanding Women's Wage Growth using Indirect Inference with Importance Sampling," *Journal of Applied Econometrics.*

van der Vaart, A. W. (2000): *Asymptotic Statistics*, Cambridge University Press.

van der Vaart, A. W. and J. A. Wellner (1996): *Weak Convergence and Empirical Processes: With Applications to Statistics*, Springer Series of Statistics.

A Auxiliary Results

A.1 Proof of Lemma 1

Lemma 1 (Uniform Convergence of Sample Criterion Function using Simulated Data)**.**

$$\max_{1\le i\le n}\sup_{(\beta,\gamma)}\left|\widehat{G}^h_{(i)}(\beta,\gamma)-G_{(i)}(\beta,\gamma)\right|\xrightarrow{p}0,$$

where

$$\begin{aligned}
\widehat{G}^h_{(i)}(\beta,\gamma) &= \frac{1}{T}\sum_{t=1}^{T}\ln f(y^h_{it}(\theta,\widehat{\alpha}_i)\mid x_{it};\beta,\gamma);\\
G_{(i)}(\beta,\gamma) &= \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\ln f(y_{it}(\theta,\alpha_{i0})\mid x_{it};\beta,\gamma).
\end{aligned}$$

Proof. The proof consists of two main steps. The first step deals with $\widehat{\alpha}_i$'s in data simulation and shows that $\widehat{G}^h_{(i)}$ is uniformly close to a criterion that uses α_{i0} to simulate the data, i.e.,

$$\widetilde{G}_{(i)}(\beta,\gamma)=\frac{1}{T}\sum_{t=1}^{T}\int_U \ln f(y_{it}(\theta,\alpha_{i0})\mid x_{it};\beta,\gamma)dF(u).$$

The second step is a uniform law of large number results showing that $\widetilde{G}_{(i)}(\beta,\gamma)$ uniformly converges to $G_{(i)}(\beta,\gamma)$.

Step 1: Given θ and a scalar τ, note that

$$\frac{1}{T}\sum_{t=1}^{T}\ln f(y^h_{it}(\theta,\tau)\mid x_{it};\beta,\gamma):=\frac{1}{T}\sum_{t=1}^{T}y^h_{it}(\theta,\tau)\ln\Phi(x'_{it}\beta+\gamma)+(1-y^h_{it}(\theta,\tau))\ln(1-\Phi(x'_{it}\beta+\gamma))$$

consists of two components: (1) an indicator function of scalar τ and (2) a smooth, bounded and monotone function of (β,γ). The indicator function $y^h_{it}(\theta,\widehat{\alpha}_i)$ belongs to type I class of Andrews (1994), which satisfies Pollard's entropy condition. The second component belongs to a class of functions satisfying bracketing entropy condition (van der Vaart and Wellner, 1996, Section 2.7.2).

Because $\frac{1}{T}\sum_{t=1}^{T}\ln f(y^h_{it}(\theta,\tau)\mid x_{it};\beta,\gamma)$ is an additive and multiplicative combination of the two classes of components, its function class also satisfies the entropy condition (Andrews, 1994), which is the primitive condition for stochastic equicontinuity. More specifically, define the following empirical process:

$$\nu_T(\tau) = \frac{1}{T}\sum_{t=1}^{T}\Big[\ln f(y_{it}^h(\theta,\tau)\mid x_{it};\beta,\gamma) - \int_U \ln f(y_{it}^h(\theta,\tau)\mid x_{it};\beta,\gamma)dF_u\Big],$$

where the integration is over the known distribution of simulation draws. By one of the equivalnent definitions of stochastic equicontinuity (i.e., Andrews, 1994, p.2252), the following condition holds: for every sequence of constants $\{\delta_T\}$ that converges to zero,

$$\sup_{(\beta,\gamma),|\tau_1-\tau_2|\le\delta_T}\sqrt{T}|\nu_T(\tau_1)-\nu_T(\tau_2)| \xrightarrow{p} 0. \tag{A.1}$$

A first–order Taylor expansion on $\int_U \ln f(y_{it}^h(\theta,\alpha_{i0})\mid x_{it};\beta,\gamma)$ with respect to α_{i0} around $\widehat{\alpha}_i$ yields

$$\begin{aligned}\int_U \ln f(y_{it}^h(\theta,\alpha_{i0})\mid x_{it};\beta,\gamma)dF_u = \int_U &\ln f(y_{it}^h(\theta,\widehat{\alpha}_i)\mid x_{it};\beta,\gamma)dF_u\\ &+\frac{\partial\int_U \ln f(y_{it}^h(\theta,\overline{\alpha}_i)\mid x_{it};\beta,\gamma)dF_u}{\partial\alpha_i}(\widehat{\alpha}_i-\alpha_{i0}).\end{aligned}$$

Combined with condition (A.1),

$$\begin{aligned}&\sqrt{T}\Big|\frac{1}{T}\sum_{t=1}^{T}\Big[\ln f(y_{it}^h(\theta,\widehat{\alpha}_i)\mid x_{it};\beta,\gamma) - \int_U \ln f(y_{it}^h(\theta,\alpha_{i0})\mid x_{it};\beta,\gamma)dF_u\Big]\Big|\\
&= \sqrt{T}\Big|\nu_T(\widehat{\alpha}_i) - \frac{\partial\int_U \ln f(y_{it}^h(\theta,\overline{\alpha}_i)\mid x_{it};\beta,\gamma)dF_u}{\partial\alpha_i}(\widehat{\alpha}_i-\alpha_{i0})]\Big|\\
&\le \sqrt{T}|\nu_T(\widehat{\alpha}_i)| + \sqrt{T}\Big|\frac{1}{T}\sum_{t=1}^{T}\frac{\partial\int_U \ln f(y_{it}^h(\theta,\overline{\alpha}_i)\mid x_{it};\beta,\gamma)dF_u}{\partial\alpha_i}(\widehat{\alpha}_i-\alpha_{i0})\Big|\\
&= \sqrt{T}|\nu_T(\alpha_{i0})+\nu_T(\widehat{\alpha}_i)-\nu_T(\alpha_{i0})| + \sqrt{T}\Big|\frac{1}{T}\sum_{t=1}^{T}\frac{\partial\int_U \ln f(y_{it}^h(\theta,\overline{\alpha}_i)\mid x_{it};\beta,\gamma)dF_u}{\partial\alpha_i}(\widehat{\alpha}_i-\alpha_{i0})\Big|\\
&\le \sqrt{T}|\nu_T(\alpha_{i0})| + \sqrt{T}|\nu_T(\widehat{\alpha}_i)-\nu_T(\alpha_{i0})| + \sqrt{T}\Big|\frac{1}{T}\sum_{t=1}^{T}\frac{\partial\int_U \ln f(y_{it}^h(\theta,\overline{\alpha}_i)\mid x_{it};\beta,\gamma)dF_u}{\partial\alpha_i}\Big|\cdot|\widehat{\alpha}_i-\alpha_{i0}|.\end{aligned}$$

where the third and last lines are due to triangular inequality. Because $\nu_T(\alpha_{i0})$ is a normalized sum of mean zero random variables, $\nu_T(\alpha_{i0}) \xrightarrow{p} 0$ by LLN. The second term is the stochastic equicontinuity condition in Eq. (A.1). Because the derivative is bounded by Assumption 4 and $\max_{1\le i\le n}|\widehat{\alpha}_i-\alpha_{i0}| = o_p(1)$ (Hahn and Kuersteiner, 2011, Theorem 4), the third term is thus $o_p(1)$.

