Math 711
Dr. Raynor

Test #2 Solutions

(1) (5 points each) True or false? Prove your response.

(a) (5 points) A space X is isometric to its completion $\tilde{X}$ iff X is complete.

Answer: True. Suppose that X is a metric space and $\tilde{X}$ is its completion. Let i be the natural embedding of X into $\tilde{X}$ as constructed in class. Recall that i is an isometry, so therefore it is injective. First, suppose that X is complete. Then we want to show that i is surjective and hence is an isomorphism. Suppose that $[(p_n)]$ is an element of $\tilde{X}$. Let (p_n) be a representative Cauchy sequence of $[(p_n)]$ in X. Since X is complete, there is an $x \in X$ such that $p_n \to x$. Consider $i(x) = [(x, x, x, \ldots)]$. Then

$$\tilde{d}(i(x), [(p_n)]) = \lim_{n\to\infty} d(i(x)_n, p_n) = \lim_{n\to\infty} d(x, p_n) = 0$$

because $p_n \to x$. Therefore $i(x) \sim (p_n)$, so $i(x) = [(p_n)]$. Therefore i is surjective as claimed. On the other hand, suppose that X is isometric to $\tilde{X}$. We proved in class that $\tilde{X}$ is complete, so X must also be complete.

(b) (5 points) Let $B = \{\vec{x} \in \mathbb{R}^2 : \|\vec{x}\| < 1\}$. Suppose that $f : B \to B$ and $\exists r < 1$ such that, $\forall \vec{x}, \vec{y} \in B$, $\|f(\vec{x}) - f(\vec{y})\| \le r\|\vec{x} - \vec{y}\|$. Then f has a unique fixed point.

Answer: False. For example, let $f : B \to B$ be the map given by

$$f(x, y) = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y).$$

Let's check that f is a contraction mapping from B to itself. Let (x, y) in B. Then

$$\begin{aligned}\|f(x,y)\|^2 &= (\frac{1}{2}x + \frac{1}{2})^2 + (\frac{1}{2}y)^2 \\ &= \frac{1}{4}(x^2 + 2x + 1 + y^2) \\ &\le \frac{1}{4}(x^2 + y^2) + \frac{1}{4}(2|x| + 1) \\ &< \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 3 \\ &= 1.\end{aligned}$$

So $f(x, y) \in B$ if (x, y) is. Also,

$$\begin{aligned}d(f(x,y), f(a,b)) &= \sqrt{\left(\frac{1}{2}x + \frac{1}{2} - (\frac{1}{2}a + \frac{1}{2})\right)^2 + \left(\frac{1}{2}y - \frac{1}{2}b\right)^2} \\ &= \frac{1}{2}\sqrt{(x-a)^2 + (y-b)^2} \\ &= \frac{1}{2}d((x,y),(a,b)),\end{aligned}$$

so f is a contraction with $r = \frac{1}{2}$. However, f does not have a fixed point in B. Notice that f is also a contraction mapping from $\mathbb{R}^2$ to itself. Since $\mathbb{R}^2$ is complete, it follows that f has a unique fixed point in $\mathbb{R}^2$. If we can check that this fixed point is not in B we will be done. But $f(1,0) = (\frac{1}{2}\cdot 1 + \frac{1}{2}, \frac{1}{2}\cdot 0) = (1,0)$, and $\|(1,0)\| = 1$, so $(1,0)$ is the fixed point and it is not in B.

(c) (5 points) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$. If a local linear approximation to f at $\vec{x}_0$ is defined, then f is differentiable at $\vec{x}_0$.
Answer: True. By the definition in the notes, a local linear approximation is a linear operator L such that $f(x) = f(x_0) + L(x - x_0) + o(x - x_0)$. Such a thing can only exist if f is differentiable. If f has partial derivatives, then one can define something that looks like a local linear transformation. However, this "tangent plane" will only be a true local linear approximation if it is an accurate approximation to f for any x near x_0, which would imply that f is differentiable and $L = Df$.

(2) (10 points) Consider the space $l^p = \{(x_n) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\}$ with the norm $\|(x_n)\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$. For the purposes of this problem, you may assume that l^p is a vector space and $\|\cdot\|_p$ is a norm on l^p. Prove that l^p is complete with respect to the norm topology.

