A Bad Introduction to Game Theory

CSC 790

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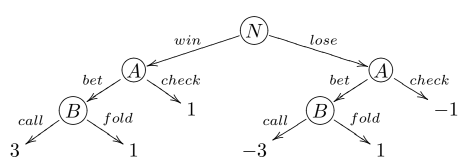
Game Theory

- Type of applied mathematics used in social sciences and economics
  - Model behavior in strategic situations, in which an individual's success in making choices depends on the choices of others
  - Three mathematical forms: extensive, normal, and characteristic
- Extensive form represents the *game tree*
  - Each node is a state of play, terminal nodes represent payoffs



Two players A and B, both but one dollar in the ante. Player A is dealt a card facedown and can bet or check. If player A checks, the the card is inspected; if its a winning card player wins ante, otherwise player A loses. If player A bets, player A must put 2 more dollars in the ante. Then player B not knowing the card must fold or call. If player B folds they lose the ante, if player B calls then player B must put 3 dollars in the ante. The card is then inspected and winner of the ante determined.

- Normal form is typically represented by a matrix
  - Matrix shows the players, strategies, and payoffs

		Player B	
		left	right
Player A	up	$(5, 3)$	$(-1, -1)$
	down	$(0, 0)$	$(3, 4)$

For example, player A can move up or down while player B can move left or right. If player A moves up and player B moves left, then player A gets payoff of 5 and player B gets payoff of 3...

- General formulation requires the game to be defined using sets

Normal or Strategic Form

- A two-person zero-sum game can be defined by the triplet (X, Y, F)
  - X is a non-empty set, set of strategies of player A
  - Y is a non-empty set, set of strategies of player B
  - F is a real-valued function defined on $X \times Y$, thus $F(x, y)$ is a real number for every $x \in X$ and $y \in Y$
- Interpretation is as follows
  - Simultaneously player A chooses $x \in X$ and player B chooses $y \in Y$
  - $F(x, y)$ is the resulting reward given to player A (if negative...)
- Although a simple definition, it can describe any finite combination game such as tic-tac-toe, chess, or #!& who smoked my stash? ...

Are You Even or Odd?

- Two players, A and B, simultaneously call out the number 1 or 2
  - If the resulting sum is odd, player A wins; otherwise player B wins
  - Amount given to the player is the sum of the numbers
- Therefore $X = \{1, 2\}$, $Y = \{1, 2\}$, and

		Player B	
		1	2
Player A	1	(-2, 2)	(3, -3)
	2	(3, -3)	(-4, 4)

can be rewritten as

		Player B	
		1	2
Player A	1	-2	+3
	2	+3	-4

Does either player have an advantage?

Consider Player A Strategies

- Suppose player A calls '1' $\frac{3}{5}$ of the time and '2' $\frac{2}{5}$ of the time
  - If player B calls '1', player A loses 2 dollars $\frac{3}{5}$ of the time and wins 3 dollars $\frac{2}{5}$ of the time, so on the average player A breaks even

$$-2\frac{3}{5} + 3\frac{2}{5} = 0$$

  - If player B calls '2', player A wins 3 dollars $\frac{3}{5}$ of the time and loses 4 dollars $\frac{2}{5}$ of the time, so on the average player A wins

$$3\frac{3}{5} - 4\frac{2}{5} = \frac{1}{5}$$

- Therefore if player A is guaranteed to at least break even no matter what player$B$ does using this mixed strategy

  Can player A play the game such that they always win?

Equalizing Strategy

- Let p denote the proportion of times player A calls '1'
  - Choose p so that player A always wins the same amount
  - If player B calls '1' then winnings are $-2p + 3(1-p)$, if player B calls '2' then winnings are $3p - 4(1-p)$

$$-2p + 3(1-p) = 3p - 4(1-p)$$

$$3 - 5p = 7p - 4$$

$$12p = 7$$

$$p = \frac{7}{12}$$

- Therefore player A should call '1' with probability $\frac{7}{12}$, otherwise '2'
  - Player A will win on average $-2\frac{7}{12} + 3\frac{5}{12} = 8\frac{1}{3}$
  - Called the **equalizing strategy**

Player A is a Winner (with or without a yellow shirt)

- The previous game is in favour of player A
  - Can do better than $8\frac{1}{3}$ per game if player B does not play properly, but player A can do worse...
  - If player B plays optimally, then they lose $\frac{1}{12}$ on average, which is the **value** of the game
- The procedure that produces the **value** is called the **optimal strategy** or a **minimax strategy**

