Math 711 Course Notes

Dr. Sarah Raynor[1]

[1]These notes were composed by Dr. Robinson last year and edited by his class. I will be editing them for our course as we proceed.

Contents

CHAPTER 1

Metric Spaces

We will begin the course by defining a metric space and look at some classic examples. This will include a review of the metric properties of $\mathbb{R}^n$.

1. Definitions

1.1. Definition of a Metric Space and Basic Examples.

DEFINITION 1.1. *A* **metric space** *is a pair* (X, d) *where* X *is a set and* $d : X \times X \to \mathbb{R}^+$ *is a function with the following properties:*

- : *(ia)* $d(x, y) \geq 0 \qquad \forall x, y \in X$
- : *(ib)* $d(x, y) = d(y, x) \qquad \forall x, y \in X$
- : *(ii)* $d(x, y) = 0 \iff x = y$
- : *(iii)* $d(x, z) \leq d(x, y) + d(y, z) \qquad \forall x, y, z \in X$

The function d *is called a* **metric**, *and it is common to refer to "the metric space* X*", when the definition of* d *is already understood.*

EXAMPLE 1.1. *The space* $\mathbb{R}$ *is a metric space with metric* $d(x, y) = |y - x|$.

EXAMPLE 1.2. *The space* $\mathbb{R}^n$ *is a metric space with metric*

$$d_2(x, y) = \|y - x\| := \sqrt{\sum_{i=1}^{n} |y_i - x_i|^2}.$$

EXAMPLE 1.3. *The space* $\mathbb{R}^n$ *is a metric space with metric*

$$d_1(x, y) = \sum_{i=1}^{n} |y_i - x_i|.$$

EXAMPLE 1.4. *The space* $\mathbb{R}^n$ *is a metric space with metric*

$$d_\infty(x, y) = \max_{i=1..n} |y_i - x_i|.$$

EXAMPLE 1.5. *The space* $\mathbb{R}$ *is a metric space with metric*

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Other examples will be discussed in class and you should record them here for future reference.

1.1.1. *Exercises.*

EXERCISE 1.1. *Consider* $\mathbb{R}^2$ *with the metric* $d_p(x,y) = \left(\sum_{i=1}^{2} |x_i - y_i|^p\right)^{\frac{1}{p}}$, *where* $1 \leq p < \infty$. *(When you are finished it should not be too difficult to generalize to* $\mathbb{R}^n$*.)*

(1) *Show that* d_p *is a metric for any* $p \geq 1$.
   (a) *Prove Young's Inequality: If* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $a, b > 0$, *then* $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$. *(Hint: Start with* $\ln(ab) = \ln(a) + \ln(b) = \cdots$. *Use the convexity of the* $\ln$ *function.)*
   (b) *Prove Hölder's Inequality: If* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*
$$\sum_1^2 |x_i y_i| \leq \left(\sum_1^2 |x_i|^p\right)^{\frac{1}{p}} \left(\sum_1^2 |y_i|^q\right)^{\frac{1}{q}}.$$
   (Hint: As a first case, assume that $||x||_p = ||y||_q = 1$.*)*
   (c) *Prove Minkowski's Inequality: If* $p > 1$ *then*
$$\left(\sum_1^2 |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_1^2 |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_1^2 |y_i|^p\right)^{\frac{1}{p}}.$$
   (Hint: Write $|x_i + y_i|^p = |x_i + y_i||x_i + y_i|^{p-1} \leq |x_i||x_i + y_i|^{p-1} + |y_i||x_i + y_i|^{p-1}$*) and apply Hölder.)*

(2) *Sketch some unit circles, i.e.* $\{(x,y) \in \mathbb{R}^2 : |x|^p + |y|^p = 1\}$, *for different* p *values.*

(3) *Investigate what happens to the metric, and the unit circles, as* $p \to \infty$. *Explain why the metric in Example 4 is called* d_∞.

(4) *If two points are* close *with respect to one of the metrics, are they also* close *with respect to the others?*

EXERCISE 1.2. *Find an appropriate generalization of Hölder's Inequality for* $\sum_1^n |x_i y_i z_i|$.

1.2. Basic Topological Definitions in a Metric Space.

DEFINITION 1.2. *A sequence in a space* X *is a function from* $\mathbb{N}$ *to* X.

DEFINITION 1.3. *A sequence* (x_n) *in a metric space* (X,d) *converges if there is an* $x \in X$ *such that* $d(x_n, x) \to 0$. *That is, given any* $\epsilon > 0$ *there is an* $N > 0$ *such that* $d(x_n, x) < \epsilon$ *for all* $n > N$.

DEFINITION 1.4. *A sequence* (x_n) *in a metric space* (X,d) *is Cauchy if given any* $\epsilon > 0$ *there is an* $N > 0$ *such that* $d(x_n, x_m) < \epsilon\ \forall n, m > N$.

PROPOSITION 1.1. *Every convergent sequence is Cauchy.*

PROOF. Let (x_n) be a sequence in a metric space X which converges to a point $x \in X$. Let $\epsilon > 0$. Then there is an N such that, $\forall n > N$, $d(x_n, x) < \frac{\epsilon}{2}$. Let $n, m > N$. Then
$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = d(x_n, x) + d(x_m, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$
by the triangle inequality and the symmetry of the metric. Therefore (x_n) is Cauchy. □

REMARK 1.1. *Not every Cauchy sequence converges. The difficulty is whether the "right" limit exists within the given space* X.

Definition 1.5. *A metric space is complete if every Cauchy sequence in the metric space converges (to an object within the metric space).*

Example 1.6. *The metric space* $\mathbb{Q}$ *is not complete. For example the sequence* $(1, 1.4, 1.41, 1.414, \ldots)$ *is Cauchy, but it converges to* $\sqrt{2}$ *which is not itself a member of* $\mathbb{Q}$. *This is an excellent example of what an "incomplete" metric space looks like. Note that the limit does exist in the larger metric space* $\mathbb{R}$. *One can say that* $\mathbb{R}$ *(which is complete as proved below) is the completion of* $\mathbb{Q}$. *(More on this later!)*

Example 1.7. *The metric space* $\mathbb{R}^n$ *is complete with the usual Euclidean metric. (Or any of the other metrics discussed above.)*

The claim in this example is a consequence of the Bolzano-Weierstrass Theorem:

Theorem 1.1. *(Bolzano-Weierstrass Theorem Version 1) Every bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

Proposition 1.2. *The following two definitions are equivalent:*

(1) *Let* X *be a metric space and let* S *be a subset of* X. *We say that* $x \in X$ *is an accumulation point of* S *if,* $\forall \epsilon > 0$, $\exists s \in S$ *such that* $s \neq x$ *and* $d(s, x) < \epsilon$.
(2) *Let* X *be a metric space and let* S *be a subset of* X. *We say that* $x \in X$ *is an accumulation point of* S *if there is a sequence* (s_n) *of elements of* S, *none of which equals* x, *such that* $s_n \to x$.

Proof. This proposition was proved in class. You may want to rewrite it here for completeness. □

The Bolzano-Weierstrass Theorem is also equivalent to the following:

Theorem 1.2. *(Bolzano-Weierstrass Theorem Version 2) Every bounded, infinite subset of* $\mathbb{R}^n$ *has an accumulation point.*

Notation 1.1. *For* $x \in X$ *and* $\epsilon > 0$, *the set* $B_\epsilon(x) := \{y \in X : d(y, x) < \epsilon\}$ *is called the open ball of radius* ϵ *centered at* x.

Definition 1.6. *A set* $U \subset X$ *is called open if, for every* $x \in U$, $\exists \epsilon > 0$ *such that* $B_\epsilon(x) \subset U$.

Definition 1.7. *Suppose that* X *is a set. A collection* $\mathcal{U}$ *of subsets of* X *is called a* topology *on* X *if*

(1) *X and $\emptyset$ are elements of $\mathcal{U}$.*
(2) *Whenever $U_i \in \mathcal{U}$ for $i = 1 \dots n$, then $\cap_{i=1}^n U_i \in \mathcal{U}$.*
(3) *For any index set A, if $U_\alpha \in \mathcal{U}$ for every $\alpha \in A$, then $\cup_{\alpha \in A} U_\alpha \in \mathcal{U}$.*

If $\mathcal{U}$ is a topology on X, then the elements of $\mathcal{U}$ are called the open *sets of X.*

Proposition 1.3. *The collection of open sets in a metric space X, as defined in Definition 1.6, form a topology on X. This topology is called the* metric topology *on X, or, to be more specific, the* topology induced by the metric d on the set X.

Definition 1.8. *A set $C \subset X$ is called closed if its complement is open in X.*

Proposition 1.4. *A set C in a metric space X is closed if and only if it contains all of its accumulation points.*

Proof. We will prove this proposition by proving first that if C is not closed then it cannot contain all of its limit points, and then that if C does not contain all of its limit points then it cannot be closed.

So, first suppose that C is not closed. This implies that $X \setminus C$ is not open. Therefore, there exists an $x \in X \setminus C$ such that $\forall \epsilon > 0$, $B_\epsilon(x) \cap C \neq \emptyset$. I claim that x is a limit point of C. Let $\epsilon = \frac{1}{n}$ and let $c_n \in B_{\frac{1}{n}}(x) \cap C$. Clearly (c_n) forms a sequence in C, and since $x \notin C$, $c_n \neq x$ for all n. Moreover, if $\delta > 0$, then for all $n > \frac{1}{\delta}$, $d(x, c_n) < \frac{1}{n} < \delta$, so $c_n \to x$. Therefore x is a limit point of C as claimed. Since x is not an element of C, this implies that C does not contain all of its limit points.

Now, suppose that C does not contain all of its limit points. Let x be a limit point of C that is not in C. Let c_n be a sequence of elements of C that converges to x. Let $\epsilon > 0$. Then there is some $n \in \mathbb{N}$ such that $d(c_n, x) < \epsilon$. Hence $B_\epsilon(x) \cap C \neq \emptyset$. Since this is true for every $\epsilon > 0$, and $x \in X \setminus C$, we may conclude that $X \setminus C$ is not open, so C is not closed. $\square$

Definition 1.9. *A set C in a metric space X is dense if for every $x \in X$ there is a sequence $(c_n) \subset C$ such that $\lim(c_n) = x$. (Alternatively, C is dense in X if given any $x \in X$ and any $\epsilon > 0$ there is a $c \in C$ such that $d(x, c) < \epsilon$.)*

Remark 1.2. *You should be able to prove that these two definitions are equivalent in the same way that you proved the equivalence of the two definitions of accumulation point above. In fact, a set C is dense in a metric space X if and only if every element of X is an accumulation point of C.*

Example 1.8. *The set $\mathbb{Q}$ is dense in $\mathbb{R}$. The set of all n-tuples of rational numbers is dense in $\mathbb{R}^n$. These facts can be justified by the Archimedean principle. If you haven't seen a proof of this fact before, you should try to construct one now.*

Definition 1.10. *A set K in the metric space X is* compact *if every open cover of K has a finite subcover.*

Definition 1.11. *A set K in the metric space X is* sequentially compact *if every sequence in K has a subsequence that converges to an element in K.*

Theorem 1.3. *If X is a metric space, then $K \subset X$ is compact if and only if K is sequentially compact.*

For a proof of this, refer to Munkres' Topology, or the introductory topology text of your choice. The forward implication is true in any topological space, but the reverse implication can fail in non-metrizable topologies. For example, the first uncountable ordinal with the order topology is sequentially compact but not compact. If you don't know what the first uncountable ordinal is, do not worry about it.

DEFINITION 1.12. *Suppose that X, Y are metric spaces and $f : X \to Y$. Then f is continuous at a point $x_0 \in X$ if given any sequence (x_n) such that $x_n \to x_0$ in X, we have $f(x_n) \to f(x_0)$ in Y. Equivalently, given any $\epsilon > 0$ there is a $\delta > 0$ such that if $d_X(x, x_0) < \delta$ then $d_Y(f(x), f(x_0)) < \epsilon$.*

DEFINITION 1.13. *Suppose that X, Y are sets and $f : X \to Y$. Let $U \subset Y$. Then the* inverse image *of U under f is $f^{-1}(U) := \{x \in X | f(x) \in U\}$.*

Note that f does not need to be invertible or even injective for $f^{-1}(U)$ to be well-defined if U is a set.

1.2.1. *Exercises.*

EXERCISE 1.3. *Assuming the B-W Theorem, prove that $\mathbb{R}^n$ is complete.*

EXERCISE 1.4. *Prove Proposition 1.3.*

EXERCISE 1.5. *Prove that the two definitions of continuity given above are equivalent. Prove that f is continuous at x for every $x \in X$ if and only if, for every $U \subset Y$ open, $f^{-1}(U)$ (as defined in Definition 1.13) is open in X.*

1.3. Definition of a Normed Linear Space. Most of the examples that we study will actually have more structure than the previous section indicates. In particular, they will be vector spaces where one can measure the length of a vector with a *norm.*

DEFINITION 1.14. *A set X equipped with two operations $+ : X \times X \to X$ (vector addition) and $\cdot : \mathbb{R} \times X \to X$ (scalar multiplication) is called a* real vector space *or* real linear space *if the following properties are satisfied:*

1. *$\forall u, v, w \in X$, $u + (v + w) = (u + v) + w$,*
2. *$\forall u, v \in X$, $u + v = v + u$,*
3. *$\exists 0 \in X$ such that $v + 0 = v$ $\forall v \in X$,*
4. *For every $v \in X$, there exists an element $-v \in X$, called the additive inverse of v, such that $v + (-v) = 0$,*
5. *$\forall a \in \mathbb{R}$, $\forall u, v \in X$, $a(u + v) = au + av$,*
6. *$\forall a, b \in \mathbb{R}$, $\forall v \in X$, $(a + b)v = av + bv$,*
7. *$\forall a, b \in \mathbb{R}$, $\forall v \in X$, $a(bv) = (ab)v$, and*
8. *$1v = v$.*

DEFINITION 1.15. *If X is a real vector space and $\| \cdot \| : X \to \mathbb{R}^+$ satisfies*

: *(ia) $\|x\| \geq 0$ for all $x \in X$*
: *(ib) $\|x\| = 0$ if and only if $x = 0$.*
: *(ii) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X$ and all $\alpha \in \mathbb{R}$*
: *(iii) (Triangle Inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$*

then $(X, \| \cdot \|)$ is called a normed linear space, and $\| \cdot \|$ is called the norm *on X.*

Definition 1.16. *Suppose that X is a normed linear space with respect to two different metrics, $\|\cdot\|_1$ and $\|\cdot\|_2$. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are* equivalent *if there exist positive constants c and C such that, for every $x \in X$, $c\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1$.*

Definition 1.17. *A* Banach space *is a complete normed linear space.*

Example 1.9. *$\mathbb{R}^n$ is a Banach space with respect to any l^p norm (including l^∞).*

Example 1.10. *The set of bounded real sequences is a Banach space with respect to the uniform norm.*

Record other examples discussed in class here.

The following theorem may be helpful in the future.

Theorem 1.4. *If $(X, ||\cdot||)$ is a normed linear space, then (X, d) is a metric space with $d(x, y) = ||x - y||$.*

1.3.1. *Exercises.*

Exercise 1.6. *Prove Theorem 1.4.*

Exercise 1.7. *Find an example of a metric space (X, d) such that d can not be associated with any norm. Bonus points if it is not an example discussed in class.*

Exercise 1.8.
(1) *Prove that all of the l^p norms on $\mathbb{R}^n$, $1 \leq p \leq \infty$ are equivalent.*
(2) *Prove that two equivalent norms generate the same topology. That is, if U is open with respect to the topology induced by the first norm, then U is open with respect to the topology induced by the second norm, and vice versa.*
(3) *Give an example of a normed linear space X and two non-equivalent norms on X.*
(4) *Suppose that X is a vector space, and let $\mathcal{N}$ be the collection of all norms on X. Prove that norm equivalence as defined above is an equivalence relation on $\mathcal{N}$.*

Remark 1.3. *Note that part 2 of this exercise means that when we have two equivalent norms, all important topological concepts–convergence, completeness, continuity, openness, closedness, compactness, etc.–will be the same with respect to both norms. Therefore when we have two equivalent norms on the same space we tend to just pick our favorite and work with that. That is one reason why we rarely worry too much about other l^p norms on $\mathbb{R}^n$ and just work with the Euclidean norm. However, when we move beyond $\mathbb{R}^n$ we will need to be more careful!*

2. Important Examples of Metric Spaces

2.1. The standard metric on $\mathbb{R}$. We will be assuming that you learned the contents of this subsection and the next in your introductory real analysis course. I encourage you to prove each of the following statements as an exercise, and then find an Elementary Real Analysis text and compare your proof to the proof in the text.

THEOREM 2.1. *$\mathbb{R}$ is a vector space and the function $|\cdot| : \mathbb{R} \to \mathbb{R}^+$ is a norm on $\mathbb{R}$.*

This is generally called the standard norm on $\mathbb{R}$, and it induces the standard metric on $\mathbb{R}$, which in turn induces the standard topology on $\mathbb{R}$.

THEOREM 2.2. *(Bolzano-Weierstrass Theorem) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

See Section 1.2 above for further discussion of the Bolzano-Weierstrass Theorem, and its consequence:

THEOREM 2.3. *$\mathbb{R}$ is a complete metric space with respect to the metric induced by the standard norm on $\mathbb{R}$.*

THEOREM 2.4. *(Heine-Borel Theorem) A subset K of $\mathbb{R}$ (equipped with the standard topology) is compact if and only if it is both closed and bounded.*

THEOREM 2.5. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

DEFINITION 2.1. *Let $K \subset \mathbb{R}$. Suppose that $f_n : K \to \mathbb{R}$ is a function for each n, and that $f : K \to \mathbb{R}$ is another function. We say that $f_n \to f$ (pointwise) or f_n* converges (pointwise) *to f if, $\forall x \in K$, $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\forall n > N$ $|f_n(x) - f(x)| < \epsilon$.*

DEFINITION 2.2. *Let $K \subset \mathbb{R}$. Suppose that $f_n : K \to \mathbb{R}$ is a function for each n, and that $f : K \to \mathbb{R}$ is another function. We say that $f_n \to f$ uniformly or f_n* converges uniformly *to f if $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\forall n > N$, $\forall x \in K$, $|f_n(x) - f(x)| < \epsilon$.*

PROPOSITION 2.1. *If each of the f_n is continuous on K and $f_n \to f$* uniformly*, then f is continuous.*

2.1.1. *Exercises.*

EXERCISE 2.1. *Explain in a paragraph the difference between pointwise and uniform convergence.*

EXERCISE 2.2. *Prove Proposition 2.1.*

EXERCISE 2.3. *Give an example of a sequence of functions which converge pointwise and for which the conclusion of Proposition 2.1 fails. Bonus points if it is not an example discussed in class.*

2.2. Generalizing to $\mathbb{R}^n$.

THEOREM 2.6. *$\mathbb{R}^n$ is a vector space and the function $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}^+$ is a norm on $\mathbb{R}^n$.*

This is generally called the standard or Euclidean norm on $\mathbb{R}^n$, and it induces the standard or Euclidean metric on $\mathbb{R}^n$, which in turn induces the standard topology on $\mathbb{R}^n$. We have already proved this result in the exercises above. We also know that there is an infinite collection of l^p norms on $\mathbb{R}^n$, each of which is equivalent to the l^2 or Euclidean norm. That means that they all induce the standard topology. We will generally work with the Euclidean norm unless I specify otherwise.

The following result is used to extend our statements about $\mathbb{R}$ above to $\mathbb{R}^n$ for $n \in \mathbb{N}$. It is a topological statement, so it holds equally in each l^p metric, as do the theorems listed below.

THEOREM 2.7. *A sequence (x_k) converges to x in $\mathbb{R}^n$ if and only if its components (x_{ki}) converge to x_i in $\mathbb{R}$ for each $i = 1, ..., n$.*

THEOREM 2.8. *(Bolzano-Weierstrass Theorem) Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

THEOREM 2.9. *$\mathbb{R}^n$ is a complete metric space with respect to the metric induced by the standard norm on $\mathbb{R}^n$.*

THEOREM 2.10. *(Heine-Borel Theorem) A subset K of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.*

THEOREM 2.11. *$\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.*

2.2.1. *Exercises.*

EXERCISE 2.4. *Prove Theorem 2.7.*

EXERCISE 2.5. *Use Theorem 2.7 and Theorems 2.2, 2.3, and 2.4 to prove the corresponding results about $\mathbb{R}^n$.*

2.3. The space of sequences l^2.

DEFINITION 2.3. *The space l^2 is defined as $\{(x_n) : \mathbb{N} \to \mathbb{R} | \sum_{n=1}^{\infty} x_n^2 < +\infty\}$. For $(x_n) \in l^2$, we define the l^2-norm of (x_n) as $\|(x_n)\|_2 := \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}}$.*

This is the space of all sequences of real numbers which are square-summable. **The elements of this space are themselves sequences.**

LEMMA 2.1. *If $(x_n) \in l^2$, and $\alpha \in \mathbb{R}$, then $\left(\sum_{n=1}^{\infty} (\alpha x_n)^2\right)^{\frac{1}{2}} = |\alpha| \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}}$.*

PROOF. This is a trivial consequence of the usual properties of infinite series. □

LEMMA 2.2. *If $\sum_{n=1}^{\infty} x_n^2 < +\infty$ and $\sum_{n=1}^{\infty} y_n^2 < +\infty$, then*

$$\left(\sum_{n=1}^{\infty}(x_n + y_n)^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} y_n^2\right)^{\frac{1}{2}}.$$

PROOF. Choose $N \in \mathbb{N}$. Then

$$\left(\sum_{n=1}^{N}(x_n + y_n)^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{N} x_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{N} y_n^2\right)^{\frac{1}{2}}$$

by Minkowski's inequality, as proved in Exercise 1.1 above. Since all of the terms in each sum are non-negative, it follows immediately that

$$\left(\sum_{n=1}^{N}(x_n+y_n)^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{\infty}x_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty}y_n^2\right)^{\frac{1}{2}}.$$

We may now take the limit as N goes to infinity. Since the right-hand side does not depend on N, it serves as an upper bound for the increasing sequence on the left. We may conclude that

$$\left(\sum_{n=1}^{\infty}(x_n+y_n)^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{\infty}x_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty}y_n^2\right)^{\frac{1}{2}}$$

as claimed. □

PROPOSITION 2.2. *The space l^2 is a vector space.*

PROOF. The sequence $(0,0,0,0,\ldots)$ functions as the zero vector in this vector space, and if $(x_1,x_2,x_3,\ldots)$ is an element of the space, and $\alpha \in \mathbb{R}$, then $\alpha(x_n) := (\alpha x_1, \alpha x_2, \alpha x_3, \ldots)$, which is again in l^2 by Lemma 2.1. If (x_n) and (y_n) are elements of l^2, we define $(x_n)+(y_n) := (x_n+y_n)$, which is again in l^2 by Lemma 2.2. With these definitions it should be clear that the usual associative, commutative, and distributive laws hold. □

PROPOSITION 2.3. *The l^2 norm as defined above is a norm on l^2.*

PROOF. We must check the three properties of the norm. First, notice that by definition the norm is a summation of nonnegative terms, so it is nonnegative for every $x \in l^2$. Also notice that $\|(0)\| = \left(\sum\limits_{n=1}^{\infty} 0^2\right)^{\frac{1}{2}} = 0$. Additionally, if $\left(\sum\limits_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}} = 0$, then $\sum\limits_{n=1}^{\infty} x_n^2 = 0$. As mentioned above, this is a sum of nonnegative terms, so it can only be zero if every term is zero. Therefore $x_n = 0$ for all $n \in \mathbb{N}$ and $x = 0$ as claimed. The homogeneity property of the norm was proved in Lemma 2.1 and the triangle inequality was proved in Lemma 2.2. Therefore the function does indeed satisfy the necessary properties of a norm. □

THEOREM 2.12. *The normed linear space l^2 is complete.*

PROOF. Suppose that $((x_n)^k)$ is a Cauchy sequence of elements of l^2. We must first find a candidate limit $x \in l^2$ and then prove that $x^k \to x$ in the l^2 norm.

