Project One

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Part I

Theoretical Part

Problem 3.1

Proof:

Obviously, since W is a matrix, then Wx is also a vector, so $\forall\, vector\, x$, $||x||_W = ||Wx|| \geq 0$, and if $x = 0$, $||x||_W = ||Wx|| = 0$. And if $||x||_W = ||Wx|| = 0$, then $Wx = 0$ by the definition of $||\centerdot||$, because W is arbitrary nonsingular matrix, which means all of row vectors are linear indenpendent, hence $x = 0$. So $||x||_W = 0$ if only and if $x = 0$;

Secondly, $\forall\, vector\, x\; and\, y$, $||x + y||_W = ||W(x + y)|| = ||Wx + Wy|| \leq ||Wx|| + ||Wy|| = ||x||_W + ||y||_W$;

Thirdly, for any scalar $\alpha \in \mathbb{C}$, $||\alpha x||_W = ||W(\alpha x)|| = ||\alpha Wx|| = \alpha||Wx|| = \alpha||x||_W$. Thus, $||\centerdot||_W$ is vector norm.

Problem 3.2

Proof:

Let λ denote arbitrary eigenvalue of A and x is the corresponding eigenvector, then we have

$$Ax = \lambda x$$

get norm of both sides, so

$$||Ax|| = ||\lambda x|| = |\lambda|||x||$$

By the definition of induced matrix norm

$$||A|| = sup_{x \neq 0} \frac{||Ax||}{||x||}$$

thus

$$|\lambda|\, ||x|| = ||Ax|| \leq ||A||\, ||x||$$

$\Rightarrow |\lambda| \leq ||A||$. Because λ is arbitrary eigenvalue of A, so $\rho(x) \leq ||A||$.

Problem 3.6

Proof:

(a).

Obviously, for any vector x, $||x||' = sup_{||y||=1}|y^*x| \geq 0$, and if $x = 0$, $||x||' = sup_{||y||=1}|y^*0| = 0$. And when $||x||' = sup_{||y||=1}|y^*x| = 0$, then

$$0 = sup_{||y||=1}|y^*x| = sup_{||y||=1}||x||\,||y||\,|\cos(\theta)|$$

where $\cos(\theta) = \frac{y^*x}{||x||\,||y||}$, so $0 = sup_{||y||=1}||x||\,||y||\,|\cos(\theta)| = ||x||$, (because $||y|| = 1$and $sup\cos(\theta) = 1$), so $x = 0$. Thus $||x||' = 0$ if and only if $x = 0$;

Secondly, for any two vector x and z, $||x+z||' = sup_{||y||=1}|y^*(x+z)| = sup_{||y||=1}|y^*x + y^*z| \leq sup_{||y||=1}|y^*x| + sup_{||y||=1}|y^*z| = ||x||' + ||z||'$;

Thirdly, for any scalar $\alpha \in \mathbb{C}$, $||\alpha x||' = sup_{||y||=1}|y^*(\alpha x)| = |\alpha|\, sup_{||y||=1}|y^*x| = |\alpha|\,||x||'$.

Hence, $||\centerdot||'$ is a norm.

(b).

Since x, y is give with $||x|| = ||y|| = 1$, then to show there exits a rank one $B = yz^*$ such that $Bx = y$ and $||B|| = 1$is to show there is such a vector z^* satisfies

Since $Bx = y \iff Bx - y = 0$, then $||Bx - y|| = 0 = ||yz^*x - y|| = ||y(z^*x - 1)||$ sinc e z^*x is scalar here, and $||y|| = 1$ by given, so $|z^*x - 1| = 0$.

By the lemma, for any x, there exits a nonzero z s.t. $|z^*x| = ||z||'||x||$, then for the given x, we have $\frac{|z^*x|}{||z||'} - 1 = 0$, to get rid of the absolute value sign, we let

$$z_o = sign(z^*x)\frac{z}{||z||'}$$

then this z_o statisfies $Bx = y$, and it is easy verify that $||B|| = \underset{||x||=1}{sup}||Bx|| = \underset{||x||=1}{sup}||y|| = 1$. Besides, since both y, $z_o \in \mathbb{C}^m$, then $B = yz_o^*$ is always a rank-one matrix, because all the row vectors of the matrix B are linear dependent, i.e. for any $i \neq j$, $1 \leq i, j \leq m$

$$\frac{\overrightarrow{b_i}}{\overrightarrow{b_j}} = \frac{y_i}{y_i}$$

where $\overrightarrow{b_i}, \overrightarrow{b_i}$ are two distinct row vectors of B and y_i, y_j are the corresponding entries of y.

