Decidable and Undecidable Languages

The Halting Problem and
The Return of Diagonalization

Friday, November 11 and Tuesday, November 15, 2011
Reading: Sipser 4; Kozen 31; Stoughton 5.2 & 5.3

CS235 Languages and Automata

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Recursively Enumerable Languages

L(M) = {w | w is accepted by the
Turing Machine M}

The **recursively enumerable**
(r.e.) languages = the set of all
languages that are the language
of some Turing Machine.

These are also called
Turing-acceptable and
Turing-recognizable languages.

We will use **RE** to name this set.

There are many languages in
RE that are not in **CFL**.

RE = Recursively Enumerable
(Turing-Recognizable/Acceptable)
Languages

$a^n b^n c^n$ ww

CFL = Context-Free Languages
$a^n b^n$ ww^R

Reg = Regular Languages
a*b*   (a+b)*bbb(a+b)*

Decidability and Semi-Decidability

A Turing Machine **decides** a language if it
rejects every string it doesn't accept – i.e., it
never loops

The **recursive** languages = the set of all
languages that are decided by some Turing
Machine = all languages described by a non-
looping TM.

These are also called the**Turing-decidable** or
decidable languages.

We will use **Dec** to name this set.

We'll soon see examples of languages that are
in **RE** but not in **Dec**. We call these languages
semi-decidable+ (Lyn's nonstandard
terminology)

Every TM for a semi-decidable+ language halts
in the accept state for strings in the language
but loops for some strings not in the language.

Any language outside **Dec** is **undecidable**.
All semi-decidable+ languages are undecidable,
but we'll see there are undecidable languages
that aren't semi-decidable+!

RE = Recursively Enumerable
(Turing-Recognizable/Acceptable)
Languages

- *semi-decidable+*

Dec = Recursive (Turing-Decidable)
Languages

$a^n b^n c^n$ ww   • *decidable*

CFL = Context-Free Languages
$a^n b^n$ ww^R

Reg = Regular Languages
a*b*   (a+b)*bbb(a+b)*

Dec vs. RE

For every language L in **Dec**, there is a **deciding machine** M
that for an input string w is guaranteed to deliver a ball to
either the accept pipe or reject pipe.

input string w → Turing Machine M for
a language L in **Dec** → accept pipe
→ reject pipe

For every language L in **RE**, there is an **accepting machine** M
that for an input string w is guaranteed to deliver a ball to
the accept pipe if w ∈ L. However, if w ∉ L, a ball *might not*
be delivered to the reject pipe (M might loop).

input string w → Turing Machine M for
a language L in **RE** → accept pipe
→ reject pipe

Game Plan for the Rest of this Lecture

Our main goal is to exhibit a language L that's semi-decidable+: L in **RE** — **Dec**.

But first:
1. we'll need to get some practice describing decidable languages that involve **language encodings**.

Then:
2. we'll define a language $HALT_{TM}$ that's in **RE** — **Dec**.

Finally:
3. we'll argue that there are languages that aren't even in **RE**!

RE = Recursively Enumerable (Turing-Recognizable/Acceptable) Languages

• *semi-decidable+*

Dec = Recursive (Turing-Decidable) Languages

$a^n b^n c^n$ ww   • *decidable*

CFL = Context-Free Languages
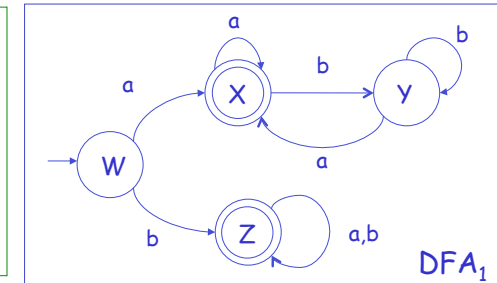
$a^n b^n$ ww^R

Reg = Regular Languages

a*b*   (a+b)*bbb(a+b)*

Language Encodings

We will consider many languages whose strings contain encodings of DFAs, FAs, NPDAs, CFGs, and TMs. Think of such encodings as Forlan-like specifications for these machines and grammars. E.g. :

<DFA₁> =
"({W,X,Y,Z},   ← states
 {a,b}   ← alphabet
 W,   ← start states
 {X,Z},   ← final states
 {W,a -> X; W,b -> Z;
 X,a -> X; X,b -> Y;
 Y,a -> X; Y,b -> Y;
 Z,a -> Z; Z,b -> Z})"

transition function

DFA_1

<CFG₁> =
"({S,A,B},   ← variables (start var first)
 {0,1},   ← terminals
 {(S,AB), (A,0A1), (A,%)
  (B,1B0), (B,%)})"   ← productions

