Final Exam

Shuowen Wei

December 17, 2011

Part I
Theoretical

Problem 24.1

Proof:

(c).

True:

Since the eigenvalues are the roots of the characteristic polynomial of A, i.e $P(\lambda) = det(\lambda I - A) = 0$, A is real matrix so all the parameters in are real, and the complex roots always appear in pairs. If λ is the root of $P(\lambda) = 0$, so is $\bar{\lambda}$, then $\bar{\lambda}$ is also the eigenvalue of A.

(f).

True:

This is the statement of **Theorem 5.5**, we just repeat the proof on Sept. 8's note here: if A is hermitian matrix, then A has a complete set of orthogonal eigenvectors and all of the eigenvectors are real. Then A has eigenvalue decomposition as $A = Q\Lambda Q^* = Q|\Lambda|sign(\Lambda)Q^*$, where the entries in the diagonal of Λ are the eigenvalues of A and Q is unitary matrix. Just denote Q as U and $sign(\Lambda)Q^*$ as V^*, then we have $\Sigma = |\Lambda|$, as desired.

Problem 24.4

Proof:

(a).

$(\Longrightarrow)$

Suppose λ is A's arbitrary eigenvalue and x is its corresponding eigenvector, then $Ax = \lambda x$, hence $\lambda^2 x = \lambda Ax = A\lambda x = AAx = A^2 x$, repeat this way we can get that $\lambda^n x = A^n x$. By using the same idea we did in Exercise 3.2 (we already proved that on **Project 1**), $|\lambda|^n \, ||x|| = ||\lambda^n x|| = ||A^n x|| \leq ||A^n|| \, ||x||$, thus we have $|\lambda^n| \leq ||A^n||$.

So if $lim_{n\to\infty}|\lambda^n| \leq lim_{n\to\infty}||A^n|| = 0$, then $lim_{n\to\infty}|\lambda^n| = 0$, then $|\lambda| < 1$, since λ is arbitrary, then $\rho(A) < 1$.

$(\Longleftarrow)$

Since $A \in \mathbb{C}^{m\times m}$ is a square matrix, by **Theorem 24.9** that A has a Schur factorization $A = QTQ^*$ where Q is unitary and T is upper-triangular. Note that A and T are similar then they have the same eigenvalues, since T is upper-triangular, then eigenvalues of A necessarily appear on the diagonal of T. Now we have

$$A^n = (QTQ^*)^n = QT^nQ^*$$

If $\rho(A) = \rho(T) < 1$, then every entries in T's diagonal is less than 1. Since T is upper triangular, then $lim_{n\to\infty}T^n = O$, which implies that $lim_{n\to\infty}||A^n|| = lim_{n\to\infty}||Q||\,||T^n||\,||Q^*|| = 0$.

(b).

The spectral abscissa of a matrix A denoted as $\alpha(A) = max\, Re(\lambda)$ where λ is the eigenvalue of A.

The definition of e^{tA} is $e^{tA} = \sum_{k=0}^{\infty}\frac{1}{k!}(tA)^k$, and it has several properties as follows:

1. $e^{t(TAT^{-1})} = Te^{tA}T^{-1}$
2. $e^{t(A+B)} = e^{tA}e^{tB}$ for all B with $AB = BA$
3. $A = diag(A_1,\, A_2, ...A_m) \Longrightarrow e^{tA} = diag(e^{tA_1}, e^{tA_2}, ..., e^{tA_m})$

Since $A \in \mathbb{C}^{m\times m}$ is a square matrix, by **Theorem 24.9** that A has a Schur factorization $A = QTQ^*$ where Q is unitary and T is upper-triangular. Then $e^{tA} = Qe^{tT}Q^*$. Since T's diagonal entries are A's eigenvalues, denote T as $T = \lambda I + N$ where N is the rest of the entries in T except the eigenvalues, it's a kind like "deficient upper triangular matrix". For any eigenvalue λ that might be complex, $\lambda = Re(\lambda) + iIm(\lambda)$, where $Re(\lambda)$ and $Im(\lambda)$ are real, then $|e^{\lambda}| = |e^{Re(\lambda)}|\,|e^{iIm(\lambda)}| = |e^{Re(\lambda)}|$.