Problem 5.3

Solotion:

(a).

Since $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$, then we denot $M \triangleq A^*A$, then

$$M=\begin{bmatrix}-2 & -10\\ 11 & 5\end{bmatrix}\begin{bmatrix}-2 & 11\\ -10 & 5\end{bmatrix}=\begin{bmatrix}104 & -72\\ -72 & 146\end{bmatrix}$$

it is easy to compute the eigenvalues λ_1, λ_2 of M by $|M-\lambda I|=0 \Rightarrow (\lambda_1-200)(\lambda_2-50)=0$, thus $\lambda_1=200>\lambda_2=50$. So the singular values in $\Sigma=\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix}$ are $\sigma_{1=}\sqrt{200}=14.1421$ and $\sigma_2=\sqrt{50}=7.0711$.

Now we compute V.

For M, find the corresponding eigenvectors of its eigenvalues λ_1, λ_2, we denote them as $X=[x_1,\, x_2]^T$ and $Y=[y_1,\, y_2]^T$:

$$(M-\lambda_1 I)\,X=0\Rightarrow\begin{bmatrix}104-200 & -72\\ -72 & 146-200\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0\Rightarrow x_1=-\frac{3}{4}x_2$$

Since $x_1^2+x_2^2=1$, then $x_1=0.6$, $x_2=-0.8$. Similarly, we can get $Y=\begin{bmatrix}0.8 & 0.6\end{bmatrix}^T$.(Under the request that V has the minimal number of signs)

Thus

$$V=\begin{bmatrix}X & Y\end{bmatrix}=\begin{bmatrix}0.6 & 0.8\\ -0.8 & 0.6\end{bmatrix}$$

Then we compute U by $U=A(V^*)^{-1}\Sigma^{-1}=AV\Sigma^{-1}=\begin{bmatrix}-2 & 11\\ -10 & 5\end{bmatrix}\begin{bmatrix}0.6 & 0.8\\ -0.8 & 0.6\end{bmatrix}\begin{bmatrix}14.1421 & 0\\ 0 & 7.0711\end{bmatrix}^{-1}$

thus $U=\begin{bmatrix}-0.7071 & 0.7071\\ -0.7071 & -0.7071\end{bmatrix}$

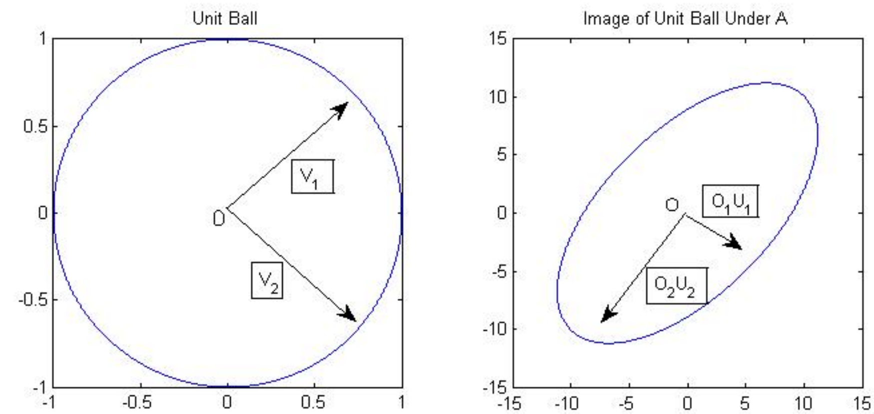
(b).

Please m-file: **5.3 .m**

The nonsingular values: $\Sigma=\begin{bmatrix}14.1421 & 0\\ 0 & 7.0711\end{bmatrix}$;

The left nonsingular vectors u_1, u_2 are: $U=[u_1|u_2]=\begin{bmatrix}-0.7071 & 0.7071\\ -0.7071 & -0.7071\end{bmatrix}$;

The right nonsingular vectors v_1, v_2 are: $V=[v_1|v_2]=\begin{bmatrix}0.6 & 0.8\\ -0.8 & 0.6\end{bmatrix}$.