Answer: Suppose that (x^k) is a Cauchy sequence of elements of l^p. We must first find a candidate limit $x \in l^p$ and then prove that $x^k \to x$ in the l^p norm.

As proved in the first part of the semester, since (x^k) is l^p-Cauchy, it is l^p-bounded. That is, $\exists M > 0$ such that $\forall k \in \mathbb{N}$, $\|x^k\|_p < M$.

Let $\epsilon > 0$. Since (x^k) is Cauchy, $\exists K \in \mathbb{N}$ such that $\forall j, k > K$, $\|x^k - x^j\|_p < \epsilon$. Recall that each element of this sequence, x^k, is itself a sequence, denoted $(x_n)^k$. Let $j, k > K$ and choose $n \in \mathbb{N}$. Note that

$$|x_n^k - x_n^j| = (|x_n^k - x_n^j|^p)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n^k - x_n^j|^p\right)^{\frac{1}{p}} < \epsilon.$$

Therefore, for each fixed n, the sequence of real numbers (x_n^k) is also Cauchy. Thus, by the completeness of $\mathbb{R}$, there exists a real number x_n so that $(x_n^k) \to x_n$. Define $x := (x_n)$.

We must show that $x \in l^p$. That is, $\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{2}} < +\infty$. Choose $N \in \mathbb{N}$ and consider $\left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}}$. For $n = 1 \ldots N$, $\exists K_n \in \mathbb{N}$ such that $\forall k > K_n$, $|x_n^k - x_n| < \frac{\epsilon}{\sqrt[p]{N}}$ by the term-wise convergence proved in the previous paragraph. Let $K = \max\{K_n : n = 1 \ldots N\}$. Let $k > K$. By the triangle inequality,

$$\left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{N} |x_n - x_n^k|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |x_n^k|^p\right)^{\frac{1}{p}}.$$

Since $k > K_n$ for each $n \le N$, we then have

$$\begin{aligned}\left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} &< \left(\sum_{n=1}^{N} \frac{\epsilon^p}{N}\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |x_n^k|^p\right)^{\frac{1}{p}} \\ &= \epsilon + \left(\sum_{n=1}^{N} |x_n^k|^p\right)^{\frac{1}{p}} \\ &\le \epsilon + \left(\sum_{n=1}^{\infty} |x_n^k|^p\right)^{\frac{1}{p}} \\ &\le \epsilon + M\end{aligned}$$

where M is the bound on $\|x^k\|_p$ found above. If we now let $\epsilon \to 0$, we find that

$$\left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} \le M.$$

Finally, letting N go to infinity, we may conclude that $\|x\|_p \le M$. Therefore $x \in l^p$ as claimed.

Finally, we must show that $(x^k) \to x$ in the l^p topology. Let $\epsilon > 0$. As above, since (x^k) is Cauchy, $\exists K \in \mathbb{N}$ such that $\forall j, k > K$, $\|x^k - x^j\|_p < \frac{\epsilon}{3}$. Let $k > K$. Choose $N \in \mathbb{N}$ and consider $\left(\sum_{n=1}^{N} |x_n^k - x_n|^p\right)^{\frac{1}{p}}$. For $n = 1 \dots N$, $\exists K_n \in \mathbb{N}$ such that $\forall k > K_n$, $|x_n^k - x_n| < \frac{\epsilon}{3\sqrt[p]{N}}$ as above. Let $J = \max\{K_n : n = 1 \dots N\}$. Let $j > \max\{J, K\}$. By the triangle inequality,

$$\left(\sum_{n=1}^{N} |x_n^k - x_n|^p\right)^{\frac{1}{p}} \le \left(\sum_{n=1}^{N} |x_n^k - x_n^j|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |x_n^j - x_n|^p\right)^{\frac{1}{p}}.$$

Since $j > K_n$ for each $n \le N$, and both j and k are greater than K, we then have

$$\begin{aligned}\left(\sum_{n=1}^{N} |x_n^k - x_n|^p\right)^{\frac{1}{p}} &< \left(\sum_{n=1}^{\infty} |x_n^k - x_n^j|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} \frac{\epsilon^p}{3^p N}\right)^{\frac{1}{p}} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3},\end{aligned}$$

Finally, letting N go to infinity, we may conclude that $\|x^k - x\|_p \le \frac{2\epsilon}{3} < \epsilon$. Therefore $x^k \to x$ as claimed.

