Math 711
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Due: February 16, 2012

Test #1 Solutions

(1) (5 points each) True or false? Justify your response with a complete argument or counterexample (not necessarily a formal proof).

(a) Let $x \in X$ and $\epsilon > 0$. Then $\overline{B_\epsilon(x)} = \{y \in X : d(x,y) \leq \epsilon\}$ is compact in X.
Answer: False. For example, in l^2, choose $x = 0$ and $\epsilon = 1$. Then $\overline{B_\epsilon(x)} = K$ as defined in the course notes, and we proved in class that K is not sequentially compact and therefore not compact.

(b) A sequence in $\mathbb{R}^n$ converges if and only if it has exactly one accumulation point.
Answer: False. This would be true if the sequence were bounded. However it is possible for an unbounded sequence to have exactly one accumulation point and not converge. For example, in $\mathbb{R}$, let

$$x_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ n & n \text{ even} \end{cases}.$$

Then $x = 0$ is the only accumulation point of (x_n), but (x_n) clearly does not converge. As a side note, the forward direction is true. Only the reverse direction is false.

(c) On the space l^2, the sup-norm and the l^2-norm are equivalent.
Answer: False. Consider the element (x^n) of l^2 given by

$$x_j^n = \begin{cases} \frac{1}{j} & j \leq n^2 \\ 0 & j > n^2 \end{cases},$$

and compute

$$\|(x^n)\|_2 = \left(\sum_{j=1}^{\infty} |x_j^n|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n^2} |\frac{1}{n}|^2 + 0\right)^{\frac{1}{2}} = (n^2 \cdot \frac{1}{n^2})^{\frac{1}{2}} = 1,$$

whereas

$$\|(x^n)\|_\infty = \sup\{|x_j^n| : j \in \mathbb{N}\} = \frac{1}{n}.$$

Now, assume that the l^2 and l^∞ norms are equivalent on l^2, so $\exists c, C > 0$ so that, forall $(x) \in l^2$, $c\|x\|_2 \leq \|x\|_\infty \leq C\|x\|_2$. Choose n sufficiently large so that $n > \frac{1}{c}$, and consider (x^n). We have $c\|x^n\|_2 \leq \|x^n\|_\infty$, but $\|x^n\|_2 = 1$ and $\|x^n\|_\infty = \frac{1}{n}$, which implies that $c \cdot 1 \leq \frac{1}{n}$, which is a contradiction. Therefore these two norms cannot be equivalent.

(d) Suppose that $f : X \to Y$ is a function which is continuous at $x_0 \in X$. Let U be some open set in Y which contains $f(x_0)$. Then $f^{-1}(U)$ must be open in X.
Answer: False. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x \end{cases}.$$

Let $x_0 = -1$. Let $\epsilon > 0$ and let $\delta = \frac{1}{2}$. If $|x - x_0| < \delta$, then $-\frac{3}{2} < x < -\frac{1}{2}$, so $f(x) = 0$, so $|f(x) - f(x_0)| = 0 < \epsilon$. Therefore f is continuous at x_0. Let $U = (-1, 1)$. Then clearly U is open in $\mathbb{R}$, and $U \ni 0 = f(x_0)$. However, $f^{-1}(U) = \{x : -1 < f(x) < 1\} = (-\infty, 0]$, which is not open in $\mathbb{R}$.

(e) If a sequence of continuous functions $f_n(x)$ converge to a continuous function $f(x)$ on a compact domain K, then they converge uniformly on K.
Answer: False. Consider the functions $f_n(x) : [0, 1] \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} nx & x \leq \frac{1}{n} \\ 2 - nx & \frac{1}{n} \leq x\frac{2}{n} \\ 0 & x \geq \frac{2}{n} \end{cases},$$

which were discussed in class. First note that the domain of these functions, $K = [0, 1]$ is indeed compact. Second, note that these functions are each continuous; they are piecewise linear, so we only need to check that they "glue" correctly. To that end, note that $n \cdot \frac{1}{n} = 1 = 2 - n \cdot \frac{1}{n}$, and $2 - n \cdot \frac{2}{n} = 0$. Therefore f_n is continuous for each n. Third, note that, for each $x \in [0, 1]$, $f_n(x) \to 0$. There are two cases to consider: if $x = 0$, then for each n $f_n(x) = n \cdot 0 = 0$, so clearly $f_n(x) \to 0$. Otherwise, if $x > 0$, then by the Archimedean principle there is an $N \in \mathbb{N}$ such that $x > \frac{2}{N}$. Then, for any $n > N$, $x > \frac{2}{n}$, so $f_n(x) = 0$. Therefore, $|f_n(x) - 0| = 0$ for $n > N$. So $f_n(x) \to 0$ as claimed. Therefore, these functions converge to a continuous limit function, $f(x) \equiv 0$. Hence we have an example of a sequence of continuous functions that converge to a continuous function on a compact domain. However, they do not converge uniformly, because for each n, $\sup_{x\in[0,1]} |f_n(x)| = f_n(\frac{1}{n}) = 1$ which does not go to 0. So these functions cannot converge uniformly.

