Jeremy Rouse's Math 711 homework

Chapter 3

1.1. The function

$$f(x,y,z) = \begin{cases} 1 & \text{if } x = y = z = 0 \\ \frac{\sin(x^2+y^2+z^2)}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0) \end{cases}$$

is a differentiable function on $\mathbb{R}^3$. This is because it is really a composition of the analytic function $\frac{\sin(s)}{s}$ with the differentiable function $x^2+y^2+z^2$ (and the chain rule proves that it is differentiable).

Math 711 students may not claim that $\frac{\sin(s)}{s}$ is analytic without proof.

The function

$$f(x,y,z) = \frac{x^2yz}{x^2+y^2+z^2}$$

is differentiable on $\mathbb{R}^3$ as well. The quotient rule proves that it is differentiable when $(x,y,z) \neq (0,0,0)$. To show that it is differentiable at $(0,0,0)$ note that

$$0 \leq \frac{x^2}{x^2+y^2+z^2} \leq 1$$

and so

$$|f(x,y,z)| \leq |yz|.$$

Noting that $y^2-2yz+z^2 \geq 0$ and $y^2+2yz+z^2 \geq 0$ we see that $|yz| \leq \frac{1}{2}(y^2+z^2) \leq \frac{1}{2}(x^2+y^2+z^2)$. Note that $x^2+y^2+z^2 = o(\sqrt{x^2+y^2+z^2}) = o(\|(x,y,z)\|)$ and so this proves $\vec{L} = (0,0,0)$ is the derivative of $f(x,y,z)$ at $(0,0,0)$.

1.2. Let

$$f(x,y,z) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \text{ or } z = 0 \\ 1 & \text{otherwise.} \end{cases}$$

It is straightforward to see that a differentiable function is continuous (a real-valued function f is continuous at $\vec{x_0}$ if $f(\vec{x}) = f(\vec{x_0}) + o(1)$). The function $f(x,y,z)$ is clearly not continuous, and so it is not differentiable. However,

$$\frac{\partial f}{\partial x}(0,0,0) = \lim_{h \to 0} \frac{f(h,0,0) - f(0,0,0)}{h} = \lim_{h \to 0} 0 = 0.$$

Similar computations show that $\frac{\partial f}{\partial y}(0,0,0) = 0$ and also $\frac{\partial f}{\partial z}(0,0,0) = 0$.

1.3. For both functions, there is no one choice of $\vec{L}$ from Definition 1.3. that works for all $\vec{x}$ close to zero. For the first function, we have $f(x,x) = -2|x|$, so for $x > 0$ we have $f(x,x) = -2x$ which forces $\vec{L} \cdot (1,1) = -2$. However, for $x < 0$ we have $f(x,x) = -2|x| = -2(-x) = 2x$ and so $\vec{L} \cdot (1,1) = -2$. This is impossible.

For the second function, we have

$$g(x,y) = \left(\frac{3x^2}{x^2+y^2}\right) y.$$

The issue here is that the expression in parentheses oscillates between 0 and 3. For example, if $y = x$, we have $g(x,x) = \frac{3}{2}x$ which forces $\vec{L}\cdot(1,1) = 3/2$. If $y = 0$, we have $g(x,y) = 0$ which forces $\vec{L}\cdot(0,1) = 0$ and if $x = 0$ we have $g(x,y) = 0$ which forces $\vec{L}\cdot(1,0) = 0$. There is no vector that simultaneously satisfies these three equations.

1.4. Suppose that $L : X \to Y$ is bounded. Then $\|Lx\|_Y \le c\|x\|_X$ for all $x \in X$. This implies that if $\|x\|_X = 1$ then $\|Lx\|_Y \le c$, which implies that $\sup\{\|Lx\|_Y : x \in X \text{ and } \|x\|_X = 1\}$ is bounded.

