Approximate Inference in Bayesian Networks

Chapter 14 – Part II

Inference by Sampling

- Key Idea
  - Draw *N* samples from a sampling distribution *S*
  - Compute an approximate posterior *P'*
  - Show that this converges to the true probability *P*

- If we could sample from a variable's (posterior probability), we could estimate its (posterior) probability

X	*count*
x_1	n_1
$\vdots$	$\vdots$
x_k	n_k
total	m

$\leftrightarrow$

X	*probability*
x_1	n_1/m
$\vdots$	$\vdots$
x_k	n_k/m

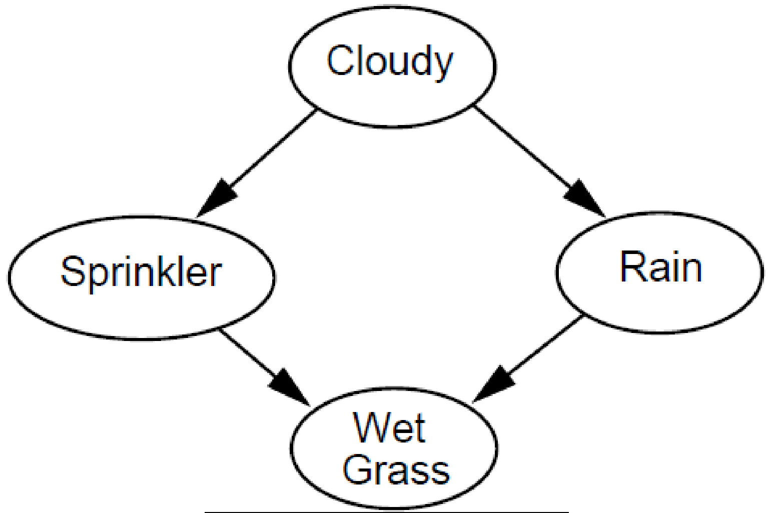
Sampling from an empty network

function PRIOR-SAMPLE(bn) **returns** an event sampled from bn
  inputs: bn, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $\mathbf{x} \leftarrow$ an event with n elements
  for $i = 1$ **to** n **do**
    $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
      given the values of $Parents(X_i)$ in $\mathbf{x}$
  return $\mathbf{x}$

Example

P(C)
.50



C	P(S|C)
T	.10
F	.50

C	P(R|C)
T	.80
F	.20

S	R	P(W|S,R)
T	T	.99
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F	T	.90
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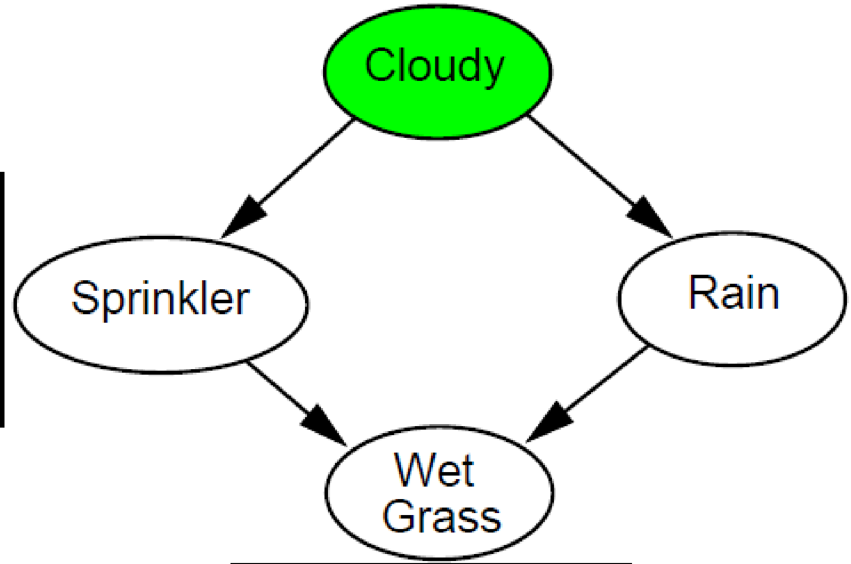

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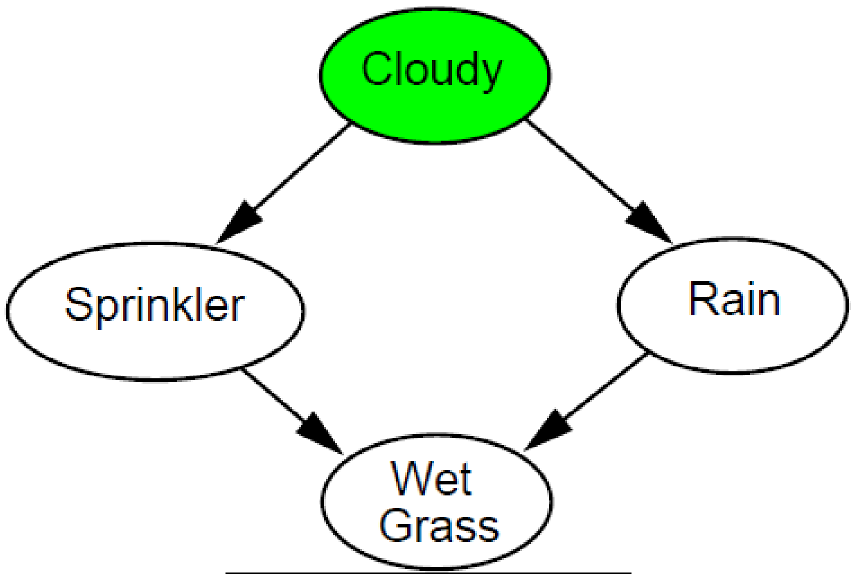

