Math 711 Study Guide

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1 Metric Spaces

1.1 Definition of a Metric Space and Basic Examples

Definition 1.1. A **metric space** is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}^+$ is a function with the following properties:

(ia) $d(x, y) \geq 0 \qquad \forall x, y \in X$

(ib) $d(x, y) = d(y, x) \qquad \forall x, y \in X$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, z) \leq d(x, y) + d(y, z) \qquad \forall x, y, z \in X$

The function d is called a **metric**, and it is common to refer to "the metric space X", when the definition of d is already understood.

Example 1.1. (Metrics on $\mathbb{R}$ and $\mathbb{R}^n$)

1. The space $\mathbb{R}$ is a metric space with metric $d(x, y) = |y - x|$.
2. The space $\mathbb{R}^n$ is a metric space with metric
$$d_1(x, y) = \sum_{i=1}^{n} |y_i - x_i|.$$
3. The space $\mathbb{R}^n$ is a metric space with metric
$$d_2(x, y) = \|y - x\| := \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2}.$$
4. The space $\mathbb{R}^n$ is a metric space with metric
$$d_\infty(x, y) = \max_{i=1..n} |y_i - x_i|.$$

5. For p with $1 \leq p < \infty$, the space $\mathbb{R}^n$ is a metric space with the metric

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

6. The space $\mathbb{R}$ is a metric space with the **discrete metric**

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Proposition 1.1. (Inequalities)

1. **(Young's Inequality)**: If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b > 0$, then

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

2. **(Hölder's Inequality)**: If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}.$$

3. **(Minkowski's Inequality)**: If $p > 1$ then

$$\left(\sum_{1}^{2} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{1}^{2} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{1}^{2} |y_i|^p\right)^{\frac{1}{p}}.$$

4. **(Generalized Hölder's Inequality)**: If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, and $x, y, z \in \mathbb{R}^n$, then

$$\sum_{1}^{n} |x_i y_i z_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |z_i|^r\right)^{\frac{1}{r}}$$

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1.2 Basic Topological Definitions in a Metric Space

Definition 1.2. (Topology): Suppose that X is a set. A collection $\mathcal{U}$ of subsets of X is called a **topology** on X if

1. X and $\emptyset$ are elements of $\mathcal{U}$.
2. Whenever $U_i \in \mathcal{U}$ for $i = 1 \ldots n$, then $\cap_{i=1}^{n} U_i \in \mathcal{U}$.
3. For any index set A, if $U_\alpha \in \mathcal{U}$ for every $\alpha \in A$, then $\cup_{\alpha \in A} U_\alpha \in \mathcal{U}$.

If $\mathcal{U}$ is a topology on X, then the elements of $\mathcal{U}$ are called the **open** sets of X.

Definition 1.3. (The Metric Topology): Let (X, d) be a metric space.

1. **(Open Ball):** For $x \in X$ and $\epsilon > 0$, the set $B_\epsilon(x) := \{y \in X : d(y, x) < \epsilon\}$ is called the **open ball of radius ϵ centered at x.**

2. **(Open Set):** A set $U \subset X$ is called **open** if, for every $x \in U$, $\exists \epsilon > 0$ such that $B_\epsilon(x) \subset U$.

3. **(Closed Set):** A set $C \subset X$ is called **closed** if its complement is open in X.

4. **(Metric Topology):** The collection of open sets in a metric space X, as defined above, form a topology on X. This topology is called the **metric topology** on X, or, to be more specific, the **topology induced by the metric d on the set X.**

Definition 1.4. (Sequences in Metric Spaces)

1. **(Sequence):** A **sequence** in a space X is a function from $\mathbb{N}$ to X.

2. **(Convergent Sequence):** A sequence (x_n) in a metric space (X, d) **converges** if there is an $x \in X$ such that $d(x_n, x) \to 0$. That is, given any $\epsilon > 0$ there is an $N > 0$ such that $d(x_n, x) < \epsilon$ for all $n > N$.

3. **(Bounded Sequence)** A sequence in a metric space X is **bounded** if there exists an open ball B in X such that $x_n \in B$ for all n

4. **(Cauchy Sequence):** A sequence (x_n) in a metric space (X, d) is **Cauchy** if given any $\epsilon > 0$ there is an $N > 0$ such that $d(x_n, x_m) < \epsilon \ \forall n, m > N$.

5. **(Complete Metric Space):** A metric space is **complete** if every Cauchy sequence in the metric space converges (to an object within the metric space).

Example 1.2. (Complete/Not Complete Metric Spaces)

1. The metric space $\mathbb{Q}$ is not complete.

2. The metric spaces $\mathbb{R}$ and $\mathbb{R}^n$ are complete with the usual Euclidean metric.

3. The set $(0, 1)$ is not complete w.r.t the standard metric.

Proposition 1.2. (More Properties of Sequences) Let x_n be a sequence in a metric space (X, d).

1. If x_n converges in X then x_n is Cauchy.

2. If x_n is Cauchy, then x_n is bounded.

Definition 1.5. (Limit/Accumulation Point):

1. **(Version 1)** Let X be a metric space and let S be a subset of X. We say that $x \in X$ is an **accumulation point** or **limit point** of S if, $\forall \epsilon > 0$, $\exists s \in S$ such that $s \neq x$ and $d(s, x) < \epsilon$.

2. **(Version 2)** Let X be a metric space and let S be a subset of X. We say that $x \in X$ is an **accumulation point** of S if there is a sequence (s_n) of elements of S, none of which equals x, such that $s_n \to x$.

Definition 1.6. (Dense Subset)

1. **(Version 1):** A set C in a metric space X is **dense** in X if for every $x \in X$ there is a sequence $(c_n) \subset C$ such that $\lim(c_n) = x$.

2. **(Version 2):** A set C is **dense** in X if given any $x \in X$ and any $\epsilon > 0$ there is a $c \in C$ such that $d(x, c) < \epsilon$.

Example 1.3. (Examples of Dense Subsets of Metric Spaces):

1. The set $\mathbb{Q}$ is dense in $\mathbb{R}$.

2. The set of all n-tuples of rational numbers is dense in $\mathbb{R}^n$.

3. The set of polynomials is dense in $C([0, 1])$

4. Every metric space is isometric to a dense subset of a complete metric

5. The set of eventually zero sequences is dense in l^2 space (its completion).

Proposition 1.3. (Limit Point Properties)

1. **(Closed Sets)** A set C in a metric space X is closed if and only if it contains all of its accumulation points.

2. **(Dense Subsets)** A set C is dense in a metric space X if and only if every element of X is an accumulation point of C.

Corollary 1.1. (Closed inherits Completeness) A closed subset of a complete metric space is complete.

Theorem 1.1. (Bolzano-Weierstrass Theorem)

1. **(Version 1)** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

2. **(Version 2):** Every bounded, infinite subset of $\mathbb{R}^n$ has an accumulation point.

Definition 1.7. (Compactness)

1. **(Open Cover):** Let $K \subset X$. A collection $\mathcal{O}$ of open sets in X is an **open cover** of K if every point $x \in K$ is in some $U \in \mathcal{O}$.

2. **(Compact):** A set K in the metric space X is **compact** if every open cover of K has a finite subcover.

