Unit 4: Signal Space Theory

EL-GY 6013: DIGITAL COMMUNICATIONS

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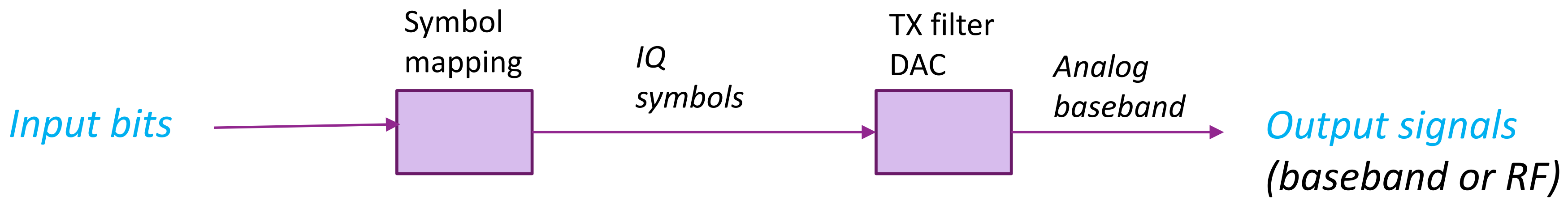
Learning Objectives

- ❑ Determine if a set is a vector space or not
- ❑ Find the dimension for a vector space or subspace. Find a basis
- ❑ Find a signal space for a set of signals.
  - Compute the degrees of freedom and rate for signal set in a signal space
  - Find the representation of a signal in signal space for a given basis
- ❑ Find the number of DoF per second of a band-limited signal
- ❑ Determine if vector or signals are orthogonal
- ❑ Find an orthonormal basis.
- ❑ Compute representations of signals in an orthonormal basis
- ❑ Find the energy per DoF in an orthonormal basis

Outline

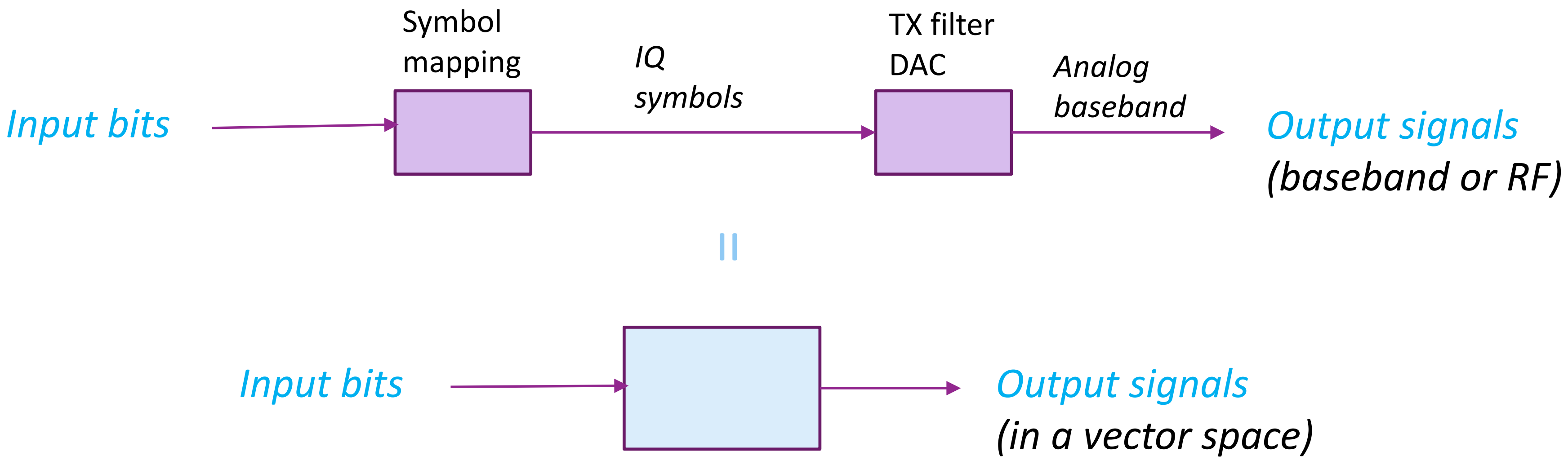
- Motivation for the signal space model
- Vector spaces, bases and dimension
- Signal spaces, rate and degrees of freedom
- Nyquist Theorem and degrees of freedom in band-limited signals
- Inner products and orthogonality
- Orthonormal bases and energy per degree of freedom

Algorithmic Model of a Typical Transmitter



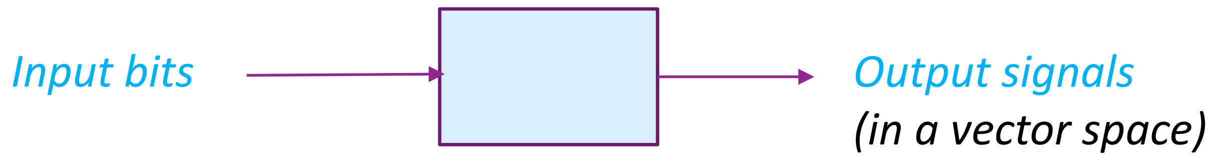
- Up to now, we have considered a standard transmitter with three steps
  - Symbol mapping
  - TX filtering
  - Upconversion

Abstract Signal Space Model of a Transmitter



- Maps bits to signals
- Signals belong to a vector space

Why Use the Signal Space Model?



- Analysis is not limited to a specific TX architecture
  - Applies to any communication system
- Describes fundamental parameters of the system
  - Rate and degrees of freedom
  - In next unit, we will also introduce a third parameter which is SNR
- Some calculations are easier in the transformed space
- Provides a geometric model to understand communication

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- ❑Motivation for the signal space model
- ❑Vector spaces, bases and dimension
- ❑Signal spaces, rate and degrees of freedom
- ❑Nyquist Theorem and degrees of freedom in band-limited signals
- ❑Inner products and orthogonality
- ❑Orthonormal bases and energy per degree of freedom

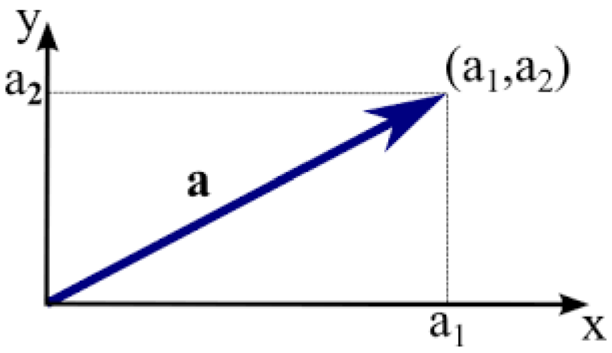
Vector Spaces

- Definition: A vector space over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$:
  A set V with operations:
  - Addition: $\boldsymbol{x}, \boldsymbol{y} \in V \Rightarrow \boldsymbol{x} + \boldsymbol{y} \in V$
  - Scalar multiplication: $\alpha \in \mathbb{F}, \boldsymbol{x} \in V \Rightarrow \alpha \boldsymbol{x} \in V$

- We will use $\mathbb{F}$ as a placeholder for the field
- Commutative, associative, distributive relationships hold
- Vector subspace: Subset that is also a vector space.

Vector Space $\mathbb{F}^N$

- Set of n-dim vectors $\boldsymbol{x} = [x_1, \ldots, x_n]$
  - Each element $x_n \in \mathbb{F}$
  - Can be row or column vectors
  - Or just vectors
- Standard addition and scalar multiplication rules
- Make sure you know how to do these operations
  - $[1,2,3] + [3,4,5] = \cdots$
  - $2[1,-2,3] = \cdots$
  - $[1+i, 2-i] - [3+2i, 4-6i] = \cdots$



Example Problem

- Which of the following are vector subspaces of $\mathbb{F}^N$
- V = set of $x = [x_1, x_2, x_3]$ s.t $x_1 = 0$
- V = set of $x = [x_1, x_2, x_3]$ s.t $2x_1 + 3x_2 = 0$
- V = set of $x = [x_1, x_2, x_3]$ s.t $2x_1 + 3x_2 = 4$
- V = set of $x = [x_1, x_2, x_3]$ s.t $x_1 \geq 3x_2$
- V = set of $x = [x_1, x_2, x_3]$ s.t $x_1^* + 3x_2 = 0$