(2) We say that a sequence (x_n) is *eventually zero* if there exists some $N \in \mathbb{N}$ so that, for all $n > N$, $x_n = 0$. Let O be the collection of all eventually zero sequences.

(a) (3 points) Show that $O \subset l^2$.
Answer: Suppose that $(x_n) \in O$. Then $\exists N \in \mathbb{N}$ such that $x_n = 0$ for all $n > N$. Therefore, $\sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=1}^{N} |x_n|^2 + 0 = \sum_{n=1}^{N} |x_n|^2$, which is a finite sum of finite terms so it is finite. Therefore $\|(x_n)\|_2 < +\infty$, so $(x_n) \in l^2$.

(b) (5 points) Show that O is dense in l^2.
Answer: We must show that, for every $x \in l^2$ and for every $\epsilon > 0$, there is a $\tilde{x} \in O$ such that $\|x - \tilde{x}\|_2 < \epsilon$. Let $x := (x_n) \in l^2$ and let $\epsilon > 0$. Since $x \in l^2$, $\sum_{n=1}^{\infty} |x_n^2| < +\infty$. Therefore, there is an $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} |x_n|^2 < \epsilon^2$. This is a property of convergent series that was proved in class. Now, define $\tilde{x}$ as follows:

$$\tilde{x}_n = \begin{cases} x_n & n \leq N \\ 0 & n > N \end{cases}.$$

Clearly, $\tilde{x} \in O$ by construction. Let's compute $\|x - \tilde{x}\|_2$:

$$\begin{aligned}
\|x-\tilde{x}\|_2^2 &= \sum_{n=1}^{\infty} |x_n - \tilde{x}_n|^2 \\
&= \sum_{n=1}^{N} |x_n - \tilde{x}_n|^2 + \sum_{n=N+1}^{\infty} |x_n - \tilde{x}_n|^2 \\
&= \sum_{n=1}^{N} |x_n - x_n|^2 + \sum_{n=N+1}^{\infty} |x_n - 0|^2 \\
&= \sum_{n=N+1}^{\infty} |x_n|^2 \\
&< \epsilon^2.
\end{aligned}$$

Therefore, $\|x - \tilde{x}\|_2 < \epsilon$, as desired.

(3) Consider the set $C([0,1])$ as defined in class, and, for $p \geq 1$, define $\|\cdot\|_p : C([0,1]) \to \mathbb{R}$ by $\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}$.

(a) (5 points) Prove that this is a norm on $C([0,1])$.

Answer: We must first verify that this is well-defined on $C([0,1])$. Let $f \in C([0,1])$. Then since f is continuous, so is $|f|^p$, and all continuous functions are integrable. Therefore, $\|f\|_p$ is well-defined.

First, note that $|f(x)| \geq 0$ for any f and x, so the $\|f\|_p \geq 0$. Also, $\|0\|_p = \left(\int_0^1 0dx\right)^{\frac{1}{p}}$, so $\|0\|_p = 0$. Finally, if $\|f\|_p = 0$, then $\int_0^1 |f(x)|^p dx = 0$. Since $|f(x)|^p$ is a nonnegative, continuous function, its integral can only be zero if it is itself zero everywhere, by Lemma 2.4 in Chapter 1 of the course notes. Since $|f(x)|^p \equiv 0$, $f(x) \equiv 0$ and thus $\|f\|_p = 0$ if and only if $f = 0$.

Now, let $\alpha \in \mathbb{R}$ and $f \in C([0,1])$, and compute

$$\begin{aligned}
\|\alpha f\|_p &= \left(\int_0^1 |\alpha f(x)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^1 |\alpha|^p |f(x)|^p dx\right)^{\frac{1}{p}} \\
&= \left(|\alpha|^p \int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} \\
&= |\alpha| \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} \\
&= |\alpha| \|f\|_p,
\end{aligned}$$

so the homogeneity property of norms is satisfied.