Conversely, suppose that $\sup\{\|Lx\|_Y : x \in X \text{ and } \|x\|_X = 1\} < \infty$. Let M be this supremum. Then, for any nonzero $x \in X$, we have $\|x/\|x\|_X\|_X = \frac{\|x\|_X}{\|x\|_X} = 1$ and so, using the linearity of L and the homogeneity of the norm, we have

$$\begin{aligned} L(x) &= L\left(\frac{\|x\|_X}{\|x\|_X}x\right) \\ &= \|x\|_X L\left(\frac{1}{\|x\|_X}x\right) \end{aligned}$$

$$\|L(x)\|_Y = \left\|\|x\|_X L\left(\frac{1}{\|x\|_X}x\right)\right\|_Y = \|x\|_X \left\|L\left(\frac{1}{\|x\|_X}x\right)\right\|_Y \le M\|x\|_X.$$

Hence, L is bounded and $\|Lx\|_Y \le \|L\|_{op}\|x\|_X$ (because $M = \|L\|_{oper}$).

1.5. Since X and Y are normed linear spaces, addition is commutative and associative, scalar multiplication is distributive and associative, multiplication by 1 is the multiplicative identity. These properties immediately yield the same results for $\mathcal{B}(X,Y)$. The map $Z : X \to Y$ given by $Z(x) = 0$ is the zero element of $\mathcal{B}(X,Y)$ and it has operator norm zero. If $L : X \to Y$ is a bounded linear map, define $-L$ by $-L(x) = -L(x)$. Then, $-L$ is the additive inverse of L. Since $\|\alpha L\|_{op} = |\alpha|\|L\|_{op}$, it follows that $\|-L\|_{op} = \|L\|_{op}$ and so $-L$ is bounded as well.

Note that by definition $\|L\|_{op} = \sup\{\|Lx\|_Y : x \in X, \|x\|_X = 1\} \ge 0$ (by the property of the norm on Y). Suppose that $\|L\|_{op} = 0$. Then, by Lemma 1.2, we have that $\|Lx\|_Y = 0$ for all $x \in X$ and so $Lx = 0$ for all $x \in X$ (by the property of $\|\cdot\|_Y$) and so $L = 0$. This completes the proof that $\mathcal{B}(X,Y)$ is a normed linear space.

Note that the limit operator L was never shown to be linear. This follows since

$$L(x+y) = \lim_{n\to\infty} L_n(x+y) = \lim_{n\to\infty} L_n(x) + L_n(y) = \lim_{n\to\infty} L_n(x) + \lim_{n\to\infty} L_n(y) = L(x) + L(y),$$

and also

$$L(\alpha x) = \lim_{n\to\infty} L_n(\alpha x) = \lim_{n\to\infty} \alpha L_n(x) = \alpha \lim_{n\to\infty} L_n(x) = \alpha L(x).$$

The map $f : Y \to \mathbb{R}$ given by $f(y) = \|y\|_Y$ is a continuous map, since if ϵ is fixed and $\|x-y\|_Y < \epsilon$, then $|f(x)-f(y)| = \|x-y\|_Y < \epsilon$. This is continuity is used in the statement that $\|L(x)\|_Y = \lim_{n\to\infty} \|L_n(x)\|_Y$.

1.6. For a positive integer n, let $f_n(x) = \sqrt{2}\sin(2\pi nx)$. Note that $f_n(x) \in C^\infty([0,1])$ and

$$\begin{aligned}\|f_n\|_2^2 &= \int_0^1 2\sin^2(2\pi nx)\,dx = \int_0^1 1-\cos(4\pi nx)\,dx \\ &= \left[x - \frac{1}{4\pi n}\sin(4\pi nx)\right]_0^1 \\ &= \left[1 - \frac{1}{4\pi n}\sin(4\pi n)\right] - \left[0 - \frac{1}{4\pi n}\sin(0)\right] \\ &= 1.\end{aligned}$$

However, $L(f_n)(x) = 2\pi n\sqrt{2}\cos(2\pi nx)$ and so

$$\begin{aligned}\|Lf_n\|_2^2 &= 4\pi^2n^2\int_0^1 2\cos^2(2\pi nx)\,dx \\ &= 4\pi^2n^2\int_0^1 (1+\cos(4\pi nx))\,dx \\ &= 4\pi^2n^2\left[x + \frac{1}{4\pi n}\sin(4\pi nx)\right]_0^1 \\ &= 4\pi^2n^2\left(\left[1 + \frac{1}{4\pi n}\sin(4\pi n)\right] - \left[0 + \frac{1}{4\pi n}\sin(0)\right]\right) = 4\pi^2n^2.\end{aligned}$$

Hence, $\|Lf_n\|_2 = 2\pi n$ and so L is not a bounded linear operator. It follows that L is not continuous.

