Jeremy Rouse's Math 711 homework

Chapter 2

1.1. Suppose that X is a complete metric space and let $i : X \to \tilde{X}$ be the map $i(x) = [(x, x, \ldots)]$ which sends x to the constant sequence consisting of x's. Proposition 1.3 from the course notes implies that i is an isometry. It is clear that i is injective. It suffices to show that i is surjective.

Let y be an arbitrary element of $\tilde{X}$ and let (z^n) be a Cauchy sequence of elements in X that is an element of y. Since X is complete, $\lim_{n\to\infty} z^n = z$ converges. This is equivalent to $\lim_{n\to\infty} d(z^n, z) = 0$. This implies that the sequence (z^n) is equivalent to the constant sequence $(z, z, z, \ldots)$. It follows that $i(z) = y$ and so i is surjective.

1.2. Let $X = \mathbb{Q}$, $\tilde{X} = \tilde{\mathbb{Q}}$ and $Y = \mathbb{R}$. Since the reals are complete as proved in Math 611, we can define a map $\phi : \tilde{X} \to Y$ by $\phi([(p_n)]) = \lim_{n\to\infty} p_n$. Note that this is well-defined, since if (p_n) and (q_n) are equivalent, then

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} (q_n - p_n) + \lim_{n\to\infty} p_n = \lim_{n\to\infty} p_n.$$

Next, I claim that ϕ is an isometry. We have that

$$\begin{aligned} d_{\tilde{X}}([(p_n)], [(q_n)]) &= \lim_{n\to\infty} d(p_n, q_n) \\ &= \lim_{n\to\infty} |p_n - q_n| \\ &= d_Y(\phi([p_n]), \phi([q_n])). \end{aligned}$$

Finally, we must show that ϕ is a bijection. If $\phi([(p_n)]) = \phi([(q_n)])$ then $\lim_{n\to\infty} p_n - q_n = 0$, which is equivalent to $d_{\tilde{X}}(p_n, q_n) \to 0$ as $n \to \infty$, in other words $[(p_n)] = [(q_n)]$. Hence ϕ is injective.

Finally, since $\mathbb{Q}$ is dense in $\mathbb{R}$, for any real number r, there is a sequence of rational numbers p_1, p_2, $\ldots$ that converges to r, and this implies that $\phi([(p_n)]) = r$.

1.3. I will interpret this question to mean functions $f \in L^2(C([0,1]))$ that are not in the image of the isometry $i : C([0,1]) \to L^2(C([0,1]))$. One such example is given by the Cauchy sequence of functions

$$f_n(x) = \begin{cases} (2x)^n & 0 \le x \le 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$$

described in a previous exercise.

Let

$$f_n(x) = \frac{4}{\pi} \sum_{k=0}^{n} \frac{1}{2k+1} \sin\left(2(2k+1)\pi x\right).$$

Then, for $n \geq m$ we have

$$(f_n(x) - f_m(x))^2 = \left(\sum_{k=m+1}^{n} \frac{1}{2k+1} \sin\left(2(2k+1)\pi x\right) \right)^2.$$

Integrating from 0 to 1 and using the fact that

$$\int_0^1 \sin(2\pi m x) \sin(2\pi n x)\, dx = 0$$

if $m \neq n$ gives that

$$\begin{aligned} \|f_n - f_m\|_2^2 &= \sum_{k=m+1}^{n} \frac{1}{(2k+1)^2} \int_0^1 \sin((4k+2)\pi x)^2\, dx \\ &\leq \sum_{k=m+1}^{n} \frac{1}{(2k+1)^2} \\ &\leq \int_m^\infty \frac{1}{x^2}\, dx \leq \frac{1}{m}. \end{aligned}$$

Hence if $\epsilon > 0$ then for $K > 1/\epsilon^2$ and $m, n \geq K$, we have $\|f_n - f_m\|_2 \leq \frac{1}{\sqrt{m}} < \epsilon$. Hence (f_n) is a Cauchy sequence.