For future use, note that since this sequence is Cauchy, it is bounded (exercise!). That is, $\exists M > 0$ such that $\forall k \in \mathbb{N}$, $\|x^k\|_2 < M$.

Let $\epsilon > 0$. Since (x^k) is Cauchy, $\exists K \in \mathbb{N}$ such that $\forall j,k > K$, $\|(x^k)-(x^j)\|_2 < \epsilon$. Let $j,k > K$ and choose $n \in \mathbb{N}$. Note that

$$|x_n^k - x_n^j| = ((x_n^k - x_n^j)^2)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{\infty}(x_n^k - x_n^j)^2\right)^{\frac{1}{2}} < \epsilon.$$

Therefore, for each fixed n, the sequence of real numbers (x_n^k) is also Cauchy. Thus, by the completeness of $\mathbb{R}$, there exists a real number x_n so that $(x_n^k) \to x_n$. Define $x := (x_n)$.

We must show that $x \in l^2$. That is, $\left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}} < +\infty$. Choose $N \in \mathbb{N}$ and consider $\left(\sum_{n=1}^{N} x_n^2\right)^{\frac{1}{2}}$. For $n = 1 \ldots N$, $\exists K_n \in \mathbb{N}$ such that $\forall k > K_n$, $|x_n^k - x_n| < \frac{\epsilon}{\sqrt{N}}$ by the term-wise convergence proved in the previous paragraph. Let $K = \max\{K_n : n = 1 \ldots N\}$. Let $k > K$. By the triangle inequality,

$$\left(\sum_{n=1}^{N} x_n^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{N} (x_n - x_n^k)^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{N} (x_n^k)^2\right)^{\frac{1}{2}}.$$

Since $k > K_n$ for each $n \leq N$, we then have

$$\begin{aligned}
\left(\sum_{n=1}^{N} x_n^2\right)^{\frac{1}{2}} &< \left(\sum_{n=1}^{N} \frac{\epsilon^2}{N}\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{N} (x_n^k)^2\right)^{\frac{1}{2}} \\
&= \epsilon + \left(\sum_{n=1}^{N} (x_n^k)^2\right)^{\frac{1}{2}} \\
&\leq \epsilon + \left(\sum_{n=1}^{\infty} (x_n^k)^2\right)^{\frac{1}{2}} \\
&\leq \epsilon + M
\end{aligned}$$

where M is the bound on $\|x^k\|_2$ found above. If we now let $\epsilon \to 0$, we find that

$$\left(\sum_{n=1}^{N} x_n^2\right)^{\frac{1}{2}} \leq M.$$

Finally, letting N go to infinity, we may conclude that $\|x\|_2 \leq M$. Therefore $x \in l^2$ as claimed.

Finally, we must show that $(x^k) \to x$ in the l^2 topology. This proof is nearly identical to the proof that $x \in l^2$, so it is left as an exercise. □

NOTATION 2.1. *We define the element $e^k \in l^2$ by $e_n^k = 1$ if $k = n$ and 0 otherwise.*

These elements are analogous to the standard basis elements of $\mathbb{R}^n$. However, notice that there are infinitely many of them.

EXAMPLE 2.1. *Notice that (e^k) is a bounded sequence in l^2 because $\|e^k\|_2 = 1$ for each k. However, if $k \neq j$ then $d_2(e^k, e^j) = \sqrt{2}$. Therefore this sequence cannot have a convergent subsequence. Also note that for any fixed n, $e_n^k \to 0$, so $(e^k) \to (0)$ termwise. Therefore, unlike in $\mathbb{R}^n$, termwise (or componentwise) convergence is not sufficient to ensure norm convergence.*

This example shows us that the Bolzano-Weierstrass theorem cannot hold for l^2 with the norm topology. Therefore there is something fundamentally different about the topology of this infinite dimensional space than of the finite dimensional spaces to which we are used.

Example 2.2. *Define* $K := \{x \in l^2 : \|x\|_2 \leq 1\}$ *to be the closed unit ball in* l^2*. Then* K *is clearly bounded, and it is also closed, which is left as an exercise. However, it fails to be sequentially compact. For example, the sequence* $(e^k)_{k=1}^{\infty}$ *defined above is contained in* K *but has no convergent subsequence.*

This example shows us that the Heine-Borel theorem also cannot hold for l^2 with the norm topology. It is possible to define other topologies on this same space for which K is compact. This is often important when proving existence theorems in analysis. Thus it is necessary to work with multiple different topologies on the same space, if the space is infinite dimensional, often within the same proof or calculation.

Definition 2.4. *We define the* Hilbert cube C *to be the collection of sequences of real numbers* (x_n) *so that,* $\forall n \in \mathbb{N}$, $-\frac{1}{n} \leq x_n \leq \frac{1}{n}$.

We will leave it as an exercise to show that the Hilbert cube is a subset of l^2.

Proposition 2.4. C *is a compact subset of* l^2*.*

Proof. The overall strategy is the following: Start with a sequence in the Hilbert Cube. Remember that this is a sequence of sequences. We need to use the fact that bounded sequences in $\mathbb{R}$ contain convergent subsequences. We can use this to get convergence in each of the components of a subsequence. Then we use the properties of the Hilbert Cube to get actual convergence in l_2.

Let $(x^k) \subset C$, and let x_n^k represent the nth component of x^k. Notice that the sequence of first components, (x_1^k), is a sequence of real numbers in $[-1, 1]$. Therefore there is a subsequence, (x^{k1}), of (x^k), such that the sequence of first components, (x_1^{k1}), converges to some number x_1. Next observe that the sequence of second components, (x_2^{k1}), is a sequence of real numbers in $[-\frac{1}{2}, \frac{1}{2}]$. Therefore there is a subsequence, (x^{k2}), of (x^{k1}), such that the second components, (x_2^{k2}), converges to some number x_2. Since (x^{k2}) is a subsequence of (x^{k1}), we also know that the sequence of first components, (x_1^{k2}), will converge to x_1. Continuing recursively to select nested subsequences we can get an nth subsequence, (x^{kn}), such that the 1st, 2nd, ..., nth, components of the subsequence each converge to $x_1, x_2, \ldots, x_n$, respectively.

In order to find a subsequence that converges in every component we use a diagonalization process: consider (x^{kk}), i.e. the kth element of the kth subsequence. Note that (x^{kk}) is a subsequence of (x^k) by construction. Also, observe that for any given n, we have that $(x^{nn}, x^{(n+1)(n+1)}, \ldots)$ is a subsequence of the nth subsequence above, (x^{kn}), and therefore converges in the nth component. Hence (x^{kk}) converges in every component.

For notational convenience we now rename x^{kk} as x^k, and proceed.

It remains to verify that (x^k) converges to $x := (x_1, x_2, x_3, \ldots)$ in l^2. Let $\epsilon > 0$ be given and choose $N > 0$ such that

$$\sum_{N+1}^{\infty} \frac{1}{k^2} < \frac{\epsilon^2}{4}.$$

Then

$$\begin{aligned}
||x^k - x||_2^2 &= \textstyle\sum_{n=1}^{\infty} |x_n^k - x_n|^2 \\
&= \textstyle\sum_{n=1}^{N} |x_n^k - x_n|^2 + \sum_{n=N+1}^{\infty} |x_n^k - x_n|^2 \\
&\leq \textstyle\sum_{1}^{N} |x_n^k - x_n|^2 + \sum_{n=N+1}^{\infty} \frac{4}{n^2} \\
&\leq \textstyle\sum_{1}^{N} |x_n^k - x_n|^2 + \epsilon^2.
\end{aligned}$$

Letting $k \to \infty$ and applying the componentwise convergence from above we have

$$\limsup_{k\to\infty} ||x^k - x||_2^2 \leq \epsilon^2.$$

It follows that $x^k \to x$ in l^2. $\square$

It is worth considering how this is different than what happened on K above, and why. Can you give a few more examples of infinite-dimensional, compact subsets of l^2?

2.3.1. *Exercises.*

Exercise 2.6. *Prove that in any normed linear space, any Cauchy sequence is bounded.*

Exercise 2.7. *Complete the proof of Theorem 2.12.*

Exercise 2.8. *Prove that l^2 is an infinite-dimensional vector space. That is, prove that there does not exist a finite collection of elements of l^2 which span l^2.*

Exercise 2.9. *Consider the set K defined in Example 2.2.*

(1) *Show that K is closed in the l^2-norm topology.*
(2) *Show that K is not compact by working directly with the open cover definition of compactness.*

Exercise 2.10. *Show that the Hilbert cube $C \subset l^2$.*

2.4. $C[0,1]$.

Definition 2.5. $C([0,1]) := \{f : [0,1] \to \mathbb{R} : f$ *is continuous*$\}$.

Proposition 2.5. *$C([0,1])$ is a real vector space.*

Proof. Exercise. $\square$

Theorem 2.13. *Let $\|\cdot\| : C([0,1]) \to \mathbb{R}$ be given by $\|f\| := \sup\{|f(x)| : x \in [0,1]\}$. Then $(C[0,1], \|\cdot\|)$ is a normed linear space.*

We will often use the notation $\|f\|_\infty$ for this norm and will call this the *sup*-norm. (Can you explain why the $||\cdot||_\infty$ notation is appropriate?)

Proof. First recall that since $[0,1]$ is a compact subset of $\mathbb{R}$, by the Extreme Value Theorem any continuous function f on $[0,1]$ takes on a finite maximum and a finite minimum value on the interval. Therefore $\|f\|$ is well-defined and finite for each f in $C([0,1])$.

Also note that by construction, because of the absolute value in the formula, $\|f\| \geq 0$ for all $f \in C([0,1])$. Moreover, $\|0\| = \sup\{0 | x \in [0,1]\} = 0$. Finally, if $\|f\| = 0$, then $\sup\{|f(x)| : x \in [0,1]\} = 0$, so $|f(x)| \leq 0$ $\forall x \in [0,1]$. Since $|f(x)| \geq 0$ by definition, we can conclude that $f(x) = 0\ \forall x \in [0,1]$. Therefore $f \equiv 0$. Hence $\|\cdot\|$ satisfies the first property of a norm.

Now, let $f \in C([0,1])$ and $\alpha \in \mathbb{R}$ and consider

$$\|\alpha f\| = \sup\{|\alpha f(x)| : x \in [0,1]\} = \sup\{|\alpha||f(x)| : x \in [0,1]\} = |\alpha| \sup\{|f(x)| : x \in [0,1]\} = |\alpha|\|f\|$$

by the properties of the sup. Therefore $\|\cdot\|$ satisfies the second property of the norm.

Finally, let $f, g \in C([0,1])$ and consider

$$\begin{aligned}\|f+g\| &= \sup\{|f(x)+g(x)| : x \in [0,1]\}\\ &\leq \sup\{|f(x)| + |g(x)| : x \in [0,1]\}\\ &\leq \sup\{|f(x)| : x \in [0,1]\} + \sup\{|g(x)| : x \in [0,1]\}\\ &= \|f\| + \|g\|\end{aligned}$$

where the second inequality is by the triangle inequality on $\mathbb{R}$ and the third is by the properties of the sup. Hence $\|\cdot\|$ satisfies the triangle inequality, which is the last property of a norm that we needed to check. Therefore it is indeed a norm on $C([0,1])$. ☐

Definition 2.6. *We say that a sequence of functions is* uniformly Cauchy *if, for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$ and $\forall x \in [0,1]$, $|f_n(x) - f_m(x)| < \epsilon$.*

Remark 2.1. *Notice that this is the same as being Cauchy with respect to the l^∞ topology.*

Lemma 2.3. *Any uniformly Cauchy sequence of functions converges uniformly.*

Proof. Suppose that (f_n) is a uniformly Cauchy sequence. Then, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, m > N$, $\sup\{|f_n(x) - f_m(x)| : x \in [0,1]\} < \epsilon$. It follows that for any fixed $x \in [0,1]$, $\forall n, m > N$, $|f_n(x) - f_m(x)| < \epsilon$. Therefore, $(f_n(x))$ is a Cauchy sequence of real numbers, so it converges by the completeness of $\mathbb{R}$. Define $f(x) := \lim_{n\to\infty} f_n(x)$. Then I claim that f is the desired limit in $C([0,1])$ so that $\lim_{n\to\infty} \|f_n - f\| = 0$.

Let $\epsilon > 0$. Choose N_0 such that if $n, m > N_0$ then $\sup\{|f_n(x) - f_m(x)| : x \in [0,1]\} < \frac{\epsilon}{2}$. Next, fix $x \in [0,1]$ and choose N_x so that $\forall n > N_x \quad |f_n(x) - f(x)| < \frac{\epsilon}{2}$. Then if $n > N_0$, and $m > \max\{N_0, N_x\}$,

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Note that although the intermediate step with f_m depended on our choice of x, n does not and neither does ϵ. Therefore, we may conclude that

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N, \forall x \in [0,1] \qquad |f_n(x) - f(x)| < \epsilon \tag{1}$$

Therefore $f_n \to f$ uniformly. ☐

Corollary 2.1. *$C([0,1])$ is complete with respect to $\|\cdot\|_\infty$.*

Proof. Let (f_n) be a Cauchy sequence in $C([0,1])$. This is the same as saying that (f_n) is uniformly Cauchy. By Lemma 2.3, it must converge uniformly. Therefore, by Proposition 2.1, we know that the limit function, f, is itself continuous. Hence $f \in C([0,1])$. Then notice that (1) is equivalent to the claim that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n > N$, $\|f_n - f\|_\infty = \sup\{|f_n(x) - f(x)| : x \in [0,1]\} < \epsilon$. Therefore we may conclude that $f_n \to f$ in the uniform-norm topology. Hence any Cauchy sequence in $C([0,1])$ must converge, so $C([0,1])$ is complete. ☐

Definition 2.7. *For $f \in C([0,1])$, define the L^2-norm of f as*

$$\|f\|_2 := \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}}. \tag{2}$$

Remark 2.2. *In general the L^2-norm can be defined on a much larger function space than $C([0,1])$. In this class we will not have time to develop the theory of Lebesgue integration, so we will not have the machinery to define explicitly the largest possible collection of functions on which this represents a norm. However, since we do know how to do standard Riemann integrals, and all continuous functions are Riemann integrable, we can certainly define this norm as a function on $C([0,1])$.*

Lemma 2.4. *If $F : [0,1] \to \mathbb{R}$ is continuous and nonnegative, and $\int_0^1 F(x)dx = 0$, then $F \equiv 0$ on $[0,1]$.*

Proof. Suppose F is a continuous, nonnegative function on $[0,1]$, and suppose that $\exists\, x_0 \in [0,1]$ such that $F(x_0) \neq 0$. Then by hypothesis $F(x_0) > 0$. Since F is continuous, by letting $\epsilon = \frac{1}{2}F(x_0)$ we can find a $\delta > 0$ such that $|F(x) - F(x_0)| < \epsilon$ on $(x_0 - \delta, x_0 + \delta) \cap [0,1]$. By the triangle inequality, this implies that $F(x) > \frac{1}{2}F(x_0)$ on $(x_0 - \delta, x_0 + \delta) \cap [0,1]$. Let $I = (x_0 - \delta, x_0 + \delta) \cap [0,1]$, and Note that I is an interval. Depending on the relative values of x_0 and δ it may have a length of anywhere from δ to 2δ but its minimum length is at least δ because at least half of $(x_0 - \delta, x_0 + \delta)$ must always lie inside $[0,1]$. Therefore,

$$\int_0^1 F(x)dx \geq \int_I F(x)dx > \int_I \frac{F(x_0)}{2}dx \geq \frac{F(x_0)\delta}{2} > 0,$$

by the basic properties of the Riemann integral. Therefore $\int_0^1 F(x)dx > 0$. Hence, by contraposition, if $\int_0^1 F(x)dx = 0$ then $F \equiv 0$. □

Proposition 2.6. *$\|\cdot\|_2$ is a norm on $C([0,1])$.*

Proof. First note that clearly $\|f\|_2 \geq 0$ for any $f \in C([0,1])$. Also, $\|0\|_2 = \sqrt{\int_0^1 0^2 dx} = 0$. Moreover, if $\|f\|_2 = 0$, then $\int_0^1 |f(x)|^2 dx = 0$. By the properties of continuous functions, $|f(x)|^2$ is continuous and it is clearly nonnegative, so by Lemma 2.4 it follows that $|f(x)|^2 \equiv 0$ on $[0,1]$. Hence $f(x) \equiv 0$ also.

Next, if $\alpha \in \mathbb{R}$ and $f \in C([0,1])$, then

$$\|\alpha f\|_2 = \sqrt{\int_0^1 |\alpha f(x)|^2 dx} = \sqrt{\int_0^1 |\alpha|^2 |f(x)|^2 dx} = \sqrt{|\alpha|^2 \int_0^1 |f(x)|^2 dx} = |\alpha| \sqrt{\int_0^1 |f(x)|^2 dx} = |\alpha| \|f\|_2,$$

as desired.

Finally, we must check the triangle inequality. This is proved following the same process as in $\mathbb{R}^n$. Let $f, g \in C([0,1])$. First, we have that if $a, b \in \mathbb{R}$ then $|ab| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$. (This is a consequence of the fact that $a^2 \pm 2ab + b^2$ are both perfect squares and hence nonnegative; it is also a special case of Young's Inequality.) Therefore, for any $x \in [0,1]$, $|f(x)g(x)| \leq \frac{1}{2}(|f(x)|^2 + |g(x)|^2)$. Now if f or g is identically zero, the claim is trivial, so we may assume they are not. Let $\tilde{f}(x) = \frac{f(x)}{\|f\|_2}$ and $\tilde{g}(x) = \frac{g(x)}{\|g\|_2}$. By the homogeneity proved above, these both have unit L^2-norm. Then

$$\int \frac{|f(x)g(x)|}{\|f\|_2\|g\|_2}dx = \int |\tilde{f}(x)\tilde{g}(x)|dx \leq \frac{1}{2}\int (|\tilde{f}(x)|^2 + |\tilde{g}(x)|^2)dx = \frac{1}{2}(1+1) = 1.$$

Thus $\int |f(x)g(x)|dx \leq \|f\|_2\|g\|_2$. Finally, compute

$$\begin{aligned}
\|f+g\|_2^2 &= \int |f(x)+g(x)|^2 dx \\
&= \int |f(x)+g(x)||f(x)+g(x)|dx \\
&\leq \int (|f(x)|+|g(x)|)|f(x)+g(x)|dx && \text{by the triangle inequality on } \mathbb{R} \\
&= \int |f(x)||f(x)+g(x)|dx + \int |g(x)||f(x)+g(x)|dx \\
&\leq \|f\|_2\|f+g\|_2 + \|g\|_2\|f+g\|_2 && \text{by the inequality proved above.}
\end{aligned}$$

Dividing through by the quantity $\|f+g\|_2$ yields the desired result. □

REMARK 2.3. *Note that this proof is exactly the same as in the $\mathbb{R}^n$ case, and the case for sequences can also be proved similarly, as can other p values.*

PROPOSITION 2.7. **Hölder's Inequality:** *For every $f, g \in C([0,1])$,*

$$|\int_0^1 f(x)g(x)dx| \leq \|f\|_2\|g\|_2. \tag{3}$$

PROOF. This was proven during the proof of the previous proposition. □

PROPOSITION 2.8. *$C([0,1])$ is not complete with respect to the $\|\cdot\|_2$-norm topology.*

PROOF. Consider the sequence of functions $(f_n)_{n=1}^\infty$, where f_n is given by

$$f_n(x) := \begin{cases} (2x)^n & x \leq \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}.$$

This is an L^2-Cauchy sequence of elements of $C([0,1])$, but it does not converge to any element of $C([0,1])$. The technicalities will be left as an exercise. □

PROPOSITION 2.9. *The conclusion of the Bolzano-Weierstrass Theorem does not hold on $C([0,1])$ with respect to either the L^∞-norm or the L^2-norm.*

PROOF. Exercise. □

The proposition above demonstrates that being closed and bounded is not enough to imply compactness in $C([0,1])$. So what additional property is needed? Well,...

DEFINITION 2.8. *A set of functions $F \subset C([0,1])$ is* uniformly bounded *if there is an $M > 0$ such that, for all $f \in F$, for all $x \in [0,1]$, $|f(x)| \leq M$.*

DEFINITION 2.9. *A set of functions $F \subset C([0,1])$ is* equicontinuous *at the point $x_0 \in [0,1]$ if given any $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$ $\forall f \in F$. We say that F is* equicontinuous on $[0,1]$ *if F is equicontinuous at every point in $[0,1]$.*

Note that when we consider a sequence of functions (f_n), we will describe it as being equicontinuous, or uniformly bounded, or some other property of sets of functions, if that same property holds for $F := \{f_n | n \in \mathbb{N}\}$.

THEOREM 2.14. **Arzela-Ascoli:** *Suppose that (f_n) is a uniformly bounded sequence of equicontinuous functions in $C([0,1])$. Then there exists a subsequence of (f_n) which converges in the L^∞-norm.*

This theorem can be rewritten in the following way:

COROLLARY 2.2. *If $K \subset C([0,1])$ is bounded, closed and equicontinuous, then K is compact.*

REMARK 2.4. *If a set has the property that every sequence in the set has a convergent subsequence (but whose limit is not necessarily in the set itself), then we call that set* precompact. *Bounded sets in $\mathbb{R}^n$ are precompact. Bounded and equicontinuous sets in $C[0,1]$ are precompact. If you add closedness to precompactness then you get compactness.*

COROLLARY 2.3. *If $K \subset C([0,1])$ is a set of uniformly bounded, differentiable functions, and if there is an $M > 0$ such that $|f'(x)| \leq M$ for all $x \in [0,1]$ and all $f \in K$, then K is precompact in $C([0,1])$.*

PROOF. A collection of functions whose derivatives are uniformly bounded is equicontinuous: For $f \in K$ and $x < y \in [0,1]$, $f(y) - f(x) = \int_x^y f'(t)dt$. Therefore, by elementary properties of the integral, $|f(x) - f(y)| \leq M|y - x|$. So if $\epsilon > 0$ is given, then whenever $|y - x| < \delta := \frac{\epsilon}{M}$, $|f(y) - f(x)| < \epsilon$. Therefore the conditions of the Arzela-Ascoli theorem hold. $\square$

Here is a proof of Theorem 2.14. The proof technique will be quite similar to the proof that the Hilbert cube is compact in l^2. However there is an extra layer of complication due to the fact that the domain of the functions in $C([0,1])$ is uncountable.

PROOF. (Arzela-Ascoli) Let $(r_i)_{i=1}^\infty$ be an enumeration of the rationals in $[0,1]$. Let (f_n) be a uniformly bounded collection of equicontinuous functions. Following the process used in the proof of Proposition 2.4, look at the values of $f_n(r_1)$. These are bounded by uniform boundedness, so there is a subsequence of this sequence of real numbers which converges. Call it $f_{n1}(r_1)$. Then, consider the sequence $f_{n1}(r_2)$, and find a sub-subsequence which converges at the point r_2. Call it f_{n2}, and note that $f_{n2}(r_1)$ also converges, because f_{n2} is a subsequence of f_{n1}. Proceeding recursively, we can find a nested collection of subsequences of f_n, called f_{nk}, such that for each k, $f_{nk}(r_i)$ converges whenever $i \leq k$. We then consider the diagonal subsequence f_{nn}. As in Proposition 2.4, we may conclude that f_{nn} converges at r_i for every natural number i.