(3) (10 points) Prove that the collection of polynomials is dense in the space $L^2([0,1])$ with respect to the topology generated by the L^2-norm.

Answer: First, note that any polynomial is continuous, so therefore each polynomial can be considered as an element of $L^2([0,1])$ by the natural imbedding i. Let $\mathcal{P}$ represent the set of polynomials on $[0,1]$. Let $[(f_n)]$ be any element of $L^2([0,1])$, and let $\epsilon > 0$. We must show that $\exists p \in \mathcal{P}$ such that $\tilde{d}([(f_n)], p) < \epsilon$. Note that, by definition, $\tilde{d}([(f_n)], p) = \lim_{n\to\infty} d(f_n, p) = \lim_{n\to\infty} \sqrt{\int_0^1 |f_n(x) - p(x)|^2 dx}$.

Recall that (f_n) is a sequence of continuous functions which is Cauchy with respect to the L^2-norm. Choose N so that, $\forall n, m > N$, $d(f_n, f_m) < \frac{\epsilon}{2}$. Consider f_{N+1}. By the Weierstrass Approximation Theorem, there exists a polynomial p so that $\sup_{x\in[0,1]}(|f_{N+1}(x) - p(x)|) < \frac{\epsilon}{2}$. Then

$$d(f_{N+1}, p) = \sqrt{\int_0^1 |f_{N+1}(x) - p(x)|^2 dx} < \sqrt{\int_0^1 \left(\frac{\epsilon}{2}\right)^2 dx} = \sqrt{\left(\frac{\epsilon}{2}\right)^2 \cdot 1)} = \frac{\epsilon}{2}.$$

Therefore, for all $n > N$, $d(f_n, p) \leq d(f_n, f_{N+1}) + d(f_{N+1}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ by the triangle inequality. Hence, $\lim_{n\to\infty} d(f_n, p) \leq \epsilon$ and $\mathcal{P}$ is dense in $L^2([0, 1])$ as claimed.

(4) Consider a first-order linear differential equation

$$\frac{dy}{dt} = p(t)y(t),$$

where p is a given continuous function. We showed in class that y is a solution of this equation and the initial condition $y(0) = 0$ if and only if y is a fixed point of the operator $F : C([0, 1]) \to C([0, 1])$ given by $F(y)(t) = \int_0^t p(s)y(s)ds$.

(a) (8 points) Prove that F is a bounded linear operator from $C([0, 1])$ to itself and compute its operator norm.

Answer: To begin with, note that if $p \equiv 0$ then F is the zero operator, which is clearly a bounded linear operator from $C([0, 1])$ to itself with operator norm 0. Therefore we will assume throughout that p is not identically zero, and thus $\|p\|_\infty > 0$.

First we must show that F is an operator from $C([0, 1])$ to itself. That is, $F(f)$ is continuous on $[0, 1]$ if f is. Clearly if pf is identically zero then $Ff \equiv 0$, so we may assume that $\|pf\|_\infty > 0$. Let $\epsilon > 0$. Choose $\delta > 0$ so that $\delta < \frac{\epsilon}{\|pf\|_\infty}$ and let $x, y \in [0, 1]$ be chosen so that $|x - y| < \delta$. Without loss of generality suppose that $x \geq y$. Then consider $F(f)(x) - F(f)(y)$:

$$\begin{aligned}
|F(f)(x) - F(f)(y)| &= \left|\int_0^x p(s)f(s)ds - \int_0^y p(s)f(s)ds\right| \\
&= \left|\int_y^x p(s)f(s)ds\right| \\
&\leq \int_y^x |p(s)||f(s)|ds \\
&\leq \|pf\|_\infty |x - y| \\
&< \|pf\|_\infty \frac{\epsilon}{\|pf\|_\infty} \\
&= \epsilon.
\end{aligned}$$

Hence $F(f)$ is uniformly continuous if f is bounded.

