A Transport Redundancy Approach to Spatial Clustering

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The Interest

In this thesis, we consider a new random walks-based method for measuring clustering on finite graphs. The method is versatile and can provide valuable insight into the locations of centers of clusters and outliers. Some results and applications are discussed.

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- In determining its next step, the walker chooses uniformly randomly among its neighbors, until such a point that a map is landed on.
- At that point, the walker is sent on a shortest path to the target.
- Of particular interest will be the expected amount of time that a randomly placed walker takes to reach a randomly chosen target among the maps.

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where $t \geq 0$ is a delay factor.

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If X_0 is taken uniformly randomly from V, we then have

$$\bar{e}(\mathcal{S}, \dagger) = \frac{1}{N}\left(\sum_{v \in V} e(\mathcal{S}, \dagger, v)\right).$$

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Define $\mathcal{E}$, via

$$\mathcal{E}(\mathcal{S}) \stackrel{def}{=} \frac{1}{n}\left(\sum_{s_i \in \mathcal{S}} \bar{e}(\mathcal{S}, s_i)\right).$$

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An important quantity, in what follows, $\mathcal{E}(\mathcal{S})$, is the expected amount of time to reach a uniformly randomly chosen target (from $\mathcal{S}$) beginning at a uniformly randomly chosen starting point.

A Simple One-Dimensional Cycle Example

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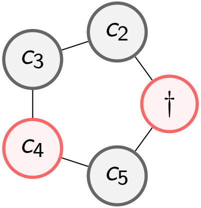
Consider a one-dimensional cycle $\mathcal{C}_5$ and $\mathcal{S} = \{c_1, c_4\}$ is the set of "maps". Let $t = 1$. First, set c_1 to be the "target". Let e_i denote $e(\mathcal{S}, \dagger, c_i)$.



We then have $e_1 = 0$, $e_2 = 1 + (e_1 + e_3)/2$, $e_3 = 1 + (e_2 + e_4)/2$, $e_4 = 2$ and $e_5 = 1 + (e_4 + e_1)/2$.

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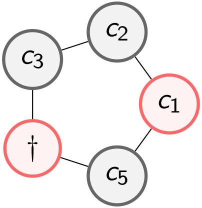
In matrix form, this gives $A\mathbf{e} = b$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}_{5\times 5} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{e} = A^{-1}\mathbf{b} = \begin{bmatrix} 0 \\ 8/3 \\ 10/3 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \bar{e}(\mathcal{S}, c_1) = \frac{1}{5}\left(\sum_{c_i \in \mathcal{C}_5} e_i\right) = 2$$

Set c_4 to be the target.



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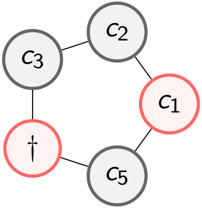

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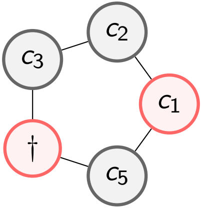

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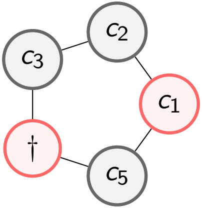

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$$\mathcal{E}(\mathcal{S}) = \frac{1}{2}\left(\sum_{s_i \in \mathcal{S}} \bar{e}(\mathcal{S}, s_i)\right) = 2$$

$\square$

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- If $c_i \notin \mathcal{S}$, then the ith row of A has two more $-1/2$ entries corresponding to the neighboring c_{i+1} and c_{i-1} on the cycle.
- The matrix becomes increasingly sparse, for a fixed graph, as the number of maps increases.