Therefore

$$\sqrt{T}\Big|\frac{1}{T}\sum_{t=1}^{T}[\ln f(y_{it}^{h}(\theta,\widehat{\alpha}_i)\mid x_{it};\beta,\gamma)-\int_U \ln f(y_{it}^{h}(\theta,\alpha_{i0})\mid x_{it};\beta,\gamma)dF_u]\Big|\xrightarrow{p}0.$$

Step 2: The second part of the proof shows that

$$\max_{1\le i\le n}\sup_{(\beta,\gamma)}\Big|\widetilde{G}_{(i)}(\beta,\gamma)-G_{(i)}(\beta,\gamma)\Big|\xrightarrow{p}0.$$

Following the the proof structure of Lemma 4 in Hahn and Kuersteiner (2011), note that

$$P\Big[\max_{1\le i\le n}\sup_{(\beta,\gamma)}\Big|\widetilde{G}_{(i)}(\beta,\gamma)-G_{(i)}(\beta,\gamma)\Big|\ge\eta\Big]\le\sum_{i=1}^{n}P\Big[\sup_{(\beta,\gamma)}\Big|\widetilde{G}_{(i)}(\beta,\gamma)-G_{(i)}(\beta,\gamma)\Big|\ge\eta\Big].$$

Since the parameter space is compact, it suffices to show that

$$\sup_{\Gamma_j}\Big|\widetilde{G}_{(i)}(\beta,\gamma)-G_{(i)}(\beta,\gamma)\Big|\to 0,$$

where Γ_j is a subset of $\mathcal{B}\times\Gamma_\gamma$ such that $\|\beta-\beta'\|\le\varepsilon$ and $|\gamma-\gamma'|\le\varepsilon$ for (β,γ) and $(\beta',\gamma')\in\Gamma_j$. By Assumption 4 on $G_{(i)}$,

$$\Big|G_{(i)}(\beta,\gamma)-G_{(i)}^{h}(\beta',\gamma')\Big|\le\mathbb{E}M(z_{it})|(\beta,\gamma)-(\beta',\gamma')|<\varepsilon\mathbb{E}M(z_{it}),$$
$$\Big|\widetilde{G}_{(i)}(\beta,\gamma)-\widetilde{G}_{(i)}(\beta',\gamma')\Big|\le\frac{1}{T}\sum_{t=1}^{T}M(z_{it})|(\beta,\gamma)-(\beta',\gamma')|<\frac{\varepsilon}{T}\sum_{t=1}^{T}M(z_{it}).$$

By the triangular inequality,

$$\begin{aligned}
&\Big|\widetilde{G}_{(i)}(\beta,\gamma)-G_{(i)}(\beta,\gamma)\Big|-\Big|\widetilde{G}_{(i)}(\beta',\gamma')-G_{(i)}(\beta',\gamma')\Big|\\
&\qquad\le\Big|\Big(\widetilde{G}_{(i)}(\beta,\gamma)-\widetilde{G}_{(i)}(\beta',\gamma')\Big)-\Big(G_{(i)}(\beta,\gamma)-G_{(i)}(\beta',\gamma')\Big)\Big|\\
&\qquad\le\Big|\widetilde{G}_{(i)}(\beta,\gamma)-\widetilde{G}_{(i)}(\beta',\gamma')\Big|+\Big|G_{(i)}(\beta,\gamma)-G_{(i)}(\beta',\gamma')\Big|\\
&\qquad<\varepsilon\mathbb{E}M(z_{it})+\frac{\varepsilon}{T}\sum_{t=1}^{T}M(z_{it})\\
&\qquad=\frac{\varepsilon}{T}\Big(\sum_{t=1}^{T}M(z_{it})-\mathbb{E}M(z_{it})\Big)+\frac{\varepsilon}{T}\mathbb{E}M(z_{it})+\varepsilon\mathbb{E}M(z_{it})\\
&\qquad<\frac{\varepsilon}{T}\Big|\sum_{t=1}^{T}M(z_{it})-\mathbb{E}M(z_{it})\Big|+2\varepsilon\mathbb{E}M(z_{it}).
\end{aligned}$$

Therefore by a rearrangement of the terms,

$$\left|\widetilde{G}_{(i)}(\beta,\gamma) - G_{(i)}(\beta,\gamma)\right| \leq \left|\widetilde{G}_{(i)}(\beta',\gamma') - G_{(i)}(\beta',\gamma')\right| + \frac{\varepsilon}{T}\left|\sum_{t=1}^{T} M(x_{it}) - \mathbb{E}M(x_{it})\right| + 2\varepsilon\mathbb{E}M(x_{it}).$$

Let ε be such that $2\varepsilon \max_i \mathbb{E}M(z_{it}) < \frac{\eta}{3}$, then

$$\begin{aligned}
&P\left[\sup_{\Gamma_j}\left|\widetilde{G}_{(i)}(\beta,\gamma) - G_{(i)}(\beta,\gamma)\right| > \eta\right] \\
&\leq P\left[\left|\widetilde{G}_{(i)}(\beta',\gamma') - G_{(i)}(\beta',\gamma')\right| > \frac{\eta}{3}\right] + P\left[\frac{1}{T}\left|\sum_{t=1}^{T} M(x_{it}) - \mathbb{E}M(x_{it})\right| > \frac{\eta}{3\varepsilon}\right] + P\left[2\varepsilon\mathbb{E}M(x_{it}) > \frac{\eta}{3}\right] \\
&= o(T^{-2}),
\end{aligned}$$

where the last line follows as the first two terms on the right–hand side are $o(T^{-2})$ by Lemma 1 in Hahn and Kuersteiner (2011) and the last term is of probability zero by construction. Since $n = O(T)$,

$$\begin{aligned}
&P\left[\max_{1\leq i\leq n}\sup_{(\beta,\gamma)}\left|\widetilde{G}_{(i)}(\beta,\gamma) - G_{(i)}(\beta,\gamma)\right| \geq \eta\right] \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{m(\varepsilon)} P\left[\sup_{\Gamma_j}\left|\widetilde{G}_{(i)}(\beta,\gamma) - G_{(i)}(\beta,\gamma)\right| \geq \eta\right] \\
&= o(T^{-1})
\end{aligned}$$

□

A.2 Proof of Lemma 2

Lemma 2 (Pointwise Consistency of Auxiliary Estimator in the Simulation World)**.** $\forall\theta \in \Theta$,

$$\widehat{\beta}^h(\theta) \xrightarrow{p} \beta(\theta),$$

where $\beta(\theta) = \theta$.

