Jeremy Rouse's Math 711 homework

Chapter 1

1.1.

(1a) Let $f(x) = \ln(x)$. Then, $f''(x) < 0$ for $x \in (0, \infty)$. For $0 < c < d$, this implies that the graph of $f(x)$ lies above the secant line through $(c, f(c))$ and $(d, f(d))$. This line can be parametrized by $(c(1-t) + dt, (1-t)f(c) + tf(d))$ for $0 \leq t \leq 1$. Hence,

$$(1-t)f(c) + tf(d) \leq f(c(1-t) + tf(d)).$$

We have

$$\ln(a) + \ln(b) = \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q).$$

Now, set $t = 1/q$, $1 - t = 1/p$, $c = a^p$ and $d = b^q$ and we get

$$\ln(a) + \ln(b) = \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) \leq \ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right).$$

Exponentiating gives $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$, as desired.

(1b) For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ define $\|\vec{x}\|_p = (\sum_i |x_i|^p)^{\frac{1}{p}}$. [You might want to define this in the problem statement, actually.] Suppose $x, y \in \mathbb{R}^n$ and $\|x\|_p = \|y\|_q = 1$. Then we have

$$\begin{aligned}\sum_{i=1}^{n} |x_i y_i| &\leq \sum_{i=1}^{n} \frac{1}{p}|x_i|^p + \frac{1}{q}|y_i|^q \\ &= \frac{1}{p}\|x\|_p^p + \frac{1}{q}\|y\|_q^q = \frac{1}{p} + \frac{1}{q} = 1,\end{aligned}$$

by Young's inequality. Now, suppose that $x, y \in \mathbb{R}^n$ are arbitrary. Note that if one of $\|x\|_p$ or $\|y\|_q$ is zero, then $x = 0$ or $y = 0$ and there is nothing to prove. Suppose therefore that $\|x\|_p, \|y\|_q > 0$. Then, $\left\|x/\|x\|_p\right\|_p = 1$ and $\left\|y/\|y\|_q\right\|_q = 1$ and so

$$\sum_{i=1}^{n} \left|\frac{x_i}{\|x\|_p} \cdot \frac{y_i}{\|y\|_q}\right| \leq 1.$$

Multiplying through by $\|x\|_p$ and $\|y\|_q$ gives

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q$$

as desired.

(1c) If $x, y \in \mathbb{R}^n$, then we have

$$\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^p &= \sum_{i=1}^{n} |x_i + y_i||x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^{n} (|x_i| + |y_i|)|x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1} \\
&\leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)}\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)}\right)^{\frac{1}{q}}
\end{aligned}$$

with $q = \frac{p}{p-1}$, by Hölder's inequality. If $\|x + y\|_p = 0$, there is nothing to prove. Otherwise note that $\frac{1}{p} + \frac{1}{q} = 1$ implies that $pq = p + q$ and so $q(p-1) = p$. This gives

$$\|x + y\|_p^p \leq (\|x\|_p + \|y\|_p)\,(\|x + y\|_p)^{\frac{p}{q}}.$$

Dividing by $\|x + y\|_p^{\frac{p}{q}}$ gives

$$\|x + y\|_p = \|x + y\|_p^{p - \frac{p}{q}} \leq \|x\|_p + \|y\|_p.$$

(2) Here's a plot of the circle for $p = 3$.

For $p = 1$, the "circle" will be a square with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. For $p \to \infty$, the unit circle will be asymptotic to the square with vertices $(\pm 1, \pm 1)$.

(3) Suppose $x \in \mathbb{R}^n$ and $x_i \geq x_j$ for all i, j. Then,

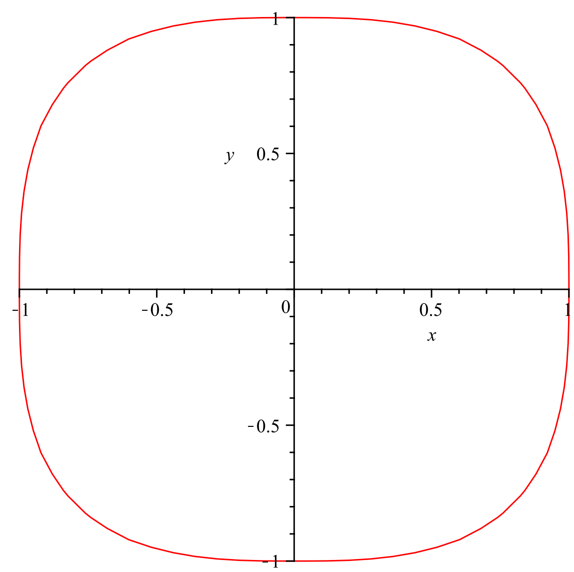
$$(|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p} = |x_i|(|x_1/x_i|^p + |x_2/x_i|^p + \cdots + 1 + \cdots |x_n/x_i|^p)^{1/p}.$$