Preliminaries and Notations for $\mathcal{C}_N$

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For convenience, for $s_1, s_2 \in \mathcal{S}$, where $s_1 < s_2$, set

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$$a \stackrel{def}{=} dist(s_1, \dagger) \quad \text{and} \quad b \stackrel{def}{=} dist(s_2, \dagger),$$

and

$$K_t \stackrel{def}{=} \frac{1}{|\mathcal{I}|} \sum_{v_i \in \mathcal{I}} e\left(\mathcal{S}, \dagger, v_i\right).$$

Existing Work for $\mathcal{C}_N$

Lemma

(Wei) Suppose $s_1, s_2 \in \mathcal{S}$, $s_1 < s_2$ and $\mathcal{I} \cap \mathcal{S} = \emptyset$. We then have

$$K_1 = \frac{d^2 + d}{6} + \frac{a + b}{2},$$

where

$$d = |\mathcal{I}| + 1.$$

Existing Work for $\mathcal{C}_N$

Lemma

(Wei) Suppose $s_1, s^*, s_2 \in \mathcal{S}$, $s_1 < s^* < s_2$, *and* $\mathcal{I} \cap \mathcal{S} = s^*$, *and* $r = dist(s^*, s_1)$*. Then,* K_1 *thought of as a function of* r*, is minimized when* $r = \dfrac{d-1}{2} + \dfrac{b-a}{2d}$*, where* $d = |\mathcal{I}| + 1$*.*

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Definition

Suppose on a cycle $\mathcal{C}_N$, $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$, with $s_1 < \ldots < s_n$. The set $\mathcal{S}$ is said to be "evenly distributed" on $\mathcal{C}_N$, if for $\mathcal{T} = \{s_2 - s_1, s_3 - s_2, \ldots, s_n - s_{n-1}, s_1 - s_n + N\}$, we have $\max(\mathcal{T}) - \min(\mathcal{T}) \leq 1$.

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Theorem

(Wei) The quantity $\mathcal{E}(\mathcal{S})$ is minimized over all n-subsets of $\mathcal{S}$ when the nodes in $\mathcal{S}$ are evenly distributed on the cycle $\mathcal{C}_N$.

Question: What about $t \neq 1$?

Example. Consider the cycle $\mathcal{C}_{10}$ and subsets $\mathcal{S}$ satisfying $|\mathcal{S}| = 5$. Consider the cases $t = 0, 1$ and 10.

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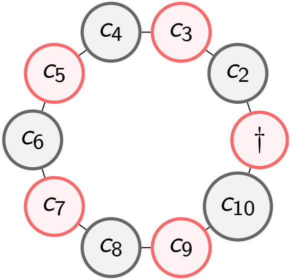

Figure : All maps are evenly distributed.

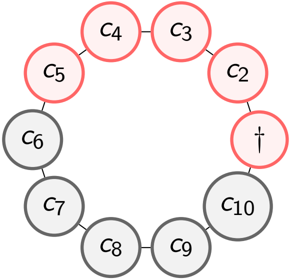

Figure : All maps are in one cluster.

	1	2	3	4	5	6	7	8	9	10	$\mathcal{E}(\mathcal{S})$
1	1	0	1	0	1	0	1	0	1	0	2.90
2	1	1	0	1	0	1	0	1	0	0	3.12
3	1	1	0	1	0	1	0	0	1	0	3.16
4	1	1	0	1	1	0	0	1	0	0	3.22
5	1	1	1	0	1	0	0	1	0	0	3.28
6	1	1	0	1	0	0	1	1	0	0	3.30
7	1	1	1	0	0	1	0	1	0	0	3.38
8	1	1	1	0	1	0	0	0	1	0	3.46
9	1	1	0	1	1	0	0	0	1	0	3.54
10	1	1	1	0	0	0	1	0	1	0	3.56
11	1	1	0	1	0	1	1	0	0	0	3.62
12	1	1	1	1	0	0	1	0	0	0	3.72
13	1	1	1	0	0	1	1	0	0	0	3.84
14	1	1	1	1	0	1	0	0	0	0	4.28
15	1	1	1	0	1	1	0	0	0	0	4.32
16	0	1	1	1	1	1	0	0	0	0	5.30

Table : Values of $\mathcal{E}(\mathcal{S})$ for $N = 10$, $n = 5$ and $t = 1$. One representative set is given for each of the sixteen distinct values of $\mathcal{E}(\mathcal{S})$.