S → AB
A → 0A1 | %
B → 1B0 | % CFG_1

Warm-Up: Some Decidable Languages

Show that the following languages are decidable by describing (at a high level) an algorithm that decides them (see more in Sipser 4.1)

string describing a pair of (1) a deterministic finite automaton DFA and (2) an input string w

• $ACCEPT_{DFA}$ = {(<DFA>, w) | w is in L(DFA)}

• $EMPTY_{DFA}$ = {<DFA> | L(DFA) = $\varnothing$}

• ALL_{DFA} = {<DFA> | L(DFA) = Σ_{DFA}*}

• EQ_{DFA} = {(<DFA₁>, <DFA₂>) | L(DFA₁) = L(DFA₂)}

• $ACCEPT_{CFG}$ = {(<CFG>, w) | w is in L(CFG)}

• $EMPTY_{CFG}$ = {<CFG> | L(CFG) = $\varnothing$}

(Warning: EQ_{CFG} and ALL_{CFG} are **not** decidable!)

What about $ACCEPT_{TM}$?

string describing a Turing Machine M and an input string w

$ACCEPT_{TM}$ = {(<M>, w) | w is in L(M) (i.e. M accepts w)}

A Turing Machine M_{UTM} can accept $ACCEPT_{TM}$ as follows:

1. Check if <M> is a well-formed Turing Machine description. If not, M_{UTM} **rejects** (<M>, w)

2. If <M> is well-formed, M_{UTM} simulates the running of M on w, e.g., via a 4-tape machine:
   • Tape 1 holds <M>.
   • Tape 2 is the tape M works on (initially w).
   • Tape 3 holds the head position for tape 2.
   • Tape 4 holds the state of M.
   In this case, M_{UTM} will:
   • **accept** (<M>, w) iff M accepts w.
   • **reject** (<M>, w) iff M rejects w.
   • **loop** iff M loops on w.

So $ACCEPT_{TM}$ is in **RE**.

Big question: is $ACCEPT_{TM}$ in **Dec**? I.e., is there another Turing Machine that **decides** $ACCEPT_{TM}$? (We'll see the answer is no.)

M_{UTM} is a Universal Turing Machine!

M_{UTM} is a **universal Turing Machine**

   = a Turing Machine interpreter written as a Turing Machine.

There's nothing strange about this:

- We can write an SML interpreter in any language, including SML.
- We can write a Java compiler in any language, including Java.
- Why not write a Turing Machine interpreter as a Turing Machine?

The tricky bit is **bootstrapping** (take CS251 for more details):

- Our *first* SML interpreter can't be written in SML;
- Our *first* Java compiler can't be written in Java.

Self-Reference is not a Problem

Consider the following:

- A decommenting program can decomment any text file, including the decommenting program itself.
- A Java compiler can compile any Java program, including one that specifies a Java compiler.
- An SML interpreter can evaluate any SML program, including one that specifies an SML interpreter.

Moral: There is nothing inherently problematic about a program being called on "itself". The program supplied as an argument is just data, and the running program P doesn't "know" that this data describes P!

You can't eat yourself, but you can eat a *description* of yourself!

What does M_{UTM} do on the input ($<M_{UTM}>$, ($<M>$,w)) ?

The Halting Problem

$HALT_{TM}$ = {($<M>$, w) | M halts on w (i.e. M decides w, cannot loop)}

Is $HALT_{TM}$ in **RE** (Turing-acceptable)?

    Yes: Simulate M running on w. Accept if M accepts or rejects w. Loop if M loops on w.

Is $HALT_{TM}$ in **Dec** (decidable)?

    No! We'll show this by diagonalization. Intuitively, the problem is that no TM for $HALT_{TM}$ can always reject ($<M>$, w) when M loops on w.

So $HALT_{TM}$ is semi-decidable+; our first example of such a language.

In the next lecture, we'll see that we can use $HALT_{TM}$ to show that other languages are semi-decidable+, including $ACCEPT_{TM}$.

Behavior of Turing Machines

The behavior of all Turing machines can be summarized by an infinite 2D table whose rows are Turing machines and whose columns are input strings. A table entry is A (accept), R (reject), or L (loop).