As $p \to \infty$, the expression inside the parentheses is bounded below by 1 and bounded above by n. Therefore, $|x_i| \leq \|x\|_p \leq \|x_i\| n^{1/p}$ and $n^{1/p} \to 1$ as $p \to \infty$. Therefore, $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$, which explains the name.

(4) If two points are close in one metric, then they are also close in another. (See Exercise 1.8(1) for a proof that all the l^p norms are equivalent on $\mathbb{R}^n$).

1.2. I claim that if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and $x, y, z \in \mathbb{R}^n$, then

$$\sum_{i=1}^{n} |x_i y_i z_i| \leq \|x\|_p \|y\|_q \|z\|_r.$$



By applying Hölder's inequality, (with p and $\frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$), we get

$$\sum_{i=1}^{n} |x_i y_i z_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i z_i|^{p/(p-1)} \right)^{\frac{p-1}{p}} .$$

Applying it again, we have

$$\left(\sum_{i=1}^{n} |y_i^{p/(p-1)} z_i^{p/(p-1)}| \right)^{\frac{p-1}{p}} \leq \left(\sum_{i=1}^{n} |y_i|^{ps/(p-1)} \right)^{(p-1)/ps} \left(\sum_{i=1}^{n} |z_i|^{pt/(p-1)} \right)^{(p-1)/pt} .$$

Here, we take $s = \frac{q(p-1)}{p}$ and

$$t = \frac{1}{1-\frac{1}{s}} = \frac{1}{1-\frac{p}{q(p-1)}}.$$

Then,

$$\frac{pt}{p-1} = \frac{p}{(p-1)-p/q} = \frac{pq}{pq-q-p} = r.$$

This yields the claimed result.

1.3. Suppose that (x_n) is a Cauchy sequence in $\mathbb{R}^n$. Since the sequence is Cauchy, there is a number N so that $d(x_n, x_m) \leq 1$ if $m, n \geq N$. Then, for any $n \geq N$

$$d(0, x_n) \leq d(0, x_N) + d(x_N, x_n) \leq d(0, x_N) + 1.$$

Therefore, for any $n \in \mathbb{N}$, $d(0, x_n) \leq \max\{d(0, x_1), \ldots, d(0, x_{N-1}), d(0, x_N) + 1\}$. Hence, the sequence is bounded. Therefore it has a subsequence x_{n_1}, x_{n_2}, $\ldots$ that converges. Let $L = \lim_{k\to\infty} x_{n_k}$. I claim that (x_n) converges to L. Fix $\epsilon > 0$. Since (x_n) is Cauchy, there is some N_2 so that if $m, n \geq N_2$, then $d(x_m, x_n) < \epsilon/2$. Choose k large enough that $n_k \geq N_2$ and $d(x_{n_k}, L) < \epsilon/2$. Then, for $n \geq N_2$,

$$d(x_n, L) \leq d(x_n, x_{n_k}) + d(x_{n_k}, L) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence, x_n converges to L.

1.4. The empty subset of X is open, since there are no points in it (and so there isn't a point that *doesn't* contain an open ball around it). If $U = X$, then for any $u \in U$, we have $B_\epsilon(u) \subseteq U$ for all ϵ, and so U is open.

Suppose that $U_1, U_2, \ldots, U_n$ are open subsets of X and $u \in \bigcap_{i=1}^n U_i$. Then by definition for each i, there is an $\epsilon_i > 0$ so that $B_{\epsilon_i}(u) \subseteq U_i$. It follows that if we let $\epsilon = \min\{\epsilon_i : 1 \leq i \leq n\}$, then $B_\epsilon(u) \subseteq \bigcap_{i=1}^n U_i$ and so $\bigcap_{i=1}^n U_i$ is open.

