Unit 5: Random Processes and Noise

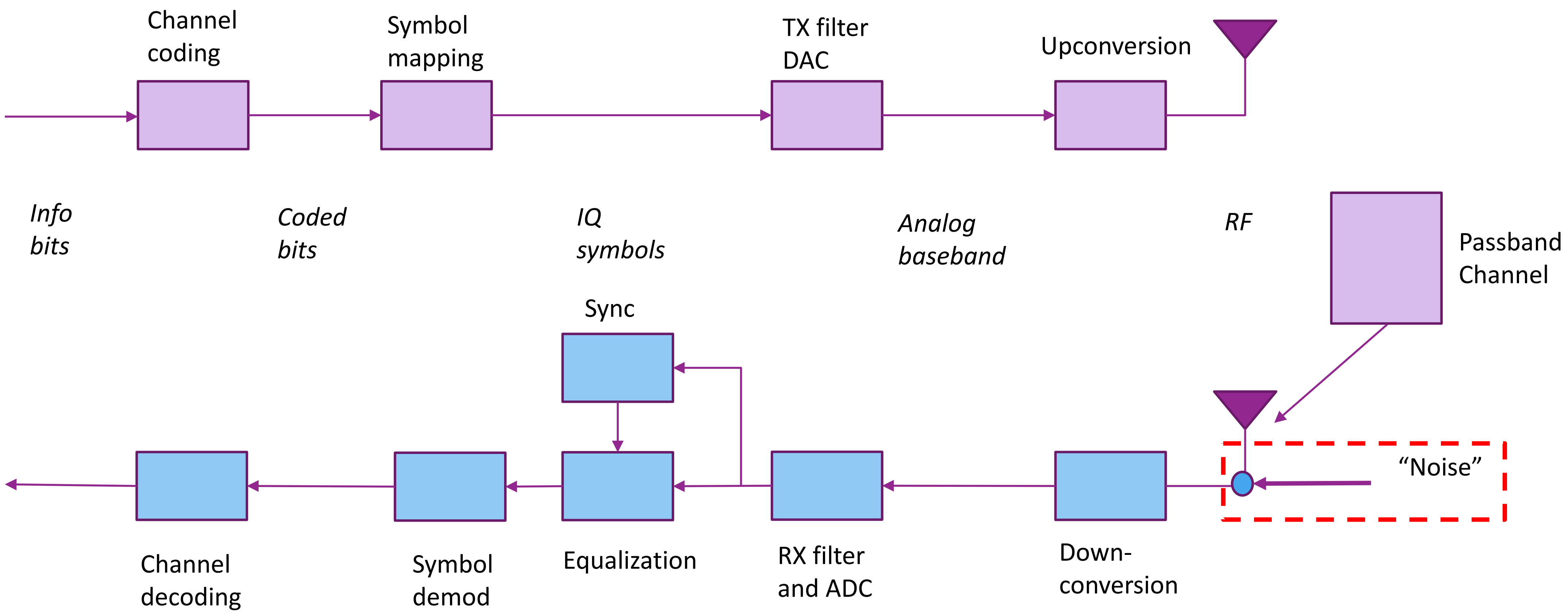
EL-GY 6013: DIGITAL COMMUNICATIONS

PROF. SUNDEEP RANGAN

Learning Objectives

- ❑Compute expectations and probabilities for random variables
- ❑Compute expectations and probabilities for real and complex random variables
- ❑Generate samples of random variables in MATLAB
  - Continuous and discrete-time
- ❑Compute probabilities of Gaussians and linear combinations of Gaussian
  - Write answers in terms of the Q-function
- ❑Compute probabilities of multiple variables with conditional distributions
  - Use total expectation, total probability, conditioning rule, …
- ❑Simulate random systems with multiple random variables
- ❑Simulate and describe random processes
- ❑Compute auto-correlation and PSD of a random process

This Unit



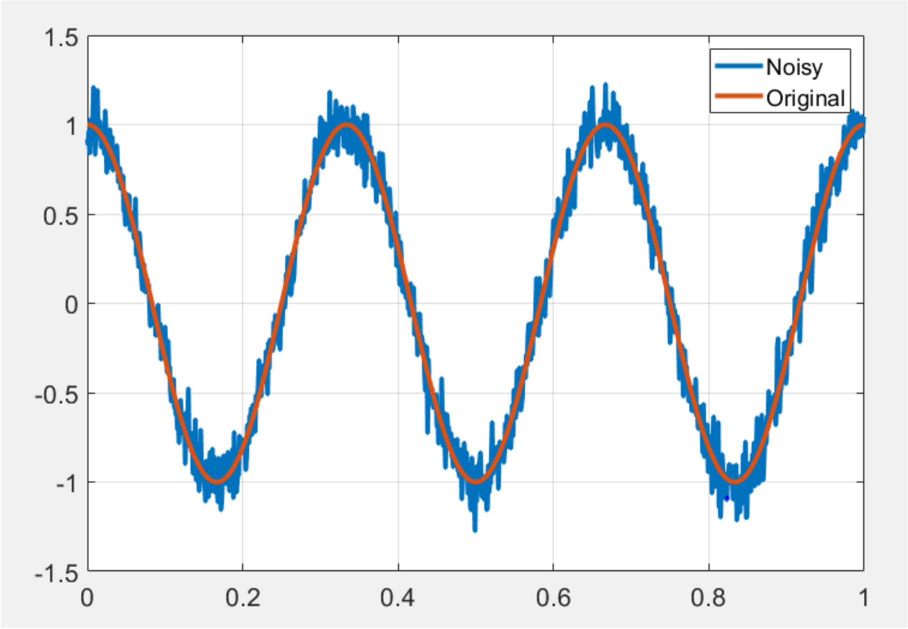
Channel coding
Symbol mapping
TX filter DAC
Upconversion
Info bits
Coded bits
IQ symbols
Analog baseband
RF
Passband Channel
Sync
"Noise"
Channel decoding
Symbol demod
Equalization
RX filter and ADC
Down-conversion

Outline

- What is noise?
- Random variables
- Simulating random variables
- Gaussian and complex Gaussian random variables
- Random vectors
- Random processes

What is Noise?

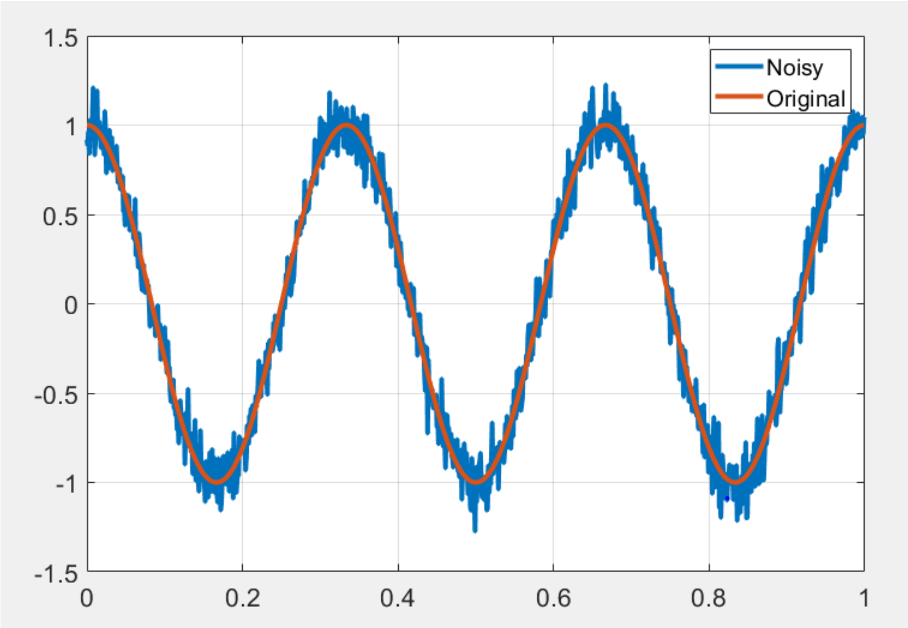
- Noise: Any unwanted component of the signal
- Key challenge in communication:
  - Estimate the transmitted signal in the presence of noise
- Two main sources of noise:
  - Thermal noise: Physical noise in the receiver
  - Interference: Signals from other sources



Statistical Models for Noise

- In communications, we model noise as a random process
  - Captures "uncertainty" in the value

- This lecture:
  - Review basics of probability and random processes
  - Describe mathematical model for noise



Outline

- What is noise?
- Random variables
- Simulating random variables
- Gaussian and complex Gaussian random variables
- Random vectors
- Random processes

Random Variables: Informal Definition

- A random variable is

  Any quantity X that can have a value that varies and/or is unknown

- Can be real, discrete, complex, …

- Used in communications to model:
  - Unknown transmitted bit, $X = 0$ or 1
  - Unknown transmitted symbol
  - Unknown channel gain, channel delay
  - Unknown “noise”

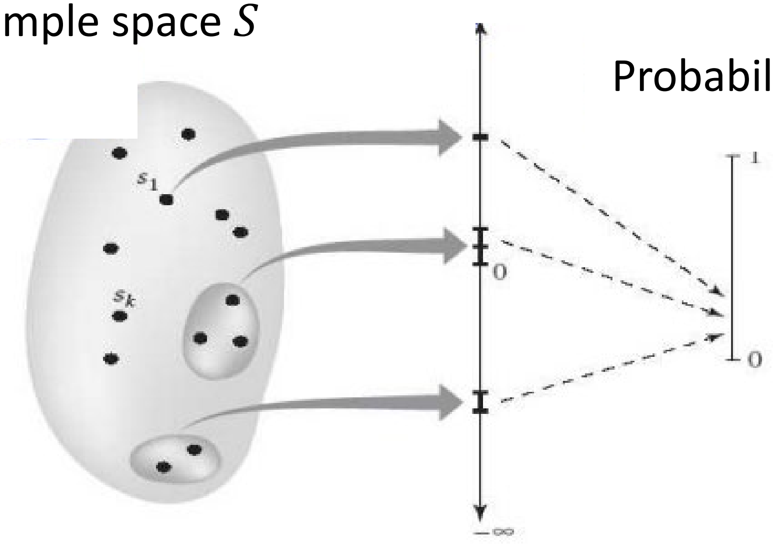
Random Variables: Formal Definition

- A probability space $(S, \mathcal{A}, P)$:
  - S = sample space. Represents set of outcomes
  - $\mathcal{A}$ = set of events (each event is a subset of S)
  - P = probability measure: Measures the probability of each event
- A random variable: A mapping $X: S \rightarrow \mathbb{R}$
  - Assigns each outcome s some value $X(s)$
- Generally, we omit dependence on s
  - Write simply X

Values $X(s) \in \mathbb{R}$

Sample space S

Probability



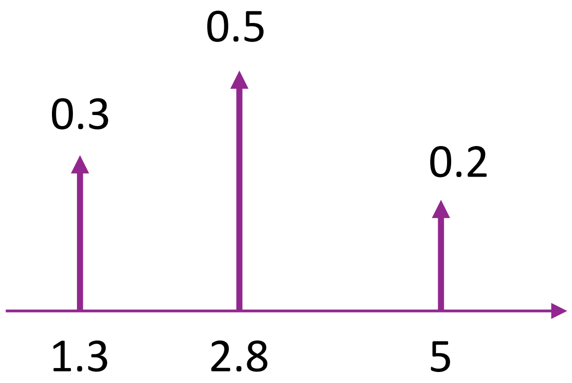
Discrete Random Variables

- Variable X takes on some set discrete values: $X \in \{x_1, x_2, \ldots, x_M\}$
  - Could also be a countable variable
- Examples in communications:
  - Value of a transmitted bit ($X \in \{0,1\}$)
  - Number of data packets that arrive for a user
  - Number of distinct paths in a communication channel
- Random variables have a probability mass function (PMF):
  - $P_X(x_i) = P(X = x_i)$