Because TM descriptions are countable, TMs are enumerable in a sequence $M_1, M_2, M_3, ...$

input strings

	%	a	...	z	aa	...	zz	...
M_1	A	R	...	A	L	...	R	...
M_2	R	L	...	R	R	...	R	...
M_3	A	A	...	R	L	...	A	...
M_4	L	L	...	A	A	...	R	...
M_5	R	A	...	R	A	...	L	...
...	...	...	...	...	...	...	...	...

Turing Machines

Towards Diagonalization

With diagonalization in mind, we focus on the subtable that results from keeping only those columns in whose input strings are valid TM descriptions.

inputs that are TM descriptions

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	...
M_1	R	A	L	A	L	...
M_2	A	A	A	A	A	...
M_3	L	L	L	L	L	...
M_4	A	L	A	A	R	...
M_5	A	R	L	A	L	...
...	...	...	...	...	...	...

Turing Machines

Decidable and Undecidable Languages   30-13

Suppose HALT$_{TM}$ Is Decidable

If HALT$_{TM}$ is decidable, then there is a Turing Machine M_{HALT} that, for all inputs ($\langle M\rangle$,w), decides if M halts on w.

inputs that are TM descriptions

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	...
M_1	R	A	L	A	L	...
M_2	A	A	A	A	A	...
M_3	L	L	L	L	L	...
M_4	A	L	A	A	R	...
M_5	A	R	L	A	L	...
...	...	...	...	...	...	...

Turing Machines

Behavior of Turing Machines on inputs that are TM descriptions

inputs that are TM descriptions

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	...
$\langle M_1\rangle$	A	A	R	A	R	...
$\langle M_2\rangle$	A	A	A	A	A	...
$\langle M_3\rangle$	R	R	R	R	R	...
$\langle M_4\rangle$	A	R	A	A	A	...
$\langle M_5\rangle$	A	A	R	A	R	...
...	...	...	...	...	...	...

TM descriptions

Behavior of M_{HALT} on inputs of the form ($\langle M_i\rangle$,$\langle M_j\rangle$)

Decidable and Undecidable Languages   30-14

Diagonalization for HALT$_{TM}$

We can perform diagonalization on the original table of TM behaviors

If M_{HALT} exists, then there is another machine M_{DIAG} defined as:

$M_{DIAG}(\langle M\rangle)$ = if $M_{HALT}($ ($\langle M\rangle$,$\langle M\rangle$)) then loop else accept

M_{DIAG} inverts every entry on the diagonal of M_{HALT}.

Because M_{DIAG} is a TM, it must appear in the list of descriptions. But it can't! What is $M_{DIAG}(\langle M_{DIAG}\rangle)$?

So the assumption that M_{HALT} exists is false, and HALT$_{TM}$ isn't decidable.

inputs that are TM descriptions

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	...	$\langle M_{DIAG}\rangle$
M_1	R	A	L	A	L	...	
M_2	A	A	A	A	A	...	
M_3	L	L	L	L	L	...	
M_4	A	L	A	A	R	...	
M_5	A	R	L	A	L	...	
...	...	...	...	...	...		
M_{DIAG}	L	L	A	L	A		???

Turing Machines

Behavior of Turing Machines on inputs that are TM descriptions

Decidable and Undecidable Languages   30-15

Alternative Diagonalization for HALT$_{TM}$

We can instead perform the diagonalization on the behavior table for M_{HALT}

If M_{HALT} exists, then there is a machine M_{DIAG} defined as:

$M_{DIAG}(\langle M\rangle)$ = if $M_{HALT}($ ($\langle M\rangle$,$\langle M\rangle$)) then loop else accept

M_{DIAG} inverts every entry on the diagonal of the table, leading to a contradiction.

inputs that are TM descriptions

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	...	$\langle M_{DIAG}\rangle$
$\langle M_1\rangle$	A	A	R	A	R	...	
$\langle M_2\rangle$	A	A	A	A	A	...	
$\langle M_3\rangle$	R	R	R	R	R	...	
$\langle M_4\rangle$	A	R	A	A	A	...	
$\langle M_5\rangle$	A	A	R	A	R	...	
...	...	...	...	...	...		
$\langle M_{DIAG}\rangle$	R	R	A	R	A		???

TM descriptions

Behavior of M_{HALT} on inputs of the form ($\langle M_i\rangle$,$\langle M_j\rangle$)

Decidable and Undecidable Languages   30-16

The Halting Problem in Scheme*

Suppose we could write a function halts? that determines if an input function f halts when applied to its argument f:

(define (halts? f x) ...)

