MAT 444 – H. Barcelo
Spring 2004

Homework 7 – Solutions

Section 6.5

Question:

4. Let G be a group of order 55.

   a) Prove that G is generated by two elements x, y, with the relations $x^{11} = 1,\ y^5 = 1,\ yxy^{-1} = x^r$, for some r, $1 \le r < 11$.

Answer:

a) Assume $|G| = 55 = 5.11$. By the third Sylow theorem we can conclude that G has one Sylow 11-subgroup and either 1 or 5 Sylow 5-subgroups.
Let H be the only Sylow 11-subgroup, then $H = \langle x \rangle$ where $|x| = 11$. Let K be a Sylow 5-subgroup, then $K = \langle y \rangle$ where $|y| = 5$. Since the orders of H and K are distinct primes we have that $H \cap K = 1$. Hence, the product map $p : H \times K \to G$ is injective, but $|HK| = 55$ and $HK \le G$, so $G \approx H \times K$. Next, since $H \lhd G$, then $yHy^{-1} = H$ which means that $yxy^{-1} = x^r$ for some $0 \le r < 11$. But if $r = 0$ we have $yxy^{-1} = 1 \Rightarrow x = 1$ a contradiction. Thus $yxy^{-1} = x^r$ for $1 \le r < 11$.

Question:

b) Prove that the following values of r are not possible: 2, 6, 7, 8, 10.

Answer:

b) Note that for any integer t, $yx^t y^{-1} = \underbrace{(yxy^{-1})(yxy^{-1})\ldots(yxy^{-1})}_{t\ \text{factors}} = (x^r)^t = x^{rt}$

$$
\begin{aligned}
x &= y^5\, xy^{-5} = y^4\, yxy^{-1}\, y^4 = y^4 x^r y^{-4}\\
\text{But}\qquad &= \quad \vdots\\
&= x^{r^5}
\end{aligned}
$$

Which means that $r^5 \equiv 1 \bmod 11$. Hence the integers 2, 6, 7, 8 and 10 are not possible solutions to this equation while 1, 3, 4, 5, or 9 are possible solutions.

Question:

c) Prove that the remaining values are possible, and there are two isomorphism classes of groups of order 55.

Answer:

c) Let $n_5 = 1$ then $K = \langle y \rangle \lhd G$ and from proposition 2.8.6 we have that $G \approx H \times K \approx C_{11} \times C_5 \approx C_{55}$ and thus G is the cyclic (abelian) group of order 55. If $n_5 \neq 1$ then clearly K is not a normal subgroup of G, and thus G is not abelian. We claim that all the groups $G_i = \langle x,\ y\ ;\ x^{11}, y^5,\ yxy^{-1}x^{-i} \rangle$ are isomorphic for $i = 3, 4, 5, 9$. One way to do this is to show that say G_i for $i = 3, 4, 5$ are isomorphic to G_9. So in G_9 we have that $yxy^{-1} = x^9$. Notice that in G_3 we have $yxy^{-1} = x^3$. Substituting y^2 for y we have that $y^2 x y^{-2} = yx^3 y^{-1} = x^9$. But y^2 is also a generator for $K = \langle y \rangle$. Thus $G_3 = \langle x,\ y';\ x^{11},\ (y')^5,\ y'x(y')^{-1} x^{-9} \rangle \approx G_9$, where $y' = y^2$. Similarly for G_4 substitute y^3 for y, and in G_5 y^4 for y, to obtain $G_4 \approx G_5 \approx G_3 \approx G_9$.

Section 6

Question:

18. Prove that if a proper normal subgroup of S_n contains a 3-cyle, it is A_n.

Answer:

Let $H \lhd S_n$ be a proper normal subgroup of S_n for $n \geq 3$ and assume that $\sigma \in H$, where σ is a 3-cycle.