Visualizing the PMF

- Represent PMF via a stem plot or table

x_i	1.3	2.8	5
$P(X = x_i)$	0.3	0.5	0.2



Expectations (Discrete r.v.s)

- In Papoulis-Pillai, covered in Chapter 5
- Mean: $\mu = E(X) = \sum_{x_i} x_i P_X(x_i)$
- Second moment: $E\left(X^2\right) = \sum_{x_i} x_i^2 P_X(x_i)$
- Variance: $var(X) = E\left(X^2\right) - E(X)^2 = E\left((X-\mu)^2\right) = \sum (x_i - \mu)^2 P_X(x_i)$
- Expectation of a function: $E\left(g(X)\right) = \sum g(x_i) P_X(x_i)$
- Probability of a set: $P(X \in A) = \sum_{x_i \in A} P(X = x_i)$

Example Problem

X	0	0.8	2.1	3.7
$P(X = x)$	0.5	0.3	0.1	0.1

- What is $E(X)$?
- $P(X < 1.5)$?
- $P(X \in [0.5, 2.5])$?
- $E|X - 1.5|$

Continuous Random Variables

- X real-valued variable
- Examples in communication:
  - The channel gain, the path loss, delay, angle of arrival, …
  - Many physical quantities
  - Noise value
- Probability density function (PDF):

$$p_X(x) = \lim_{\delta \to 0} \frac{1}{\delta} P(X \in [x, x + \delta))$$

  - Represents probability per unit area
- X is called a continuous random variable when the limit exists
- For continuous RVs, probability of an individual point is zero: $P(X = x_0) = 0$

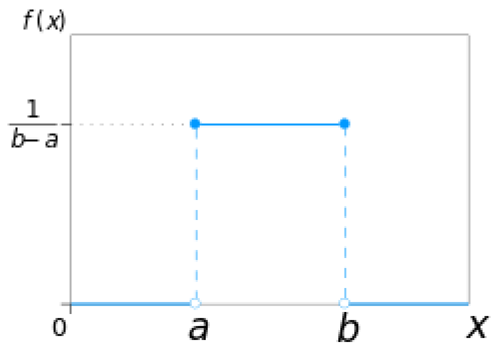
Expectation (Continuous r.v.s)

- In Pillai, also covered in Chapter 5
- Mean: $\mu = E(X) = \int x p_X(x) dx$
- Variance: $var(X) = E\left(X^2\right) - E(X)^2 = E\left((X-\mu)^2\right) = \sum (X-\mu)^2 p_X(x) dx$
- Expectation: $E\left(g(X)\right) = \int g(x) p_X(x) dx$
- Cumulative Distribution Function: $F_X(x) = P(X \le x) = \int_{-\infty}^{x} p_X(u) du$
- Similar to discrete-random variables except sum is replaced by integral
  - Matches if we use “delta” functions

Examples

- Uniform on $[a, b]$

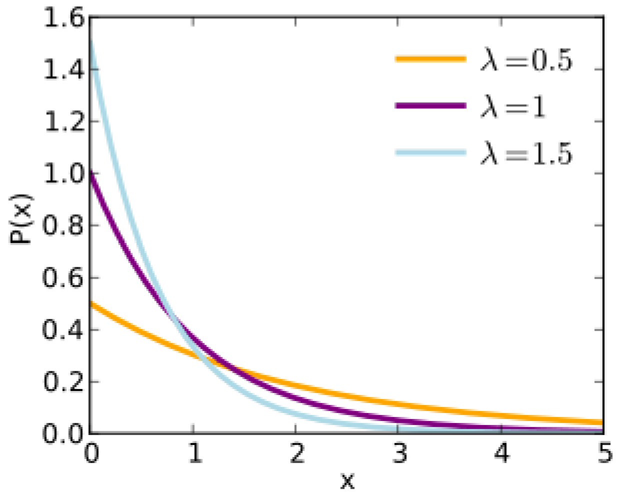
$$p_X(x) = \frac{1}{b-a} 1_{\{x \in [a,b]\}}$$



- Exponential

$$p_X(x) = \lambda e^{-\lambda x} 1_{\{x \geq 0\}}$$

  - Mean $E(X) = \lambda^{-1}$
  - Variance $\mathrm{var}(X) = \lambda^{-2}$



Example Problem

- Suppose that the delay X on some signal is exponential with $E(X) = 100$ ns.
- What is the CDF of X. Draw it
- What is $P(X < 50)$?
- Suppose the power of the signal is $Q = e^{-aX}$. What is $E(Q)$?

Outline

- What is noise?
- Random variables
- Simulating random variables
- Gaussian and complex Gaussian random variables
- Random vectors
- Random processes

Sampling Random Variables

- Often need to generate independent samples of random variables in MATLAB
- Most importantly, to simulate systems with random models
- MATLAB has many routines to generate random samples
  - Discrete random variables
  - Continuous random variables
  - Most standard distributions

Ex: Discrete Uniform Random Variables

- Generating 1000 discrete uniform random variables $X \in \{1,2,..,5\}$

```
nvals = 5;
n = 1000;
x = unidrnd(nvals, [n,1]);
```

- Display first 10 samples

```
disp(x(1:10)');
```

```
     2     5     5     5     1     1     3     3     3     4
```

Ex 2. Discrete Uniform on a Set

- Generating random variables on an arbitrary set: $X \in \{1,2,4,6,10\}$

```
vals = [1,2,4,6,10]';
ind = unidrnd(nvals, [n,1]);
x = vals(ind);

disp(x(1:10)');
```

```
     2    10     4     4     6     2     2    10     1     1
```

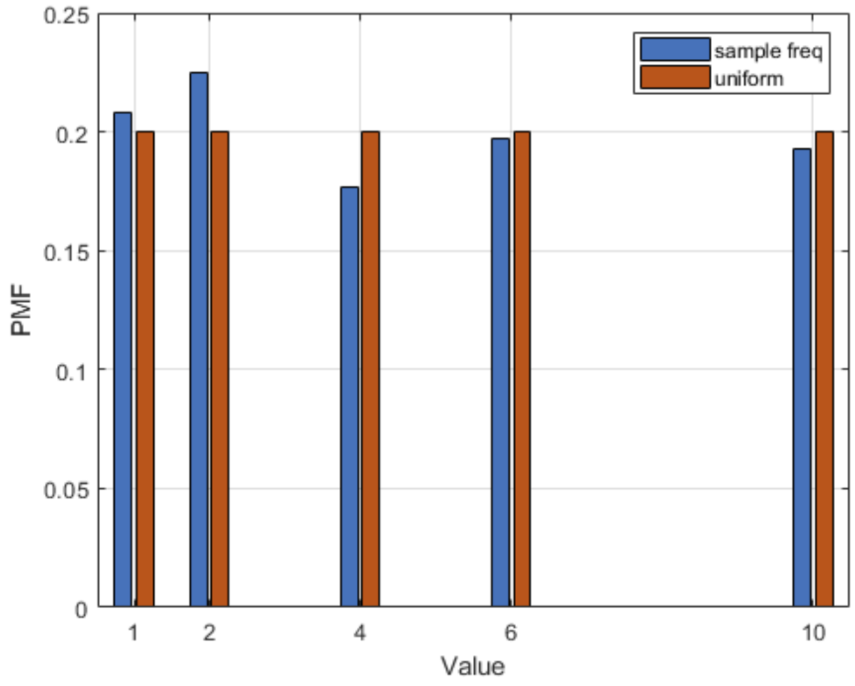
Measuring and Plotting the PMF

- Sample PMF:
  - Given samples $x_i, i = 1, \dots, N$ with values $x_i \in \{v_1, \dots, v_M\}$
  - Sample PMF: $\hat{P}(v_k) = \frac{1}{N} \#\{x_i = v_k\}$ =fraction of samples $x_i = v_k$
  - If $x_i \sim X$ is i.i.d. then $\hat{P}(v_k) \to P(X = v_k)$

```
% Get counts in each value
cnts = histcounts(categorical(x), categorical(vals));

% Estimate sample frequency
psamp = cnts' / n;
punif = ones(nvals,1)/nvals;
```

```
bar(vals, [psamp punif]);
grid();
xlabel('Value');
ylabel('PMF');
legend('sample freq', 'uniform');
```



Ex 3: Non-Uniform Discrete RV

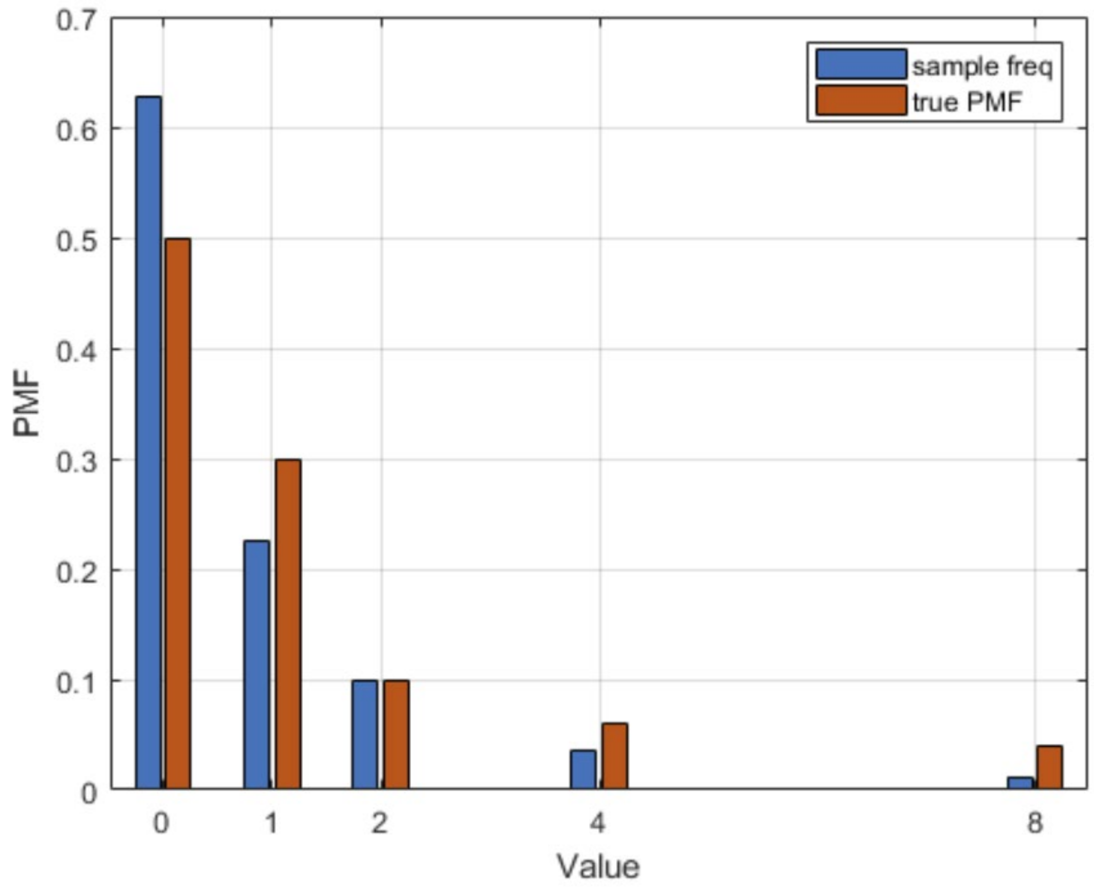
- Generating a random variable with a given PMF

```
vals  = [0,1,2,4,8]';
ptrue = [0.5,0.3,0.1,0.06,0.04]';
nvals = length(vals);

% Compute the CDF
pcdf = [0; cumsum(p)];

% Generate the random samples
[~,ind] = histc(rand(n,1),pcdf);
x = vals(ind);

disp(x(1:10)');
```