Similarly, let $g_n(x) = \sin(2\pi nx)$. Then, $\|g_n\|_\infty = 1$ and $L(g_n) = 2\pi n\cos(2\pi nx)$ and so $\|L(g_n)\|_\infty = 2\pi n$. This shows that $L : C^1([0,1]) \to C([0,1])$ is not bounded and so (by Lemma 1.4) it is not continuous.

1.7. Clearly if $L : X \to Y$ is continuous at every point of X, then L is continuous at zero.

Suppose that $L : X \to Y$ is continuous at zero and let $x_0 \in X$. Fix $\epsilon > 0$ and choose $\delta > 0$ so that if $\|z\|_X < \delta$ then $\|Lz\|_Y < \epsilon$. If $\|x - x_0\|_X < \delta$, then $\|L(x-x_0)\|_Y < \epsilon$ by the statement above. Since L is linear, this implies that $\|L(x) - L(x_0)\|_Y < \epsilon$ and so L is continuous at x_0 too, as desired. (In fact, L is uniformly Lipschitz continuous.)

1.8. Fix $f \in L^2([0,1])$. For $h \in L^2([0,1])$ we have that

$$\int_0^1 2f(t)h(t)\,dt$$

is well-defined, since by Hölder's inequality we have $\|2fh\|_1 \leq 2\|f\|_2\|h\|_2$. Hence $DF(f) : L^2([0,1]) \to \mathbb{R}$ is a well-defined function. Also

$$\begin{aligned} DF(f)[h_1+h_2] &= \int_0^1 2f(t)(h_1(t)+h_2(t))\,dt = \int_0^1 2f(t)h_1(t)\,dt + \int_0^1 2f(t)h_2(t)\,dt \\ &= DF(f)[h_1] + DF(f)[h_2] \\ DF(f)[\alpha h] &= \int_0^1 2\alpha f(t)h(t)\,dt = \alpha \int_0^1 2f(t)h(t)\,dt \\ &= \alpha DF(f)[h] \end{aligned}$$

and so $DF(f) : L^2([0,1]) \to \mathbb{R}$ is a linear functional.

Now, suppose that $g \in L^2([0,1])$. Then, we have

$$\begin{aligned} F(f+g) - F(f) &= \int_0^1 (f(t)+g(t))^2\,dt - \int_0^1 (f(t))^2\,dt \\ &= \int_0^1 \left(2f(t)g(t) + g(t)^2\right)\,dt \\ &= DF(f)[g] + \int_0^1 g(t)^2\,dt \\ &= DF(f)[g] + \|g\|_2^2. \end{aligned}$$

Since $\|g\|_2^2 = o(\|g\|)$, it follows that F is differentiable. The Hölder's inequality calculation above also shows that $\|DF(f)[g]\| \leq 2\|f\|\|g\|$, so $DF(f)$ is a bounded operator and hence F is Fréchet differentiable.

1.9. Theorem 1.2 states that

$$f(x) - \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i = \int_{x_0}^{x} \frac{f^{(k+1)}(s)}{k!}(x-s)^k\,ds.$$

We have

$$\begin{aligned} \left| \int_{x_0}^{x} \frac{f^{(k+1)}(s)}{k!}|x-s|^k\,ds \right| &\leq \frac{1}{k!}\int_{x_0}^{x} |f^{(k+1)}(s)||x-s|^k\,ds \\ &\leq \frac{1}{k!}|x-x_0|^k \int_{x_0}^{x} |f^{(k+1)}(s)|\,ds. \end{aligned}$$

Since f is C^{k+1}, we have that $f^{(k+1)}(s)$ is continuous on $[x_0,x]$ and by the extreme value theorem, there is a $c \in [x_0,x]$ so that $|f^{(k+1)}(c)| = \sup\{|f^{(k+1)}(s)| : x_0 \leq s \leq x\}$. Therefore $|f^{(k+1)}(s)| \leq |f^{(k+1)}(c)|$ for all $s \in [x_0,x]$ and so

$$\int_{x_0}^{x} |f^{(k+1)}(s)|\,ds \leq \int_{x_0}^{x} |f^{(k+1)}(c)|\,ds = |f^{(k+1)}(c)||x-x_0|.$$

The desired result follows.