Then we have $e^{tT} = e^{t\lambda}e^{tN} = e^{t\lambda}(1 + tN + \cdots + \frac{1}{(m-1)!}(tN)^{m-1})$, thus

$$||e^{tA}|| = ||Qe^{tT}Q^*|| = |e^{t\lambda}|\,||e^{tN}|| = |e^{tRe(\lambda)}|(1+t||N||+\cdots+\frac{1}{(m-1)!}t^{m-1}||N||^{m-1})$$

Then $\forall\lambda$, $\exists M > 0$, such that $||e^{tA}|| = Me^{tRe(\lambda)} \leq Me^{t\alpha(A)}$,

Since λ is arbitrary eigenvalue of A, thus $lim_{t\to\infty}||e^{tA}|| = 0 \Longleftrightarrow \alpha(A) < 0$.

Problem 25.1(a)

Proof:

Since A is tridiagonal and hermitian, then A is full rank. For any $\lambda \in \mathbb{C}$, we have

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & 0 & \cdots & 0 \\ a_{12} & a_{22}-\lambda & a_{23} & \cdots & 0 \\ 0 & a_{23} & a_3-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & a_{m-1,m} \\ 0 & 0 & 0 & a_{m-1,m} & a_{mm}-\lambda \end{bmatrix}$$

since the subdiagonal and superdiagonal entries a_{12}, a_{23}, $...a_{m-1,m}$ are nonzeros, then after row operation we can still get at least $m-1$ rows of the matrix with first entry of each row is nonzero, thus $A-\lambda I$ has at least rank $m-1$.

Since A is hermitian, by **Theorem 24.7**, A is unitarily diagonalizable, then we have that $A = Q\Lambda Q^*$, where Q is unitary matrix and Λ is diagonal matrix with enreies are the eigenvalues of A , let $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_m)$, so

$$A - \lambda I = Q\Lambda Q^* - Q\lambda I Q^* = Q \begin{bmatrix} \lambda_1 - \lambda & 0 & \cdots & 0 \\ 0 & \lambda_2 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m - \lambda \end{bmatrix} Q^*$$

Since λ is arbitrary in $\mathbb{C}$, so if A has any two same eigenvalues, say λ_i, λ_j, then let $\lambda = \lambda_i$, then $Q(\Lambda - \lambda I)Q^*$ will has rank at most $m-2$, which contradicts that it has at least rank $m-1$, so every eigenvalues of A are distinct.

Problem 4

Solution:

For any matrix $A \in C^{m\times n}$, $m > n$, the condition number in terms of the singular value of A is

$$k(A) = \| A \| \| A^\dagger \| = \frac{\sigma_1}{\sigma_n}$$

where $\sigma_1$is largest singular value of A and σ_n is the smallest singular value of A. If A is not full rank, then $\sigma_n = 0$, thus $k(A) = \infty$.

Problem 5

Proof:

I think something is wrong with this question. Why the conjugancy of A by two vectors is still a matrix? And what's the subspace of a vector space?

Problem 6

Proof:

Suppose the matrix A's rank is $r\,(\leq min(m,n))$, then by **Theorem 5.9**, that for any k with $0 \leq k \leq r$, we have the best rank-k approximation of A is

$$||A - A_k||_F = \inf_{B\in\mathbb{C}^{m\times n}} ||A - B||_F = \sqrt{\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_r} \quad (*)$$

Since A has SVD decomposition $A = \sum_{i=1}^n \sigma_i u_i v_i^*$ and for $1 \leq k \leq r$ we have $A_k = \sum_{i=1}^k \sigma_i u_i v_i^*$, and each $\sigma_i u_i v_i^*$ is a rank one matrix. The SVD of A is unique if the sign of U or V is fixed, since for any k, $\sigma_k \geq \sigma_{k+1}$. If $\sigma_k = \sigma_{k+1}$, then we will get two matrix A_1 and A_2 just with the last column different, one is $\sigma_k u_k v_k^*$,the other one is $\sigma_{k+1} u_{k+1} v_{k+1}^*$ such that, by formula (*)

$$||A-A_k||_F = \sqrt{\sigma_{k+1}+\sigma_{k+2}+\cdots+\sigma_r} = \sqrt{\sigma_k+\sigma_{k+2}+\cdots+\sigma_r} = ||A-A_{k+1}||_F$$

Hence the besk rank-k approximation is not unique. So A_k is the unique solution to formula (*) to minimise the problem if and only if $\sigma_k > \sigma_{k+1}$.