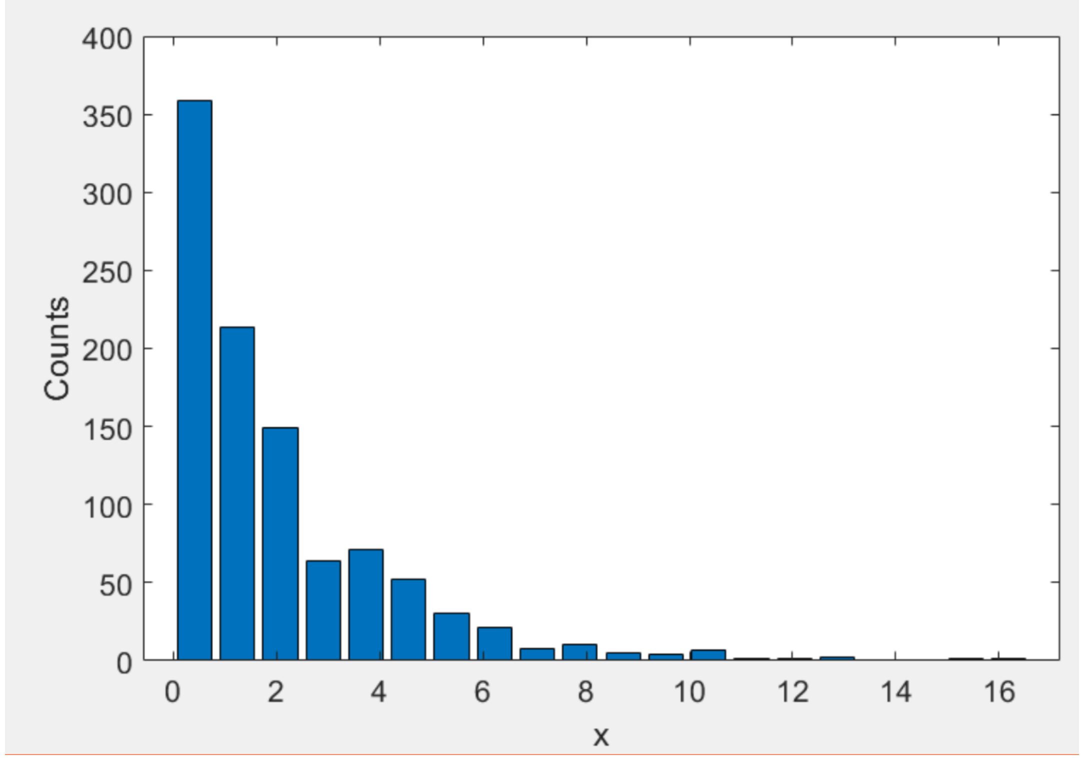


Ex 4. Exponential Random Variable

- Can generate many standard random variables

```
%% Exponential random variable
% Now, we consider a continuous random variable:
% We generate 1000 samples of an exponential with |mu = 2|.
mu = 2;
n = 1000;
x = exprnd(mu,[n,1]);

% Plotting the histogram
nbins = 20;
[cnts,edges] = histcounts(x,nbins);
binCenter = (edges(1:nbins)+edges(2:nbins+1))/2;
bar(binCenter, cnts);
xlabel('x');
ylabel('Counts');
set(gca,'Fontsize',16);
```



Estimating a PDF

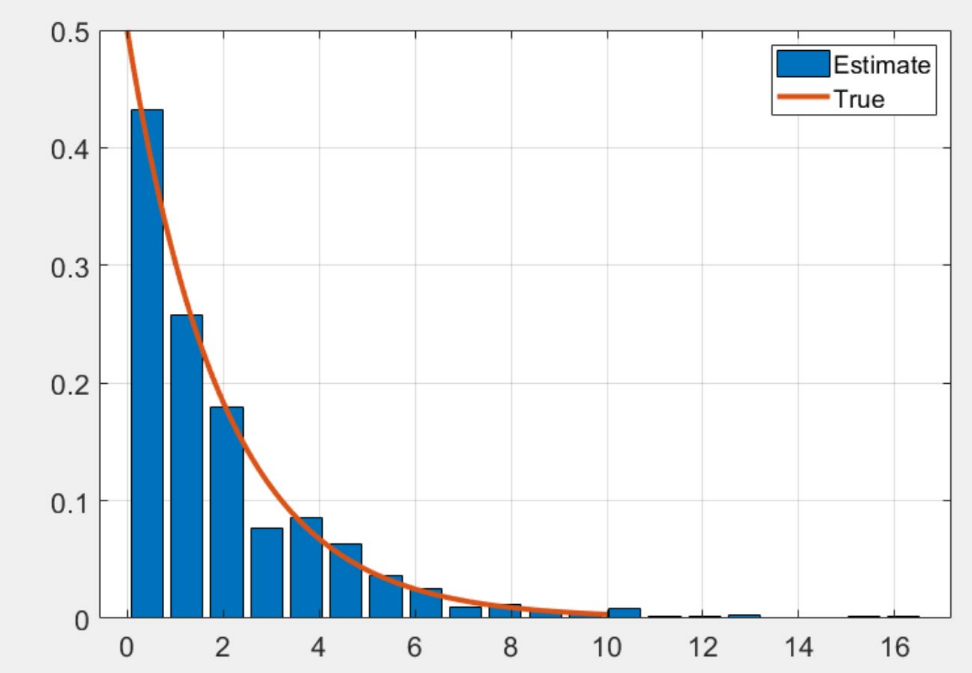
- We can estimate a PDF from histogram:
  - Fix any x. Let $[a, b]$ =bin interval containing x
  - Density estimate is fraction of sample/bin width:

$$\hat{p}(x) = \frac{\text{sample fraction}}{\text{bin width}} = \frac{1}{N(b-a)} \#\{i \,|x_i \in [a,b]\}$$

```
% We can estimate the PDF via
%    pest(x) = fraction of samples in bin/bin width
binWid = edges(2)-edges(1);
pest = cnts/n/binWid;

% Compute true PDF
xplot = linspace(0,5*mu,100)';
ptrue = exppdf(xplot,mu);

% Plot
bar(binCenter, pest);
hold on;
plot(xplot,ptrue,'-','Linewidth',3);
grid();
hold off;
```



Outline

- What is noise?
- Random variables
- Simulating random variables
- Gaussian and complex Gaussian random variables
- Random vectors
- Random processes

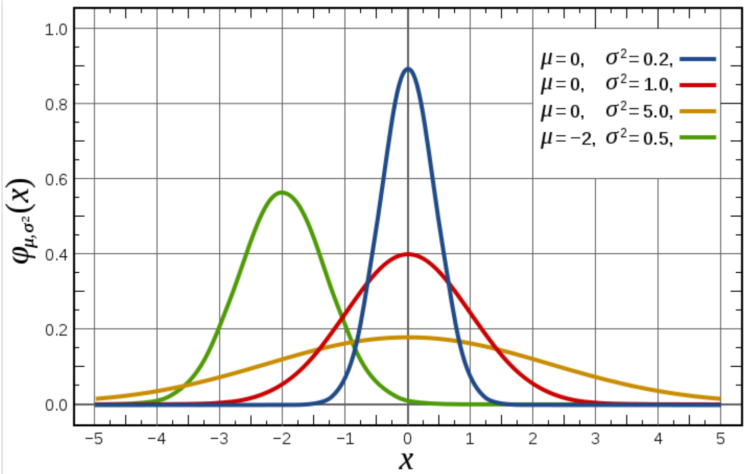
Gaussian Random Variables

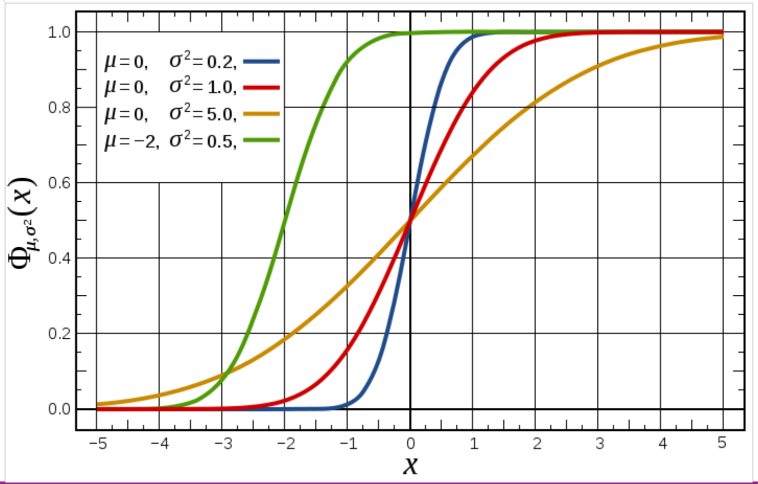
- Gaussian $X \sim N(\mu, \sigma^2)$
  - $EX = \mu, var(X) = \sigma^2$
- Probability density function (PDF):

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$$

- Cumulative distribution function (CDF):

$$F_X(x) = \int_{-\infty}^{x} p_X(u)du$$





Linear Transformations of Gaussians

- Suppose $X \sim N(\mu_X, \sigma_X^2)$
- Consider linear transformation: $Y = aX + b$
- Then $Y \sim N(\mu_Y, \sigma_Y^2)$ with $\mu_Y = a\mu_X + b,\ \ \sigma_Y^2 = a^2\sigma_X^2$

- Why? Consider CDF of Y and use change of variables:

$$F_Y(y) = P(Y \le y) = P\left(X \le \frac{y-b}{a}\right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{(y-b)/a} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right) dx = \frac{1}{\sqrt{2\pi}\sigma_Y} \int_{-\infty}^{y} \exp\left(-\frac{(u-\mu_Y)^2}{2\sigma_Y^2}\right) du$$

Gaussian CDF and Q Function

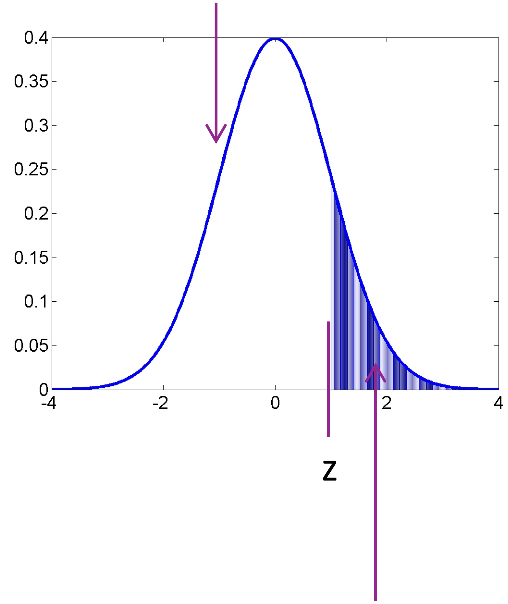
- Gaussian CDF has no closed-form expression.
- Write CDF in terms of unit variance Gaussian:
  - Suppose $X \sim N(\mu, \sigma^2)$.
  - Define z-score: $Z = (X - \mu)/\sigma$
  - Then: $X = \mu + \sigma Z$
  - CDF of X often expressed in terms of Marcum Q-Function on z:

$$Q(z) = P(Z \geq z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2} du$$

- Then:

$$F_X(x) = P(X \leq x) = P\left(Z \leq \frac{x-\mu}{\sigma}\right) = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$$