Proof. The proof structure follows from that for Theorem 3 in Hahn and Kuersteiner (2011), with minor modification of notations. Fix $\eta > 0$ and set

$$\varepsilon = \inf_i \left[G_{(i)}(\theta, \alpha_{i0}) - \sup_{\{(\beta,\gamma):\|(\beta,\gamma)-(\theta,\alpha_{i0})\|>\eta\}} G_{(i)}(\beta, \gamma) \right] > 0$$

With probability $1 - o(T^{-1})$,

$$\begin{aligned}
\max_{\|\beta-\theta\|>\eta,\gamma_1,\ldots,\gamma_n} \frac{1}{n}\sum_{i=1}^{n} \widehat{G}^h_{(i)}(\beta,\gamma_i) &\leq \max_{\|(\beta,\gamma_i)-(\theta,\alpha_{i0})\|>\eta} \frac{1}{n}\sum_{i=1}^{n} \widehat{G}^h_{(i)}(\beta,\gamma_i) \\
&\leq \max_{\|(\beta,\gamma_i)-(\theta,\alpha_{i0})\|>\eta} \frac{1}{n}\sum_{i=1}^{n} G_{(i)}(\beta,\gamma_i) + \frac{1}{3}\varepsilon \\
&< \frac{1}{n}\sum_{i=1}^{n} G_{(i)}(\theta,\alpha_{i0}) - \frac{2}{3}\varepsilon \\
&< \frac{1}{n}\sum_{i=1}^{n} \widehat{G}^h_{(i)}(\theta,\alpha_{i0}) - \frac{1}{3}\varepsilon,
\end{aligned}$$

where the second and last inequalities are due to Lemma 1. By definition

$$\max_{\beta,\gamma_1,\ldots,\gamma_n} \frac{1}{n}\sum_{i=1}^{n} \widehat{G}^h_{(i)}(\beta,\gamma_i) \geq \frac{1}{n}\sum_{i=1}^{n} G^h_{(i)}(\theta,\alpha_{i0}).$$

Hence

$$P\left[\|\widehat{\beta}^h(\theta) - \beta(\theta)\| \geq \eta \right] = o(T^{-1}).$$

□

B Proofs of Main Results

B.1 Proof of Proposition 1 (Uniform Consistency)

Proof. The main structure of the proof follows Theorem 1 in Newey (1991). The parameter space Θ is compact by assumption. The limiting function $\beta(\theta)$ is continuous since it is an identity function. Lemma 2 establishes the pointwise convergence result using simulated data: $\forall\theta \in \Theta$, $\widehat{\beta}^h(\theta) \xrightarrow{p} \beta(\theta)$. Therefore, it suffices to prove that $\widehat{\beta}^h(\theta)$ is stochastic equicontinuous.

By Markov inequality, $\forall \eta > 0$,

$$Pr\Big(\sup_{\theta\in\Theta}\|\widehat{\beta}^h(\theta) - \beta(\theta)\| > \eta\Big) \leq \frac{1}{\eta}\mathbb{E}\Big(\sup_{\theta\in\Theta}\|\widehat{\beta}^h(\theta) - \beta(\theta)\|\Big).$$

Combined with the compactness assumption, it suffices to show that

$$\mathbb{E}\Big(\sup_{\|\theta_1-\theta_2\|\leq\delta}\|\widehat{\beta}^h(\theta_1) - \widehat{\beta}^h(\theta_2)\|\Big) \leq C\delta, \tag{B.1}$$

where δ denotes a positive scalar that is arbitrarily small and C is a constant. The rest of the proof consists of three parts. Firstly, a representation of $\widehat{\beta}^h(\theta_1) - \widehat{\theta}^h(\theta_2)$ in terms of profiled likelihood is established. Then, the question is transformed to bounding terms related to components of profiled likelihood. Lastly, difference pieces are glued together to give an expression of C.

Step 1: Let $\widehat{Q}(\widehat{\beta}^h(\theta); \theta)$ denote the profiled log likelihood function using simulated data h,

$$\widehat{Q}(\beta;\theta) = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} y_{it}^h(\theta,\widehat{\alpha}_i)\ln\Big(\Phi(x_{it}'\beta + \widehat{\gamma}_i(\beta(\theta))\Big) + (1-y_{it}^h(\theta,\widehat{\alpha}_i))\ln\Big(1-\Phi(x_{it}'\beta+\widehat{\gamma}_i(\beta(\theta)))\Big) \tag{B.2}$$

Then by definition, $\widehat{\beta}^h(\theta_1)$ and $\widehat{\beta}^h(\theta_1)$ satisfy the first-order conditions,

$$\frac{\partial\widehat{Q}(\widehat{\beta}^h(\theta_1);\theta_1)}{\partial\beta} = 0, \quad \frac{\partial\widehat{Q}(\widehat{\beta}^h(\theta_2);\theta_2)}{\partial\beta} = 0$$

A first-order Taylor expansion yields

$$\frac{\partial\widehat{Q}(\widehat{\beta}^h(\theta_1);\theta_1)}{\partial\beta} = 0 = \frac{\partial\widehat{Q}(\widehat{\beta}^h(\theta_2);\theta_1)}{\partial\beta} + \frac{\partial^2\widehat{Q}(\widetilde{\beta};\theta_1)}{\partial\beta\partial\beta'}(\widehat{\beta}^h(\theta_1) - \widehat{\beta}^h(\theta_2)),$$

where $\widetilde{\beta}$ is between $\widehat{\beta}^h(\theta_1)$ and $\widehat{\beta}^h(\theta_2)$. Therefore,

$$\frac{\partial^2 \widehat{Q}(\widetilde{\beta};\theta_1)}{\partial\beta\partial\beta'}(\widehat{\beta}^h(\theta_1)-\widehat{\beta}^h(\theta_2)) = \frac{\partial \widehat{Q}(\widehat{\beta}^h(\theta_2);\theta_2)}{\partial\beta} - \frac{\partial \widehat{Q}(\widehat{\beta}^h(\theta_2);\theta_1)}{\partial\beta}.$$

Let λ_s denote the smallest eigenvalue of the Hessian of the profiled likelihood, then a quadratic inequality leads to

$$\lambda_s\|\widehat{\beta}^h(\theta_1)-\widehat{\beta}^h(\theta_2)\| \leq \Big|\frac{\partial \widehat{Q}(\widehat{\beta}^h(\theta_2);\theta_2)}{\partial\beta} - \frac{\partial \widehat{Q}(\widehat{\beta}^h(\theta_2);\theta_1)}{\partial\beta}\Big|.$$

For binary response panel probit models, some algebra leads to the following expression of the right–hand–side term in the absolute sign,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\Big(y_{it}^h(\theta_1,\widehat{\alpha}_i)-y_{it}^h(\theta_2,\widehat{\alpha}_i)\Big)\Big(\frac{\phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))(x_{it}+\frac{\partial\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))}{\partial\beta})}{\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))[1-\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))]}\Big), \tag{B.3}$$

where $y_{it}^h(\theta) = \mathbf{1}\{x_{it}'\theta+\widehat{\alpha}_i \geq u_{it}^h\}$ and u_{it}^h is from the standard normal distribution. Therefore, to establish Condition (B.1), it suffices to focus on Eq. (B.3).