	1	2	3	4	5	6	7	8	9	10	$\mathcal{E}(\mathcal{S})$
1	1	0	1	0	1	0	1	0	1	0	0.5
2	1	1	0	1	0	1	0	1	0	0	0.7
3	1	1	1	0	1	0	0	1	0	0	0.9
4	1	1	1	0	1	0	1	0	0	0	1.2
5	1	1	1	1	0	0	1	0	0	0	1.4
6	1	1	1	1	0	1	0	0	0	0	2.1
7	0	1	1	1	1	1	0	0	0	0	3.5

Table : Values of $\mathcal{E}(\mathcal{S})$ for $N = 10$, $n = 5$ and $t = 0$. One representative set is given for each of the seven distinct values of $\mathcal{E}(\mathcal{S})$.

	1	2	3	4	5	6	7	8	9	10	$\mathcal{E}(\mathcal{S})$
1	0	1	1	1	1	1	0	0	0	0	21.5
2	1	1	1	0	1	0	0	0	1	0	23.8
3	1	1	1	1	0	1	0	0	0	0	23.9
4	1	1	0	1	1	0	0	1	0	0	24.1
5	1	1	1	0	1	1	0	0	0	0	24.3
6	1	0	1	0	1	0	1	0	1	0	24.5
7	1	1	1	1	0	0	1	0	0	0	24.6
8	1	1	1	0	1	0	0	1	0	0	24.7
9	1	1	1	0	1	0	1	0	0	0	24.8
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Large delay factors, t, can greatly influence cluster detection. This will be evident for the two-dimensional grid.

- Consider a two-dimensional square toroidal grid on $m \times m$ nodes (i.e. $N = m^2$), $G(V, E) = [(i, j)]$ with (i, j) representing the vertex in the ith row and jth column of the grid.

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- For convenience, we map the pair (i, j) to one dimension, via

$$\phi(i, j) = (i - 1)\, m + j, \qquad \textit{for } i, j = 1, 2, \ldots, m.$$

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- Similar to before, let $e_i = E[T | \mathcal{S}, \dagger, \phi^{-1}(i)]$ denote the expected amount of time for a walker starting at $\phi^{-1}(i)$ to reach the target $\dagger$. Suppose the set $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ contains the designated maps while $\dagger \in \mathcal{S}$ is the target.

It is worth noting that if the pair (i, j) satisfies either i or j is equal to 1 or m, then by the toroidal effect, it still has four neighbors.

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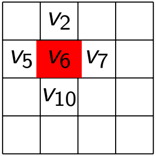

Figure : Neighbors of v_6.

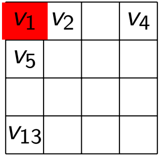

Figure : Neighbors of v_1.

For convenience in what follows, we will at times refer to the pair $\phi^{-1}(i)$ simply as i or potentially v_i.

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$$r_{i,j} = \frac{\left(\bar{e}\left(\left(\mathcal{S} \setminus s_j\right), s_i\right)\right)}{\left(\bar{e}\left(\mathcal{S}, s_i\right)\right)} \qquad i, j = 1, 2, \ldots, n.$$

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We refer to the matrix $[\rho_{ij}]$ as the *value ratio matrix*, $\mathcal{V}_{rat}$ and denote this by

$$\mathcal{V}_{rat} \stackrel{def}{=} [\rho_{i,j}] = \left[\frac{r_{i,j}}{r_{j,i}}\right].$$

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The greater w_j is, the more valuable s_j is to the other maps in $\mathcal{S}$, as a whole.

Two-Dimensional Toroidal Grid Examples

Suppose we have a two-dimensional square toroidal grid with $N = 100$ (10×10), and that $\mathcal{S} \in V$ is the set of maps. Consider cases where $t = 1, 100$. Plots with maps shaded via the ranks of the shading values are given below; all the shaded locations represent maps. The darker the color is, the smaller the corresponding shading value is. The overall quantity $\mathcal{E}(\mathcal{S})$ is also included.