Finally, if $\{U_\alpha\}_{\alpha\in A}\}$ is an arbitrary collection of open sets and $u \in \bigcup_{\alpha\in A} U_\alpha$ then $u \in U_\alpha$ for some α. Hence, there is an $\epsilon > 0$ so that $B_\epsilon(u) \subseteq U_\alpha \subseteq \bigcup_{\alpha\in A} U_\alpha$. Hence, $\bigcup_{\alpha\in A} U_\alpha$ is open.

This proves that the open sets in a metric space form a topology.

1.5. ($\epsilon - \delta$ continuity $\implies$ sequential continuity) Fix $\epsilon > 0$ and choose δ small enough so that if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \epsilon$. Suppose that $x_n \to x_0$ in X. Then there is an N so that if $x_n > N$, then $d_X(x_n, x_0) < \delta$. Therefore, $d_Y(f(x_n), f(x_0)) < \epsilon$. This proves that $f(x_n) \to f(x_0)$.

(sequential continuity $\implies$ $\epsilon - \delta$ continuity) We will prove the contrapositive. Suppose that f is not $\epsilon - \delta$ continuous at x_0. Then, there is some $\epsilon > 0$ so that for all $\delta > 0$ there is an x_n with $d_X(x_n, x_0) < \delta$ so that $d_Y(f(x_n), f(x_0)) > \epsilon$. Hence, for all $n \geq 1$, there is some x_n so that $d(x_n, x_0) < 1/2^n$ but $d(f(x_n), f(x_0)) > \epsilon$. Then $x_n \to x_0$ because if $\epsilon_0 > 0$ and N is chosen so that $\frac{1}{2^N} < \epsilon_0$, then for $n > N$, $d(x_n, x_0) < \frac{1}{2^n} < \epsilon_0$. However, $\lim_{n\to\infty} f(x_n)$ does not equal $f(x_0)$, because there is no N so that for all $n > N$ $d(f(x_n), f(x_0)) < \epsilon$. This shows that f is not sequentially continuous.

($\epsilon - \delta$ continuity $\implies$ topological space continuity) Suppose that U is an open subset of Y. Pick some $x \in f^{-1}(U)$. Then $f(x) \in U$ and since U is open, there exists an $\epsilon > 0$ so that $B_\epsilon(f(x)) \subseteq U$. Then, there is a $\delta > 0$ so that if $d(x, y) < \delta$, $d(f(y), f(x)) < \epsilon$ and so $B_\delta(x) \subseteq f^{-1}(U)$ and so $f^{-1}(U)$ is open.

(topological space continuity $\implies$ $\epsilon - \delta$ continuity) Fix $\epsilon > 0$ and a point $x_0 \in X$. Let $y = f(x_0)$. Let $U = B_\epsilon(y) \subseteq Y$. Since $f^{-1}(U)$ is open, there is a $\delta > 0$ so that $B_\delta(x) \subseteq f^{-1}(U)$. Then if $d(x, y) < \delta$, then $f(y) \in U$ and so $d(f(y), f(0)) < \epsilon$. This proves the desired result.

1.6. We must verify the four axioms that a metric satisfies. We have $d(x, y) = \|x - y\| \geq 0$. Clearly $d(x, x) = \|x - x\| = 0$ for any $x \in X$. If $d(x, y) = 0$, then $\|x - y\| = 0$ and so $x - y = 0$ and hence $x = y$. We have $d(y, x) = \|y-x\| = |-1|\|y-x\| = \|(-1)(y-x)\| = \|x-y\| = d(x, y)$. Finally, the triangle inequality holds for norms, and so

$$d(x, z) = \|x - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z).$$

This proves that d is a metric.

1.7. Suppose we consider the case of $X = \mathbb{R}^n$. If d is a metric that comes from a norm, then $d(cx, cy) = \|cx - cy\| = |c|\|x - y\| = |c|d(x, y)$. However, the discrete metric does not satisfy this, since if $x \neq y$, then $d(x, y) = 1$ and so $d(x/2, y/2) = 1 \neq 1/2$.

1.8. (1) By 1.8 (4), equivalence of norms is, in fact, an equivalence relation. It suffices therefore to prove that all of these norms are equivalent to, say, the l^1 norm. (My argument for 1.8 (4) does not rely on 1.8 (1)).