- Solutions on board

Vector Spaces of Functions

- In communications, we often need to consider vector spaces of functions
- Example: $V = C[a, b]$
  - Set of continuous functions on $[a, b]$
  - Each "vector" is a function $f(t), t \in [a, b]$
  - Can be complex or real-valued
- We can add and subtract functions as usual:
- Ex: $f(t) = t^2,\ g(t) = t^3$
  - If $h = 2f + 3g$ then $h(t) = 2t^2 + 3t^3$

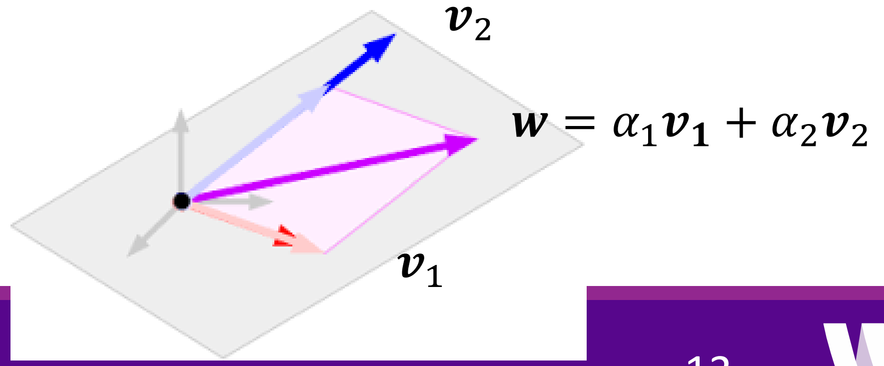
Example Problem

- Which of the following are vector subspaces $C[a, b]$
- Set of differentiable functions
- Set of functions $f(t) \geq 0$
- Set of polynomial functions
- Fix $g(t)$. Let V = set of functions $f(t)$ with $\int g(t)f(t) = 0$

- Solutions on board

Linear Span

- Let V some vector space and $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_d\} \subset V$
- A linear combination of $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_d\}$ is a vector of the form:
$$\boldsymbol{w} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_d \boldsymbol{v}_d, \qquad \alpha_i \in \mathbb{F}$$
- Span: The set of all linear combinations of the vectors:
$$\mathrm{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_d\} = \{\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_d \boldsymbol{v}_d, \alpha_i \in \mathbb{F}\} \subseteq V$$
  - This is a vector subspace of V
  - $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_d\}$ are said to span W if $W = \mathrm{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_d\}$
- Example: In $V = \mathbb{R}^3$, $\mathrm{span}\{v_1, v_2\}$ is the plane that contains the two vectors



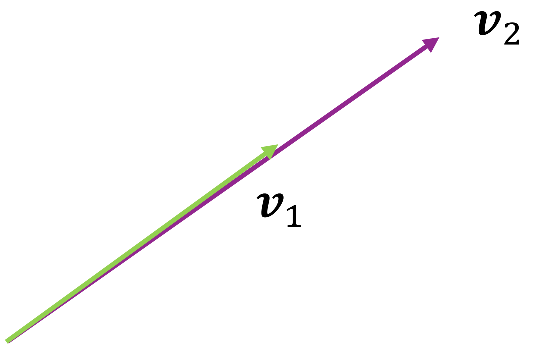
Linear Independence

- Vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_d\}$ are linearly independent if:

$$\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_d \boldsymbol{v}_d = 0 \Rightarrow \alpha_i = 0 \text{ for all } i$$

  - Any non-zero combination of the vectors $\boldsymbol{v}_j$ is non-zero.
  - If vectors are not linearly independent, then they are linearly dependent

- Example: Consider case of two vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$
  - $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent if and only if $\boldsymbol{v}_1 = \beta \boldsymbol{v}_2$ or $\boldsymbol{v}_2 = \beta \boldsymbol{v}_1$
  - That is, vectors are in the same direction



Basis and Dimension

- Given vector space V
- Vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are a basis for V if
  - $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are linearly independent and span V
- This means that every $\boldsymbol{x} \in V$ can be uniquely written as:

$$\boldsymbol{x} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_d \boldsymbol{v}_d$$

- Say $[\alpha_1, \dots, \alpha_d]$ are the coordinates of x in the basis
- Dimension of $V = n$

Simple Vector Example

- Vectors [1,2], [3,6] are linearly dependent since:

$$[3,6] = 3[1,2] \Rightarrow [3,6] - 3[1,2] = 0$$

- Vectors $v_1 = [1,0], v_2 = [2,3]$ are linearly independent:
  - Why? Suppose that $\alpha_1 v_1 + \alpha_2 v_2 = 0$
  - Then: $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
  - Matrix $\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is invertible. So this system of equations has a unique soln: $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
  - Hence $\boldsymbol{v}_1 = [1,0], \boldsymbol{v}_2 = [2,3]$ are a basis.

- Write $w = [5,6]$ as a linear combination of v_1, v_2: Solve $\boldsymbol{w} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2$
  - $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \Rightarrow \ \ldots \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- Performing these calculations for larger dimension matrices by hand is generally difficult

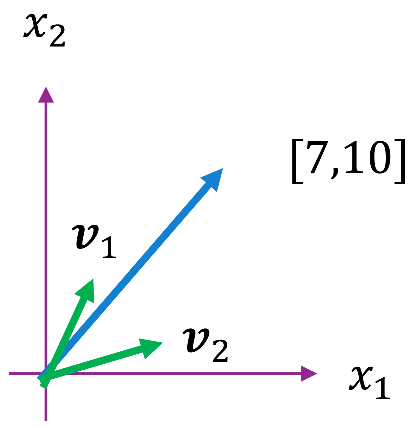
Standard Basis and Change of Basis

- Standard basis of $\mathbb{F}^N$: $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$
  - $\boldsymbol{e}_i = [0, \ldots, 0,1,0, \ldots 0]$. A "1" in position i
- Any vector $\boldsymbol{x} = [x_1, \ldots, x_N] = \sum_i x_i \boldsymbol{e}_i$
  - Hence $[x_1, \ldots, x_N]$ are the coordinates in the standard basis
- Change of basis:
  - Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ is another basis
  - Want to find coordinates $\boldsymbol{x} = \sum_i \alpha_i \boldsymbol{v}_i$

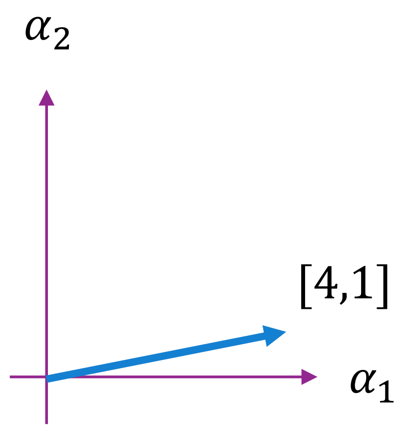
Example

- Suppose $\boldsymbol{v}_1 = [1,2],\ \boldsymbol{v}_2 = [3,2]$
- $x = [7,10]$ in the standard basis
- Find coefficients of $\boldsymbol{x}$ in $\boldsymbol{v}_1, \boldsymbol{v}_2$ basis
- Solution:
  - Solve $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$
  - Get $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Standard basis



New basis



Function Example

- Consider V = set of continuous functions $f: [0,1] \rightarrow \mathbb{R}$
- Let $f_j(x) = x^j, j = 0, \dots, d$ for some d
- Then $f_j(x)$ are linearly independent.
  - Why? Suppose that $g = \sum_i \alpha_i f_i = 0$
  - Then: $g(x) = \sum_i \alpha_i x^i = 0$ for all $x \in [0,1]$ (sometimes we say $g(x)$ is identically zero)
  - But $g(x)$ is a polynomial of degree d
  - A polynomial $g(x) = 0$ for all x if and only all the coefficients are zero
    (Follows from the fact that any non-zero polynomial of degree d has at most d real roots)
- Since we can make d arbitrarily large, V has infinite dimension
  - There are an arbitrary number of linearly independent vectors

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Abstract Transmitter and the Signal Set

Input bits
$(b_1, \dots, b_K)$

Output signals
$s(t) \in \{s_1(t), \dots, s_M(t)\}$

- Abstract transmitter:
  - Input is a set of bits $(b_1, \dots, b_K)$
  - Outputs one of a finite set of signals $s_1(t), \dots, s_M(t)$
  - For now, we will consider only a finite set of bits
- In general there are $M = 2^K$ possible output signals
- Definition: We call $\mathcal{S} = \{s_1(t), \dots, s_M(t)\}$ the signal set
  - We say $K = \log_2 M$ is the number of bits in $\mathcal{S}$
- This model is not specific to any architecture

Example

- Suppose we linearly map N QPSK symbols:

$$u(t) = \sum_{n=0}^{N-1} s[n]p(t - nT), \qquad s[n] = \pm A \pm jA$$

- There are $\mathrm{K} = 2N$ input bits
- There are $M = 2^{2N}$ output signals $u(t)$
- Each set of bits $b = (b_1, \ldots, b_K)$
  - Each bit pair (b_{2n+1}, b_{2n+2}) gets mapped to a symbol $s[n]$
  - We obtain $N = \frac{K}{2}$ symbols $(s[0], .., s[N-1])$
  - Results in signal $u(t)$ after linear modulation

Signal Space

- Consider a signal set $\mathcal{S} = \{s_1(t), s_2(t), \ldots, s_M(t)\}$
  - Can be either a set of received signals or transmitted signals
  - Can be in discrete-time, analog baseband or analog RF
  - For now, we consider only finite signal set, $M < \infty$
  - But signals $s_i(t)$ can be infinite in duration or finite
- A signal space for $\mathcal{S}$ is any vector space V that contains $\mathcal{S}$
- Note: $\mathcal{S}$ is not itself a vector space
  - It is a discrete set
  - But, it is contained in a vector space.

