Tutorial 8

Preface: This tutorial is intended as the round-up of the first part of the course dealing with computability theory. It is an excellent opportunity to return to some of the exercises that you did not manage to solve during the previous tutorial sessions. We will be ready to answer any of your questions concerning the material covered so far. Please, use this opportunity.

In case you do not have many questions, go ahead and solve the exercises on this sheet. They should give you further practice in formulating your own proofs by reduction. Some of these reductions you already saw during the lectures/tutorials but try to solve them without any support material one more time. We will be glad to check your answers during the tutorial.

Exercise 1 (compulsory)

Consider the following claim that some of you used during the second test.

Claim: If a language is co-recognizable then it cannot be decidable.

Does this claim hold or not? Give arguments for your answer.

Solution:
The claim is of course wrong. If a language is co-recognizable then it can still be decidable (but not all co-recognizable languages are of course decidable). In fact, every decidable language is by definition also recognizable and co-recognizable.

You might have gotten the impression that if a problem is co-recognizable then it is "sort of more difficult than being recognizable", but this is not the right intuition. In fact both recognizable and co-recognizable problems are essentially on the same "undecidability level", they just differ in the point whether we have a recognizer that accepts the positive or the negative instances of the problem.

Exercise 2 (for further practice on the simplest reductions)

1. Prove that $HALT_{TM} \leq_m A_{TM}$. (Note that your task is to find a mapping reduction in the opposite direction than the one provided in the book in Example 5.24 on page 212).

2. Prove that the language

$$EPSILON_{TM} \stackrel{\text{def}}{=} \{\langle M\rangle \mid \ M \text{ is a TM and } \epsilon \in L(M)\ \}$$

   is undecidable. Is $EPSILON_{TM}$ recognizable? Is $EPSILON_{TM}$ co-recognizable?

3. Prove that E_{TM} is undecidable. First, define the problem and then provide either the standard reduction or mapping reduction from a suitable undecidable problem.

Solution:

1. We will prove that $HALT_{TM} \leq_m A_{TM}$ by constructing a computable function f which on input $\langle M, w\rangle$ returns $\langle M', w\rangle$ such that

$$\langle M, w\rangle \in HALT_{TM} \text{ if and only if } \langle M', w\rangle \in A_{TM}.$$

   The idea is that we modify the machine M into a new machine M' such that M' will accept w if and only if M halted on w. The following TM M_f computes the function f.

M_f = "On input $\langle M, w\rangle$:
  1. Construct the following machine M':
    M' = "On input x:
      1. Run M on x.
      2. If M accepted, then M' accepts.
      3. If M rejected, then M' accepts."
  2. Output $\langle M', w\rangle$."

Clearly M_f computes the function f with the required property. Hence $HALT_{TM}$ is mapping reducible to A_{TM}.

2. Complete analogy with Exercise 3 from Exercise Set 5 (just replace 0010 with ϵ).

3. Lecture 5, slide 7.

Exercise 3 (for even further practice)

Prove that the problem

$$INFINITE_{TM} \stackrel{\text{def}}{=} \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ contains infinitely many strings }\}$$

is undecidable.

Exercise 4 (only for "feinschmeckers")

The credit for this exercise goes to Morten Dahl and Morten Kühnrich.

Assume that $L \subseteq \{0,1\}^*$ is an undecidable language. Prove that $L' \stackrel{\text{def}}{=} L \cup F$ remains undecidable for any finite language $F \subseteq \{0,1\}^*$. Is this the case also if we allow F to be an infinite language?

Hint: Prove the undecidability claim by reduction from L to L'. The details of the proof are rather delicate and in some sense a part of the proof is non-constructive.