## Review of Select Quiz Questions

## Linear Algebra

1. Consider the matrix
$$A = \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$$
What is the transpose of $A$ (denote the transpose of a matrix $A$ by $A'$)?
The transpose of A is
$$A' = \begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix}.$$

2. Obtain explicitly the matrix $C = A'A$
$$C = A'A = \begin{pmatrix} 1 & b \\ b & b^2 + c^2 \end{pmatrix}.$$

3. Suppose for this problem that $b = 0$ and $c = 1$. What is the rank of the matrix $A$? Find the inverse of this matrix.
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
The rank is of A is 2.
$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. Consider now the matrix $A$ but now with $c = 0$ and $b = 0$. What is the rank of this matrix? Is it invertible? What is the relationship between the rank of a square matrix and the existence of its inverse?
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
The rank of A is 1 and the matrix is not invertible. A matrix that is not full rank is not invertible.

5. Consider the following equations and expressions
$$\beta_2 = 0$$
$$\beta_1 + \beta_3 = 3$$
$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$
Rewrite these equations in matrix form (i.e. of the form $A\beta = b$ for some matrix $A$ that you need to find and the matrix $\beta$ defined above and for some $2 \times 1$ vector $b$ that you need to find as well).
The equations can be written as

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

6. Consider the two $m \times 1$ vectors given by

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

What is $\mathbf{a}'\mathbf{b}$? What is $\mathbf{b}'\mathbf{a}$ ?
$\mathbf{a}'\mathbf{b} = \mathbf{b}'\mathbf{a} = \sum_{i=1}^{m} a_i b_i$.

7. Consider the $r \times m$ matrix $\mathbf{A}$ written in terms of its $m$ columns

$$\mathbf{A}_{r\times m} = \begin{pmatrix} \mathbf{a}_1 & . & . & \mathbf{a}_m \end{pmatrix}$$

where $\mathbf{a}_j$ is a $r \times 1$ vector and denotes the $j^{th}$ column of $\mathbf{A}$. Consider the $m \times d$ matrix $\mathbf{B}$ expressed in terms of its rows

$$\mathbf{B}_{m\times d} = \begin{pmatrix} \mathbf{b}_1 \\ . \\ . \\ \mathbf{b}_m \end{pmatrix}$$

where $\mathbf{b}_j$ is a $1 \times d$ vector that denotes the $j^{th}$ row of $\mathbf{B}$.

(a) What is the dimension of the matrix $\mathbf{AB}$
The dimension of $\mathbf{AB}$ is $r \times d$.

(b) Obtain an expression for $\mathbf{AB}$ in terms of the vectors above.
$\mathbf{AB} = \sum_{i=1}^{m} \mathbf{a}_i \mathbf{b}_i$.

## Probability and Statistics

1. Consider a random variable $X$ that takes on the value 0 with probability $p$ and the value 1 with probability $1 - p$

(a) Compute the expectation of $X$, denoted by $\mathbb{E}(X)$

$$\mathbb{E}(X) = 1 - p$$

(b) Compute the variance of $X$, denoted by $Var(X)$

$$Var(X) = \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2 = (1-p) - (1-p)^2 = p(1-p)$$

(c) Suppose you define a new random variable

$$Z = 1 + 2X$$

Repeat parts (a) and (b) for the random variable $Z$.

$$\begin{aligned} \mathbb{E}(Z) &= 1 + 2\mathbb{E}(X) = 1 + 2(1-p) \\ Var(X) &= 4Var(X) = 4p(1-p) \end{aligned}$$

(d) Compute the covariance between $X$ and $Z$.

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\begin{aligned}
Cov(X,Z) &= \mathbb{E}(XZ) - \mathbb{E}(X)\mathbb{E}(Z) \\
&= \mathbb{E}\left(X + 2X^2\right) - (1-p)(1+2(1-p)) \\
&= \mathbb{E}(X) + 2\mathbb{E}\left(X^2\right) - (1-p) - 2(1-p)^2 \\
&= (1-p) + 2(1-p) - (1-p) - 2(1-p)^2 \\
&= 2p(1-p)
\end{aligned}
$$

2. Let the random variable $X$ denote the outcome of the toss of a six sided die. That is, $X$ can take on the values $\{1,2,3,4,5,6\}$. Suppose that the die is fair so that each number in $\{1,2,3,4,5,6\}$ has an equal probability of occurring.