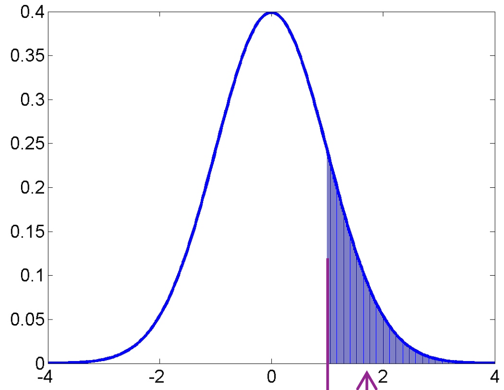
$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$



Q(z) = area under curve

Properties of the Q Function

- Suppose $X \sim N(\mu, \sigma^2)$
- Then (make sure you know how to do these):
  - $P(X \geq x) = Q\left(\frac{x-\mu}{\sigma}\right)$
  - $P(X \leq x) = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$
  - $P(X \in [a, b]) = Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$
  - $P(|X - \mu| \geq t) = 2Q\left(\frac{t}{\sigma}\right)$
- Also:
  - $Q(-\infty) = 1, Q(\infty) = 0, Q(0) = \frac{1}{2}$
  - $Q(-z) = 1 - Q(z)$



z

$P(Z > z) = Q(z)$
= area under curve

Error Function

- In many other programs, you cannot directly call the Q function.
- Typically, use the error function (erf) and complementary error function (erfc):

$$\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-x^2} dx\,, \qquad \mathrm{erfc}(u) = 1 - \mathrm{erf}(u)$$

- With a change of variables can show

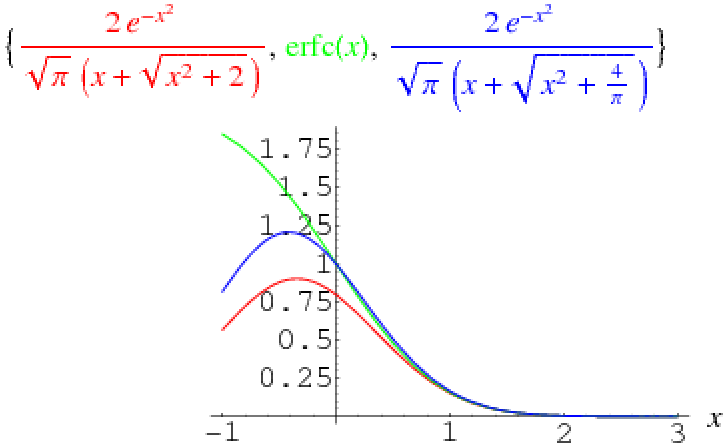
$$Q(z) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{z}{\sqrt{2}}\right)$$

- In MATLAB, you can use `qfunc` and `qfuncinv`.

Q Function Bounds

- Nclosed form expression for erfc(z) and Q(z)
- But, good approximations available for large z:
- For z > 0:

$$\frac{1}{\sqrt{2\pi}z}e^{-z^2/2} > Q(z) > \frac{1}{\sqrt{2\pi}z}\left(1-\frac{1}{z^2}\right)e^{-z^2/2}$$



- When z >> 0, bounds converge.
- When z > 1 can further approximate upper bound as:

$$Q(z) < \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

- When z << 0, use the relation $Q(z) = 1 - Q(-z)$

Example

- Suppose $X \sim N(\mu, \sigma^2)$ with μ=2, σ=1. What is P(X² < 16)?
- Let Z=(X-μ)/σ so that
  - $P(X^2 < t) = P((\mu + \sigma Z)^2 \leq t) = P(Z \in [z_1, z_2])$
  - $z_1 = (-\sqrt{t} - \mu)/\sigma = -6, \quad z_2 = (\sqrt{t} - \mu)/\sigma = 2$
  - $P(X^2 < t) = P(Z \in [z_1, z_2]) = Q(z_1) - Q(z_2)$
- For approximations:
  - $Q(z_1) = Q(-6) = 1 - Q(6) \approx 1$
  - $Q(z_2) = Q(2) \approx \frac{1}{\sqrt{2\pi} z_2} e^{-2^2/2}$

```
t=16;
sigma = 1;
mu = 2;
z1 = (-sqrt(t)-mu)/sigma;
z2 = (sqrt(t)-mu)/sigma;
P = qfunc(z1) - qfunc(z2);

Q2_approx = 1/sqrt(2*pi)/z2*exp(-z2^2/2);
P_approx = 1-Q2_approx;

fprintf(1,'P(X^2 < t):  True= %f approx=%f\n', P, P_approx);
```

```
P(X^2 < t):  True= 0.977250 approx=0.973005
```

Complex Random Variables

- Complex random variable: $Z = Z_r + jZ_i$
  - Z_r and Z_i are real random variables

- Most terms similar to real case with natural modifications:
  - $\mu = E(Z) = E(Z_r) + jE(Z_s)$
  - $\sigma^2 = var(Z) = E|Z - \mu|^2$

- Note that variance is positive

Complex Gaussian

- $Z = Z_r + jZ_r$ is a complex Gaussian random variable if:
  - Z_r and Z_r are independent; and
  - $Z_r \sim N(a, \sigma^2/2), Z_r \sim N(b, \sigma^2/2)$
    for some a, b and σ^2
- Write $Z \sim CN\left(\mu, \sigma^2\right),\ \mu = a + jb$
- $E(Z) = \mu,\ var(Z) = \sigma^2$
- PDF:

$$p(z) = \frac{1}{\pi\sigma^2} \exp\left[-\frac{1}{\sigma^2}|z - \mu|^2\right]$$

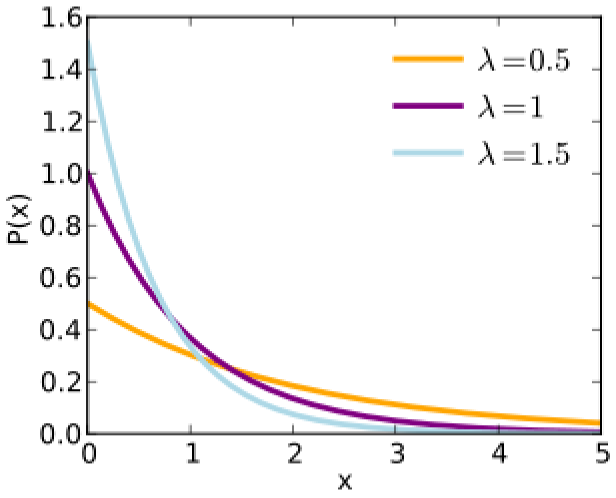
  - Note scaling factors slightly different from real Gaussian

Distributions on Magnitude

- Suppose $Z \sim CN(0, \sigma^2)$
- $R = |Z|$ is Rayleigh with scale parameter $\sigma^2/2$

$$p(r) = \frac{2r}{\sigma^2} e^{-r^2/\sigma^2}$$

- $U = |Z|^2$ is exponential with mean σ^2
  - $p(u) = \frac{1}{\sigma^2} e^{-u/\sigma^2}$
  - $F(u) = 1 - e^{-u/\sigma^2}$



Exponential distribution

Example Calculations (On board)

- Probability in an box
- Probability on $|Z|^2$

Outline

- What is noise?
- Random variables
- Simulating random variables
- Gaussian and complex Gaussian random variables
- Random vectors
- Random processes

Random Vectors

- Random vector: A vector $\boldsymbol{X} = [X_1, \dots, X_n]$ where each component X_i is a random variable
  - Can be a complex or real-valued random vector
- Why do we need random vectors?
- In communications many model involve several related random variables:
  - Multiple transmitted symbols, bits
  - Values of a channel characteristic at different times
  - Need to model their statistical relations
- Random vectors are not independent trials!
  - We should think of one outcome leading to n values
  - Not n different outcomes
  - Formally, each $s \in S$ gives rise to values $\boldsymbol{X}(s) = [X_1(s), \dots, X_n(s)]$

Joint PDF and PMF

- Suppose that $\boldsymbol{X} = [X_1, \ldots, X_n]$ is a random vector
- Discrete random variables: $X \in \{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(M)}\}$
  - Each possible value is a vector: $\boldsymbol{x}^{(m)} = [x_1^{(m)}, \ldots, x_n^{(m)}]$
  - Describe by a joint PMF: $P(\boldsymbol{X} = \boldsymbol{x}) = P(X_1 = x_1, \ldots, X_n = x_n)$
- Continuous random vectors: Components are continuous-valued
  - Described by a joint PDF is:

$$p(\boldsymbol{x}) = \lim_{vol(A) \to 0} \frac{1}{vol(A)} P(\boldsymbol{X} \in A)$$

  - Limit is over sets $A \subset \mathbb{F}^n$ that contain $\boldsymbol{x}$
  - Random variable is continuous if this limit exists

Properties of the Joint PDF

- Normalization:
  - $p(\boldsymbol{x}) \geq 0$ and $\int p(\boldsymbol{x})d\boldsymbol{x} = \int p(x_1, \ldots, x_n)dx_1 \cdots dx_n = 1$
- Probability in a set:

$$P(X \in A) = \int_{x \in A} p(x_1, \ldots, x_n)dx_1 \cdots dx_n$$

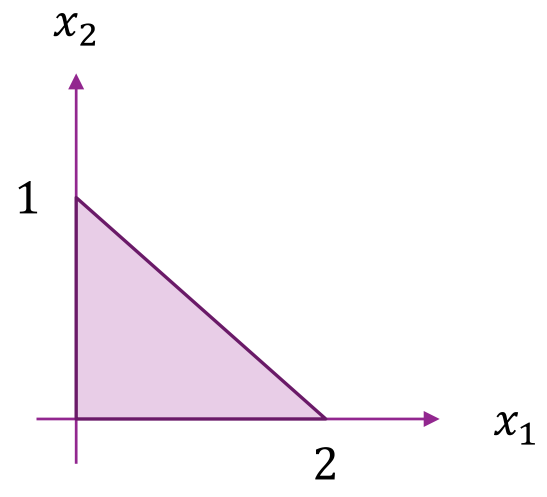
- Expectation: Given a vector-valued function

$$E\big(g(X)\big) = \int g(x_1, \ldots, x_n)p(x_1, \ldots, x_n)dx_1 \cdots dx_n$$

- Marginal PDFs: "Integrating out" the other variables
  - Example: Given $p(x_1, x_2, x_3)$ then $p(x_1) = \iint p(x_1, x_2, x_3)dx_2dx_3$
- Similar properties for the PMF

Sample Problem

- Suppose $p(x_1, x_2) = Cx_1x_2$ on the triangular region shown
- Find C
- Find $p(x_1)$
- Find $E(X_1X_2)$
- Find $P(X_2 > 0.5X_1)$
- Solution on board



Conditional Density

- Conditional probabilities:
  - Describe how random variable influences another
- Conditional probability of events: $P(A|B) = \frac{P(A\cap B)}{P(B)}$
- Condition PDF: $p_{X|Y}(x|y) = \frac{p(x,y)}{p(y)}$
- Conditional PMF: $P(X = x|Y = y) = \frac{P(X=x,Y=y)}{P(Y=y)}$
- Conditional probability rule: $p_{XY}(x,y) = p_X(x)p_{Y|X}(y|x) = p_Y(y)p_{X|Y}(x|y)$
- Conditional expectations: $E(X|Y) = \int xp(x|y)dx$
- Note these are all functions of Y

Ex 1: Power and Distance

- Given a distance D from a source, the received power Y is exponentially distributed with

$$p_{Y|D}(y|d) = \lambda(d)\exp(-\lambda(d)y), \qquad y \ge 0, \qquad \lambda(d) = \frac{d^2}{C}$$

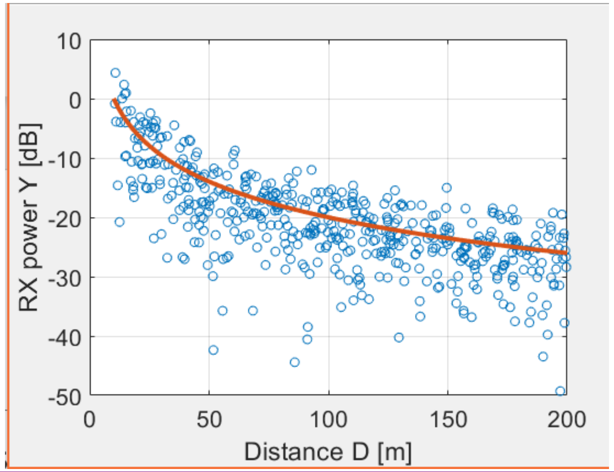
  - The average power $E(Y|D) = \frac{C}{d^2}$ decreases with distance.

