Math 711
Dr. Raynor
Due: February 16, 2012

Test #1

You may refer to the written materials from our course for this exam, such as the typed course notes, your notes from lecture, and your homework (including any help I gave you). You may use your Math 611 (or equivalent) textbook. You may not use any other written or recorded materials. You may not use the internet. You may not speak with any person about this exam (except me).

You may assume any theorems proved in the 611 textbook (or equivalent), but you must footnote any result that you use. You may assume all theorems actually proven in class or in the course notes. However, you must prove any exercises or other statements that you wish to use, including statements made without proof in the notes.

This test is out of **60 points**. It is due at **noon on Thursday, February 16**. You must sign the honor pledge and turn it in with your exam.

Pledge: I pledge on my honor that I have neither given nor received any assistance on this exam, nor have I used any dishonest means to obtain my results. I have not referred to any forbidden materials in completing this exam.

Signature: ______________________________

Throughout this test you may assume that (X, d) and (Y, ρ) are metric spaces.

(1) (5 points each) True or false? Justify your response with a complete argument or counterexample (not necessarily a formal proof).
  (a) Let $x \in X$ and $\epsilon > 0$. Then $\overline{B_\epsilon(x)} = \{y \in X : d(x, y) \leq \epsilon\}$ is compact in X.
  (b) A sequence in $\mathbb{R}^n$ converges if and only if it has exactly one accumulation point.
  (c) On the space l^2, the sup-norm and the l^2-norm are equivalent.
  (d) Suppose that $f : X \to Y$ is a function which is continuous at $x_0 \in X$. Let U be some open set in Y which contains $f(x_0)$. Then $f^{-1}(U)$ must be open in X.
  (e) If a sequence of continuous functions $f_n(x)$ converge to a continuous function $f(x)$ on a compact domain K, then they converge uniformly on K.

(2) We say that a sequence (x_n) is *eventually zero* if there exists some $N \in \mathbb{N}$ so that, for all $n > N$, $x_n = 0$. Let O be the collection of all eventually zero sequences.
  (a) (3 points) Show that $O \subset l^2$.
  (b) (5 points) Show that O is dense in l^2.

(3) Consider the set $C([0, 1])$ as defined in class, and, for $p \geq 1$, define $\|\cdot\|_p : C([0, 1]) \to \mathbb{R}$ by $\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}$.
  (a) (5 points) Prove that this is a norm on $C([0, 1])$.
  (b) (6 points) Prove that if $\frac{1}{p} + \frac{1}{q} = 1$, then $\forall f, g \in C([0, 1])$, $\int_0^1 |f(x)g(x)|dx \leq \|f\|_p \|g\|_q$. (Hint: This is extremely similar to Exercise 1.1.)
  (c) (6 points) Is $C([0, 1])$ complete with respect to this norm? Either prove it or give a counterexample.

(4) (10 points) The following is a correct proof that the Nested Interval Property is equivalent to the Bolzano-Weierstrass Theorem on $\mathbb{R}$. However, the sentences in

each half of the proof have been put in random order. All you have to do is to put them into the correct order to form a logically valid proof. Note that your goal is to put MY proof in order, not to create your own. For maximum partial credit you will want to copy out your answer in full sentences so I can see how well it fits together.

For your reference, the Nested Interval Property states that if $(I_n)_{n=1}^{\infty}$ is a sequence of nonempty, closed, bounded intervals in $\mathbb{R}$ so that, for each n, $I_n \supset I_{n+1}$, then $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem 1. *The nested interval property of the reals is equivalent to the Bolzano-Weierstrass Theorem on $\mathbb{R}$.*

Proof. $(\Rightarrow)$

(a) Suppose that for every nested sequence $(F_n)_{n=1}^{\infty}$ of nonempty, closed, bounded intervals in the reals, $\cap_n F_n \neq \emptyset$.

(b) Also note that the length of the interval F_j is $\frac{M}{2^{j-1}}$.

(c) Because (x_n) is bounded, there exists $M > 0$ such that $-M < x_n < M$ $\forall n \in \mathbb{N}$.

(d) Then we define F_{j+1} to be either the left half or the right half of F_j, whichever half contains an infinite number of sequence terms x_n.