3. **(Sequentially Compact):** A set K in the metric space X is **sequentially compact** if every sequence in K has a subsequence that converges to an element in K.

4. **(Equivalence of Sequential/Open Cover Compactness):** If X is a metric space, then $K \subset X$ is compact if and only if K is sequentially compact.

5. **(General Topological Spaces)** If X is a general topological space (not necessarily a metric space), then we can only say that sequential compactness implies compactness, but not the converse.

Definition 1.8. (Continuity at a point):

1. **(Sequential continuity):** Suppose that X, Y are metric spaces and $f : X \to Y$. Then f is **continuous** at a point $x_0 \in X$ if given any sequence (x_n) such that $x_n \to x_0$ in X, we have $f(x_n) \to f(x_0)$ in Y.

2. **(Epsilon-delta continuity):** Suppose that X, Y are metric spaces and $f : X \to Y$. Then f is **continuous** at a point $x_0 \in X$ if given any $\epsilon > 0$ there is a $\delta > 0$ such that if $d_X(x, x_0) < \delta$ then $d_Y(f(x), f(x_0)) < \epsilon$.

Definition 1.9. (Continuity on a domain):

1. **(Continuity on X)** Suppose that X, Y are metric spaces and $f : X \to Y$. Then f is **continuous** on X if f is continuous at x for all $x \in X$.

2. **(Inverse Image)** Suppose that X, Y are sets and $f : X \to Y$. Let $U \subset Y$. Then the **inverse image** of U under f is $f^{-1}(U) := \{x \in X | f(x) \in U\}$.

3. **(Open-Set Definition)** Suppose that X, Y are metric spaces and $f : X \to Y$. Then f is continuous on X if and only if, for every $U \subset Y$ open, $f^{-1}(U)$ is open in X.

Example 1.4. (Discontinuous Functions):

1. **(Continuous at only one point):** The function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}$$

is only continuous at $x = 0$.

2. **(Discontinuous Everywhere):** The function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not continuous anywhere.

Theorem 1.2. (Extreme Value Theorem)

1. **(Continuous Image of Compact Sets)** Let X, Y be metric spaces and $f : X \to Y$ be continuous. If X is a compact space, then $f(X)$ is compact.

2. **(Corollary: Extreme Value Theorem)** If X is a compact metric space and $f : X \to \mathbb{R}$ is continuous then f is bounded and attains its minimum and maximum values on X.

1.3 Definition of a Normed Linear Space

Definition 1.10. (Real Vector Space): A set X equipped with two operations $+ : X \times X \to X$ (vector addition) and $\cdot : \mathbb{R} \times X \to X$ (scalar multiplication) is called a **real vector space** or **real linear space** if the following properties are satisfied:

1. $\forall u, v, w \in X$, $u + (v + w) = (u + v) + w$,
2. $\forall u, v \in X$, $u + v = v + u$,
3. $\exists 0 \in X$ such that $v + 0 = v \ \forall v \in X$,
4. For every $v \in X$, there exists an element $-v \in X$, called the additive inverse of v, such that $v + (-v) = 0$,
5. $\forall a \in \mathbb{R}$, $\forall u, v \in X$, $a(u + v) = au + av$,
6. $\forall a, b \in \mathbb{R}$, $\forall v \in X$, $(a + b)v = av + bv$,
7. $\forall a, b \in \mathbb{R}$, $\forall v \in X$, $a(bv) = (ab)v$, and
8. $1v = v$.

Definition 1.11. (Norm on a Real Vector Space): If X is a real vector space and $\| \cdot \| : X \to \mathbb{R}^+$ satisfies

(ia) $\|x\| \geq 0$ for all $x \in X$

(ib) $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X$ and all $\alpha \in \mathbb{R}$

(iii) (Triangle Inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

then $(X, \| \cdot \|)$ is called a **normed linear space**, and $\| \cdot \|$ is called the **norm** on X.

Definition 1.12. (Banach Spaces) A **Banach space** is a complete normed linear space.

Example 1.5. (Examples of Banach Spaces):

1. $\mathbb{R}^n$ is a Banach space with respect to any l^p norm (including l^∞).
2. The set of bounded real sequences is a Banach space with respect to the uniform norm.
3. The space l^2 is a Banach space w.r.t the l^2 norm.
4. The space $C([0, 1])$ is a Banach space w.r.t to the uniform norm.
5. The space $C^1([0, 1])$ is a Banach space w.r.t to the norm given by:

$$\|f\| = \|f\|_\infty + \|f'\|_\infty$$

Definition 1.13. (Inner Product Space) An **inner product** on a real vector space X is a function that takes each ordered pair (u, v) of elements of X to a number $\langle u, v\rangle \in \mathbb{R}$ such that $\langle , \rangle$ has the following properties:

(ia) (Positivity) $\langle v, v\rangle \geq 0$ for all $v \in X$

(ib) (Definiteness) $\langle v, v\rangle = 0$ if and only if $v = 0$.

(ii) (Homogeneity) $\langle au, v\rangle = a\langle u, v\rangle = \langle u, av\rangle$ for all $u, v \in X$ and all $a \in \mathbb{R}$

(iiia) (Additivity)$\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$ for all $u, v, w \in X$

(iiib) (Additivity) $\langle u, v + w\rangle = \langle u, v\rangle + \langle uw\rangle$ for all $u, v, w \in X$

(iv) (Symmetry) $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in X$

An **inner-product space** is a real vector space X with an inner product defined on it.

Definition 1.14. (Norms induced by an Inner Product) If V is an inner product space, then $(V, \|\cdot\|)$ is a normed linear space with $\|x\| = \sqrt{\langle x, x\rangle}$.

Theorem 1.3. (Metrics Induced by Norms) If $(X, ||\cdot||)$ is a normed linear space, then (X, d) is a metric space with $d(x, y) = ||x - y||$.

Definition 1.15. (Hilbert Space) A complete inner-product space is called a **Hilbert Space**.

Example 1.6. (Hilbert Spaces)

1. The space l^2 is a Hilbert space. It is the only l^p space that is also an inner-product space.

2. $\mathbb{R}^n$ is a Hilbert space with the dot product (or Euclidean inner product). This induces the Euclidean norm, Euclidean metric, etc.

3. The space L^2 is a Hilbert space with $\langle , \rangle$ given by:

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$$

Example 1.7. (Metric Spaces that are Not Normed Linear Spaces):

1. Let $(X, \|\cdot\|)$ be a normed linear space and d be the metric induced by $\|\cdot\|$. If W is not a subspace of X, then (W, d) is not a normed linear space.

2. Any set can be equipped with discrete metric. For example, let $X = \{1, 2, 3\}$ with the metric
$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Definition 1.16. (Equivalent Norms on a Normed Linear Space):

1. **(Equivalent Norms):** Suppose that X is a normed linear space with respect to two different metrics, $\|\cdot\|_1$ and $\|\cdot\|_2$. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are **equivalent** if there exist positive constants c and C such that, for every $x \in X$, $c\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1$.

2. **(Topological Properties):** Two equivalent norms generate the same topology. That is, if U is open with respect to the topology induced by the first norm, then U is open with respect to the topology induced by the second norm, and vice versa.