Step 2: By Cauchy–Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\|\theta_1-\theta_2\|\leq\delta}\Big|\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\Big(y_{it}^h(\theta_1,\widehat{\alpha}_i)-y_{it}^h(\theta_2,\widehat{\alpha}_i)\Big)\Big(\frac{\phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))(x_{it}+\frac{\partial\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))}{\partial\beta})}{\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))[1-\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))]}\Big)\Big|\Big)\\
&\leq \sqrt{\mathbb{E}\Big(\sup_{\|\theta_1-\theta_2\|\leq\delta}\Big|\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}y_{it}^h(\theta_1,\widehat{\alpha}_i)-y_{it}^h(\theta_2,\widehat{\alpha}_i)\Big|^2\Big)}\times\\
&\sqrt{\mathbb{E}\Big(\Big|\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))(x_{it}+\frac{\partial\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))}{\partial\beta})}{\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))[1-\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))]}\Big|^2\Big)}.
\end{aligned}$$

For each i and t, the following two L^2–smoothness conditions hold:

$$\sqrt{\mathbb{E}\Big(\sup_{\|\theta_1-\theta_2\|\leq\delta}|y_{it}^h(\theta_1,\widehat{\alpha}_i)-y_{it}^h(\theta_2,\widehat{\alpha}_i)|^2\Big)} \leq \sqrt{\frac{\mathbb{E}\|x_{it}\|_2}{\sqrt{2\pi}}}\sqrt{\delta}, \tag{B.4}$$

$$\sqrt{\mathbb{E}\Big(\Big|\frac{\phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))(x_{it}+\frac{\partial\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))}{\beta})}{\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))[1-\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))]}\Big|^2\Big)} \leq K_2, \tag{B.5}$$

where $\|x\|_2$ denotes the L_2-norm $\sqrt{\mathbb{E}|x|^2}$. This corresponds to type IV class in Andrews (1994).

Proving condition (B.4): Denote $\Delta\theta := \theta_2 - \theta_1$ and note that

$$\sup_{\|\theta_1-\theta_2\|\le\delta} |y_{it}^h(\theta_1) - y_{it}^h(\theta_2)| = \sup_{\|\Delta\theta\|\le\delta} |\mathbf{1}\{x_{it}'\theta_1 + \widehat{\alpha}_i \ge u_{it}^h\} - \mathbf{1}\{x_{it}'(\theta_1 + \Delta\theta) + \widehat{\alpha}_i \ge u_{it}^h\}|.$$

The direction that obtains the supremum is given by

$$\Delta\theta = \pm\frac{\delta}{\|x_{it}\|_2} x_{it}.$$

Therefore

$$\mathbb{E}\Big[\sup_{\|\theta_1-\theta_2\|\le\delta} |y_{it}^h(\theta_1) - y_{it}^h(\theta_2)|\Big] \le \mathbb{E}\Big(\mathbf{1}\{x_{it}'\theta_1 + \widehat{\alpha}_i \ge u_{it}^h\} - \mathbf{1}\{x_{it}'\theta_1 - \|x_{it}\|_2\delta + \widehat{\alpha}_i \ge u_{it}^h\}\Big). \tag{B.6}$$

Because δ is a scalar, a proof strategy à la Chen et al. (2003) is employed to bound right–hand–side term in Equation (B.6). More specifically, note that

$$\mathbf{1}\{x_{it}'\theta_1 + \widehat{\alpha}_i \ge u_{it}^h\} - \mathbf{1}\{x_{it}'\theta_1 - \|x_{it}\|_2\delta + \widehat{\alpha}_i \ge u_{it}^h\}$$

takes value either 1 or 0, and the expectation is the probability that the following event occurs:

$$x_{it}'\theta_1 + \widehat{\alpha}_i \ge u_{it}^h \ge x_{it}'\theta_1 - \|x_{it}\|_2\delta + \widehat{\alpha}_i.$$

Applying law of iterated expectation on the right–hand–side term and first–order Taylor expansion around δ,

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}\Big(\mathbf{1}\{x_{it}'\theta_1 + \widehat{\alpha}_i \ge u_{it}^h\} - \mathbf{1}\{x_{it}'\theta_1 - \|x_{it}\|_2\delta + \widehat{\alpha}_i \ge u_{it}^h\} \mid x_{it}', \widehat{\alpha}_i\Big)\Big] \\
=&\mathbb{E}\Big[\Phi(x_{it}'\theta_1 + \widehat{\alpha}_i) - \Phi(x_{it}'\theta_1 - \|x_{it}\|_2\delta + \widehat{\alpha}_i)\Big] \\
=&\mathbb{E}\Big[\phi(x_{it}'\theta + \widehat{\alpha}_i)\|x_{it}\|_2\Big]\delta
\end{aligned}$$

Therefore,

$$\mathbb{E}\Big[\sup_{\|\theta_1-\theta_2\|\le\delta} |y_{it}^h(\theta_1) - y_{it}^h(\theta_2)|\Big] \le \mathbb{E}\Big[\phi(x_{it}'\theta + \widehat{\alpha}_i)\|x_{it}\|_2\Big]\delta \le \frac{\mathbb{E}\|x_{it}\|_2}{\sqrt{2\pi}}\delta,$$

where the last inequality uses the fact that $\phi(\cdot) \le \frac{1}{\sqrt{2\pi}}$.

Proving condition (B.5): Note that

$$\sqrt{\mathbb{E}\Big(\Big|\frac{\phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))(x_{it}+\frac{\partial\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))}{\beta})}{\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))[1-\Phi(x_{it}'\widehat{\beta}^h(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}^h(\theta_2))]}\Big|^2\Big)}$$

is no greater than

$$\sqrt{\mathbb{E}\Big(\sup_{(\beta,\gamma)}\Big|\frac{\phi(x_{it}'\beta+\gamma)x_{it}}{\Phi(x_{it}'\beta+\gamma)[1-\Phi(x_{it}'\beta+\gamma)]}\Big|^2\Big)},$$

which is bounded based on Lipschitz condition.

