Homework 4

Due 11/8/2012 2 PM

1. (4 points): **Question 13.8**

   (a) This asks for the probability that Toothache is true. $P(toothache) = 0.108+0.012+0.016+0.064 = 0.2$

   (b) This asks for the vector of probability values for the random variable Cavity. It has two values, which we list in the order $< true, false >$. First add up $0.108 + 0.012 + 0.072 + 0.008 = 0.2$. Then we have $P(Cavity) = < 0.2, 0.8 >$

   (c) This asks for the vector of probability values for Toothache, given that Cavity is true. $P(Toothache|cavity) = < (.108 + .012)/0.2, (0.072 + 0.008)/0.2 > = < 0.6, 0.4 >$

   (d) This asks for the vector of probability values for Cavity, given that either Toothache or Catch is true. First compute $P(toothache \vee catch) = 0.108 + 0.012 + 0.016 + 0.064 + 0.072 + 0.144 = 0.416$. Then $P(Cavity|toothache \vee catch) = < (0.108+0.012+0.072)/0.416, (0.016+0.064+0.144)/0.416 > = < 0.4615, 0.5384 >$

2. (4 points): **Question 13.17** There is a bug in this question – $P(B|X,Z)$ should be $P(Y|X,Z)$ i.e., B should be replaced with Y. So your goal is to prove that $P(X,Y|Z) = P(X|Z)P(Y|Z)$ is equivalent to each of $P(X|Y,Z) = P(X|Z)$ and $P(Y|X,Z) = P(Y|Z)$.

   (Answer) We with the given statement,

   $$P(X,Y|Z) = P(X|Z)P(Y|Z)$$

   Now we apply the definition of conditional probability.

   $$P(X,Y|Z) = \frac{P(X,Y,Z)}{P(Z)} \text{and} P(Y|Z) = \frac{P(Y,Z)}{P(Z)}$$

   Now,

   $$\frac{P(X,Y,Z)}{P(Z)} = P(X|Z)\frac{P(Y,Z)}{P(Z)}$$

   Since $P(X,Y,Z) = P(X|Y,Z)P(Y,Z)$, we substitute the RHS for $P(X,Y,Z)$ to get:

   $$\frac{P(X|Y,Z)P(Y,Z)}{P(Z)} = P(X|Z)\frac{P(Y,Z)}{P(Z)}$$
   $$\Rightarrow P(X|Y,Z) = P(X|Z) \tag{1}$$

   The second part of the exercise follows from by a similar derivation, or by noticing that X and Y are interchangeable in the original statement (because multiplication is commutative and X,Y means the same as Y,X).

3. (10 points): **Question 14.6** Parts **a** through **d**.

a. (c) matches the equation. The equation describes absolute independence of the three genes, which requires no links among them.

b. (a) and (b). The assertions are the absent links; the extra links in (b) may be unnecessary but they do not assert an actual dependence. (c) asserts independence of genes which contradicts the inheritance scenario.

c. (a) is best. (b) has spurious links among the H variables, which are not directly causally connected in the scenario described. (In reality, handedness may also be passed down by example/training.)

d. Notice that the $l \to r$ and $r \to l$ mutations cancel when the parents have different genes, so we still get 0.5.

4. (3 points): **Question 14.14** Part **a**. The network asserts (ii) and (iii). (For (iii), consider the Markov blanket of M.)

5. (9 points): **Question 14.15** Parts **a,b,c**.

a.

$$
\begin{aligned}
&P(B|j,m) \\
&= \alpha P(B) \sum_e P(e) \sum_a P(a|b,e)P(j|a)P(m|a) \\
&= \alpha P(B) \sum_e P(e) \left[.9 \times .7 \times \begin{pmatrix} .95 & .29 \\ .94 & .001 \end{pmatrix} + .05 \times .01 \times \begin{pmatrix} .05 & .71 \\ .06 & .999 \end{pmatrix} \right] \\
&= \alpha P(B) \sum_e P(e) \begin{pmatrix} .598525 & .183055 \\ .59223 & .0011295 \end{pmatrix} \\
&= \alpha P(B) \left[0.002 \times \begin{pmatrix} .598525 \\ .183055 \end{pmatrix} + 0.998 \times \begin{pmatrix} .59223 \\ .0011295 \end{pmatrix} \right] \\
&= \alpha \begin{pmatrix} .001 \\ .999 \end{pmatrix} \times \begin{pmatrix} .59224259 \\ .001493351 \end{pmatrix} \\
&= \alpha \begin{pmatrix} .00059224259 \\ .0014918576 \end{pmatrix} \\
&\approx \langle .284, .716 \rangle
\end{aligned}
\tag{2}
$$

b. Including the normalization step, there are 7 additions, 16 multiplications, and 2 divisions. The enumeration algorithm has two extra multiplications.

c. To compute $\mathbf{P}(X_1|X_n = true)$ using enumeration, we have to evaluate two complete binary trees (one for each value of X_1), each of depth $n-2$, so the total work is $O(2^n)$. Using variable elimination, the factors never grow beyond two variables. For example, the first step is

$$
\begin{aligned}
&\mathbf{P}(X_1|X_n = true) \\
&\qquad = \alpha \mathbf{P}(X_1) \ldots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) \sum_{x_{n-1}} P(x_{n-1}|x_{n-2}) P(X_n = true|x_{n-1}) \\
&\qquad = \alpha \mathbf{P}(X_1) \ldots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) \sum_{x_{n-1}} f_{X_{n-1}}(x_{n-1}, x_{n-2}) f_{X_n}(x_{n-1}) \\
&\qquad = \alpha \mathbf{P}(X_1) \ldots \sum_{x_{n-2}} P(x_{n-2}|x_{n-3}) f_{\overline{X_{n-1}}X_n}(x_{n-2})
\end{aligned}
\tag{3}
$$

The last line is isomorphic to the problem with $n-1$ variables instead of n; the work done on the first step is a constant independent of n, hence (by induction on n, if you want to be formal) the total work is $O(n)$.