(c).
$||A||_1 = 16$
$||A||_2 = \sqrt{\rho(A^T A)} = \sqrt{200} = 14.1421$
$||A||_\infty = 15$
(d).
Since $A = U\Sigma V^*$, then $(A)^{-1} = (U\Sigma V^*)^{-1} = (V^*)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^*$, then

$$A = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 1/\sqrt{200} & 0 \\ 0 & 1/\sqrt{50} \end{bmatrix} \begin{bmatrix} -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{bmatrix} = \begin{bmatrix} 0.05 & -0.11 \\ 0.1 & -0.02 \end{bmatrix}$$

(e).
Denote λ be the eigenvalues of matrix A, then let $|A - \lambda I| = 0 \Rightarrow \left| \begin{matrix} -2-\lambda & 11 \\ -10 & 5-\lambda \end{matrix} \right| = 0 \Rightarrow (\lambda - 5)(\lambda + 2) + 100 = 0$, then solve the equation, the roots are the eigenvalues of A : $\lambda_{1,2} = \frac{3 \pm \sqrt{-391}}{2} = 1.5 \pm 9.8869i$
(f).
Eigenvalues of A are $\lambda_{1,2} = \frac{3 \pm \sqrt{-391}}{2} = 1.5 \pm 9.8869i$, so

$$\lambda\lambda_2 = \frac{3 + \sqrt{-391}}{2} \cdot \frac{3 - \sqrt{-391}}{2} = 100$$

while$det(A) = -2 \times 5 - (-10) \times 11 = 100$
thus

$$det(A) = \lambda_1 \lambda_2$$

And $|det(A)| = 100$, and $\sigma_1\sigma_2 = \sqrt{200}\sqrt{50} = 100$, so $|det(A)| = \sigma_1\sigma_2$.
(g).
The length of the long axis is $||\sigma_1 u_1||_2 = 10\sqrt{2}$ and the length of the short axis is $||\sigma_2 u_2||_2 = 5\sqrt{2}$, so the eare is $S = \pi ab = 100\pi$.

Problem 5.4

Solution:

Suppose the eigenvalues of the hermitian matrix $M = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}_{2m\times 2m}$ in block form are $\lambda_i = \begin{bmatrix} \lambda_{(i-1)m+1} & 0 & \cdots & 0 \\ 0 & \lambda_{(i-1)m+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{(i-1)m+m} \end{bmatrix}_{m\times m}$, $i = 1, 2$, and denote $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2m\times 2m}$ be the corresponding eigenmatrix of the eigenvalues matrix of $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where a, b, $c\,d$ are all $m \times m$ block matrixs, then the eigenvalue decomposition of M is $M = X\Lambda X^{-1}$. So our goal is to find such Λ and its corresponding X.

Since M is hermitian matrix, then by Theorem 5.5, the eigenvalues of M are real and the eigenmatrixs are unitary, which implies $X^{-1} = X^*$, so $M = X\Lambda X^*$. To find Λ, we compute the roots of $det(M - \Lambda I) = 0 = det(\begin{bmatrix} -\lambda & A^* \\ A & -\lambda \end{bmatrix}) \Rightarrow \lambda^2 = \Sigma^2$, so $\lambda_1 = \Sigma\, and\, \lambda_2 = -\Sigma$ since λ_1, λ_2 are both real.

Next, we try to find X. We compute MM separately.

Since U, V are unitary, so in one way, we have:

$$MM = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix} = \begin{bmatrix} V\Sigma^2V^* & 0 \\ 0 & U\Sigma^2U* \end{bmatrix}$$

In the other way, since X is also unitary, so we have:

$$MM = (X\Lambda X^*)(X\Lambda X^*) = X\Lambda^2X^*$$

thus

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}\begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} a\Sigma^2a^* + b\Sigma^2b^* & a\Sigma^2c^* + b\Sigma^2d^* \\ c\Sigma^2a^* + d\Sigma^2b^* & c\Sigma^2c^* + d\Sigma^2d^* \end{bmatrix}$$

so

$$\begin{bmatrix} V\Sigma^2V^* & 0 \\ 0 & U\Sigma^2U* \end{bmatrix} = \begin{bmatrix} a\Sigma^2a^* + b\Sigma^2b^* & a\Sigma^2c^* + b\Sigma^2d^* \\ c\Sigma^2a^* + d\Sigma^2b^* & c\Sigma^2c^* + d\Sigma^2d^* \end{bmatrix}$$

