Project 3

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Part I
Theoretical

Problem 20.1

Proof:

($\Rightarrow$)

Since $A \in \mathbb{C}^{m\times m}$ is nonsingular, then $det(A) \neq 0$, by (20.1) we have $L_{m-1}...L_2L_1A = U$ with each L_i is unit lower-triangular matrix and U is upper-triangular, then $det(L_i) = 1 \neq 0$. So

$$0 \neq det(A) = det(L_1^{-1})det(L_2^{-1})...det(L_{m-1}^{-1})det(U)$$

Thus $det(U) \neq 0$, so U is nonsingular.

Since the Gaussian elimination didn't change the determinants $det(A_{1:k,1:k})$, thus for $1 \leq k \leq m$, $det(A_{1:k,1:k}) = det(U_{1:k,1:k})$, since U is upper-triangular and nonsingular, then every entries in the diagonal of U is nonzero. Then $det(A_{1:k,1:k}) = det(U_{1:k,1:k}) \neq 0$, thus every upper-left $k \times k$ block $A_{1:k,1:k}$ is nonsingular.

($\Leftarrow$) Since every upper-left $k \times k$ block of A is nonsingular, then at every step of Gaussian elimination, after the row operation it still leaves the matrix nonsingular, thus matrix A has such a LU factorization.

By the theorem on note of 15/11, since A is nonsingular and hence invertible, if we require that the diagonal of L or U consists of ones, then the LU decomposition is unique.

Problem 20.3

Solution:

(a). It is very obvious:

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} IA_{11} + 0A_{21} & IA_{12} + 0A_{22} \\ -A_{21}A_{11}^{-1}A_{11} + IA_{21} & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

(b). Suppose $A_{21} = [\ c_1, \quad ..., \quad c_{n-1}, \quad c_n\ ]$ and each $c_i \in \mathbb{C}^{m-n}$, then by n steps of Gaussian elimination, we have the L_1, L_2, $...L_n$ as follows:

$$\begin{bmatrix} I & \\ [0, ..., -c_n]\, A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} I & \\ [0..., -c_{n-1}, 0]\, A_{11}^{-1} & I \end{bmatrix} \cdots \begin{bmatrix} I & \\ [-c_1, 0...0]\, A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} I & \\ [0, ..., -c_n]\, A_{11}^{-1} & I \end{bmatrix} \cdots \begin{bmatrix} I & \\ [0, -c_2, 0...0]\, A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ [-c_1, 0..., 0] + A_{21} & A_{22} - c_1 A_{11}^{-1} A_{12} \end{bmatrix}$$

$$......$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ [-c_1, -c_2..., -c_n] + A_{21} & A_{22} - (c_1 + c_2... + c_n)A_{11}^{-1}A_{12} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

thus the bottom-right $(m-n) \times (m-n)$ block of the result is again $A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Problem 21.1

Solution:

(1). Determine of A from (20.5) is $det(A) = det(L)det(U)$, since L is a unit lower-triangular matrix and U is a upper-triangular matrix, thus

$$det(A) = 1 \cdot 2 \cdot 1 \cdot 2 \cdot 2 = 8$$

(2). Determine of A from (20.5) is $det(PA) = deet(P)det(A) = det(L)det(U)$, the same with (1) except we need to calculate $det(P) = -1$, thus

$$-1 \cdot det(A) = 1 \cdot 8 \cdot \frac{7}{4} \cdot (-\frac{6}{7}) \cdot \frac{2}{3} = -8$$

then $det(A) = 8$.

(3). Since interchanging two rows or columns will change the sign of the determinant of A, then after Gaussian elimination with pivoting we get $PA = LU$, since $det(L) = 1$ and $det(U)$ is the product of its diagonal entries, and $det(P) = (-1^n)$ where n is the number of the times of interchanging two rows of matrix A (partial pivoting), thus

$$det(A) = det(U) * (-1)^n$$

Problem 21.3

Proof:

(a). Since both row operation and column operation don't change the singularity of the matrix, thus AQ is still nonsingular since A is nonsingluare, hence AQ is invertible, then by the Theorem on 15/11's note that an invertible matrix admits a LU factorization, we know that such a factorization $AQ = LU$ always exists.

(b). To show such a factorization does not always exist is to give an example. Since A is singular, let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and just let $Q = I$, thus $L_1 = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -7 & & 1 \end{bmatrix}$, then we have

$$L_1 AQ = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -7 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -6 & -12 \end{bmatrix} = U_1$$

thus a factorization does not always exist.

Problem 23.1

Proof:

It is true that $R = U$.

Since A is nonsingular square matrix, then the SVD of A is $A = U\Sigma V^*$, so $A^*A = V^*\Sigma^2 V$, whose eigenvalues are the diagonal entries of Σ^2, i.e. positive, and since $(A^*A)^* = A^*A$, then A^*A is hermitian positive definite.

By the QR factorization of A that $A = QR$ where Q is unitray, then $A^*A = R^*Q^*QR = R^*R$, since $r_{jj} > 0$, then such a QR factorization is unique, and R is upper triangular matrix.

While A^*A has Cholesky Factorization $A^*A = U^*U$ with $u_{jj} > 0$, by **Theorem 23.1** that, every hermitian positive definite matrix has a unique Cholesky Factorization, thus $R = U$.

Part II
Numerical Experiments

1.

Please run the m-file: **data.m**

Gaussian elimination without pivoting: **nplu.m**

Gaussian elimination with partial pivoting: **plu.m** (with **maxposition.m** and **interchange.m**)

23.3

Answer:

(1). The purpose of carrying out this experiment is to compare the computation time of using "\" to solve $Ax = b$ when given different input matrix A. The computation time actually reflects the steps of each part because they are all under the same computer machine, we can see that it varies depends on the condition of given matrix A.

(2). After checking "help chol" and "help slash" in MATLAB, we see that the "\" is better than inv and better than gaussian elimination. Also, it was much more faster because it consumes less steps.

We guess that for hermitian positive definite matrix of each part, "\" prefer to use Cholesky decomposition to get the inverse of matrix A, the result shows that it costs the least steps to compute the inverse of matrix A in (f) because $A5$ itself is upper triangular matrix; and it costs much more time than others in (e) because it is not positive definite; and it costs much time in (c) and (g) because they are nor hermitian any more; for (a) (b) and (d), it's OK because they are just changes a little in their entries.