Step 3: Because the supremum of sum is no greater than sum of the supremum,

$$\begin{aligned}\mathbb{E}\Big(\sup_{\|\theta_1-\theta_2\|\leq\delta}\Big|\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}y_{it}^h(\theta_1,\widehat{\alpha}_i)-y_{it}^h(\theta_2,\widehat{\alpha}_i)\Big|^2\Big) &\leq \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\mathbb{E}\Big(\sup_{\|\theta_1-\theta_2\|\leq\delta}|y_{it}^h(\theta_1,\widehat{\alpha}_i)-y_{it}^h(\theta_2,\widehat{\alpha}_i)|^2\Big)\\ &\leq \frac{\delta}{\sqrt{2\pi}}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\mathbb{E}\|x_{it}\|_2,\end{aligned}$$

$$\mathbb{E}\Big(\Big|\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\phi(x_{it}'\widehat{\beta}(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}(\theta_2))x_{it}}{\Phi(x_{it}'\widehat{\beta}(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}(\theta_2))[1-\Phi(x_{it}'\widehat{\beta}(\theta_2)+\widehat{\gamma}_i(\widehat{\beta}(\theta_2))]}\Big|^2\Big)\leq\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\mathbb{E}K_{it}.$$

Therefore,

$$\mathbb{E}\Big(\sup_{\|\theta_1-\theta_2\|\leq\delta}\|\widehat{\beta}(\theta_1)-\widehat{\beta}(\theta_2)\|\Big)\leq\frac{\sqrt{\delta}}{(2\pi)^{1/4}}\sqrt{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\mathbb{E}\|x_{it}\|_2K_{it}}.$$

This verifies condition (B.1) and hence establishes the stochastic equicontinuity condition.

Step 4: By Theorem 1 in Newey (1991), $\widehat{\beta}^h(\theta)$ converges to $\beta(\theta)$ uniformly over $\theta\in\Theta$. □

B.2 Proof of Theorem 1 (Consistency)

Proof. Following the argument as in Appendix 1 of Gouriéroux et al. (1993),, consistency of $\widetilde{\theta}^H$ requires the following three conditions to hold:

1. the function $\beta(\theta)$ is invertible;
2. $\widehat{\theta}$ pointwise converges to $\beta(\theta_0)$;
3. $\widehat{\beta}^h(\theta)$ converges to $\beta(\theta)$ uniformly over $\theta \in \Theta$.

The first condition is satisfied because function is an identity. The second condition only involves fixed effects estimator using observed data, and is a standard result in large$-T$ panel literature (e.g, Hahn and Kuersteiner, 2011, Theorem 3). The third condition is verified by Proposition 1.

Recall that $\widetilde{\theta}^H$ is the solution to the optimization problem:

$$\widetilde{\theta}^H = \arg\min_{\theta \in \Theta} [\widehat{\theta} - \widehat{\beta}^H(\theta)]'[\widehat{\theta} - \widehat{\beta}^H(\theta)],$$

where $\widehat{\beta}^H(\theta) := \frac{1}{H}\sum_{h=1}^{H} \widehat{\beta}^h(\theta)$. Therefore. the limit of the optimization problem becomes

$$\min_{\theta \in \Theta} [\theta_0 - \theta]'[\theta_0 - \theta],$$

which has a unique solution θ_0. Therefore,

$$\widetilde{\theta}^H \xrightarrow{p} \theta_0.$$

□

B.3 Proof of Theorem 2 (Asymptotic Normality)

Proof. By Assumption 6 and consistency of $\widetilde{\theta}^H$,

$$\widehat{\theta} = \widehat{\beta}^H(\widetilde{\theta}^H) = \widehat{\beta}^H(\theta_0) + \mathbb{E}(\widehat{\beta}^H(\widetilde{\theta}^H) - \widehat{\beta}^H(\theta_0)) + o_p\Big(\frac{1}{\sqrt{nT}}\Big).$$

By mean–value theorem,

$$\mathbb{E}(\widehat{\beta}^H(\widetilde{\theta}^H) - \widehat{\beta}^H(\theta_0)) = \frac{\partial \mathbb{E}\widehat{\beta}^H(\bar{\theta})}{\partial \theta}(\widetilde{\theta}^H - \theta_0),$$

where $\bar{\theta}$ is between θ_0 and $\widetilde{\theta}^H$. Therefore,

$$\begin{aligned}
\sqrt{nT}(\widetilde{\theta}^H - \theta_0) &= -\Big[\frac{\partial \mathbb{E}\widehat{\beta}^H(\bar{\theta})}{\partial \theta}\Big]^{-1}\sqrt{nT}\Big(\widehat{\beta}^H(\theta_0) - \widehat{\theta}\Big) \\
&= \sqrt{nT}\Big(\widehat{\theta} - \widehat{\beta}^H(\theta_0)\Big) + o_p(1),
\end{aligned}$$

where the last equality uses the property that $\beta(\theta) = \theta$. Therefore, it suffices to focus on $\sqrt{nT}(\widehat{\theta} - \widehat{\beta}^H(\theta_0))$. Hahn and Kuersteiner (2011) derive the representation of $\widehat{\theta} - \theta_0$ as follows:

$$\widehat{\theta} - \theta_0 = \frac{A(\theta_0, \alpha_0)}{\sqrt{nT}} + \frac{B(\theta_0, \alpha_0)}{T} + o\Big(\frac{1}{T}\Big),$$

Because the same regression is run on simulated data,

$$\widehat{\beta}^h(\theta_0) - \theta_0 = \frac{A^h(\theta_0, \widehat{\alpha})}{\sqrt{nT}} + \frac{B^h(\theta_0, \widehat{\alpha})}{T} + o\Big(\frac{1}{T}\Big),$$

where $\widehat{\alpha} := (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_n)$. This implies

$$\widehat{\beta}^H(\theta_0) = \beta(\theta_0) + \frac{1}{H}\sum_{h=1}^{H}\frac{A^h(\theta_0, \widehat{\alpha}_i)}{\sqrt{nT}} + \frac{1}{H}\sum_{h=1}^{H}\frac{B^h(\theta_0, \widehat{\alpha}_i)}{T} + o\Big(\frac{1}{T}\Big).$$

Therefore,

$$\begin{aligned}
\sqrt{nT}\Big(\widehat{\theta} - \widehat{\beta}^H(\theta_0)\Big) &= \Big(A(\theta_0, \alpha_0) - \frac{1}{H}\sum_{h=1}^{H}A^h(\theta_0, \widehat{\alpha})\Big) \\
&\quad + \sqrt{\frac{n}{T}}\Big(B(\theta_0, \alpha_0) - \frac{1}{H}\sum_{h=1}^{H}B^h(\theta_0, \widehat{\alpha})\Big) + o\Big(\sqrt{\frac{n}{T^3}}\Big).
\end{aligned}$$

The rest of the proof shows that bias term cancels out and the asymptotic normality holds. To simplify notation, the rest of the proof proceeds by setting $H = 1$.