Note however that the $f_n(x)$ is a partial sum for the Fourier series of

$$f(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ 0 & 1/2 \leq x \leq 1. \end{cases}$$

Also, by the Riesz-Fischer theorem, for any function in L^2, the Fourier series converges to it (in the L^2-norm). Hence $\|f_n - f\|_2 \to 0$ as $n \to \infty$. Again, this implies (as in the same argument in the proof of Proposition 2.9) that there is no function $g \in C([0,1])$ so that $\|f_n - g\|_2 \to 0$.

1.4. First, since the function $f(x) = x^{1/p}$ is continuous, $\left(\lim_{n\to\infty} \int_0^1 |f_n(x)|^p\, dx\right)^{1/p} = \lim_{n\to\infty} \|f_n\|_p$.

Suppose that (f_n) and (g_n) are two equivalent Cauchy sequences. Then $\|f_n - g_n\|_p \to 0$ as $n \to \infty$. We have that

$$\|f_n\|_p - \|g_n - f_n\|_p \leq \|g_n\|_p \leq \|f_n\|_p + \|g_n - f_n\|_p$$

and the squeeze theorem now implies that $\lim_{n\to\infty} \|f_n\|_p = \lim_{n\to\infty} \|g_n\| p$. Hence, $\|\cdot\|_p$ is well-defined on $L^p([0,1])$.

If (f_n) and (h_n) are in L^p, then

$$\begin{aligned}\|[(f_n+h_n)]\|_p &= \lim_{n\to\infty}\|f_n+h_n\|_p \\ &\le \lim_{n\to\infty}\|f_n\|_p+\|h_n\|_p \\ &= \lim_{n\to\infty}\|f_n\|_p+\lim_{n\to\infty}\|h_n\|_p \\ &= \|[(f_n)]\|_p+\|[(h_n)]\|_p.\end{aligned}$$

It is also easy to see that $\|[(\alpha f_n)]\|_p = |\alpha|\|[(f_n)]\|_p$.

Finally, if $\lim_{n\to\infty}\|f_n\|_p = 0$, then $[(f_n)]$ is equivalent to $[(0)]$ because

$$\tilde{d}([(f_n)],[(0)]) = \lim_{n\to\infty} d(f_n,0) = \lim_{n\to\infty}\|f_n\|^p = 0$$

and hence $\|\cdot\|_p$ is a norm. Also, the metric on $L^p([0,1])$ is defined by $\lim_{n\to\infty} d(p_n,q_n) = \lim_{n\to\infty}\|p_n-q_n\|_p$ which is $\|[(p_n)]-[(q_n)]\|_p$, so this norm does indeed generate the metric topology.

1.5. Any continuous function on S will be in $L^2(S)$. Also, $f(x,y)=\frac{1}{\sqrt{1-xy}}$ will be in $L^2(S)$, even though f is not continuous. We have

$$\iint_S f(x,y)^2\,dA = \int_0^1\int_0^1 \frac{1}{1-xy}\,dy\,dx.$$

Expanding

$$\frac{1}{1-xy} = \sum_{n=0}^{\infty}(xy)^n$$

and integrating termwise (which can be justified since Taylor series for analytic functions converge normally), we get that

$$\int_0^1\int_0^1\frac{1}{1-xy}\,dy\,dx = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Let

$$f_n(x,y) = \begin{cases}\frac{1}{\sqrt{1-xy}} & \text{if } xy \le \frac{n-1}{n}\\ \sqrt{n} & \text{if } xy \ge \frac{n-1}{n}.\end{cases}$$

Note that the area of $\{(x,y) \in S : xy \geq (n-1)/n$ is

$$\begin{aligned}\int_{\frac{n-1}{n}}^{1} \left(1 - \frac{n-1}{nx}\right) dx &= \left[x - \frac{n-1}{n}\ln(x)\right]_{\frac{n-1}{n}}^{1} \\ &= 1 - \frac{n-1}{n}\ln(1) - \frac{n-1}{n} + \frac{n-1}{n}\ln\left(\frac{n-1}{n}\right) \\ &= \frac{1}{n} + \left(1 - \frac{1}{n}\right)\ln\left(1 - \frac{1}{n}\right) \\ &= \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \\ &\leq \frac{C}{n^2}\end{aligned}$$

for some absolute constant C. In the last step we used the Taylor series expansion of $\ln(1-x)$. It follows that if $m \geq n$ then

$$\|f_n - f_m\|_2 \leq \sqrt{\int_{xy \geq (n-1)/n} n \, dA} \leq \frac{C}{\sqrt{n}}.$$

It follows that the $f_n(x,y)$ are a Cauchy sequence of functions in $C(S)$ that converge pointwise to $f(x,y)$.