- Suppose that D is uniformly distributed in $d \in [d_1, d_2]$

- Generate 500 samples of (D, Y)

```
n = 500;
dmin = 10;
dmax = 200;
d = unifrnd(dmin,dmax,[n,1]);

c = 100;
ymean = c./(d.^2);
y = exprnd(ymean);
ydB = 10*log10(y);
```



Ex 1: Power and Distance, Continued

- From previous slide: $p_{Y|D}(y|d) = \lambda(d)\exp(-\lambda(d)y)\,,\;\; y \geq 0,\; \lambda(d) = \frac{d^2}{C},\; D{\sim}Unif(d_1, d_2)$
- Find $E(Y)$
  - From the conditional distribution: $E(Y|D=d) = \frac{1}{\lambda(d)} = \frac{C}{d^2}$
  - Since $D{\sim}Unif(d_1, d_2) \Rightarrow p(d) = \frac{1}{d_2-d_1},\;\; d \in [d_1, d_2]$
  - From total expectation: $E(Y) = E[\,E(Y|D)] = \frac{1}{d_2-d_1}\int_{d_1}^{d_2} \frac{C}{d^2} dd = \frac{C}{d_2-d_1}\left[\frac{1}{d_1} - \frac{1}{d_2}\right] = \frac{C}{d_1 d_2}$
- From $p(y)$
  - The $p(y) = \int p(d)p(y|d)dd = \frac{1}{d_2-d_1}\int_{d_1}^{d_2} \frac{d^2}{C} \exp(-\frac{yd^2}{c})\, dd$

Ex 2: Additive Channel

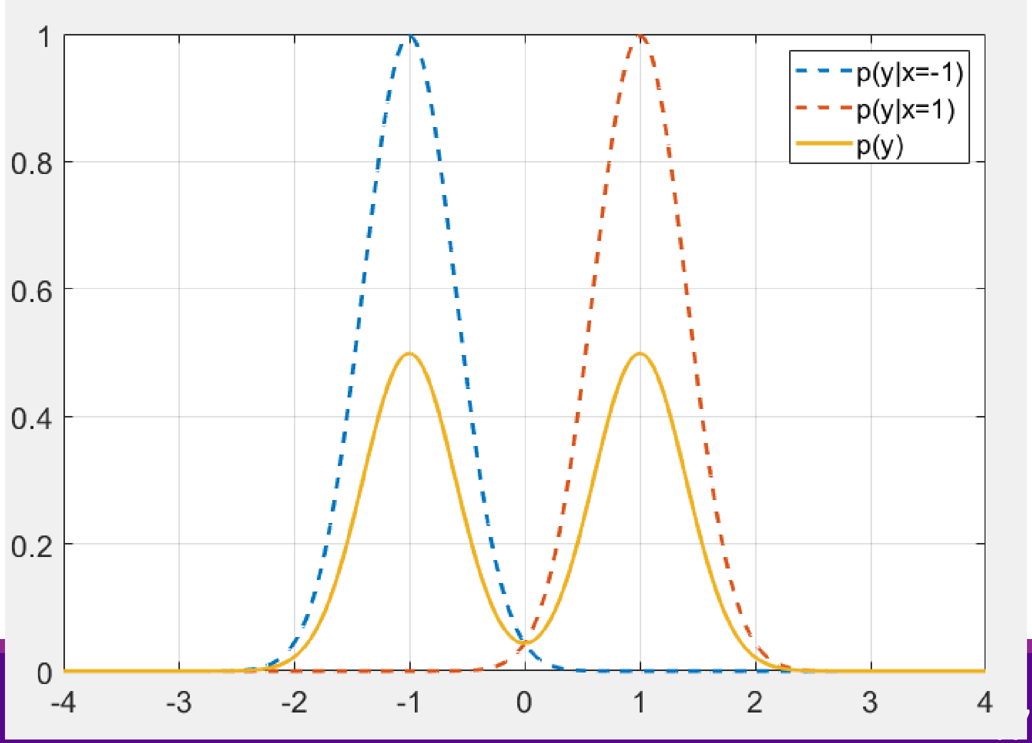
- Suppose that $Y = X + W$ and X, W are independent
- Then $p_{Y|X}(y|x) = p_W(y - x)$
- Why?
  - Look at CDF: $F_{Y|X}(y|x) = P(Y \leq y|X = x) = P(X + W \leq y|X = x) = P(W \leq y - x|X = x)$
  - Since X, W are independent: $F_{Y|X}(y|x) = P(W \leq y - x) = F_W(y - x)$
  - Take derivatives:

$$p_{Y|X}(y|x) = \frac{\partial}{\partial y} F_{Y|X}(y|x) = \frac{\partial}{\partial y} F_W(y - x) = p_W(y - x)$$

Ex 2. Additive Channel

- Example: $Y = X + W$
  - $X = \pm 1$ equiprobable
  - $W \sim N(0, \sigma^2)$, $\sigma = 0.3$
- Conditional PDF:
  - $p(y|x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-x)^2/(2\sigma^2)}$
- Marginal PDF of Y:
  - $p(y) = p(y|x=1)P(x=1) + p(y|x=-1)P(X=-1)$
  - Sum of two Gaussians

```
% Plotting the PDF
sig = 0.4;
x = linspace(-4,4,1000)';
px0 = normpdf(x,-1,sig);
px1 = normpdf(x,1,sig);
px = 0.5*(px0 + px1);
plot(x,[px0 px1],'--', 'Linewidth', 2);
hold on;
plot(x,px,'-', 'Linewidth', 2);
hold off;
legend('p(y|x=-1)', 'p(y|x=1)', 'p(y)');
set(gca,'Fontsize',16);
grid()
```

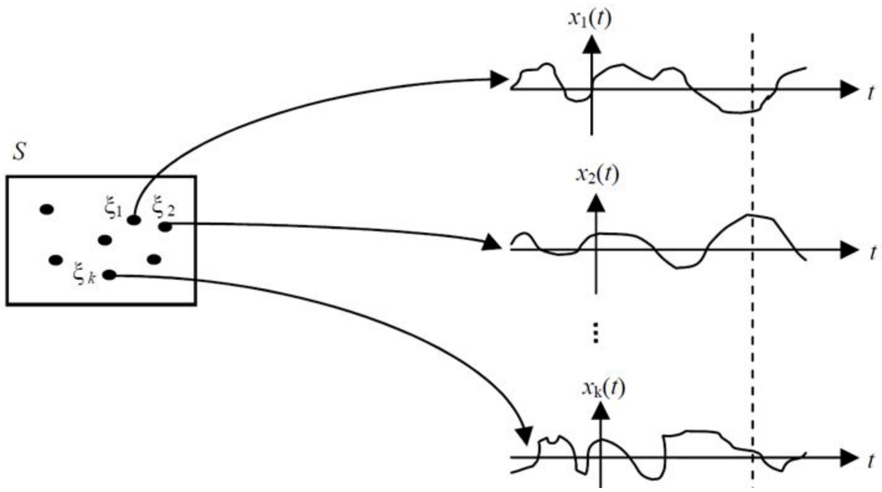


Outline

- ❑What is noise?
- ❑Random variables
- ❑Simulating random variables
- ❑Gaussian and complex Gaussian random variables
- ❑Random vectors
- ❑Random processes

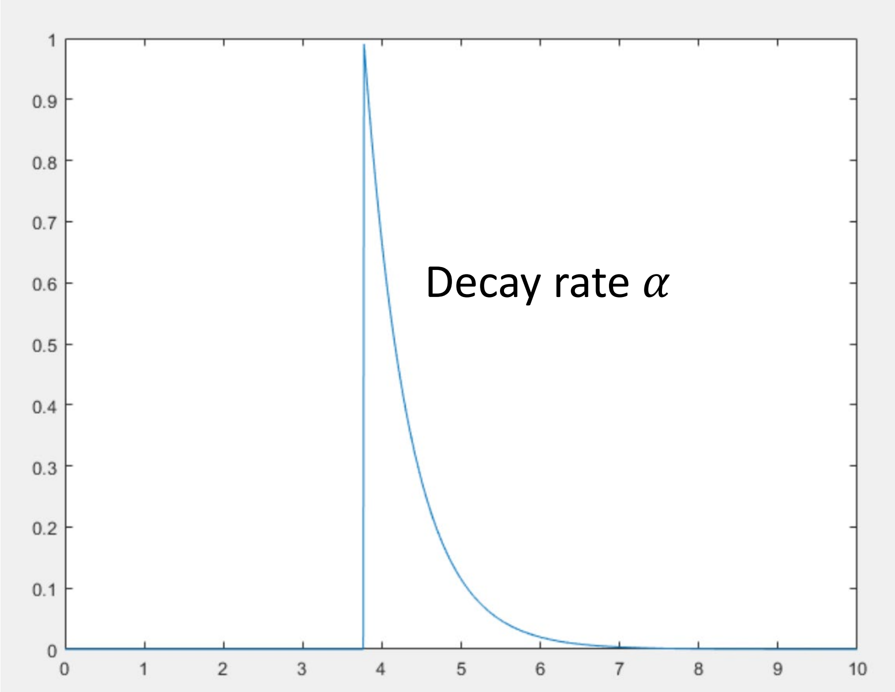
Random Process

- Communications commonly models signals via random processes
- Informal definition:
  - Any quantity $x(t)$ that varies with time,
  - …and variations have uncertainty
- Formal definition:
  - A set of random variables $x(t)$
  - All the random variables are on the same probability space
- Sample space model:
  - Consider a single outcome ω from sample space S
  - Each outcome results in entire sequence $s \mapsto x(t,s)$
  - The resulting sequence is called a realization
  - Often drop dependence on s and write $x(t) = x(t,s)$



Example 1: A Two Parameter Process

- Suppose $x(t) = e^{-\alpha(t-\tau)} 1_{t \geq \tau}$
  - $\tau = \text{Uniform}[0,5]$: A random initial start time
  - $\alpha = \text{Uniform}[0.1,3]$: A random decay rate
  - α, τ are independent
- Then $x(t)$ is a random process
  - Each τ, α creates a realization
- Write MATLAB code to generate $M = 10$ random realizations
- Solutions: Next slide