The CLT and bias terms are a combination of high–order derivatives of the log likelihood function and takes the following form (Hahn and Kuersteiner, 2011):

$$A(\theta_0, \alpha_0) = \Big(\frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_i\Big)^{-1} \frac{1}{\sqrt{nT}} \sum_{i=1}^{n}\sum_{t=1}^{T} U_{it}, \tag{B.7}$$

$$B(\theta_0, \alpha_0) = -\Big(\frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_i\Big)^{-1} \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{V_{it}}{\mathbb{E}\Big[\frac{\partial V_i}{\partial \alpha_i}\Big]}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\Big(\frac{\partial U_i}{\partial \alpha_i} - \frac{\mathbb{E}\Big[\frac{\partial^2 U_{it}}{\partial \alpha_i^2}\Big]}{2\mathbb{E}\Big[\frac{\partial V_i}{\partial \alpha_i}\Big]} V_{it}\Big)\right], \tag{B.8}$$

where

$$\mathcal{I}_i := -\mathbb{E}\left[\frac{\partial U_{it}}{\partial \theta'}\right], \quad U_{it} := U_i(y_{it}, x_{it}, \theta_0, \alpha_{i0}), \quad V_{it} := V_i(y_{it}, x_{it}, \theta_0, \alpha_{i0}),$$

and

$$\begin{aligned}
U_i(y_{it}, x_{it}; \theta, \alpha_i) &= \frac{\partial \ln f(y_{it} \mid x_{it}; \theta, \alpha_i)}{\partial \theta} - \rho_{i0} \cdot \frac{\partial \ln f(y_{it} \mid x_{it}; \theta, \alpha_i)}{\partial \alpha_i}, \\
V_i(y_{it}, x_{it}; \theta, \alpha_i) &= \frac{\partial \ln f(y_{it} \mid x_{it}; \theta, \alpha_i)}{\partial \alpha_i}, \\
\rho_{i0} &= \mathbb{E}\left[\frac{\partial^2 \ln f(y_{it} \mid x_{it}; \theta_0, \alpha_{i0})}{\partial \theta \partial \alpha_i}\right] \Big/ \mathbb{E}\left[\frac{\partial^2 \ln f(y_{it} \mid x_{it}; \theta_0, \alpha_{i0})}{\partial \alpha_i^2}\right].
\end{aligned}$$

The simulation analog that uses θ and $\tau := (\tau_1, \ldots, \tau_n)$ for data generation is

$$\begin{aligned}
B^h(\theta, \tau) &= -\Big(\frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_i(\theta, \tau)\Big)^{-1} \times \\
&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{V_{it}^h(\theta, \tau_i)}{\int_X\int_U \Big[\frac{\partial V_i^h(\theta,\tau_i)}{\partial \gamma_i}\Big] dF_u dF_x}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\Big(\frac{\partial U_i^h(\theta, \tau_i)}{\partial \gamma_i} - \frac{\int_X\int_U \Big[\frac{\partial^2 U_{it}^h}{\partial \gamma_i^2}\Big] dF_u dF_x}{2\int_X\int_U \Big[\frac{\partial V_i^h(\theta,\tau_i)}{\partial \gamma_i}\Big] dF_u dF_x} V_{it}^h(\theta, \tau_i)\Big)\right],
\end{aligned} \tag{B.9}$$

where

$$\begin{aligned}
&\mathcal{I}_i(\theta, \tau_i) := -\int_X\int_U \frac{\partial U_{it}^h}{\partial \beta'} dF_u dF_x, \quad U_{it}^h(\theta, \tau_i) := U_i(y_{it}^h(\theta, \tau_i), x_{it}, \theta, \tau_i), \\
&V_{it}^h(\theta, \tau_i) := V_i(y_{it}^h(\theta, \tau_i), x_{it}, \theta, \tau_i),
\end{aligned}$$

and

$$U_i(y_{it}^h(\theta,\tau_i),x_{it};\beta,\gamma_i)=\frac{\partial \ln f(y_{it}^h(\theta,\tau_i)\mid x_{it};\beta,\gamma_i)}{\partial\beta}-\rho_i(\theta,\tau_i)\cdot\frac{\partial \ln f(y_{it}^h(\theta,\tau_i)\mid x_{it};\beta,\gamma_i)}{\partial\gamma_i},$$

$$V_i(y_{it}^h(\theta,\tau_i),x_{it};\beta,\gamma_i)=\frac{\partial \ln f(y_{it}^h(\theta,\tau_i)\mid x_{it};\beta,\gamma_i)}{\partial\gamma_i},$$

$$\rho_i(\theta,\tau_i)=\left[\int_X\int_U \frac{\partial^2 \ln f(y_{it}^h(\theta,\tau_i)\mid x_{it};\theta,\tau_i)}{\partial\theta\partial\alpha_i}dF_u dF_x\right]\Bigg/\left[\int_X\int_U \frac{\partial^2 \ln f(y_{it}^h(\theta,\tau_i)\mid x_{it};\theta,\tau_i)}{\partial\alpha_i^2}dF_u dF_x\right].$$

In the simulation world, the integration is taken over the distribution of sampling of observed explanatory variables and the distribution of simulation draws. Define

$$\begin{aligned}B_i^h(\theta,\tau_i):=&\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{V_{it}^h(\theta,\tau_i)}{\int_X\int_U\left[\frac{\partial V_i^h(\theta,\tau_i)}{\partial\gamma_i}\right]dF_u dF_x}\right]\times\\ &\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\frac{\partial U_i^h(\theta,\tau_i)}{\partial\gamma_i}-\frac{\int_X\int_U\left[\frac{\partial^2 U_{it}^h(\theta,\tau_i)}{\partial\gamma_i^2}\right]dF_u dF_x}{2\int_X\int_U\left[\frac{\partial V_i^h(\theta,\tau_i)}{\partial\gamma_i}\right]dF_u dF_x}V_{it}^h(\theta,\tau_i)\right)\right].\end{aligned}\tag{B.10}$$

Step 1: Bias correction is established in Proposition 2.

Step 2: The proof sketch is provided here. The simulation analog of the CLT term is

$$A^h(\theta_0,\widehat{\alpha})=\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\widehat{\alpha}_i)\right)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}U_{it}^h(\theta,\widehat{\alpha}_i)$$

First for iid data, find an envelope for $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}U_{it}^h(\theta,\widehat{\alpha}_i)$, then by Corollary 19.35 in van der Vaart (2000). Then the expectation of each simulated CLT term can be bounded a finite uniform entropy integral. These individual bounds are then aggregated to form a measure of approximation error that is asymptotically negligible.