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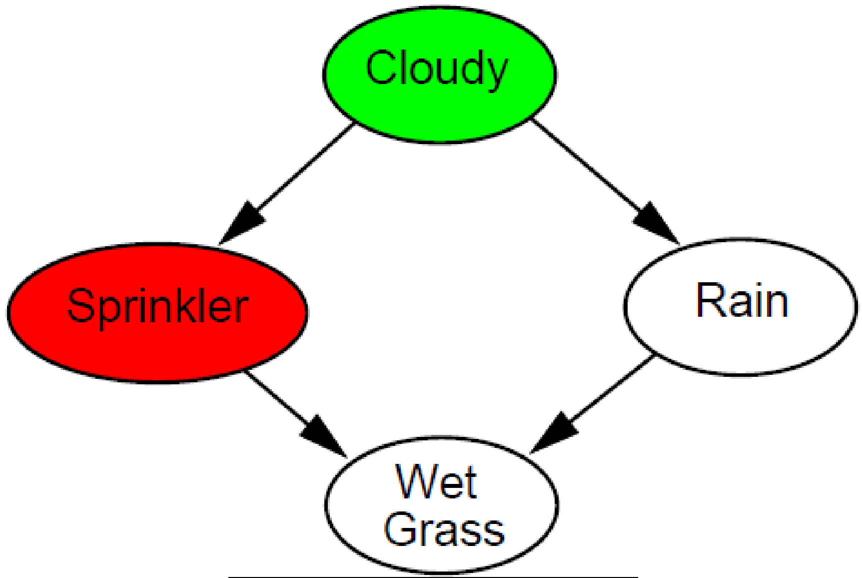

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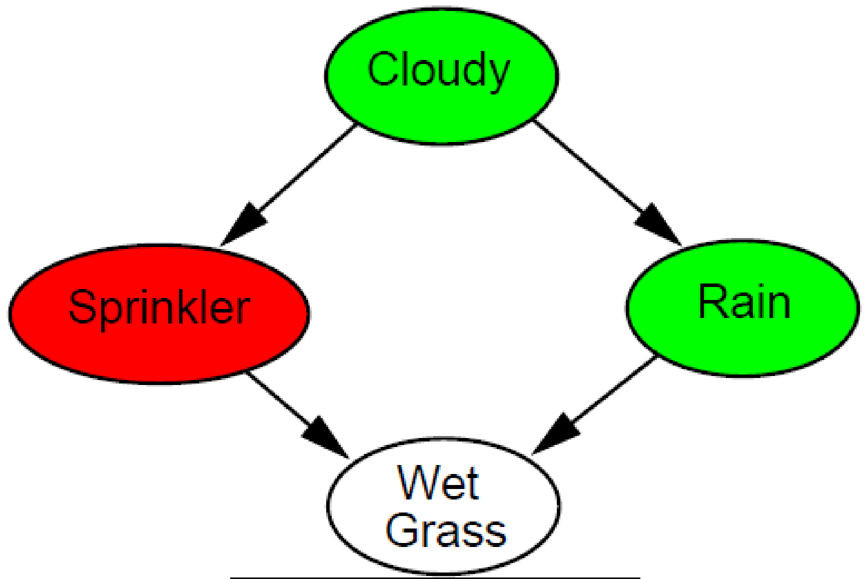

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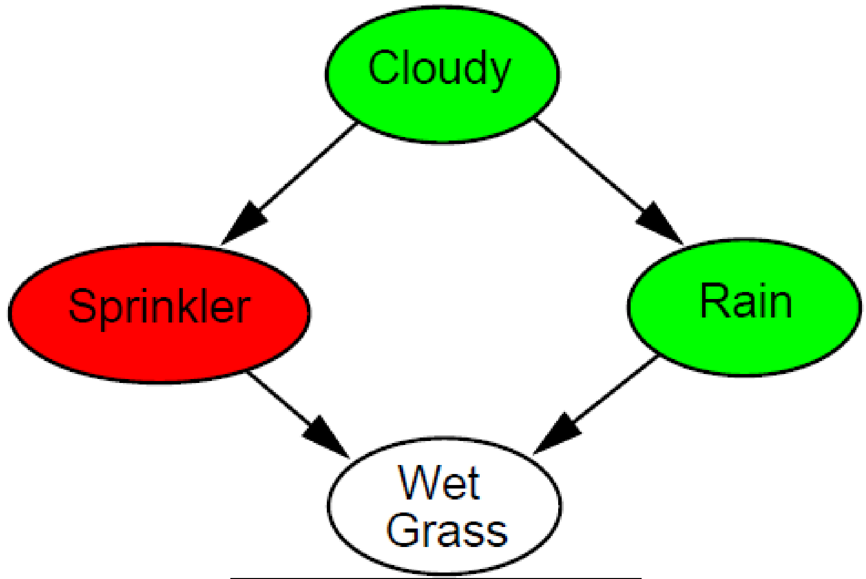