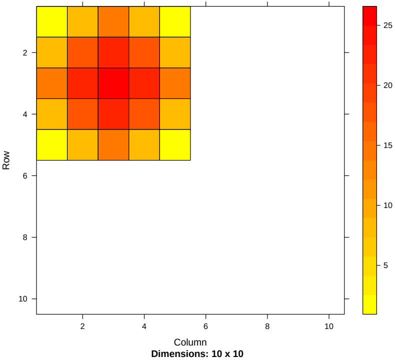


Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 21.13$.

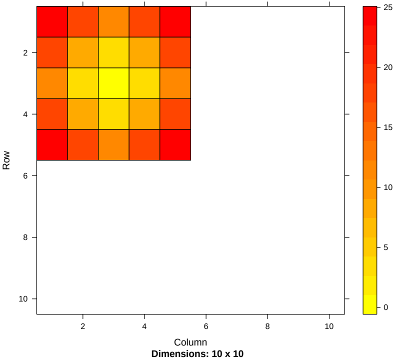


Figure : Here, $t = 100$, $\mathcal{E}(\mathcal{S}) = 369.86$.

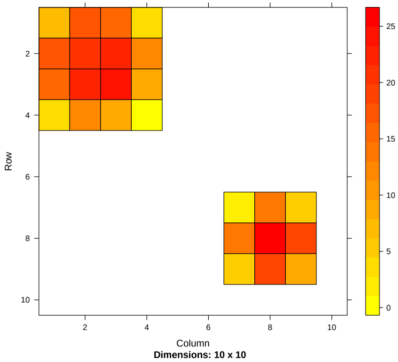


Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 15.69$.

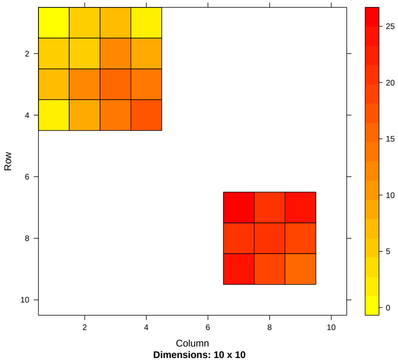


Figure : Here, $t = 100$, $\mathcal{E}(\mathcal{S}) = 498.27$.

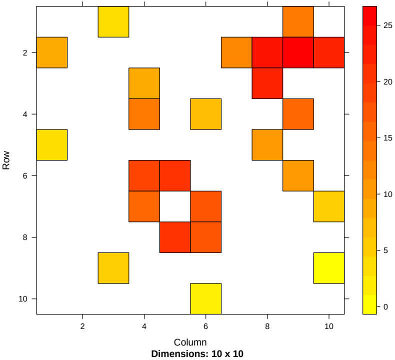


Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 8.62$.

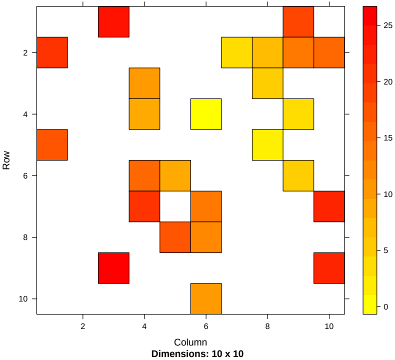


Figure : Here, $t = 100$, $\mathcal{E}(\mathcal{S}) = 501.07$.

Some additional plots for the case $t = 1$ with $n = 34$ maps. Note the change in $\mathcal{E}(\mathcal{S})$ as the maps become less "clustered".