If $1 < p < \infty$ then, setting $q = \frac{p}{p-1}$, by Hölder's inequality, we get

$$\|x\|_1 = \sum |x_i| = \sum |x_i| \cdot 1 \leq \|x\|_p \left(\sum_{i=1}^n 1 \right)^{\frac{1}{q}}$$

and so $\|x\|_1 \leq n^{1/q}\|x\|_p$.

We have that for $x > 0$, $f(x) = x^p$ is concave up, and so $f'(x)$ is increasing. Thus,

$$(x + y)^p = \int_0^{x+y} pt^{p-1}\, dt = \int_0^x pt^{p-1} + \int_x^{x+y} pt^{p-1}\, dt \geq \int_0^x pt^{p-1}\, dt + \int_0^y pt^{p-1}\, dt = x^p + y^p.$$

A simple induction implies then that

$$\left(\sum_{i=1}^n |x_i| \right)^p \geq \sum_{i=1}^n |x_i|^p.$$

It follows from this that $\|x\|_1 \geq \|x\|_p$. This shows that l^1 and l^p are equivalent.

Finally, it is easy to see that

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty.$$

Thus, all the l^p norms are equivalent on $\mathbb{R}^n$.

(2) Suppose that U is open in the topology generated by $\| \cdot \|_1$. If $x \in U$, then there is an $\epsilon > 0$ so that $\{y : \|x - y\|_1 < \epsilon\} \subseteq U$. Now $\{y : \|x - y\|_1 < \epsilon\} \supseteq \{y : \|x - y\|_2 < c\epsilon\}$, and so there is an open ball (of radius ϵ/C) in the $\| \cdot \|_2$-topology containing x, and so U is open in the topology generated by $\| \cdot \|_2$. Replacing $\| \cdot \|_1$ and $\| \cdot \|_2$ gives the reverse implication.

(3) Let X be the set of all sequences (a_n) with all but finitely many terms of the (a_n) equal to zero. Define

$$\|a\|_1 = \sum_{n=1}^{\infty} |a_n|$$

and

$$\|a\|_2 = \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2}.$$

For $a, b \in X$, there is a positive integer n so that a and b can both be thought of as elements of $\mathbb{R}^n$, and the fact that the two functions above are both norms on $\mathbb{R}^n$ implies that they are norms on X.

Now, let a be the sequence whose first k terms are 2, and all of the rest of the terms are zero. Then, $\|a\|_1 = 2k$ and $\|a\|_2 = 2\sqrt{k}$. If there is a constant $C > 0$ so that $\|a\|_1 \leq C\|a_2\|$, then $2k \leq 2C\sqrt{k}$ for all positive integers k. This is a contradiction for $k > C^2$.

(4) If $n_1 \in \mathcal{N}$, then $n_1 \sim n_1$ since

$$\|x\|_{n_1} \leq \|x\|_{n_1} \leq \|x\|_{n_2}.$$

If $n_1, n_2 \in \mathcal{N}$ and $n_1 \sim n_2$, then there are constants c and C so that

$$c\|x\|_{n_1} \leq \|x\|_{n_2} \leq C\|x\|_{n_1}$$

for all $x \in X$. Dividing the left half by c yields

$$\|x\|_{n_1} \leq \frac{1}{c}\|x\|_{n_2},$$

whereas dividing the right half by C yields

$$\frac{1}{C}\|x\|_{n_2} \leq \|x\|_{n_1},$$

so we may conclude that

$$\frac{1}{C}\|x\|_{n_2} \leq \|x\|_{n_1} \leq \frac{1}{c}\|x\|_{n_2}.$$

This proves that $n_2 \sim n_1$. Finally, If $n_1, n_2, n_3 \in \mathcal{N}$, with $n_1 \sim n_2$ and $n_2 \sim n_3$, then there are positive constants c, C, d and D so that

$$c\|x\|_{n_1} \leq \|x\|_{n_2} \leq C\|x\|_{n_1}, \text{ and}$$
$$d\|x\|_{n_2} \leq \|x\|_{n_3} \leq D\|x\|_{n_2}.$$

Plugging the first inequality into the second gives

$$cd\|x\|_{n_1} \leq \|x\|_{n_3} \leq CD\|x\|_{n_1},$$

and so $n_1 \sim n_3$. Thus, equivalence of norms is an equivalence relation on $\mathcal{N}$.