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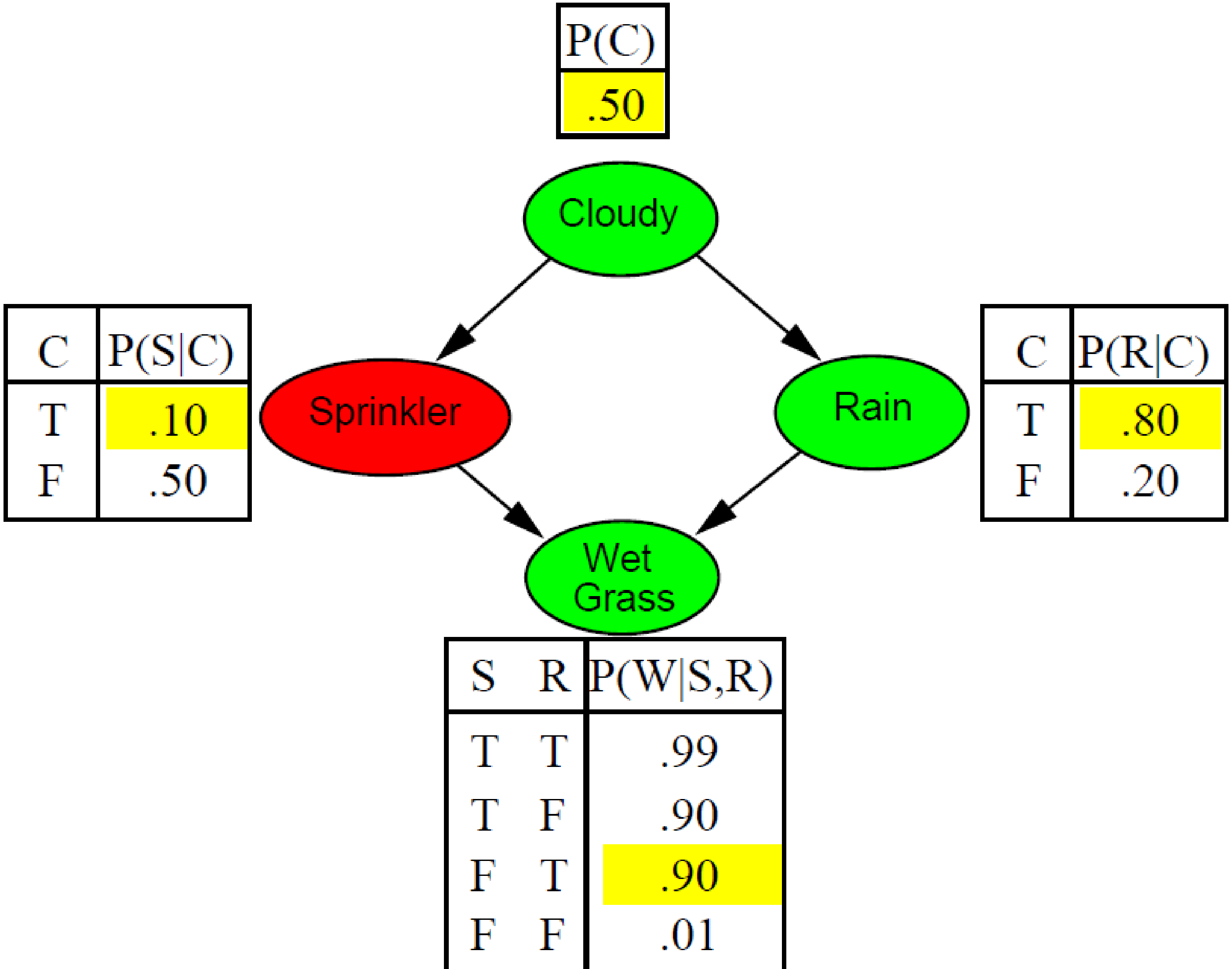


P(C)
.50
Cloudy
C
P(S|C)
T
.10
F
.50
Sprinkler
Rain
C
P(R|C)
T
.80
F
.20
Wet
Grass
S
R
P(W|S,R)
T
T
.99
T
F
.90
F
T
.90
F
F
.01

Sampling from an empty network

Probability that PriorSample generates a particular event

$$S_{PS}(x_1 \ldots x_n) = \Pi_{i=1}^{n} P(x_i | parents(X_i)) = P(x_1 \ldots x_n)$$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned} \lim_{N\to\infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N\to\infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

That is, estimates derived from PriorSample are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection Sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

```
function REJECTION-SAMPLING(X, e, bn, N) returns an estimate of P(X|e)
   local variables: N, a vector of counts over X, initially zero

   for j = 1 to N do
       x ← PRIOR-SAMPLE(bn)
       if x is consistent with e then
           N[x] ← N[x]+1 where x is the value of X in x
   return NORMALIZE(N[X])
```

E.g., estimate $\mathbf{P}(Rain|Sprinkler=true)$ using 100 samples

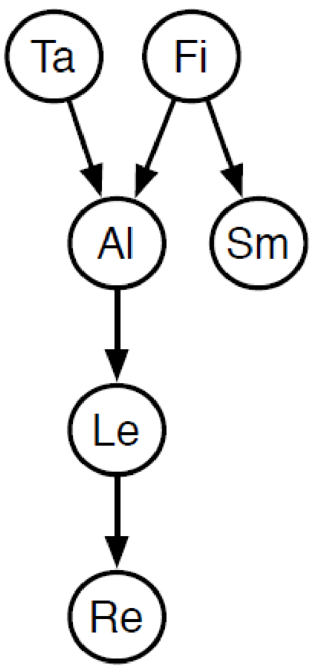
27 samples have $Sprinkler=true$

Of these, 8 have $Rain=true$ and 19 have $Rain=false$.

$\hat{\mathbf{P}}(Rain|Sprinkler=true) = \text{NORMALIZE}(\langle 8, 19\rangle) = \langle 0.296, 0.704\rangle$

Similar to a basic real-world empirical estimation procedure

Rejection Sampling - Example



Observe $Sm = true, Re = true$

	Ta	Fi	Al	Sm	Le	Re	
s_1	false	true	false	true	false	false	✗
s_2	false	true	true	true	true	true	✓
s_3	true	false	true	false	—	—	✗
s_4	true	true	true	true	true	true	✓
...							
s_{1000}	false	false	false	false	—	—	✗

$P(sm) = 0.02$
$P(re|sm) = 0.32$
How many samples are rejected?
How many samples are used?