The function $f(x,y) = \frac{1}{x^2+y^2}$ is not in $L^2(S)$, since

$$\iint_S f(x,y)^2 \, dA \geq \int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)^2 \, dy \, dx = \frac{\pi}{4} \int_0^1 \frac{1}{r^4} r \, dr$$

diverges.

1.6. Suppose that (x_n^k) is a Cauchy sequence in l^p. Since $|x_n^k - x_n^j| \leq \|x^k - x^j\|_p$, it follows that for a fixed n, (x_n^k) is a Cauchy sequence in $\mathbb{R}$. Define $x_n = \lim_{n \to \infty} x_n^k$. We must show that $x \in l^p$ and that $x^k \to x$ in l^p.

Fix a positive integer N and $\epsilon > 0$. For $1 \leq n \leq N$, let K_n be such that $|x_n^k - x_n| < \epsilon / N^{1/p}$. Let $K = \max\{K_n : 1 \leq n \leq N\}$. Then for $k > N$, we have

$$\begin{aligned}\left(\sum_{n=1}^{N} x_n^p\right)^{\frac{1}{p}} &\leq \left(\sum_{n=1}^{N} |x_n - x_n^k|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |x_n^k|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{N} \frac{\epsilon^p}{N}\right)^{\frac{1}{p}} + \|x^k\|_p \\ &\leq \epsilon + \|x^k\|_p.\end{aligned}$$

I use Minkowski's inequality to derive the first inequality.

Since $\{x^k\}$ is Cauchy, it is bounded, and it follows (by taking $\epsilon = 1$ and $M = \max_k \|x^k\|_p$) that

$$\left(\sum_{n=1}^{N} x_n^p\right)^{\frac{1}{p}} \leq 1 + M.$$

This is true for all N, and so $\|x\|_p \leq 1 + M$ and so $x \in l^p$.

Now, we'll show that $x^k \to x$ in l^p. Fix $\epsilon > 0$ and choose K so that if $j, k \geq K$, then $\|x^k - x^j\| < \epsilon/2$. Fix a positive integer N. We will show that

$$\left(\sum_{n=1}^{N} |x_n - x_n^k|^p\right)^{1/p} < \epsilon.$$

Each of the sequences (x_1^k), (x_2^k), …, (x_N^k) converge, and so we can find a positive integer $j \geq K$ so that $|x_n - x_n^j| < \epsilon/(2N^{1/p})$ for $1 \leq n \leq N$. We then have

$$\begin{aligned}
\left(\sum_{n=1}^{N} |x_n - x_n^k|^p\right)^{\frac{1}{p}} &\leq \left(\sum_{n=1}^{N} |x_n - x_n^j|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |x_n^j - x_n^k|^p\right)^{\frac{1}{p}} \\
&< (\epsilon^p/2^p)^{1/p} + \|x^j - x^k\|_p \\
&< (\epsilon/2) + (\epsilon/2) < \epsilon.
\end{aligned}$$

Hence $x^k \to x$ in l^p.

1.7. If $f \in X$, let $\|f\| = \|f\|_1 + \|f\|_2$. We have

$$\begin{aligned}
\|\alpha f\| &= \|\alpha f\|_1 + \|\alpha f\|_2 \\
&= |\alpha|\|f\|_1 + |\alpha|\|f\|_2 \\
&= |\alpha|\|f\|.
\end{aligned}$$

If $f = 0$, then $\|f\| = \|f\|_1 + \|f\|_2 = 0$. For a general f we have $\|f\| = \|f\|_1 + \|f\|_2 \geq 0$. If $\|f\| = 0$, then $\|f\|_1 = 0$ which implies that $f = 0$, since $\|\cdot\|_1$ is a norm.

