HW 6 Sols

MATH 358/658 Assignment 6

Due March 7. Hand in to my mailbox if you are leaving town early for the break.

- Page 416 #6
- (0 pts) Page 416 #7 By "consistent", they mean that the Bayes Estimator converges in probability to θ
- Page 416 #12
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- Page 472 #12

Give +2 if people got 416 #7 correct, but it's worth 0 otherwise.

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p. 416 #6

$$f(x|\theta) = \text{Poisson}(\theta) = \frac{e^{-\theta}\,\theta^{x_i}}{x_i!}, \quad x_i = 0, 1, \ldots ; \ \theta > 0.$$

$$\xi(\theta) = \text{Gamma}(\alpha, \beta)$$

By conjugacy, $\xi(\theta|x) = \text{Gamma}(\alpha + \Sigma x_i, \beta + n)$

$$E(\theta|x) = \frac{\alpha + \Sigma x_i}{\beta + n}.$$

$$= \frac{\alpha}{\beta + n} + \frac{\Sigma x_i}{\beta + n}$$

$$= \frac{(\alpha/\beta)}{1 + (n/\beta)} + \frac{(\Sigma x_i/n)}{(\beta/n) + 1}$$

$$= \left(\frac{1}{1 + n/\beta}\right) \cdot \mu_0 + \left(\frac{1}{1 + \beta/n}\right) \cdot \overline{X}$$

$\therefore$ let $\gamma_n = \dfrac{1}{1 + \beta/n}$. $\lim_{n \to \infty} \gamma_n = 1.$

$$1 - \gamma_n = \frac{1 + \beta/n}{1 + \beta/n} - \frac{1}{1 + \beta/n} = \frac{\beta/n}{1 + \beta/n}$$

$$= \frac{1}{1 + n/\beta}. \Rightarrow E(\theta|x) = (1 - \gamma_n)\mu_0 + \gamma_n \cdot \overline{X}.$$

#7 (lots of solutions I guess, here's the simplest)

$$\hat{\theta} = E(\theta|x) = (1-\gamma_n)\mu_0 + \gamma_n \cdot \bar{X}$$

$$\lim_{n\to\infty} \hat{\theta} = \bar{X} \quad \text{since} \quad \lim_{n\to\infty} \gamma_n = 1$$

By LLN, $\bar{X} \xrightarrow{P} E(\bar{x}) = \theta.$

p. 416 #12

$\xi_A(\theta) = 2\theta$ for $0 < \theta < 1$

$\xi_B(\theta) = 4\theta^3$ for $0 < \theta < 1$

it'll save time if you recognized

$\xi_A(\theta) = \text{Beta}(\alpha = 2, \beta = 1)$

$\xi_B(\theta) = \text{Beta}(\alpha = 4, \beta = 1)$, but not necessary.

By conjugacy, $\xi_A(\theta | x) = \text{Beta}(712, 291)$

$\xi_B(\theta | x) = \text{Beta}(714, 291)$

b.) For A, $\hat{\theta}_{bayes} = \dfrac{712}{712 + 291} = .7098$

For B, $\hat{\theta}_{bayes} = \dfrac{714}{714 + 291} = .7104$

c.) Let $n = 1000$, but don't specify $\sum x_i$.

For A, $\hat{\theta}_{bayes} = \dfrac{2 + \sum x_i}{1003}$

For B, $\hat{\theta}_{bayes} = \dfrac{4 + \sum x_i}{1005}$

numerators will always differ by only 2, so estimators differ $< \dfrac{2}{1003} < \dfrac{2}{1\ldots}$

P. 472 #2

$$f(x) = \frac{1}{2^{m/2}\,\Gamma(m/2)}\, x^{(m/2-1)}\, e^{-x/2}$$

set $f'(x) = 0$ to find mode.

$$f'(x) = -\frac{1}{2}\cdot x^{(m/2-1)}\, e^{-x/2} + e^{-x/2}\left(\frac{m}{2}-1\right)\cdot x^{(\frac{m}{2}-2)} = 0$$

$$\Rightarrow -\frac{1}{2}\cdot x + \left(\frac{m}{2}-1\right) = 0$$

$$\Rightarrow x = m-2$$

clearly, mode $= x = m-2$ requires $m \geq 2$.

#4 $D = \sqrt{X^2+Y^2}$, find $P(D \leq d) = 0.99$ for some d.

$$\Leftrightarrow P(D^2 \leq d^2) = .99$$

$$\Leftrightarrow P(X^2+Y^2 \leq d^2) = .99 \quad \text{where } X^2+Y^2 \sim \chi^2_2.$$

$$P(\chi^2_2 \leq d^2) = .99 \Rightarrow d^2 = 9.2103$$

$$\boxed{d = 3.0348}$$

p.472 #5 $D = \sqrt{X^2+Y^2+Z^2}$

Find $P(D \le 1)$

$= P(D^2 \le 1)$

$= P(\chi^2_3 \le 1) = \boxed{.1987}$

p.472 #6 $X \sim N(0, \sigma^2 t)$

At $t=2$, $X \sim N(0, 2\sigma^2)$

$\Rightarrow \frac{X}{\sqrt{2}\sigma} \sim N(0,1)$

$\Rightarrow \frac{X^2}{2\sigma^2} \sim \chi^2_1$

$\frac{X^2}{2\sigma^2} + \frac{Y^2}{2\sigma^2} + \frac{Z^2}{2\sigma^2} \sim \chi^2_3$

Let $D^2 = X^2 + Y^2 + Z^2$.

$P(D^2 \le 16\cdot\sigma^2) = P\left(\frac{D^2}{2\sigma^2} \le \frac{16\cdot\sigma^2}{2\sigma^2}\right)$

$= P(\chi^2_3 \le 8)$

$= \boxed{0.9539}$

p. 472 # 12

$$P(Y \le 0.09) = P\left(\frac{\sigma^2}{k} \cdot \chi^2_{10} \le 0.09\right)$$

$$= P\left(\chi^2_{10} \le \frac{.9}{\sigma^2}\right) = .9$$

only if $\frac{.9}{\sigma^2} \geq 15.99$

or $\sigma^2 \le \frac{.9}{15.99} = 0.0562.$