Start time τ

Example 1: A Two Parameter Process

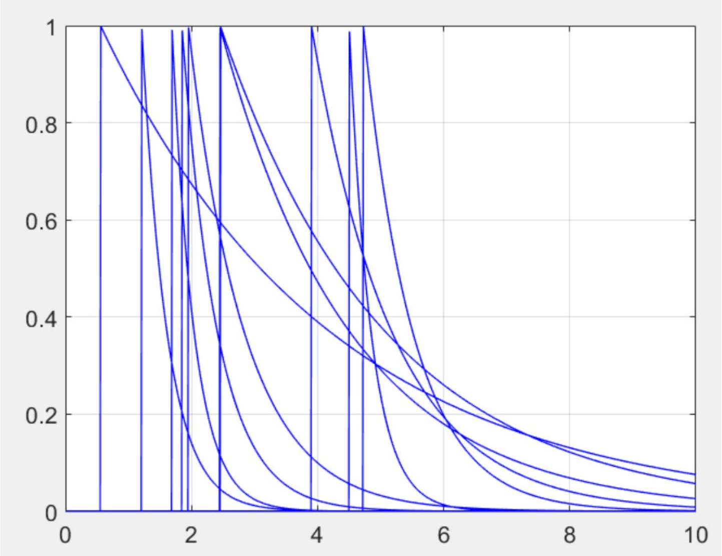
- MATLAB code to generate realizations

```
%% Two parameter random process
% Times to plot the signals
nt = 1000;
t = linspace(0,10,nt)';

% Generate the random parameters
m = 10;   % number of realizations
t0 = unifrnd(0,5,m,1);
alpha = unifrnd(0.1,3,m,1);

% Create a matrix of the realization
x = zeros(nt,m);
for i = 1:m
    x(:,i) = exp(-alpha(i)*(t-t0(i))).*(t > t0(i));
end
```

```
% Plot the realizations
plot(t,x,'b-', 'Linewidth', 1);
grid();
set(gca, 'Fontsize', 16);
```

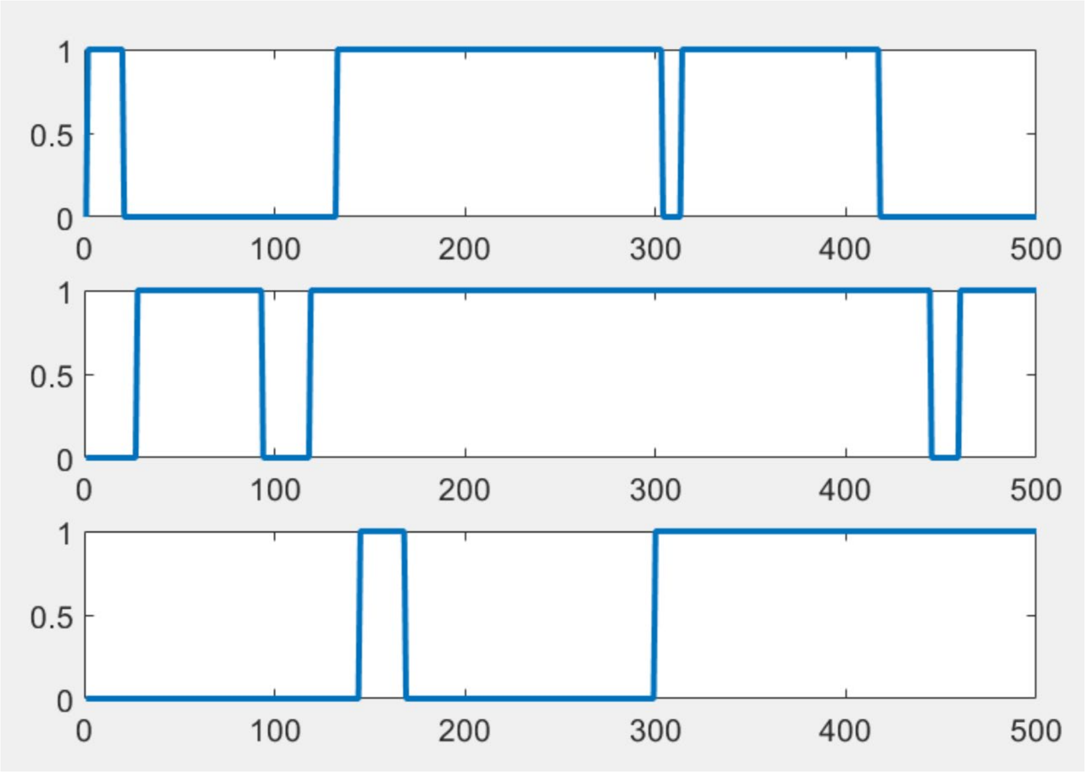


Ex 2: Discrete Ex: Switching Process

- Suppose $x[n] = 0$ or 1
- Switches with probability p in each time step
- Common model for many binary random processes
  - Ex: Channel state is good or bad

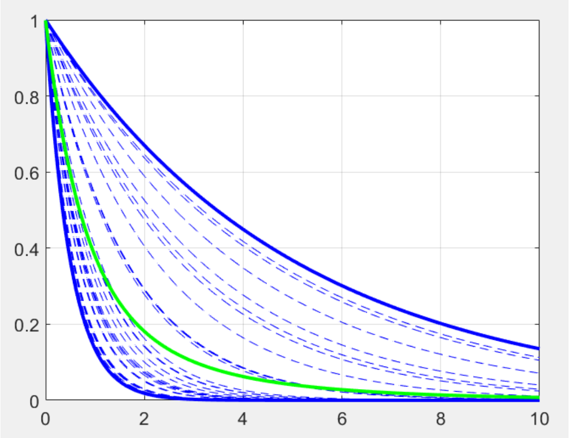
```
p = 0.01;
nt = 500;
m = 3;
x = zeros(nt,m);
v = (rand(nt-1,m) < p);
for t = 1:nt-1
    x(t+1,:) = mod(x(t,:) + v(t,:), 2);
end
```

$M = 3$ random realizations with $p = 0.01$



Mean of a Random Process

- The mean of a process: $\mu(t) = E(x(t))$
  - The average value over all realizations
- Example: $x(t) = e^{-\alpha t}, \quad t \geq 0$
  - $\alpha = \text{Uniform}[a, b]$
  - $\mu(t) = E\big(x(t)\big) = E(e^{-\alpha t}) = \frac{1}{b-a}\int_a^b e^{-\alpha t} d\alpha = \frac{1}{(b-a)t}\left[e^{-at} - e^{-bt}\right]$



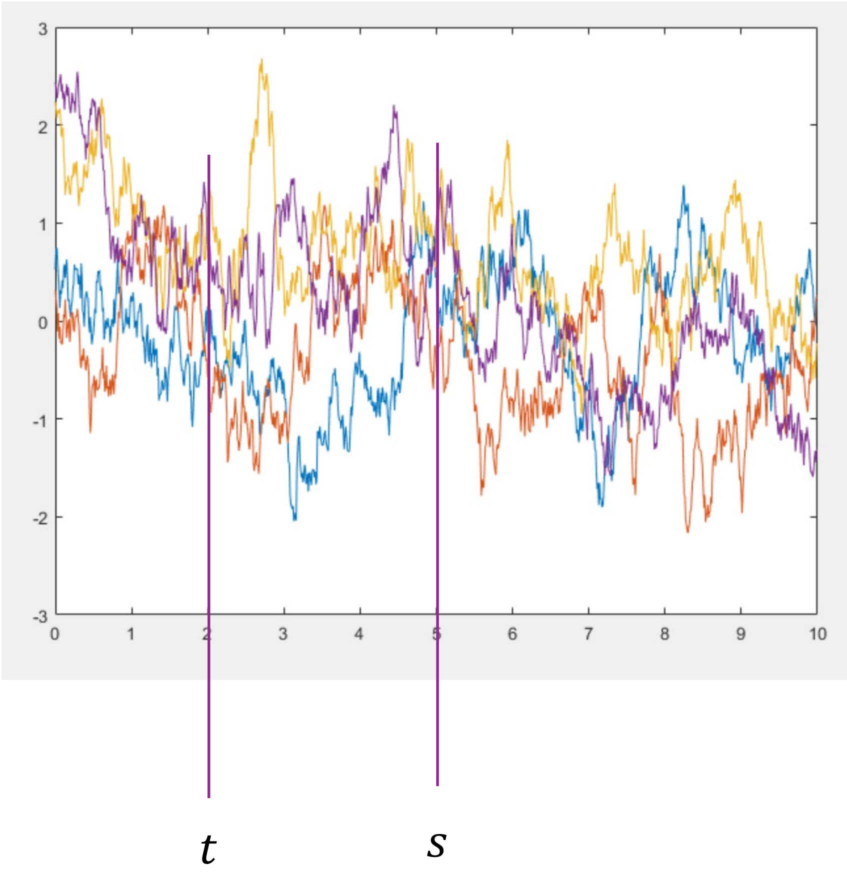
$a = 0.2, \ b = 2$

Dashed lines: Realizations $x(t)$ for different α

Green line: $\mu(t) = E(x(t))$

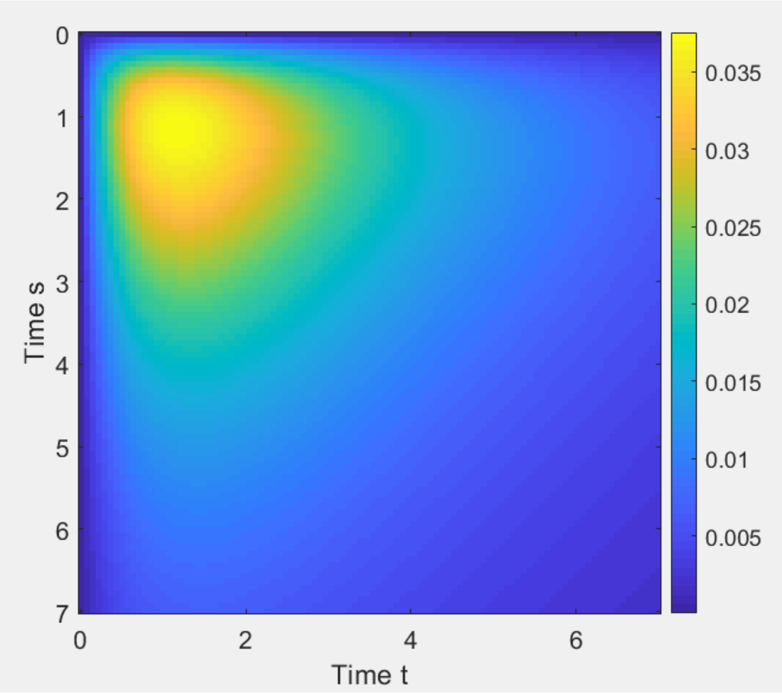
Auto-Correlation and Auto-Covariance

- Given a random process $x(t)$ (real or complex valued)
- Auto-correlation: $R(t,s) = E(x(t)x^*(s))$
- Auto-covariance: $C(t,s) = E\left[\left(x(t)-\mu(t)\right)\left(x(s)-\mu(s)\right)^*\right]$
- Note:
  - The times t and s are fixed
  - We are taking the ensemble average over $x(t)$ and $x(s)$
- Describes the correlation between two different times t and s



Example: Single Parameter Process

- Previous example: $x(t) = e^{-\alpha t}, \quad t \geq 0, \; \alpha = \text{Uniform}[a, b]$
  - $\mu(t) = E\big(x(t)\big) = E(e^{-\alpha t}) = \frac{1}{(b-a)t}\left[e^{-at} - e^{-bt}\right]$
  - $R(t,s) = E\big(x(t)x(s)\big) = E\big(e^{-\alpha(t+s)}\big)$
    $= \frac{1}{(b-a)(t+s)}\left[e^{-a(t+s)} - e^{-b(t+s)}\right]$
  - $C(t,s) = R(t,s) - \mu(t)\mu(s)$
- Figure to right: $a = 0.2, \; b = 2$
  - Covariance is low for $t = s \approx 0$ since $x(t) \approx 1$
    Low variance in $x(t)$
  - Covariance is also low for t or $s \approx \infty$
    Since $x(t) \approx 0$ with low variance