3. **(Equivalence Relation):** Suppose that X is a vector space, and let $\mathcal{N}$ be the collection of all norms on X. Then, norm equivalence as defined above is an equivalence relation on $\mathcal{N}$.

Example 1.8. (Equivalent and Non-Equivalent Norms):

1. The space $C^1([0,1])$ with the sup-norm $\|f\|_\infty$ and the norm given by

$$\|f\| = \|f\|_\infty + \|f'\|_\infty$$

2. The space $C([0,1])$ with norms $\|f\|_1 = \int_0^1 |f|dx$ and $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$

3. The space l^2 with norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$

4. All of the l^p norms on $\mathbb{R}^n$, $1 \leq p \leq \infty$ are equivalent.

5. In a finite-dimensional space, all norms are equivalent.

Proposition 1.4. (Adding Two Norms) If $\|\cdot\|_1$ and $\|\cdot\|_2$ are both norms on a given space X, then $\|\cdot\|_1 + \|\cdot\|_2$ is also a norm on X.

2 Important Examples of Metric Spaces

2.1 The standard metric on $\mathbb{R}$

Theorem 2.1. (Properties of $\mathbb{R}$):

1. **(Normed Vector Space):** $\mathbb{R}$ is a vector space and the function $|\cdot| : \mathbb{R} \to \mathbb{R}^+$ is a norm on $\mathbb{R}$. This is generally called the **standard norm** on $\mathbb{R}$, and it induces the **standard metric** on $\mathbb{R}$, which in turn induces the standard topology on $\mathbb{R}$.

2. **(Bolzano-Weierstrass Property):** Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

3. **(Completeness):** $\mathbb{R}$ is a complete metric space with respect to the metric induced by the standard norm on $\mathbb{R}$. $\mathbb{R}$ is a Banach space.

4. **(Heine-Borel Property):** A subset K of $\mathbb{R}$ (equipped with the standard topology) is compact if and only if it is both closed and bounded.

5. **(Nested Interval Property):** If $(I_n)_{n=1}^\infty$ is a sequence of nonempty, closed, bounded intervals in $\mathbb{R}$ so that, for each n, $I_n \supset I_{n+1}$, then $\cap_{n=1}^\infty I_n \neq \emptyset$.

6. **(Dense Subset):** $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 2.1. (Convergence of Sequences of Functions):

1. **(Point-wise convergence)** Let $K \subset \mathbb{R}$. Suppose that $f_n : K \to \mathbb{R}$ is a function for each n, and that $f : K \to \mathbb{R}$ is another function. We say that $f_n \to f$ point-wise or f_n **converges (point-wise) to** f if, $\forall x \in K$, $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\forall n > N$ $|f_n(x) - f(x)| < \epsilon$.

2. **(Uniform convergence)** Let $K \subset \mathbb{R}$. Suppose that $f_n : K \to \mathbb{R}$ is a function for each n, and that $f : K \to \mathbb{R}$ is another function. We say that $f_n \to f$ uniformly or f_n **converges uniformly to** f if $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\forall n > N$, $\forall x \in K$, $|f_n(x) - f(x)| < \epsilon$.

Proposition 2.1. (Uniform limit of continuous functions). If each of the f_n is continuous on K and $f_n \to f$ uniformly, then f is continuous.

Corollary 2.1. (Discontinuous Limit) Let f_n be a sequence of continuous functions on K. If f_n converges pointwise to a discontinuous function f, then f_n cannot converge uniformly to f.

Example 2.1. (Sequences of Functions)

1. Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^n$. Each f_n is continuous, however f_n converges point-wise to a discontinuous function.

2. The sequence of functions g_n on $[0,1]$ defined by:

$$g_n(x) = \begin{cases} (2x)^n & \text{if } x \leq \frac{1}{2} \\ 1 & \text{if } x \geq \frac{1}{2} \end{cases}$$

   converges point-wise to discontinuous function.

3. Define $h_n : [0,1] \to \mathbb{R}$ by

$$h_n(x) = \frac{1}{1 + n^2x^2}$$

   h_n converges point-wise to a discontinuous function

4. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{1}{n}\sin(nx)$$

   then $f_n \to 0$ uniformly on $[0,1]$.

5. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{x}{1 + nx^2}$$

   then $f_n \to 0$ uniformly on $[0,1]$.

2.2 Generalizing to $\mathbb{R}^n$

Theorem 2.2. (Properties of $\mathbb{R}^n$)

1. **(Normed Linear Space)** $\mathbb{R}^n$ is a vector space and the function $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}^+$ is a norm on $\mathbb{R}^n$. This is generally called the **standard or Euclidean norm** on $\mathbb{R}^n$, and it induces the **standard or Euclidean metric** on $\mathbb{R}^n$, which in turn induces the standard topology on $\mathbb{R}^n$.

2. **(Convergence)** A sequence (x_k) in $\mathbb{R}^n$ converges to x in $\mathbb{R}^n$ if and only if its components (x_{ki}) converge to x_i in $\mathbb{R}$ for each $i = 1, ..., n$.

3. **(Bolzano-Weierstrass Property)** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

4. **(Completeness)** $\mathbb{R}^n$ is a complete metric space with respect to the metric induced by the standard norm on $\mathbb{R}^n$.

5. **(Heine-Borel Property)** A subset K of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.

6. **(Dense Subset)** $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

2.3 The space of sequences l^2

Definition 2.2. The space l^2 is defined as

$$l^2 := \{(x_n) : \mathbb{N} \to \mathbb{R} | \sum_{n=1}^{\infty} x_n^2 < +\infty\}.$$

This is the space of all sequences of real numbers which are square-summable. The elements of this space are themselves sequences.

Theorem 2.3. (Properties of l^2)

1. **(Linear Space)** The space l^2 is a real vector space with zero vector $0 = (0, 0, 0, \cdots)$, vector addition defined by $(x_n) + (y_n) := (x_n + y_n)$ and scalar multiplication defined by $(\alpha x_n) := \alpha(x_n)$.

2. **(Inner Product)** The function $\langle , \rangle$ given by

   $$\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n y_n$$

   for all $(x_n), (y_n) \in l^2$ is an inner product on l^2.

3. **(Norm)** For $(x_n) \in l^2$, we define the l^2-norm of (x_n) as

   $$\|(x_n)\|_2 := \left(\sum_{n=1}^{\infty} x_n^2 \right)^{\frac{1}{2}}.$$

4. **(Completeness)** The normed linear space l^2 is complete. That is, l^2 is a Banach space.

5. **(Infinite-dimensional)** l^2 has no finite basis.

6. **(No Bolzano-Weierstass Property)** There exists bounded sequences in l^2 with no convergent subsequence.

7. **(No Heine-Borel Property)** There exists closed and bounded subsets of l^2 that are not compact.

Example 2.2. (Examples/Counterexamples)

1. **(Bounded, but no Convergent Subsequence)** Define the element $e^k \in l^2$ by $e^k_n = 1$ if $k = n$ and 0 otherwise. Notice that (e^k) is a bounded sequence in l^2 because $\|e^k\|_2 = 1$ for each k. However, if $k \neq j$ then $d_2(e^k, e^j) = \sqrt{2}$. Therefore this sequence cannot have a convergent subsequence.