Pure and Mixed

- There are *pure* and *mixed* strategies
  - Refer to the elements of X and Y as pure strategies
  - Choosing at random among pure strategies is a **mixed strategy**
- In the previous game, player A used a mixed strategy
  - Also assumed the player only cares about the long term average
  - For example, the player would be indifferent between 5 million dollars guaranteed, versus flipping a coin and receiving 10 million with probability $\frac{1}{2}$ and nothing with probability $\frac{1}{2}$
- Utility theory is a better basis for the expected payoff
  - Premise is that a player's value of money is not linear (*it's what makes game shows interesting after all...*)
  - A utility function maps happiness/satisfication to an outcome

Minimax

- A two-person zero-sum game (X, Y, F) is finite if X and Y are finite
  - Therefore the even-odd game was finite
- Fundamental theorem of game theory for zero-sum finite games
  - There is a number V, called the value of the game
  - There is a mixed strategy for player A such that A's average gain is at least V no matter what B does, and
  - There is a mixed strategy for player B such that B's average loss is at most V no matter what A does
  - *Thanks von Neumann...*

What is the implication if V is zero, positive, or negative?

Matrix Games

- Finite two-person zero-sum game in a strategic form is a matrix game
  - Payoff function F can be represented as a matrix

$$F = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{n,m} \end{pmatrix}$$

    where $a_{i,j} = F(x_i, y_j)$
  - In this form player A chooses a row, player B chooses a column
- Mixed strategy for player A can be represented as a m-tuple, $p = (p_1, p_2, ..., p_m)$ of probabilities that sum to 1
- Mixed strategy for player B can be represented as a n-tuple, $q = (q_1, q_2, ..., q_m)$ of probabilities that sum to 1

- Using the matrix, the payoffs can be determined
  - If player A uses mixed strategy p and player B chooses column j (pure strategy) then payoff to A is

$$\sum_{i=1}^{m} p_i a_{i,j}$$

  - If player A uses p and player B uses q then the payoff to A is

$$p^\mathsf{T} A q = \sum_{1=1}^{m} \sum_{j=1}^{n} p_i a_{i,j} q_j$$

- Using this representation it may be possible to *solve* the game, which finds the optimal strategy

Saddles, yee haw...

- Occasionally it is easy to solve the game, an entry $a_{i,j}$ in F is called a *saddle point* if it has the following properties
  - $a_{i,j}$ is the minimum of the i^{th} row, and
  - $a_{i,j}$ is the maximum of the j^{th} column
- At the saddle point player A can win at least $a_{i,j}$ by choosing row i and player B can keep the loses to at most $a_{i,j}$ by choosing column j
- For example,

$$F = \begin{pmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{pmatrix}$$

  - The entry $a_{1,1} = 2$ is the saddle point
  - Value of game is 2, and $(0, 1, 0)$ is optimal mixed strategy for both

- For large $m \times n$ matrices it can be tedious to check entries
  - Easier to compute min of each row and max of each column, then check if they match

Example 1

					row min
	3	2	1	0	0
	0	1	2	0	0
	1	0	2	1	0
	3	1	2	2	1
col max	3	2	2	2	

Example 2

					row min
	3	1	1	0	0
	0	1	2	0	0
	1	0	2	1	0
	3	1	2	2	1
col max	3	1	2	2	

  - In the first example, no row minimum is equal to any column maximum, so no saddle point
  - In the second example, the minimum of the fourth row is equal to the maximum of the second column, so $a_{3,1} = 1$ is the saddle point *(assume start counting at zero...)*

The Solution to All 2×2 Matrix Games

- To solve for the general 2×2 matrix games, try
  - Test for saddle point
  - If no saddle point, solve by finding equalization strategies
- But many games (more interesting ones) are larger than 2×2
  - Sometimes these can be reduced to an equivalent 2×2 game
  - Delete rows and columns which are obviously bad for a player

  Dominates: Row i of F *dominates* row k if $a_{i,j} \geq a_{k,j}, \forall j$. Row i *strictly dominates* row k if if $a_{i,j} > a_{k,j}, \forall j$. Similarly, the column j of A dominates column k if $a_{i,j} \leq a_{i,k}, \forall i$ (strictly $a_{i,j} < a_{i,k}$).

- Anything that player A can achieve using a dominated row can be achieve at least as well using the row it dominates, therefore dominated rows can be removed. Similar argument for player B.

- For example,

$$F = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$$

- Last column is dominated by the middle so remove, giving

$$F = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{pmatrix}$$

- Top row is dominated by the bottom row so remove, giving

$$F = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$