2.1. A sequence of functions converging pointwise means that if ϵ is fixed, there is an N (depending on x!) so that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$. Uniform convergence is the same except there is a value of N that is valid simultaneously for all $x \in K$.

2.2. Fix $x \in K$ and $\epsilon > 0$. Choose N large enough so that $|f(x) - f_N(x)| < \epsilon/3$ for all $x \in K$. Since f_N is continuous at x, $\exists \delta > 0$ so that $|f_N(x) - f_N(y)| < \epsilon/3$ if $|x - y| < \delta$. Then for $|x - y| < \delta$,

$$\begin{aligned}|f(x) - f(y)| &< |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

Hence, f is continuous.

2.3. Let $f_n(x) = \sin^n(\pi x)$ with $K = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Then, for $x \neq 1/2$, $0 \leq \sin(\pi x) < 1$ and so $\lim_{n\to\infty} \sin^n(\pi x) = 0$. For $x = 1/2$, $f_n(x) = \sin^n(\pi/2) = 1$. Thus, f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1/2 \\ 1 & \text{if } x = 1/2 \end{cases}$$

which is clearly not continuous.

2.4. Suppose that $x_{ki} \to x_i$ for $1 \leq i \leq n$. Fix $\epsilon > 0$. For each i, there is an N_i so that $|x_{ki} - x_i| < \frac{\epsilon}{\sqrt{N}}$ for $k > N_i$. Then for $k > \max\{N_1, N_2, \dots, N_n\}$, we have

$$\begin{aligned}\|x_k - x\|_2 &= \sqrt{\sum_{i=1}^{n} |x_{ki} - x_i|^2} < \sqrt{\sum_{i=1}^{n} \left(\frac{\epsilon}{\sqrt{N}}\right)^2} \\ &= \epsilon.\end{aligned}$$

Hence, $x_k \to x$.

Conversely, suppose that $\|x_k - x\|_2 \to 0$. Fix $\epsilon > 0$ and choose N so that if $k \geq N$, $\|x_k - x\| < \epsilon$. Then

$$\sum_{i=1}^{n} |x_{ki} - x_i|^2 < \epsilon^2$$

and this implies that $|x_{ki} - x_i| < \epsilon$ for all i, $1 \leq i \leq N$. Hence $x_{ki} \to x_i$ for each i.

2.5. (i) Suppose (x_i) is a bounded sequence in $\mathbb{R}^n$. Then (x_{1i}) is a bounded sequence in $\mathbb{R}$ and so it has a convergent subsequence (x_{1j}), $j \in S_1 \subseteq \mathbb{N}$. We may choose a subset $S_2 \subseteq S_1$ so that (x_{2j}), $j \in S_2$ converges. Continuing this process we may choose $S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_1$ so that (x_{kj}) converges for all $1 \leq k \leq n$ for $j \in S_n$. By Theorem 2.7, this implies that x_j, $j \in S_n$, converges.

(ii) Suppose (x_i) is a Cauchy sequence in $\mathbb{R}^n$. Then (x_{ki}) is a Cauchy sequence in $\mathbb{R}$ for $1 \leq k \leq n$ and so $x_{ki} \to L_k$ as $i \to \infty$. Theorem 2.7 then implies that $x_k \to L = (L_1, L_2, \dots, L_n)$. Hence, $\mathbb{R}^n$ is complete.

(iii) In any metric space a compact set is closed and bounded. [I will prove this in the two paragraphs below.]

If K is compact and y is an accumulation point of K, there is a sequence of points $y_i \in K$ that converges to y. Since K is sequentially compact, there is a subsequence that converges in K, and this subsequence also converges to y, and so $y \in K$. Thus, K is closed.

If K is compact and for each $k \in K$, we let $U_k = \{x \in X : d(x,k) < 1\}$ then $\bigcup_{k\in K} U_k$ is an open cover for K. It has a finite subcover and this implies that it is bounded.

It suffices to prove that if $K \subseteq \mathbb{R}^n$ is closed and bounded, then it is compact. For $1 \leq i \leq n$, let $K_i = \{x_i : (x_1, x_2, \ldots, x_n) \in K\}$. Since K is bounded, K_i is bounded for $1 \leq i \leq n$. Since K is closed, if y is an accumulation point of K_i, there is a sequence (y_i) in K so that y_{1i}, y_{2i}, $\ldots$ converges to the ith component of y. Since K is bounded, we may choose a convergent subsequence $y'_1, y'_2, \ldots$ (that must converge to y). Then $y'_{1i}, y'_{2i}, \ldots$ converges to the ith component of y and so the ith component of y is in K_i and K_i is closed. Hence $K_i \subseteq \mathbb{R}$ is compact.