2. **(Closed & Bounded, but not Compact)** Define $K := \{x \in l^2 : \|x\|_2 \leq 1\}$ to be the closed unit ball in l^2. Then K is clearly bounded, and it is also closed. However, it fails to be sequentially compact. For example, the sequence $(e^k)_{k=1}^{\infty}$ defined above is contained in K but has no convergent subsequence.

3. **(A Compact Subset of l^2)**We define the **Hilbert cube** C to be the collection of sequences of real numbers (x_n) so that, $\forall n \in \mathbb{N}$, $-\frac{1}{n} \leq x_n \leq \frac{1}{n}$.

2.4 The vector space of continuous functions: $C[0,1]$

Definition 2.3. The space $C([0,1])$ is defined as follows:

$$C([0,1]) := \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}.$$

$C([0,1])$ is a real vector space.

Definition 2.4. (Equipping $C([0,1])$ with the sup-norm (L^∞))

1. **(The Sup-Norm)** Let $\|\cdot\| : C([0,1]) \to \mathbb{R}$ be given by $\|f\| := \sup\{|f(x)| : x \in [0,1]\}$. Then $(C[0,1], \|\cdot\|)$ is a normed linear space.

2. **(Cauchy Sequences)** We say that a sequence of functions is **uniformly Cauchy** if, for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$ and $\forall x \in [0,1]$, $|f_n(x) - f_m(x)| < \epsilon$.

3. **(Uniform Convergence)** Any uniformly Cauchy sequence of functions converges uniformly.

4. **(Completeness)** The space $C([0,1])$ is complete w.r.t. the uniform norm. Therefore, $(C([0,1]), \|\cdot\|_\infty)$ is a Banach space.

5. **(No Bolzano-Weierstrass Property)**

Definition 2.5. (Equipping $C([0,1])$ with the L^2-norm)

1. **(Integrability on $C([0,1])$):** All continuous functions are Riemann integrable.

2. **(Vanishing Property):** If $F : [0,1] \to \mathbb{R}$ is continuous and nonnegative, and $\int_0^1 F(x)dx = 0$, then $F \equiv 0$ on $[0,1]$.

3. **(L^2-norm)** For $f \in C([0,1])$, define the L^2-norm of f as
$$\|f\|_2 := \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}}$$

4. **(Hölder's Inequality)** For every $f, g \in C([0,1])$,
$$\left|\int_0^1 f(x)g(x)dx\right| \leq \|f\|_2\|g\|_2.$$

5. **(Incompleteness)** $C([0,1])$ is not complete with respect to the $\|\cdot\|_2$-norm topology.

6. **(No Bolzano-Weierstrass Property)**

Example 2.3. (Examples/Counterexamples in $C([0,1])$)

1. **(L^2-Cauchy, Not Convergent):** Consider the sequence of functions $(f_n)_{n=1}^{\infty}$, where f_n is given by
$$f_n(x) := \begin{cases} (2x)^n & x \leq \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}.$$
This is an L^2-Cauchy sequence of elements of $C([0,1])$, but it does not converge to any element of $C([0,1])$.

2.

Definition 2.6. (Sequences in $C([0,1])$)

1. **(Uniform Boundedness)** A set of functions $F \subset C([0,1])$ is **uniformly bounded** if there is an $M > 0$ such that, for all $f \in F$, for all $x \in [0,1]$, $|f(x)| \leq M$.

2. **(Equicontinuity)** A set of functions $F \subset C([0,1])$ is **equicontinuous** at the point $x_0 \in [0,1]$ if given any $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon \ \forall f \in F$. We say that F is **equicontinuous on** $[0,1]$ if F is equicontinuous at every point in $[0,1]$.

Theorem 2.4. (Arzela-Ascoli): Suppose that (f_n) is a uniformly bounded sequence of equicontinuous functions in $C([0,1])$. Then there exists a subsequence of (f_n) which converges in the L^∞-norm.

Corollary 2.2. (Compactness) If $K \subset C([0,1])$ is bounded, closed and equicontinuous, then K is compact.

Definition 2.7. (Precompact) If a set has the property that every sequence in the set has a convergent subsequence (but whose limit is not necessarily in the set itself), then we call that set **precompact**.

Example 2.4. Precompact Sets

1. Bounded and equicontinuous sets in $C[0,1]$ are precompact.
2. Bounded sets in $\mathbb{R}^n$ are precompact.
3. If you add closedness to precompactness then you get compactness.

Corollary 2.3. (Uniformly Bounded Derivatives & Precompactness) If $K \subset C([0,1])$ is a set of uniformly bounded, differentiable functions, and if there is an $M > 0$ such that $|f'(x)| \le M$ for all $x \in [0,1]$ and all $f \in K$, then K is precompact in $C([0,1])$.

Example 2.5. (Equicontinuous, Uniformly Bounded, etc.)

2.5 The space of continuously differentiable functions: $C^1([0,1])$

Our next example is a modification of $C([0,1])$ allowing for the function to be differentiable:

Definition 2.8. The space $C^1([0,1])$ is defined as follows:

$$C^1([0,1]) := \{f : [0,1] \to \mathbb{R} : f \text{ is differentiable on } [0,1] \text{ and } \frac{df}{dx} \in C([0,1])\},$$

Definition 2.9. (Equipping $C^1([0,1])$ with a norm)

1. **(Real Vector Space)** $C^1([0,1])$ is a real vector space.
2. **(Norm)** $C^1([0,1])$ is a normed linear space with the norm defined by:
$$\|f\| = \|f\|_\infty + \|f'\|_\infty$$
3. **(Metrizable)** Let $d : C^1([0,1]) \times C^1([0,1]) \to \mathbb{R}$ be given by
$$d(f,g) = ||f-g||_\infty + ||f'-g'||_\infty$$
. Then $(C^1([0,1]), d)$ is a metric space, d is the metric induced by $\|\cdot\|$
4. **(Completeness)** $C^1([0,1])$ is complete with respect to d. $C^1([0,1])$ is therefore a Banach space.

Lemma 2.1. Suppose that f is a continuously differentiable function on $[0,1]$, and $\sup_{x\in[0,1]} |f'(x)| = M$. Then, for all $x, y \in [0,1]$, $|f(x) - f(y)| \le M|x-y|$.

Definition 2.10. ($C^1([0,1])$ with the $W^{1,2}$-norm) For $f \in C^1([0,1])$, define the $W^{1,2}$-norm of f as

$$\|f\|_{1,2} := \left(\int_0^1 |f(x)|^2 dx + \int_0^1 |f'(x)|^2 dx\right)^{\frac{1}{2}}. \tag{1}$$

1. **(Not Complete)** $C^1([0,1])$ is not complete with respect to the $W^{1,2}$-norm.

3 Several Important Constructions in Metric Spaces

4 The Completion of a Metric Space

Theorem 4.1. (Existence of the Completion $\tilde{X}$) Let (X, d) be a metric space. Then there exists a complete metric space $(\tilde{X}, \tilde{d})$, called the **completion** of X, and a natural embedding $i : X \to \tilde{X}$ such that $\forall x, y \in X$, $\tilde{d}(i(x), i(y)) = d(x, y)$. Moreover, $i(X)$ is dense in $\tilde{X}$.