I claim that in fact f_{nn} converges uniformly on $[0,1]$. To see this, let $\epsilon > 0$, and for each $i \in \mathbb{N}$, let N_i be chosen so that $\forall n, m > N_i$, $|f_{nn}(r_i) - f_{mm}(r_i)| < \frac{\epsilon}{3}$. Such an N_i exists because $f_{nn}(r_i)$ is a convergent, and hence Cauchy, sequence of real numbers. Note, though, that right now N_i depends on r_i so we still have some work to do.

Finally, for each $x_0 \in [0,1]$, by equicontinuity there is an open set U_{x_0} containing x_0 such that if $x, y \in U_{x_0}$ and $n \in \mathbb{N}$, then $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. Note that $[0,1]$ is compact, and this collection forms an open cover of $[0,1]$. Therefore, there is a finite subcover. Call it $\{U_{x_1}, \ldots, U_{x_J}\}$. Then, because the rationals are dense in the reals, each of these sets contains a rational number. Call it r_j. Let $N = \max\{N_j | j = 1 \ldots J\}$. Let $n, m > N$, and let $x \in [0, 1]$. Clearly $x \in U_{x_j}$ for at least one j. Then, by the triangle inequality, for that choice of r_j,

$$|f_{nn}(x) - f_{mm}(x)| \leq |f_{nn}(x) - f_{nn}(r_j)| + |f_{nn}(r_j) - f_{mm}(r_j)| + |f_{mm}(r_j) - f_{mm}(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

where here we have applied the equicontinuity twice, to control the first and third terms, and the convergence at r_j to control the second term. Therefore we may conclude that (f_{nn}) is a uniformly Cauchy sequence, and hence, as proved in Lemma 2.3, it must also converge uniformly, i.e. converge in $C([0, 1])$. $\square$

2.4.1. *Exercises.*

Exercise 2.11. *Prove Propositions 2.5, 2.8, and 2.9.*

Exercise 2.12. *Provide an example of a sequence of equicontinuous functions on* $[0, 1]$ *which do not have a convergent subsequence.*

2.5. $C^1([0, 1])$**.** Our next example is a modification of $C([0, 1])$ allowing for the function to be differentiable:

Definition 2.10.

$$C^1([0, 1]) := \{f : [0, 1] \to \mathbb{R} : f \textit{ is differentiable on } [0, 1] \textit{ and } \frac{df}{dx} \in C([0, 1])\},$$

Proposition 2.10. $C^1([0, 1])$ *is a real vector space.*

Proof. Exercise. $\square$

Theorem 2.15. *Let* $d : C^1([0, 1]) \times C^1([0, 1]) \to \mathbb{R}$ *be given by* $d(f, g) = ||f - g||_\infty + ||f' - g'||_\infty$. *Then* $(C^1([0, 1]), d)$ *is a metric space.*

Proof. Exercise. $\square$

Lemma 2.5. *Suppose that* f *is a continuously differentiable function on* $[0, 1]$*, and* $\sup_{x \in [0,1]} |f'(x)| = M$. *Then, for all* $x, y \in [0, 1]$, $|f(x) - f(y)| \leq M|x - y|$.

Proof. Let f, x, y be as in the statement of the theorem. By the mean value theorem, there is a c with $x \leq c \leq y$ so that $\frac{f(x) - f(y)}{x - y} = f'(c)$. Therefore,

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|.$$

$\square$

Proposition 2.11. $C^1([0, 1])$ *is complete with respect to* d.

Proof. Suppose that (f_n) is a Cauchy sequence in $C^1([0, 1])$. Then, since $\|f_n - f_m\|_\infty \leq d(f_n, f_m)$, (f_n) is also Cauchy in $C([0, 1])$. Therefore, $\exists f \in C([0, 1])$ so that $f_n \to f$ uniformly. Similarly, since $\|f'_n - f'_m\|_\infty \leq d(f_n, f_m)$, $\exists g \in C([0, 1])$ so that $f'_n \to g$ uniformly.

Fix $x \in [0, 1]$. Consider the difference quotient $D_n(h) = \frac{f_n(x+h) - f_n(x)}{h}$. Then I claim that $(D_n(h))$ is a uniformly Cauchy sequence with respect to h. That is, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $\forall n, m > N, |D_n(h) - D_m(h)| < \epsilon$. Let $\epsilon > 0$. Because $\|f'_n - f'_m\|_\infty \to 0$,

$\exists N \in \mathbb{N}$ such that $\forall n, m > N$, $\sup_{[0,1]} |(f_n - f_m)'| < \epsilon$. By Lemma 2.5, then, for all $x, y \in [0,1]$ it follows that $|(f_n - f_m)(x - y)| < \epsilon|x - y|$. Now, for $n, m > N$ and $h > 0$,

$$\begin{aligned}
|D_n(h) - D_m(h)| &= \left| \frac{f_n(x+h) - f_n(x)}{h} - \frac{f_m(x+h) - f_m(x)}{h} \right| \\
&= \frac{1}{h} |f_n(x+h) - f_m(x+h) - (f_n(x) - f_m(x))| \\
&= \frac{1}{h} |(f_n - f_m)(x+h) - (f_n - f_m)(x)| \\
&< \frac{1}{h} \epsilon |x + h - x| \\
&= \epsilon
\end{aligned}$$

as claimed. Hence the functions $(D_n(h))$ are uniformly Cauchy. Therefore, there is a function $D(h)$ to which they converge. But if we consider a fixed h, then it is clear that $D_n(h)$ must converge (pointwise) to $\frac{f(x+h)-f(x)}{h}$ by the (pointwise) convergence of f_n to f. Hence, by the uniqueness of limits, $D_n(h)$ converges uniformly to $D(h) := \frac{f(x+h)-f(x)}{h}$.

Now, recall that $(f_n'(x))$ is a uniformly Cauchy sequence which converges to some function $g(x)$. We want to show that $\lim_{h \to 0} D(h) = g(x)$. This will show that $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = g(x)$, i.e. f is differentiable and f' is g, the uniform limit of (f_n'), which is what we set out to show.

So, let $\epsilon > 0$, and choose n sufficiently large so that $\|D_n(h) - D(h)\|_\infty < \frac{\epsilon}{3}$. *and* $|f_n'(x) - g(x)| < \frac{\epsilon}{3}$. Then, for this fixed n, choose δ so that if $|h| < \delta$, then $|D_n(h) - f_n'(x)| < \frac{\epsilon}{3}$. Finally, by the triangle inequality, we have that

$$|D(h) - g(x)| \leq |D(h) - D_n(h)| + |D_n(h) - f_n'(x)| + |f_n'(x) - g(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

and the claim follows. ☐

DEFINITION 2.11. *For $f \in C^1([0,1])$, define the $W^{1,2}$-norm of f as*

$$\|f\|_{1,2} := \left(\int_0^1 |f(x)|^2 dx + \int_0^1 |f'(x)|^2 dx \right)^{\frac{1}{2}}. \tag{4}$$

PROPOSITION 2.12. *$\|\cdot\|_{1,2}$ is a norm on $C^1([0,1])$.*

PROOF. Exercise. ☐

PROPOSITION 2.13. *$C^1([0,1])$ is not complete with respect to the $W^{1,2}$-norm.*

PROOF. Exercise. ☐

2.5.1. *Exercises.*

EXERCISE 2.13. *Prove Proposition 2.10, Theorem 2.15, and Propositions 2.12 and 2.13. Hint: all of the heavy lifting has been done, so be sure to take advantage of what has already been proved.*

CHAPTER 2

Several Important Constructions in Metric Spaces

1. The Completion of a Metric Space

Let (X, d) be a metric space. We have discussed the importance of completeness in a metric space several times. Completeness is of particular interest if we are trying to construct a solution a problem via the convergence of approximate solutions. So what should we do if the metric space with which we want to work is not complete? For example, the L^2 and $W^{1,2}$ norms that were introduced in the previous chapter are both of interest due to their relationship with the energy in physical problems. However, we know that neither norm generates a complete metric space on the spaces $C^k([0,1])$. It is possible, by introducing the Lebesgue integral and a suitable equivalence relation on integrable functions, to explicitly construct the natural, complete metric spaces for working with these norms, but it is a rather involved construction. Instead, we will prove an abstract theorem stating that every metric space can be "completed," and define the desired complete metric spaces based on this construction.

THEOREM 1.1. *Let (X, d) be a metric space. Then there exists a complete metric space $(\tilde{X}, \tilde{d})$, called the* completion *of X, and a natural embedding $i : X \to \tilde{X}$ such that $\forall x, y \in X$, $\tilde{d}(i(x), i(y)) = d(x, y)$. Moreover, $i(X)$ is dense in $\tilde{X}$.*

In order to prove this theorem, we need first to construct an appropriate choice of $\tilde{X}$ and $\tilde{d}$ and then show that they have the desired properties.

DEFINITION 1.1. *Suppose that (p_n) and (q_n) are both Cauchy sequences in X. Then we say that (p_n) is* equivalent *to (q_n), denoted $(p_n) \sim (q_n)$, if $\lim_{n\to\infty} d(p_n, q_n) = 0$.*

LEMMA 1.1. *The relation defined in Definition 1.1 is an equivalence relation.*

PROOF. Let (p_n),(q_n) and (r_n) be arbitrary Cauchy sequences in X. Clearly, for each n $d(p_n, p_n) = 0$, so $\lim_{n\to\infty} d(p_n, p_n) = 0$, so $(p_n) \sim (p_n)$ and the relation is reflexive.

Now, suppose that $(p_n) \sim (q_n)$. Since the metric is symmetric, $d(p_n, q_n) = d(q_n, p_n)$ for every $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} d(q_n, p_n) = \lim_{n\to\infty} d(p_n, q_n) = 0$, so $(q_n) \sim (p_n)$ and the relation is symmetric.

Finally, suppose that $(p_n) \sim (q_n)$ and $(q_n) \sim (r_n)$. Then

$$\lim_{n\to\infty} d(p_n, r_n) \leq \lim_{n\to\infty} (d(p_n, q_n) + d(q_n, r_n)) = \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) = 0 + 0 = 0,$$

by the triangle inequality, the properties of limits, and the hypotheses. Therefore $(p_n) \sim (r_n)$ and the relation is transitive. □

We are now in a position to define $\tilde{X}$ and $\tilde{d}$:

Definition 1.2. *Define Y to be the collection of all Cauchy sequences in X, and define $\tilde{X}$ to be the quotient of Y by the equivalence relation defined above.*

Lemma 1.2. *Suppose that $(p_n), (\tilde{p}_n), (q_n)$, and $(\tilde{q}_n)$ are Cauchy sequences in X and $(p_n) \sim (\tilde{p}_n)$ and $(q_n) \sim (\tilde{q}_n)$. Then $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(\tilde{p}_n, \tilde{q}_n)$.*

Proof. Let $(p_n), (\tilde{p}_n), (q_n)$, and $(\tilde{q}_n)$ satisfy the hypotheses of the lemma. Then

$$\begin{aligned}
\lim_{n\to\infty} d(p_n, q_n) &\le \lim_{n\to\infty} \left(d(p_n, \tilde{p}_n) + d(\tilde{p}_n, \tilde{q}_n) + d(\tilde{q}_n, q_n)\right) \\
&= \lim_{n\to\infty} d(p_n, \tilde{p}_n) + \lim_{n\to\infty} d(\tilde{p}_n, \tilde{q}_n) + \lim_{n\to\infty} d(\tilde{q}_n, q_n) \\
&= 0 + \lim_{n\to\infty} d(\tilde{p}_n, \tilde{q}_n) + 0 \\
&= \lim_{n\to\infty} d(\tilde{p}_n, \tilde{q}_n).
\end{aligned}$$

Therefore, $\lim_{n\to\infty} d(p_n, q_n) \le \lim_{n\to\infty} d(\tilde{p}_n, \tilde{q}_n)$. By an identical argument, we can also show that $\lim_{n\to\infty} d(\tilde{p}_n, \tilde{q}_n) \le \lim_{n\to\infty} d(p_n, q_n)$. Hence $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(\tilde{p}_n, \tilde{q}_n)$ as claimed. $\square$

Proposition 1.1. *The function $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ given by $\tilde{d}([(p_n)], [(q_n)]) = \lim_{n\to\infty} d(p_n, q_n)$ is a metric on $\tilde{X}$.*

Proof. First, note that, by the lemma above, the value of $\tilde{d}([(p_n)], [(q_n)])$ is independent of the choice of representatives (p_n) and (q_n) in each equivalence class, so it is a well-defined function on $\tilde{X}$. Also note that it is a nonnegative function by construction, and that, by definition, $\tilde{d}([(p_n)], [(q_n)]) = 0$ if and only if $(p_n) \sim (q_n)$, i.e. $[(p_n)] = [(q_n)]$. Finally, the symmetry and triangle inequality properties are trivial consequences of the same properties on d and the fact that $\tilde{d}$ is well-defined. $\square$

Proposition 1.2. *The metric space $(\tilde{X}, \tilde{d})$ is complete.*

Proof. Suppose that $([p]^m)$ is a Cauchy sequence of elements of $\tilde{X}$. Thus, each $[p]^m$ can be represented by a sequence (p^m_n) in X which is itself Cauchy with respect to d. Choose N_1 so that $\forall j, k > N_1$, $d(p^1_k, p^1_j) < \frac{1}{2}$ and choose some $n_1 > N_1$. Now, suppose that, for $i = 1..I$ a value n_i has been chosen, and then consider the sequence (p^{I+1}_n). Choose N_{I+1} so that, for all $j, k > N_{I+1}$, $d(p^{I+1}_j, p^{I+1}_k) < \frac{1}{2^{I+1}}$, which exists because (p^{I+1}_n) is a Cauchy sequence with respect to n. Then choose $n_{i+1} > \max\{n_I, N_{I+1}\}$.. Now, consider the sequence $(p^m_{n_m})_{m=1}^\infty$. This is a sequence of elements of X, and I claim that it is Cauchy and therefore its equivalence class represents an element of $\tilde{X}$.

To see that $(p^m_{n_m})_{m=1}^\infty$ is Cauchy, let $\epsilon > 0$. Choose M so that whenever $m_1, m_2 > M$, then $d([p^{m_1}], [p^{m_2}]) < \frac{\epsilon}{4}$, and also so that $\frac{1}{2^M} < \frac{\epsilon}{4}$. Let $m_1, m_2 > M$. Then,
$d([p^{m_1}], [p^{m_2}]) < \frac{\epsilon}{4}$. Therefore, $\lim_{n\to\infty} d(p^{m_1}_n, p^{m_2}_n) < \frac{\epsilon}{4}$, so there exists an $N \in \mathbb{N}$ so that when $n_0 > N$,

$$|d(p^{m_1}_{n_0}, p^{m_2}_{n_0}) - \lim_{n\to\infty} d(p^{m_1}_n, p^{m_2}_n)| < \frac{\epsilon}{4},$$

which implies that for $n_0 > N$, $d(p^{m_1}_{n_0}, p^{m_1}_{n_0}) < \frac{\epsilon}{2}$. Now, fix $n > \max\{N, N_{m_1}, N_{m_2}\}$, and consider the following:

$$d(p^{m_1}_{n_{m_1}}, p^{m_2}_{n_{m_2}}) \le d(p^{m_1}_{n_{m_1}}, p^{m_1}_n) + d(p^{m_1}_n, p^{m_2}_n) + d(p^{m_2}_n, p^{m_2}_{n_{m_2}}) < \frac{1}{2^{m_1}} + \frac{\epsilon}{2} + \frac{1}{2^{m_2}} < \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon,$$

where the first and third terms are controlled by the Cauchy nature of (p^{m_1}), and (p^{m_2}) in X, respectively, and the middle term is controlled by the Cauchy nature of $([(p_n)^m])$ in $\tilde{X}$. Therefore, we may conclude that the sequence $(p^m_{n_m})$ is Cauchy in X. For simplicity, let us call this sequence $p = (p_n)$ from here on.

Now, I claim that $([p^m]) \to [p]$ in the $\tilde{X}$ topology. We need to show that $\lim_{m\to\infty} \tilde{d}([p^m], [p]) = 0$, i.e. that $\lim_{m\to\infty} \lim_{n\to\infty} d(p^m_n, p_n) = 0$. Let $\epsilon > 0$. Choose M so that for $n, m > M$, $d(p_n, p_m) < \frac{\epsilon}{2}$, and so that $\frac{1}{2^M} < \frac{\epsilon}{2}$. Now let $m > M$. Since (p^m_n) is Cauchy, we have the N_m found above so that $\forall j, k > N_m$, $d(p^m_j, p^m_k) < \frac{1}{2^m} < \frac{1}{2^M} < \frac{\epsilon}{2}$, and also recall that by construction $n_m > N_m$. Then, for all $n > \max\{M, N_m\}$, we have that

$$d(p^m_n, p_n) \leq d(p^m_n, p^m_{n_m}) + d(p^m_{n_m}, p_n) = d(p^m_n, p^m_{n_m}) + d(p_m, p_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

by our choice of n and the construction of p. Hence, for all $m > M$, $\lim_{n\to\infty} d(p^m_n, p_n) \leq \epsilon$. Thus $\tilde{d}([p^m], [p]) < \epsilon$. Therefore, $\lim_{m\to\infty} \tilde{d}([p^m], [p]) = 0$ as claimed. This means that every Cauchy sequence in $\tilde{X}$ has a limit in $\tilde{X}$, so $\tilde{X}$ is indeed complete. □

PROPOSITION 1.3. *Let $i : X \to \tilde{X}$ be given by $i(x) = [(x, x, \ldots, x, \ldots)]$. Then i is an* isometry. *That is, for every $x, y \in X$, $\tilde{d}(i(x), i(y)) = d(x, y)$.*

PROOF. First note that clearly $(x, x, \ldots)$, being a constant sequence, is Cauchy, so it generates a well-defined equivalence class in $\tilde{X}$. Then, compute

$$\tilde{d}(i(x), i(y)) = \lim_{n\to\infty} d(i(x)_n, i(y)_n) = \lim_{n\to\infty} d(x, y) = d(x, y),$$

as claimed. □

PROPOSITION 1.4. *The image $i(X)$ is dense in $\tilde{X}$.*

PROOF. Let $[(p_n)]$ be any element of $\tilde{X}$, and let $\epsilon > 0$. Since (p_n) is a Cauchy sequence in X, there is an $N \in \mathbb{N}$ so that, for all $n, k > N$, $d(p_n, p_k) < \epsilon$. Fix a single $k > N$, and let $x = p_k$. Then $i(x) = [(x, x, \ldots)]$. Consider

$$\tilde{d}([(p_n)], i(x)) = \lim_{n\to\infty} d(p_n, i(x)_n) = \lim_{n\to\infty} d(p_n, p_k).$$

For n sufficiently large, $d(p_n, p_k) < \epsilon$ by the above calculation, so $\tilde{d}([(p_n)], i(x)) < \epsilon$. Therefore, $i(X)$ is dense in $\tilde{X}$ as claimed. □

This completes the proof of Theorem 1.1.

1.0.2. *Exercises.*

EXERCISE 1.1. *Suppose that X is a complete metric space. Show that X and $\tilde{X}$ are isometric. (That is, the map i constructed above is surjective.)*

EXERCISE 1.2. *Compute the completion of the rationals with respect to the standard topology and justify your work.*

EXERCISE 1.3. *Give several examples of functions in the completion of $C([0,1])$ with respect to the L^2 norm that are not themselves in $C([0,1])$. For each, show an example of a Cauchy sequence whose equivalence class generates the function.*

1.1. The Space L^p.

DEFINITION 1.3. *$L^2([0,1])$ is the completion of $C([0,1])$ with respect to the L^2-metric defined in* (2)*.*

DEFINITION 1.4. *$L^p([0,1])$ is the completion of $C([0,1])$ with respect to the L^p-metric defined on Test #1:*

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}}. \tag{5}$$

Our definition of L^p is rather abstract, so let's look at what it means for something to be an element of L^p.

- We normally think of the elements of L^p as being functions.
- Give some examples of elements of L^p:

- Give some examples of elements of L^p that are not in $C([0,1])$:

- What does the equivalence relation in the definition of $\tilde{X}$ mean in this context?
- Give an example of two equivalent functions in L^2. What makes them equivalent?

- Give some examples of elements of L^1 that are not in L^2. What about vice versa? What about, say, L^4?

- If $p \neq q$, how are L^p and L^q related? Do they contain the same functions? Why or why not? Compare and contrast with the situation for l^p and l^q.

  LEMMA 1.3. **(Hölder's Inequality)** *Suppose that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p([0,1])$ and $g \in L^q([0,1])$. Then*
  $$\int_0^1 |f(x)g(x)|dx \leq \|f\|_p \|g\|_q.$$

PROOF. By Young's Inequality, we have that if $a, b \in \mathbb{R}$ then $|ab| \leq \frac{1}{p}a^p + \frac{1}{q}b^q$. Therefore, for any $x \in [0,1]$, $|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$. Now if f or g is identically zero, the claim is trivial, so we may assume they are not. Let $\tilde{f}(x) = \frac{f(x)}{\|f\|_p}$ and $\tilde{g}(x) = \frac{g(x)}{\|g\|_q}$. By the homogeneity of the L^p norms, these have unit L^p and L^q norms respectively. Then

$$\int \frac{|f(x)g(x)|}{\|f\|_p\|g\|_q}dx = \int |\tilde{f}(x)\tilde{g}(x)|dx \leq \int (\frac{1}{p}|\tilde{f}(x)|^p + \frac{1}{q}|\tilde{g}(x)|^q)dx = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus $\int |f(x)g(x)|dx \leq \|f\|_p\|g\|_q$. ☐

PROPOSITION 1.5. *Suppose that* $1 \leq p < q < \infty$. *Then* $L^q([0,1]) \subset L^p([0,1])$ *and if* $f \in L^q([0,1])$, *then* $\|f\|_p \leq \|f\|_q$.

PROOF. Suppose $f \in L^q([0,1])$. Then there is a sequence (f_n) of continuous functions so that $\|f_n - f\|_q \to 0$. We will show that $\|f_n - f\|_p \to 0$ as well, which will imply that f is in L^p. Now, define $r = \frac{q}{p}$ and $s = \frac{q}{q-p}$. Note that since $q > p$, these numbers are both greater than 1, and also $\frac{1}{r} + \frac{1}{s} = 1$. Then compute:

$$\begin{aligned}
\|f_n - f\|_p^p &= \int_0^1 |f_n(x) - f(x)|^p dx \\
&= \int_0^1 1 \cdot |f_n(x) - f(x)|^p dx \\
&\leq \|1\|_s \||f_n(x) - f(x)|^p\|_r,
\end{aligned}$$

by Hölder's inequality (Lemma 1.3). Then $\|1\|_s = (\int 1^s dx)^{\frac{1}{s}} = 1$, and

$$\||f_n(x) - f(x)|^p\|_r = (\int_0^1 |f_n(x) - f(x)|^{p\frac{q}{p}} dx)^{\frac{p}{q}} = (\int_0^1 |f_n(x) - f(x)|^q dx)^{\frac{1}{q}p} = \|f_n - f\|_q^p.$$

Taking pth roots, we conclude that $\|f_n - f\|_p \leq \|f_n - f\|_q \to 0$. So, $f \in L^p$ as claimed.