Problem 7

Proof:

By the definition of F-norm, $||A||_F^2 = tr(A^*A)$, then first of all, we will prove that for any matrix B and C, $||AB+(I-AA^\dagger)C||_F^2 = ||AB||_F^2+||(I-AA^\dagger)C||_F^2$ holds.

$$||AB+(I-AA^\dagger)C||_F^2$$

$$= (AB+(I-AA^\dagger)C))^*(AB+(I-AA^\dagger)C))$$

$$= (AB)^*(AB)+((I-AA^\dagger)C)*((I-AA^\dagger)C)+(AB)^*(I-AA^\dagger)C+((I-AA^\dagger)C)^*(AB)$$

Since Moore-Penrose peseudoinverse $A^\dagger$ has properties $A^*AA^\dagger = A^*$ and $A^{\dagger *}A^*A = A$, then

$$(AB)^*(I-AA^\dagger)C = O \, and \, ((I-AA^\dagger)C)^*(AB) = O$$

thus

$$||AB+(I-AA^\dagger)C||_F^2 = ||AB||_F^2+||(I-AA^\dagger)C||_F^2$$

Then for any n-by-m matrix X, we have

$$||AX-I||_F^2 = ||A(X-A^\dagger)+(I-AA^\dagger)(-I)||_F^2$$

$$= ||A(X-A^\dagger)||_F^2+||(I-AA^\dagger)(-I)||_F^2$$

$$\geq ||AA^\dagger - I||_F^2$$

Thus $X = A^\dagger$ minimizes $||AX-I||_F$ over all n-by-m matrices, and we immediately get the value of this minimum is $||AA^\dagger - I||_F$.

Part II
Numerical Experiments

1.

Please run the m-file: **1.m**

we will get lamdas at iteration k=2,5,10,15,20,30,50,100 is:

ans =

Columns 1 through 4

5.214312617702448 5.214319743375389 5.214319743377534 5.214319743377536

Columns 5 through 8

5.214319743377536 5.214319743377536 5.214319743377536 5.214319743377536

2.

Please run the m-file: **2.m**

we will get lamdas at iteration k=2,5,10,15,20,30,50,100 is:

ans =

Columns 1 through 4

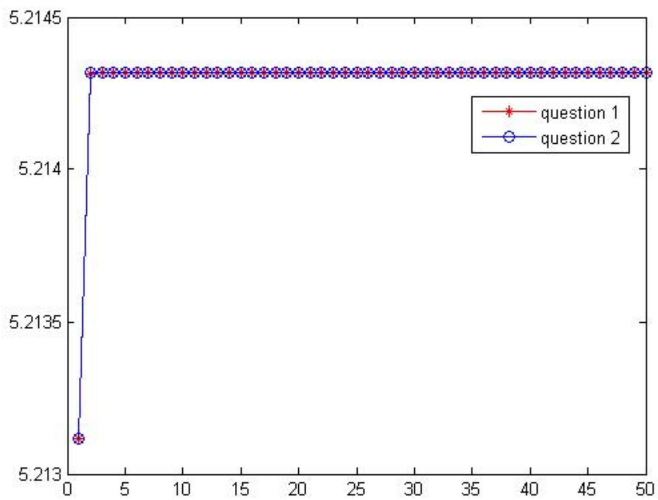
5.214319743184033 5.214319743377535 5.214319743377534 5.214319743377535

Columns 5 through 8

5.214319743377534 5.214319743377534 5.214319743377534 5.214319743377534

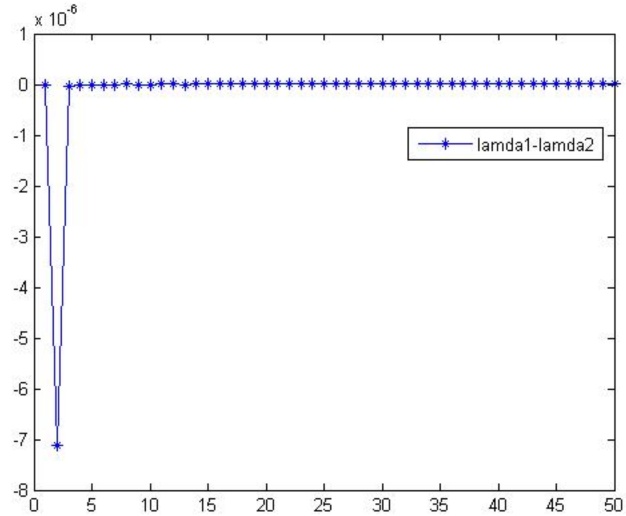
3.

Please run the m-file: **3.m**



comments:

Both algorithms are very fast to get the right eigenvalues of A, and initial point is also the same, both are 5. But from the graph of $\lambda_1 - \lambda_2$ below, we can see that algorithm 2 use less steps to get the final answer, while its computting obviously comsumes mch more time.


x 10^-6
1
0
-1
-2
-3
-4
-5
-6
-7
-8
0
5
10
15
20
25
30
35
40
45
50
lamda1-lamda2