Finally, for $f, g \in X$, the triangle inequalities for $\|\cdot\|_1$ and $\|\cdot\|_2$ imply that

$$\begin{aligned}
\|f + g\| &= \|f + g\|_1 + \|f + g\|_2 \\
&\leq \|f\|_1 + \|g\|_1 + \|f\|_2 + \|g\|_2 \\
&\leq \|f\| + \|g\|,
\end{aligned}$$

as desired. Thus, $\|f\|$ is a norm.

2.1. Fix $\epsilon > 0$ and $x \in X$ and let $\delta = \epsilon/r$. If $d(x, y) < \delta$, then

$$d(f(x), f(y)) \leq r d(x, y) < r\delta = \epsilon.$$

Hence f is continuous. (In fact, f is uniformly continuous.)

2.2. Let $y(t) = \frac{1}{1/2-t}$. Then $y(0) = 1/(1/2) = 2$ and

$$y'(t) = \frac{0 \cdot (1/2 - t) - (-1) \cdot 1}{(1/2 - t)^2} = \frac{1}{(1/2 - t)^2} = y(t)^2.$$

2.3. If $y(t) = t^2/4$, then $y'(t) = 2t/4 = t/2 = \sqrt{y(t)}$. Also, if $y(t) = 0$, then $y'(t) = 0 = \sqrt{y(t)}$.

2.4. Let X be complete and $Y \subseteq X$ be closed. Let (y_n) be a Cauchy sequence in Y. Since X is complete, the sequence (y_n) converges in X and so there is an $x \in X$ with $\lim_{n\to\infty} y_n = x$. We must now prove that $x \in Y$. If $y_n = x$ for some $n \geq 1$, then $x = y_n \in Y$ and we're done. If $y_n \neq x$ for any $n \geq 1$, then x is an accumulation point of Y (since it is the limit of a sequence of points none of which equals x). Since Y is closed, this implies that $x \in Y$, as desired. This proves that Y is complete.

3.1. See the attached Maple printout.

3.2. The definition of "integrable" in this Chapter is too fluffy for me to know precisely what is meant. For this reason, I will not attempt to show that

$$\int_{-\infty}^{\infty} |f * g(x)|\, dx$$

is well-defined. We then have

$$\begin{aligned} \int_{-\infty}^{\infty} |f * g(x)|\, dx &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(y) g(x-y)\, dy \right| dx \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)||g(x-y)|\, dy\, dx. \end{aligned}$$

Applying Fubini's theorem gives

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)||g(x-y)|\, dx\, dy \\ &= \int_{-\infty}^{\infty} |f(y)| \int_{-\infty}^{\infty} |g(x-y)|\, dx\, dy \\ &= \int_{-\infty}^{\infty} |f(y)| \left(\int_{-\infty}^{\infty} |g(u)|\, du \right) dy \\ &= \left(\int_{-\infty}^{\infty} |f(y)|\, dy \right) \cdot \left(\int_{-\infty}^{\infty} |g(u)|\, du \right) \end{aligned}$$

which is the desired result.

Evaluation of c_n: I wanted to point out that

$$c_n = \frac{2^{2n+3} n! (n+1)!}{(2n+2)!}.$$

This follows by setting $r = x/2$ and finding that

$$c_n = 2\int_{-1}^{1} (1-r^2)^n \, dr.$$

If we let $f(m,n) = \int_{-1}^{1} r^{2m}(1-r^2)^n \, dr$ then integrating by parts shows that for $n > 0$ we have

$$f(m,n) = \frac{2n}{2m+1} f(m+1, n-1)$$

and $f(m,0) = \frac{2}{2m+1}$. Combining this gives that

$$c_n = 2f(n,0) = 2 \cdot \left(\frac{2n \cdot (2n-2) \cdot (2n-4) \cdots 2 \cdot 2}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \right).$$