Signal Space Bases

- Let $\mathcal{S} = \{s_1(t), s_2(t), \ldots, s_M(t)\}$ be a signal set
- Suppose $\mathcal{S} \subset V, V =$ signal space containing $\mathcal{S}$
- Since V is a vector space it has a basis: $\phi_1(t), \ldots, \phi_N(t)$
- Then, each signal $s_i(t)$ can be expanded as: $s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t)$
- The vector $\boldsymbol{s}_i = [s_{i1}, \ldots, s_{iM}]$ is the coordinate vector of $s_i(t)$ in the basis
  - The components s_{ij} are the coordinates

Degrees of Freedom and Rate

- Let $\mathcal{S} = \{s_1(t), s_2(t), \dots, s_M(t)\}$ be a signal set
- Signal space theory has two key parameters
- Degrees of freedom:
  - The smallest dimension of a signal space V with $\mathcal{S} \subset V$
  - If N=#degrees of freedom, then there are N basis vectors $\phi_1(t), \dots, \phi_N(t)$ with

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t)$$

- Rate: The rate of a signal set in a signal space V with dimension N is:

$$R = \frac{\text{bits}}{\text{DoF}} = \frac{\log_2(M)}{N}$$

  - Represents the amount of information transmitted per DoF

Complex vs. Real DoF

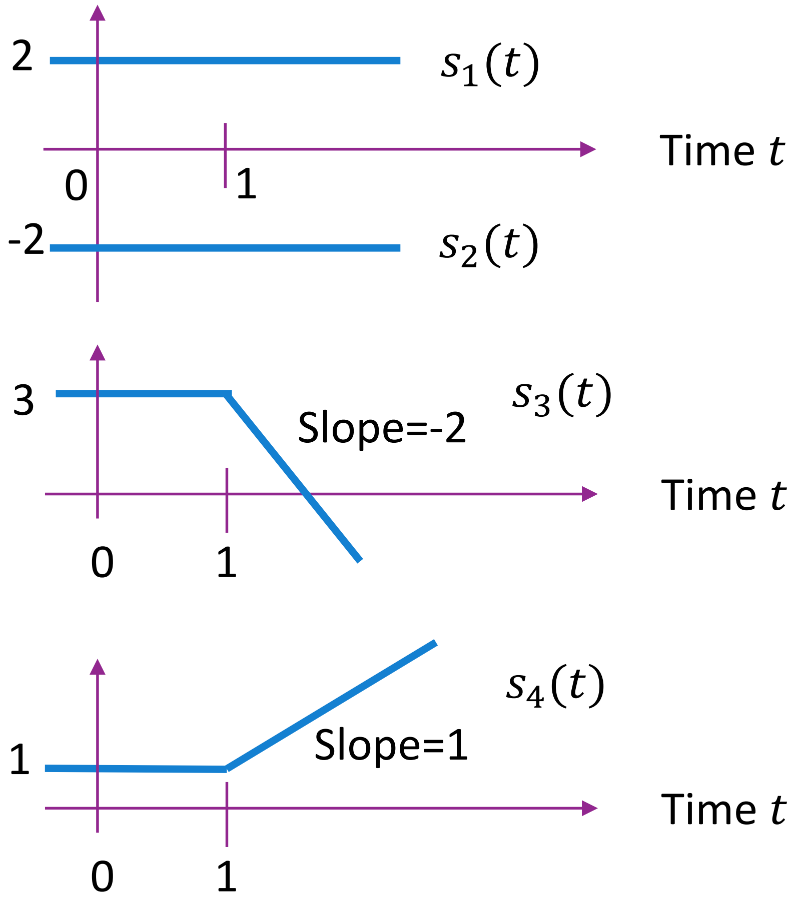
- Often signals $s_i(t)$ or $s_i[n]$ are complex-valued
- Typically use a complex vector space
  - $s_i(t) = \sum_{n=1}^{N} s_{in}\phi_n(t)$,
  - Coordinates will be complex $s_{ij} \in \mathbb{C}$
  - We say there are N complex degrees of freedom
- Or, we can convert to a vector space over $\mathbb{R}$
  - Define basis with $2N$ vectors: $\phi_1(t), \ldots, \phi_N(t)$ and $j\phi_1(t), \ldots, j\phi_N(t)$
  - Then, we can write:

$$s_i(t) = \sum_{n=1}^{N} a_{in}\phi_n(t) + b_{in}j\phi_n(t)$$

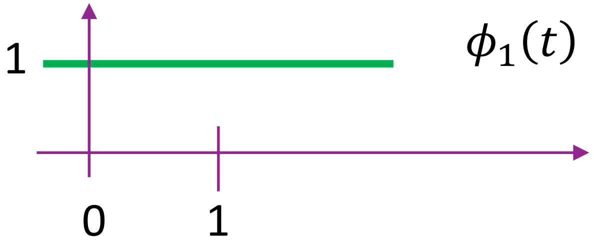
  - So, there are $2N$ real degrees of freedom

Example

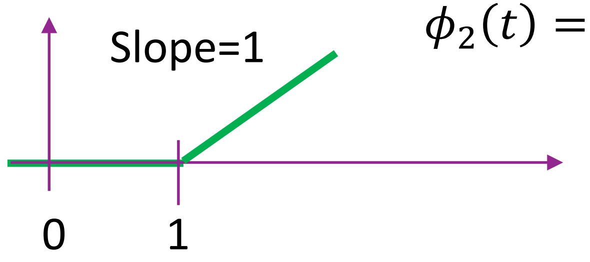
- Consider the four functions shown to the right
- Find N =degrees of freedom of $\mathcal{S}$
- Find a signal space V for $\mathcal{S}$ with dimension N
- Find a basis for V
- Find the coordinate vector of each $s_i(t)$ in the basis



Example Solution

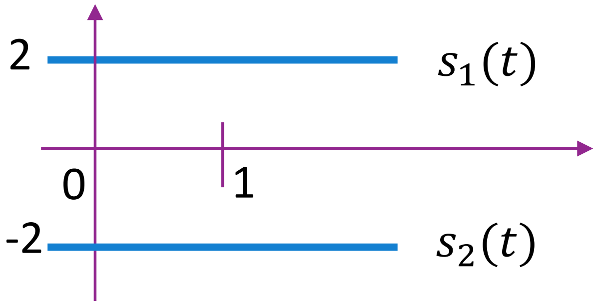


$\phi_1(t) = 1$



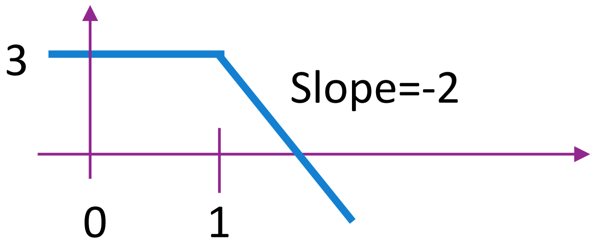
$\phi_2(t) = \max(0, t-1)$

- Can use $N = 2$ basis vectors
- So, there are $N = 2$ degrees of freedom
- $K = \log_2(4) = 2$ bits
- Rate $R = \frac{K}{N} = 1$ bit / DoF