(a) Consider the events $A = \{1,3,5\}$, $B = \{1,4\}$, $C = \{2,4,6\}$ and $D = \{3,5\}$ and compute their probabilities.

$$\mathbb{P}(A) = \mathbb{P}(C) = \frac{1}{2}$$
$$\mathbb{P}(B) = \mathbb{P}(D) = \frac{1}{3}$$

(b) Compute the probabilities $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A \cup B)$.

$$\mathbb{P}(A \cap B) = \frac{1}{6}$$
$$\mathbb{P}(A \cup B) = \frac{2}{3}$$

(c) Are the events $A$ and $C$ independent?

$$\mathbb{P}(A \cap C) = 0 \neq \mathbb{P}(A)\mathbb{P}(C) = \frac{1}{4}$$

Thus $A$ and $C$ are not independent.

(d) Compute the conditional probability $\mathbb{P}(D|A)$

$$\mathbb{P}(D|A) = \frac{\mathbb{P}(D \cap A)}{\mathbb{P}(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

3. Consider two random variables $X$ and $Y$ each of which can take on two values $\{0,1\}$. The joint distribution is given by

$$\mathbb{P}(X=0, Y=0) = p$$
$$\mathbb{P}(X=0, Y=1) = q$$
$$\mathbb{P}(X=1, Y=0) = r$$

(a) Express $\mathbb{P}(X=1, Y=1)$ as a function of $(p,q,r)$

$$\mathbb{P}(X=1, Y=1) = 1 - p - q - r.$$

(b) Express the covariance between $X$ and $Y$ as a function of $(p, q, r)$. Is the covariance between $X$ and $Y$ the same as the covariance between $Y$ and $X$?

$$cov(X,Y) = E[XY] - E[X]E[Y] = (1-p-q-r) - (1-p-q)(1-p-r) = p - p^2 - pr - pq - qr,$$

where

$$\begin{aligned} E[X] &= 0 \cdot (p+q) + 1 \cdot (1-p-q) = 1-p-q \\ E[Y] &= 0 \cdot (p+r) + 1 \cdot (1-p-r) = 1-p-r \\ E[XY] &= 0 \cdot (p+q+r) + 1 \cdot (1-p-q-r) = 1-p-q-r. \end{aligned}$$

(c) Express the correlation between $X$ and $Y$ as a function of $(p, q, r)$

$$\rho_{XY} = \frac{cov(X,Y)}{\sigma_X \sigma_Y} = \frac{p - p^2 - pr - pq - qr}{\sqrt{(p+q)(p+r)(1-p-r)(1-p-q)}},$$

where

$$\begin{aligned} E[X^2] = 0^2 \cdot (p+q) + 1^2 \cdot (1-p-q) = 1-p-q \\ E[Y^2] = 0^2 \cdot (p+r) + 1^2 \cdot (1-p-r) = 1-p-r \\ \sigma_X^2 = E[X^2] - (E[X])^2 = (1-p-q) - (1-p-q)^2 = (p+q)(1-p-q) \\ \sigma_Y^2 = E[Y^2] - (E[Y])^2 = (1-p-r) - (1-p-r)^2 = (p+r)(1-p-r). \end{aligned}$$

(d) Compute the expectation of $Y$, $\mathbb{E}(Y)$ this is also sometimes called the unconditional expectation of $Y$.
Expectation computed in part (c)

(e) Compute the conditional expectation of $Y$ given that $X$ is equal to 1 $\mathbb{E}(Y|X=1)$. That is, what would you expect the average value of $Y$ to be given that $X$ is equal to 1.

The probability distribution for $Y$ when $X$ is equal to 1 is

$$P(Y=1|X=1) = \frac{1-p-q-r}{1-p-q}$$

$$P(Y=0|X=1) = 1 - P(Y=1|X=1)$$

so

$$E[Y|X=1] = 1 \cdot \frac{(1-p-q-r)}{1-p-q} + 0\left(1 - \frac{(1-p-q-r)}{1-p-q}\right) = \frac{1-p-q-r}{1-p-q}$$

4. Suppose that $\{Y_i\}_{i=1}^n$ is an i.i.d. sample from a distribution with mean $\mu$ and variance $\sigma^2$

(a) Let $\bar{Y}$ denote the sample mean. Characterize the behaviour of this quantity when the sample size is large and state the theorem you use.
By the Law of Large numbers (Stock and Watson, page 50), $\bar{Y}$ converges in probability to $\mu$.

(b) Consider the object $\sqrt{n}(\bar{Y} - \mu)/\sigma$. Characterize the limiting distribution of this object and state the theorem you use to characterize it.
By the Central Limit Theorem (Stock and Watson, page 55), $\sqrt{n}(\bar{Y} - \mu)/\sigma$ converges distribution to a standard normal.