

Convolution

Let $f(x)$ and $g(x)$ be continuous real-valued functions for $x \in \mathbb{R}$ and assume that f or g is zero outside some bounded set (this assumption can be relaxed a bit). Define the *convolution*

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)g(y) dy \quad (1)$$

Since f or g is zero outside a bounded set, this integral is only over a bounded interval. One simple yet useful observation is

$$f * g = g * f. \quad (2)$$

To prove this make the change of variable $t = x - y$ in the integral (1).

One small observation. If $f * g(x) = \int_a^b f(y)g(x-y) dy$, then only the values of g on the interval $[x-b, x-a]$ are used. Thus if $x \in [a, b]$, then the convolution only involves the values of g on $[a-b, b-a]$. By a simple rescaling, we will often use $b-a=1$.

SMOOTHNESS OF $f * g$.

Theorem 1 *If $f \in C^1(\mathbb{R})$ then $f * g \in C^1(\mathbb{R})$. Better yet, if $f \in C^k(\mathbb{R})$ and $g \in C^\ell(\mathbb{R})$, then $f * g \in C^{k+\ell}(\mathbb{R})$.*

PROOF This is clearer if we write $h(x) = f * g(x)$. Then

$$\frac{h(x) - h(x_0)}{x - x_0} = \int_{-\infty}^{\infty} \frac{f(x-y) - f(x_0-y)}{x - x_0} g(y) dx. \quad (3)$$

We will be done if we can show that $f(x-y) - f(x_0-y)/(x-x_0)$ converges uniformly to $f'(x_0-y)$. To do this we use the integral form of the mean value theorem:

$$\begin{aligned} f(x-y) - f(x_0-y) &= \int_0^1 \frac{df(x_0-y+t(x-x_0))}{dt} dt \\ &= \left[\int_0^1 f'(x_0-y+t(x-x_0)) dt \right] (x-x_0). \end{aligned}$$

Then

$$\frac{f(x-y) - f(x_0-y)}{x - x_0} - f'(x_0-y) = \int_0^1 [f'(x_0-y+t(x-x_0)) - f'(x_0-y)] dt \quad (4)$$

Since f' is assumed continuous and is zero outside of a bounded set, it is uniformly continuous. Thus, given any $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x-x_0| < \delta$ then

$$|f'(z+t(x-x_0)) - f'(z)| < \varepsilon$$

for all values of z . In our case $z = x_0 - y$. Thus the left side of (4) tends to zero uniformly for all choices of x_0 and y . Thus, $h \in C^1(\mathbb{R})$.

Repeating this we conclude that if $f \in C^k$ then $h \in C^k$. Because of (2) $f^{(k)} * g = g * f^{(k)}$, so we can repeat this reasoning to show that $g * f^{(k)} \in C^\ell$. Thus $f * g \in C^{k+\ell}$. Note that although g might not be zero outside a bounded set, because f is zero outside a bounded set, the integration in $g * f^{(k)}$ is only over a bounded set – in which the derivatives of g are uniformly continuous.

APPROXIMATE IDENTITIES

Let $\varphi_n(t)$ be a sequence of smooth real-valued functions with the properties

$$(a) \varphi_n(t) \geq 0, \quad (b) \varphi_n(t) = 0 \text{ for } |t| \geq 1/n, \quad (c) \int_{-\infty}^{\infty} \varphi_n(t) dt = 1. \quad (5)$$

Note: because of (b), this integral is only over $-1/n \leq t \leq 1/n$.

Assume $f(x)$ is uniformly continuous for all $x \in \mathbb{R}$ and zero outside a bounded set. Define

$$f_n(x) := f * \varphi_n(x) = \int_{-\infty}^{\infty} f(x-t) \varphi_n(t) dt. \quad (6)$$

Theorem 2 $f_n(x) \in C^\infty$ converges uniformly to $f(x)$ for all $x \in \mathbb{R}$. Thus, on a compact set any continuous function can be approximated arbitrarily closely in the uniform norm by a smooth function.

PROOF The smoothness of the approximations f_n is an immediate consequence of Theorem 1.

Since $f(x) = f(x) \left(\int_{-\infty}^{\infty} \varphi_n(t) dt \right) = \int_{-\infty}^{\infty} f(x) \varphi_n(t) dt$,

$$f_n(x) - f(x) = \int_{|t| \leq 1/n} [f(x-t) - f(x)] \varphi_n(t) dt. \quad (7)$$

Since f is uniformly continuous, given any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|t| < \delta$ then $|f(x-t) - f(x)| < \varepsilon$ for all x . Thus, if $1/n < \delta$, then by (5c)

$$|f_n(x) - f(x)| < \varepsilon \int_{|t| \leq 1/n} \varphi_n(t) dt = \varepsilon. \quad (8)$$

Since the right side is independent of x this shows that in the uniform norm $\|f_n - f\|_\infty < \varepsilon$.

Since the operators $T_n(f) := f * \varphi_n \rightarrow f$, so in this sense T_n converges to the identity operator I , we sometime call the T_n (or the φ_n) *approximate identities*.

EXAMPLE Assume $f(x)$ is continuous on the interval $[a, b]$. Then $\int_a^b f(x) \sin \lambda x dx \rightarrow 0$.

SOLUTION If $f \in \mathbb{C}^1([a, b])$ this is easy to show by an integration by parts, using $|f'(x)| \leq M$ for some constant M . If f is only continuous, use Theorem 2 to find a smooth $g(x)$ with $\|f - g\|_\infty < \varepsilon$ on $[a, b]$. Then

$$\left| \int_a^b f(x) \sin \lambda x dx \right| \leq \left| \int_a^b [f(x) - g(x)] \sin \lambda x dx \right| + \left| \int_a^b g(x) \sin \lambda x dx \right|.$$

Since $\|f - g\|_\infty < \varepsilon$, the first term on the right is small. Because g is smooth, the second term goes to zero as $\lambda \rightarrow \infty$.