(e) By the nested interval property, $\cap_j F_j \neq \emptyset$.

(f) Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers.

(g) Let $\epsilon > 0$.

(h) Let (x_{n_1}) be any term of (x_n) which is in F_1.

(i) Suppose that we have already selected $x_{n_1}, \ldots, x_{n_k}$.

(j) Choose J sufficiently large so that $\frac{M}{2^{J-1}} < \epsilon$.

(k) Then we claim that there is a subsequence of (x_n) which converges to x_0:

(l) Let x_0 be an element of $\cap_j F_j$.

(m) Clearly, either $[-M, 0]$ or $[0, M]$ contains an infinite number of the x_n.

(n) Then $x_{n_j} \in F_j$ by construction, and $F_j \subset F_J$ by the nestedness of (F_j).

(o) Now, suppose that we have chosen intervals F_i for $i = 0 \ldots j$, so that each F_{i+1} is a subset of F_i with half the length of F_i and so that each F_i contains an infinite number of the x_n.

(p) Then choose $x_{n_{k+1}}$ to be any term of (x_n) in F_{k+1} such that $n_{k+1} > n_k$.

(q) Therefore $|x_{n_j} - x_0| < \epsilon$.

(r) This is possible because F_{k+1} contains infinitely many terms of (x_n) so there must be one beyond x_{n_k}.

(s) Define F_1 to be whichever half contains an infinite number of the x_n.

(t) Let $j > J$.

(u) This allows us to recursively generate a sequence of intervals (F_j) such that, $\forall j$, $F_j \supset F_{j+1}$ and there are an infinite number of terms of (x_n) in F_j.

(v) Since x_0 and x_{n_j} are both elements of F_J, they cannot be further apart than the total length of F_J.

(w) Recall that x_0 is also an element of F_J for each J and that the total length of F_J is $\frac{M}{2^{J-1}} < \epsilon$.

(x) Hence $x_{n_k} \to x_0$ and (x_n) has a convergent subsequence as claimed.

(y) Clearly this process constructs a subsequence (x_{n_k}) of (x_n), so it only remains to prove that $x_{n_k} \to x_0$.

(z) Define $F_0 = [-M, M]$.

($\Leftarrow$):

(a) Hence no subsequence of (a_n) can possibly converge to a, which is a contradiction.
(b) Write $F_n = [a_n, b_n]$ for all n.
(c) If not, then there is a k such that $b_k < a$.
(d) We claim that $a \in \cap_n [a_n, b_n]$.
(e) This is a contradiction, as every a_n must be less than every b_k for every k and n by the nestedness of (F_n).
(f) Finally, we must show that $a \leq b_n$ for each n.
(g) Hence the sequence (a_n) is a sequence of real numbers which is bounded between a_1 and b_1.
(h) Since $F_n \supset F_{n+1}\ \forall n$, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n\ \forall n$.
(i) Since a subsequence of the (a_n) converges to a, there is some a_{n_i} such that $|a_{n_i} - a| < \epsilon$ which implies that $a_{n_i} > a - \epsilon = \frac{a+b_k}{2} > b_k$.
(j) Let $\epsilon = \frac{a-b_k}{2}$.
(k) Suppose not.
(l) Let F_n be a nested sequence of nonempty, closed, bounded intervals in the reals.
(m) Therefore $a \geq a_n$ for each n.
(n) Let $\epsilon = \frac{a_k - a}{2}$.
(o) Suppose that every bounded sequence of real numbers has a convergent subsequence.
(p) But then since the (a_n) are an increasing sequence, $a < a_j$ for all $j > k$.
(q) First, we will show that $a \geq a_n$ for all n.
(r) Therefore $a \in \cap_n F_n$ and $\cap_n F_n \neq \emptyset$, as claimed.
(s) Then $\exists k$ such that $a < a_k$
(t) By the Bolzano-Weierstrass theorem, (a_n) has a convergent subsequence (a_{n_k}) which converges to some number a.
(u) Then, $\forall j > k, |a_j - a| \geq |a_k - a| > \epsilon$.
(v) Hence we may conclude that $a_n \leq a \leq b_n$ for all n , i.e. $a \in F_n$ for all n.

$\square$