Analysis of Rejection Sampling

$$\begin{aligned}
\hat{\mathbf{P}}(X|\mathbf{e}) &= \alpha \mathbf{N}_{PS}(X,\mathbf{e}) && \text{(algorithm defn.)} \\
&= \mathbf{N}_{PS}(X,\mathbf{e})/N_{PS}(\mathbf{e}) && \text{(normalized by } N_{PS}(\mathbf{e})\text{)} \\
&\approx \mathbf{P}(X,\mathbf{e})/P(\mathbf{e}) && \text{(property of } \textsc{PriorSample}\text{)} \\
&= \mathbf{P}(X|\mathbf{e}) && \text{(defn. of conditional probability)}
\end{aligned}$$

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood weighting
aka Importance Sampling

Idea: fix evidence variables, sample only nonevidence variables, and weight each sample by the likelihood it accords the evidence

function LIKELIHOOD-WEIGHTING($X, \mathbf{e}, bn, N$) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{W}$, a vector of weighted counts over X, initially zero

  for $j = 1$ to N **do**
    $\mathbf{x}, w \leftarrow$ WEIGHTED-SAMPLE(bn)
    $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{W}[X]$)

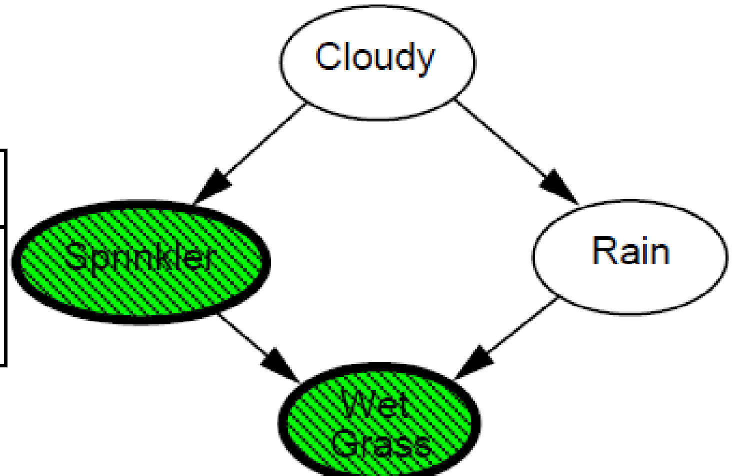
---

function WEIGHTED-SAMPLE($bn, \mathbf{e}$) **returns** an event and a weight

  $\mathbf{x} \leftarrow$ an event with n elements; $w \leftarrow 1$
  for $i = 1$ **to** n **do**
    if X_i has a value x_i in $\mathbf{e}$
      then $w \leftarrow w \times P(X_i = x_i \mid parents(X_i))$
      else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
  return $\mathbf{x}$, w

Likelihood Weighting Example

P(C)
.50



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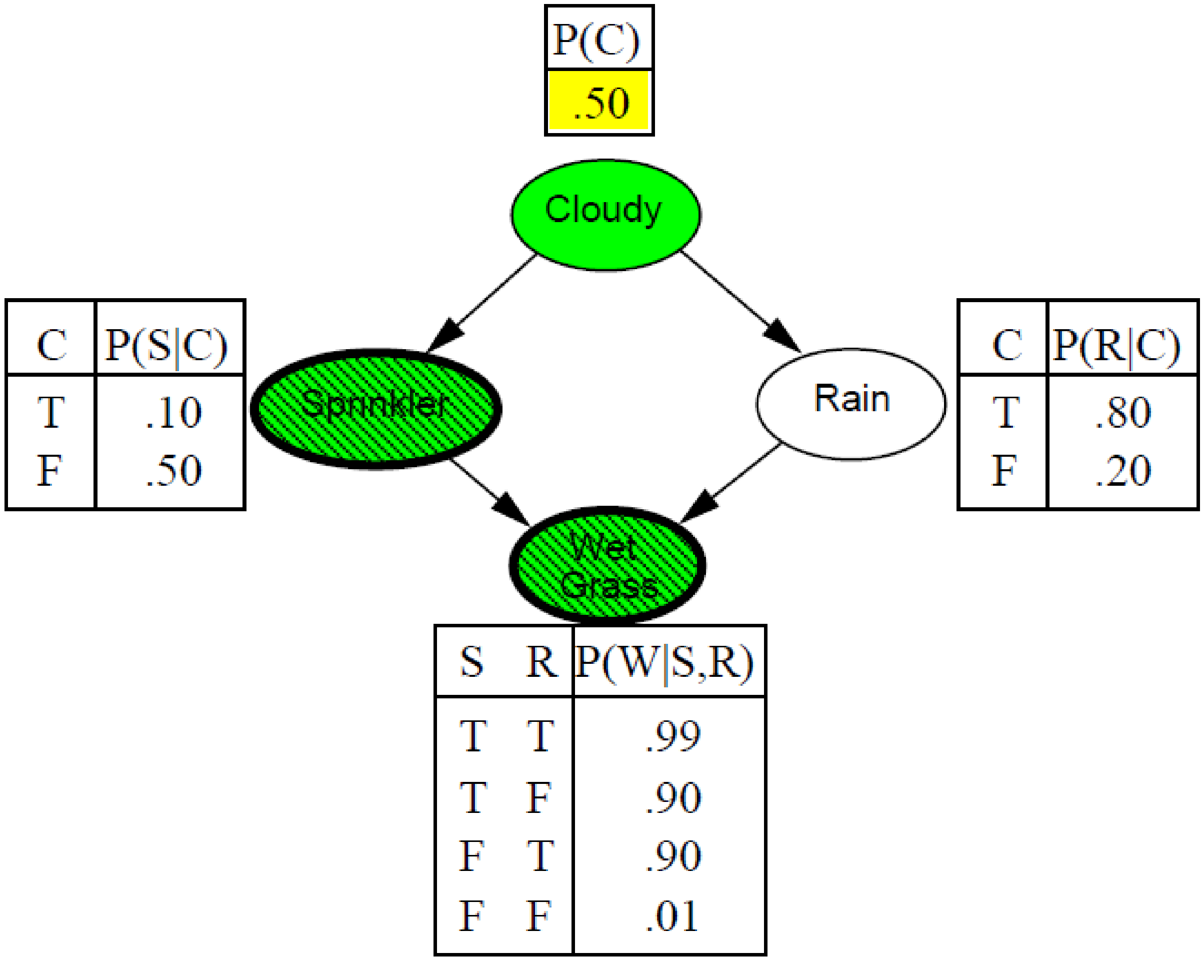
$w = 1.0$