In many applications the restriction (5b) is too restrictive.

Theorem 3 Theorem 1 is valid if you replace (5b) with:

$$\text{For every } \delta > 0, \quad \lim_{n \rightarrow \infty} \int_{|t| > \delta} \varphi_n(t) dt = 0. \quad (5b')$$

PROOF Replace (7) by

$$\begin{aligned} f_n(x) - f(x) &= \int_{|t| \leq \delta} [f(x-t) - f(x)] \varphi_n(t) dt + \int_{|t| > \delta} [f(x-t) - f(x)] \varphi_n(t) dt \\ &= J_1 + J_2. \end{aligned}$$

Given $\varepsilon > 0$, pick δ as was done above. Then

$$|J_1| \leq \varepsilon \int_{|t| \leq \delta} \varphi_n(t) dt \leq \varepsilon.$$

To estimate J_2 , say $|f(x)| \leq M$ for all x . Then by our assumption on the φ_n ,

$$|J_2| \leq 2M \int_{|t| > \delta} \varphi_n(t) dt \rightarrow 0.$$

This proves that $\|f_n - f\|_\infty \rightarrow 0$.

Weierstrass used essentially this argument with

$$u(x, t) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y) dy,$$

thinking of $t = 1/n \rightarrow 0$. Then $\lim_{t \rightarrow 0} u(x, t) \rightarrow f(x)$. This classical formula was well-known since $u(x, t)$ is the solution of the *heat equation* $u_t = u_{xx}$ for $x \in \mathbb{R}$, $t > 0$ with initial temperature $u(x, 0) = f(x)$.

We'll use this to prove

Theorem 4 (WEIERSTRASS APPROXIMATION THEOREM) *Let f be a continuous function. Then on any compact set it can be approximated uniformly by a polynomial.*

PROOF If in (6) the functions φ_n are polynomials, then the approximations $f * \varphi_n$ are also polynomials. However, polynomials will never satisfy the restrictions required of the φ_n . Our $\varphi_n(x)$, defined below, will be polynomials for $|x| \leq 1$ and zero for $|x| \geq 1$. As we observed in the paragraph after (2), if $f(x) = 0$ for $|x| \geq 1/2$, then the convolution $f_n(x) = f * \varphi_n$ will only use the values of $\varphi_n(x)$ for $|x| \leq 1$ so the approximations $f_n(x)$ will be polynomials. Note that by scaling the x -axis, we may assume that $f(x) = 0$ for $|x| \geq 1/2$.

Define the functions $\varphi(x)$ as

$$\varphi_n(x) = \begin{cases} \frac{(1-x^2)^n}{c_n} & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases},$$

where

$$c_n = \int_{-1}^1 (1-x^2)^n dx.$$

Thus ϕ_n satisfies the conditions (5a, c). We will verify the modified property of Theorem 3 by showing that for any $\delta > 0$ in the region $|x| > \delta$ the functions $g_n(x)$ converge uniformly to zero. By computing the second derivatives it is easy to verify that

$$(1 - x^2)^n \geq 1 - nx^2 \quad \text{for } |x| \leq 1.$$

and that $1 - nx^2 \geq 1 - (n/4n) = 3/4$ for $|x| \leq 1/(2\sqrt{n})$. Thus

$$c_n \geq \int_{|x| \leq 1/(2\sqrt{n})} (1 - nx^2) dx > \frac{3}{4\sqrt{n}}.$$

Consequently, if $1 \geq |x| > \delta$, then

$$0 \leq g_n(x) \leq \frac{4\sqrt{n}}{3}(1 - x^2)^n \leq \frac{4\sqrt{n}}{3}(1 - \delta^2)^n.$$

This has the form $a\sqrt{n}b^n \rightarrow 0$ since $0 < b < 1$.

Thus we have verified the assumptions of Theorem 3, so our approximations $f_n(x)$, which are polynomials for $|x| \leq 1/2$, converge uniformly to $f(x)$.

EXAMPLE In the function space $L_2([0, 1])$ the norm comes from an inner product

$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)} dx \quad \text{so} \quad \|f\|_2 = \sqrt{\langle f, f \rangle}.$$

We say f and g are *orthogonal* if $\langle f, g \rangle = 0$. Assume that the continuous function f is orthogonal to $1, x, x^2, \dots$, so

$$\int_0^1 f(x)x^k dx = 0, \quad k = 0, 1, 2, \dots$$

We claim the only possibility is that $f(x) \equiv 0$ for all $x \in [0, 1]$. In brief, this is because f is orthogonal to all polynomials p , but by the Weierstrass approximation theorem, polynomials are dense in $L_2([0, 1])$ so f is essentially orthogonal to itself. Thus $f \equiv 0$. In greater detail, find a polynomial p cfso that $\|f - p\|_\infty < \varepsilon$ in $[0, 1]$. Then $\langle f, p \rangle = 0$ so by the Schwarz inequality

$$\|f\|_2^2 = \langle f, f - p \rangle + \langle f, p \rangle \leq \|f\|_2 \|f - p\|_2 \leq \varepsilon \|f\|_2.$$

If $f(x) \not\equiv 0$, then $\|f\|_2 \leq \varepsilon$ for any $\varepsilon > 0$. This gives a contradiction.