Moreover, note that $\|f\|_p^p \leq \|1\|_s\|f\|_q^p$ by the same Hölder calculation as above, so it follows that $\|f\|_p \leq \|f\|_q$. ☐

DEFINITION 1.5. *The space* l^p *is defined as* $\{(x_n)_{n=1}^\infty | \sum_{n=1}^\infty |x_n|^p < +\infty\}$, *and equipped with the norm* $\|(x_n)\|_p = (\sum_{n=1}^\infty |x_n|^p)^{\frac{1}{p}}$.

LEMMA 1.4. *The space* $(l^p, \|\cdot\|_p)$ *is a Banach space (complete normed vector space).*

PROOF. This proof is a combination of the proofs for l^2 and L^p. ☐

PROPOSITION 1.6. *If* $1 \leq p < q < \infty$, *then* $l^p \subset l^q$.

PROOF. Suppose that $(x_n) \in l^p$. Then $\sum_{n=1}^\infty |x_n|^p < +\infty$. Therefore, by the divergence test, $\lim_{n\to\infty} x_n = 0$. Hence there is an $N \in \mathbb{N}$ such that $\forall n > N$

$|x_n| < 1$. Select such an N. Now, consider $\sum_{n=1}^{\infty} |x_n|^q$:

$$\begin{aligned}\sum_{n=1}^{\infty} |x_n|^q &= \sum_{n=1}^{N} |x_n|^q + \sum_{n=N+1}^{\infty} |x_n|^q \\ &= \sum_{n=1}^{N} |x_n|^q + \sum_{n=N+1}^{\infty} (|x_n|^p |x_n|^{q-p}) \\ &< \sum_{n=1}^{N} |x_n|^q + \sum_{n=N+1}^{\infty} (|x_n|^p 1^{q-p}) \\ &= \sum_{n=1}^{N} |x_n|^q + \sum_{n=N+1}^{\infty} |x_n|^p \\ &< +\infty\end{aligned}$$

Here, the third line is due to the fact that $q - p > 0$ and $|x_n| < 1$ for all $n \geq N + 1$, so $|x_n|^{q-p} < 1^{q-p}$ for all such n. The final line then follows from the fact that the first term on the right-hand side is a finite sum and hence finite, and the second term on the right-hand side is finite by the hypothesis that $(x_n) \in l^p$. Therefore, it follows that $(x_n) \in l^q$ as claimed. $\square$

Notice that the situations in l^p and $L^p([0, 1])$ are opposites.

- Compare and contrast the case for $L^p([0, 1])$ and $L^p(\mathbb{R})$.

  Suppose that $p < q$. In $\mathbb{R}$, we can construct functions that are in L^p but not L^q **and** vice versa. Therefore there is no containment result like the ones above. This is because the examples from $L^p([0, 1])$ and the examples from l^p are both in play (if defined carefully). This was discussed in class and you might want to record the examples here for future reference.

1.1.1. *Exercises.*

Exercise 1.4. *Show that* $\|[(f_n)]\|_p := \left(\lim_{n\to\infty} \int_0^1 |f_n(x)|^p dx\right)^{\frac{1}{p}}$ *is a norm on* $L^p([0, 1])$ *which generates the metric defined through the completion.*

Exercise 1.5. *Consider the square* $S := [0, 1] \times [0, 1] \subset \mathbb{R}^2$. *Give some examples of functions that are and are not in* $L^2(S)$, *and explain your reasoning.*

Exercise 1.6. *Show that* l^p *is a Banach space.*

1.2. The Space $W^{1,2}$. Consider the space $C^1([0, 1])$ with the norm defined in Definition 2.12. As discussed above, this space is not complete. But we can compute its completion! Define

$$W^{1,2}([0, 1]) := \text{ the completion of } C^1([0, 1]) \text{ with respect to } \|\cdot\|_{W^{1,2}}.$$

- What kind of functions can be in $W^{1,2}$?
- Can a function in $W^{1,2}$ be discontinuous?
- Can a function in $W^{1,2}$ have a discontinuous derivative?
- Can a function in $W^{1,2}$ have a cusp?
- How continuous does a function in $W^{1,2}$ have to be?
- How does this compare to $W^{1,p}$?
- What happens if we require more derivatives?
- What happens if we change our domain from $[0,1]$ to $\mathbb{R}$?

1.2.1. *Exercises.*

EXERCISE 1.7. *Show that if* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are both norms on a given space* X*, then* $\|\cdot\|_1 + \|\cdot\|_2$ *is also a norm on* X*.*

2. The Contraction Mapping Principle

DEFINITION 2.1. *A point* $x \in X$ *is called a* fixed point *of a function* $f : X \to X$ *if* $f(x) = x$*.*

DEFINITION 2.2. *A function* $f : X \to X$ *is called a* contraction mapping *if there exists a constant* r *with* $0 \leq r < 1$ *such that for all* $x, y \in X$

$$d(f(x), f(y)) \leq r\ d(x, y).$$

THEOREM 2.1 (Contraction Mapping Theorem). *Let* f *be a contraction mapping on a complete nonempty metric space,* X*. Then* f *has a unique fixed point.*

PROOF. Let X be a complete nonempty metric space and $f : X \to X$ be a contraction mapping.

[Uniqueness] Assume $x, y \in X$ are fixed points. This implies $d(x, y) = d(f(x), f(y))$. But since f is a contraction we have $d(x, y) = d(f(x), f(y)) \leq r\ d(x, y)$ for $0 \leq r < 1$. This is impossible if $d(x, y) > 0$. Thus $d(x, y) = 0$. Therefore $x = y$.

[Existence] Let $x_0 \in X$. Define the sequence $\{x_n\}$ recursively by

$$x_{n+1} = f(x_n)$$

for $n \geq 0$. We claim $\{x_n\}$ is a Cauchy sequence:

Let $a = d(x_0, x_1)$. Now consider $d(x_{n+1}, x_n)$. Since f is a contraction mapping, we have

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq r d(x_n, x_{n-1}).$$

By repeating this n times, we see that $d(x_{n+1}, x_n) \leq r^n d(x_0, x_1) = ar^n$. Now if $m < n$, then

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{j=m}^{n-1} d(x_{j+1}, x_j) \\ &\leq \sum_{j=m}^{n-1} ar^j \\ &= \frac{ar^m - ar^n}{1-r} \\ &= \frac{ar^m(1 - r^{n-m})}{1-r} \\ &\leq \frac{ar^m}{1-r}. \end{aligned}$$

We used the triangle inequality to obtain the first inequality. For the second inequality, we used the previous calculation that $d(x_{n+1}, x_n) \leq ar^n$. The two subsequent equalities are from the geometric series summation formula. The last inequality holds since $m < n$ and $r < 1$, so we have $1 - r^{n-m} < 1$.

Let $\epsilon > 0$. Choose N large enough such that $\dfrac{ar^N}{1-r} < \epsilon$. Now for $n \geq m > N$ we have $d(x_m, x_n) \leq \dfrac{ar^m}{1-r} < \dfrac{ar^N}{1-r} < \epsilon$. Thus $\{x_n\}$ is a Cauchy sequence.

Now since X is complete and $\{x_n\}$ is Cauchy we know that $\{x_n\}$ converges in X. Let $x = \lim_{n\to\infty} x_n$. Since f is continuous, we have that $f(x) = \lim_{n\to\infty} f(x_n)$. But since $f(x_n) = x_{n+1}$ we have $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$. Thus $f(x) = x$. Therefore x is a fixed point. □

This proof actually gives a constructive method to find the fixed point, by iteration of the map.

Corollary 2.1. *Let f be a contraction mapping on a complete nonempty metric space X. If x_0 is any point of X, and $x_{n+1} = f(x_n)$ for $n \geq 0$ then the sequence $\{x_n\}$ converges to the fixed point of f.*

2.0.2. *Exercises.*

Exercise 2.1. *Show that any contraction mapping is continuous.*

2.1. Existence and Uniqueness for Solutions to Differential Equations. In this section we apply the Contraction Mapping Theorem to derive a fundamental result for ordinary differential equations. You may recall from your undergraduate ODE class that you learned to solve *initial value problems* such as

Example 2.1.

$$y'(t) = (y(t))^2, y(0) = 2.$$

If you consider other examples such as

EXAMPLE 2.2.

$$y'(t) = (y(t))^{\frac{1}{2}}, y(0) = 0,$$

then you will discover that this problem has more than one solution.

How can we generalize what we observe in these examples? It would be extremely useful to be able to look at a problem and quickly determine whether or not it has a solution. For applied problems it is also very important to know that the problem has just one solution. Otherwise how do you know which solution is the one that appears in the *real world*?

We begin by noticing that the general initial value problem (IVP), can be stated as

$$(IVP): \quad y'(t) = f(y), y(0) = y_0. \tag{6}$$

From the examples above it is clear that having a continuous f is not enough to guarantee the existence of exactly one solution. We need something that is a little bit better than continuous, such as ...

DEFINITION 2.3. $f : [a, b] \to \mathbb{R}$ *is* Lipschitz continuous *if there is a* $k > 0$ *so that,* $\forall x, y \in [a, b]$, $|f(x) - f(y)| \leq k|x - y|$.

Note that Lemma 2.5 states that if a function has a bounded derivative then it is Lipschitz. However, Lipschitz functions do not need to have a derivative at every point in their domain.

DEFINITION 2.4. $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ *is* uniformly Lipschitz continuous *if there is a* $k > 0$ *so that,* $\forall t \in [a, b]$, $\forall x, y \in \mathbb{R}$, $|f(t, x) - f(t, y)| \leq k|x - y|$.

THEOREM 2.2. *If* f *is uniformly Lipschitz continuous on an open rectangle containing* $(0, y_0)$*, then there is an* $\epsilon > 0$ *and a differentiable function* $y \in C^1([0, \epsilon])$ *such that* y *is the unique solution of* (6) *on* $[0, \epsilon]$.

The plan is to transform the problem into a fixed point problem and then apply the contraction mapping theorem.

PROOF. First, note that y is a C^1 solution to (6) on $[0, \epsilon]$ if and only if it solves $y(0) = y_0$ and

$$\int_0^t y'(s)ds = \int_0^t f(s, y(s))ds$$

for all $t \in [0, \epsilon]$. This equation was obtained by integrating both sides of the differential equation from 0 to t. Next, we apply the Fundamental Theorem of Calculus and the initial value to get

$$y(t) - y_0 = \int_0^t f(s, y(s))ds.$$

Thus we have a fixed point problem $y = F(y)$, where

$$F(y)(t) = y_0 + \int_0^t f(s, y(s))ds.$$

Rather than seek a fixed point in all of $C([0, 1])$, we will define an appropriate closed ball on which to work. Let

$$B := B_1(y_0) = \{y \in C([0, \epsilon]) : \sup\{|y(t) - y_0| : 0 \leq t \leq \epsilon\} \leq \ 1\}.$$

Since B is a closed subset of a complete metric space, $C([0, 1])$, it is also complete. (Exercise.) For the moment we have not yet chosen ϵ.

We must check that, for an appropriate choice of ϵ, F is a contraction mapping on B. First we must check that at least F maps B to itself. So, suppose that $y \in B$. Then, forall $t \in [0, \epsilon]$, $|y(t) - y_0| < 1$. Therefore,

$$
\begin{aligned}
|F(y)(t) - y_0| &= |y_0 + \int_0^t f(s, y(s))ds - y_0| \\
&\leq \int_0^t |f(s, y(s))|ds \\
&\leq \int_0^t |f(s, y(s)) - f(s, y_0)|ds + \int_0^t |f(s, y_0)|ds \\
&\leq \int_0^t k|y(s) - y_0|ds + \sup_{t \in [0,1]} |f(t, y_0)|\epsilon \\
&\leq k\epsilon + \sup_{t \in [0,1]} |f(t, y_0)|\epsilon
\end{aligned}
$$

So, $\sup\{|F(y)(t) - y(0)| : 0 \leq t \leq \epsilon\} \leq \epsilon(k + \sup_{t \in [0,1]} |f(t, y_0)|)$. In order for $F(y)$ to be in B, we need this to be at most 1, and it is clear that we can achieve that for $\epsilon = \frac{1}{2(k + \sup_{t \in [0,1]} |f(t, y_0)|}$. Hence for ϵ sufficiently small, F maps B to itself.

It remains to show that F is a contraction mapping on B. Let $y, z \in B$ and consider

$$
\begin{aligned}
|F(y)(t) - F(z)(t)| &= |y_0 + \int_0^t f(s, y(s))ds - (y_0 + \int_0^t f(s, z(s))ds| \\
&\leq \int_0^t |f(s, y(s)) - f(s, z(s))|ds \\
&\leq \int_0^t k|y(s) - z(s)|ds \\
&\leq k\epsilon \sup_{0 \leq t \leq \epsilon} |y(t) - z(t)| \\
&\leq \frac{k}{2(k + \sup_{t \in [0,1]} |f(t, y_0)|)} \sup_{0 \leq t \leq \epsilon} |y(t) - z(t)|.
\end{aligned}
$$

Letting $r = \frac{k}{2(k + \sup_{t \in [0,1]} |f(t, y_0)|)}$, which is clearly positive and strictly less than one, we conclude that $\|F(y) - F(z)\|_\infty \leq r\|y - z\|_\infty$ if $y, z \in B$. Therefore F is a contraction mapping on B as claimed.

Then by the Contraction Mapping Principle there is a unique fixed point of F in B. Therefore (6) has a unique solution on $[0, \epsilon]$. $\square$

2.1.1. *Exercises.*

EXERCISE 2.2. *Find a solution to Example 2.1. Notice that you can find infinitely many functions that solve the ODE, but only one which solves the ODE* **and** *satisfies the initial value. Also notice that the solution is only valid on a finite interval, which can be somewhat surprising given that the problem looks* nice.

EXERCISE 2.3. *Find at least two solutions to Example 2.2, both solving the same given initial condition.*

EXERCISE 2.4. *Prove that a closed subset of a complete metric space is complete.*

3. Function Approximation

It is often helpful to know that functions that are not so nice can be approximated by functions that are very nice. Another way to express this is to say that the set of *nice* functions is dense in the set of *not so nice* functions. (How is that for being precise?!)

The term *nice* can mean a lot of things. Being infinitely differentiable is nice. Being analytic, *i.e.* a power series, is nicer. Being a polynomial is perhaps the nicest. In this section we will prove that every function in $C([0,1])$ can be approximated to any degree of accuracy by a polynomial, *i.e.* that polynomials are dense in $C([0,1])$. But before getting to this famous theorem of Weierstrass we will discuss function approximation in general.

Suppose that we are trying to approximate the discontinuous function

$$f(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}$$

We would like the approximating function to be a little bit nicer, *i.e.* continuous. For the moment we won't be too picky about the metric being used, we just want to find a nice continuous function that approximates f in an intuitive way. Moreover, we would like to be able to adjust the approximation so that it can be improved, *i.e.* we want to create a sequence of approximations, (f_n), that get better as $n \to \infty$.

We already know one sequence of functions that meets these requirements, namely

$$f_n(x) = \begin{cases} (2x)^n & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}.$$

Here is another choice, g_n:

$$g_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}(x - \frac{1}{2} + \frac{1}{n}) & \frac{1}{2} - \frac{1}{n} \le x < \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$

You should sketch some graphs of f, f_n, and g_n. Notice that (g_n) is an approximation of f by piecewise linear functions, which is quite a bit easier (and more natural) to work with than (f_n).

How can we best capture this idea using analysis? First, notice that $g_n(x)$ is the average of $f(x)$ over the interval $[x - \frac{1}{n}, x + \frac{1}{n}]$. This is an important conceptual point. Averaging in analysis is a way of smoothing out irregularities and is most often achieved via integration. Recall from your calculus experience that the average of a function $f(x)$ over an interval $[a,b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

Thus we can rewrite our formula for $g_n(x)$ as

$$g_n(x) = \frac{n}{2}\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(y)dy.$$

Notice that there is a small technical problem with this formulation. If x is close to 1, and we attempt to average over the interval $[x - \frac{1}{n}, x + \frac{1}{n}]$, then it could be that $x + \frac{1}{n} > 1$, so that part of our averaging happens in an area where $f(x)$ is not defined. This problem is easily fixed by *extending* $f(x)$ to a function that has values outside of $[0, 1]$, its original domain. There is more than one way to do this. Most commonly we will extend the function so that it is eventually 0 outside of some larger interval.

There is a second technical problem that, at first, might not seem like a problem. How can all of the averaging information be consolidated into one convenient package inside the integral? Right now the information lies both in the limits of integration and in the constant in front of the integral. If we wanted to modify the averaging process in any way, it is not easy to see how we might do this using the current form. Denote by $\chi_S(x)$ the indicator function of a set S. That is, $\chi_S(x) = 1$ if $x \in S$ and 0 otherwise. Here is a standard way to rewrite the integral...

$$\begin{aligned} \tfrac{n}{2}\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(y)dy &= \tfrac{n}{2}\int_{-\infty}^{\infty} \chi_{[x-\frac{1}{n}, x+\frac{1}{n}]} f(y)dy \\ &= \tfrac{n}{2}\int_{-\infty}^{\infty} \chi_{[-\frac{1}{n}, \frac{1}{n}]}(y-x) f(y)dy \\ &= \int_{-\infty}^{\infty} K_n(y-x) f(y)dy, \end{aligned}$$

where $K_n(y) := \frac{n}{2}\chi_{[-\frac{1}{n}, \frac{1}{n}]}$. K_n is often called the *mollifier* or the *convolution kernel.* Notice that it captures all of the information necessary to describe the approximation scheme. The approximation process can be summarized as follows. To approximate a function $f(y)$ near a point x ...

- **:** (i) Extend f to a function whose domain is all of $\mathbb{R}$.
- **:** (ii) Create an appropriate kernel $K_n(y)$.
- **:** (iii) Shift $K_n(y)$ so that it is centered at x, *i.e.* $K_n(y - x)$.
- **:** (iv) Multiply $f(y)$ by $K_n(y - x)$.
- **:** (v) Integrate over $\mathbb{R}$.

As a technical detail notice that K_n is even, *i.e.* $K_n(-y) = K_n(y)$, so we can switch $K_n(y - x)$ with $K_n(x - y)$ in the integral without changing anything. This ends up being a technical convenience in later definitions.

The key properties that we abstract from the process above are that

Definition 3.1. *A sequence of functions* (K_n) *are called mollifiers if:*

- **:** nonnegativity: $K_n \geq 0$,
- **:** unit area: $\int_{-\infty}^{\infty} K_n(y)dy = 1$, *and*
- **:** concentration: $K_n(y)$ concentrates *at* 0, i.e. *given any* $\delta > 0$, *we have*

$$\lim_{n\to\infty}\left(\int_{-\infty}^{-\delta} K_n(y)dy + \int_{\delta}^{\infty} K_n(y)dy\right) = 0.$$

We will use the formula

$$f_n(x) := (f * K_n)(x) = \int_{-\infty}^{\infty} f(y)K_n(x - y)dy,$$

to create a sequence of approximations.

3.1. Convolution and Mollification.

Definition 3.2. *Suppose that* f, g *are functions. The* convolution *of* f *and* g *is defined to be*

$$f * g(x) := \int_{-\infty}^{\infty} f(y)g(x-y)dy,$$

as long as this integral is well-defined.

Remark 3.1. *Given what we know in this course so far, it is difficult to give a technically precise definition of the broadest cases where this convolution operation is well-defined. It will turn out that as long as* f *is in one of the* L^p *spaces for some* $p \geq 1$ *and* g *is "nice" (e.g. continuous, and compactly supported), then the convolution will be well defined. For the purposes of approximation, this is all that we need. In fact,* g *does not even need to be quite that nice.*

Definition 3.3. *We say that a function* $f : \mathbb{R} \to \mathbb{R}$ *is* integrable *if* $\int_{-\infty}^{\infty} f(x)dx$ *is well-defined and finite.*

Definition 3.4. *A function* $g : \mathbb{R} \to \mathbb{R}$ *is called* compactly supported *if* $\exists M > 0$ *such that* $g(x) = 0$ *for every* x *such that* $|x| > M$.

Lemma 3.1. *The convolution of* f *and* g *satisfies the following properties:*

(1) $f * g(x) = g * f(x)$.
(2) *If* f *is integrable,* g *is bounded, and the convolution is well-defined, then* $f * g$ *is bounded.*
(3) *If* f *is integrable and* g *is continuous and compactly supported, then* $f*g$ *is uniformly continuous.*
(4) *If* f *is integrable and* g *is continuously differentiable and compactly supported, then* $f * g$ *is differentiable and* $(f * g)' = f * (g')$.

Note that the last part of this lemma can be applied repeatedly to obtain higher levels of differentiability for $f * g$ if g is smooth.

Proof. For part 1, compute, making the u-substitution $u = x-y$, $y = x-u$, $du = -dy$,

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} f(y)g(x-y)dy \\ &= \int_{\infty}^{-\infty} f(x-u)g(u)(-du) \\ &= \int_{-\infty}^{\infty} f(x-u)g(u)du \\ &= g * f(x) \end{aligned}$$

as claimed.

For part 2, first note that, since f is integrable, $0 \leq \int_{-\infty}^{\infty} |f(x)|dx < +\infty$. Let $M = \int_{-\infty}^{\infty} |f(x)|dx$. Also, since g is continuous with compact support, it is bounded, so there is a $C > 0$ so that, $\forall x \in \mathbb{R}$, $|g(x)| \leq C$. Then, for $x \in \mathbb{R}$,

$$|f * g(x)| \leq \int_{-\infty}^{\infty} |f(y)||g(x-y)|dy \leq C \int_{-\infty}^{\infty} |f(y)|dy \leq CM.$$

Therefore, $f * g$ is bounded.

For part 3, we want to show that $f * g$ is uniformly continuous. Note that if g is continuous and compactly supported, then g is uniformly continuous, so $\forall \epsilon > 0\ \exists \delta > 0$ so that if $|x_1 - x_2| < \delta$ then $|g(x_1) - g(x_2)| < \epsilon$. Let $\epsilon > 0$, and choose $\delta > 0$ so that if $|x_1 - x_2| < \delta$ then $|g(x_1) - g(x_2)| < \frac{\epsilon}{M}$. Then suppose that $x_1, x_2 \in \mathbb{R}$ with $|x_1 - x_2| < \delta$, and note that this implies that, $\forall y \in \mathbb{R}$, $|(x_1 - y) - (x_2 - y)| < \delta$. Then compute:

$$\begin{aligned}
|(f * g)(x_1) - (f * g)(x_2)| &= \left| \int_{-\infty}^{\infty} f(y)g(x_1 - y)dy - \int_{-\infty}^{\infty} f(y)g(x_2 - y)dy \right| \\
&= \left| \int_{-\infty}^{\infty} f(y)(g(x_1 - y) - g(x_2 - y))dy \right| \\
&\leq \int_{-\infty}^{\infty} |f(y)||g(x_1 - y) - g(x_2 - y)|dy \\
&< \int_{-\infty}^{\infty} |f(y)| \frac{\epsilon}{M} dy \\
&= M\frac{\epsilon}{M} = \epsilon.
\end{aligned}$$

Hence it follows that $f * g$ is uniformly continuous.

