Math 721 - Final Exam Version 1 - December 13, 2011

Name: ______________________________

Question Number	Possible Points	Score
1	19	
2	19	
3	19	
4	19	
5	19	
6	19	
7	19	
8	19	
9	19	
10	19	
11	10	
Total	200	

Instructions:

- You MAY NOT use the book, notes, other people's work, or other people's brains in the course of taking this exam.
- You MAY quote the result of any homework problem that has been assigned in this course.
- You MAY have fun.
- You MAY have an excellent winter break.
- 42.

1. (19 points).

(a) (6 points). State the definition of a group.

(b) (4 points). What does it mean for a group to be abelian?

(c) (4 points). What does it mean for a group to be cyclic?

(d) (5 points). True or false: In an abelian group G with order 2^n that contains an element x of order 2^k, then for any element y of order 2^{k-1}, there is an element $z \in G$ so that $z^2 = y$.

2. (19 points).

(a) (5 points). If g_1 and g_2 are elements of a group G, what does it mean to say that g_1 and g_2 are conjugate? Prove that conjugacy is an equivalence relation.

(b) (6 points). Prove that if g_1 and g_2 are conjugate, then g_1 and g_2 have the same order.

(c) (8 points). Suppose that G is a finite group and $g \in G$ has exactly m conjugates. Prove that m divides $|G|$. Prove that G has a subgroup of index m.

3. (19 points).

(a) (6 points). State the definition of a ring homomorphism. State the definition of an ideal (you may state whichever version of the definition you like).

(b) (7 points). Explain why the kernel of a ring homomorphism is an ideal. Is every ideal the kernel of some ring homomorphism?

(c) (6 points). State the correspondence theorem for rings.

4. (19 points). Is there a finite group G containing two elements x and y so that

- x and y both have order 2
- xy has order 4?

Give an example of such a group G and elements x and y, or prove that no such objects exist.

5. (19 points).

(a) (4 points). State Lagrange's theorem.

(b) (15 points). Prove Lagrange's theorem. [Prove any properties of cosets you use.]

6. (19 points). Suppose that G acts transitively on a set S and s_1 and s_2 are two elements of S. Let K_1 be the stabilizer of s_1 and K_2 be the stabilizer of s_2. Prove that K_1 and K_2 are conjugate subgroups of G. [You may assume that K_1 and K_2 are in fact subgroups.]

7. (19 points). Prove the second Sylow theorem. This states that if G is a finite group and p divides $|G|$, then all the Sylow p-subgroups of G are conjugate, and every subgroup of G whose order is a power of p is contained in some Sylow p-subgroup. [You may use the first Sylow theorem in your proof.]

8. (19 points). Prove that every group of order 721 is cyclic. [Hint: The number 103 is prime.
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9. (19 points). Let R be the set of continuous functions $f : [0,1] \to \mathbb{R}$, and define addition and multiplication on R by setting

$$(f+g)(x) = f(x) + g(x), (f \cdot g)(x) = f(x) \cdot g(x).$$

(a) (8 points). List the properties that would need to be checked to prove that R is a ring.

(b) (6 points). Let $\phi : R \to \mathbb{R}$ be given by $\phi(f) = f(1/2)$. Prove that ϕ is a homomorphism.

(c) (5 points). Is $\ker \phi$ a principal ideal of R? [Hint: You may use that if $f(x) \in R$, then $f(x)^{1/3} \in R$.]

10. (19 points). Suppose that R and S are rings and $\phi : R \to S$ is a surjective ring homomorphism with $\ker \phi = I$. Prove that if J is a *prime* ideal of R containing I, then $\phi(J)$ is a prime ideal of S.

11. (10 points). Please wait to answer this question until you have finished your work on all other problems on the exam. You may also work on this question once the three hours alloted for the final exam have finished.

Please comment on your understanding of each of the first ten problems on this exam. Did you finish the problem? If so, are you confident that your work is correct? [One point will be for each problem. If your work is correct and you are confident, then you'll earn 1 point for that problem. If either your work is correct and you're not confident, or you say you're confident and your work is not correct, you don't get the point. If you don't finish or you say you're not confident and your work is incorrect, you get the point.]