HW #7 solutions

3. $f'(x_0) = [\alpha_0 f(x_0) + \alpha_1 f(x_0+h) + \alpha_2 f(x_0+2h) + \alpha_3 f(x_0+3h) + \alpha_4 f(x_0+4h)] / h$

$f(x_0+h) = f(x_0) + hf'(x_0) + h^2/2!\, f''(x_0) + h^3/3!\, f'''(x_0) + h^4/4!\, f^{iv}(x_0) + h^5/5!\, f^{(v)}(\xi_1)$, $\xi_1 \in [x_0, x_0+h]$

$f(x_0+2h) = f(x_0) + 2hf'(x_0) + 4/2!\, f''(x_0)h^2 + 8/3!\, h^3 f'''(x_0) + 16/4!\, h^4 f^{iv}(x_0) + \frac{32h^5}{5!} f^{(v)}(\xi_2)$, $\xi_2 \in [x_0, x_0+2h]$

$f(x_0+3h) = f(x_0) + 3hf'(x_0) + 9/2!\, f''(x_0)h^2 + 27/3!\, h^3 f'''(x_0) + 81/4!\, h^4 f^{iv}(x_0) + \frac{243}{5!} h^5 f^{(v)}(\xi_3)$, $\xi_3 \in [x_0, x_0+3h]$

$f(x_0+4h) = f(x_0) + 4hf'(x_0) + 16/2!\, f''(x_0)h^2 + 64/3!\, h^3 f'''(x_0) + \frac{256}{4!} h^4 f^{iv}(x_0) + \frac{1024h^5}{5!} f^{(v)}(\xi_4)$, $\xi_4 \in [x_0, x_0+4h]$

Set $f'(x_0) = [\alpha_0 f(x_0) + \alpha_1 f(x_0+h) + \alpha_2 f(x_0+2h) + \alpha_3 f(x_0+3h) + \alpha_4 f(x_0+4h)] / h$

GATHER similar terms:

$f(x_0)$ terms: $[\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4]/h = 0 \Rightarrow \boxed{\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0}$

$f'(x_0)$ terms: $[h\alpha_1 + 2h\alpha_2 + 3h\alpha_3 + 4h\alpha_4]/h = 1 \Rightarrow \boxed{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 1}$

$f''(x_0)$ terms: $[\alpha_1 h^2/2! + \alpha_2 4/2!\, h^2 + 9/2!\, h^2\alpha_3 + 16/2!\, h^2\alpha_4]/h = 0 \Rightarrow \boxed{\frac{\alpha_1}{2} + 2\alpha_2 + \frac{9}{2}\alpha_3 + 8\alpha_4 = 0}$

$f'''(x_0)$ terms: $[\alpha_1 h^3/3! + \alpha_2 8/2!\, h^3 + \alpha_3 27/3!\, h^3 + \alpha_4 \frac{64}{3!} h^3]/h = 0 \Rightarrow \boxed{\alpha_1/6 + 8/6\,\alpha_2 + 27/6\,\alpha_3 + 64/6\,\alpha_4 = 0}$

$f^{iv}(x_0)$ term: $[\alpha_1 h^4/4! + \alpha_2 16/4!\, h^4 + \alpha_3 \frac{81}{4!} h^4 + \frac{256}{4!}\alpha_4 h^4]/h = 0 \Rightarrow \boxed{1/24\,\alpha_1 + 16/24\,\alpha_2 + \frac{81}{24}\alpha_3 + \frac{256}{24}\alpha_4 = 0}$

5 equations and 5 unknowns. Solve to get:

$\alpha_0 = -25/12, \quad \alpha_1 = 4, \quad \alpha_2 = -3, \quad \alpha_3 = 16/12, \quad \alpha_4 = -3/12$

So, $f'(x_0) = [-25f(x_0) + 4f(x_0+h) - 3f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)] / 12h$
$+ O(h^4)$

The $O(h^4)$ term is: $\left[4\, h^5/5!\, f^{v}(\xi_1) + -3 \cdot \frac{32}{5!} h^5 f^{v}(\xi_2) + 16/12 \cdot \frac{243}{5!} h^5 f^{v}(\xi_3) - \frac{3}{12} \cdot \frac{1024h^5}{5!} f^{v}(\xi_4)\right] / h$

$= \left[\frac{4}{5!} f^{v}(\xi_1) + \frac{32 \cdot 4}{5!} f^{(v)}(\xi_2) + \frac{16}{12} \cdot \frac{243}{5!} f^{v}(\xi_3) - \frac{3}{12} \cdot \frac{1024}{5!} f^{(v)}(\xi_4)\right] h^4$

where $\xi_1 \in [x_0, x_0+h]$, $\xi_2 \in [x_0, x_0+2h]$, $\xi_3 \in [x_0, x_0+3h]$, $\xi_4 \in [x_0, x_0+4h]$.

4. $f(x+h) = f(x) + hf'(x) + h^2/2!\, f''(x) + h^3/3!\, f'''(\xi_1)$, $\xi_1 \in [x, x+h]$

$f(x+2h) = f(x) + 2hf'(x) + 4h^2/2!\, f''(x) + 8h^3/3!\, f'''(\xi_2)$, $\xi_2 \in [x, x+2h]$

$$f'(x) = \frac{1}{2h}\Big[-3f(x) + 4\big[f(x) + hf'(x) + h^2/2!\, f''(x) + h^3/3!\, f'''(\xi_1)\big] - \big[f(x) + 2hf'(x) + 4h^2/2!\, f''(x) + 8h^3/3!\, f'''(\xi_2)\big]\Big]$$

$$\Rightarrow f'(x) = 0 \cdot f(x) + \frac{2h}{2h} f'(x) + \left(\frac{4h^2}{2!} - 4h^2/2!\right)\frac{1}{2h} f''(x) + \frac{1}{2h}\left[\frac{h^3}{3!} \cdot 4 f'''(\xi_1) - \frac{8h^3}{3!} f'''(\xi_2)\right]$$

$$\Rightarrow f'(x) = f'(x) + \underline{\underline{\Theta(h^2)}}$$

error