Finally, for the fourth claim, suppose that g is a continuously differentiable function. Note that this means that g' is a continuous. Let $x \in \mathbb{R}$ and consider the difference quotient $D_h(x) = \frac{f*g(x+h) - f*g(x)}{h}$. We want to show that $\lim_{h\to 0} D_h(x)$ exists and equals $f * (g')(x)$. But

$$\begin{aligned}
D_h(x) &= \frac{f * g(x+h) - f * g(x)}{h} \\
&= \frac{1}{h}\left(\int_{-\infty}^{\infty} f(y)g(x + h - y)dy - \int_{-\infty}^{\infty} f(y)g(x - y)dy \right) \\
&= \frac{1}{h}\left(\int_{-\infty}^{\infty} f(y)[g(x + h - y) - g(x - y)]dy \right) \\
&= \int_{-\infty}^{\infty} f(y) \left[\frac{g(x + h - y) - g(x - y)}{h} \right] dy.
\end{aligned}$$

Therefore, since the integrals are uniformly bounded in h (because g and the derivative of g are uniformly bounded),

$$\begin{aligned}
\lim_{h\to 0} D_h(x) &= \lim_{h\to 0} \int_{-\infty}^{\infty} f(y) \left[\frac{g(x + h - y) - g(x - y)}{h} \right] dy \\
&= \int_{-\infty}^{\infty} f(y) \left[\lim_{h\to 0} \frac{g(x + h - y) - g(x - y)}{h} \right] dy \\
&= \int_{-\infty}^{\infty} f(y)g'(x - y)dy \\
&= f * (g')(x)
\end{aligned}$$

as claimed. $\square$

Based on this lemma, we see that $f*g$ inherits the smoothness properties of the smoothest function (f or g) in the convolution. Suppose that, as in the introduction to this section, we have a function $f : [0,1] \to \mathbb{R}$ that is "not nice." Extend f to $\mathbb{R}$ so that it is compactly supported, and hence integrable. If g is "nice," then $f*g$ will be just as "nice." Our example earlier in the section tells us that, in fact, $f*g$ may be nicer than *both* f and g, since g_n is continuous for each n even though f is not, and $K_n = \frac{n}{2}\chi_{[-\frac{1}{n},\frac{1}{n}]}$ is also not continuous.

Finally, we need to confirm that, if we convolve f with a sequence of averaging kernels K_n, then as $n \to \infty$, $f_n = K_n * f$ does indeed converge to f. For the purposes of the Weierstrass Approximation Theorem, the following lemma will suffice:

LEMMA 3.2. *Suppose that f is a bounded, integrable function on $\mathbb{R}$, and that S is a compact subset of $\mathbb{R}$ on which f is continuous. Then, if K_n is a sequence of convolution kernels satisfying Definition 3.1, the functions $f_n := f * K_n$ converge to f uniformly on S.*

PROOF. We have that $f_n(x) = \int_{-\infty}^{\infty} f(x-t)K_n(t)dt$ by part 1 of Lemma 3.1. Additionally, because $\int_{-\infty}^{\infty} K_n(x)dx = 1$, we have that $f(x) = \int_{-\infty}^{\infty} f(x)K_n(t)dt$. So,

$$f_n(x) - f(x) = \int_{-\infty}^{\infty} [f(x-t) - f(x)]K_n(t)dt.$$

Let $\epsilon > 0$. We want to show that there is an N so that $\forall n > N$, $\forall x \in S$, $|f_n(x) - f(x)| < \epsilon$. Since S is compact and f is continuous on S, f is uniformly continuous on S. Therefore, there exists a $\delta > 0$ so that, if $x \in S$ and $|t| < \delta$, then $|f(x-t) - f(x)| < \frac{\epsilon}{2}$. Let $M = \sup_{\mathbb{R}} |f(x)|$, which is finite because f is bounded. Since $\lim_{n\to\infty} \left(\int_{-\infty}^{-\delta} K_n(t)dt + \int_{\delta}^{\infty} K_n(t)dt \right) = 0$, there is an $N \in \mathbb{N}$ so that $\forall n > N$, $\int_{-\infty}^{-\delta} K_n(t)dt + \int_{\delta}^{\infty} K_n(t)dt < \frac{\epsilon}{4M}$. Finally, let $n > N$ and compute, for any $x \in S$,

$$\begin{aligned}
|f_n(x) - f(x)| &= \left| \int_{-\infty}^{-\delta} [f(x-t) - f(x)]K_n(t)dt + \int_{-\delta}^{\delta} [f(x-t) - f(x)]K_n(t)dt + \int_{\delta}^{\infty} [f(x-t) - f(x)]K_n(t)dt \right| \\
&\le \int_{-\infty}^{-\delta} |f(x-t) - f(x)|K_n(t)dt + \int_{-\delta}^{\delta} |f(x-t) - f(x)|K_n(t)dt + \int_{\delta}^{\infty} |f(x-t) - f(x)|K_n(t)dt \\
&= \int_{-\infty}^{-\delta} |f(x-t) - f(x)|K_n(t)dt + \int_{\delta}^{\infty} |f(x-t) - f(x)|K_n(t)dt + \int_{-\delta}^{\delta} |f(x-t) - f(x)|K_n(t)dt \\
&< (2M)\frac{\epsilon}{4M} + \frac{\epsilon}{2} \cdot 1 = \epsilon,
\end{aligned}$$

where we have used our choices of δ and n above, and the fact that $\int_{-\delta}^{\delta} K_n(t)dt \le \int_{-\infty}^{\infty} |f(x-t) - f(x)|K_n(t)dt = 1$. Since x was arbitrary, it follows that $f_n \to f$ uniformly on S as claimed. ☐

3.1.1. *Exercises.*

EXERCISE 3.1. *Suppose that K_n is $\frac{n}{2}\chi_{[-\frac{1}{n},\frac{1}{n}]}(x)$ (the example discussed above). For each of the following functions, draw pictures of f and $f * K_n$ for several values of n.*

-
$$f(x) = \begin{cases} 1 & \frac{1}{2} \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

- $$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

- $$f(x) = \begin{cases} 4(x-\frac{1}{2})^2 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

(You may want to use Maple to visualize this.)

EXERCISE 3.2. *Show that if f and g are both integrable functions, then $f * g$ is also integrable, and*

$$\int_{-\infty}^{\infty} |f * g(x)| dx \le \left(\int_{-\infty}^{\infty} |f(x)| dx\right) \left(\int_{-\infty}^{\infty} |g(x)| dx\right).$$

This is a special case of Young's inequality *for convolutions.*

3.2. The Weierstrass Approximation Theorem.

THEOREM 3.1. *(Weierstrass Approximation Theorem) The set of polynomials is dense in $C[0,1]$.*

In order to prove this theorem, we need to apply our work in the previous section to a well-chosen sequence of convolution kernels which will produce polynomials.

Define

$$q_n(x) = \begin{cases} (1-\frac{x^2}{4})^n & -2 \le x \le 2 \\ 0 & |x| > 2 \end{cases},$$

and $c_n := \int_{-\infty}^{\infty} q_n(x) dx$. Then let $p_n(x) = \frac{1}{c_n} q_n(x)$. Notice that I have not computed c_n, and in fact I never need to while proving the statements below. I would have to compute c_n if I wanted to find explicit formulas for the approximations.

LEMMA 3.3. *The (p_n) form a sequence of convolution kernels satisfying the conditions listed in 3.1.*

PROOF. The nonnegativity and evenness of p_n should be clear from the definition. The unit area condition is guaranteed by our choice of c_n. We just need to check the concentration property. This requires a little bit of calculus. Compute $q_n'(x) = \frac{-2nx}{4}\left(1-\frac{x^2}{4}\right)^{n-1}$, and $q_n''(x) = n\left(1-\frac{x^2}{4}\right)^{n-2}[\frac{x^2}{8}(2n-1)-\frac{1}{2}]$. Notice that this is always greater than or equal to its value at $x=0$, which is $-\frac{n}{2}$. Therefore the function $q_n(x)$ is greater than or equal to $g_n := 1-\frac{n}{4}x^2$ for all x in the interval. This parabola is the quadratic Taylor series for q_n at 0. Notice next that $g_n = 0$ for $x = \pm\frac{2}{\sqrt{n}}$. Therefore,

$$c_n = \int_{-2}^{2} q_n(x) dx \ge \int_{-\frac{2}{\sqrt{n}}}^{\frac{2}{\sqrt{n}}} q_n(x) dx \ge \int_{-\frac{2}{\sqrt{n}}}^{\frac{2}{\sqrt{n}}} g_n(x) dx = (x - \frac{nx^3}{12})|_{-\frac{2}{\sqrt{n}}}^{\frac{2}{\sqrt{n}}} = \frac{8}{3\sqrt{n}}.$$

Then, for $\delta > 0$,

$$\begin{aligned} \int_{-\infty}^{-\delta} p_n(x)dx + \int_{\delta}^{\infty} p_n(x)dx &= \frac{2}{c_n}\int_{\delta}^{2} q_n(x)dx \\ &\leq \frac{3\sqrt{n}}{4}\int_{\delta}^{2}(1-\frac{\delta^2}{4})^n dx \\ &\leq \frac{3\sqrt{n}}{2}(1-\frac{\delta^2}{4})^n. \end{aligned}$$

Set $r = 1 - \frac{\delta^2}{4}$, and note that $r < 1$. Therefore $\lim_{n\to\infty} 3\sqrt{n}r^n = 0$ by an application of L'Hopital's Rule (exercise!), and the concentration property holds as claimed. □

Now that we have constructed p_n, we are ready to prove the theorem:

THEOREM 3.2. *(Weierstrass Approximation Theorem, Version 2) Let $f \in C([0,1])$. Then there is a sequence of polynomials (f_n) on $[0,1]$ so that $f_n \to f$ uniformly on $[0,1]$.*

PROOF. Let $f \in C([0,1])$. We need f to have a domain of $\mathbb{R}$ and to be bounded and piecewise continuous on $\mathbb{R}$. So, extend f to $[-1,2]$ in any fashion which makes f continuous on $[-1,2]$ and $f(-1) = f(2) = 0$. Then extend f to the rest of $\mathbb{R}$ by 0. Clearly, then, f will be bounded and piecewise continuous on $\mathbb{R}$. Note also that $[0,1]$ is a compact subset of $\mathbb{R}$ on which f is continuous.

Let (p_n) be as constructed above. By Lemma 3.3, these are a sequence of mollifiers. Therefore, by Lemma 3.2, the functions $f_n := p_n * f$ converge uniformly to f. It remains only to show that f_n is a polynomial on $[0,1]$ for any n.

Recall that $f_n(x) = \int_{-\infty}^{\infty} f(t)p_n(x-t)dt$. Also, since $f \equiv 0$ outside $(-1,2)$, and $0 \leq x \leq 1$, the only nonzero contributions to this integral occur when $-2 < x - t < 2$. On this domain, p_n is a polynomial, so $p_n(x-t)$ can be written as $\sum_{i=0}^{k} a_i(x-t)^i$ for some power k and coefficients $\{a_i\}$. By the binomial theorem, this is $\sum_{i=0}^{k}\sum_{j=0}^{i} a_i(-1)^{i-j}\binom{i}{j}x^j t^{i-j}$. So,

$$\begin{aligned} f_n(x) &= \int_{-\infty}^{\infty} f(t)p_n(x-t)dt \\ &= \int_{-1}^{2} f(t)p_n(x-t)dt \\ &= \int_{-1}^{2} f(t)\left[\sum_{i=0}^{k}\sum_{j=0}^{i} a_i(-1)^{i-j}\binom{i}{j}x^j t^{i-j}\right]dt \\ &= \sum_{i=0}^{k}\sum_{j=0}^{i} x^j\left[a_i(-1)^{i-j}\binom{i}{j}\int_{-1}^{2} f(t)t^{i-j}dt\right], \end{aligned}$$

which is clearly a polynomial in x. □

3.2.1. *Exercises.*

EXERCISE 3.3. *Check that $\lim_{n\to\infty} 3\sqrt{n}r^n = 0$ if $r < 1$ as claimed in the proof above.*

EXERCISE 3.4. *Check that the two versions of the Weierstrass Approximation Theorem are equivalent.*

Exercise 3.5. *Sketch graphs of p_n for $n = 1, 2, 3, \ldots$.*

Exercise 3.6. *Use a computer to compute the first few f_n for each of the following choices of f and confirm that they are polynomials and that they are good uniform approximations of f.*

- $f(x) = x$.
- $f(x) = e^x$.
- $f(x) = \sin(\pi x)$.

Exercise 3.7. *Look up the proof of the Weierstrass Approximation Theorem that uses Bernstein Polynomials. Can you see that this is also just a form of averaging? (You don't have to memorize the proof; you don't have to turn this problem in, it is a reading and thinking assignment.)*

CHAPTER 3

Calculus in Normed Vector Spaces

Having defined metric spaces and normed linear spaces, and spent some time looking at the fundamental topological properties thereon, we are ready to do some calculus. We will begin by discussing differentiability.

1. Differentiability

1.1. Review of Differentiability on $\mathbb{R}^n$.

Definition 1.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is* differentiable *at $x_0 \in \mathbb{R}$ if $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists. If so, we define $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$.*

Definition 1.2. *The "little o" notation $o(x - x_0)$ represents an error term with the property that $\lim_{x\to x_0} \frac{o(x-x_0)}{(x-x_0)} = 0$.*

Lemma 1.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 if and only if there is a real number L such that $f(x) = f(x_0) + L(x - x_0) + o(x - x_0)$.*

Remark 1.1. *If the number L exists, then we say that L is the derivative of f at x_0 and we write $L = f'(x_0)$.*

Proof. We proved this lemma together in class on Monday, March 19. You should record the proof here for your future reference.

□

The alternate definition of differentiability given by this lemma expresses the notion of *local linear approximation* in a natural way, and this is the notion that serves as a defining property for derivatives in more general settings. We recall from multivariable calculus the following definition:

DEFINITION 1.3. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{x}_0$ if there is a vector $\vec{L} \in \mathbb{R}^n$ such that $f(\vec{x}) = f(\vec{x}_0) + \vec{L} \cdot (\vec{x} - \vec{x}_0) + o(\|\vec{x} - \vec{x}_0\|)$. If f is differentiable at x_0, then the function $L(\vec{x}) = f(\vec{x}_0) + \vec{L} \cdot (\vec{x} - \vec{x}_0)$ is called the* local linear approximation *or* tangent plane approximation *to f at $\vec{x}_0$.*

EXAMPLE 1.1. *On $\mathbb{R}^2$, the functions $f(x, y) = |(|x| - |y|)| - |x| - |y|$ and $g(x, y) = \frac{3x^2y}{x^2+y^2}$ are not differentiable at $(0, 0)$ even though $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial g}{\partial x}$, and $\frac{\partial g}{\partial y}$ all exist (and equal 0) at $(0, 0)$.*

1.1.1. *Exercises.*

EXERCISE 1.1. *Construct at least two different examples of differentiable functions on $\mathbb{R}^3$.*

EXERCISE 1.2. *Construct a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that, at $\vec{x}_0 = (0, 0, 0)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ all exist, but f is not differentiable at x_0.*

EXERCISE 1.3. *Explain, in paragraph form, why the functions in Example 1.1 are not differentiable; make an explicit comparison with Definition 1.3*

1.2. Linear Operators on Normed Linear Spaces. In $\mathbb{R}^n$, we know what it means for a function to be differentiable, because we have a clear idea of what a "local linear approximation" should look like. Before we can define differentiability for functions on a normed linear space, we must first discuss what it means for such a function to be linear.

DEFINITION 1.4. *Let X, Y be normed linear spaces. $L : X \to Y$ is a* linear operator *if*

(1) *$L(cx) = cL(x)$ for all $c \in \mathbb{R}$ and all $x \in X$.*
(2) *$L(x_1 + x_2) = L(x_1) + L(x_2)$ for all $x_1, x_2 \in X$.*

An important question to ask about any new classification of functions is whether or not they are continuous. It turns out that not all linear operators are continuous. This is a surprising statement, given that linear operators are usually considered to be about as *nice* as a function can be.

EXAMPLE 1.2. *Let $X = Y = C^\infty([0, 1])$ = the space of all infinitely differentiable functions on $[0, 1]$. Equip X with the L^2 norm. Then let $L : X \to Y$ be given by $L(f) = \frac{df}{dx}$. Then L is linear but not continuous.*

EXAMPLE 1.3. *Let $X = C^1([0, 1])$, and $Y = C([0, 1])$ and equip* **both** *X and Y with the sup-norm. Then let $L : X \to Y$ be given by $L(f) = \frac{df}{dx}$. Then L is linear but not continuous.*

Note that this is a subtlety that is not encountered in finite dimensions.

DEFINITION 1.5. *Suppose that $L : X \to Y$ is a linear operator as in the previous definition. We say that L is a* bounded *linear operator if there is a constant $c > 0$ such that $||Lx||_Y \leq c||x||_X$ for all $x \in X$.*

Lemma 1.2. *A linear operator* $L : X \to Y$ *is bounded if and only if* $||L||_{op} := \sup\{||L(x)||_Y : x \in X, ||x||_X = 1\} < +\infty$*. If so, then,* $\forall x \in X$, $\|Lx\|_Y \le \|L\|_{op}\|x\|_X$.

Proof. Exercise. $\square$

Theorem 1.1. *Let* $\mathcal{B}(X,Y)$ *be the set of all bounded linear operators from* X *to* Y*. Then* $(\mathcal{B}(X,Y), ||\cdot||_{op})$ *is a normed linear space. If* Y *is complete, then so is* $\mathcal{B}(X,Y)$*.*

Proof. We define addition and scalar multiplication as follows. If L_1 and L_2 are in $\mathcal{B}(X,Y)$, then $(L_1 + L_2) : X \to Y : (L_1 + L_2)(x) = L_1(x) + L_2(x)$. If $(\alpha \in \mathbb{R})$, then $\alpha L_1 : X \to Y : (\alpha L_1)(x) = \alpha(L_1(x))$. Notice that for any $x \in X$ with $||x||_X \le 1$ we have $||(L_1 + L_2)(x)||_Y = ||L_1(x) + L_2(x)||_Y \le ||L_1(x)||_Y + ||L_2(x)||_Y \le ||L_1||_{op} + ||L_2||_{op}$. Hence $(L_1 + L_2)$ is bounded with $||L_1 + L_2||_{op} \le ||L_1||_{op} + ||L_2||_{op}$. A similar, and simpler, argument confirms that αL_1 is bounded with $||\alpha L_1||_{op} = |\alpha| ||L_1||_{op}$. We have simultaneously checked that $\mathcal{B}(X,Y)$ is closed under addition and scalar multiplication, so it is a linear space, and that $||\cdot||_{op}$ satisfies two important properties of a norm. The other properties are easily checked. Thus $\mathcal{B}(X,Y)$ is a normed linear space.

Let $(L_n) \subset \mathcal{B}(X,Y)$ be a Cauchy sequence. Let $x \in X$. Then

$$||L_n(x) - L_m(x)||_Y = ||(L_n - L_m)x||_Y \le ||L_n - L_m||_{op}||x||_X.$$

Given $\epsilon > 0$ we can choose $N > 0$ such that $||L_n - L_m||_{op} \le \frac{\epsilon}{||x||}$ for $n, m > N$, and so $||L_n(x) - L_m(x)||_Y \le \epsilon$ for $n, m > N$. Hence $(L_n(x))$ is Cauchy in Y. We assumed that Y is complete, so this sequence must converge to some element of Y. Let $L(x) := \lim(L_n(x))$.

Now we must show that $L \in \mathcal{B}(X,Y)$ and that $||L_n - L||_{op} \to 0$. Since (L_n) is Cauchy we know that it is a bounded sequence in $\mathcal{B}(X,Y)$, so assume $||L_n||_{op} \le K$ for all n. If $||x|| \le 1$, then $||L_n(x)||_Y \le K$ for all n, so $||L(x)||_Y = \lim ||L_n(x)|| \le K$. Hence L is bounded with $||L||_{op} \le K$. If $||x||_X \le 1$ we also have $||(L - L_n)(x)||_Y \le ||(L - L_m)(x)||_Y + ||(L_m - L_n)(x)||_Y$. Let $\epsilon > 0$ be given and choose N such that $||L_n - L_m||_{op} \le \epsilon$ for all $n, m > N$. Then $||(L - L_n)(x)||_{op} \le ||(L - L_m)(x)||_Y + \epsilon$ for all $n, m > N$. Let $m \to \infty$ and use the fact that $L_m(x) \to L(x)$ in Y to get $||(L - L_n)(x)|| \le \epsilon$ for all $n > N$. This statement is true for all $x \in X$ such that $||x||_X \le 1$, so we can take a supremum of the left hand side of the inequality to get $||L - L_n||_{op} \le \epsilon$ for all $n > N$. Hence $L_n \to L$ in $\mathcal{B}(X,Y)$. The proof is done. $\square$

Lemma 1.3. *If* $L : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear operator, then it is a bounded linear operator.*

Proof: Consider the standard basis $\{e_1, ..., e_n\}$. Then

$$||L(x)|| = ||L(x_1e_1 + ... + x_ne_n)|| \le \sum_{i=1}^n |x_i| ||L(e_i)|| \le \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n ||L(e_i)||^2\right)^{\frac{1}{2}} = c||x||,$$

where $c = (\sum_{i=1}^n ||L(e_i)||^2)^{\frac{1}{2}}$.

Lemma 1.4. *A linear operator* $L : X \to Y$ *is continuous on* X *if and only if it is bounded.*

Proof. $\Rightarrow$) Suppose L is continuous on X. Then L is continuous at 0, so $\forall \epsilon > 0$, $\exists \delta > 0$ so that if $x \in X$ with $\|x\| < \delta$, then $\|Lx\| < \epsilon$. Let $\epsilon = 1$ and choose a corresponding δ. Let

$x \in X$ with $\|x\| = 1$. By linearity, $L(\frac{\delta}{2}x) = \frac{\delta}{2}L(x)$. Also, $\|\frac{\delta}{2}x\| < \delta$, so $\|L(\frac{\delta}{2}x)\| < 1$. So $\|L(x)\| < \frac{2}{\delta}$. So, $\sup\{\|Lx\| : \|x\| = 1\} \leq \frac{2}{\delta}$ and L is bounded by Lemma 1.2.

$\Leftarrow$) Suppose L is bounded. Let $\|L\| = M$. Let $\epsilon > 0$ and $x_0 \in X$. Let $\delta = \frac{\epsilon}{M}$ and let $x \in X$ satisfy $\|x - x_0\| < \delta$. Then, using linearity and Lemma 1.2, we obtain

$$\|Lx - Lx_0\| = \|L(x - x_0)\| \leq \|L\|\|x - x_0\| = M\|x - x_0\| < M\delta = M\frac{\epsilon}{M} = \epsilon.$$

Therefore, L is continuous at every x_0 in X. □

1.2.1. *Exercises.*

Exercise 1.4. *Prove Lemma 1.2.*

Exercise 1.5. *Fill in the missing details in the proof of Theorem 1.1.*

Exercise 1.6. *Use Lemma 1.4 to justify the claims in Examples 1.2 and 1.3.*

Exercise 1.7. *Prove that a linear operator is continuous at every point in its domain if and only if it is continuous at* 0.

1.3. Fréchet Differentiation. Now that we know a bit more about bounded linear operators, we are ready for the general definition of differentiable.

Definition 1.6. *Let X, Y be normed linear spaces, and let $F : X \to Y$. We say that F is differentiable at $x_0 \in X$ if there is a linear operator $L : X \to Y$ such that $F(x) = F(x_0) + L(x - x_0) + o(x - x_0)$. If such an L exists, then we say the L is the derivative of F at x_0 and write $L = DF(x_0)$. If $L \in \mathcal{B}(X, Y)$, then we say that F is Fréchet differentiable at x_0.*

In all that follows the term *differentiable* will mean *Fréchet differentiable.* There are some theorems that do not require the derivative to be a bounded linear operator, but in these notes we will not spend time arguing about those differences. For our purposes we will want our derivatives to be bounded linear operators. You are, of course, welcome to consider this difference on your own.