1.10. The notion of Frechet differentiability is an extension of the usual notion of differentiability. For this reason the function $i : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ given by $i(x) = \frac{1}{x}$ is Frechet differentiable with $Di(x) = -\frac{1}{x^2}$. Therefore, if $f : X \to \mathbb{R}$ is Frechet differentiable and $f(x_0) \neq 0$, then $i \circ f : X \to \mathbb{R}$ is Frechet differentiable by the chain rule (Lemma 1.9) and

$$D(i \circ f)(x_0) = Di(f(x_0)) \cdot Df(x_0) = -\frac{1}{f(x_0)^2} Df(x_0).$$

Note that $i(f(x)) = \frac{1}{f(x)}$. Now, the product rule (Lemma 1.7) implies that $(1/f) \cdot F : X \to Y$ is Frechet differentiable at x_0 (again provided $f(x_0) \neq 0$), and

$$\begin{aligned} D((1/f) \cdot F)(x_0) &= \frac{1}{f(x_0)} DF(x_0) + F(x_0) D(1/f)(x_0) \\ &= \frac{1}{f(x_0)} DF(x_0) + F(x_0) \cdot \left(-\frac{Df(x_0)}{f(x_0)^2} \right) \\ &= \frac{f(x_0)DF(x_0) - F(x_0)Df(x_0)}{f(x_0)^2}. \end{aligned}$$

1.11. First, I claim that if $f : \mathbb{R}^n \to \mathbb{R}^m$ and $f = (f_1, f_2, \ldots, f_m)$ then if the f_i are all differentiable, then so is f. [Since $f_i = \pi_i \circ f$ where π_i is the projection onto the ith coordinate, the chain rule easily implies that the converse of this statement is true.]

Fix some $x \in \mathbb{R}^n$. Since f_i is differentiable at x, we have

$$f_i(x + h) = f_i(x) + \nabla f_i(x) \cdot h + o(h).$$

Then

$$f(x + h) - f(x) = \langle \nabla f_1(x) \cdot h + o(h), \nabla f_2(x) \cdot h + o(h), \ldots, \nabla f_n(x) \cdot h + o(h) \rangle.$$

Letting L be the linear transformation represented by the matrix whose rows are the $\nabla f_i(x)$, we have

$$\frac{f(x + h) - f(x) - L(x)(h)}{\|h\|} = \left\langle \frac{r_1(h)}{\|h\|}, \frac{r_2(h)}{\|h\|}, \ldots, \frac{r_n(h)}{\|h\|} \right\rangle$$

where each $r_i(h) = o(h)$. This implies that $\lim_{\|h\| \to 0} \frac{r_i(h)}{\|h\|} = 0$ for all i, $1 \leq i \leq n$. Therefore, Theorem 2.7 from Chapter 1 implies that

$$\left\| \frac{f(x + h) - f(x) - L(x)(h)}{\|h\|} \right\| \to 0$$

as $\|h\| \to 0$ and f is differentiable. [To get the conversion from Theorem 2.7 of Chapter 1 to the statement above, one must note that in a metric space X, we have that $\lim_{x \to x_0} f(x) = L$ if and only if for every sequence $(x_i) \subseteq X$ with $\lim_{i \to \infty} x_i = x_0$ we have $\lim_{i \to \infty} f(x_i) = L$.]

It suffices therefore to prove the the result in the case of $f : \mathbb{R}^n \to \mathbb{R}$.

