HW 8 Solutions

1. Let $X_1, \dots, X_n \sim Exp(\beta)$.

(a.) If $\hat{\beta}_1 = X_1$, $E(\hat{\beta}_1) = E(X_1) = \frac{1}{\beta}$

$$var(\hat{\beta}_1) = \frac{1}{\beta^2}$$

$$\Rightarrow bias(\hat{\beta}_1) = \frac{1}{\beta} - \frac{1}{\beta} = 0$$

$$\boxed{MSE(\hat{\beta}_1) = \frac{1}{\beta^2}.}$$

(b.) If $\hat{\beta}_2 = \frac{1}{n} \sum X_i$

$$E(\hat{\beta}_2) = \frac{1}{n} \cdot n \frac{1}{\beta} = \frac{1}{\beta} \quad (bias = 0).$$

$$V(\hat{\beta}_2) = \frac{1}{n^2} \cdot n \cdot \frac{1}{\beta^2} = \frac{1}{n \cdot \beta^2}$$

$$\Rightarrow \boxed{MSE(\hat{\beta}_2) = \frac{1}{n\beta^2}}$$

(c.) Let $\hat{\beta}_3 = n \cdot \min X_i$.

$$P(\hat{\beta}_3 > y) = P(n \cdot \min X_i > y)$$

$$= P(\min X_i > \frac{y}{n})$$

$$= P\left(X_i > \frac{y}{n}\right)^n$$

$$= \left(e^{-\beta \frac{y}{n}}\right)^n$$

$$= e^{-\beta \cdot y} \Rightarrow \hat{\beta}_3 \sim Exp(\beta).$$

$$E(\hat{\beta}_3) = \frac{1}{\beta} \quad \text{unbiased}$$

$$V(\hat{\beta}_3) = \frac{1}{\beta^2}, \quad \boxed{MSE(\hat{\beta}_3) = \frac{1}{\beta^2}.}$$

$$\boxed{\tilde{\beta}_2 \text{ is best.}}$$

p. 513 #8 $\quad f(x) = p(1-p)^x \quad x = 0, 1, 2, \ldots$

$$= (1-t) \cdot t^x \quad x = 0, 1, 2, \ldots$$

$$E[\delta(x)] = \sum_{x=0}^{\infty} \delta(x)(1-t) \cdot t^x$$

$$= (1-t) \underbrace{\sum_{x=0}^{\infty} \delta(x) \cdot t^x}_{\text{must } = 1 \text{ for all } t.}$$

$$\sum_{x=0}^{\infty} \delta(x) t^x = \delta(0) + \delta(1) \cdot t + \delta(2) t^2 + \ldots$$

$= 1$ iff $\delta(0) = 1$

$\delta(x) = 0$ for $x > 0$.

p. 513 # 11

$X_1, \ldots, X_m \sim$ Type A $(\theta, 4\sigma^2)$

$Y_1, \ldots, Y_n \sim$ Type B (θ, σ^2)

Let $\tilde{\theta} = \alpha \overline{X}_m + (1-\alpha) \overline{Y}_n$

(a.) $E(\tilde{\theta}) = \alpha E(\overline{X}) + (1-\alpha) E(\overline{Y})$

$= \alpha \cdot \theta + (1-\alpha)\theta$

$= \boxed{\theta}$ (for all n, m, α).

(b.) $\operatorname{var}(\tilde{\theta}) = \alpha^2 \operatorname{var}(\overline{X}_m) + (1-\alpha)^2 \operatorname{var}(\overline{Y})$

$= \alpha^2 \cdot 4 \dfrac{\sigma^2}{m} + (1-\alpha)^2 \dfrac{\sigma^2}{n}$

$\dfrac{d}{d\alpha} \operatorname{var}(\tilde{\theta}) = 8 \cdot \alpha \cdot \dfrac{\sigma^2}{m} + 2(1-\alpha)(-1)\dfrac{\sigma^2}{n} = 0$

$\Rightarrow \dfrac{4\alpha}{m} + \dfrac{-2(1-\alpha)}{n} = 0$

$\Rightarrow \boxed{\alpha = \dfrac{m}{m+4n}}$

p. 527 #2

Let $X \sim Geo(p)$

$\Rightarrow f(x|p) = p(1-p)^x$

$\lambda(x|p) = \log p + x\log(1-p)$

$$\lambda'(x|p) = \frac{1}{p} + \frac{-x}{1-p} = \frac{1}{p} - \frac{x}{1-p}$$

$$\lambda''(x|p) = \frac{-1}{p^2} - \frac{x}{(1-p)^2}$$

$$-E[\lambda''(x|p)] = \frac{1}{p^2} + \frac{(\cancel{1-p})}{p(1-p)^{\cancel{2}}}$$

$$= \frac{(1-p)}{p^2(1-p)} + \frac{p}{p^2(1-p)}$$

$$= \boxed{\frac{1}{p^2(1-p)}} = I(p)$$

p. 527 #4 $\quad X \sim N(0, \sigma^2)$

$$f(x|\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

$$\lambda(x|\sigma) = -\log(\sqrt{2\pi}) - \log\sigma - \frac{x^2}{2\sigma^2}$$

$$\lambda'(x|\sigma) = \frac{-1}{\sigma} + \frac{x^2}{\sigma^3}$$

$$\lambda''(x|\sigma) = \frac{1}{\sigma^2} - \frac{3x^2}{\sigma^4}$$

$$I(\sigma) = -E(\lambda''(x|\sigma)) = \frac{-1}{\sigma^2} + \frac{3}{\sigma^4} E(x^2)$$

$$= \frac{-1}{\sigma^2} + \frac{3}{\sigma^4}\left(\operatorname{var}(x) + Ex^2\right)$$

$$= \frac{-1}{\sigma^2} + \frac{3}{\sigma^2} = \boxed{\frac{2}{\sigma^2}}$$