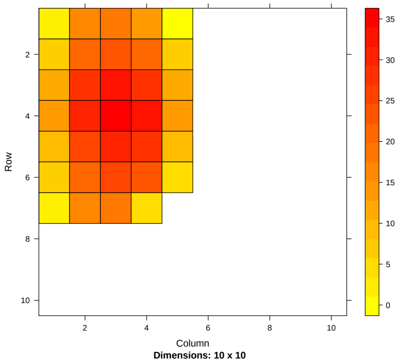


Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 15.04$.

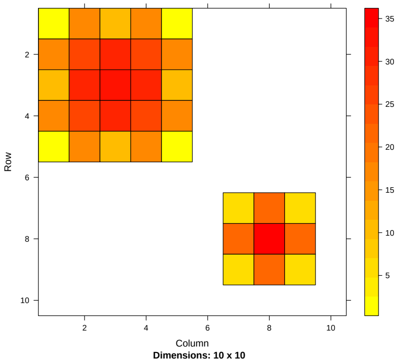


Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 11.66$.

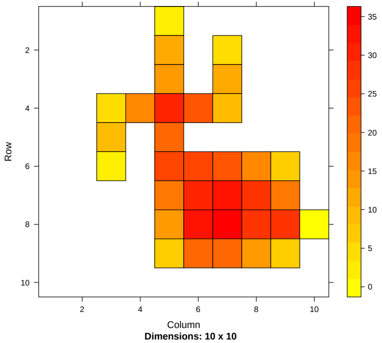


Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 11.61$.

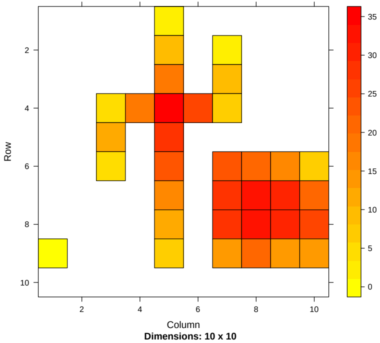


Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 10.44$.

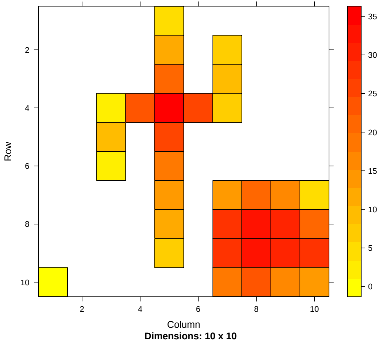
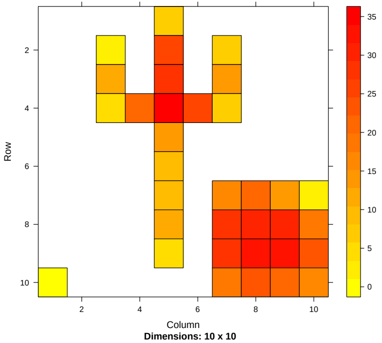


Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 10.24$.



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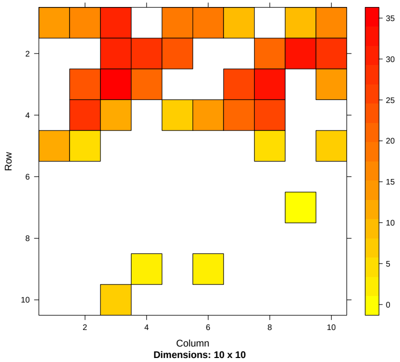
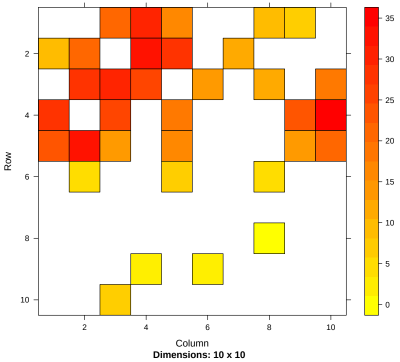


Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 9.42$.