$\Rightarrow a\Sigma^2 a^* + b\Sigma^2 b^* = V\Sigma^2 V^*$, $a\Sigma^2 c^* + b\Sigma^2 d^* = 0$, $c\Sigma^2 a^* + d\Sigma^2 b^* = 0$, $c\Sigma^2 c^* + d\Sigma^2 d^* = U\Sigma^2 U*$. Since X is also a unitary matrix, which means $\begin{bmatrix} a & c \end{bmatrix}^T$ and $\begin{bmatrix} b & d \end{bmatrix}^T$ are mutually orthogonal and their norms should be 1. So we find :

$$X = \begin{bmatrix} \frac{\sqrt{2}}{2}V & \frac{\sqrt{2}}{2}V \\ \frac{\sqrt{2}}{2}U & -\frac{\sqrt{2}}{2}U \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} V & V \\ U & -U \end{bmatrix}$$

Thus we get $M = X\Lambda X^{-1}$

Problem 6.3

Proof:

Since $A \in \mathbb{C}^{m\times n}$ is not a square matrix (suppose $m > n$), so we use the reduced singular value decomposition (SVD) of A, which is $A = \hat{U}\hat{\Sigma}V^*$ where $\hat{U} \in \mathbb{C}^{m\times n}$ and $V \in \mathbb{C}^{n\times n}$ are both unitary and $\hat{\Sigma} \in \mathbb{R}^{n\times n}$ is diagonal. Then

$$A^*A = (U\hat{\Sigma}V^*)^*(U\hat{\Sigma}V^*) = V\hat{\Sigma}^2 V^*$$

($\Leftarrow$) : If A^*A is nonsingular, then $det(A^*A) = det(V\hat{\Sigma}^2V^*) = det(Vdet(\hat{\Sigma}^2)det(V^*) = det(\hat{\Sigma}^2) \neq 0$, then all the n entries of $\hat{\Sigma}$ are nonzero, which implies all of the n singular values of$A$ are nonzero, then$A$ is full rank.

($\Rightarrow$) :Since A is full rank, then A has n nonzero singular values, then $det(A^*A) = det(\hat{\Sigma}^2) \neq 0$, so A^*A is nonsingular.

Problem 7.5

Proof:

(a).

($\Rightarrow$) : Since A has rank n, then A is rull rank, so by Theorem 7.2 we can easily conclude that the diagonal entries of $\hat{R}$ are nonzero (positive).

($\Leftarrow$) : From the Gram-Schmidt iteration, (7.8) and (7.6), we have

$$|r_{jj}| = \|a_j - \Sigma_{i=1}^{j-1} r_{ij} q_i\|$$

where

$$q_i = \frac{a_i - \Sigma_{k=1}^{i-1} r_{ki} q_k}{r_{ii}}$$

If all diagonal entries r_{jj} of $\hat{R}$ are nonzero, then for any $j = 1, 2, 3... n$, $\|a_j - \Sigma_{i=1}^{j-1} r_{ij} q_i\| \neq 0$, which implies that a_j cannot be written as the compositon of $q_1, q_2, ...q_{j-1}$, and because each q_i can be written as composition of $a_1, a_2, ...a_i$, let $i = j - 1$, then a_j cannot be written as the compositon of $a_1, a_2, ...a_{j-1}$, i.e. a_j is linear independent with $a_1, a_2, ...a_{j-1}$, thus all the n column vectors $a_1, a_2, ...a_n$of A are mutually linear independent. Thus A has full rank.

(b).

$\hat{R}$ has k nonzero diagonal entries, then the rank of A is exactly k.

This follows from the ($\Leftarrow$) of part (a). Suppose that only $r_{11}, r_{22}, ...r_{kk}$ are nonzero. Then we can find, similar to part (a), at least $a_1, a_2, ...a_k$ are mutually linear independent. Now we consider a_{k+1}:

Since$r_{k+1,k+1} = 0$,and $|r_{k+1,k+1}| = \|a_{k+1} - \Sigma_{i=1}^{k} r_{i,k+1} q_i\| = 0$, so

$$a_{k+1} = \Sigma_{i=1}^{k} r_{i,k+1} q_i$$

where $q_i = \frac{a_i - \Sigma_{k=1}^{i-1} r_{ki} q_k}{r_{ii}}$, since every q_i can be written as composition of $a_1, a_2, ...a_i$, so let $i = k$, then a_{k+1} can be written as the composition of $a_1, a_2, ...a_k$, thus

$$a_i \in < a_1, a_2, ...a_k > \; for \, k+1 \leq i \leq n$$

Thus A has exactly rank k.

2.

