## Mapping Reducibility (Section 5.3)

As mentioned previously, there exist different types of reductions.

A type of reducibility that is often used is what Sisper calls *mapping reducibility*. This is usually called *many-one reducibility*.

**Definition 5.20:**
A language $A$ is *mapping reducible* to language $B$, written as $A \leq_m B$, if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every $w$,

$$w \in A \Longleftrightarrow f(w) \in B.$$

The function $f$ is called the *reduction* from $A$ to $B$.

What's a computable function? [Definition 5.17.]



**Properties of mapping reductions**

If $A \leq_m B$ and $B$ is decidable, then $A$ is decidable [Theorem 5.22].

Proof: Let $M$ be a decider for $B$ and let $f$ be a mapping reduction from $A$ to $B$. We describe a decider $N$ for $A$ as follows:
$N =$ "On input $w$

1. Compute $f(w)$.
2. Run $M$ on input $f(w)$. If $M$ accepts, *accept*; otherwise *reject*."

As an immediate corollary, we get:
If $A \leq_m B$ and $A$ is undecidable, then $B$ is undecidable [Corollary 5.23].



**Examples**

Some of the undecidability results from earlier in this chapter give examples of mapping reducibilities.

For example, $A_{\mathsf{TM}} \leq_m$ *HALT*$_{\mathsf{TM}}$ [Example 5.24]. Give the reduction!

Also, *HALT*$_{\mathsf{TM}} \leq_m$ *ALWAYSHALT*$_{\mathsf{TM}}$.
In this case, the reduction $f$ maps $\langle M, w\rangle$ to $\langle M_w\rangle$, where $M_w$ is constructed as in the undecidability proof of *ALWAYSHALT*$_{\mathsf{TM}}$ from the notes.



Not all undecidability proofs from this chapter give examples of mapping reductions.

For example, look at the proof that $E_{\mathsf{LBA}}$ is undecidable.

The proof uses a reduction from $A_{\mathsf{TM}}$ to $E_{\mathsf{LBA}}$, but it is not a mapping reduction.

Note that $\langle M, w\rangle \in A_{\mathsf{TM}}$ iff $\langle B\rangle \notin E_{\mathsf{LBA}}$, where $B$ is defined as in the proof of Theorem 5.10.

So, this is a mapping reduction from $A_{\mathsf{TM}}$ to $\overline{E_{\mathsf{LBA}}}$ (equivalently, a mapping reduction from $\overline{A_{\mathsf{TM}}}$ to $E_{\mathsf{LBA}}$).

Show that $A_{\mathsf{TM}}$ is *not* mapping reducible to $E_{\mathsf{LBA}}$.

## Questions

If $A$ and $B$ are undecidable, does that imply that $A \leq_m B$?

If $A$ and $B$ are decidable, does that imply that $A \leq_m B$?

If $A \leq_m B$ and $B$ is regular, does that imply that $A$ is regular?

If $A \leq_m B$ and $A$ is decidable, does that imply that $B$ is decidable?



## Reductions and Turing-recognizable sets

If $A \leq_m B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable [Theorem 5.28].

Proof: Simple modification of the proof of Theorem 5.22.

As an immediate corollary, we get:
If $A \leq_m B$ and $A$ is not Turing-recognizable, then $B$ is not Turing-recognizable. [Corollary 5.29].

So, for example, if $\overline{A_{\mathsf{TM}}} \leq_m B$, then $B$ is not Turing-recognizable, and if $A_{\mathsf{TM}} \leq_m B$, then $B$ is not co-Turing-recognizable.

See Theorem 5.30 ($EQ_{\mathsf{TM}}$ is neither Turing-recognizable nor co-Turing-recognizable) for an application.