Next, we need to check that F is linear. Let $f, g \in C([0,1])$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned} F(af+bg)(t) &= \int_0^t p(s)[af(s)+bg(s)]ds \\ &= a\int_0^t p(s)f(s)ds + b\int_0^t p(s)g(s)ds \\ &= aF(f)+bF(g) \end{aligned}$$

by the linearity properties of the integral, so F is indeed a linear operator.
Finally, we need to show that F is bounded and compute its operator norm. Let $f \in C([0,1])$. Then $F(f)[t] = \int_0^t p(s)f(s)ds \le \|f\|_\infty \int_0^t |p(s)|ds$. Since p is continuous, $\int_0^1 |p(s)|ds$ is finite. Therefore,

$$\|F(f)\|_\infty \le \sup_{t\in[0,1]} \left(\|f\|_\infty \int_0^t |p(s)|ds\right) \le \left(\int_0^1 |p(s)|ds\right)\|f\|_\infty,$$

and F is a bounded operator from $C([0,1])$ to itself. I claim that $\|F\|_{oper} = \int_0^1 |p(s)|ds$. The calculation just completed shows that $\|F\|_{oper} \le \int_0^1 |p(s)|ds$. To see the opposite inequality, let $f(t) = \frac{|p(t)|}{p(t)}$ for t such that $p(t) \neq 0$, and $f(t) = 0$ if $p(t) = 0$. The function f, which is called $sgn(p)$, is 1 where $p > 0$, 0 where $p = 0$, and -1 where $p < 0$. It may not be continuous, but we proved in class that since it has at most countably many jump discontinuities, for any $\epsilon > 0$, it is possible to come up with a function g which is continuous so that $\int_0^1 |f(t) - g(t)|dt < \frac{\epsilon}{\|p\|_\infty}$. This g will equal f whenever p is either positive or negative but will close the gap across the jumps with straight line segment. Therefore, unless $p \equiv 0$, $\|g\|_\infty = 1$. Consider such a g. Then

$$\begin{aligned} F(g)[1] &= \int_0^1 p(s)g(s)ds \\ &= \int_0^1 p(s)f(s)ds + \int_0^1 p(s)(g(s)-f(s))ds \\ &\ge \int_0^1 p(s)f(s)ds - \int_0^1 |p(s)||g(s)-f(s)|ds \\ &\ge \int_0^1 p(s)f(s)ds - \|p(s)\|_\infty \int_0^1 |g(s)-f(s)|ds \\ &> \int_0^1 p(s)f(s)ds - \|p(s)\|_\infty \frac{\epsilon}{\|p\|_\infty} \\ &= \int_0^1 p(s)f(s)ds - \epsilon \\ &= \int_0^1 p(s)\frac{|p(s)|}{p(s)}ds - \epsilon \\ &= \int_0^1 |p(s)|ds - \epsilon \end{aligned}$$

Therefore, $\int_0^1 |p(s)|ds - \epsilon \leq \|F(g)\|_\infty$. Since $\|g\|_\infty = 1$, we may conclude that $\|F\|_{oper} = \sup\{\|Fg\|_\infty : \|g\|_\infty = 1\} \geq \int_0^1 |p(s)|ds - \epsilon$. Hence $\int_0^1 |p(s)|ds - \epsilon \leq \|F\|_{oper} \leq \int_0^1 |p(s)|ds$. Letting $\epsilon \to 0$, we may conclude that $\|F\|_{oper} = \int_0^1 |p(s)|ds$ as claimed.

(b) (5 points) Prove directly, using the $\epsilon - \delta$ definition of the limit, that F is a continuous operator from $C([0,1])$ to itself.

Answer: As above, note that if $p \equiv 0$, then $F(f) \equiv 0$ for any $f \in C([0,1])$, so, for any $\epsilon > 0$ and any $f, g \in C([0,1])$, $\|F(f) - F(g)\|_\infty = \|0 - 0\|_\infty = 0 < \epsilon$, so in this case F is trivially continuous. Therefore we may assume that p is not identically zero and hence that $\int_0^1 |p(s)|ds > 0$. Define $M = \int_0^1 |p(s)|ds$.
Let $\epsilon > 0$. Let $f \in C([0,1])$ and let $\delta = \frac{\epsilon}{M}$. Suppose that $g \in C([0,1])$ with $\|f - g\|_\infty < \delta$. Then

$$\begin{aligned}
\|F(f) - F(g)\|_\infty &= \|F(f-g)\|_\infty \\
&= \sup_{0 \leq t \leq 1} \left| \int_0^t p(s)[f(s) - g(s)]ds \right| \\
&\leq \sup_{0 \leq t \leq 1} \int_0^t |p(s)||f(s) - g(s)|ds \\
&\leq \int_0^1 |p(s)||f(s) - g(s)|ds \\
&\leq \|f - g\|_\infty \int_0^1 |p(s)|ds \\
&< \frac{\epsilon}{M} M \\
&= \epsilon
\end{aligned}$$

Hence $\|Ff - Fg\|_\infty < \epsilon$ as desired, so F is indeed continuous.