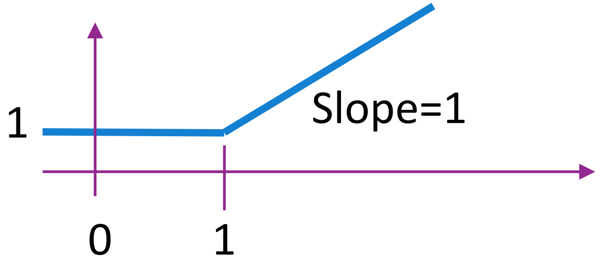


$s_1(t) = 2\phi_1(t)$
$\boldsymbol{s}_1 = [2,0]$

$s_2(t) = -2\phi_1(t)$
$\boldsymbol{s}_2 = [-2,0]$

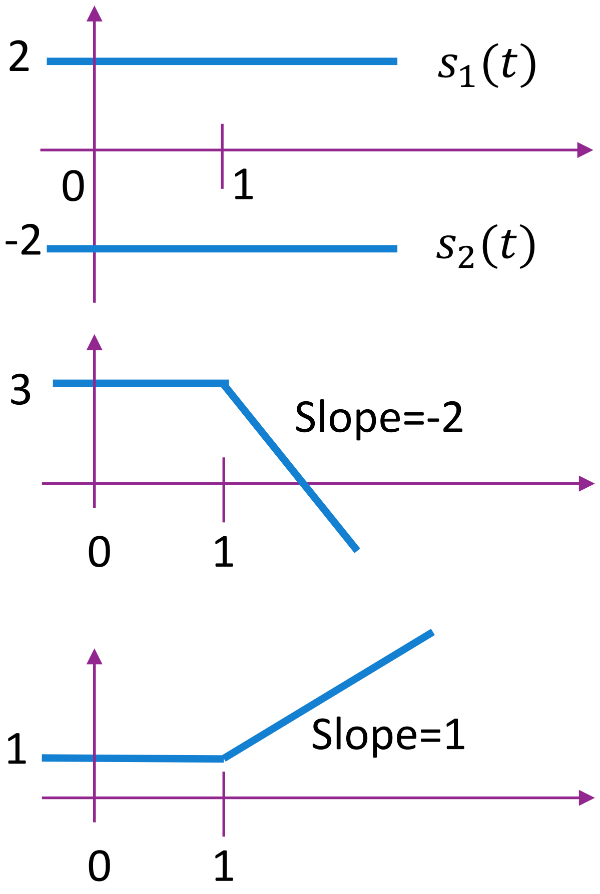


$s_3(t) = 3\phi_1(t) - 2\phi_2(t)$
$\boldsymbol{s}_3 = [3,-2]$



$s_4(t) = \phi_1(t) + \phi_2(t)$
$\boldsymbol{s}_4 = [1,1]$

Visualizing the Coordinate Vectors



$s_1(t) = 2\phi_1(t)$
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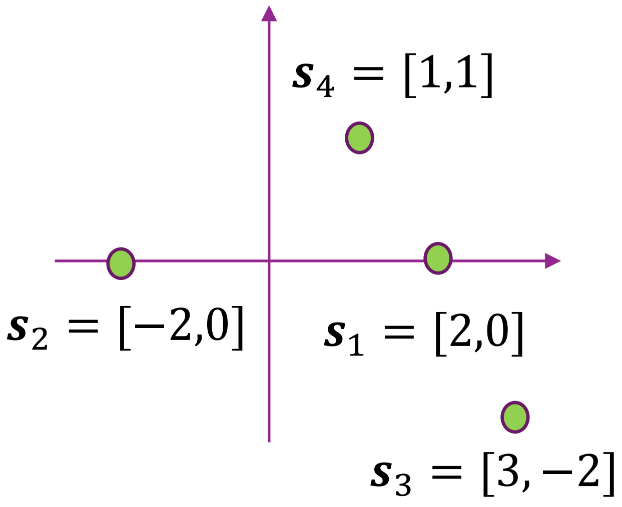
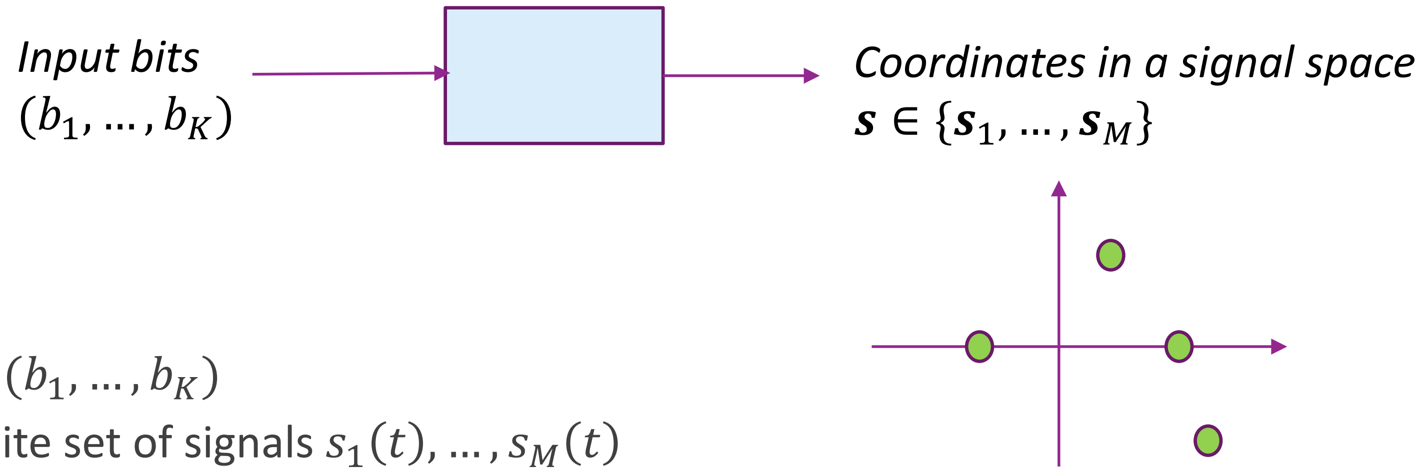
$s_2(t) = -2\phi_1(t)$
$\boldsymbol{s}_2 = [-2,0]$

$s_3(t) = 3\phi_1(t) - 2\phi_2(t)$
$\boldsymbol{s}_3 = [3,-2]$

$s_4(t) = \phi_1(t) + \phi_2(t)$
$\boldsymbol{s}_4 = [1,1]$



Signal Space View of Transmitter



- Signal space view
  - Input is a set of bits $(b_1, \dots, b_K)$
  - Outputs one of a finite set of signals $s_1(t), \dots, s_M(t)$
  - Each output has a coordinate vector $\boldsymbol{s}_1, \dots, \boldsymbol{s}_M \in \mathbb{F}^N$
- So, abstractly transmitter maps:

$$bits \mapsto vectors\ in\ \mathbb{F}^N$$

Signal Space for Pulse Shaping

- Suppose that we transmit N symbols via linear modulation:

$$u(t) = \sum_{n=0}^{N-1} s[n]p(t - nT)$$

- Take basis: $\phi_n(t) = p(t - nT), n = 0, \dots, N - 1$
- Then: $u(t) = \sum_{n=0}^{M-1} s[n]\phi_n(t)$
- So, $u(t)$ are contained in a signal space with N (complex) degrees of freedom
- Coordinate of $u(t)$ in basis are they symbols $[s[0], \dots, s[N - 1]]$

Example for 16-QAM

- Transmit: $u(t) = \sum_{n=0}^{M-1} s[n]p(t - nT)$
- Suppose each symbol is 16-QAM.
  - $s[n] = a[n] + ib[n],\ \ a[n], b[n] \in \{-3, -1, 1, 3\}$
  - $R_{mod} = 4$ bits per symbol
- Number of bits per signal: $K = R_{mod}N$
- Rate is $R = \frac{K}{N} = R_{mod}$ bits per DoF

Unique Decodability

- Suppose that we use linear modulation with functions $\phi_n(t)$

$$u(t) = \sum_{n=0}^{N-1} s[n]\phi_n(t)$$

- Definition: We say modulation is uniquely decodable if, for any $u(t)$, we can uniquely determine $s[n]$

- Uniquely decodable is the minimum we can expect of a modulation scheme:
  - If modulation is not uniquely decodable, there are at least two different $s[n]$ that give rise to $u(t)$
  - Hence, a receiver will not know which $s[n]$ was transmitted
  - So, there is no hope of estimating the correct symbols reliability without further information
  - …And we haven't even considered any impairments like noise

Unique Decodability and Degrees of Freedom

- Theorem: Consider linear modulation $u(t) = \sum_{n=0}^{N-1} s[n]\phi_n(t)$
  - Then, modulation is uniquely decodable if and only if $\phi_n(t)$ are linearly independent
- Proof: This is an immediate consequence of linear independence
- Implication:
  - Suppose we use linear modulation in a signal space where $u(t) \in V$
  - Suppose that V has N degrees of freedom
  - Then, we can transmit at most N symbols unique
- Caveats:
  - Even though DoF limits number of symbols, it does not limit number of bits
  - We can transmit multiple bits per symbol.