Then we could write the following:

(define (loop) (loop))

(define (diag f) (if (halts? f f) (loop) #t))

Suppose (diag diag) halts. Then it should loop!

Suppose (diag diag) loops. Then it should halt with #t!

This accurately captures the diagonalization dilemma, but is a bit hokey since a Scheme halts? function can't examine the structure of the input function in the way that a Turing Machine M_{HALT} can examine the structure of a TM description.

* It's tougher to show this in SML because its type system prohibits self-application.

So What? Does the Halting Problem Matter?

Yes! Compilers and other programs that analyze programs often want to perform **termination analysis** to determine whether or not the evaluation of a particular subexpression will terminate.

E.g., it's often helpful to evaluate a subexpression, but only safe to do so if evaluation will terminate (otherwise, the compiler might not terminate!)

Sadly, the halting problem says there is no iron-clad way to determine in advance whether or not subexpression evaluation will terminate.

We'll see later that various heuristics can be used, but they can only conservatively approximate exact termination analysis.

We'll also see that most forms of program analysis suffer from the same problems as termination analysis.

With Great Power Comes Great Uncomputability

Turing machines and equivalent models of computation (lambda calculus, Java, SML, etc.) are far more powerful than finite automata and pushdown automata.

But the power is gained via features that can cause programs to loop infinitely. If we want the power, we must live with the looping.

Programs that loop vs. taking a long time

How do we distinguish programs that run a long time from ones that loop?

E.g. 3x+1 problem:

$$f(x) = \begin{cases} 3x + 1, & \text{if } x \text{ is odd} \\ x/2, & \text{if } x \text{ is even} \end{cases}$$

Problem: for all n, is there some i s.t. $f^i(n) = 1$? I.e., is it the case that iterating f at a starting point never loops?
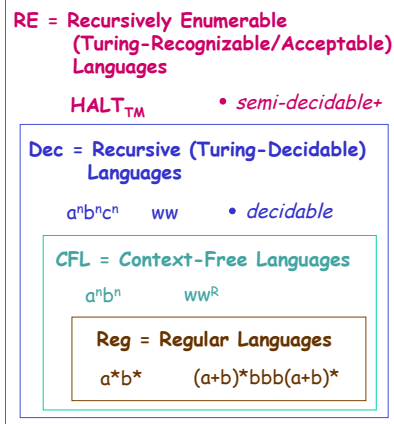
No one knows! This is an open problem!

Are there non-RE Languages?

Our next big question: are there any languages that are not **RE**?

We'll see the answer is yes.
In fact, **way** yes (lots of them!)

- *Is there any language out here?*

RE = Recursively Enumerable (Turing-Recognizable/Acceptable) Languages

HALT$_{TM}$ • *semi-decidable+*

Dec = Recursive (Turing-Decidable) Languages

$a^n b^n c^n$ ww • *decidable*

CFL = Context-Free Languages

$a^n b^n$ ww^R

Reg = Regular Languages

$a*b*$ $(a+b)*bbb(a+b)*$

Dec is Closed Under Complement

Suppose M is a Turing Machine that decides L.

We can construct a machine $\overline{M}$ that decides $\overline{L}$:
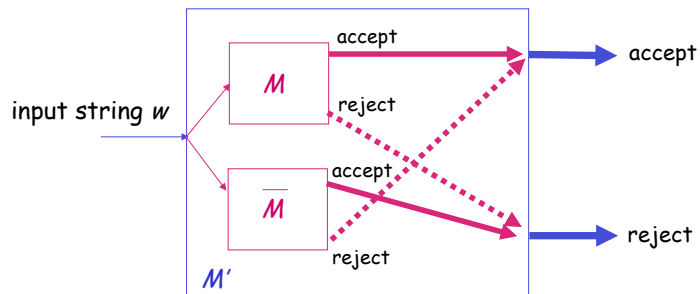

input string w → M → accept / reject (crossed) → accept / reject

$\overline{M}$
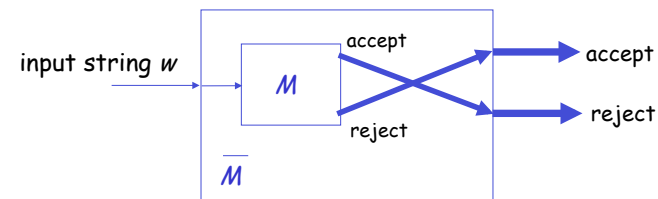
RE is **NOT** Closed Under Complement

Suppose M accepts L and $\overline{M}$ accepts $\overline{L}$.