Then use Berbee's Lemma (Dedecker and Louhichi, 2002, Lemma 4.1) to extend the previous results to mixing data by constructing i.i.d sequences that approximate its simulated time series with a controlled remainder. The complication is to take care of the extra approximation errors. □

B.4 Proof of Proposition 2

Proof. Consider an infeasible fixed effects estimator $\widehat{\beta}^h(\theta_0, \alpha_0)$ that is obtained from data simulated by (θ_0, α_0). Then the representation of $\widehat{\beta}^h(\theta_0, \alpha_0) - \theta_0$ takes the form

$$\widehat{\beta}^h(\theta_0, \alpha_0) - \theta_0 = \frac{A^h(\theta_0, \alpha_0)}{\sqrt{nT}} + \frac{B^h(\theta_0, \alpha_0)}{T} + o\Big(\frac{1}{T}\Big),$$

where the superscript h denotes the fact that the dependent variable in $B^h(\theta_0, \alpha_0)$ is $y^h_{it}(\theta_0, \alpha_{i0})$. Because $B(\theta_0, \alpha_0)$ and $B^h(\theta_0, \alpha_0)$ have the same probability limit, they converge to the same expectation, which is the asymptotic bias. Therefore, it suffices to show that $B^h(\theta_0, \widehat{\alpha})$ uniformly well approximates $B^h(\theta_0, \alpha_0)$.

Now prove bias correction of the following form:

$$|B^h(\theta, \widehat{\alpha}) - B(\theta, \alpha_0)| \xrightarrow{p} 0.$$

By Markov inequality, $\forall \eta > 0$,

$$Pr(|B^h(\theta, \widehat{\alpha}) - B(\theta, \alpha_0)| \geq \eta) \leq \frac{1}{\eta}\mathbb{E}(|B^h(\theta, \widehat{\alpha}) - B(\theta, \alpha_0)|).$$

Therefore it suffices to bound the RHS term. By triangular inequality,

$$\mathbb{E}(|B^h(\theta, \widehat{\alpha}) - B(\theta, \alpha_0)|) \leq \mathbb{E}(|B^h(\theta, \widehat{\alpha}) - B^h(\theta, \alpha_0)|) + \mathbb{E}(|B^h(\theta, \alpha_0) - B(\theta, \alpha_0)|). \tag{B.11}$$

The second RHS term in equation (B.11) is $o_p(1)$ because $B^h(\theta, \alpha_0)$ and $B(\theta, \alpha_0)$ have the same probability limit. Regarding the first RHS term, by triangular inequality,

$$\begin{aligned}
\mathbb{E}|B^h(\theta, \widehat{\alpha}) - B^h(\theta, \alpha_0)| \leq \mathbb{E}\Big|&\Big(\frac{1}{n}\sum_{i=1}^n \mathcal{I}_i(\theta_0, \widehat{\alpha}_i)\Big)^{-1}\frac{1}{n}\sum_{i=1}^n B^h_i(\theta_0, \widehat{\alpha}_i) - \Big(\frac{1}{n}\sum_{i=1}^n \mathcal{I}_i(\theta_0, \widehat{\alpha}_i)\Big)^{-1}\frac{1}{n}\sum_{i=1}^n B^h_i(\theta_0, \alpha_{i0}) \\
&+ \Big(\frac{1}{n}\sum_{i=1}^n \mathcal{I}_i(\theta_0, \widehat{\alpha}_i)\Big)^{-1}\frac{1}{n}\sum_{i=1}^n B^h_i(\theta_0, \alpha_{i0}) - \Big(\frac{1}{n}\sum_{i=1}^n \mathcal{I}_i(\theta_0, \alpha_{i0})\Big)^{-1}\frac{1}{n}\sum_{i=1}^n B^h_i(\theta_0, \alpha_{i0})\Big| \\
\leq \mathbb{E}&\Big|\Big(\frac{1}{n}\sum_{i=1}^n \mathcal{I}_i(\theta_0, \widehat{\alpha}_i)\Big)^{-1}\Big| \cdot \Big|\frac{1}{n}\sum_{i=1}^n \big[B^h_i(\theta_0, \widehat{\alpha}_i) - B^h_i(\theta_0, \alpha_{i0})\big]\Big| \qquad \text{(B.12)} \\
+ \mathbb{E}&\Big|\Big(\frac{1}{n}\sum_{i=1}^n \mathcal{I}_i(\theta_0, \widehat{\alpha}_i)\Big)^{-1} - \Big(\frac{1}{n}\sum_{i=1}^n \mathcal{I}_i(\theta_0, \alpha_{i0})\Big)^{-1}\Big| \cdot \Big|\frac{1}{n}\sum_{i=1}^n B^h_i(\theta_0, \alpha_{i0})\Big|. \qquad \text{(B.13)}
\end{aligned}$$

Therefore, it suffices to focus on bounding terms (B.12) and (B.13).

For term (B.13), note that

$$
\begin{aligned}
&\Big(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\widehat{\alpha}_i)\Big)^{-1}-\Big(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\alpha_{i0})\Big)^{-1}\\
=&\Big(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\widehat{\alpha}_i)\Big)^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n}\big[\mathcal{I}_i(\theta_0,\alpha_{i0})-\mathcal{I}_i(\theta_0,\widehat{\alpha}_i)\big]\Big)\Big(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\alpha_{i0})\Big)^{-1}.
\end{aligned}
$$

By continuous mapping theorem, $\mathcal{I}_i(\theta_0,\widehat{\alpha}_i)\xrightarrow{p}\mathcal{I}_i(\theta_0,\alpha_{i0})$ for each i. Therefore,

$$
\mathbb{E}\Big|\Big(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\widehat{\alpha}_i)\Big)^{-1}-\Big(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\alpha_{i0})\Big)^{-1}\Big|\cdot\Big|\frac{1}{n}\sum_{i=1}^{n}B_i^h(\theta_0,\alpha_{i0})\Big|\xrightarrow{p}0.
$$

For term (B.12), note that

$$
\begin{aligned}
\Big|\frac{1}{n}\sum_{i=1}^{n}\big[B_i^h(\theta_0,\widehat{\alpha}_i)-B_i^h(\theta_0,\alpha_{i0})\big]\Big| &\le \frac{1}{n}\sum_{i=1}^{n}|B_i^h(\theta_0,\widehat{\alpha}_i)-B_i^h(\theta_0,\alpha_{i0})|\\
&\le \max_{1\le i\le n}|B_i^h(\theta_0,\widehat{\alpha}_i)-B_i^h(\theta_0,\alpha_{i0})|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\Big|\Big(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\widehat{\alpha}_i)\Big)^{-1}\Big|\cdot\Big|\frac{1}{n}\sum_{i=1}^{n}\big[B_i^h(\theta_0,\widehat{\alpha}_i)-B_i^h(\theta_0,\alpha_{i0})\big]\Big|\\
\le&\mathbb{E}\Big|\Big(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\widehat{\alpha}_i)\Big)^{-1}\Big|\cdot\max_{1\le i\le n}|B_i^h(\theta_0,\widehat{\alpha}_i)-B_i^h(\theta_0,\alpha_{i0})|\\
\le&\sqrt{\mathbb{E}\Big|\Big(\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(\theta_0,\widehat{\alpha}_i)\Big)^{-1}\Big|^2}\cdot\sqrt{\mathbb{E}\max_{1\le i\le n}\Big|B_i^h(\theta_0,\widehat{\alpha}_i)-B_i^h(\theta_0,\alpha_{i0})\Big|^2},
\end{aligned}
$$

where the second inequality is due to Cauchy–Schwarz inequality. Note that $B_i^h(\theta,\tau_i)$ has a finite number of parameters. Therefore, it suffices to bound

$$
\mathbb{E}\max_{1\le i\le n}\Big|B_i^h(\theta_0,\widehat{\alpha}_i)-B_i^h(\theta_0,\alpha_{i0})\Big|^2.
$$