Fix $\vec{x}_0 \in \mathbb{R}^n$ and let $\vec{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n})$. For $\vec{x} \in \mathbb{R}^n$ write $\vec{x} = (x_1, x_2, \ldots, x_n)$ and define for $1 \leq i \leq n$, the vectors

$$\vec{y_i} = (x_1, x_2, \ldots, x_{i-1}, x_{0,i}, x_{0,i+1}, \ldots, x_{0,n}).$$

Define $\vec{y}_{n+1} = \vec{x}$. We have

$$f(\vec{x}_0) - f(\vec{x}) = \sum_{i=1}^{n} (f(\vec{y_i}) - f(\vec{y}_{i+1})).$$

Applying the mean value theorem to each term, we get

$$f(\vec{x}_0) - f(\vec{x}) = \sum_{i=1}^{n} -\frac{\partial f}{\partial x_i}(x_1, \ldots, x_{i-1}, \tilde{x}_i, x_{0,i+1}, \ldots, x_{0,n})(x_i - x_{0,i}).$$

Here $\tilde{x}_i$ is a real number between x_i and $x_{0,i}$. For $1 \leq i \leq n$, define

$$\vec{z_i} = (x_1, \ldots, x_{i-1}, \tilde{x}_i, x_{0,i+1}, \ldots, x_{0,n})$$

and define $\vec{z}_{n+1} = \vec{x}$. Then,

$$f(\vec{x}_0) - f(\vec{x}) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(\vec{x}_0)(x_i - x_{0,i}) - \frac{\partial f}{\partial x_i}(\vec{z_i})(x_i - x_{0,i}) \right).$$

By the continuity of the partial derivatives,

$$\frac{\partial f}{\partial x_i}(\vec{x_0}) - \frac{\partial f}{\partial x_i}(\vec{z_i}) = o(1)$$

(as $x_i \to x_{0,i}$) and this implies (again by Theorem 2.7 of Chapter 1) that

$$f(\vec{x}_0) - f(\vec{x}) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = o(\|\vec{x} - \vec{x}_0\|)$$

and so f is differentiable at $\vec{x}_0$ with derivative $\nabla f(\vec{x}_0)$.

1.12. We have

$$\begin{aligned}
D_{i,h} D_{j,l} f(\vec{x}) &= \frac{D_{j,l}(f)(\vec{x} + h\vec{e}_i) - D_{j,l}(f)(\vec{x})}{h} \\
&= \frac{\frac{f(\vec{x}+h\vec{e}_i+l\vec{e}_j)-f(\vec{x}+h\vec{e}_i)}{l} - \frac{f(\vec{x}+l\vec{e}_j)-f(\vec{x})}{l}}{h} \\
&= \frac{f(\vec{x} + h\vec{e}_i + l\vec{e}_j) - f(\vec{x} + h\vec{e}_i) - f(\vec{x} + l\vec{e}_j) + f(\vec{x})}{hl}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
D_{j,l}D_{i,h}f(\vec{x}) &= \frac{D_{i,h}(f)(\vec{x}+l\vec{e}_j) - D_{i,h}(f)(\vec{x})}{l} \\
&= \frac{\frac{f(\vec{x}+l\vec{e}_j+h\vec{e}_i)-f(\vec{x}+l\vec{e}_j)}{h} - \frac{f(\vec{x}+h\vec{e}_i)-f(\vec{x})}{h}}{l} \\
&= \frac{f(\vec{x}+l\vec{e}_j+h\vec{e}_i) - f(\vec{x}+l\vec{e}_j) - f(\vec{x}+h\vec{e}_i) - f(\vec{x})}{hl} \\
&= \frac{f(\vec{x}+h\vec{e}_i+l\vec{e}_j) - f(\vec{x}+h\vec{e}_i) - f(\vec{x}+l\vec{e}_j) - f(\vec{x})}{hl} \\
&= D_{i,h}D_{j,l}f(\vec{x}).
\end{aligned}$$

1.13. I will construct a function $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and points $\vec{x}_1$ and $\vec{x}_2$ in $\mathbb{R}^2$ so that there is no value of c, $0 \le c \le 1$ so that

$$f(\vec{x}_2) - f(\vec{x}_1) = Df(\vec{x}_1 + c(\vec{x}_2 - \vec{x}_1))(\vec{x}_2 - \vec{x}_1).$$

Let $f(x, y) = (x^2, x^2y)$, $\vec{x}_1 = (1, 0)$ and $\vec{x}_2 = (0, 1)$. Then, $f(\vec{x}_2) - f(\vec{x}_1) = (0, 0) - (1, 0) = (-1, 0)$.

