Problems: Signal Space

Prof. Sundeep Rangan

1. *Vector spaces and bases in* $\mathbb{F}^N$. For each set V below, identify if it is a vector space or not. If it is a vector space, find a basis. If not, state the property that fails to occur.

   (a) $V =$ the set of (x_1, x_2, x_3) such that $2x_1 + x_2 = 0$.

   (b) $V =$ the set of $\mathbf{x} \in \mathbb{R}^3$ with $\|\mathbf{x}\| \leq 1$.

2. *Vector spaces of functions.* For each set V below, state if V is a subspace or not. Explain.

   (a) Let $T > 0$ be some sampling period. $V =$ the set of $f(t)$ such that $f(nT) = 0$ for all n.

   (b) Let $f_{max} > 0$. V is the set of $s(t)$ that are bandlimited so that $S(f) = 0$ for $|f| > f_{max}$.

   (c) $V =$ set of functions on $[0, \infty)$ of the form, $f(t) = Ae^{-(t-\tau)}$ for some A and τ.

   (d) $V =$ set of functions on $[0, \infty)$ of the form, $f(t) = Ae^{-Bt}$ for some A and B.

3. *Signal set and signal space.* Let N and K be constants and consider the signal set $\mathcal{S}$ consisting of signals $s[n]$ such that $s[n] = 1$ in *exactly* K times $n \in [0, 1, \ldots, N-1]$. For all other n, $s[n] = 0$.

   (a) Find M, the number of signals in $\mathcal{S}$.

   (b) Find the number of degrees of freedom.

   (c) Find the rate of signal set.

   This type of signal set can encode information by the position of the non-zero elements.

4. *Signal set and signal space.* Consider the following four functions:

   $$s(t) = e^{-At+B}, t \geq 0,$$

   where $A = 1$ or 2 and $B = 0$ or 1.

   (a) Find a basis for a signal space containing the signal set. Use a basis with a minimum number of signals.

   (b) Find the coordinates of each signals in the basis.

5. *Bandlimited channels.* Suppose that a communication system is allocated a channel 2.29 to 2.31 GHz and has 10% overhead.

   (a) What are the (complex) degrees of freedom per second?

   (b) What is the spectral efficiency required for 40 Mbps?

   (c) What is the rate if the system uses 16-QAM on every degree of freedom?

(d) If the signal is received at -100 dBm, what is the average energy per degree of freedom.

6. *Orthonormal bases* Suppose that a signal space has a basis $s_1(t)$, $s_2(t)$ with

$$\|s_1\|^2 = \|s_2\|^2 = 1, \langle s_1, s_2 \rangle = \rho.$$

(a) Using Gram-Schmidt, find an orthonormal basis u_1, u_2 for the signal space.

(b) Write s_1 and s_2 in terms of u_1 and u_2.

(c) Suppose a signal is transmitted as,

$$s(t) = a_1 s_1(t) + a_2 s_2(t).$$

Find the coordinates of $s(t)$ in the $u_1(t), u_2(t)$ basis. That is, find b_1, b_2 in terms of (a_1, a_2) such that $s(t) = b_1 u_1(t) + b_2 u_2(t)$

(d) Find b_1, b_2 for the four constellation points $a_1 = a_2 = \pm 1$ and $\rho = 0.2$.