(c) (2 points) Explain in complete sentences why the result of part (b) is not surprising.

Answer: First, we proved in part (a) that F is a bounded linear operator. In class we proved that all bounded linear operators are continuous. Second, in class we proved that, at least for small times, F is a contraction mapping. In class we proved that all contraction mappings are continuous. (If we apply that argument to larger times we will see that F is always Lipschitz.) Therefore we have several reasons to expect that F is continuous without referring directly to the $\epsilon - \delta$ definition.

(5) (10 points) The following is a true statement:

Theorem 1. *Suppose that $[(f_n)] \in W^{1,2}([0,1])$. Then $[(f_n)]$ has a representative f which is continuous. (That is, $\exists f : [0,1] \to \mathbb{R}$ such that $\lim_{n\to\infty} \|f_n - f\|_{W^{1,2}} = 0$ and f is continuous.)*

Corrected Proof; Corrections in Red. Let $[(f_n)] \in W^{1,2}([0,1])$, and choose a representative sequence (f_n) of $[(f_n)]$. Then (f_n) is Cauchy with respect to the $W^{1,2}$-norm, so $\|f_n\|_{W^{1,2}}$ is bounded by some constant M for all n. Also, by definition of $W^{1,2}$,

$f_n \in C^1([0,1])$ for each n. Fix n. Let $x, y \in [0,1]$. Without loss of generality, assume $x \geq y$. Then consider $|f_n(x) - f_n(y)|$:

$$\begin{aligned}
|f_n(x) - f_n(y)| &= |\int_y^x f_n'(t)dt| \\
&\leq \int_y^x |1 \cdot f_n'(t)|dt \\
&\leq \sqrt{\int_y^x |1|^2 dt \int_y^x |f_n'(t)|^2 dt} \\
&\leq \sqrt{|x-y|}\|f_n'\|_{L^2} \\
&\leq M\sqrt{|x-y|}
\end{aligned}$$

using Hölder's inequality. Now, if $\epsilon > 0$, let $\delta = (\frac{\epsilon}{M})^2$, and let $x, y \in [0,1]$ with $|x-y| < \delta$. Then by the calculations above, regardless of n,

$$|f_n(x) - f_n(y)| \leq M\sqrt{|x-y|} < M\sqrt{\delta} = M\sqrt{(\frac{\epsilon}{M})^2} = M\frac{\epsilon}{M} = \epsilon,$$

so the functions (f_n) are equicontinuous. Next, let $x \in [0,1]$ and note that

$$|f_n(x)| \leq |f_n(x) - f_n(y)| + |f_n(y)|$$

for any $y \in [0,1]$ by the triangle inequality. So,

$$\begin{aligned}
|f_n(x)| &= \int_0^1 |f_n(x)|dy \\
&\leq \int_0^1 |f_n(x) - f_n(y)|dy + \int_0^1 |f_n(y)|dy \\
&\leq \int_0^1 M\sqrt{|x-y|}dy + \sqrt{\int_0^1 |f_n(y)|^2 dy} \\
&\leq M + M \\
&= 2M
\end{aligned}$$

by Hölder's inequality again. This inequality tells us that the functions (f_n) are uniformly bounded, since this bound does not depend on n. Hence, by the Arzela-Ascoli theorem, there exists a subsequence (f_{n_k}) of (f_n) which converges uniformly to some function f. Then since f_{n_k} is continuous for all k, f is continuous. Since (f_n) is Cauchy and (f_{n_k}) is a subsequence of (f_n), $\lim_{k\to\infty} d(f_k, f_{n_k}) = 0$, so $[(f_n)] = [(f_{n_k})]$ and hence f is a valid, continuous representative of the given equivalence class. $\square$