Definition 4.1. (Properties of $\tilde{X}$)

1. **(Equivalence of Cauchy sequences)** Suppose that (p_n) and (q_n) are both Cauchy sequences in X. Then we say that (p_n) is **equivalent** to (q_n), denoted $(p_n) \sim (q_n)$, if $\lim_{n\to\infty} d(p_n, q_n) = 0$.

2. **(Equivalence Relation)** Define Y to be the collection of all Cauchy sequences in X, then the relation $\sim$ is an equivalence relation on the set Y.

3. **($\tilde{X}$ as a Quotient)** Define $\tilde{X}$ to be the quotient of Y by the equivalence relation $\sim$ defined above.

4. **(The Metric on $\tilde{X}$)** The function $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ given by

$$\tilde{d}([(p_n)], [(q_n)]) = \lim_{n\to\infty} d(p_n, q_n)$$

   is a metric on $\tilde{X}$.

5. **(Completeness of $\tilde{X}$)** The metric space $(\tilde{X}, \tilde{d})$ is complete.

6. **(Isometry)** Let $i : X \to \tilde{X}$ be given by $i(x) = [(x, x, \ldots, x, \ldots)]$. Then i is an **isometry**. That is, for every $x, y \in X$, $\tilde{d}(i(x), i(y)) = d(x, y)$.

7. **($i(X)$ is dense in $\tilde{X}$)** Every metric space X can be isometrically embedded into its completion. This embedding is dense in $\tilde{X}$.

8. **(The Completion of a Complete space)** A space X is isometric to $\tilde{X}$ if and only if X is complete.

4.1 The L^p and l^p Spaces

Definition 4.2. (The L^p spaces)

1. The space $L^2([0, 1])$ is the completion of $C([0, 1])$ with respect to the metric induced by L^2-norm:

$$\|f\|_2 := \left(\int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}}$$

2. **((Completion of $C([0,1])$)) The space $L^p([0,1])$ is the completion of $C([0,1])$ with respect to the metric induced by the L^p-norms:**

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}. \tag{2}$$

3. **(Completion Metric) $\|[(f_n)]\|_p := \left(\lim_{n\to\infty} \int_0^1 |f_n(x)|^p dx\right)^{\frac{1}{p}}$ is a norm on $L^p([0,1])$ which generates the metric defined through the completion.**

4. **(Hölder's Inequality) Suppose that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p([0,1])$ and $g \in L^q([0,1])$. Then**

$$\int_0^1 |f(x)g(x)|dx \leq \|f\|_p \|g\|_q.$$

5. **(Relation of L^p and L^q) Suppose that $1 \leq p < q < \infty$. Then $L^q([0,1]) \subset L^p([0,1])$ and if $f \in L^q([0,1])$, then $\|f\|_p \leq \|f\|_q$.**

Definition 4.3. (The l^p spaces)

1. The space l^p is defined as $\{(x_n)_{n=1}^\infty | \sum_{n=1}^\infty |x_n|^p < +\infty\}$, and equipped with the norm:

$$\|(x_n)\|_p = (\sum_{n=1}^\infty |x_n|^p)^{\frac{1}{p}}.$$

2. **(Banach Space)** The space $(l^p, \|\cdot\|_p)$ is a Banach space.

3. **(Relation between l^p and l^q)** If $1 \leq p < q < \infty$, then $l^p \subset l^q$.

Remark 4.1. ($L^p([0,1])$ and $L^p(\mathbb{R})$) Suppose that $p < q$. We can construct functions that are in L^p but not L^q and vice versa. Therefore there is no containment result like the ones above. This is because the examples from $L^p([0,1])$ and the examples from l^p are both in play (if defined carefully).

Example 4.1. (Elements of L^p and l^p)

1. **(In L^2 but not $C([0,1])$)** Recall the L^2-Cauchy sequence f_n:

$$f_n(x) := \begin{cases} (2x)^n & x \leq \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$$

This sequence converges in L^2 to the function f given by:

$$f(x) := \begin{cases} 0 & x \leq \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$$

2. Piece-wise continuous functions are elements of L^2

3. **(In L^p but not L^q)** Suppose $1 \leq p < q$, then for α with $\frac{1}{q} < \alpha < \frac{1}{p}$ the function $f(x) = x^{-\alpha}$ is in L^p but not L^q.

4. **(In l^q but not l^p)** Suppose $1 \leq p < q$, then for α with $\frac{1}{q} < \frac{1}{\alpha} < \frac{1}{p}$ the sequence (x_n) given by:

$$x_n = n^{-1/\alpha}$$

is in l^q but not l^p.

4.2 The Space $W^{1,2}$

Definition 4.4. (The Space $W^{1,2}$)

1. **(The $W^{1,2}$-norm)** Recall that the $W^{1,2}$-norm on $C([0,1])$ is given by:

$$\|f\|_{1,2} := \left(\int_0^1 |f(x)|^2 dx + \int_0^1 |f'(x)|^2 dx \right)^{\frac{1}{2}}$$

2. **(The Space $W^{1,2}$)** Define

$$W^{1,2}([0,1]) := \text{ the completion of } C^1([0,1]) \text{ with respect to } \| \cdot \|_{W^{1,2}}.$$

- What kind of functions can be in $W^{1,2}$? Functions with bounded derivatives.
- Can a function in $W^{1,2}$ be discontinuous? No, every element is continuous
- Can a function in $W^{1,2}$ have a discontinuous derivative? Yes, but no jump discontinuities
- Can a function in $W^{1,2}$ have a cusp? Yes
- How continuous does a function in $W^{1,2}$ have to be?
- How does this compare to $W^{1,p}$?
- What happens if we require more derivatives?
- What happens if we change our domain from $[0,1]$ to $\mathbb{R}$?

5 The Contraction Mapping Principle

Definition 5.1. (Contracting Mappings)

1. **(Fixed Points)** A point $x \in X$ is called a **fixed point** of a function $f : X \to X$ if $f(x) = x$.

2. **(Contraction Mapping)** A function $f : X \to X$ is called a **contraction mapping** if there exists a constant r with $0 \leq r < 1$ such that for all $x, y \in X$

$$d(f(x), f(y)) \leq r \, d(x, y).$$

3. **(Continuity)** A contraction mapping on X is clearly continuous on X.

Theorem 5.1 (Contraction Mapping Theorem)**.** Let f be a contraction mapping on a complete nonempty metric space, X. Then f has a unique fixed point.

Corollary 5.1. Let f be a contraction mapping on a complete nonempty metric space X. If x_0 is any point of X, and $x_{n+1} = f(x_n)$ for $n \geq 0$ then the sequence $\{x_n\}$ converges to the fixed point of f.

5.1 Existence and Uniqueness for Solutions to Differential Equations

Definition 5.2. (Ordinary Differential Equations)

1. **(Initial Value Problems)** The general initial value problem (IVP), can be stated as
$$(IVP): \quad y'(t) = f(y), y(0) = y_0. \tag{3}$$

2. **(Lipschitz Continuous)** $f : [a,b] \to \mathbb{R}$ is **Lipschitz continuous** if there is a $k > 0$ so that, $\forall x, y \in [a,b]$, $|f(x) - f(y)| \leq k|x - y|$.