Then L is decidable (by M' below)!

input string w → M (accept/reject) and $\overline{M}$ (accept/reject) → accept / reject

M'

Important detail: M and $\overline{M}$ must be run in parallel, not sequentially!
See next slide.

So if L is semi-decidable+ (L in **RE** - **Dec**), then $\overline{L}$ *can't* be in **RE**!


Digression: How to Run Two TMs in Parallel?

We often want to run two accepting machines M_1 and M_2 on the same or different inputs.

The machines should not be run *sequentially* (say M_1 before M_2) because if M_1 loops, M_2 will never run.

Instead, the machines are run *in parallel* by performing one step of M_1 followed by one step of M_2, alternating between the two machines.

After each step, we can check whether either machine is in its accept state or reject state. So it's possible to run M_1 and M_2 in parallel until one accepts, both accept, one rejects, or both reject.

input string w_1 → M_1 → accept / reject
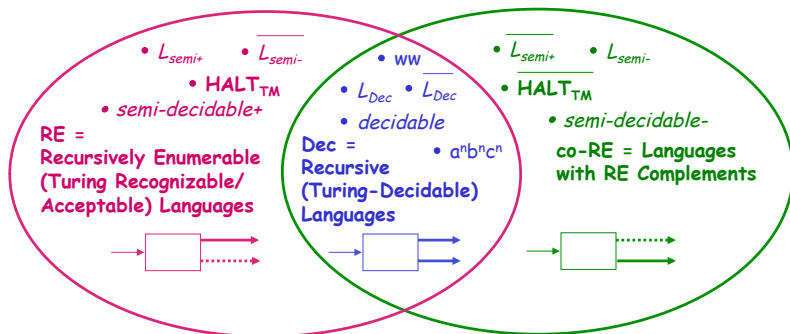
input string w_2 → M_2 → accept / reject
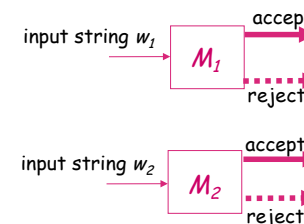
Complementary, My Dear Watson

co-RE is the set of languages whose complements are **RE**.

If L is semi-decidable+ (in **RE** — **Dec**),

then $\bar{L}$ is **semi-decidable-** (in **co-RE** — **Dec**).

We'll say L is **semi-decidable** if it's semi-decidable+ or semi-decidable-

- L_{semi+}
- $\overline{L_{semi-}}$
- **HALT$_{TM}$**
- *semi-decidable+*

**RE =
Recursively Enumerable
(Turing Recognizable/
Acceptable) Languages**

- ww
- L_{Dec} $\overline{L_{Dec}}$
- *decidable*
- $a^n b^n c^n$

**Dec =
Recursive
(Turing-Decidable)
Languages**

- $\overline{L_{semi+}}$
- L_{semi-}
- $\overline{\textbf{HALT}_{TM}}$
- *semi-decidable-*

**co-RE = Languages
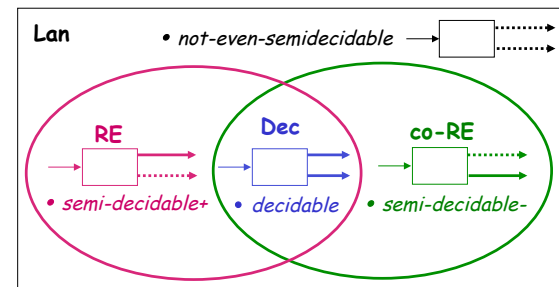with RE Complements**

Not-Even-Semidecidable Languages

Any Turing Machine can be described as a string in Σ^* (for an appropriate Σ),

$states \# inputAlphabet \# tapeAlphabet \# transitions \# start \# accept \# reject$

We can enumerate all possible Turing Machine descriptions, so
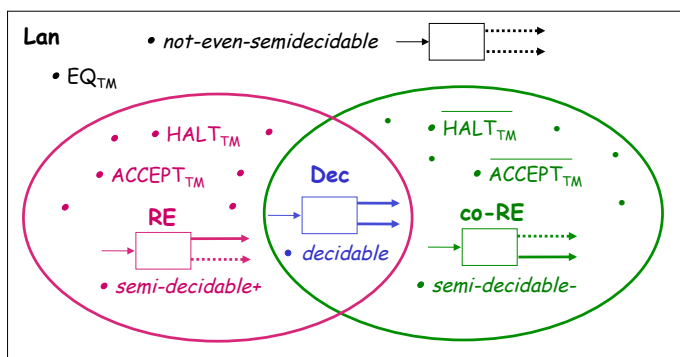**RE** and **co-RE** must be *countable*.