Likelihood Weighting Example



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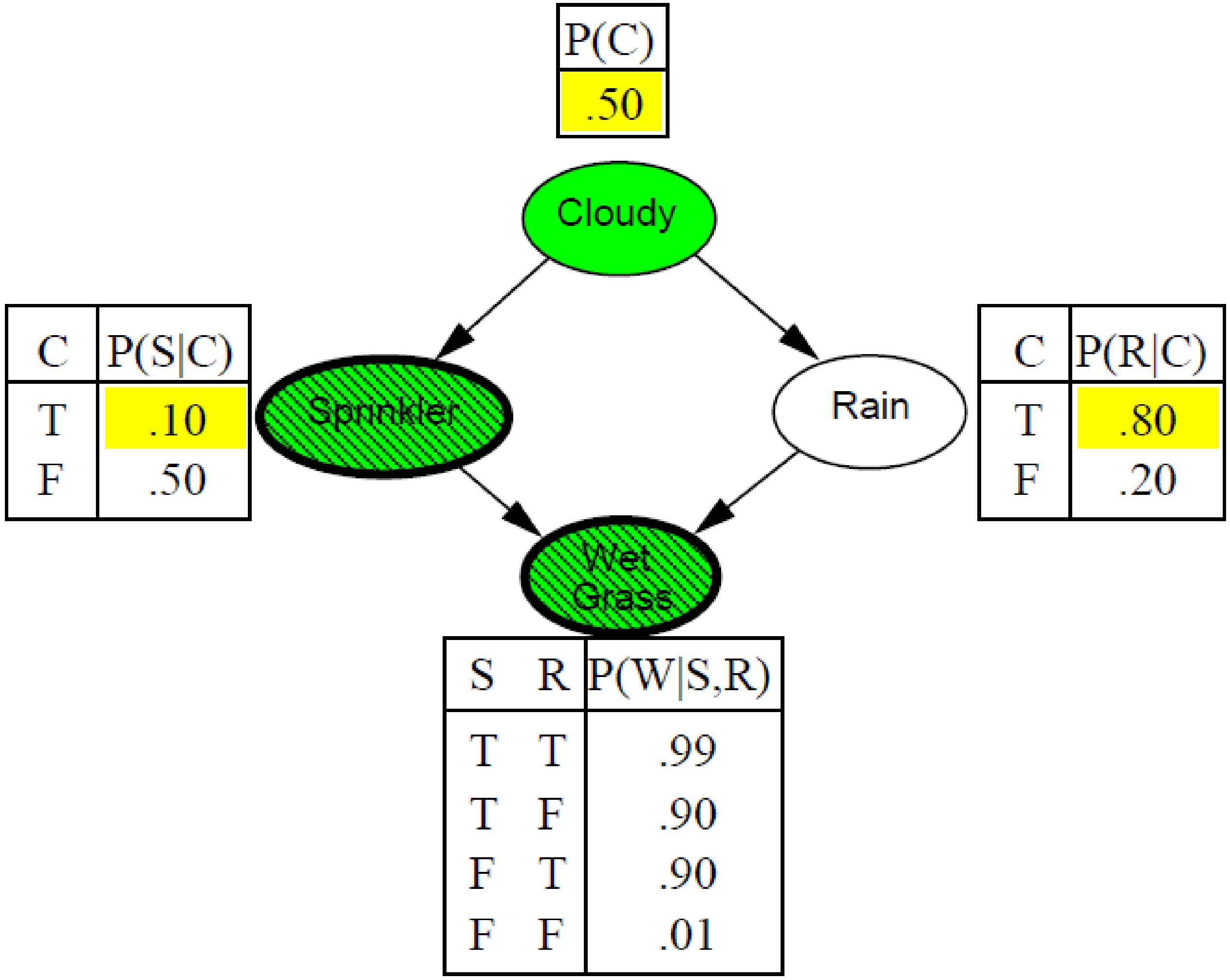
Likelihood Weighting Example



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Likelihood Weighting

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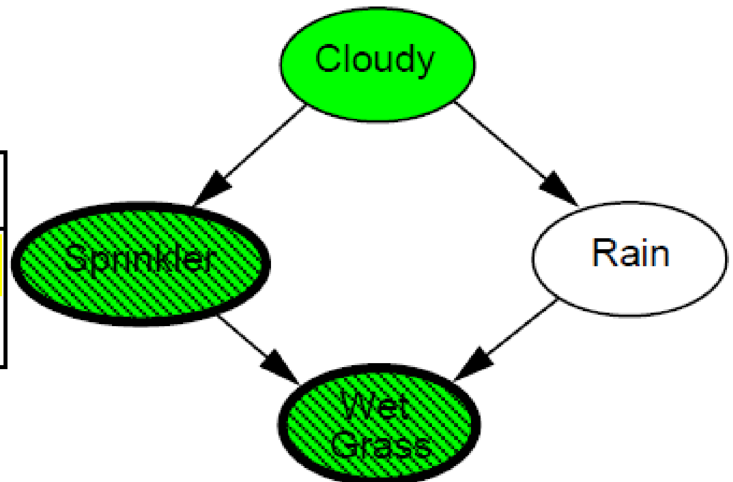

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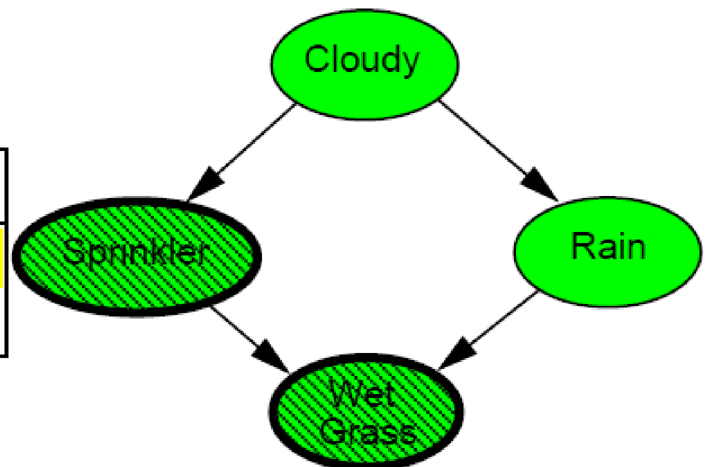
$w = 1.0 \times 0.1$