Recall from expression (B.9) that some elements in $B_i^h(\theta,\widehat{\alpha}_i)$ are not smooth, namely $\partial U_{it}^h(\theta,\tau_i)/\partial\gamma_i$ and $V_{it}^h(\theta,\tau_i)$ that contain $\mathbf{1}\{x_{it}'\theta+\widehat{\alpha}_i\ge u_{it}^h\}$, which is a function of a scalar $\widehat{\alpha}_i$. The terms $\int_X\int_U\Big[\frac{\partial V_i^h(\theta,\tau_i)}{\partial\gamma_i}\Big]dF_u dF_x$, $\int_X\int_U\Big[\frac{\partial^2 U_{it}^h(\theta,\tau_i)}{\partial\gamma_i^2}\Big]dF_u dF_x$ and $\int_X\int_U\Big[\frac{\partial V_i^h(\theta,\tau_i)}{\partial\gamma_i}\Big]dF_u dF_x$ are differentiable. For panel probit models, algebra derivation of $B_i^h(\theta_0,\widehat{\alpha}_i)-B_i^h(\theta_0,\alpha_{i0})$ can lead to bounding $\mathbb{E}\sup_{|\widehat{\alpha}_i-\alpha_{i0}|\le\delta}\mathbf{1}\{x_{it}'\theta+\widehat{\alpha}_i\ge u_{it}^h\}-\mathbf{1}\{x_{it}'\theta+\alpha_{i0}\ge u_{it}^h\}$, which is bounded using the argument in

Chen et al. (2003).

Hence, $B^h(\theta_0, \widehat{\alpha}_i)$ uniformly approximates $B^h(\theta, \alpha_0)$. Because $B^h(\theta_0, \alpha_0)$ and $B(\theta_0, \alpha_0)$ have the same probability limit $\mathbb{E}(B(\theta_0, \alpha_0))$, which is the asymptotic bias term, $B^h(\theta_0, \widehat{\alpha})$ also does, and bias correction is completed. $\square$

C Computation Appendix

C.1 Calibration Procedures

Simulation procedures for the labor force participation application.

1. Run the regression on the LFP data to obtain $\widetilde{\theta}$ and $\widetilde{\alpha}_i$'s. These are treated as true coefficients for the calibration exercise.

2. For each simulation $s = 1, \ldots, S$, create a synthetic panel data based on the equation

$$y_{it}^s = \mathbf{1}\{X_{it}\widetilde{\theta} + \widetilde{\alpha}_i > u_{it}^s\},$$

   where $u_{it}^s \sim iid\mathcal{N}(0, 1)$. The data $\{(y_{it}^s, X_{it})\}$ are considered as the observed data for simulation s.

3. **Implementing the estimation:**

   (a) Run Probit regression on $\{(y_{it}^s, X_{it})\}$ and obtain $\widehat{\theta}^s$ and $\widehat{\alpha}_i^s$. This denotes the fixed effects estimators using observed data.

   (b) **Data simulation**:

      i. Given a set of parameter θ, simulate dependent variable using

$$y_{it}^h(\theta) = \mathbf{1}\{X_{it}'\theta + \widehat{\alpha}_i^s > \varepsilon_{it}^h\}, \quad \varepsilon_{it}^h \sim iid\mathcal{N}(0, 1)$$

      Run Probit regression on $\{y_{it}^h(\theta), X_{it}\}$to obtain $\widehat{\beta}^h(\theta)$.

      ii. Repeat step (i) for $H = 10$ times and compute

$$\widehat{\beta}^H(\theta) = \frac{1}{H}\sum_{h=1}^{H}\widehat{\beta}^h(\theta).$$

      iii. Compute the indirect inference estimator $\widetilde{\theta}^H$ by solving the following equation

$$\widehat{\theta}^s = \widehat{\beta}^H(\widetilde{\theta}^H).$$

4. Repeat steps 2 and 3 for $S = 500$ times.

C.2 Simulations for Dynamic Labor Force Participation

This subsection introduces dynamics into the specification and compare the performance of indirect fixed effects estimators with other estimators.

Positive serial correlation observed in employment outcomes motivates the question of identifying state dependence, i.e., the causal impact of past employment on future employment for married women. However, the positive correlation can also be driven by individual–specific unobserved heterogeneity such as willingness to work. Therefore, an important question of interest is to distinguish between state dependence and persistent unobserved heterogeneity.

Following the empirical specification in Fernández-Val (2009), this paper controls for time–invariant unobserved heterogeneity by adding individual fixed effects,

$$y_{it} = \mathbf{1}\{X_{it}'\theta + \alpha_i \geq u_{it}\}, \quad u_{it} \sim \mathcal{N}(0,1), \tag{C.1}$$

where the vector of pre–determined covariates $X_{it} := (x_{it}, y_{i,t-1})$ now contains an extra variable: $y_{i,t-1}$, which denotes the lagged dependent variable. The first year of the sample is excluded for use as the initial condition in the dynamic model. In the data simulation step, the dependent variable at time t has the following representation:

$$y_{it}^h(\theta, \widehat{\alpha}_i) = \mathbf{1}\{\theta_1 y_{i,t-1}^h(\theta, \widehat{\alpha}_i) + x_{it}'\theta_{-1} + \widehat{\alpha}_i \geq u_{it}^h\}, \quad u_{it}^h \sim \mathcal{N}(0,1). \tag{C.2}$$

where θ_{-1} denotes parameters other than the one for $y_{i,t-1}^h$. Tables (3) and (4) tabulate the results of calibration exercise. Compared to the static case Tables (1) and (2), adding dynamics into the regression further deteriorates fixed effects estimators of strictly exogenous covariates, which are comparable with the standard deviations. On the other hand, indirect fixed effects estimators correct the bias significantly.

Table 3: FE Estimators for Dynamic LFP

	laglfp	kids0_2	kids3_5	kids6_17	loghusinc	age	age2
Bias	-53.53	33.75	48.01	50.45	23.76	29.17	28.97
Std Dev	5.82	14.40	24.24	73.93	29.10	24.57	27.11
RMSE	53.84	36.69	53.77	89.45	37.55	38.12	39.66

Table 4: Indirect–FE Estimators for Dynamic LFP

	laglfp	kids0_2	kids3_5	kids6_17	loghusinc	age	age2
Bias	3.59	-5.75	-9.33	-26.93	6.32	1.38	-2.33
Std Dev	6.41	10.33	18.60	55.90	48.22	5.47	22.83
RMSE	7.34	11.82	20.79	62.00	48.59	5.64	22.92