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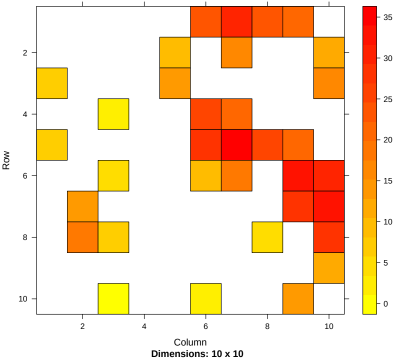

Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 7.76$.

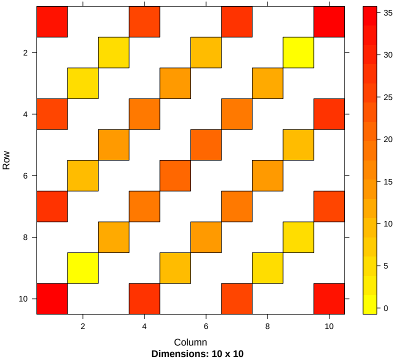

Figure : Here, $t = 1$, $\mathcal{E}(\mathcal{S}) = 6.48$.

Returning to the Cycle Case.

Recall $\{X_0, X_1, \ldots, X_{T_0}, \ldots\}$ be a random walk starting at $X_0 = x_0$. We then have

$$\begin{aligned} T_0 &= \min\{i \geq 0 | X_i \in \mathcal{S}\}, \\ T_1 &= dist(X_{T_0}, \dagger) \qquad \text{and} \\ T &= T_0 + tT_1. \end{aligned}$$

where $t \geq 0$ is a delay factor.

Recall for $s_1, s_2 \in \mathcal{S}$, where $s_1 < s_2$, set

$$\mathcal{I} = \mathcal{I}_{s_1,s_2} \overset{def}{=} (s_1, s_2) = \{s_1 + 1, s_1 + 2, \ldots, s_2 - 1\},$$

$$a \overset{def}{=} dist(s_1, \dagger) \quad \text{and} \quad b \overset{def}{=} dist(s_2, \dagger),$$

and

$$K_t \overset{def}{=} \frac{1}{|\mathcal{I}|} \sum_{v_i \in \mathcal{I}} e\left(\mathcal{S}, \dagger, v_i\right).$$

Minimization with $t \neq 1$

Lemma

Suppose $\mathcal{I} \cap \mathcal{S} = \emptyset$ *and* $t \geq 0$*, is the delay factor. Then, with* $d = |\mathcal{I}| + 1$*,*

$$K_t = \frac{d^2 + d}{6} + \frac{(a+b)t}{2}.$$

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Lemma

Suppose $\mathcal{I} \cap \mathcal{S} = s^*$, $r = dist(s^*, s_1)$ and $t \geq 0$, is the delay factor. Then K_t is minimized when

$$r = \frac{d}{2} + \frac{(b-a-d)t}{2d}.$$

Maximization

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Suppose $\mathcal{I} \cap \mathcal{S} = s^*$, $r = dist(s^*, s_1)$ and $t \geq 0$, is the delay factor. Then K_t is maximized when

$$r = \begin{cases} d-1 & \text{if } t \geq \frac{2}{d} \\ 1 & \text{if } t \leq \frac{2}{d} \end{cases}$$

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Corollary

Suppose $\mathcal{I} \cap \mathcal{S} = s^*$, $r = dist(s^*, s_1)$. Then K_1 is maximized when $r = d-1$.

Example

$\dagger$	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	$\bar{e}(\mathcal{S}, c_1)$
1		1						1	1			1	6.03

Example

†	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	$\bar{e}(\mathcal{S}, c_1)$
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1							1	1	1	1			7.34

Lemma

Suppose on a cycle $\mathcal{C}_N$, $\mathcal{S}_1$ and $\mathcal{S}_2$ are two differing map sets. In addition, suppose $P \notin \mathcal{I}$,

$$1 \leq l < k < s_2 - s_1 - 1 = b - a - 1 = d - 1,$$

$$\mathcal{I} \cap \mathcal{S}_1 = \{s_1 + 1, s_1 + 2, \dots, s_1 + k\}$$

and

$$\mathcal{S}_2 \cap \mathcal{I} = \{s_1 + 1, \dots, s_1 + l\} \cup \{s_2 - (k - l), \dots, s_2 - 1\}.$$