Now, let $y_1, y_2, \ldots$ be any sequence in K. Choose a subsequence $y^1_1, y^1_2, \ldots$, so that $y^1_{11}, y^1_{21}, \ldots$ converges. For $2 \leq i \leq n$, we choose a subsequence y^i_r of (y^{i-1}_r) so that y^i_{ri} converges as $r \to \infty$. Then, y^n_{ri} converges for $1 \leq i \leq n$ by Theorem 2.7. Moreover, it must converge to an element of K since K is closed. Therefore, K is sequentially compact.

2.6. In any metric space, a Cauchy sequence is bounded. Let x_1, x_2, $\ldots$, be a Cauchy sequence. Then there exists and N so that if $n, m \geq N$, then $d(x_n, x_m) \leq 1$. Let where $M = 1 + \max\{d(x_j, x_N) : 1 \leq j \leq N\}$. Then, for any i and j,

$$d(x_i, x_j) \leq d(x_i, x_N) + d(x_N, x_j) \leq 2M$$

by the same calculation as in problem 1.3. Hence, the sequence (x_i) is bounded.

2.7. Fix $\epsilon > 0$ and choose K so that if $k, \ell \geq K$, then $\|x^k - x^\ell\| < \epsilon/2$. Also fix a positive integer N and consider

$$\left(\sum_{n=1}^{N} |x_n - x^k_n|^2\right)^{1/2}.$$

The sequence $(x^r_1, x^r_2, \ldots, x^r_N)$ converges to $(x_1, x_2, \ldots, x_N)$ (by Theorem 2.7), and so there is an $L \geq K$ so that if $\ell \geq L$, then $|x_r - x^\ell_r| < \epsilon/(2\sqrt{N})$ for $1 \leq r \leq N$. Then

$$\begin{aligned}
\left(\sum_{n=1}^{N} |x_n - x^k_n|^2\right)^{1/2} &\leq \left(\sum_{n=1}^{N} |x_n - x^L_n|^2\right)^{1/2} + \left(\sum_{n=1}^{N} |x^L_n - x^k_n|^2\right)^{1/2} \\
&< \left(\sum_{n=1}^{N} \frac{\epsilon^2}{2N}\right)^{1/2} + \|x^k_n - x^L_n\|_2 \\
&< \epsilon/2 + \epsilon/2 < \epsilon.
\end{aligned}$$

2.8. Note that $\{(a_1, a_2, \ldots) : a_i = 0$ if i is large enough $\}$ is a subspace of l^2. If l^2 is finite-dimensional (with dimension N), then any linearly independent set has dimension less than or equal to N. This is a contradiction since if we let

$$u_i^j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j, \end{cases}$$

then $\{u_1, u_2, \ldots, u_{N+1}\}$ is a linearly independent subset of l^2 for any $N \geq 1$. To see this, suppose that there are constants $c_1, \ldots, c_{N+1}$ such that

$$0 = c_1 u_1 + \ldots + c_{N+1} u_{N+1} = (c_1, c_2, c_3, \ldots, c_{N+1}, 0, \ldots).$$

Clearly this is only possible if $c_1 = c_2 = \ldots = c_{N+1} = 0$. Hence these vectors are indeed linearly independent.

2.9. (1) It is easy to see that the function $f : l^2 \to \mathbb{R}$ given by $f(x) = \|x\|_2$ is continuous. Then, $K^c = \{x \in l^2 : \|x\|_2 > 1\} = f^{-1}((1, \infty))$ and hence K^c is open and so K is closed.

(2) Suppose to the contrary that K is compact. Let $U = \bigcup_{x \in K} \{x \in K : \|x - k\|_2 < 1/2\}$ is an open cover for K and so it must have a finite subcover. Since the set $\{e^i : i \geq 1\}$ is infinite, there must be $i \neq j$ and $k \in K$ so that $e^i, e^j \in \{x \in K : \|x - k\|_2 < 1/2\}$. This implies that $\|e^i - e^j\|_2 \leq \|e^i - k\|_2 + \|k - e^j\|_2 < \frac{1}{2} + \frac{1}{2} = 1$ and this is a contradiction, since $\|e^i - e^j\|_2 = \sqrt{2} > 1$.