Likelihood Weighting



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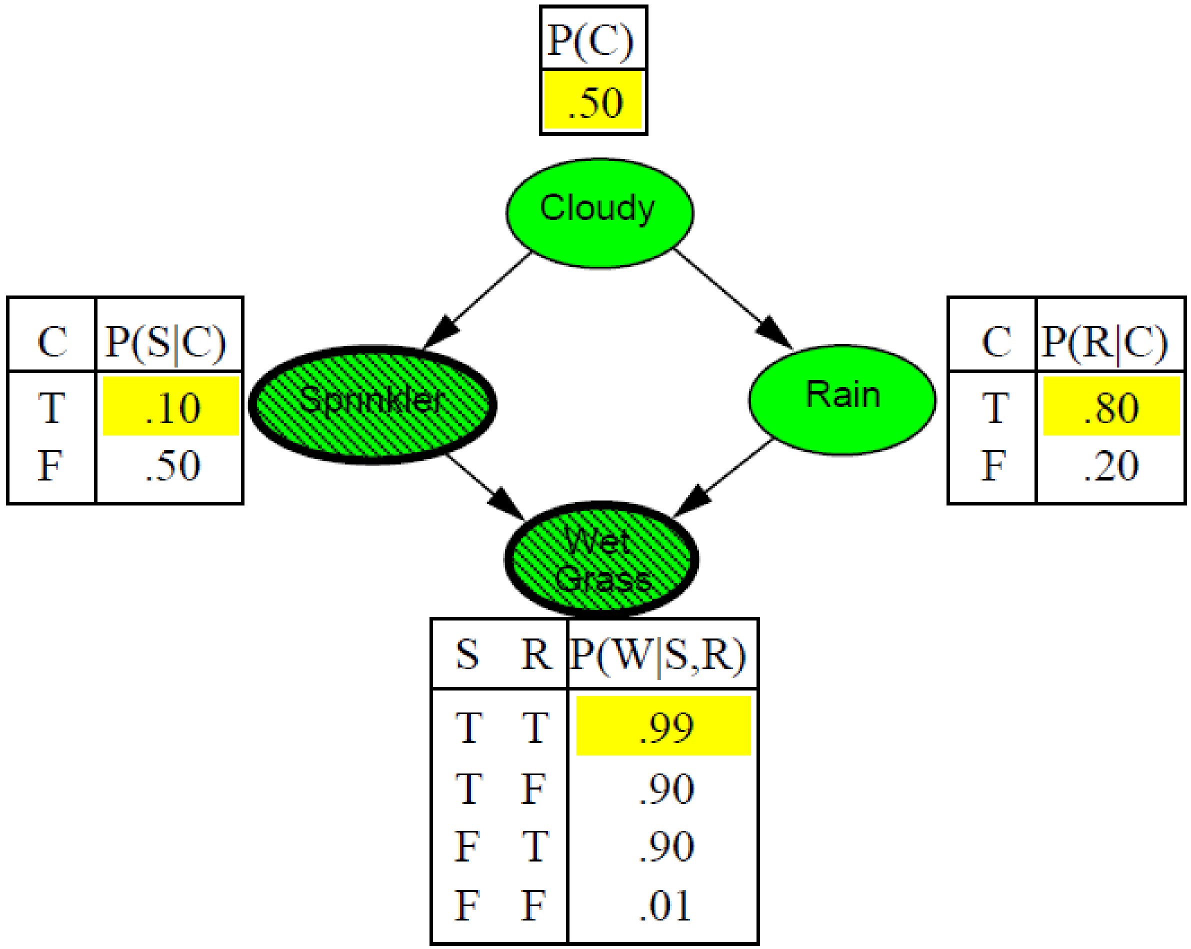
Likelihood Weighting