Lemma 1.5. *Derivatives are unique.*

Proof. Assume that L_1 and L_2 are both derivatives of F at x_0 as in the definition above. To prove that $L_1 = L_2$ we must show that $L_1 v = L_2 v$ for any $v \in X$. We know that

$$\begin{aligned} F(x) &= F(x_0) + L_1(x - x_0) + o_1(x - x_0) \\ F(x) &= F(x_0) + L_2(x - x_0) + o_2(x - x_0). \end{aligned}$$

Thus $L_1(x - x_0) - L_2(x - x_0) = o_2(x - x_0) - o_1(x - x_0)$ for all x. Let $v \in X$ be arbitrary and let $t > 0$. Since the statement above is true for all x we can choose x so that $x - x_0 = tv$. This leads to $L_1(tv) - L_2(tv) = o_2(tv) - o_1(tv)$. Using the linearity properties, we can divide both sides by t to get $L_1(v) - L_2(v) = \frac{o_2(tv)}{t||v||}||v|| - \frac{o_1(tv)}{t||v||}||v||$. Letting $t \to 0$ we get $L_1(v) = L_2(v)$. □

Example 1.4. *Consider $F : L^2([0,1]) \to \mathbb{R} : F(u) = \int_0^1 u^2$. Then F is Fréchet differentiable, and for f in $L^2([0,1])$, $DF(f)$ is the linear functional from $L^2([0,1])$ to $\mathbb{R}$ given by $DF(f)[h] = \int_0^1 2f(t)h(t)dt \ \forall h \in L^2([0,1])$.*

We now take a small detour into single variable calculus in order to introduce our next example.

THEOREM 1.2. **[Taylor's Theorem with Remainder]** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a $k+1$ times continuously differentiable function for $k \geq 1$. Let $x_0 \in \mathbb{R}$ and let $P_k(x) = \sum_{i=0}^{k} \frac{f^{(i)}}{i!}(x-x_0)^i$ be the kth Taylor polynomial for f centered at x_0. Then, for any $x \in \mathbb{R}$,*

$$f(x) - P_k(x) = \int_{x_0}^{x} \frac{f^{k+1}(t)}{k!}(x-t)^k dt.$$

PROOF. Cauchy's Mean Value Theorem states that if F and G are continuously differentiable functions on the interval $[x_0, x]$, then there exists some $c \in (x_0, x)$ so that

$$(F(b) - F(a))G'(c) = (G(b) - G(a))F'(c).$$

This is an extension of the usual mean value theorem. Let $F(t) = \sum_{i=0}^{k} \frac{f^{(i)}(t)}{i!}(x-t)^i$, and let $G(t) = \int_{x_0}^{t} \frac{f^{k+1}(s)}{(k+1)!}(x-s)^k ds$. Then note that

$$F'(t) = f'(t) + \sum_{i=1}^{k} \left(\frac{f^{(i+1)}(t)}{i!}(x-t)^i - \frac{f^{(i)}(t)}{(i-1)!}(x-t)^{i-1} \right) = \frac{f^{k+1}(t)}{k!}(x-t)^k,$$

because the sum telescopes. Also,

$$G'(t) = \frac{f^{k+1}(t)}{(k+1)!}(x-t)^k$$

by the Fundamental Theorem of Calculus. Plugging into the Cauchy MVT, with $a = x_0$ and $b = x$, we obtain that there is some $c \in (x_0, x)$ so that

$$\left(f(x) - \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i \right) \cdot \frac{f^{k+1}(c)}{(k+1)!}(x-c)^k = \left(\int_{x_0}^{x} \frac{f^{k+1}(s)}{(k+1)!}(x-s)^k ds - 0 \right) \cdot \frac{f^{k+1}(c)}{k!}(x-c)^k.$$

Cancelling terms, we conclude that

$$f(x) - \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i = \int_{x_0}^{x} \frac{f^{k+1}(s)}{(k)!}(x-s)^k ds$$

as claimed. $\square$

COROLLARY 1.1. *Under the hypotheses of Theorem 1.2, there is some $c \in [x_0, x]$ such that*

$$|f(x) - \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i| \leq \frac{|f^{k+1}(c)|}{(k+1)!}|x-x_0|^{k+1}.$$

PROOF. Exercise. $\square$

EXAMPLE 1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-continuously differentiable function so that f'' is bounded. Then the functional $F : L^2([0,1]) \to \mathbb{R}$ given by $F(u) = \int_0^1 f(u(x))dx$ is Fréchet differentiable with derivative $DF(u)[h] = \int_0^1 f'(u(x))h(x)dx$.*

Proof. Let f be as in the statement of the example and let $M = \sup_{\mathbb{R}} |f''(x)|$. Let $h \in L^2([0,1])$, and consider

$$\frac{1}{\|h\|_2}\left(F(u+h) - F(u) - \int_0^1 f'(u(x))h(x)dx\right) = \int_0^1 \frac{f(u+h) - f(u) - f'(u)h}{\|h\|_2}dx.$$

By the corollary to Taylor's Theorem above, for each x there is a $t \in [0,1]$ so that $|f(u(x) + h(x)) - f(u(x)) - f'(u(x))h(x)| \leq \frac{1}{2}|f''(t)||h(x)|^2 \leq \frac{M}{2}|h(x)|^2$.

Therefore,

$$\left|\frac{1}{\|h\|_2}\left(F(u+h) - F(u) - \int_0^1 f'(u(x))h(x)dx\right)\right| \leq \frac{1}{2}\int_0^1 \frac{M|h(x)|^2}{\|h\|_2}dx = \frac{M}{2}\|h\|_2.$$

So,

$$\lim_{\|h\|\to 0}\left|\frac{1}{\|h\|_2}\left(F(u+h) - F(u) - \int_0^1 f'(u(x))h(x)dx\right)\right| \leq \lim_{\|h\|\to 0}\frac{M}{2}\|h\|_2.$$

Therefore the definition of differentiability is verified, using the formula for the derivative given above. □

As in calculus, it is valuable to have some rules for differentiation.

Lemma 1.6. *Suppose that $F, G : X \to Y$ are both differentiable at $x_0 \in X$. Then $F + G$ is differentiable at x_0 with $D(F+G)(x_0) = DF(x_0) + DG(x_0)$.*

Proof. We proved this together in class on Monday, April 2. Record the proof here for your records.

□

Lemma 1.7. *Suppose that $F : X \to Y$ and $f : X \to \mathbb{R}$ are both Fréchet differentiable at x_0. Then $fF : X \to Y$ is also Fréchet differentiable at x_0 with $D(fF)(x_0) = f(x_0)DF(x_0) + F(x_0)Df(x_0)$.*

PROOF. Because F and f are both Fréchet differentiable at x_0, there exist bounded linear operators $DF(x_0) : X \to Y$ and $Df(x_0) : X \to \mathbb{R}$ such that

$$\lim_{\|x-x_0\|\to 0} \frac{F(x) - F(x_0) - DF(x_0)[x - x_0]}{\|x - x_0\|} = 0 \tag{7}$$

and

$$\lim_{\|x-x_0\|\to 0} \frac{f(x) - f(x_0) - Df(x_0)[x - x_0]}{\|x - x_0\|} = 0. \tag{8}$$

We compute:

$$\begin{aligned} f(x)F(x) - f(x_0)F(x_0) &= f(x)F(x) - f(x)F(x_0) + f(x)F(x_0) - f(x_0)F(x_0) \\ &= f(x)[F(x) - F(x_0)] + F(x_0)[f(x) - f(x_0)], \end{aligned}$$

so

$$\begin{aligned} f(x)F(x) - &f(x_0)F(x_0) - f(x_0)DF(x_0)[x - x_0] - Df(x_0)[x - x_0]F(x_0) \\ &= f(x)[F(x) - F(x_0) - DF(x_0)[x - x_0]] + [f(x) - f(x_0)]DF(x_0)[x - x_0] - \\ &\quad - F(x_0)[f(x) - f(x_0) - Df(x_0)[x - x_0]]. \end{aligned}$$

The first term is $f(x)$ multiplied by a term that is $o(\|x - x_0\|)$, and $f(x)$ is bounded as $x \to x_0$, so it is $o(\|x - x_0\|)$. The third term, similarly, is a constant $(F(x_0))$ times a term which is $o(\|x - x_0\|)$, so it is $o(\|x - x_0\|)$. For the middle term, compute

$$\begin{aligned} [f(x) - f(x_0)]DF(x_0)[x - x_0] &= (Df(x_0)[x - x_0] + o(\|x - x_0\|))DF(x_0)[x - x_0] \\ &= (Df(x_0)[x - x_0])DF(x_0)[x - x_0] + o(\|x - x_0\|)DF(x_0)[x - x_0]. \end{aligned}$$

The latter term is again clearly $o(\|x - x_0\|)$ times something bounded. Finally, notice that

$$\begin{aligned} \|(Df(x_0)[x - x_0])DF(x_0)[x - x_0]\| &\leq \|Df(x_0)\|_{oper}\|x - x_0\|\|DF(x_0)\|_{oper}\|x - x_0\| \\ &\leq C\|x - x_0\|^2 = o(\|x - x_0\|). \end{aligned}$$

Hence all of the error terms are $o(\|x - x_0\|)$, so by definition fF is differentiable with derivative $D(fF)(x_0) = f(x_0)DF(x_0) + F(x_0)Df(x_0)$. Finally notice that by the rules of linear operators, $D(fF)$ is again a bounded linear operator, so fF is in fact Fréchet differentiable. □

Before proving the chain rule it is helpful to note the following fact.

LEMMA 1.8. *If $F : X \to Y$ is differentiable at x_0, then F is Lipschitz continuous at x_0. That is, $\exists M > 0$ and $\exists \delta > 0$ so that if $\|x - x_0\|_X < \delta$, then $\|F(x) - F(x_0)\|_Y \leq M\|x - x_0\|_X$.*

PROOF. We know that

$$F(x) - F(x_0) = DF(x_0)(x - x_0) + o(x - x_0).$$

Let $\epsilon = 1$ and choose $\delta > 0$ such that $\dfrac{\|F(x) - F(x_0) - DF(x_0)[x - x_0]\|_Y}{\|x - x_0\|_X} \le \epsilon = 1$. It follows that for $\|x - x_0\| \le \delta$ we have

$$\begin{aligned}\|F(x) - F(x_0)\|_Y &\le \|DF(x_0)(x - x_0)\|_Y + \|F(x) - F(x_0) - DF(x_0)[x - x_0]\|_Y \\ &\le \|DF(x_0)\|_{oper}\|x - x_0\|_X + \|x - x_0\|_X \\ &\le (\|DF(x_0)\|_{oper} + 1)\|x - x_0\|_X\end{aligned}$$

Therefore the definition of Lipschitz continuity at x_0 is satisfied with
$M = \|DF(x_0)\|_{oper} + 1$. □

LEMMA 1.9. *Suppose that $F : X \to Y$ is differentiable at x_0 and that $G : Y \to Z$ is differentiable at $F(x_0)$. Then $G \circ F : X \to Z$ is differentiable at x_0 with $D(G \circ F)(x_0) = DG(F(x_0)) \cdot DF(x_0)$.*

PROOF. Write

$$\begin{aligned}o(\|F(x) - F(x_0)\|) &= G \circ F(x) - G \circ F(x_0) - DG(F(x_0))[F(x) - F(x_0)] \\ &= G \circ F(x) - G \circ F(x_0) - DG(F(x_0))[F(x) - F(x_0) - DF(x_0)[x - x_0]] + \\ &\quad + DG(F(x_0))DF(x_0)[x - x_0]\end{aligned}$$

So,

$$\begin{aligned}&G \circ F(x) - G \circ F(x_0) - DG(F(x_0)DF(x_0)[x - x_0] \\ &\qquad = DG(F(x_0))[F(x) - F(x_0) - DF(x_0)[x - x_0]] + o(\|F(x) - F(x_0)\|.\end{aligned}$$

We need to check that each of these error terms is $o(\|x - x_0\|)$. For the first one, since $DG(F(x_0))$ is a bounded linear operator,

$$\begin{aligned}&\|DG(F(x_0))[F(x) - F(x_0) - DF(x_0)[x - x_0]\| \\ &\qquad \le \|DG(F(x_0))\|_{oper}\|[F(x) - F(x_0) - DF(x_0)[x - x_0]\| \\ &\qquad \le \|DG(F(x_0))\|_{oper} o(\|x - x_0\|) \\ &\qquad = o(\|x - x_0\|).\end{aligned}$$

For the second one, recall that, by the previous lemma, $\|F(x) - F(x_0)\| \le M\|x - x_0\|$ for $\|x - x_0\|$ sufficiently small. Therefore,

$$\begin{aligned}\lim_{\|x - x_0\| \to 0} \frac{o(\|F(x) - F(x_0)\|)}{\|x - x_0\|} &= \frac{o(\|F(x) - F(x_0)\|)}{\|F(x) - F(x_0)\|} \frac{\|F(x) - F(x_0)\|}{\|x - x_0\|} \\ &\le M \frac{o(\|F(x) - F(x_0)\|)}{\|F(x) - F(x_0)\|} \\ &= 0,\end{aligned}$$

so this is, indeed, $o(\|x - x_0\|)$. Therefore, by definition of differentiability, $G \circ F$ is differentiable and, since the composition of two bounded linear operators is a bounded linear operator, $D(G \circ F)(x_0) = DG(F(x_0)) \circ DF(x_0)$ is bounded so $G \circ F$ is Fréchet differentiable. □

1.3.1. *Exercises.*

EXERCISE 1.8. *In Example 1.4, confirm that F is Fréchet differentiable, and verify the formula for DF.*

EXERCISE 1.9. *Prove Corollary 1.1*

EXERCISE 1.10. *Prove the following quotient rule: Suppose that $F : X \to Y$ is Fréchet differentiable at x_0 and that $f : X \to \mathbb{R}$ is Fréchet differentiable at x_0 and $f(x_0) \neq 0$. Then $\frac{F}{f} : X \to Y$ is differentiable at x_0 and, for $h \in X$, $D\frac{F}{f}(x_0)(h) = \frac{f(x_0)DF(x_0)(h) - Df(x_0)(h)F(x_0)}{f(x_0)^2}$.*

1.4. A Detour into $\mathbb{R}^n$.

LEMMA 1.10. *If $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x_0, then $DF(x_0)$ can be represented by a* Jacobian *matrix* $\left[\frac{\partial F_i}{\partial x_j}\right]$.

PROOF. Recall that any linear operator $L : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by the matrix $[L_{ij}]$ where the entry in the ith row and jth column is given by $\langle L(e_j), f_i\rangle$, where $\{e_1, ..., e_n\}$ is a standard basis for $\mathbb{R}^n$ and $\{f_1, ..., f_m\}$ is a standard basis for $\mathbb{R}^m$.

Substitute $L = DF(x_0)$ into the previous formula and use the fact that

$$\begin{aligned} DF(x_0)(e_j) &= D_{e_j}F(x_0) \\ &= \lim_{h\to 0} \frac{F(x_0+he_j)-F(x_0)}{h} \\ &= \begin{bmatrix} \lim_{h\to 0} \frac{F_1(x_0+he_j)-F_1(x_0)}{h} \\ \lim_{h\to 0} \frac{F_2(x_0+he_j)-F_2(x_0)}{h} \\ \vdots \\ \lim_{h\to 0} \frac{F_m(x_0+he_j)-F_m(x_0)}{h} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial F_1}{\partial x_j} \\ \frac{\partial F_2}{\partial x_j} \\ \vdots \\ \frac{\partial F_m}{\partial x_j} \end{bmatrix} \end{aligned}$$

It follows that $\langle DF(x_0)(e_j), f_i\rangle = \frac{\partial F_i}{\partial x_j}$. □

The Mean Value Theorem is one of the most useful tools in one-variable calculus. It is fairly easy to generalize to any case where the codomain is $\mathbb{R}$:

THEOREM 1.3. *Mean Value Theorem: Assume that $f : X \to \mathbb{R}$ is differentiable. Given any $x_1, x_2 \in X$ there is a $c \in (0, 1)$ such that $f(x_2) - f(x_1) = Df(x_1 + c(x_2 - x_1))(x_2 - x_1)$.*

PROOF. We reduce the general case to the one variable case by considering $g(t) = f(x_1 + t(x_2 - x_1))$, which is a differentiable function of one variable with $g'(t) = Df(x_1 + t(x_2 - x_1))(x_2 - x_1)$ by the chain rule. By the one variable MVT we have $g(1) - g(0) = g'(c)(1 - 0)$ for some $c \in (0, 1)$. But this is precisely what we wanted to prove. □

Here are two nice applications of the one-variable Mean Value Theorem:

THEOREM 1.4. *If the first partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}^m$ exist and are continuous on a neighborhood of x_0, then f is differentiable at x_0.*

PROOF. In class we stuck to the case where $f : \mathbb{R}^2 \to \mathbb{R}$. Let $\vec{x}_0 = (x_0, y_0) \in \mathbb{R}^2$. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives in some neighborhood of $\vec{x}_0$. Write

$$\begin{aligned} f(\vec{x}) - f(\vec{x}_0) = f(x, y) - f(x_0, y_0) &= f(x, y) - f(x, y_0) + f(x, y_0) - f(x_0, y_0) \\ &= \frac{\partial f}{\partial y}(x, \tilde{y})(y - y_0) + \frac{\partial f}{\partial x}(\tilde{x}, y_0)(x - x_0), \end{aligned}$$

for some $\tilde{x}$ between x and x_0 and some $\tilde{y}$ between y and y_0, by the one dimensional Mean Value Theorem. Then

$$\begin{aligned} f(\vec{x}) - f(\vec{x}_0) &- \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \\ &= \frac{\partial f}{\partial x}(\tilde{x}, y_0)(x - x_0) - \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x, \tilde{y})(y - y_0) - \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \\ &= \left(\frac{\partial f}{\partial x}(\tilde{x}, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x, \tilde{y}) - \frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) \end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{\vec{x} \to \vec{x}_0} \frac{|f(\vec{x}) - f(\vec{x}_0) - \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)|}{\|\vec{x} - \vec{x}_0\|} \\ &\quad = \lim_{\vec{x} \to \vec{x}_0} \frac{\left|\left(\frac{\partial f}{\partial x}(\tilde{x}, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x, \tilde{y}) - \frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0)\right|}{|x - x_0| + |y - y_0|} \\ &\quad \leq \lim_{\vec{x} \to \vec{x}_0} \left|\frac{\partial f}{\partial x}(\tilde{x}, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)\right| + \left|\frac{\partial f}{\partial y}(x, \tilde{y}) - \frac{\partial f}{\partial y}(x_0, y_0)\right| \\ &\quad = 0 \end{aligned}$$

by the continuity of the partial derivatives. Hence,

$$f(\vec{x}) - f(\vec{x}_0) - \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) = o(\|\vec{x} - \vec{x}_0\|),$$

so f is differentiable at $\vec{x}_0$ as claimed. $\square$

THEOREM 1.5. *If the second partial derivatives of* $f : \mathbb{R}^n \to \mathbb{R}^m$ *exist and are continuous on a neighborhood of* x_0*, then* $\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x_0)$.

PROOF. Suppose that $\vec{f}$ is as in the statement of the theorem. If $\vec{f}(x) = (f_1(x), \ldots, f_m(x))$, then the claim can be checked for each $k = 1 \ldots m$ separately. Therefore, without loss of generality we may assume that $f : \mathbb{R}^n \to \mathbb{R}$. Define $D_{i,h} f(\vec{x}) = \frac{f(\vec{x} + h\vec{e}_i) - f(\vec{x})}{h}$ to be the difference quotient of f in the ith direction (that is, the ratio which will become the partial derivative if $h \to 0$). For $i \neq j$, and fixed $h, l > 0$, note that $D_{i,h} D_{j,l} f(\vec{x}) = D_{j,l} D_{i,h} f(\vec{x})$ by direct computation. (Exercise!) By the Mean Value Theorem, we have that for some t in

$(0, h)$,

$$\begin{aligned} D_{i,h} D_{j,l} f(\vec{x}) &= \frac{\partial}{\partial x_i} \left(D_{j,l} f(\vec{x} + t\vec{e}_i) \right) \\ &= \frac{1}{l} \left(\frac{\partial f}{\partial x_i}(\vec{x} + t\vec{e}_i + l\vec{e}_j) - \frac{\partial f}{\partial x_i}(\vec{x} + t\vec{e}_i) \right) \\ &= \frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x} + t\vec{e}_i + s\vec{e}_j), \end{aligned}$$

for some s in $(0, l)$, by the Mean Value Theorem again. By an identical calculation, $D_{j,l} D_{i,h} f(\vec{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x} + r\vec{e}_i + u\vec{e}_j)$ for some $r \in (0, h)$ and some $u \in (0, l)$. Since $D_{i,h} D_{j,l} f(\vec{x}) = D_{j,l} D_{i,h} f(\vec{x})$, we conclude that

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x} + t\vec{e}_i + s\vec{e}_j) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x} + r\vec{e}_i + u\vec{e}_j).$$

Taking the limit as h, $l \to 0$ and using the continuity of these second derivatives yield the claim. □

The geometry of $\mathbb{R}$ is fundamentally special, and that is what makes the Mean Value Theorem true. What goes wrong if we try to extend its proof to a differentiable function with more than one output, *i.e.* $f : \mathbb{R}^n \to \mathbb{R}^m$?

EXAMPLE 1.6. *Consider the function* $\vec{f} : \mathbb{R} \to \mathbb{R}^2$ *given by* $f(t) = \langle \cos(t), \sin(t) \rangle$. *Does the mean value theorem hold for* $x_1 = 0$ *and* $x_2 = 2\pi$*?*

It turns out that many applications of the MVT only require the following weaker statement, which does hold for general normed linear space codomains:

THEOREM 1.6. *Mean Value Inequality: Assume that* $f : X \to Y$ *is Fréchet differentiable. Given any* $x_1, x_2 \in X$, *set* $M := \sup\{||Df(x_1 + t(x_2 - x_1))||_{op} : 0 \le t \le 1\}$. *Then* $||f(x_2) - f(x_1)||_Y \le M||x_2 - x_1||_X$.

PROOF. Once again we will reduce to a single variable by considering $g(t) = f(x_1 + t(x_2 - x_1))$, so that $g'(t) = Df(x_1 + t(x_2 - x_1))(x_2 - x_1)$. If you reexamine our proof that differentiable functions are locally Lipschitz, you will see that for each $t \in [0, 1]$ there is a δ_t such that $||g(s) - g(t)||_Y \le (||Dg(t)||_{op} + 1)|s - t|$ when $|s - t| \le \delta_t$. In fact, the 1 in the previous inequality was an arbitrarily chosen positive number. The same proof technique will show that for any $\epsilon > 0$ and any $t \in [0, 1]$ there is a $\delta_t > 0$ such that $||g(s) - g(t)||_Y \le (||Dg(t)||_{op} + \epsilon)|s - t|$ when $|s - t| \le \delta_t$. You should check this as an exercise. For each t let $I_t := (t - \frac{\delta_t}{2}, t + \frac{\delta_t}{2})$. It is clear that $\{I_t : t \in [0, 1]\}$ is an open cover of $[0, 1]$. But $[0, 1]$ is compact, so there is a finite cover $\{I_{t_1}, I_{t_2}, ..., I_{t_n}\}$. In fact we can arrange it so that $0 = t_1 < t_2 < \cdots < t_n = 1$. Now we have

$$\begin{aligned} |g(1) - g(0)| &\le |g(t_n) - g(t_{n-1})| + \cdots + |g(t_2) - g(t_1)| \\ &\le (||Dg(t)||_{op} + \epsilon)|t_n - t_{n-1}| + \cdots + (M + \epsilon)|t_2 - t_1| \\ &= (||Dg(t)||_{op} + \epsilon)(|t_n - t_{n-1}| + \cdots + |t_2 - t_1|) \\ &= (||Dg(t)||_{op} + \epsilon). \end{aligned}$$

Since $\|Dg(t)\|_{oper} \le M\|x_2 - x_1\|_X$ and ϵ was arbitrary we conclude that $|g(1) - g(0)| \le M$ and thus $||f(x_2) - f(x_1)||_Y \le M||x_2 - x_1||_X$. □

1.4.1. *Exercises.*

EXERCISE 1.11. *Generalize the proof of Theorem 1.4 from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$.