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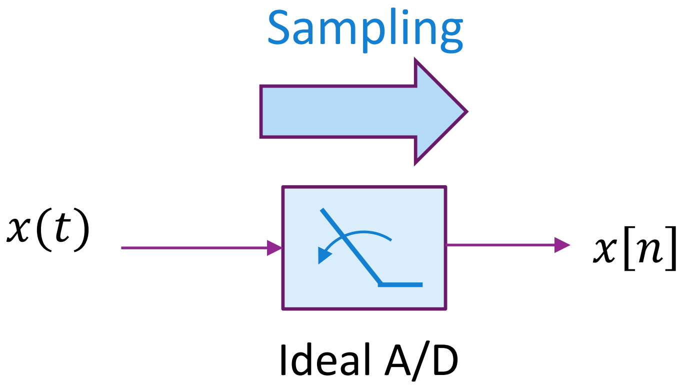
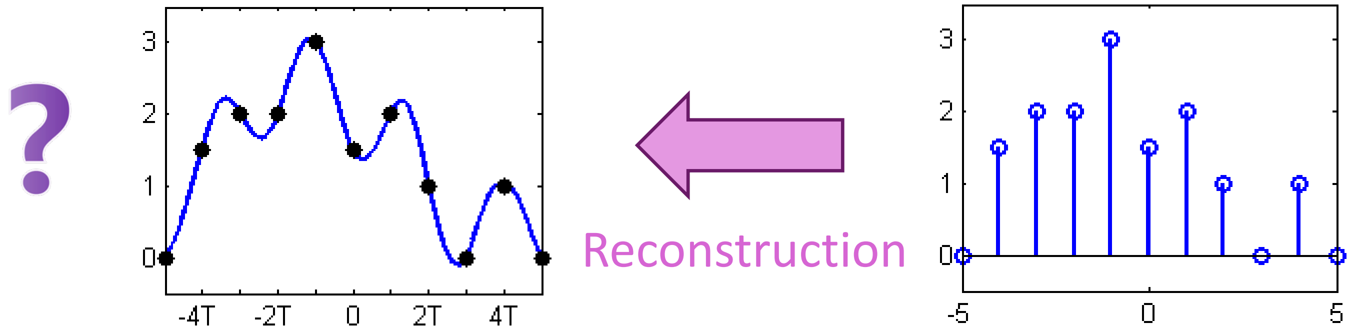
Reconstructing a Signal from Samples

- ❑ Let $x(t)$ be a continuous-time signal
- ❑ Suppose we are given samples $x[n] = x(nT)$
- ❑ Reconstruction problem:
  - Estimate $x(t)$ from sample $x[n]$
- ❑ A classic problem in signal processing





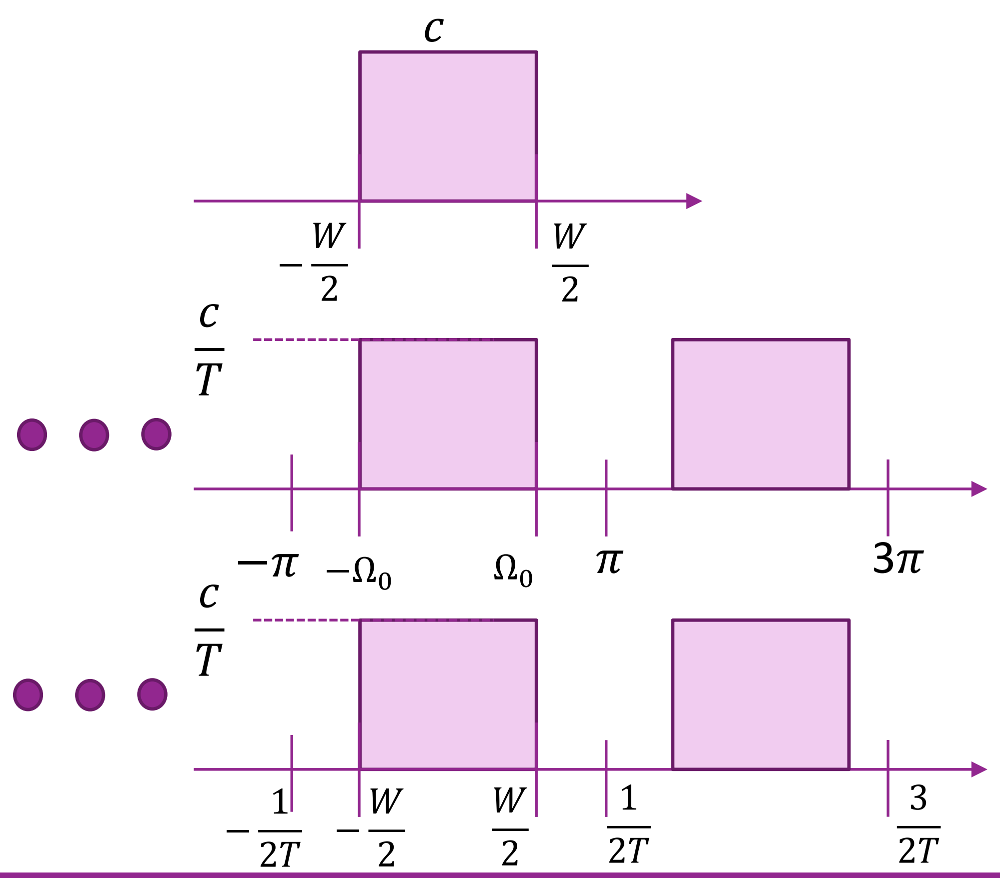
Nyquist Theorem

- Suppose $x(t)$ is band-limited: $X(f) = 0$ for $|f| > \frac{W}{2}$
  - $\frac{W}{2}$ is the single-sided bandwidth
- We are given samples $x_d[n] = x(nT)$
- Nyquist theorem: If $W < \frac{1}{T}$ then, we can reconstruct $x(t)$ by:

$$x(t) = \sum_{n=-\infty}^{\infty} x_d[n] \operatorname{sinc}\left(\frac{t - nT}{T}\right)$$

Nyquist Theorem Proof

- A simple picture proof in frequency-domain

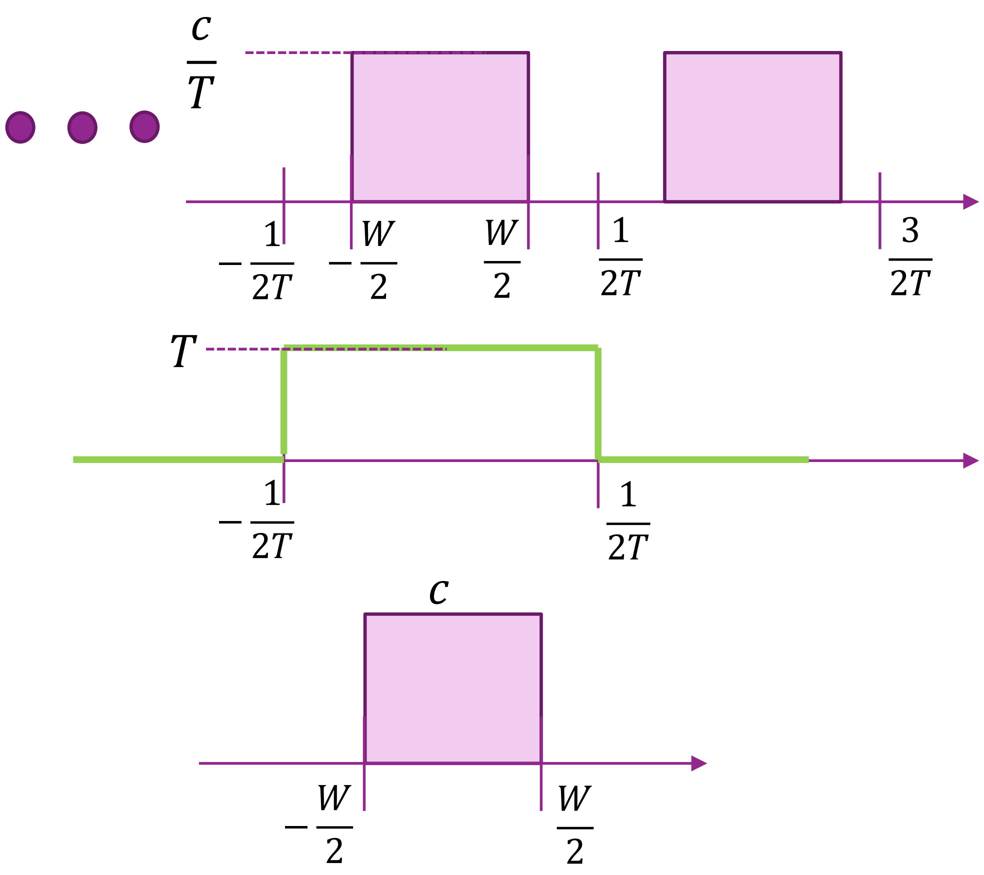


FT $X(f)$ of original bandlimited $x(t)$

DTFT $X_d(\Omega)$ of sampled signal $x_d[n] = x(nT)$
$f = \frac{W}{2}$ gets mapped to $\Omega_0 = \frac{WT}{2}(2\pi) < \pi$