Then K_t thought of as a function of map sets satisfies,

$$K_t(\mathcal{S}_1) = K_t(\mathcal{S}_2).$$

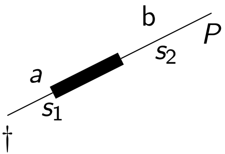

Figure : The height indicates the distance to †. The bold parts indicate maps $(\mathcal{S}_1 \cap \mathcal{I})$.

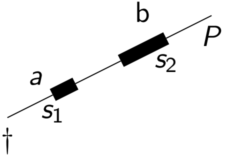

Figure : The height indicates the distance to †. The bold parts indicate maps $(\mathcal{S}_2 \cap \mathcal{I})$.

Lemma

Suppose $\mathcal{I} \cap \mathcal{S} = \{s^*, s^* + 1, ..., s^* + k - 1\}$ *and* $dist(s_1, s^*) = r$. *Then* $K_1(r)$ *is maximized when* $r = d - k$, *where* $d = |\mathcal{I}| + 1$.

Proof.

Proof. Suppose N is even (the case N is odd is similar). Assume $b \geq a$ and denote by P, the furthest vertex from $\dagger$. Set $p = dist(P, \dagger)$.

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Case 1: Assume $b = a + d$.

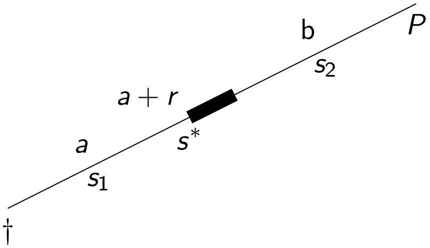

Figure : The case $b = a + d$. The height is intended to indicate the distance to $\dagger$. The bold lines indicate map clusters.

Case 2: Assume $b < a + d$.

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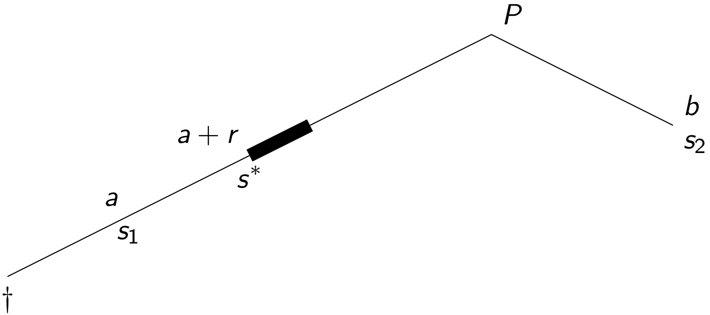

Figure : The case $b \leq d + a - 1$. The height is intended to indicate distance to $\dagger$. Here, the whole cluster lies on the left arm. The bold line indicates maps.

Subcase 1: Neither arm is able to hold the whole cluster.

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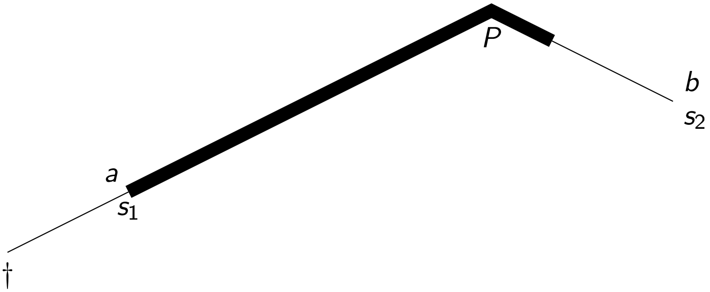

Figure : The case $b \leq d + a - 1$. The height is intended to indicate distance to $\dagger$. The bold lines indicate map clusters.

Subcase 2: The short arm is not able to hold the whole cluster.