EXERCISE 1.12. *Check that for* $i \neq j$, *and fixed* $h, l > 0$, *note that* $D_{i,h}D_{j,l}f(\vec{x}) = D_{j,l}D_{i,h}f(\vec{x})$ *as claimed in the proof of Theorem 1.5.*

EXERCISE 1.13. *Give an example of a continuously differentiable function from* $\mathbb{R}^2$ *to itself and points* $\vec{x}_1, \vec{x}_2$ *for which the mean value theorem is not satisfied. Compute the Jacobian matrix of your example function. Demonstrate that the mean value inequality is satisfied.*

2. The Inverse Function Theorem

PROPOSITION 2.1. *Assume that* $f \in C^1(\mathbb{R})$ *and* $x_0 \in \mathbb{R}$ *with* $f'(x_0) \neq 0$. *Then there is an interval containing* x_0 *such that* f *is invertible from that interval to its image. Further, the inverse* f^{-1} *is continuously differentiable.*

PROOF. Without loss of generality, assume that $f'(x_0) > 0$. Since $f'(x)$ is continuous, there is a $\delta > 0$ so that, if $|x - x_0| < \delta$, then $|f'(x) - f'(x_0)| < \frac{|f'(x_0)|}{2}$, which implies that for all such x, $f'(x) > f'(x_0) - \frac{f'(x_0)}{2} = \frac{f'(x_0)}{2} > 0$. Hence $f'(x) > 0$ on $(x_0 - \delta, x_0 + \delta)$. As proved in Math 611, this implies that f is injective on the interval $I := (x_0 - \delta, x_0 + \delta)$. Let J be the interval $f(I)$. (J is an interval by the Intermediate Value Theorem.) Then $f : I \to J$ is a bijective function which is therefore invertible. Now, let $y_0 \in J$ and consider $\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0)}$. Let $x_0 = f^{-1}(y_0)$ and $x = f^{-1}(y)$. The function f^{-1} is injective, so if $y \neq y_0$ then $x \neq x_0$. Therefore

$$\begin{aligned}
\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} &= \lim_{y \to y_0} \frac{x - x_0}{f(x) - f(x_0)} \\
&= \lim_{y \to y_0} \frac{1}{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)} \\
&= \frac{1}{\lim_{f(x) \to f(x_0)} \left(\frac{f(x) - f(x_0)}{x - x_0}\right)} \\
&= \frac{1}{\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\right)} \\
&= \frac{1}{f'(x_0)},
\end{aligned}$$

where we have used the fact that $f'(x_0) \neq 0$ so the limit can be moved inside of the ratio, and the fact that f is a homeomorphism, so $f(x) \to f(x_0)$ if and only if $x \to x_0$. Hence we may conclude that $(f^{-1})'(y_0)$ exists and equals $\frac{1}{f'(x_0)}$ as we learned in calculus. Finally note that since $f \in C^1([0, 1])$, f' is continuous. Moreover, by construction $f' > 0$ on J, and the function g given by $g(x) = \frac{1}{x}$ is continuous whenever $x \neq 0$. Therefore, $(f^{-1})' = g \circ f' \circ f^{-1}$ is a composition of continuous functions and hence continuous. $\square$

We want to extend this simple argument to the normed linear space setting. We will need completeness (to apply the contraction mapping theorem), so we require X and Y to be Banach spaces.

DEFINITION 2.1. *Suppose that $f : X \to Y$. We say that $f \in C^1(X;Y)$, or f is C^1, if, $\forall x \in X$, f is Fréchet differentiable at x, and, moreover, $Df(x)$ is continuous as a function of x. That is, $\forall x_0 \in X$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|x - x_0\|_X < \delta$ implies that $\|Df(x) - Df(x_0)\|_{oper} < \epsilon$.*

THEOREM 2.1. **Inverse Function Theorem:** *Assume that X and Y are Banach spaces and that $f \in C^1(X;Y)$. Suppose $x_0 \in X$. If $Df(x_0)$ is invertible, then there are neighborhoods U of x_0 and V of $y_0 = f(x_0)$ such that $f : U \to V$ is an invertible function with $f^{-1} \in C^1(V;U)$.*

We will break this proof down into several parts. Here is the outline:

(1) Special Case: We first assume that $X = Y$, $x_0 = f(x_0) = 0$ and $Df(x_0) = I$. This means that the local linear approximation to f near $x_0 = 0$ is $f(x) = x + r(x)$, where $\lim_{x\to 0} \frac{\|r(x)\|_X}{\|x\|_X} = 0$. Within this case we prove the theorem as follows:
  (a) Check that f is bijective, hence invertible, in some neighborhood U of 0.
  (b) Check that f^{-1} is C^1:
    (i) Check that f^{-1} is Lipschitz continuous on $f(U) =: V$.
    (ii) Check that $Df(x)$ is an invertible linear operator for each $x \in U$.
    (iii) Check that f^{-1} is differentiable at each $y \in V$.
    (iv) Check that $D(f^{-1})$ is a continuous function on V.
(2) Extension to the general case.

We now begin the full proof:

PROOF. We begin with the simplifying assumptions that $X = Y$, $x_0 = f(x_0) = 0$ and $Df(x_0) = I$, the identity operator. Then the local linear approximation of f can then be written simply as $f(x) = x + r(x)$ where $\lim_{x\to 0} \frac{\|r(x)\|_X}{\|x\|_X} = 0$. Let $y \in X$ be a fixed value. We will show that if y is close to the origin, then we can solve for x in $y = x + r(x)$, *i.e.* we can find an inverse. This can quickly be changed into a fixed point problem by rewriting as $x = y - r(x)$ and letting $T(x) := y - r(x)$. Hence, solving for x in $y = x + r(x)$ is equivalent to finding a fixed point for T.

Since $r(x) = f(x) - x$ we have that $Dr(x) = Df(x) - I$. By the continuity of Df and the fact that $Df(0) = I$ we know that given any $\epsilon > 0$ there is a $\delta > 0$ such that $||Dr(x)|| = ||Df(x) - I|| \le \epsilon$ for all $||x|| \le \delta$. By the Mean Value Inequality we have that $||r(x_2) - r(x_1)|| \le \epsilon ||x_2 - x_1||$ for all $x_1, x_2 \in \overline{B_\delta(0)}$. Set $\epsilon = \frac{1}{2}$. (Recall that $\overline{B_\delta(0)}$ is the closed ball of radius δ centered at 0. Since X is a Banach space, $\overline{B_\delta(0)}$ is a complete metric space.)

Now consider T. If $||x|| \le \delta$ then $||T(x)|| = ||y - r(x)|| \le ||y|| + ||r(x)|| \le ||y|| + \frac{1}{2}\delta$. Therefore if $||y|| \le \frac{\delta}{2}$, then $||T(x)|| \le \delta$ and so $T : \overline{B_\delta(0)} \to \overline{B_\delta(0)}$. If $x_1, x_2 \in \overline{B_\delta(0)}$, then $||T(x_2) - T(x_1)|| = ||r(x_2) - r(x_1)|| \le \frac{1}{2}||x_2 - x_1||$. Hence $T : \overline{B_\delta(0)} \to \overline{B_\delta(0)}$ is a contraction. By the Contraction Mapping Theorem there is a unique $x \in \overline{B_\delta(0)}$ such that $T(x) = x$

and thus $f(x) = y$. This defines $x = f^{-1}(y)$ for any $y \in \overline{B_{\frac{\delta}{2}}(0)}$. We let $V = B_{\frac{\delta}{2}}(0)$ and $U = f^{-1}(B_{\frac{\delta}{2}}(0)) \bigcap B_\delta(0)$.

It remains to show that the inverse is a C^1 function. We do this in several steps. The first step is to show that f^{-1} is Lipschitz continuous. Observe that for $y \in B_{\frac{\delta}{2}}(0)$ we have $f^{-1}(y) = y - r(f^{-1}(y))$. Suppose that $y_1, y_2 \in B_{\frac{\delta}{2}}(0)$. Then $||f^{-1}(y_2) - f^{-1}(y_1)|| = ||(y_2 - y_1) - (r(f^{-1}(y_2)) - r(f^{-1}(y_1)))|| \le ||y_2 - y_1|| + ||r(f^{-1}(y_2)) - r(f^{-1}(y_1))||$. Recall that r is Lipschitz continuous on U with constant $\frac{1}{2}$. Hence $||f^{-1}(y_2) - f^{-1}(y_1)|| \le ||y_2 - y_1|| + \frac{1}{2}||f^{-1}(y_2) - f^{-1}(y_1)||$. Therefore, by collecting common terms together, we get $||f^{-1}(y_2) - f^{-1}(y_1)|| \le 2||y_2 - y_1||$, and so f^{-1} is Lipschitz continuous.

We would now like to show that f^{-1} is differentiable at each point in V. It is reasonable to expect that $Df^{-1}(y) = (Df(x))^{-1}$ when $f(x) = y$. However, before we can verify this fact we need to know that $Df(x)$ is actually invertible at each $x \in U$. Here is a rather clever proof of this fact. This is a nice proof technique which relies on the idea that Taylor series can be formed for functions of linear operators, just as they can for functions of real numbers. We set $L = I - Df(x)$, so $Df(x) = I - L$. Since $f(x) = x + r(x)$ we know that $Df(x) = I + Dr(x)$, and so $||L|| = ||I - Df(x)|| = ||Dr(x)|| \le \frac{1}{2}$. By analogy to geometric series we consider the sum $\sum_{n=0}^{\infty} L^{\circ n}$. A standard argument shows that the partial sums are Cauchy in $\mathcal{B}(X, X)$. Since X is complete we know that $\mathcal{B}(X, X)$ is complete. This means that the partial sums must converge to some object G in $\mathcal{B}(X, X)$. Notice that $G \circ (I - L) = \lim_{n\to\infty}(I + L + L^{\circ 2} + \cdots + L^{\circ n}) \circ (I - L) = \lim_{n\to\infty}(I - L^{\circ(n+1)}) = I$, since $L^{\circ n} \to 0$. This shows that G is an inverse for $I - L = Df(x)$. A further consequence is that $||Df(x)^{-1}|| \le \sum_{n=0}^{\infty} ||L||^n \le 2$.

Now that we know that $Df(x)$ is invertible for all $x \in U$ we can show that f^{-1} is differentiable at all points of V and that $Df^{-1}(y) = (Df(x))^{-1}$. Start with the linear approximation $f(x) = f(x_0) + Df(x_0)(x - x_0) + r(x - x_0)$. Rewriting this in terms of $y = f(x)$ we get $y = y_0 + Df(x_0)(f^{-1}(y) - f^{-1}(y_0)) + r(f^{-1}(y) - f^{-1}(y_0))$, which can be rearranged to get $f^{-1}(y) = f^{-1}(y_0) + (Df(x_0))^{-1}(y - y_0) - (Df(x_0))^{-1}r(f^{-1}(y) - f^{-1}(y_0))$. Finally, we get that

$$\begin{aligned}\tfrac{||(Df(x_0))^{-1}(r(f^{-1}(y)-f^{-1}(y_0))||}{||y-y_0||} &\le ||(Df(x_0))^{-1}||\tfrac{||-r(f^{-1}(y)-f^{-1}(y_0))||}{||f^{-1}(y)-f^{-1}(y_0)||}\tfrac{||f^{-1}(y)-f^{-1}(y_0)||}{||y-y_0||} \\ &\le 2||(Df(x_0))^{-1}||\tfrac{||r(f^{-1}(y)-f^{-1}(y_0))||}{||f^{-1}(y)-f^{-1}(y_0)||},\end{aligned}$$

using the Lipschitz continuity of f^{-1} as proved above. By the continuity of f^{-1} we know that $f^{-1}(y) \to f^{-1}(y_0)$ when $y \to y_0$. Hence the last fraction in the inequality above goes to 0 as $y \to y_0$. Therefore $(Df(x_0))^{-1}(-r(f^{-1}(y) - f^{-1}(y_0))) = o(y - y_0)$. Hence, we have proved that f^{-1} is differentiable at each y_0 with $Df^{-1}(y_0) = (Df(x_0))^{-1}$.

All that remains is to show that $Df^{-1}(y)$ is continuous. Consider $||Df^{-1}(y) - Df^{-1}(y_0)|| = ||Df^{-1}(y)(I - Df(x)Df^{-1}(y_0))|| \le 2||I - Df(x)Df^{-1}(y_0)||$ because $\|Df^{-1}(y)\|_{oper} \le 2$. In this last norm we know that $Df(x) \to Df(x_0) = (Df^{-1}(y_0))^{-1}$ as $x \to x_0$. Hence, as $y \to y_0$ we know that $x \to x_0$ and so $(I - Df(x)Df^{-1}(y_0) \to (I - Df(x_0)Df^{-1}(y_0)) = I - I = 0$. Thus $Df^{-1}(y) \to Df^{-1}(y_0)$, and we have proved continuity of Df^{-1}.

We have finished the proof for the special case. It is important to realize that all of the hard work is now done, because it is not hard to convert the general case to the special case proved above. Consider $\overline{f}(x) := (Df(x_0))^{-1}(f(x + x_0) - f(x_0))$, and check that $\overline{f}$ satisfies the special case above. Once we know that $\overline{f}$ is invertible, it is a matter of algebra and an application of the chain rule to show that f has a C^1 inverse. $\square$

One of the most important uses of the Inverse Function theorem is changes of coordinates. Consider a function $T : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that $(x_1, \ldots, x_n)$ is an element of the domain $\mathbb{R}^n$, and express $T(\vec{x}) = (u_1(\vec{x}), \ldots, u_n(\vec{x}))$. Then we can think of $\vec{x}$ as being the original coordinates in $\mathbb{R}^n$ and $\vec{u}$ as being the new coordinates. T is the function which expresses how one can convert a point from $\vec{x}$ terms into $\vec{u}$ terms. Then the inverse function theorem says that T is invertible, i.e. $\vec{x}$ and $\vec{u}$ are equivalent coordinates for $\mathbb{R}^n$ and we can solve for $\vec{x}$ in terms of $\vec{u}$, if DT is invertible. Here, DT is the $n \times n$ matrix of first derivatives of T, usually called the *Jacobian matrix* of T. Then since DT is a square matrix, by standard linear algebraic results, T is invertible if $det(DT) \neq 0$. The function $det(DT)$, or sometimes $|det(DT)|$ is usually called the *Jacobian* of T. It is the same quantity which appears in the change of variables formula for multiple integrals (that is, u-substitution).

EXAMPLE 2.1. *Consider polar coordinates on* $\mathbb{R}^2$. *We can express* x *and* y *in terms of* r *and* θ *as follows:* $x = r\cos(\theta)$, $y = r\sin(\theta)$. *Then our transformation* T *is given by* $T(r, \theta) = (r\cos(\theta), r\sin(\theta))$, *so*

$$DT = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix},$$

and $det(DT) = r$. *It therefore follows that the transformation from polar to Cartesian coordinates is invertible unless* $r = 0$, *i.e. except at the origin. If* $r > 0$, *then we obtain the following formula for* DT^{-1}:

$$DT^{-1} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{pmatrix}.$$

EXAMPLE 2.2. *We may similarly consider spherical coordinates on* $\mathbb{R}^3$. *We can express the transformation as follows:*

$$T(\rho, \theta, \phi) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)),$$

where θ *is the horizontal angle from the positive* x*-axis (longitude), which varies from* 0 *to* 2ϕ, *and* ϕ *is the angle from the* z*-axis (azimuthal angle; latitude), which varies from* 0 *to* π; ρ *is the distance from the origin, which can be any nonnegative real number. So*

$$DT = \begin{pmatrix} \cos(\theta)\sin(\phi) & -\rho\cos(\theta)\sin(\phi) & \rho\cos(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{pmatrix}.$$

It therefore follows that $det(DT) = -\rho^2\sin(\phi)$, *so* T *is invertible unless* $\rho = 0$ *or* $\phi = 0$ *or* 2π. *This makes sense–spherical coordinates, just like polar coordinates, are singular at the origin where the angles become undefined. They are also singular at the north and south pole where* ϕ *makes sense, but* θ *does not.*

2.0.2. *Exercises.*

EXERCISE 2.1. *Write a formal proof that* $\{G_N := \sum_{n=0}^{N} L^{\circ n}\}$ *is a Cauchy sequence with respect to the operator norm.*

EXERCISE 2.2. *Show that* $G \circ (I - L) = (I - L) \circ G = I$.

EXERCISE 2.3. *Consider the function* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *given by* $f(x, y, z) = (yz, xz, xy)$. *Determine all points* (x_0, y_0, z_0) *in* $\mathbb{R}^3$ *such that* f *is invertible in a neighborhood of* (x_0, y_0, z_0). *Compute the derivative of the inverse function at the point* $(1, 1, 1)$.

EXERCISE 2.4. *Consider the function* $L : C^1([0, 1]) \to C^0([0, 1])$ *given by* $L(f) = \frac{df}{dx} + f^2$.

(1) *Show that* L *is a Fréchet differentiable at* f_0 *for any fixed* $f_0 \in C^1([0, 1])$, *and compute* $DL(f_0)$.
(2) *Show that* $L \in C^1(C^1([0, 1]); C^0([0, 1]))$.
(3) *Does* L *satisfy the conditions of the inverse function theorem? Why or why not?*
(4) *Set* $f_0(x) \equiv 1$. *Is* L *invertible in a neighborhood of* f_0*? Why or why not? If it is, find its inverse and show that it is* C^1. *If not, give an explicit example of the failure of injectivity.*

EXERCISE 2.5. *Check the details to show that the general case of the Inverse Function Theorem can be reduced the special case via the transformation* $\overline{f}(x) := (Df(x_0))^{-1}(f(x + x_0) - f(x_0))$ *as claimed in the proof above.*

EXERCISE 2.6. *Consider the transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $T(u, v) = (u, uv)$. *Draw a picture of the image of the square* $[0, 1] \times [0, 1]$ *under this transformation. Determine when* T *is invertible. Find the area element for the transformation* T.

EXERCISE 2.7. *Consider the transformation* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *which converts cylindrical coordinates to cartesian coordinates. Following the examples above, express* T *in terms of coordinate functions, find* DT *and* $\det(DT)$ *and use this information and the inverse function theorem to determine on which parts of* $\mathbb{R}^3$ *the conversion is invertible.*

2.1. The Implicit Function Theorem.

THEOREM 2.2. *Let* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ *be continuously differentiable. Let* $\vec{y} \in \mathbb{R}^{n+m}$ *be represented as* $(\vec{x}, \vec{z})$, *with* $\vec{x} \in \mathbb{R}^n$ *and* $\vec{z} \in \mathbb{R}^m$, *and suppose that* $F(\vec{x}_0, \vec{z}_0) = \vec{0}$ *and* $D_{\vec{z}}F(\vec{x}_0, \vec{z}_0)$ *is an invertible map in* $\mathcal{B}(\mathbb{R}^m, \mathbb{R}^m)$. *Then there is a neighborhood* $U \subset \mathbb{R}^n$ *of* $\vec{x}_0$, *and a neighborhood* $V \subset \mathbb{R}^m$ *of* z_0 *and a function* $g \in C^1(U; V)$ *so that, for each* $\vec{x} \in U$, $F(\vec{x}, g(\vec{x})) = 0$.

Basically, this theorem says that if you have m equations in $n + m$ unknowns, and the equations are not degenerate, then the last m variables can be solved in terms of the first n, at least locally.

PROOF. Suppose that F, x_0 and z_0 are as in the statement of the theorem. Define $\tilde{F} : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by $\tilde{F}(\vec{x}, \vec{z}) = (\vec{x}, F(\vec{x}, \vec{z}))$. Then $D\tilde{F}$ has the form:

$$D\tilde{F}(\vec{x}_0, \vec{z}_0) = \left(\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ ------- & ------- \\ D_{\vec{x}}F(\vec{x}_0, \vec{z}_0) & D_{\vec{z}}F(\vec{x}_0, \vec{z}_0) \end{array} \right).$$

By linear algebra, $D\tilde{F}$ has the same determinant as DF, so $D\tilde{F}$ is invertible if and only if DF is. Therefore, under the hypotheses of this theorem, by the Inverse Function Theorem, there exist neighborhoods $\tilde{U} \subset \mathbb{R}^{n+m}$ of $(\vec{x}_0, \vec{z}_0)$ and $\tilde{V} \subset \mathbb{R}^{n+m}$ of $(\vec{x}_0, F(\vec{x}_0, \vec{z}_0))$, respectively, so that $\tilde{F}$ is invertible from $\tilde{U}$ to $\tilde{V}$, and $\tilde{F}^{-1} \in C^1(\tilde{V}; \tilde{U})$. Notice that, since the first components of $\tilde{F}$ are the identity, we have that $\tilde{F}^{-1}(\vec{x}, \vec{a}) = (\vec{x}, G(\vec{x}, \vec{a}))$ for some $G : \tilde{V} \to \mathbb{R}^m$. Define

$g(\vec{x}) = G(\vec{x}, \vec{0})$. Select U so that $U \times \{0\} \subset \tilde{V}$, which is possible because the topology on $\mathbb{R}^{n+m}$ is a product topology, and set $V = g(U)$. Then it is clear that $g \in C^1(U; V)$, by the properties of $\tilde{F}^{-1}$ and therefore G. Finally, if $\vec{x} \in U$, then $F(\vec{x}, g(\vec{x})) = F(\vec{x}, G(\vec{x}, \vec{0})) = F(\tilde{F}^{-1}(\vec{x}, \vec{0})) = \vec{0}$ by the construction of $\tilde{F}$. Therefore the theorem is proven. $\square$

Suppose that $f : \mathbb{R}^{n+1} \to \mathbb{R}$. We will express elements of $\mathbb{R}^{n+1}$ using the notation $(z, \vec{x})$, where $\vec{x} \in \mathbb{R}^n$. Suppose that $(z_0, \vec{x}_0) \in \mathbb{R}^{n+1}$ such that $F(z_0, \vec{x}_0) = 0$. Then the Implicit Function Theorem states that if $\frac{\partial f}{\partial z}(z_0, \vec{x}_0) \neq 0$, then there exist neighborhoods $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}$ so that $\vec{x}_0 \in U$ and $z_0 \in V$ and a function $g : U \to V$ so that, for any $\vec{x} \in U$, $f(g(\vec{x}), \vec{x}) = 0$. That is, the level set of the function f, which is defined to be $f^{-1}(0) = \{(z, \vec{x}) \in \mathbb{R}^{n+1} : f(z, \vec{x}) = 0\}$, can be expressed as the graph of the function $z = g(\vec{x})$, at least locally near the point $(z_0, \vec{x}_0)$. Another way to say this is that the surface defined by $F(z, \vec{x}) = 0$ may not be the graph of a function globally in all of $\mathbb{R}^{n+1}$, and it may not be possible at all to solve explicitly for z as a function of $\vec{x}$, but it is *usually* possible to implicitly express z as a function of $\vec{x}$ is small neighborhoods.