FT of upsampled signal $x_\delta(t) = \sum_n x_d[n]\delta(t - nT)$

Nyquist Theorem Proof



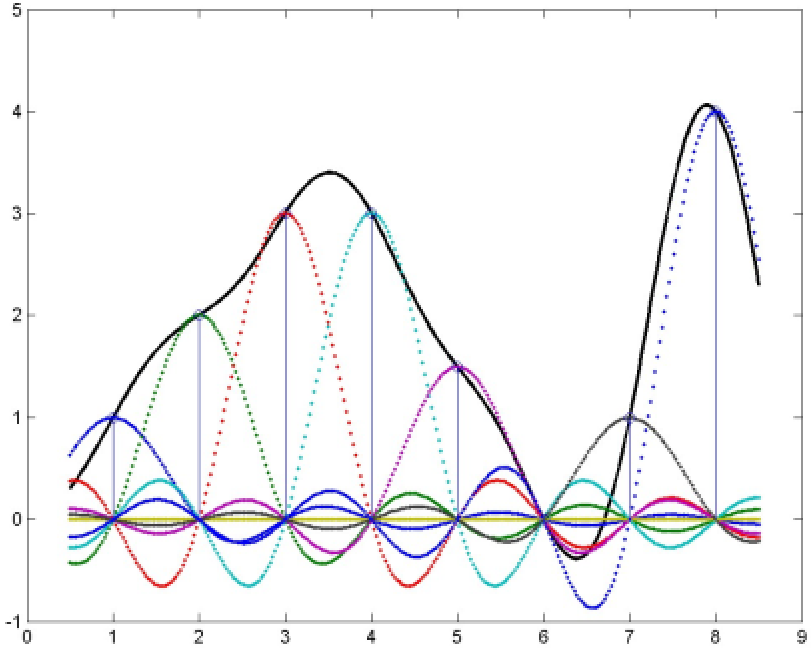
From previous slide
FT of upsampled signal $x_\delta(t) = \sum_n x_d[n]\delta(t - nT)$

Sinc filter $h(t) = sinc\left(\frac{t}{T}\right) \Rightarrow H(f) = TRect(fT)$

FT of filtered upsampled signal:

$$X(f) = H(f)X_\delta(f) \Rightarrow x(t) = \sum_n x_d[n] sinc\left(\frac{t - nT}{T}\right)$$

Sinc Interpolation Visualized



- Sinc interpolation:

$$x(t) = \sum_{n=-\infty}^{\infty} x_d[n] \operatorname{sinc}\left(\frac{t - nT}{T}\right)$$

- Around each sample $x_d[n] = x(nT)$
  - Take a sinc function
  - Shift to center at nT
  - Scale by $x(nT)$

DoF for Bandlimited Signals

- The following is an important consequence of Nyquist Theorem
- Theorem (Loose non-rigorous statement): Suppose that V is a signal space such that:
  - $s(t)$ is band-limited to $|f| < \frac{W}{2}$ for all $s(t) \in V$
  - $s(t) \approx 0$ for $|t| > \frac{L}{2}$ (i.e. time-limited to L seconds)

  Then V has at most $\approx LW$ degrees of freedom
- Thus, DoF≈Bandwidth $\times$ Time
- We say there are W DoF per second
- This is non-rigorous statement
  - Rigorous statement would properly bound $s(t) \approx 0$ for $|t| > \frac{L}{2}$

“Proof” of DoF Statement

- If $s(t) \in V$ then
  - $s(t)$ is band-limited to $|f| < \frac{W}{2}$; and
  - $s(t) \approx 0$ for $|t| > \frac{L}{2}$ (i.e. time-limited to L seconds)
- Since it is band-limited, $s(t) = \sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right) \operatorname{sinc}(Wt - n)$
- Since it is time-limited, $s\left(\frac{n}{W}\right) \approx 0$ for $\frac{|n|}{W} > \frac{L}{2}$
- Thus, $s(t) \approx \sum_{|n|<\frac{LW}{2}} s\left(\frac{n}{W}\right) \operatorname{sinc}(Wt - n)$
- This summation has only LW terms
- So all signal can be expressed as span of LW terms

Example

- Suppose you download data
  - Download time is L =10 seconds
  - Bandwidth is $W = 20$ MHz (a standard bandwidth for the largest single LTE channel)
  - Rate is 1.5 bits / DoF available
- How much do you download?
- Solution:
  - There are $N = LW = (10)(20)(10)^6 = 2(10)^8$ DoF
  - You can download $B = RN = (1.5)2(10)^8 = 300$ Mbits
  - This is 43.75 MB of data

Outline

- ❑Motivation for the signal space model
- ❑Vector spaces, bases and dimension
- ❑Signal spaces, rate and degrees of freedom
- ❑Nyquist Theorem and degrees of freedom in band-limited signals
- ❑Inner products and orthogonality
- ❑Orthonormal bases and energy per degree of freedom

Inner Product in $\mathbb{F}^n$

- Inner product in $\mathbb{F}^n$: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \coloneqq \sum_{i=1}^{n} \bar{x}_i y_i$
  - $\bar{x}_i$= complex conjugate of x_i
  - Note: Some texts use $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \coloneqq \sum_{i=1}^{n} x_i \bar{y}_i$
- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are column vectors, then

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = [\bar{x}_1, \cdots, \bar{x}_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{x}^* \boldsymbol{y}$$

- Examples:
  - $\boldsymbol{x} = [1,2]$ and $\boldsymbol{y} = [3,4] \Rightarrow \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 1(3) + 2(4) = 11$
  - $\boldsymbol{x} = [1 + 2i, 3 + 4i]$ and $\boldsymbol{y} = [5, 6 + 7i]$
    $\Rightarrow \langle \boldsymbol{x}, \boldsymbol{y} \rangle = (1 - 2i)5 + (3 - 4i)(6 + 7i) = 5 - 10i + 18 + 28 - 24i + 21i$

Inner Product on a General Vector Space

- Suppose V is a vector space over $\mathbb{F}$
- Definition: An inner product on V is a function $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \mapsto \mathbb{F}$ such that:
  - Conjugate symmetry: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}$
  - Linearity in the second argument: $\langle \boldsymbol{x}, \alpha_1 \boldsymbol{y}_1 + \alpha_2 \boldsymbol{y}_2 \rangle = \alpha_1 \langle \boldsymbol{x}, \boldsymbol{y}_1 \rangle + \alpha_2 \langle \boldsymbol{x}, \boldsymbol{y}_2 \rangle$
  - Positive definiteness: $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$ and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \Leftrightarrow \boldsymbol{x} = 0$
- A vector space with an inner product is called an inner product space
- Regular inner product in $\mathbb{F}^n$ satisfies the above properties
- Derived property: Conjugate linearity in the first argument:

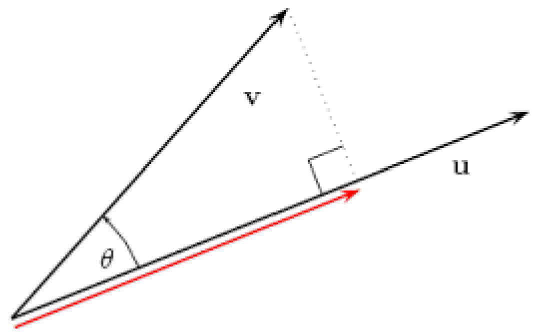
$$\langle \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 \rangle}$$
$$= \bar{\alpha}_1 \overline{\langle \boldsymbol{y}, \boldsymbol{x}_1 \rangle} + \bar{\alpha}_2 \overline{\langle \boldsymbol{y}, \boldsymbol{x}_2 \rangle} = \bar{\alpha}_1 \langle \boldsymbol{x}_1, \boldsymbol{y} \rangle + \bar{\alpha}_2 \langle \boldsymbol{x}_2, \boldsymbol{y} \rangle$$