3. **(Uniformly Lipschitz Continuous)** $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is *uniformly Lipschitz continuous* if there is a $k > 0$ so that, $\forall t \in [a,b]$, $\forall x, y \in \mathbb{R}$, $|f(t,x) - f(t,y)| \leq k|x - y|$.

Theorem 5.2. (Existence and Uniqueness of Solutions to IVPs) If f is uniformly Lipschitz continuous on an open rectangle containing $(0, y_0)$, then there is an $\epsilon > 0$ and a differentiable function $y \in C^1([0,\epsilon])$ such that y is the unique solution of (3) on $[0,\epsilon]$.

Example 5.1. (Examples of Initial Value Problems)

1. **(Has unique solution)**
$$y'(t) = (y(t))^2, y(0) = 2.$$

2. **(No unique solution)**
$$y'(t) = (y(t))^{\frac{1}{2}}, y(0) = 0,$$

6 Function Approximation

Remark 6.1. (Niceness Hierarchy)

1. Being infinitely differentiable is nice.

2. Being analytic, *i.e.* a power series, is nicer.

3. Being a polynomial is perhaps the nicest.

4. Every function in $C([0,1])$ can be approximated to any degree of accuracy by a polynomial

Example 6.1. (Function Approximation) Suppose that we are trying to approximate the discontinuous function

$$f(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}$$

1. **(A L^2-Cauchy in $C([0,1])$ that $\to f$.)**

$$f_n(x) = \begin{cases} (2x)^n & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}.$$

2. **(Approximation by Piecewise-Linear Function)** A sequence of functions that *averages* $f(x)$ over over the interval $[x - \frac{1}{n}, x + \frac{1}{n}]$:

$$g_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}(x - \frac{1}{2} + \frac{1}{n}) & \frac{1}{2} - \frac{1}{n} \le x < \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$

Definition 6.1. (Averaging Kernels)

1. **(The Average of a Function)** Recall from your calculus experience that the average of a function $f(x)$ over an interval $[a, b]$ is given by

$$\frac{1}{b-a} \int_a^b f(x)dx.$$

2. **(The Average of a Function on $[x - \frac{1}{n}, x + \frac{1}{n}]$)** we can rewrite our formula for $g_n(x)$ from the previous example as:

$$g_n(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(y)dy.$$

3. **(The Indicator Function)** The indicator function of a set S, denoted by $\chi_S(x)$ is given by:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

4. **(Rewriting the g_n)** We can rewrite g_n again as:

$$\begin{aligned} \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(y)dy &= \frac{n}{2} \int_{-\infty}^{\infty} \chi_{[x-\frac{1}{n}, x+\frac{1}{n}]} f(y)dy \\ &= \frac{n}{2} \int_{-\infty}^{\infty} \chi_{[-\frac{1}{n}, \frac{1}{n}]}(y-x) f(y)dy \\ &= \int_{-\infty}^{\infty} K_n(y-x) f(y)dy, \end{aligned}$$

   where $K_n(y) := \frac{n}{2}\chi_{[-\frac{1}{n}, \frac{1}{n}]}$

5. **(The Convolution Kernel K_n)** The K_n is often called the *mollifier* or the *convolution kernel.*

The key properties that we abstract from the process above are that

Definition 6.2. (Mollification) A sequence of functions (K_n) are called **mollifiers** if:

(Nonnegativity): $K_n \geq 0$,

(Unit Area): $\int_{-\infty}^{\infty} K_n(y)dy = 1$, and

(Concentration): $K_n(y)$ **concentrates** at 0, *i.e.* given any $\delta > 0$, we have

$$\lim_{n\to\infty} \left(\int_{-\infty}^{-\delta} K_n(y)dy + \int_{\delta}^{\infty} K_n(y)dy \right) = 0.$$

Remark 6.2. We will use the formula

$$f_n(x) := (f * K_n)(x) = \int_{-\infty}^{\infty} f(y)K_n(x-y)dy,$$

to create a sequence of approximations.

6.1 Convolution and Mollification

Definition 6.3. (Convolution and Mollifiers)

(Convolution) Suppose that f, g are functions. The **convolution** of f and g is defined to be

$$f * g(x) := \int_{-\infty}^{\infty} f(y)g(x-y)dy,$$

as long as this integral is well-defined.

2. **(Integrable)** We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **integrable** if $\int_{-\infty}^{\infty} f(x)dx$ is well-defined and finite.

3. **(Compact Support)** A function $g : \mathbb{R} \to \mathbb{R}$ is called **compactly supported** if $\exists M > 0$ such that $g(x) = 0$ for every x such that $|x| > M$.

Lemma 6.1. (Properties of the Convolution) The convolution of f and g satisfies the following properties:

1. **(Commutativity)** $f * g(x) = g * f(x)$.

2. **(Boundedness)** If f is integrable, g is bounded, and the convolution is well-defined, then $f * g$ is bounded.

3. **(Uniform Continuity)**If f is integrable and g is continuous and compactly supported, then $f * g$ is uniformly continuous.

4. **(Differentiability)** If f is integrable and g is continuously differentiable and compactly supported, then $f * g$ is differentiable and $(f * g)' = f * (g')$.

5. **(Young's Inequality)** If f and g are both integrable functions, then $f * g$ is also integrable, and

$$\int_{-\infty}^{\infty} |f * g(x)| dx \leq \left(\int_{-\infty}^{\infty} |f(x)| dx \right) \left(\int_{-\infty}^{\infty} |g(x)| dx \right).$$

Note that the last part of this lemma can be applied repeatedly to obtain higher levels of differentiability for $f * g$ if g is smooth.

Lemma 6.2. (Uniform Convergence) Suppose that f is a bounded, integrable function on $\mathbb{R}$, and that S is a compact subset of $\mathbb{R}$ on which f is continuous. Then, if K_n is a sequence of convolution kernels satisfying Definition 6.2, the functions $f_n := f * K_n$ converge to f uniformly on S.

6.2 The Weierstrass Approximation Theorem

Theorem 6.1. (Weierstrass Approximation Theorem)

1. **(Version 1)** The set of polynomials is dense in $C[0, 1]$.

2. **(Version 2)** Let $f \in C([0, 1])$. Then there is a sequence of polynomials (f_n) on $[0, 1]$ so that $f_n \to f$ uniformly on $[0, 1]$.

Lemma 6.3. (Choosing Mollifiers) Define

$$q_n(x) = \begin{cases} (1 - \frac{x^2}{4})^n & -2 \leq x \leq 2 \\ 0 & |x| > 2 \end{cases},$$

and $c_n := \int_{-\infty}^{\infty} q_n(x) dx$. Then let $p_n(x) = \frac{1}{c_n} q_n(x)$. The (p_n) form a sequence of convolution kernels satisfying the required conditions for mollifiers.

7 Calculus in Normed Vector Spaces

8 Differentiability

8.1 Review of Differentiability on $\mathbb{R}^n$

Proposition 8.1. (Review of Derivatives)

1. **(Derivatives of functions on $\mathbb{R}$)** A function $f : \mathbb{R} \to \mathbb{R}$ is **differentiable** at $x_0 \in \mathbb{R}$ if $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists. If so, we define $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

2. **(Little-o)**The "little o" notation $o(x - x_0)$ represents an error term with the property that $\lim_{x \to x_0} \frac{o(x-x_0)}{(x-x_0)} = 0$.