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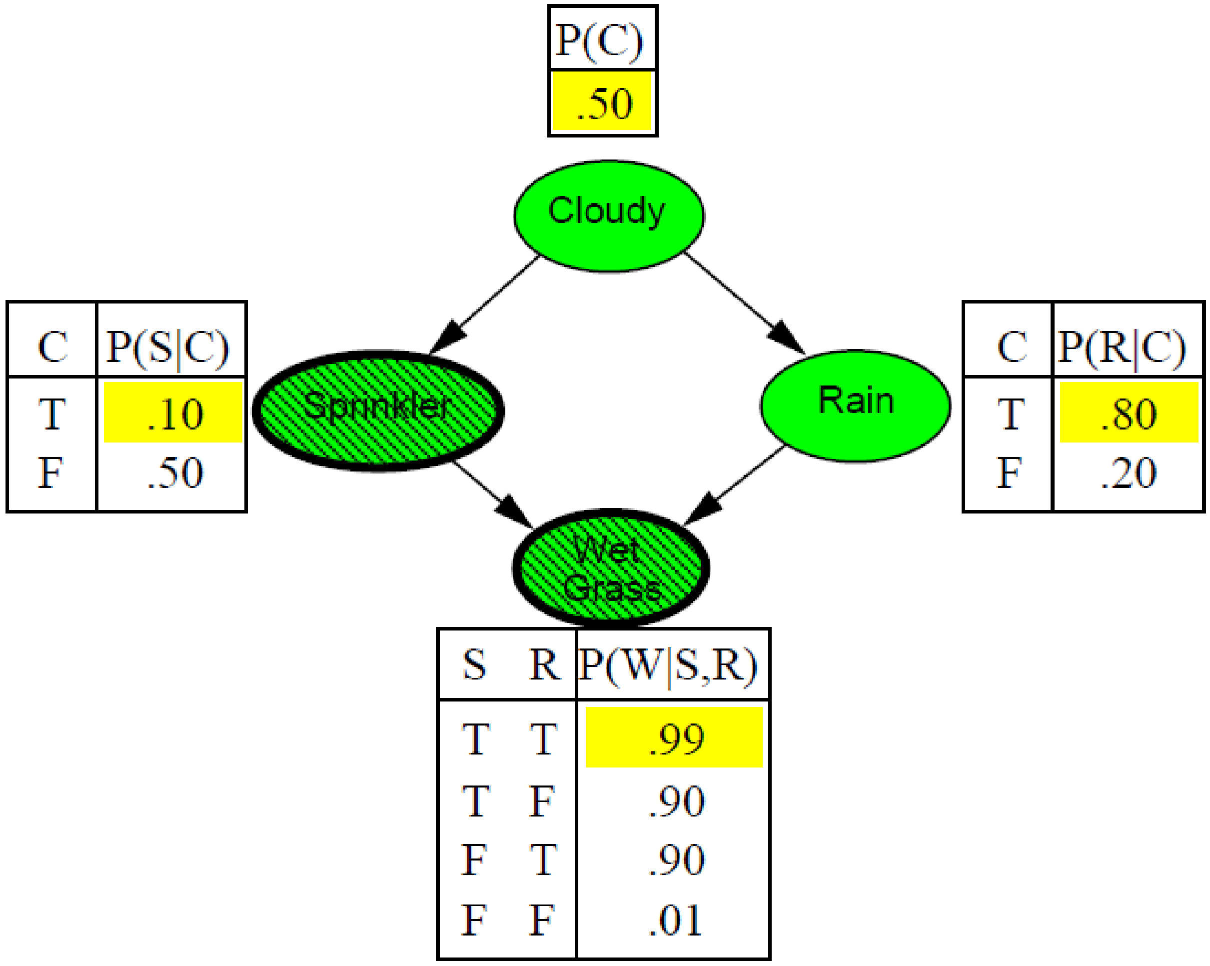
Likelihood Weighting



$$w = 1.0 \times 0.1 \times 0.99 = 0.099$$

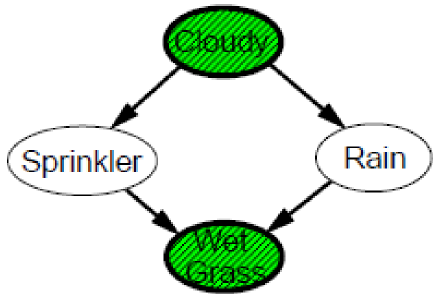
Analysis of Likelihood Weighting

Sampling probability for WEIGHTEDSAMPLE is

$$S_{WS}(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{l} P(z_i | parents(Z_i))$$

Note: pays attention to evidence in **ancestors** only

$\Rightarrow$ somewhere "in between" prior and posterior distribution



Weight for a given sample $\mathbf{z}, \mathbf{e}$ is

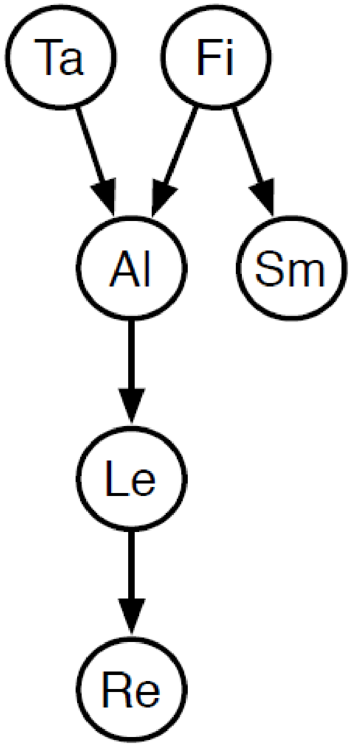
$$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i | parents(E_i))$$

Weighted sampling probability is

$$S_{WS}(\mathbf{z}, \mathbf{e}) w(\mathbf{z}, \mathbf{e})$$
$$= \prod_{i=1}^{l} P(z_i | parents(Z_i)) \ \prod_{i=1}^{m} P(e_i | parents(E_i))$$
$$= P(\mathbf{z}, \mathbf{e}) \text{ (by standard global semantics of network)}$$

Hence likelihood weighting returns consistent estimates but performance still degrades with many evidence variables because a few samples have nearly all the total weight

Likelihood Weighting Example



	Ta	Fi	Al	Le	Weight
s_1	true	false	true	false	0.01×0.01
s_2	false	true	false	false	0.9×0.01
s_3	false	true	true	true	0.9×0.75
s_4	true	true	true	true	0.9×0.75
$\ldots$					
s_{1000}	false	false	true	true	0.01×0.75

$$P(sm|fi) = 0.9$$
$$P(sm|\neg fi) = 0.01$$
$$P(re|le) = 0.75$$
$$P(re|\neg le) = 0.01$$

Markov Chain Monte Carlo

"State" of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket
Sample each variable in turn, keeping evidence fixed