Since $H \lhd S_n$ we have that $\tau H \tau^{-1} = H,\ \forall \tau \in S_n$, which means that $\tau \sigma \tau^{-1} \in H,\ \forall \tau \in S_n$, hence that H contains all 3-cyles of S_n since 3-cycles are all conjugate to one another. Thus, we have that $A_n \leq H$; but since $[S_n : A_n] = 2$ and since H is a proper subgroup of S_n, we have that $A_n = H$. Note, in this proof we are assuming the result of exercise #17 that says that A_n is generated by 3-cycles.

Section 8

Question:

9. The *commutator subgroup C* of a group G is the smallest subgroup containing all commutators.
   a) Prove that the commutator subgroup is a characteristic subgroup.

Answer:

a) Let G be a group and $C = \langle \{ aba^{-1}b^{-1} \mid a,\ b \in G \} \rangle$ be the smallest subgroup of G that contains C; this means that C is generated by elements of the form $(aba^{-1}b^{-1})^i$ for $i = 1,$ or -1, note that if $i = -1$ $(aba^{-1}b^{-1})^{-1} = bab^{-1}a^{-1}$ which is of the form $cdc^{-1}d^{-1}$. Let $\gamma \in \mathrm{Aut}(G)$. Then $\gamma(aba^{-1}b^{-1}) = \gamma(a)\gamma(b)\gamma(a)^{-1}\gamma(b)^{-1} \in C$. Since γ is a group automorphism if the generators of C are mapped into C then $\gamma(C) = C$.

Question:

b) Prove that G/C is an abelian group.

Answer:

b) Since $\gamma \in \text{Aut}(G)$, then for all inner automorphisms I_g we have that $I_g(C) = gCg^{-1} = C$. Thus $C \triangleleft G$. Next, consider $G/C \cdot \forall\ a,b \in G$ we have that

$$
\begin{aligned}
aba^{-1}b^{-1} \in C \Rightarrow aba^{-1}b^{-1}C &= C \\
\Rightarrow ab(ba)^{-1}C &= C \\
\Rightarrow abC(ba)^{-1}C &= C \\
\Rightarrow (ab)C &= (ba)C \\
\Rightarrow aC \cdot bC &= bC \cdot aC
\end{aligned}
$$

which means that G/C is abelian

Question:

14. Let *N* be a normal subgroup of a group *G*. Prove that G/N is abelian if and only if *N* contains the commutator subgroup of *G*.

Answer:

Let $N \triangleleft G$. First we show that if $N \supseteq C$, where *C* is the commutator subgroup of *G*, then G/N is abelian. Since $N \supset C$, then for all $a,b \in G$ we have that $aba^{-1}b^{-1} \in N$. Then proceed as in the previous exercise.

Next we show that if G/N is abelian then *N* contains *C*. $\forall\ a,b \in G$ we have that

$$
\begin{aligned}
aNbN = bNaN \Rightarrow abN &= baN \\
\Rightarrow (ba)\,abN &= N \\
\Rightarrow a^{-1}b^{-1}abN &= N \\
\Rightarrow a^{-1}b^{-1}ab &\in N\ \forall\ a,b \in G
\end{aligned}
$$

$\Rightarrow N$ contains C.

Miscellaneous Problems

Question:

7. Let H, N be subgroups of a group G, and assume that N is a normal subgroup.
   a) Determine the kernels of the restrictions of the canonical homomorphism $\pi : G \to G/N$ to the subgroups H and HN.
   b) Apply the First Isomorphism Theorem to these restrictions to prove the *Second Isomorphism Theorem*: $H/H(H \cap N)$ is isomorphic to $(HN)/N$.

Answers:

See Fraleigh's book: *A first course in Abstract Algebra*, page 211.

Question:

8. Let H, N be normal subgroups of a group G such that $H \supset N$, and let $\overline{H} = H/N$, $\overline{G} = G/N$.
   a) Prove that $\overline{H}$ is a normal subgroup of $\overline{G}$.

   c) Use the composed homomorphism $G \to \overline{G} \to \overline{G}/\overline{H}$ to prove the *Third Isomorphism Theorem*: G/H is isomorphic to $\overline{G}/\overline{H}$.

Answers:

See Fraleigh's book: *A first course in Abstract Algebra*, page 211.