3. **(Alternate definition of derivatives)**A function $f : \mathbb{R} \to \mathbb{R}$ is **differentiable** at x_0 if and only if there is a real number L such that $f(x) = f(x_0) + L(x - x_0) + o(x - x_0)$. If the number L exists, then we say that L is the derivative of f at x_0 and we write $L = f'(x_0)$.

4. **(Real-valued functions of n variables)** A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{x}_0$ if there is a vector $\vec{L} \in \mathbb{R}^n$ such that $f(\vec{x}) = f(\vec{x}_0) + \vec{L} \cdot (\vec{x} - \vec{x}_0) + o(\|\vec{x} - \vec{x}_0\|)$. If f is differentiable at x_0, then the function $L(\vec{x}) = f(\vec{x}_0) + \vec{L} \cdot (\vec{x} - \vec{x}_0)$ is called the **local linear approximation** or **tangent plane approximation** to f at $\vec{x}_0$.

5. **(Directional Derivatives)** Let $f : \mathbb{R}^n \to \mathbb{R}$, then the **directional derivative** of f at $\vec{x_0}$ in the direction of $\vec{u}$, denoted $D_{\vec{u}}(\vec{x_0})$ is given by:

$$D_{\vec{u}}(\vec{x_0}) = \lim_{h \to 0} \frac{f(\vec{x_0} + h\vec{u}) - f(\vec{x_0})}{h}$$

6. **(Partial Derivatives)** Let $f : \mathbb{R}^n \to \mathbb{R}$, then the **ith partial derivative** of f at $\vec{x_0}$, denoted $D_i(\vec{x_0})$ or $\frac{\partial f}{\partial x_i}(\vec{x_0})$ is given by:

$$D_i(\vec{x_0}) = \frac{\partial f}{\partial x_i}(\vec{x_0}) = \lim_{h \to 0} \frac{f(\vec{x_0} + h\vec{e_i}) - f(\vec{x_0})}{h}$$

Example 8.1. (Differentiable and Nondifferentiable Functions)

1. **(Not Differentiable)** On $\mathbb{R}^2$, the functions $f(x, y) = |(|x| - |y|)| - |x| - |y|$ and $g(x, y) = \frac{3x^2y}{x^2+y^2}$ are not differentiable at $(0, 0)$ even though $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial g}{\partial x}$, and $\frac{\partial g}{\partial y}$ all exist (and equal 0) at $(0, 0)$.

2. **(Not Differentiable)** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \text{ or } z = 0 \\ 1 & \text{otherwise} \end{cases}$$

then at $(0, 0, 0)$, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$, but f is not differentiable at x_0 since it is not continuous at $(0, 0, 0)$.

3. **(Differentiable)** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be linear, then we have:

$$\lim_{h \to 0} \frac{|f(x + h) - f(x) - f(h)|}{|h|} = \lim_{h \to 0} \frac{|f(x) + f(h) - f(x) - f(h)|}{|h|} = 0$$

thus f is differentiable with $Df(x) = f$.

4. **(Differentiable)** Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $g(x, y, z) = (e^x, e^y, e^z)$ then for each $(a, b, c) \in \mathbb{R}^3$, $Df(a, b, c)$ is given by:

$$Df(a, b, c) = \begin{bmatrix} e^a & 0 & 0 \\ 0 & e^b & 0 \\ 0 & 0 & e^c \end{bmatrix}$$

8.2 Linear Operators on Normed Linear Spaces

Definition 8.1. (Linear Operators) Let X, Y be normed linear spaces. $L : X \to Y$ is a **linear operator** if

1. $L(cx) = cL(x)$ for all $c \in \mathbb{R}$ and all $x \in X$.
2. $L(x_1 + x_2) = L(x_1) + L(x_2)$ for all $x_1, x_2 \in X$.

Definition 8.2. (Bounded Linear Operator) Suppose that $L : X \to Y$ is a linear operator as in the previous definition. We say that L is a *bounded* linear operator if there is a constant $c > 0$ such that $||Lx||_Y \le c||x||_X$ for all $x \in X$.

Lemma 8.1. (Bounded Operators)

1. A linear operator $L : X \to Y$ is bounded if and only if $||L||_{op} := \sup\{||L(x)||_Y : x \in X, ||x||_X = 1\} < +\infty$. If so, then, $\forall x \in X$, $\|Lx\|_Y \le \|L\|_{op}\|x\|_X$.
2. If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator, then it is a bounded linear operator.
3. A linear operator $L : X \to Y$ is continuous on X if and only if it is bounded.
4. A linear operator is continuous at every point in its domain if and only if it is continuous at 0.

Theorem 8.1. Let $\mathcal{B}(X, Y)$ be the set of all bounded linear operators from X to Y. Then $(\mathcal{B}(X, Y), || \cdot ||_{op})$ is a normed linear space. If Y is complete, then $\mathcal{B}(X, Y)$ is also complete (and therefore a Banach space).

Example 8.2. (Linear operators)

1. **(Not Continuous)** Let $X = Y = C^\infty([0, 1])$ = the space of all infinitely differentiable functions on $[0, 1]$. Equip X with the L^2 norm. Then let $L : X \to Y$ be given by $L(f) = \frac{df}{dx}$. Then L is linear but not continuous.
2. **(Not Continuous)** Let $X = C^1([0, 1])$, and $Y = C([0, 1])$ and equip **both** X and Y with the sup-norm. Then let $L : X \to Y$ be given by $L(f) = \frac{df}{dx}$. Then L is linear but not continuous.
3. **(Differentiation vs Integration)** Let $X = C^1([0, 1])$, and $Y = C([0, 1])$ and equip both X and Y with the sup-norm. Let $L : X \to Y$ be given by $L(f) = \frac{df}{dx}$ and let $Q : Y \to X$ be given by $Q(f) = \int_0^x f(t)dt$. Both Q and L are linear but not inverses of each other: $L(Q(f)) = f$ but $Q(L(f)) = f + C$.

8.3 Fréchet Differentiation

Definition 8.3. (Differentiable and Fréchet Differentiable) Let X, Y be normed linear spaces, and let $F : X \to Y$. We say that F is **differentiable** at $x_0 \in X$ if there is a linear operator $L : X \to Y$ such that $F(x) = F(x_0) + L(x - x_0) + o(x - x_0)$. If such an L exists, then we say the L is the derivative of F at x_0 and write $L = DF(x_0)$. If $L \in \mathcal{B}(X, Y)$, then we say that F is **Fréchet differentiable** at x_0.

Proposition 8.2. (Properties of Derivatives)

1. **(Uniqueness)** Derivatives are unique.

2. **(Sums)** Suppose that $F, G : X \to Y$ are both differentiable at $x_0 \in X$. Then $F+G$ is differentiable at x_0 with $D(F+G)(x_0) = DF(x_0) + DG(x_0)$.

3. **(Product Rule)** Suppose that $F : X \to Y$ and $f : X \to \mathbb{R}$ are both Fréchet differentiable at x_0. Then $fF : X \to Y$ is also Fréchet differentiable at x_0 with $D(fF)(x_0) = f(x_0)DF(x_0) + F(x_0)Df(x_0)$.