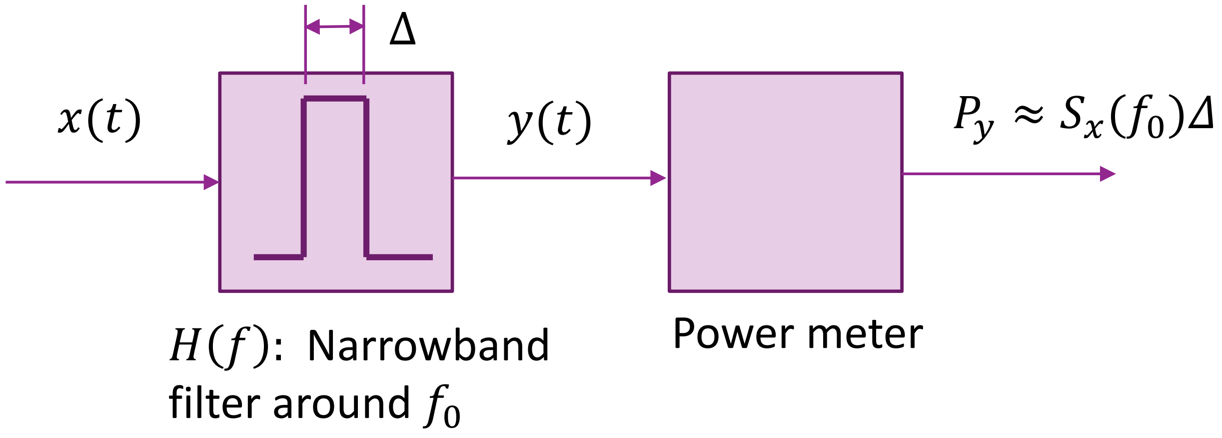


Wide Sense Stationary

- A random process $x(t)$ is wide-sense stationary (WSS) if:
  - $\mu(t) = E\big(x(t)\big) = \mu$: Mean does not change with time
  - $R(t, s) = E\big(x(t)x^*(s)\big) = R(t - s)$: Correlation depends only on difference $t - s$
- Similar definition for discrete-time random process $x[n]$:
  - $\mu[n] = E(x[n]) = \mu$
  - $R[n, m] = E(x[n]x^*[m]) = R[n - m]$
- Many processes in communication can be modeled well as WSS:
  - Particular time does not change the statistics

PSD: Filtering Definition

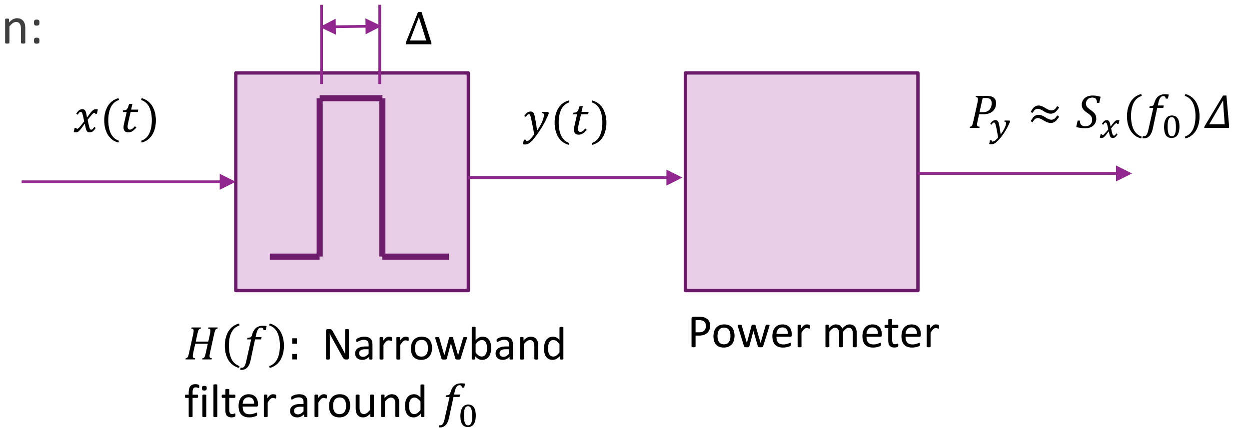
- Recall earlier definition of PSD.
- Let $x(t)$ be a power signal
- Select frequency f_0 to measure PSD
- Filter with narrowband filter
  - $y(t) = h(t) * x(t)$
  - $H(f) = 1$ for $|f - f_0| \leq \Delta/2$
- Measure power P_y



- PSD at f_0 is defined as $S_x(f_0) \coloneqq \lim_{\Delta \to 0} \frac{1}{\Delta} P_y$

PSD for WSS Processes

- PSD at f_0 is defined as $S_x(f_0) \coloneqq \lim_{\Delta \to 0} \frac{1}{\Delta} P_y$
- Theorem: Suppose that $x(t)$ is WSS. Then:
  - $S_x(f_0)$ limit exists for all f_0
  - $S_x(f_0) =$ Fourier transform of $R_x(t)$



- For WSS signals:
  - Can compute PSD from auto-correlation
  - Relates time-domain correlation to PSD
  - Faster varying signals have PSD at higher frequencies

Gaussian Random Process

- Definition: A random process $x(t)$ is a Gaussian random process if

$$[x(t_1), \dots, x(t_K)]$$

is a Gaussian random vector for every finite set of times $t_1, \dots, t_K$.

  - Can be complex or real Gaussian
  - Can be discrete-time or continuous-time

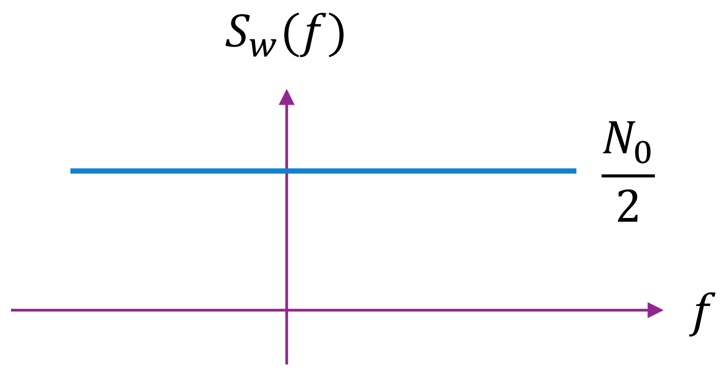
- All probabilities can be determined from the second order statistics:
  - Mean $\mu(t)$ and autocorrelation $R(t, s)$

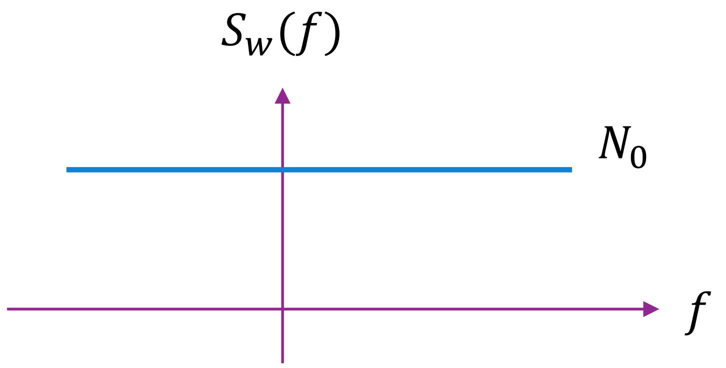
Example Problem

- Suppose that $x(t)$ is a real-valued WSS Gaussian random process with
  - $\mu = 0$ and $R(t) = P_0 e^{-a|t|}$
- Find $E\big(x(t) - x(s)\big)^2$
  - $E\big(x(t) - x(s)\big)^2 = E\big(x(t)^2\big) - 2E\big(x(t)x(s)\big) + E\big(x(s)^2\big)$
    $= R(0) - 2R(t-s) + R(0) = 2P_0(1 - e^{-a|t-s|})$
- What is $P(x(t) > x(s) + c)$?
  - Let $V = x(t) - x(s)$
  - Since $x(t)$ is a Guassian random process, V is Gaussian
  - $E(V) = 0,\ var(V) = 2P_0(1 - e^{-a|t-s|})$
  - $P(x(t) > x(s) + c) = P(V > c) = Q\left(\dfrac{c}{\sqrt{2P_0(1-e^{-a|t-s|})}}\right)$

White Gaussian Noise

- Definition: A real-valued WSS random process $w(t)$ is white if:
  - $E\big(w(t)\big) = 0,\ R(s) = E\big(w(t)w(t-s)\big) = \frac{N_0}{2}\delta(s)$
- This implies PSD is flat: $S_w(f) = \frac{N_0}{2}$
  - Constant power across all frequencies
  - Infinite energy
  - Note the factor of 2
- Complex white process:
  - $E\big(w(t)\big) = 0,\ R(s) = E\big(w(t)w^*(t-s)\big) = N_0\delta(s)$
- Discrete-time complex white WSS:
  - $E(w[n]) = 0,\ R(n) = E(w[m]w^*[m-n]) = N_0\delta(n)$





Filtered Gaussian Processes

- Theorem: Suppose that $x(t)$ is WSS Gaussian and $y(t) = h(t) * x(t)$.
  - Then $y(t)$ is WSS Gaussian
- Notes:
  - Applies to real and complex-valued processes, discrete and continuous-time
- PSD and autocorrelation
  - PSD given by: $S_y(f) = |H(f)|^2 S_x(f)$
  - Autocorrelation: $R_y(t) = h(t) * \tilde{h}(t) * R_x(t)$ where $\tilde{h}(t) = h^*(-t)$
- Any WSS Gaussian process can be obtained as $y(t) = h(t) * w(t)$ where $w(t)$ is white.
  - Why? Find a stable, causal filter $H(f)$ s.t. $S_y(f) = |H(f)|^2$
  - Take $w(t)$ to be white with $S_w(f) = 1$

Example: First Order Filter

- Suppose $x[n]$ is a real-valued, sample rate $\frac{1}{T} = 100$ MHz with

$$x[n+1] = ax[n] + bw[n]$$

  - $w[n]$ is white $Ew[n]^2 = \sigma_w^2$
- Find $\sigma_x^2 = E|x[n]|^2$
  - $x[n]$ and $w[n]$ are uncorrelated since $x[n]$ is a function of $w[n-1], w[n-2], \ldots$
  - Hence

$$\sigma_x^2 = Ex[n+1]^2 = E(ax[n] + bw[n]) = a^2Ex[n]^2 + b^2Ew[n]^2 \;= a^2\sigma_x^2 + b^2\sigma_w^2$$

  - Therefore: $\sigma_x^2 = \frac{b\sigma_w^2}{1-a^2}$
- Find $R[n]$,
  - $x[n] = bw[n-1] + abw[n-2] + \cdots + a^{n-1}bw[0] + a^n x[0]$
  - Then: $R[n] = E(x[n]x[0]) = a^n E\left(x[0]^2\right) = a^n\sigma^2$ for $n > 0$
  - Similarly, $R[n] = a^{-n}\sigma_x^2$ for $n \leq 0$
  - Overall $R[n] = a^{-|n|}\sigma_x^2$