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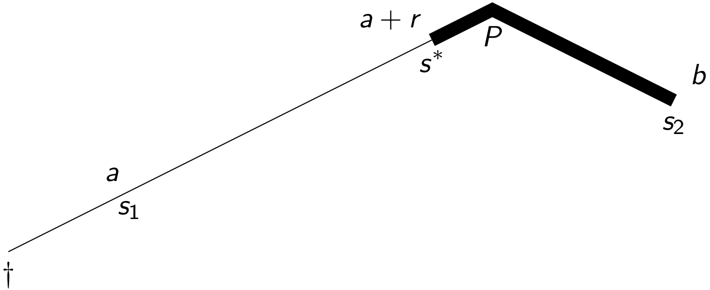

Figure : The case $b \leq d + a - 1$. The height is intended to indicate distance to $\dagger$. The bold lines indicate map clusters.

Subcase 3: Both arms are long enough to hold the whole cluster.

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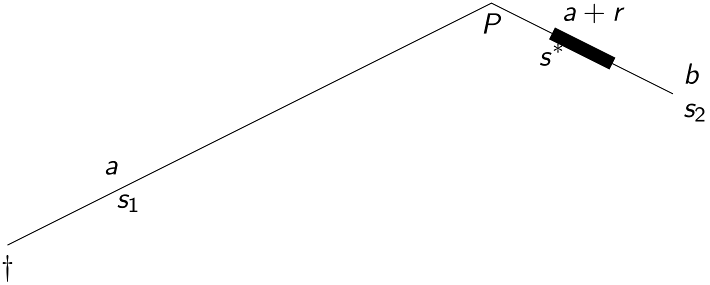

Figure : The case $b \leq d + a - 1$. The height is intended to indicate distance to $\dagger$. The bold lines indicate a map cluster.

Theorem

Suppose $0 \leq t \leq 1$. The quantity $\mathcal{E}(\mathcal{S})$ is maximized over all n-subsets of $\mathcal{S}$ when the nodes in $\mathcal{S}$ are consecutively clustered on the cycle $\mathcal{C}_N$.

Proof

- Suppose $\mathcal{S} = \{s_1, \ldots, s_n\}$ is a subset of $\mathcal{C}_N$, which does not consist of consecutive nodes in $\mathcal{C}_N$.

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- For each $1 \leq i \leq n$, we employ the preceding lemmas to construct a chain of n-sets, $\mathcal{S}_{i,1}, \mathcal{S}_{i,2}, \ldots, \mathcal{S}_{i,\nu_i}$, with the last, S_{i,ν_i}, consisting of two disjoint non-neighboring sets of consecutive nodes in $\mathcal{C}_N$, satisfying

  $$\bar{e}(\mathcal{S}, s_i) \leq \bar{e}(\mathcal{S}_{i,1}, s_i) \leq \cdots \leq \bar{e}(\mathcal{S}_{i,\nu_i}, s_i) \leq \bar{e}(\{1, 2, \ldots, n\}, i)$$

  where $\mathcal{S}_\nu$ is comprised of consecutive nodes in $\mathcal{C}_N$.

- Finally,

$$\begin{aligned}
\mathcal{E}(\mathcal{S}) &= \frac{1}{n}\sum_{1\leq i\leq n}\bar{e}(\mathcal{S}, s_i) \\
&\leq \frac{1}{n}\sum_{1\leq i\leq n}\bar{e}(\{1,2,\ldots,n\}, i) \\
&\leq \mathcal{E}(\{1,2,\ldots,n\}),
\end{aligned}$$

and the theorem is proven.

Example

$\dagger$	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	$\bar{e}(\mathcal{S}, c_1)$
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1		1						1	1			1	6.03

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1		1						1	1			1	6.03

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1							1	1	1			1	7.34

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1		1						1	1			1	6.03

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1							1	1	1			1	7.34

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1							1	1	1	1			7.34

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1	1	1	1	1									11.08

Thank you!