Proof:

Since $A \in \mathbb{C}^{m \times n}$ and $b \in \mathbb{C}^m$, so $b \notin range(A)$, thus this rectangular system of equations is overdetermined. Denote $r = b - Ax$ called the residual, then by Theorem 11.1 a vector x minimizes the residual norm $||r||_2 = ||b - Ax||_2$ if and only if $r \perp range(A)$, that is $A^*r = 0$. Thus $A^*r = 0 \Leftrightarrow A^*Ax = A^*b$.

Let's compute x now. In proof of **problem 6.3** (see above), we have already proved that if A has full rank, then A^*A is nonsingular, thus A^*A 's inverse exits. By the given reduced SVD of $A = \hat{U}\hat{\Sigma}\hat{V}^*$, then we find x:

$$x = (A^*A)^{-1}A^*b = (\hat{V}\Sigma\hat{U}^*\hat{U}\hat{\Sigma}\hat{V}^*)^{-1}(\hat{U}\hat{\Sigma}\hat{V}^*)^*b = \hat{V}\Sigma^{-2}\hat{V}^*\hat{V}\Sigma\hat{U}^*b = \hat{V}\Sigma^{-1}\hat{U}^*b$$

thus

Part II
Numerical Experiments

1.

Please see the m-file: **Q1 of coding part.m**

Just copy the whole code and run it, we will get as follows:

*****to verify the unitary of Qs from clgs and msg *****

–All of column vectors are unit in Qs from clgs

****

–All of column vectors are unit in Qs from mgs

****

*****to verify the orthogonality of Qs from clgs and msg *****

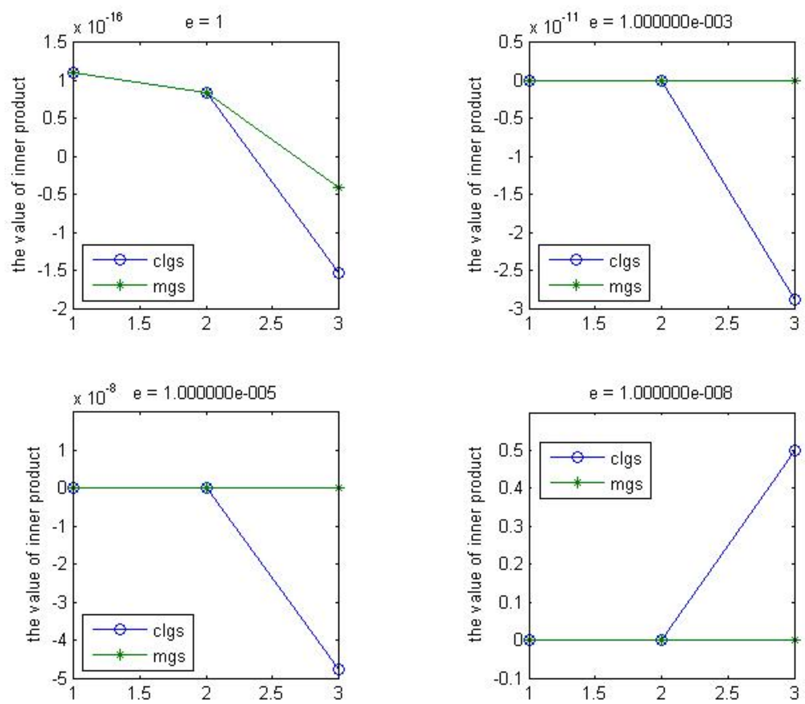
1.Qs from clgs are not orthogonal

it is not orthogonal when e = 1.000000e-008

the inner product of the two nonorthogonal vector is 5.000000e-001, which is far away from 0

2.Qs from mgs are orthogonal

The observations are cited insides the gragh below:



Conclusion: Classical G-S algorithm is not stable when the matrix contains some tiny values, that's to say the matrix is ill conditioned. While Modified G-S algorithm is more stable.

Problem 10.2

Please see the m-files: **house.m** and **formQ.m**

Actually, the R we get from house.m is m-by-n matrix.

Problem 10.3

Please see the m-file: **10.3.m**

Differences:

1. The diagonal entries of $R1$ by **mgs** are all positive, but the diagonal entries of $R2$ and $R3$ exist some negative ones.

2. $R2$ has the same column and row numbers as Z because householder projections are just kind of row operations, they don't change the shape of the original matrix. While $R1$ and $R3$'s column and rom numbers are same with each other but not the same with Z.