4. **(Lipschitz Continuity)** If $F : X \to Y$ is differentiable at x_0, then F is Lipschitz continuous at x_0. That is, $\exists M > 0$ and $\exists \delta > 0$ so that if $\|x - x_0\|_X < \delta$, then $\|F(x) - F(x_0)\|_Y \leq M\|x - x_0\|_X$.

5. **(Chain Rule)** Suppose that $F : X \to Y$ is differentiable at x_0 and that $G : Y \to Z$ is differentiable at $F(x_0)$. Then $G \circ F : X \to Z$ is differentiable at x_0 with

$$D(G \circ F)(x_0) = DG(F(x_0)) \cdot DF(x_0).$$

6. **(Quotient Rule)** Suppose that $F : X \to Y$ is Fréchet differentiable at x_0 and that $f : X \to \mathbb{R}$ is Fréchet differentiable at x_0 and $f(x_0) \neq 0$. Then $\frac{F}{f} : X \to Y$ is differentiable at x_0 and, for $h \in X$,

$$D\frac{F}{f}(x_0)(h) = \frac{f(x_0)DF(x_0)(h) - Df(x_0)(h)F(x_0)}{f(x_0)^2}.$$

Example 8.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-continuously differentiable function so that f'' is bounded. Then the functional $F : L^2([0,1]) \to \mathbb{R}$ given by $F(u) = \int_0^1 f(u(x))dx$ is Fréchet differentiable with derivative $DF(u)[h] = \int_0^1 f'(u(x))h(x)dx$.

Example 8.4. Consider $F : L^2([0,1]) \to \mathbb{R} : F(u) = \int_0^1 u^2$. Then F is Fréchet differentiable, and for f in $L^2([0,1])$, $DF(f)$ is the linear functional from $L^2([0,1])$ to $\mathbb{R}$ given by $DF(f)[h] = \int_0^1 2f(t)h(t)dt \ \forall h \in L^2([0,1])$.

8.4 Taylor's Theorem

Definition 8.4. (Taylor Expansion/Taylor Polynomial) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a $k+1$ times continuously differentiable function for some $k \geq 1$. Let $x_0 \in \mathbb{R}$, then the **Taylor expansion** of f at x_0 is given by:

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}}{n!}(x - x_0)^n$$

Theorem 8.2. (Taylor's Theorem) Let f be C^k in a neighborhood of x_0. Then $f - P_k = o(\|x - x_0\|^k)$ as $x \to x_0$.

Theorem 8.3. [Taylor's Theorem with Remainder] Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a $k+1$ times continuously differentiable function for $k \geq 1$. Let $x_0 \in \mathbb{R}$ and let $P_k(x) = \sum_{i=0}^{k} \frac{f^{(i)}}{i!}(x-x_0)^i$ be the kth Taylor polynomial for f centered at x_0. Then, for any $x \in \mathbb{R}$,

$$f(x) - P_k(x) = \int_{x_0}^{x} \frac{f^{k+1}(t)}{k!}(x-t)^k dt.$$

Corollary 8.1. Under the hypotheses of Theorem 8.3, there is some $c \in [x_0, x]$ such that

$$|f(x) - \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i| \leq \frac{|f^{k+1}(c)|}{(k+1)!}|x-x_0|^{k+1}.$$

8.5 A Detour into $\mathbb{R}^n$

Lemma 8.2. (The Jacobian Matrix) If $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x_0, then $DF(x_0)$ can be represented by a **Jacobian** matrix $\left[\frac{\partial F_i}{\partial x_j}\right]$:

$$DF(x_0) = \begin{bmatrix} \nabla F_1(x_0) \\ \vdots \\ \nabla F_m(x_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x_0) & \cdots & \frac{\partial F_1}{\partial x_m}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1}(x_0) & \cdots & \frac{\partial F_m}{\partial x_m}(x_0) \end{bmatrix}$$

Theorem 8.4. (Mean Value Theorem): Assume that $f : X \to \mathbb{R}$ is differentiable. Given any $x_1, x_2 \in X$ there is a $c \in (0,1)$ such that $f(x_2) - f(x_1) = Df(x_1 + c(x_2 - x_1))(x_2 - x_1)$.

Theorem 8.5. (Continuous Partial Derivatives) If the first partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}^m$ exist and are continuous on a neighborhood of x_0, then f is differentiable at x_0.

Definition 8.5. (Difference Quotient) Define the **difference quotient** $D_{i,h}f_k(\vec{x})$ by:

$$D_{i,h}f_k(\vec{x}) = \frac{f_k(\vec{x_0} + h\vec{e_i}) - f(\vec{x_0})}{h}$$

Note that if we take $h \to 0$, $D_{i,h}f_k(\vec{x}) \to D_i f_k(\vec{x})$.

Theorem 8.6. (Equality of Mixed Partials) If the second partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}^m$ exist and are continuous on a neighborhood of x_0, then $\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x_0)$.

Theorem 8.7. (Mean Value Inequality:) Assume that $f : X \to Y$ is Fréchet differentiable. Given any $x_1, x_2 \in X$, set $M := \sup\{||Df(x_1 + t(x_2 - x_1))||_{op} : 0 \leq t \leq 1\}$. Then

$$||f(x_2) - f(x_1)||_Y \leq M||x_2 - x_1||_X$$

Example 8.5. Consider the function $\vec{f} : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = \langle \cos(t), \sin(t) \rangle$. Does the mean value theorem hold for $x_1 = 0$ and $x_2 = 2\pi$?

Example 8.6. Give an example of a continuously differentiable function from $\mathbb{R}^2$ to itself and points $\vec{x}_1, \vec{x}_2$ for which the mean value theorem is not satisfied. Compute the Jacobian matrix of your example function. Demonstrate that the mean value inequality is satisfied.

9 The Inverse and Implicit Function Theorems

Proposition 9.1. (Inverse Functions in $C^1(\mathbb{R})$) Assume that $f \in C^1(\mathbb{R})$ and $x_0 \in \mathbb{R}$ with $f'(x_0) \neq 0$. Then there is an interval containing x_0 such that f is invertible from that interval to its image. Further, the inverse f^{-1} is continuously differentiable.

Definition 9.1. Suppose that $f : X \to Y$. We say that $f \in C^1(X; Y)$, or f is C^1, if, $\forall x \in X$, f is Fréchet differentiable at x, and, moreover, $Df(x)$ is continuous as a function of x. That is, $\forall x_0 \in X$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|x - x_0\|_X < \delta$ implies that $\|Df(x) - Df(x_0)\|_{oper} < \epsilon$.

Theorem 9.1. Inverse Function Theorem: Assume that X and Y are Banach spaces and that $f \in C^1(X; Y)$. Suppose $x_0 \in X$. If $Df(x_0)$ is invertible, then there are neighborhoods U of x_0 and V of $y_0 = f(x_0)$ such that $f : U \to V$ is an invertible function with $f^{-1} \in C^1(V; U)$.

Theorem 9.2. Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be continuously differentiable and suppose that $D_{\vec{x}} F(\vec{x}_0, \lambda_0)$ is an invertible map in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Then there is an $\epsilon > 0$ and a C^1 function $\vec{x} : (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \to \mathbb{R}^n$ such that $F(\vec{x}(\lambda), \lambda) = F(\vec{x}_0, \lambda_0)$ for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$.