Finally, we need to prove the triangle inequality. To do so, we must first verify Hölder's inequality for L^p and L^q, so let $f \in L^p$ and $g \in L^q$, where $q = pp - 1$ so that $\frac{1}{p} + \frac{1}{q} = 1$. By Young's Inequality, we have that if $a, b \in \mathbb{R}$ then $|ab| \leq$

$\frac{1}{p}a^p + \frac{1}{q}b^q$. Therefore, for any $x \in [0,1]$, $|f(x)g(x)| \leq \frac{1}{p}|f(x)|^2 + \frac{1}{q}|g(x)|^2$ Now if f or g is identically zero, the claim is trivial, so we may assume they are not. Let $\tilde{f}(x) = \frac{f(x)}{\|f\|_p}$ and $\tilde{g}(x) = \frac{g(x)}{\|g\|_q}$. By the homogeneity proved above these have unit L^p and L^q norms respectively. Then

$$\int \frac{|f(x)g(x)|}{\|f\|_p\|g\|_q}dx = \int |\tilde{f}(x)\tilde{g}(x)|dx \leq \int (\frac{1}{p}|\tilde{f}(x)|^p + \frac{1}{q}|\tilde{g}(x)|^q)dx = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus $\int |f(x)g(x)|dx \leq \|f\|_p\|g\|_q$.

We are now ready to verify the triangle inequality. So, let $f, g \in C([0,1])$. Note that $f+g$ is then also continuous, so $\|f+g\|_p$ is well-defined for any p. Then compute:

$$\begin{aligned}
\|f+g\|_p^p &= \int |f(x)+g(x)|^p dx \\
&= \int |f(x)+g(x)||f(x)+g(x)|^{p-1}dx \\
&\leq \int (|f(x)|+|g(x)|)|f(x)+g(x)|^{p-1}dx && \text{by the triangle inequality on } \mathbb{R} \\
&= \int |f(x)||f(x)+g(x)|^{p-1}dx + \int |g(x)||f(x)+g(x)|^{p-1}dx \\
&\leq \|f\|_p\|(f+g)^{p-1}\|_q + \|g\|_p\|(f+g)^{p-1}\|_q && \text{by Hölder's inequality, proved above.}
\end{aligned}$$

Note that $\|(f+g)^{p-1}\|_q = \left(\int_0^1 |f(x)+g(x)|^{(p-1)q}dx\right)^{\frac{1}{q}}$. By our choice of q, $(p-1)q = (p-1)\frac{p}{p-1} = p$, and $\frac{1}{q} = \frac{p-1}{p} = (p-1)\frac{1}{p}$, so $\|(f+g)^{p-1}\|_q = \|(f+g)\|_p^{p-1}$. Hence, we have that

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)\,\|f+g\|_p^{p-1}.$$

Dividing through by the quantity $\|f+g\|_p^{p-1}$ yields the desired result. Therefore, $\|\cdot\|_p$ is a norm on $C([0,1])$.

(b) (6 points) Prove that if $\frac{1}{p} + \frac{1}{q} = 1$, then $\forall f, g \in C([0,1])$, $\int_0^1 |f(x)g(x)|dx \leq \|f\|_p\|g\|_q$. (Hint: This is extremely similar to Exercise 1.1.)

Answer: See the proof of this in the answer to (a).

(c) (6 points) Is $C([0,1])$ complete with respect to this norm? Either prove it or give a counterexample.

Answer: No. For example, consider the functions

$$f_n(x) := \begin{cases} (2x)^n & x \leq \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}.$$

These functions are Cauchy in L^p for any p, and they converge to the function

$$f(x) := \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases},$$

which is not an element of $C([0,1])$. The proof of these facts is the same as what we discussed for L^2.

(4) (10 points) The following is a correct proof that the Nested Interval Property is equivalent to the Bolzano-Weierstrass Theorem on $\mathbb{R}$. However, the sentences in each half of the proof have been put in random order. All you have to do is to put them into the correct order to form a logically valid proof. Note that your goal is to put MY proof in order, not to create your own. For maximum partial credit you will want to copy out your answer in full sentences so I can see how well it fits together.

For your reference, the Nested Interval Property states that if $(I_n)_{n=1}^{\infty}$ is a sequence of nonempty, closed, bounded intervals in $\mathbb{R}$ so that, for each n, $I_n \supset I_{n+1}$, then $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem 1. *The nested interval property of the reals is equivalent to the Bolzano-Weierstrass Theorem on* $\mathbb{R}$.

Proof. ($\Rightarrow$)

(a) (a) Suppose that for every nested sequence $(F_n)_{n=1}^{\infty}$ of nonempty, closed, bounded intervals in the reals, $\cap_n F_n \neq \emptyset$.

(b) (f) Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers.

(c) (c) Because (x_n) is bounded, there exists $M > 0$ such that $-M < x_n < M$ $\forall n \in \mathbb{N}$.

(d) (z) Define $F_0 = [-M, M]$.