2.10. If $x \in C$, then

$$\begin{aligned} \|x\|_2 &= \sqrt{\sum_{n=1}^{\infty} |x_n|^2} \\ &\leq \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \\ &= \sqrt{\frac{\pi^2}{6}} = \frac{\pi}{\sqrt{6}}. \end{aligned}$$

Thus, $\|x\|_2 \leq \pi/\sqrt{6}$ and so $x \in l^2$.

2.11. (a) It is well-known that sums and products of continuous functions are continuous. The constant functions $f(x) = c$ are continuous, and these properties (together with the usual commutative, associative and distributive laws for the reals) imply that $C([0,1])$ is a real vector space.

(b) We have

$$\begin{aligned}
\|f_n - f_m\|_2 &= \sqrt{\int_0^{1/2} (2^n x^n - 2^m x^m)^2 \, ds} \\
&= \sqrt{\int_0^{1/2} 2^{2n} x^{2n} - 2^{m+n+1} x^{m+n} + 2^{2m} x^{2m} \, dx} \\
&= \sqrt{\left[\frac{2^{2n} x^{2n+1}}{2n+1} - \frac{2^{m+n+1} x^{m+n+1}}{m+n+1} + \frac{2^{2m} x^{2m+1}}{2m+1}\right]_0^{1/2}} \\
&= \sqrt{\frac{1}{2(2n+1)} - \frac{1}{m+n+1} + \frac{1}{2(2m+1)}}.
\end{aligned}$$

Each term inside the square root above tends to zero as m and n both tend to infinity, and so for any $\epsilon > 0$, there is an N so that for $m, n \geq N$, we have $\|f_n - f_m\|_2 < \epsilon$. In other words, the sequence is indeed Cauchy.

Let

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1. \end{cases}$$

I claim that $\|f_n - g\|_2 \to 0$ as $n \to \infty$. This is easy to verify as

$$\begin{aligned}
\|f_n - g\|_2^2 &= \int_0^{1/2} (2x)^{2n} \, dx \\
&= \left[\frac{2^{2n} x^{2n+1}}{2n+1}\right]_0^{1/2} \\
&= \frac{1}{2(2n+1)} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Suppose to the contrary that there is a continuous function f so that $f_n \to f$ in the L^2-norm. Then, for any $\epsilon > 0$, there is an N so that for $n \geq N$, $\|f_n - f\|_2 < \epsilon/2$ and $\|f_n - g\|_2 < \epsilon/2$ and this implies that

$$\|f - g\|_2 \leq \|f - f_n\|_2 + \|f_n - g\|_2 < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since ϵ was arbitrary, it follows that

$$\int_0^1 (f(x) - g(x))^2 \, dx = 0.$$

We will show that this is a contradiction (although not to Lemma 2.4, since $f - g$ is not continuous). Let $c = f(1/2)$ and let $\delta > 0$ be such that $|f(x) - c| < 1/4$ for $1/2 - \delta < x < 1/2 + \delta$.

We have then that

$$|g(x) - f(x)| \geq |g(x) - c| - |c - f(x)|.$$

The integral of the second term over $[1/2 - \delta, 1/2 + \delta]$ is less than $\delta/2$. However, the integral of the first term over $[1/2 - \delta, 1/2 + \delta]$ is

$$\int_{1/2-\delta}^{1/2} |c|\, dx + \int_{1/2}^{1/2+\delta} |1 - c| = (\frac{|c| + |1 - c|}{2}) \cdot \delta \geq \delta/2.$$

This shows that $\int_{1/2-\delta}^{1/2+\delta} (f(x) - g(x))^2\, dx > 0$ and this is a contradiction.