We have $\vec{x}_1 + c(\vec{x}_2 - \vec{x}_1) = (1 - c, c)$. We have

$$\begin{aligned}
Df &= \begin{bmatrix} 2x & 0 \\ 2xy & x^2 \end{bmatrix} \\
Df(1-c, c) &= \begin{bmatrix} 2-2c & 0 \\ 2c-2c^2 & c^2-2c+1 \end{bmatrix} \\
Df(1-c, c) \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} &= (2c-2, -2c+2c^2+(c^2-2c+1)) \\
&= (2c-2, 3c^2-4c+1).
\end{aligned}$$

If $2c - 2 = -1$ then $2c = 1$ and so $c = 1/2$. However, $3(1/2)^2 - 4(1/2) + 1 = -1/4$, and so there is no value of c so that $Df(\vec{x}_1 + c(\vec{x}_2 - \vec{x}_1))(\vec{x}_2 - \vec{x}_1)(\vec{x}_2 - \vec{x}_1) = f(\vec{x}_2) - f(\vec{x}_1)$.

The Jacobian matrix of my example function is given above. Fix c with $0 \le c \le 1$. The matrix $Df(1 - c, c)$ is diagonalizable with eigenvalues $\lambda_1 = 2 - 2c$ and $\lambda_2 = c^2 - 2c + 1$. (These are only equal when $c = 1$ and for this value of c, we have that $Df(1 - c, c)$ is the zero matrix.) Let $\vec{v}_1$ and $\vec{v}_2$ be the corresponding linearly independent eigenvectors. For any vector $\vec{x} \in \mathbb{R}^2$, we may write

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$$

and we have

$$Df(1 - c, c)\vec{x} = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2.$$

For $\vec{x} \neq 0$, we have

$$\begin{aligned}
\frac{\|Df(1-c,c)\vec{x}\|_2^2}{\|\vec{x}\|_2^2} &= \frac{\|c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2\|^2}{\|c_1\vec{v}_1 + c_2\vec{v}_2\|} \\
&= \frac{c_1^2\lambda_1^2\vec{v}_1\cdot\vec{v}_1 + 2c_1c_2\lambda_1\lambda_2\vec{v}_1\cdot\vec{v}_2 + c_2^2\lambda_2^2\vec{v}_2\cdot\vec{v}_2}{c_1^2\vec{v}_1\cdot\vec{v}_1 + 2c_1c_2\vec{v}_1\cdot\vec{v}_2 + c_2^2\vec{v}_2\cdot\vec{v}_2} \\
&\leq \frac{(\max\{\lambda_1^2,\lambda_2^2\})\left(c_1^2\vec{v}_1\cdot\vec{v}_1 + 2c_1c_2\vec{v}_1\cdot\vec{v}_2 + c_2^2\vec{v}_2\cdot\vec{v}_2\right)}{c_1^2\vec{v}_1\cdot\vec{v}_1 + 2c_1c_2\vec{v}_1\cdot\vec{v}_2 + c_2^2\vec{v}_2\cdot\vec{v}_2} \\
&= \max\{\lambda_1^2,\lambda_2^2\}.
\end{aligned}$$

Therefore, we have that if $\|\vec{x}\|_2 = 1$, then

$$\|Df(1-c,c)\vec{x}\|_2 \leq \max\{|\lambda_1|,|\lambda_2|\}.$$

Moreover, we have equality if we take $\vec{x}$ to be parallel to one of $\vec{v}_1$ or $\vec{v}_2$ (whichever corresponds to the eigenvalue with largest absolute value). It follows that

$$\|Df(1-c,c)\|_{op} = \max\{|\lambda_1|,|\lambda_2|\}.$$

We have $|\lambda_1| = 2-2c \leq 2$ and $|\lambda_2| = (c-1)^2 \leq 1$. Therefore, $M = \sup\{\|Df(1-c,c)\|_{op} : 0 \leq c \leq 1\} = 2$. We have $\|f(\vec{x}_2) - f(\vec{x}_1)\| = 1$ and $\|\vec{x}_2 - \vec{x}_1\| = 1$ and so the mean value inequality reads

$$1 = \|f(\vec{x}_2) - f(\vec{x}_1)\| \leq 2\|\vec{x}_2 - \vec{x}_1\| = 2.$$