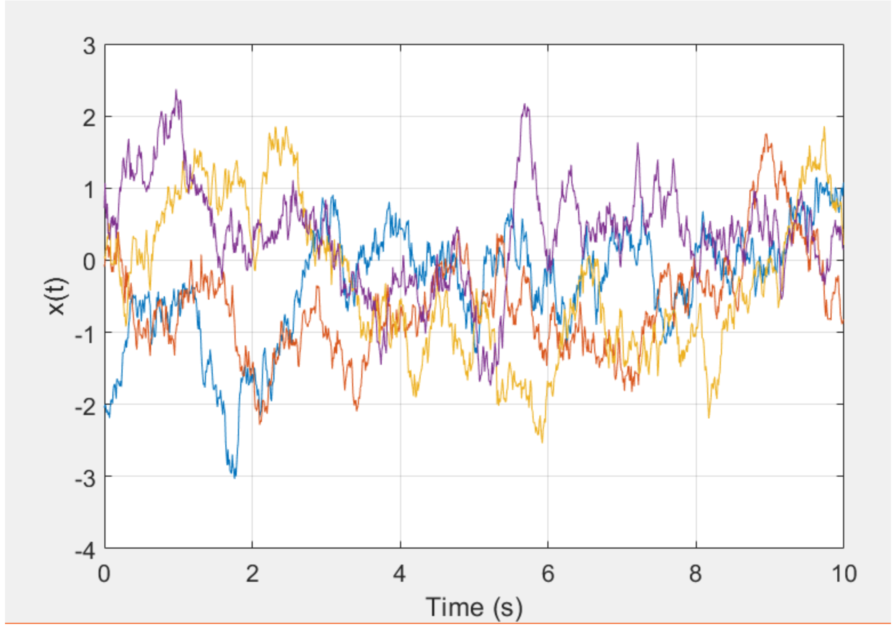
Example: First Order Filter

- Suppose $x[n]$ is a real-valued, sample rate $\frac{1}{T} = 100$ MHz with
$$x[n+1] = ax[n] + bw[n]$$
  - $w[n]$ is white $Ew[n]^2 = \sigma_w^2$
- Autocorrelation can also be computed via FT:
  - $H(\Omega) = \frac{X(\Omega)}{W(\Omega)} = \frac{b}{1-ae^{-j\Omega}} \Rightarrow |H(\Omega)|^2 = \frac{b^2}{|1-ae^{-j\Omega}|^2} = \frac{b^2}{1-a^2-2a\cos(\Omega)}$
  - Hence: $S_x(\Omega) = |H(\Omega)|^2 S_w(\Omega) = \frac{b^2\sigma_w^2}{1-a^2-2a\cos(\Omega)}$
  - Take inverse DTFT (use table):
$$R_x[n] = \mathcal{F}^{-1}\left(S_x(\Omega)\right) = \frac{b^2 a^{-|n|}\sigma_w^2}{1-a^2}$$

Ex: First Order Filter Simulation

- Suppose $x[n]$ is a real-valued, sample rate $\frac{1}{T} = 100$ Hz with $x[n+1] = ax[n] + bw[n]$
  - $w[n]$ is white $Ew[n]^2 = \sigma_w^2$
- Four realizations with $a = 0.99, \sigma_x^2 = \sigma_w^2 = 1$

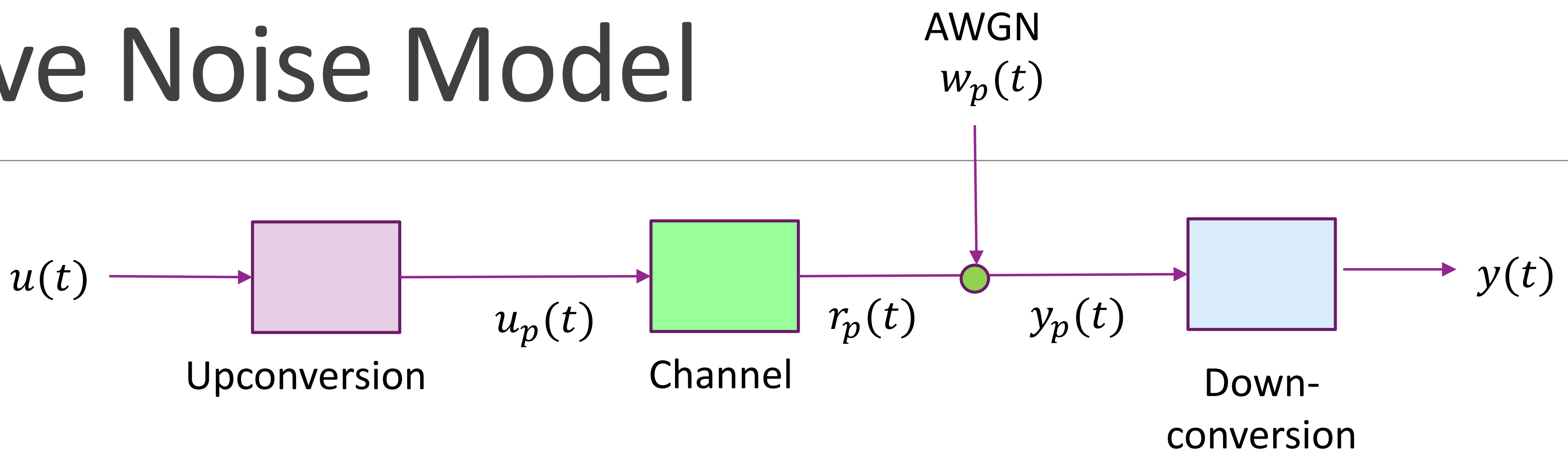
```
%% First order filter
T = 0.01;
nt = 1000;
m = 4;
t = (0:nt-1)'*T;
a = 0.99;
b = sqrt(1-a^2);
w = randn(nt,m);
x0 = randn(1,m);
x = filter(b,[1 -a],w, x0);
plot(t,x);
xlabel('Time (s)');
ylabel('x(t)');
grid();
set(gca,'Fontsize',16);
```



Outline

- What is noise?
- Random variables
- Simulating random variables
- Gaussian and complex Gaussian random variables
- Random vectors
- Random processes
- AWGN models

Additive Noise Model



- We first look at modeling thermal noise
- Thermal noise:
  - Due to random fluctuations of electrons in the receiver
  - Called “thermal” since the level of the fluctuations increases with temperature
- Common Additive White Gaussian Noise (AWGN) model: $y_p(t) = r_p(t) + w_p(t)$
  - $w_p(t)$ is real Gaussian WSS noise with PSD $\frac{N_0}{2}$

Scaling Up- and DownConversion

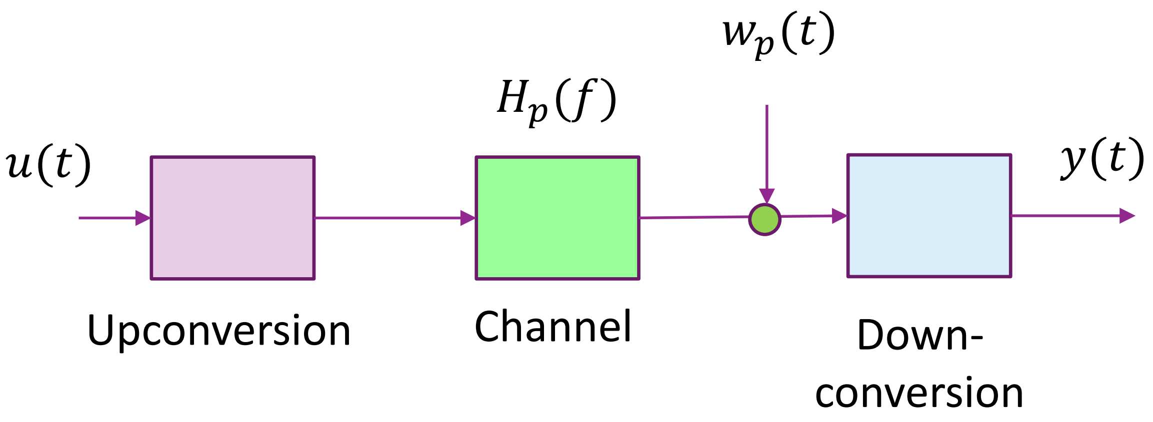
- For noise modeling, it is convenient to use a different scaling convention
- Modified scaling will keep powers in passband and baseband equal
- Note: Proakis uses original scaling and has a factor of 2 in the conversion

	Earlier scaling	Current scaling
Upconversion	$u_p(t) = Real(u(t)e^{j\omega_c t})$	$u_p(t) = \sqrt{2}Real(u(t)e^{j\omega_c t})$
Downconversion	$v(t) = 2u(t)e^{-j\omega_c t}$ $u(t) = h_{LPF}(t) * v(t)$	$v(t) = \sqrt{2}u(t)e^{-j\omega_c t}$ $u(t) = h_{LPF}(t) * v(t)$

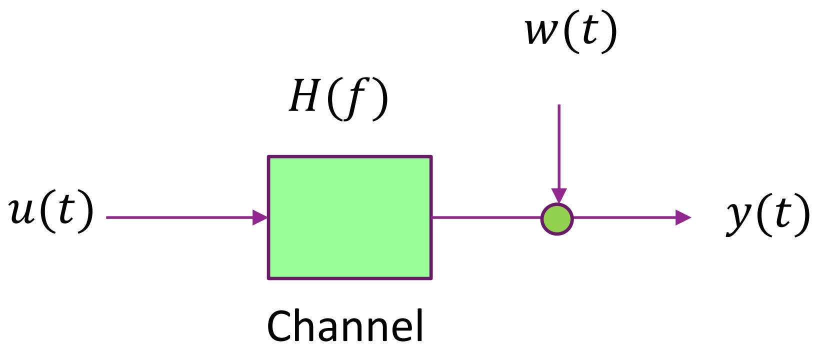
Downconverting Noise

- Suppose that $w_p(t)$ is real-valued WSS noise with PSD $\frac{N_0}{2}$
- Consider downconversion (with modified scaling factor):
  - $v(t) = \sqrt{2}e^{-j\omega_c t} w_p(t)$
  - $y(t) = h_{LPF}(t) * v(t)$
- Theorem: PSD of $y(t)$ is $S_y(t) = N_0 |H_{LPF}(f)|^2$
- Why?
  - $E\left(v(t)v^*(s)\right) = 2e^{-j\omega_c(t-s)} E\left(w_p(t)w_p(s)\right) = 2e^{-j\omega_c(t-s)}\delta(t-s)\frac{N_0}{2} = N_0\delta(t-s)$
  - So $v(t)$ is complex white WSS with PSD N_0. $S_v(f) = N_0$
  - $S_y(f) = |H_{LPF}(f)|^2\, S_v(f) = |H_{LPF}(f)|^2 N_0$

Equivalent Channel with Noise



- Passband model:
  - $y_p(t) = h_p(t) * u_p(t) + w_p(t)$
  - $w_p(t)$: additive noise in passband
  - Noise PSD = $\frac{N_0}{2}$



- Complex baseband equivalent model:
  - $y(t) = h(t) * u(t) + w(t)$
  - PSD of effective baseband noise:

$$S_w(t) = N_0 |H_{LPF}(f)|^2$$

Effective Baseband Noise ≈ White

- Prev. slide: PSD of effective baseband noise is:

$$S_w(f) = N_0 |H_{LPF}(f)|^2$$

- Suppose that $|H_{LPF}(f)| \approx 1$ for $|f| \leq \frac{W}{2}$
  - Approximately constant in band of interest
- Hence: $S_w(f) \approx N_0$
- Effective baseband PSD is approximately flat
- Can be well modeled as additive white noise