If $P = 1 \cdot 3 \cdot 5 \cdots (2n+1)$, then

$$2^{n+1}(n+1)!P = 1 \cdot 2 \cdot 3 \cdots (2n+1) \cdot (2n+2) = (2n+2)!$$

and so $P = \frac{(2n+2)!}{2^{n+1}(n+1)!}$. Thus, we have

$$c_n = 2 \cdot \frac{2^n n! \cdot 2}{(2n+2)!/(2^{n+1}(n+1)!)} = \frac{2^{2n+3} n!(n+1)!}{(2n+2)!}.$$

Stirling's approximation gives that $c_n \sim \frac{2\sqrt{\pi}}{\sqrt{n}}$. In fact, $c_n \le \frac{2\sqrt{\pi}}{\sqrt{n}}$. This follows from noting that $1 - \frac{x^2}{4} \le e^{-x^2/4}$ for $|x| \le 2$ using the alternating series test for the Taylor series for $e^{-x^2/4}$. This gives

$$\begin{aligned} c_n &\le \int_{-2}^{2} (e^{-x^2/4})^n \, dx \\ &\le \int_{-\infty}^{\infty} e^{-nx^2/4} \, dx. \end{aligned}$$

Setting $u = \sqrt{n}/2x$ we get

$$c_n \le \frac{2}{\sqrt{n}} \int_{-\infty}^{\infty} e^{-u^2} \, du = \frac{2\sqrt{\pi}}{\sqrt{n}},$$

as claimed.

3.3. I didn't want to use L'Hopital's rule.

Lemma: For $x > 0$ we have $\ln(x) \le \frac{2}{e}\sqrt{x}$.

Proof: Let $f(x) = \frac{\ln(x)}{\sqrt{x}}$. Then,

$$\begin{aligned} f'(x) &= \frac{(1/x)\sqrt{x} - \frac{1}{2}x^{-1/2}\ln(x)}{x} \\ &= \frac{1 - \frac{1}{2}\ln(x)}{x^{3/2}}. \end{aligned}$$

Hence, $f'(x) > 0$ if $x < e^2$ and $f'(x) < 0$ if $x > e^2$ and so the maximum value of $f(x)$ is $f(e^2) = 2/e$. QED Lemma

Now, if $f(n) = 3\sqrt{n}r^n$, then

$$\begin{aligned} \ln f(n) &= \ln(3) + \frac{1}{2}\ln(n) + n\ln(r) \\ &\leq \ln(3) + \frac{1}{e}\sqrt{n} + n\ln(r). \end{aligned}$$

This implies that $\lim_{n\to\infty} \frac{\ln f(n)}{n} = \ln(r)$. Fix $\epsilon > 0$ and choose N large enough so that (i) for $n \geq N$, $\ln f(n)/n \leq (1/2)\ln(r)$ and (ii) $N \geq \frac{2\ln(\epsilon)}{\ln(r)}$. Then,

$$\ln f(N) \leq \frac{1}{2}N\ln(r) \leq \ln(\epsilon)$$

and so $f(N) < \epsilon$. This proves that $\lim_{n\to\infty} f(n) = 0$.

3.4. Suppose that $f \in C([0,1])$. Let (f_n) be a sequence of polynomials that converges uniformly to f on $[0,1]$. Fix $\epsilon > 0$ and choose f_n so that $|f(x) - f_n(x)| < \epsilon$ for all $x \in [0,1]$. Then $\|f - f_n\|_\infty < \epsilon$. This implies that the polynomials are dense in $C([0,1])$.

Conversely, suppose that the polynomials are dense in $C([0,1])$. Fix $f \in C([0,1])$ and for each positive integer n choose a polynomial f_n so that $\|f_n - f\|_\infty < 1/n$. For any $\epsilon > 0$, if we choose N so that $\frac{1}{N} < \epsilon$ then for any $n \geq N$ and $x \in [0,1]$ we have

$$|f_n(x) - f(x)| \leq \|f_n - f\|_\infty \leq \frac{1}{N} < \epsilon$$

and so $f_n \to f$ uniformly in $[0,1]$.

3.5. See the attached Maple printouts.

3.6. See the attached Maple printouts.

3.7. I looked up the proof of the Weierstrass approximation theorem using Bernstein polynomials in Ross. The averaging in this case involves values of f multiplied by polynomials "concentrated" at the corresponding input to f.