(e) (m) Clearly, either $[-M, 0]$ or $[0, M]$ contains an infinite number of the x_n.

(f) (s) Define F_1 to be whichever half contains an infinite number of the x_n.

(g) (o) Now, suppose that we have chosen intervals F_i for $i = 0 \ldots j$, so that each F_{i+1} is a subset of F_i with half the length of F_i and so that each F_i contains an infinite number of the x_n.

(h) (d) Then we define F_{j+1} to be either the left half or the right half of F_j, whichever half contains an infinite number of sequence terms x_n.

(i) (u) This allows us to recursively generate a sequence of intervals (F_j) such that, $\forall j$,
$F_j \supset F_{j+1}$ and there are an infinite number of terms of (x_n) in F_j.

(j) (b) Also note that the length of the interval F_j is $\frac{M}{2^{j-1}}$.

(k) (e) By the nested interval property, $\cap_j F_j \neq \emptyset$.

(l) (l) Let x_0 be an element of $\cap_j F_j$.

(m) (k) Then we claim that there is a subsequence of (x_n) which converges to x_0:

(n) (h) Let (x_{n_1}) be any term of (x_n) which is in F_1.

(o) (i) Suppose that we have already selected $x_{n_1}, \ldots, x_{n_k}$.

(p) (p) Then choose $x_{n_{k+1}}$ to be any term of (x_n) in F_{k+1} such that $n_{k+1} > n_k$.

(q) (r) This is possible because F_{k+1} contains infinitely many terms of (x_n) so there must be one beyond x_{n_k}.

(r) (y) Clearly this process constructs a subsequence (x_{n_k}) of (x_n), so it only remains to prove that $x_{n_k} \to x_0$.

(s) (g) Let $\epsilon > 0$.

(t) (j) Choose J sufficiently large so that $\frac{M}{2^{J-1}} < \epsilon$.

(u) (t) Let $j > J$.

(v) (n) Then $x_{n_j} \in F_j$ by construction, and $F_j \subset F_J$ by the nestedness of (F_j).

(w) (w) Recall that x_0 is also an element of F_J for each J and that the total length of F_J is $\frac{M}{2^{J-1}} < \epsilon$.

(x) (v) Since x_0 and x_{n_j} are both elements of F_J, they cannot be further apart than the total length of F_J.
(y) (q) Therefore $|x_{n_j} - x_0| < \epsilon$.
(z) (x) Hence $x_{n_k} \to x_0$ and (x_n) has a convergent subsequence as claimed.

($\Leftarrow$):

(a) (o) Suppose that every bounded sequence of real numbers has a convergent subsequence.
(b) (l) Let F_n be a nested sequence of nonempty, closed, bounded intervals in the reals.
(c) (b) Write $F_n = [a_n, b_n]$ for all n.
(d) (h) Since $F_n \supset F_{n+1}\ \forall n$, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n\ \forall n$.
(e) (g) Hence the sequence (a_n) is a sequence of real numbers which is bounded between a_1 and b_1.
(f) (t) By the Bolzano-Weierstrass theorem, (a_n) has a convergent subsequence (a_{n_k}) which converges to some number a.
(g) (d) We claim that $a \in \cap_n [a_n, b_n]$.
(h) (q) First, we will show that $a \geq a_n$ for all n.
(i) (k) Suppose not.
(j) (s) Then $\exists k$ such that $a < a_k$
(k) (p) But then since the (a_n) are an increasing sequence, $a < a_j$ for all $j > k$.
(l) (n) Let $\epsilon = \frac{a_k - a}{2}$.
(m) (u) Then, $\forall j > k, |a_j - a| \geq |a_k - a| > \epsilon$.
(n) (a) Hence no subsequence of (a_n) can possibly converge to a, which is a contradiction.
(o) (m) Therefore $a \geq a_n$ for each n.
(p) (f) Finally, we must show that $a \leq b_n$ for each n.
(q) (c) If not, then there is a k such that $b_k < a$.
(r) (j) Let $\epsilon = \frac{a - b_k}{2}$.
(s) (i) Since a subsequence of the (a_n) converges to a, there is some a_{n_i} such that $|a_{n_i} - a| < \epsilon$ which implies that $a_{n_i} > a - \epsilon = \frac{a + b_k}{2} > b_k$.
(t) (e) This is a contradiction, as every a_n must be less than every b_k for every k and n by the nestedness of (F_n).
(u) (v) Hence we may conclude that $a_n \leq a \leq b_n$ for all n , i.e. $a \in F_n$ for all n.
(v) (r) Therefore $a \in \cap_n F_n$ and $\cap_n F_n \neq \emptyset$, as claimed.

$\square$