Inner Product Geometry

- Any inner product induces a norm: $\|x\| = \sqrt{\langle x, x\rangle}$
  - Note $\langle x, x\rangle \geq 0$ so the square root can be taken
  - Easy to verify that it is satisfies the required norm properties
- For $x \in \mathbb{F}^n$, this is the standard 2-norm: $\|x\|^2 = \langle x, x\rangle = \sum_i |x_i|^2$
- Triangle inequality: $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$
- Cauchy-Schwartz inequality:

$$|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|$$

- Write: $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \|\boldsymbol{x}\|\|\boldsymbol{y}\|\cos(\theta)$
  - θ =angle between vectors
  - Represents “similarity” between vectors
  - For real vetors, $\mathbf{x}^{\mathrm{T}}\mathbf{y} = \langle \boldsymbol{x}, \boldsymbol{y}\rangle = \|\boldsymbol{x}\|\|\boldsymbol{y}\|\cos(\theta)$



Inner Products on Functions

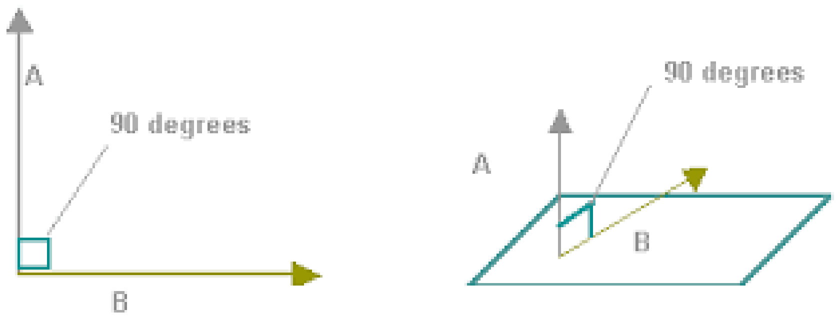
- Let $f(t), g(t)$ be two functions on $[a, b]$
  - Possibly complex-valued
- Define inner product: $\langle f, g\rangle = \int f(t)^* g(t) dt$
  - Same as inner product in $\mathbb{F}^n$ with sum replaced by integral
- Norm squared is the energy:
  - $\mathcal{E} = \|f\|^2 = \int |f(t)|^2 dt$ Signal energy
- Parseval's theorem
  - $\langle x(t), y(t)\rangle = \langle X(f), Y(f)\rangle$
  - $E_x = \langle x(t), y(t)\rangle = \langle X(f), Y(f)\rangle$
- The space $L_2[a, b]$ = set of $f(t)$ with finite energy $\|f\|^2$

Example Problem

- Let $f_i(t) = e^{-\alpha_i t},\ \ t \geq 0, i = 1,2, \ldots$
  - Assume $\alpha_i \geq 0$
- Find $\langle f_i, f_j \rangle$
- Find $||f_i||$
- Solution:
  - $\langle f_i, f_j \rangle = \int_0^\infty f_i(t) f_j(t) dt = \int_0^\infty e^{-(\alpha_1+\alpha_2)t} dt = \frac{1}{\alpha_1+\alpha_2}$
  - $||f_i||^2 = \langle f_i, f_i \rangle = \frac{1}{2\alpha_i}$
  - $||f_i|| = \frac{1}{\sqrt{2\alpha_i}}$

Orthogonality

- Orthogonal vectors: $\langle x, y \rangle = 0$
  - Pythagoras Theorem: $\|x+y\|^2 = \|x\|^2 + \|y\|^2$
  - Write $\boldsymbol{x} \perp \boldsymbol{y}$.
- For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$:
  - $\boldsymbol{x} \perp \boldsymbol{y} \Leftrightarrow \boldsymbol{x}, \boldsymbol{y}$ are right angles to one another
- If $f, g \in L_2[a, b]$, the signals are orthogonal if:
  - $\langle f, g \rangle = \int f(t)^* g(t) dt = 0$



- A set of vectors $x_1, x_2, \ldots$ is orthonormal if:
  - Pairwise orthogonal: $x_i \perp x_j$ when $i \neq j$
  - Unit energy: $\|x_i\| = 1$
  - Could be a finite or infinite set

Examples of Orthogonal Vectors and Signals

- In $V = \mathbb{R}^2$
  - $\boldsymbol{x} = [1,3]$, $\boldsymbol{y} = [-6,2]$
  - $\boldsymbol{x}^*\boldsymbol{y} = (1)(-6) + 3(2) = 0$
- In $L_2[a,b]$: Any two signals $f(t), g(t)$ that have non-overlapping support
- Complex exponentials: $f_k(t) = e^{2\pi jkt/T}$ for $t \in [0, T]$
- Sinc functions: $\phi_k(t) = \frac{1}{\sqrt{T}} sinc\left(\frac{t-nT}{T}\right)$
  - We showed these are orthogonal last unit

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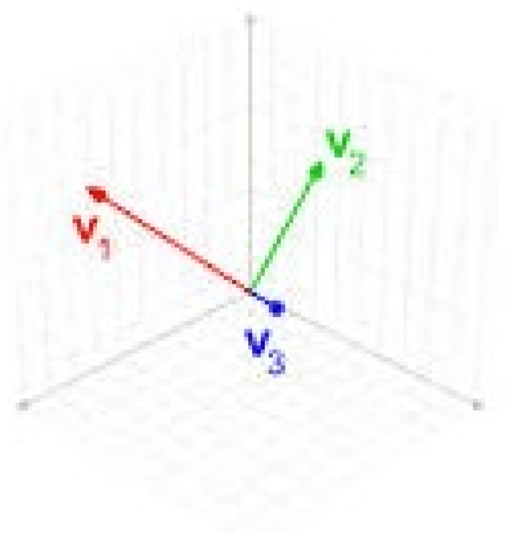
Orthonormal Basis

- Let V be any vector space
- Definition: A set $\boldsymbol{v}_1, \boldsymbol{v}_2 \ldots$ is orthonormal if:
  - $\|\boldsymbol{v}\| = 1$ for all n (all vectors have unit energy)
  - $\langle \boldsymbol{v}_n, \boldsymbol{v}_m \rangle = 0$ for all $n \neq m$ (different vectors are orthogonal)
- Orthonormal $\Rightarrow$linearly independent.
  - Why? Suppose that $\sum_i \alpha_i \boldsymbol{v}_i = 0$
  - Multiplying by $\boldsymbol{v}_j^*$ we get $0 = \sum_i \alpha_i \boldsymbol{v}_j^* \boldsymbol{v}_i = \alpha_j$ for all j (since $\boldsymbol{v}_j^* \boldsymbol{v}_i = \delta_i$
- Consequence: if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N \in \mathbb{F}^N$ are orthonormal, then they are a basis for $\mathbb{F}^N$
  - Called an orthonormal basis



Example

- Show the vectors are orthonormal:

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{11}}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \frac{1}{\sqrt{66}}\begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$$

- Solution:
  - $\boldsymbol{v}_1^T \boldsymbol{v}_2 = \frac{1}{\sqrt{66}}(-3 + 2 + 1) = 0$
  - $\boldsymbol{v}_1^T \boldsymbol{v}_3 = \frac{1}{\sqrt{66(11)}}(-3 - 4 + 7) = 0$
  - $\boldsymbol{v}_2^T \boldsymbol{v}_3 = \frac{1}{\sqrt{66(6)}}(1 - 8 + 7) = 0$
  - $\|\boldsymbol{v}_1\|^2 = \frac{1}{11}(9 + 1 + 1) = 1, \quad \|\boldsymbol{v}_2\|^2 = \frac{1}{6}(1 + 4 + 1) = 1,$
  - $\|\boldsymbol{v}_3\|^2 = \frac{1}{66}(1 + 16 + 49) = 1,$

Unitary Matrices

- A matrix $\boldsymbol{U} \in \mathbb{C}^{N\times N}$ is unitary if $\boldsymbol{U}^*\boldsymbol{U} = I$
  - $\boldsymbol{U}$ called orthogonal if it is real-valued (i.e. $\boldsymbol{U} \in \mathbb{R}^{N\times N}$ and $\boldsymbol{U}^T\boldsymbol{U} = I$)