But we know that **Lan** = $P(\Sigma^*)$ is an *uncountable* set.

So **Lan** – (**RE** ∪ **co-RE**) must contain *many* languages. We'll call these languages
not-even-semidecidable. We'll see concrete examples of these next lecture.

For an not-even-semidecidable
language, every Turing
Machine must loop for
some strings in the
language and some not
in the language.

Lan
- *not-even-semidecidable*

RE   **Dec**   **co-RE**

- *semi-decidable+*   • *decidable*   • *semi-decidable-*

What We're Aiming For

Lan
- *not-even-semidecidable*
- EQ$_{TM}$

- HALT$_{TM}$
- ACCEPT$_{TM}$

RE
- *semi-decidable+*

Dec
- *decidable*

- $\overline{\text{HALT}_{TM}}$
- $\overline{\text{ACCEPT}_{TM}}$

co-RE
- *semi-decidable-*

Note: **undecidable** =
semi-decidable+ ∪ semi-decidable- ∪ not-even-semidecidable

Closure Properties of **Dec** and **RE**

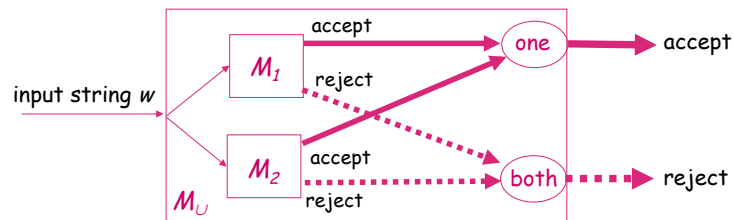| **Dec** is closed under:	**RE** is closed under:
• union	• union
• intersection	• intersection
• concatenation	• concatenation
• Kleene star	• Kleene star
• **complement**	

We've already seen the complement story:
- L in **Dec** implies $\bar{L}$ in **Dec**.
- L and $\bar{L}$ are in **RE** implies L and $\bar{L}$ in **Dec**.
- L is semi-decidable+ (in **RE** — **Dec**)
  and $\bar{L}$ semi-decidable- (in **co-RE** — **Dec**)

Now we'll study the other properties

RE is Closed Under Union

Suppose L_1, L_2 are accepted by machines M_1, M_2, respectively.
$L_1 \cup L_2$ is accepted by the following machine M_U, which
runs M_1 and M_2 in parallel until one accepts or both reject.


It is essential to run M_1 and M_2 in *parallel*. Why?

A similar diagram shows **Dec** is closed under union,
  but in that case M_1 and M_2 can be run *sequentially*. Why?

Similar arguments (PS10) show **RE**, **Dec** are closed under intersection.

RE is Closed Under Concatenation

Suppose L_1, L_2 are accepted by machines M_1, M_2, respectively. $L_1 @ L_2$ is
accepted by the following machine $M_@$, which runs copies of M_1 and M_2 in
parallel on all possible decompositions of w into pairs of substrings until one
pair accepts or all reject.
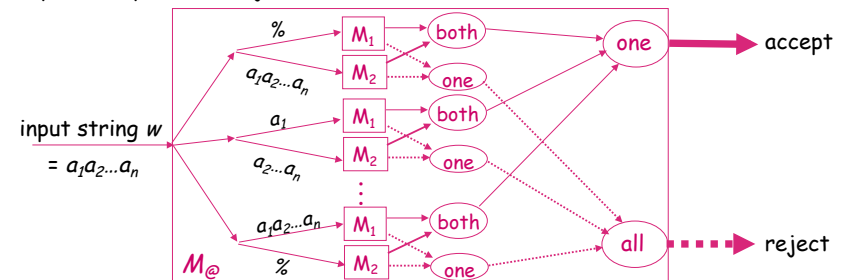
This diagram can be implemented by a Turing Machine program that
loops over all possible decompositions, interleaving the steps for each.

Similar ideas show that **Dec** is closed under concatenation and that both
RE and **Dec** are closed under Kleene star.