Moreover, the Implicit Function Theorem also states that at any such point where $\frac{\partial f}{\partial z} \neq 0$, then $\nabla_{\vec{x}} g(\vec{x}) = \frac{-1}{\frac{\partial f}{\partial z}} \nabla_{\vec{x}} f(z, \vec{x})$.

Let's look at how this works in a simple example:

EXAMPLE 2.3. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(z, x) = z^2 + x^2 - 1$. Then the set where $f(z, x) = 0$ is the unit circle $z^2 + x^2 = 1$. Clearly, this set is not, in its entirety, the graph of a function, because it does not pass the vertical line test. However, either the upper or lower semicircle is a function. Therefore, if we start at a point (z_0, x_0), then the level curve is a function in a neighborhood of (z_0, x_0) unless $(z_0, x_0) = (0, 1)$ or $(0, -1)$–the two points where the graph has a vertical tangent line and then turns back on itself. In this case, the equation is sufficiently simple that we can actually solve for z in terms of x: in the upper semicircle we have $z = \sqrt{1 - x^2}$, and in the lower semicircle we have $z = -\sqrt{1 - x^2}$. In both cases it is possible to check (as discussed in class), that $\frac{dz}{dx} = -\frac{x}{z}$.*

Now let's look at this from the perspective of the Implicit Function Theorem. We expect to be able to solve for z in terms of x as long as $\frac{\partial f}{\partial z} \neq 0$. We have that $\frac{\partial f}{\partial z} = 2z \neq 0$ so long as $z \neq 0$. Hence this is possible except at the points $(0, 1)$ and $(0, -1)$, exactly as found above. Moreover, in this case we expect that $\frac{dg}{dx} = \frac{-1}{\frac{\partial f}{\partial z}} \frac{\partial f}{\partial x} = -\frac{x}{z}$, again as computed above. Therefore, our direct calculations agree with the conclusions of the implicit function theorem.

As the dimension increases, we see a little more complication, but the ideas are fundamentally the same.

EXAMPLE 2.4. *Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is given by $f(z, x) = z^2 + x^2 + y^2 - 9$. Then the set where $f(z, x, y) = 0$ is the sphere $z^2 + x^2 + y^2 = 9$. This again does not pass the vertical line test. However, either the upper or lower hemisphere is a function. Therefore, if we start at a point (z_0, x_0, y_0), then the level surface is a function in a neighborhood of (z_0, x_0, y_0) unless $z_0 = 0$, i.e.$x_0^2 + y_0^2 = 9$–the equator, where the graph has a vertical tangent plane and then turns back on itself. In this case, the equation is sufficiently simple that we can actually solve for z in terms of x: in the upper hemisphere we have $z = \sqrt{9 - x^2 - y^2}$, and in the lower hemisphere we have $z = -\sqrt{9 - x^2 - y^2}$. In both cases it is possible to check (as discussed in class), that $\frac{\partial z}{\partial x} = -\frac{x}{z}$ and $\frac{\partial z}{\partial y} = -\frac{y}{z}$.*

Now let's look at this from the perspective of the Implicit Function Theorem. We expect to be able to solve for z in terms of x and y as long as $\frac{\partial f}{\partial z} \neq 0$. We have that $\frac{\partial f}{\partial z} = 2z \neq 0$ so long as $z \neq 0$. Hence this is possible except at the equator, exactly as discussed above. Moreover, in this case we expect that $\frac{\partial g}{\partial x} = \frac{-1}{\frac{\partial f}{\partial z}}\frac{\partial f}{\partial x} = -\frac{x}{z}$ and $\frac{\partial g}{\partial y} = \frac{-1}{\frac{\partial f}{\partial z}}\frac{\partial f}{\partial y} = -\frac{y}{z}$, again as computed above. Therefore, our direct calculations agree with the conclusions of the implicit function theorem.

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions. Suppose that we want to maximize the value of the function f subject to the constraint that g equals a given constant c. That is, we want to find the maximum value of $f(x, y)$ on the level curve given by $g(x, y) = c$. This is a very common problem in applications of calculus. For example, we may want to maximize the volume of a cylindrical can subject to a fixed surface area. In that case, the function f is the volume of the can and the function g is the surface area.

This type of problem is usually solved via a technique known as Lagrange multipliers. The main theorem is as follows:

THEOREM 2.3. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions. Let $c \in \mathbb{R}$. Suppose that $g(x_0, y_0) = c$ and $f(x_0, y_0) \geq f(x, y)$ for every $(x, y) \in \mathbb{R}^2$ such that $g(x, y) = c$. Then if $\nabla g(x_0, y_0) \neq \vec{0}$, it follows that $\nabla f(x_0, y_0)$ is a scalar multiple of $\nabla g(x_0, y_0)$.*

REMARK 2.1. *Notice that this is not an existence theorem–it is certainly possible for no maximum to exist, depending on the shapes of f and g.*

REMARK 2.2. *Notice also that this is not a uniqueness theorem–it is possible for this technique to produce critical points which are not maxima or even extrema. Moreover, it is possible for there to be critical points at which $\nabla g = \vec{0}$, which this method will not find.*

REMARK 2.3. *The conclusion of this theorem is identical in higher dimensions, but for the purposes of space we will prove it only in two dimensions.*

PROOF. Suppose that f, g, c, x_0, and y_0 are as in the hypotheses of the theorem. Consider the set $\{(x, y) \in \mathbb{R}^2 : g(x, y) = c$. We have assumed that $\nabla g \neq \vec{0}$, so without loss of generality we may assume that $\frac{\partial g}{\partial y} \neq 0$. (Otherwise we can use the same argument as below but interchange the roles of x and y.) Then by the Implicit Function Theorem there exists a function $h : \mathbb{R} \to \mathbb{R}$ so that, for all x in some neighborhood U of x_0, $g(x, h(x)) = c$, and moreover $\frac{dh}{dx} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}$ Let $\vec{v}$ be any vector in $\mathbb{R}^2$ for which $\vec{v} \cdot \nabla g(x_0, y_0) = 0$. Now, define the curve $\vec{\gamma}(t)$ by $\vec{\gamma}(t) = (x_0 + tv_1, h(x_0 + tv_1))$. Then, by the chain rule,

$$\frac{dg(\gamma(t))}{dt} = \nabla g \cdot (v_1, \frac{dh}{dx}v_1) = \nabla g \cdot (v_1, v_1\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}) = \frac{v_1}{\frac{\partial g}{\partial y}}(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}) \cdot (-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}) = 0.$$

Hence $g(\gamma(t)) = c$ for all times t. Therefore, if $f(x_0, y_0)$ is the maximum value of f on the constraint set $g = c$, it follows that $f(x_0, y_0)$ is also the maximum value of f along the graph of γ. Therefore, it must be the case that $f(\gamma(t))$ has a critical point at $t = 0$. But $\frac{df(\gamma(t))}{dt} = \nabla f \cdot (v_1, v_2)$ at $t = 0$. So, ∇f is orthogonal to any vector to which ∇g is orthogonal. Hence, at (x_0, y_0), the two vectors ∇f and ∇g are parallel, i.e. scalar multiples of one another. $\square$

EXAMPLE 2.5. *Suppose that you want to make a cylindrical can which has a fixed surface area of 2π units, with maximum volume. Let r be the radius of the can, and let h be its height. Then the volume of the can is $\pi r^2 h$ and the surface area is $2\pi rh + 2\pi r^2 = 2\pi r(r+h)$. Therefore what we want is to maximize the function $\pi r^2 h$ under the constraint $r(r+h) = 1$. In the language of Lagrange Multipliers, $f(r,h) = \pi r^2 h$, and $g(r,h) = r^2 + rh$. By the theorem proved above, any nondegenerate maximum satisfies the equation $\nabla f = \lambda \nabla g$ for some scalar λ. Note that $\nabla f = (2\pi rh, \pi r^2)$, and $\nabla g = (2r+h, r)$. Therefore we must solve the following three equations:*

$$2\pi rh = \lambda(2r+h) \tag{9}$$

$$\pi r^2 = \lambda r \tag{10}$$

$$r(r+h) = 1 \tag{11}$$

for the three unknowns r, h, and λ. The second equation tells me that $\lambda = \pi r$. Plugging into the first equation, $2\pi rh = 2\pi r^2 + \pi rh$, so $\pi rh = 2\pi r^2$, so $h = 2r$. Finally, plugging into the third equation, $r(3r) = 1$, so $r = \frac{1}{\sqrt{3}}$ and $h = \frac{2}{\sqrt{3}}$. Since this is the only critical point found, and also since the volume goes to 0 if r or h goes to 0, we infer that this is the maximum (although this is not a proof). We conclude that the maximum volume under the constraint is $\pi r^2 h = \frac{2\pi}{3\sqrt{3}}$.

2.1.1. *Exercises.*

EXERCISE 2.8. *Consider the surface $f(z,x,y) = x^3 + 3y^2 + 8xz^3 - 3z^3 y = 1$. Determine near which points (x_0, y_0, z_0) in $\mathbb{R}^3$ it is possible to express z as function of x and y locally, and justify your work via the Implicit Function Theorem. At such points, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.*

EXERCISE 2.9. *Maximize the function $x^4 + y^4$ subject to the constraint $x^2 + y^2 = 1$. Are there any non-maximum critical points?*

3. The Calculus of Variations

3.1. Geodesics. Suppose that $\vec{a}$ and $\vec{b}$ are two points in $\mathbb{R}^2$. We would like to answer the following question: Does there exist a path from $\vec{a}$ to $\vec{b}$ which is shorter than any other path between these points? Is this shortest path unique? What is the length of the shortest path.

In order to answer these questions, we are going to give a brief introduction to the calculus of variations, the area of analysis concerned with minimizing functionals on metric spaces. Define $\mathcal{A} = \{\vec{c} \in C^2([0,1];\mathbb{R}^2) : \vec{c}(0) = \vec{a} \text{ and } \vec{c}(0) = \vec{b}\}$. This is the set of *admissible* path, that is, paths which really start at $\vec{a}$ and end at $\vec{b}$, and which are smooth enough for the calculations which we would like to do. Also define the set $\mathcal{P} = \{\vec{c} \in C^2([0,1];\mathbb{R}^2) : \vec{c}(0) = \vec{0} \text{ and } \vec{c}(0) = \vec{0}\}$. This is the set of *perturbations*, which we will use later. Notice that $\mathcal{P}$ is a closed linear subspace of the complete normed vector space $C^2([0,1];\mathbb{R}^2)$, so it is itself a complete normed vector space. The set $\mathcal{A}$ is not itself a linear subspace of $C^2([0,1];\mathbb{R}^2)$, but it is an *affine* space: if $\vec{c} \in \mathcal{A}$ and $\vec{h} \in \mathcal{P}$, then $\vec{c} + \vec{h} \in \mathcal{A}$.

Now, consider the functional $J : \mathcal{A} \to \mathbb{R}$ given by

$$J(\vec{c}) = \int_0^1 \sqrt{\left(\frac{dc_1}{dt}\right)^2 + \left(\frac{dc_2}{dt}\right)^2}\, dt,$$

which outputs the arclength of any curve $\vec{c}(t) = (c_1(t), c_2(t)) \in \mathcal{A}$. Mathematically, the question listed above is:

QUESTION 3.1. *Does J have a unique minimizer on $\mathcal{A}$? If so, what is the value of the minimum of J over $\mathcal{A}$?*

We will not be able to answer this question in its entirety in this course, but we *can* obtain a lot of information from our newfound ability to differentiate functionals. Suppose that there exists some $\vec{\gamma} \in \mathcal{A}$ so that $J(\vec{\gamma}) = \inf_{\vec{c}\in\mathcal{A}} J(\vec{c})$. Then it follows, as in regular calculus, that $\vec{\gamma}$ must be a critical point of J. That is, $DJ(\vec{\gamma})[\vec{h}] = 0$ for any $\vec{h} \in \mathcal{P}$.[1] Let's compute DJ. Let $\vec{c} \in \mathcal{A}$ and $\vec{h} = (h_1, h_2) \in \mathcal{P}$ and compute:

$$J(\vec{c}+\vec{h}) - J(\vec{c}) = \int_0^1 \sqrt{\left(\frac{d(c_1+h_1)}{dt}\right)^2 + \left(\frac{d(c_2+h_2)}{dt}\right)^2}\,dt - \int_0^1 \sqrt{\left(\frac{dc_1}{dt}\right)^2 + \left(\frac{dc_2}{dt}\right)^2}\,dt.$$

In class we computed this and extracted the linear curves using the tangent line approximation for the square root function. You may want to record the result here: We found at the

end of the calculation that we could make a simplifying assumption and ignore the square root inside the integral. For clarity of exposition I will make that assumption now:

ASSUMPTION 3.1. *A function $\vec{\gamma} \in \mathcal{A}$ is a critical point of J if and only if $\vec{\gamma}$ is a critical point of the functional $K : C^2([0,1];\mathbb{R}^2) \to \mathbb{R}$ given by $K(\vec{c}) = \int_0^1 \left(\left(\frac{dc_1}{dt}\right)^2 + \left(\frac{dc_2}{dt}\right)^2\right) dt$.*

Therefore we can proceed by finding DK instead:

$$\begin{aligned}K(\vec{c}+\vec{h})-K(\vec{c}) &= \int_0^1 \left(\left(\frac{d(c_1+h_1)}{dt}\right)^2 + \left(\frac{d(c_2+h_2)}{dt}\right)^2\right) dt - \int_0^1 \left(\left(\frac{dc_1}{dt}\right)^2 + \left(\frac{dc_2}{dt}\right)^2\right) dt \\ &= \int_0^1 \left((\frac{dc_1}{dt})^2 + 2\frac{dc_1}{dt}\frac{dh_1}{dt} + (\frac{dh_1}{dt})^2 + (\frac{dc_2}{dt})^2 + 2\frac{dc_2}{dt}\frac{dh_2}{dt} + (\frac{dh_2}{dt})^2 - (\frac{dc_1}{dt})^2 - (\frac{dc_2}{dt})^2\right) dt \\ &= \int_0^1 \left(2\frac{dc_1}{dt}\frac{dh_1}{dt} + 2\frac{dc_2}{dt}\frac{dh_2}{dt}\right) dt + o(\|\vec{h}\|)\end{aligned}$$

It therefore follows that $DK(\vec{c})[\vec{h}] = int_0^1 \left(2\frac{dc_1}{dt}\frac{dh_1}{dt} + 2\frac{dc_2}{dt}\frac{dh_2}{dt}\right) dt$. So, if $\vec{\gamma}$ is a critical point for K, then, for every $\vec{h} \in \mathcal{P}$, we have that $int_0^1 \left(2\frac{d\gamma_1}{dt}\frac{dh_1}{dt} + 2\frac{d\gamma_2}{dt}\frac{dh_2}{dt}\right) dt = 0$. Since, $\vec{h} = \vec{0}$ at

[1] Note that we need $h \in \mathcal{P}$, not $\mathcal{A}$, because $\mathcal{A}$ is an affine space as discussed above. Therefore if we want to look at things like $J(\vec{c}+\vec{h}) - J(\vec{c})$, then we need $J(\vec{c}+\vec{h})$ to make sense, so we need $\vec{c}+\vec{h}$ to be in $\mathcal{A}$.

both endpoints, we may integrate by parts to find that $-2\int_0^1(\frac{d^2\gamma_1}{dt^2}h_1+\frac{d^2\gamma_2}{dt^2}h_2)dt=0$ for any $\vec{h}\in\mathcal{P}$. It turns out that this can only hold if $\frac{d^2\gamma_1}{dt^2}=\frac{d^2\gamma_2}{dt^2}\equiv 0$. Hence, $\vec{\gamma}$ is a straight line. As we expected, the straight line segment from $\vec{a}$ to $\vec{b}$ is the unique critical point of J.

The second order differential equation obtained from seeking critical points of J is called the **Euler-Lagrange Equation** of J. It turns out that if a critical point of a functional J exists and is sufficiently smooth, then it will always be a solution of the corresponding Euler-Lagrange equation. However, existence and smoothness of critical points are both hard problems that are beyond the scope of this class.

3.2. Weak Solutions to Differential Equations. One thing we can do to alleviate the question of smoothness, at least, is to look at what are called *weak* solutions to the Euler-Lagrange Equation:

DEFINITION 3.1. *Suppose that $c\in W^{1,2}([0,1])$ and $c(0)=a$ and $c(1)=b$. If, for every $\phi\in W^{1,2}([0,1])$ such that $\phi(0)=\phi(1)=0$, we have that*

$$\int_0^1 (c'(t)\phi'(t)+f(c(t))\phi(t))\,dt=0,$$

then we say that c is a **weak solution** *of the boundary value problem $-c''(t)+f(c(t))=0$ on $[0,1]$ with boundary conditions $c(0)=a$ and $c(1)=b$.*

Notice that this sort of weak solution is exactly what we obtained when deriving the Euler-Lagrange equation above. Therefore we do not need our c to be in C^2 as previously assumed, but only in $W^{1,2}$, in order to make sense of the problem. If c happens to actually be in C^2, then it is a weak solution of the differential equation if and only if it is a solution in the classical sense. There are several reasons to consider weak solutions to differential equations:

- As we saw above, the definition of a weak solution is a natural extension of the calculus of variations. Any time we are dealing with a problem that comes from an energy-based application, and are minimizing an integral, weak solutions will naturally appear.
- The space $W^{1,2}$ is the largest space on which J and K make sense, so if we want a solution to exist we ought to allow ourselves to consider as many options as possible.
- Rough functions occur frequently in applications; for example, random functions tend to be in spaces like $W^{1,2}$, but not C^2 or even C^1.
- The space $W^{1,2}$ is a Hilbert space and therefore has advantageous structural properties from a linear algebraic perspective.

3.3. The Second Variation. In the geodesic problem above, we found that J had a unique critical point. But how do we know that it is a local and hopefully global minimizer? Just like in calculus, we can do a second derivative test to see if it is a local min. To compute the second derivative of K, we differentiate $DK(\vec{c})[\vec{h}]$ with respect to $\vec{c}$: Let $\vec{\epsilon}=(\epsilon_1,\epsilon_2)\in\mathcal{P}$

and compute

$$
\begin{aligned}
&DK(\vec{c}+\vec{\epsilon})[\vec{h}] - DK(\vec{c})[\vec{h}] \\
&\qquad = \int_0^1 \left(2\frac{d(c_1+\epsilon_1)}{dt}\frac{dh_1}{dt} + 2\frac{d(c_2+\epsilon_2)}{dt}\frac{dh_2}{dt}\right)dt - \int_0^1 \left(2\frac{dc_1}{dt}\frac{dh_1}{dt} + 2\frac{dc_2}{dt}\frac{dh_2}{dt}\right)dt \\
&\qquad = \int_0^1 \left(2\frac{d\epsilon_1}{dt}\frac{dh_1}{dt} + 2\frac{d\epsilon_2}{dt}\frac{dh_2}{dt}\right)dt.
\end{aligned}
$$

This entire quantity is linear in $\vec{\epsilon}$. Therefore, we conclude that

$$D^2K(\vec{c})[\vec{\epsilon},\vec{h}] = \int_0^1 \left(2\frac{d\epsilon_1}{dt}\frac{dh_1}{dt} + 2\frac{d\epsilon_2}{dt}\frac{dh_2}{dt}\right)dt.$$

Notice that this does not depend on $\vec{c}$, so if we were to take the third derivative it would be zero. Also notice that, for any $\vec{h} \in \mathcal{P}$, $|D^2K[\vec{h},\vec{h}]| = 2\|\vec{h'}\|_{L^2}^2$. Therefore, the function is uniformly concave up in every possible direction, and it follows that $\vec{\gamma}$ is indeed a min. Since this didn't depend on our particular choice of $\vec{c}$, this also tells us that the functional K is uniformly convex on its entire domain.

Finally, let's check that $\vec{\gamma}$ is actually a global min. Since K is uniformly convex, it is enough to check that K is *coercive*. That is, as $\|\vec{c}\| \to \infty$, $K(\vec{c}) \sim \|\vec{c}\|^2$. But this is clear from the definition of K, since $K(\vec{c}) = \|c'\|_{L^2}^2$. Therefore, much as an upward-facing parabola in one dimension can have only one critical point which must be a global min, this function K can have only one critical point, which is a global min. Therefore, the straight line path is indeed the shortest path between any two points in $\mathbb{R}^2$, and any other path will have strictly larger length.

On a general Riemannian manifold, these same calculations can be done, and the resulting curve (whose shape depends on the curvature tensor of the manifold) is called a *geodesic*. The existence, uniqueness, smoothness, and other properties of geodesics are quite important to the study of Riemannian geometry.

3.4. More Examples. Another interesting example of a physical situation in which an energy is minimized is the *Minimal Surface Equation*. The minimal surface equation models the behavior of elastic surfaces, such as a soap film. An elastic surface gains potential energy when it is stretched, so when seeking an equilibrium position it will seek to minimize its total surface area. That is, if B is a closed, connected, smooth domain in $\mathbb{R}^2$, and $z = f(x,y)$ is a function in $W^{1,2}(B)$, such that z is equal to a fixed function $\phi(x,y)$ on ∂B. (This fixed function, or *boundary condition*, tells us what shape the wire to which the soap film is attached takes.) Then the surface area of the graph of f is given by

$$J(f) = \int_B \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}dxdy.$$

As above, if we seek a minimizer $u(x,y)$ of this functional, then we expect that minimizer to be a critical point. That is, we expect $DJ(u)[h] = 0$ for any $h \in W^{1,2}(B)$ such that $h = 0$

on ∂B. We compute:

$$
\begin{aligned}
J(f+h) - J(f) &= \int_B \left[\sqrt{1 + (\frac{\partial(f+h)}{\partial x})^2 + (\frac{\partial(f+h)}{\partial y})^2} - \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}\right] dxdy \\
&= \int_B \frac{1}{2\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}} \left(\frac{\partial f}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial y}\right) dxdy + o(\|h\|)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
DJ(f)[h] &= \int_B \frac{1}{2\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}} \left(\frac{\partial f}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial y}\right) dxdy \\
&= \int_B \frac{1}{2\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}} \nabla f \cdot \nabla h dxdy \\
&= \int_B \nabla \cdot \left(\frac{1}{2\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}} \nabla f\right) h dxdy.
\end{aligned}
$$

Thus, if u solves $DJ(u)[h] = 0$ for all valid test functions h, then it follows that

$$
\nabla \cdot \left(\frac{1}{2\sqrt{1 + (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2}} \nabla u\right) = 0
$$

in the domain B. This is the minimal surface equation. Notice that it is highly nonlinear. In the context of Riemannian geometry, this equation can be written to concern only the intrinsic curvatures of the surface in which we are interested.

3.4.1. *Exercises.*

EXERCISE 3.1. *Consider the functional* $J : W^{1,p}(\mathbb{R}^2) \to \mathbb{R}$ *given by* $J(u) = \int_{\mathbb{R}^2} \|\nabla u\|^p dxdy$.

(1) *Compute* DJ. *Is* J *Fréchet differentiable on this space?*
(2) *Find the Euler-Lagrange equation for* J.
(3) *What function is the global minimizer of* J *on this space and why?*