(c) The above argument shows, in fact, that the sequence f_n in $C([0,1])$ with the L^2-norm does not have a convergent subsequence. Similarly, this implies that the sequence f_n in $C([0,1])$ with the L^∞-norm does not have a convergent subsequence, because if $\|f - g\|_\infty < \epsilon$, then

$$\|f - g\|_2 = \sqrt{\int_0^1 |f - g|^2\, dx} < \sqrt{\epsilon^2} = \epsilon.$$

2.12. The sequence $f_n(x) = x + n$ for $n \geq 1$ comes to mind. We have $|f_n'(x)| = 1$ and so $|f_n(x) - f_n(y)| \leq |x - y|$ for all $n \geq 1$. Therefore, the $f_n(x)$ are equicontinuous (by Corollary 2.3). However, there is no subsequence (f_{n_k}) that converges. Clearly, we have

$$\lim_{k\to\infty} x + n_k$$

does not exist for any $x \in [0,1]$ and any subsequence n_k of $(1, 2, 3, \ldots)$.

2.13.

(a) Proof of Prop. 2.10: It suffices to prove that $C^1([0,1])$ is closed under addition and scalar multiplication. It is clear that if f and g are C^1, then so is $f + g$, and also that if $c \in \mathbb{R}$ and $f \in C^1$, so is cf. This proves that $C^1([0,1])$ is a real vector space.

(b) Proof of Thm 2.15: For any $f, g \in C^1([0,1])$, we have $\|f - g\|_\infty + \|f' - g'\|_\infty \geq 0$. If it equals zero, then $\|f - g\|_\infty = 0$ and so $f = g$. It is also clear that

$$\|f + g\|_\infty + \|f' + g'\|_\infty \leq \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty$$

and this implies the triangle inequality in $C^1([0,1])$. Finally, $d(f,g) = d(g,f)$ is clear.

(c) Proof of Prop 2.12: We have $\|f\|_{1,2} = \sqrt{\|f\|_2^2 + \|f'\|_2^2}$. We have that $\|f\|_{1,2} \geq 0$ and $\|0\|_{1,2} = 0$. Also, if $\|f\|_{1,2} = 0$, then $\|f\|_2^2 = 0$. This implies that $\int_0^1 |f(x)|^2\, dx = 0$, which (by Lemma 2.4) implies that $f = 0$ (since $f \in C([0,1])$). We also have $\|\alpha f\|_{1,2} = \sqrt{\|\alpha f\|_2^2 + \|\alpha f'\|_2^2} = \sqrt{\alpha^2\|f\|_2^2 + \alpha^2\|f'\|_2^2} = |\alpha|\|f\|_{1,2}$, and finally, if we consider the two dimensional vectors $\vec{a} = \langle \|f\|_2, \|f'\|_2 \rangle$ and $\vec{b} = \langle \|g\|_2, \|g'\|_2 \rangle$, then Minkowski's inequality tells us that $\|\vec{a} + \vec{b}\|_2 \leq \|\vec{a}|_2 +$

$\|\vec{b}\|_2$. In terms of f and g, we have

$$\begin{aligned}
\|f+g\|_{1,2} &= \sqrt{\|f+g\|_2^2 + \|f'+g'\|_2^2} \\
&\leq \sqrt{(\|f\|_2 + \|g\|_2)^2 + (\|f'\|_2 + \|g'\|_2)^2} \\
&= \|\vec{a}+\vec{b}\|_2 \\
&\leq \|\vec{a}|_2 + \|\vec{b}\|_2 \qquad\qquad = \sqrt{\|f\|_2 + \|f'\|_2} + \sqrt{\|g\|_2 + \|g'\|_2} \\
&= \|f\|_{1,2} + \|g\|_{1,2}
\end{aligned}$$

as claimed.

(d) Proof of Prop 2.13: Take the example from Prop 2.8 of the $f_n(x)$ and let $g_n(x) = \int_0^x f_n(t)\,dt$. Then, we know from Prop 2.8 that the sequence $f_n(x)$ is Cauchy with the L^2 norm, and we have

$$\begin{aligned}
|g_m(x) - g_n(x)|^2 &= \left| \int_0^x f_m(t) - f_n(t)\,dt \right|^2 \\
&\leq \int_0^x |f_m(t) - f_n(t)|^2\,dt \\
&\leq \|f_m - f_n\|_2^2.
\end{aligned}$$

Integrating from 0 to 1 gives $\|g_m - g_n\|_2^2 \leq \|f_m - f_n\|_2^2$. Thus, $\|g_m - g_n\|_{1,2} \leq 2\|f_m - f_n\|$. However, the sequence does not converge, since the sequence f_n is a Cauchy sequence in $C([0,1])$ with the L^2-norm, but does not converge.