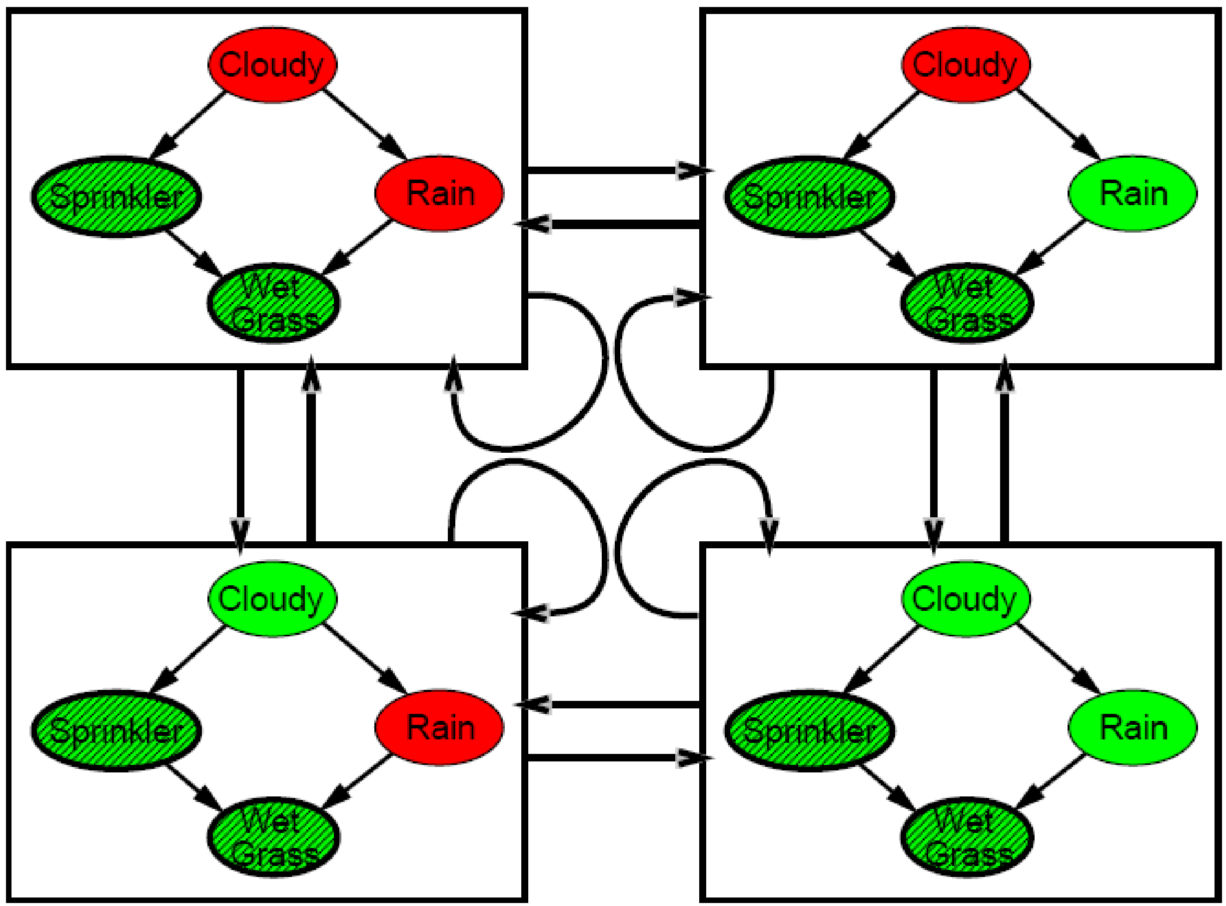
function MCMC-ASK(X, e, bn, N) **returns** an estimate of $P(X|\mathrm{e})$
  local variables: $\mathbf{N}[X]$, a vector of counts over X, initially zero
    $\mathbf{Z}$, the nonevidence variables in bn
    $\mathbf{x}$, the current state of the network, initially copied from e

  initialize $\mathbf{x}$ with random values for the variables in $\mathbf{Y}$
  for j = 1 to N **do**
    for each Z_i in $\mathbf{Z}$ **do**
      sample the value of Z_i in $\mathbf{x}$ from $\mathbf{P}(Z_i|mb(Z_i))$
        given the values of $MB(Z_i)$ in $\mathbf{x}$
      $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{N}[X]$)

Can also choose a variable to sample at random each time

Markov Chain

With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

MCMC

Estimate $\mathbf{P}(Rain|Sprinkler=true, WetGrass=true)$

Sample $Cloudy$ or $Rain$ given its Markov blanket, repeat.
Count number of times $Rain$ is true and false in the samples.

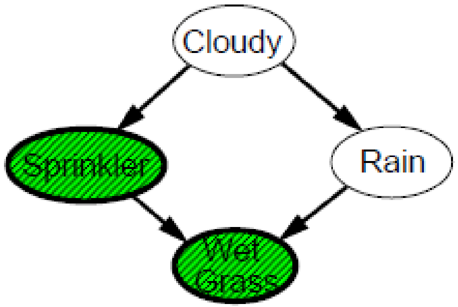
E.g., visit 100 states
  31 have $Rain=true$, 69 have $Rain=false$

$$\hat{\mathbf{P}}(Rain|Sprinkler=true, WetGrass=true)$$
$$= \textsc{Normalize}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle$$

Theorem: chain approaches stationary distribution:
  long-run fraction of time spent in each state is exactly proportional to its posterior probability

Sampling the Markov Network

Markov blanket of $Cloudy$ is
$Sprinkler$ and $Rain$
Markov blanket of $Rain$ is
$Cloudy$, $Sprinkler$, and $WetGrass$



Probability given the Markov blanket is calculated as follows:

$$P(x_i'|mb(X_i)) = P(x_i'|parents(X_i))\prod_{Z_j \in Children(X_i)} P(z_j|parents(Z_j))$$

Easily implemented in message-passing parallel systems, brains

Main computational problems:

1) Difficult to tell if convergence has been achieved
2) Can be wasteful if Markov blanket is large:
$P(X_i|mb(X_i))$ won't change much (law of large numbers)

Summary of Inference Methods

Exact inference by variable elimination:

- polytime on polytrees, NP-hard on general graphs
- space = time, very sensitive to topology

Approximate inference by LW, MCMC:

- LW does poorly when there is lots of (downstream) evidence
- LW, MCMC generally insensitive to topology
- Convergence can be very slow with probabilities close to 1 or 0
- Can handle arbitrary combinations of discrete and continuous variables