- $\boldsymbol{U}$ is unitary/orthogonal if and only if one of the following equivalent properties are true:
  - $U^* = U^{-1}$
  - The rows of U are an orthonormal set
  - The columns of U are an orthonormal set
  - $\|Ux\|^2 = \|x\|^2$ for all $x \in \mathbb{F}^N$
  - $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{F}^N$

Proof of the Unitary Matrix Properties

- U unitary $\Leftrightarrow U^{-1} = U^*$: Follows from the definition $UU^* = U^*U = I$
- U unitary $\Leftrightarrow$ columns of U are orthonormal
  - Write $U = [u_1, \dots, u_N]$ where u_i is the i-th column
  - Then: $(U^*U)_{ij} = u_i^* u_j$
  - Since $U^*U = I,\ u_i^* u_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$
  - Hence:$u_i^* u_j = 0$ for $i \neq j$ and $\|u_i\|^2 = u_i^* u_i = 1$ for all i
- U unitary $\Leftrightarrow$ row of U are orthonormal. Same proof as previous but use $UU^* = I$
- U unitary $\Rightarrow \|Ux\|^2 = \|x\|^2$: $\|Ux\|^2 = x^* U^* U x = x^* x = \|x\|^2$

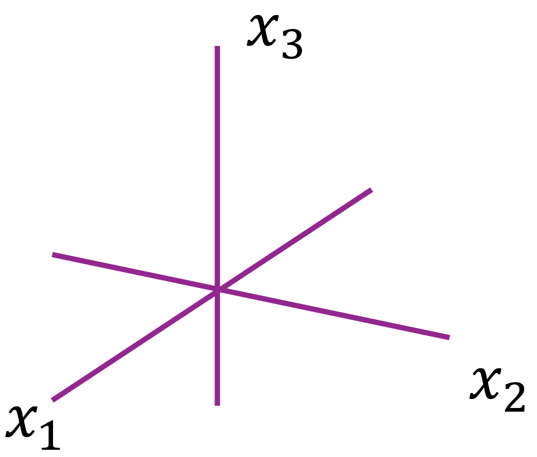
Coefficients in an Orthonormal Basis

- Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N \in \mathbb{F}^N$ be an orthonormal basis
- Theorem: Given any $\boldsymbol{x} \in \mathbb{F}^N$, $\boldsymbol{x} = \sum_j z_j \boldsymbol{v}_j$ where $z_j = \langle \boldsymbol{v}_j, \boldsymbol{x} \rangle$
- Consequence: Find coefficients of x in an orthonormal basis is easy
  - Just take inner products $z_j = \langle \boldsymbol{v}_j, \boldsymbol{x} \rangle$.
  - NO matrix inverse
- Proof of theorem
  - Since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ is orthonormal, $\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N]$ is unitary
  - Since $z_j = \langle \boldsymbol{v}_j, \boldsymbol{x} \rangle$ we can write $\boldsymbol{z} = \boldsymbol{V}^* \boldsymbol{x}$
  - But then $\sum_j z_j \boldsymbol{v}_j = \boldsymbol{V}\boldsymbol{z} = \boldsymbol{V}\boldsymbol{V}^* \boldsymbol{x} = \boldsymbol{x}$

Orthogonal Matrix Transformations

Coefficient in standard basis

$$\boldsymbol{x} = [x_1, \dots, x_N]$$

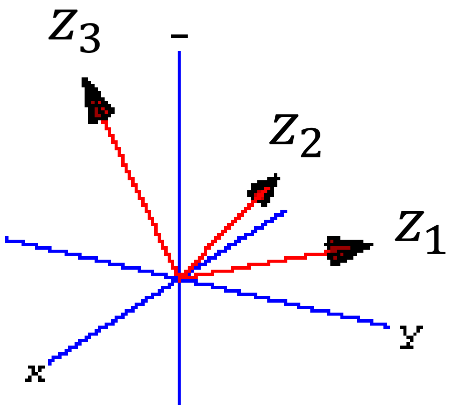


$$\boldsymbol{z} = \boldsymbol{V}^* \boldsymbol{x}$$

$$\boldsymbol{x} = \boldsymbol{V} \boldsymbol{z}$$

Coefficients in orthonormal basis

$$\boldsymbol{z} = [z_1, \dots, z_N]$$



Example

- Consider the orthonormal set from before:

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{11}}\begin{bmatrix}3\\1\\1\end{bmatrix},\ \boldsymbol{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix},\ \boldsymbol{v}_3 = \frac{1}{\sqrt{66}}\begin{bmatrix}-1\\-4\\7\end{bmatrix}$$

- Let $\boldsymbol{x} = [0,1,2]^T$. Find the coefficient expansion in the orthonormal bases
- Solution:
  - $z_1 = \boldsymbol{v}_1^*\boldsymbol{x} = \frac{1}{\sqrt{11}}(1+2) = \frac{3}{\sqrt{11}}$
  - $z_2 = \boldsymbol{v}_2^*\boldsymbol{x} = \frac{1}{\sqrt{6}}(2+2) = \frac{4}{\sqrt{6}}$
  - $z_3 = \boldsymbol{v}_2^*\boldsymbol{x} = \frac{1}{\sqrt{66}}(-4+14) = \frac{10}{\sqrt{66}}$
  - Then $\boldsymbol{x} = z_1\boldsymbol{v}_1 + z_2\boldsymbol{v}_2 + z_3\boldsymbol{v}_3$

Finding an orthonormal basis: Gram Schmidt

- Let $\boldsymbol{v}_1, .., \boldsymbol{v}_K$ be linearly independent
- There exists an orthonormal set $\boldsymbol{u}_1, .., \boldsymbol{u}_K$ such that

$$span\{\boldsymbol{v}_1, .., \boldsymbol{v}_K\} = span\{\boldsymbol{u}_1, .., \boldsymbol{u}_K\}$$

- Conclusion: Every subspace has an orthonormal basis
- Gram-Schmidt Procedure for finding $\boldsymbol{u}_1, .., \boldsymbol{u}_K$
  - $\boldsymbol{u}_1 = \boldsymbol{v}_1 / \|\boldsymbol{v}_1\|$
  - $\boldsymbol{w}_2 = \boldsymbol{v}_2 - \langle \boldsymbol{u}_1, \boldsymbol{v}_2 \rangle \boldsymbol{u}_1, \boldsymbol{u}_2 = \boldsymbol{w}_2 / \|\boldsymbol{w}_2\|$
  - …
  - $\boldsymbol{w}_k = \boldsymbol{v}_k - \sum_{i=1}^{k-1} \langle \boldsymbol{u}_i, \boldsymbol{v}_k \rangle \boldsymbol{u}_i\,,\ \boldsymbol{u}_k = \boldsymbol{w}_k / \|\boldsymbol{w}_k\|$
  - …

Gram Schmidt Example

- $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
- $\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
- $\boldsymbol{w}_2 = \boldsymbol{v}_2 - \langle \boldsymbol{u}_1, \boldsymbol{v}_1 \rangle \boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{1}{10}(6+2) \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 \\ 6 \end{bmatrix}$
- $\boldsymbol{u}_2 = \frac{1}{\sqrt{4+36}} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \frac{1}{\sqrt{40}} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

QR Factorization

- Gram-Schmidt is mostly done on a computer
- Use the QR factorization.
- Given matrix $A = [a_1, \ldots, a_K]$, $a_i \in \mathbb{F}^N$
  - Columns are the vectors that we want to apply the GS procedure to
- QR factorization: $A = QR$
  - $Q = [q_1, \ldots, q_K] \in \mathbb{F}^{N \times K}$, q_i orthonormal set from the GS procedure
  - $R =$ lower triangular matrix has coefficients of vectors $\boldsymbol{a}_j$ in terms of $\boldsymbol{q}_i$:

$$\boldsymbol{a}_j = \sum_{i=1}^{j} \boldsymbol{q}_i R_{ij}$$

- LU is another factorization

Orthonormal Basis of Functions

- Up to now, we have defined orthonormal bases for finite-dimensional spaces
- Let $\phi_k(t), k = 0,1, \ldots$ be a set of functions in $L_2(a, b)$
  - Could be indexed from $k = \cdots, -1,0,1, \ldots$ also
  - (a, b) could be a bounded or infinite interval
- Definition: We say $\{\phi_k(t)\}$ is an orthonormal basis if:
  - $\phi_k(t), k = 0,1, \ldots$ is an orthonormal set
  - For any $f(t) \in L_2(a, b)$

$$\lim_{k \to \infty} \left\| f - \sum_{j=0}^{k} \langle \phi_j, f \rangle \phi_j \right\| = 0$$

- This means the approximation $\mathrm{f(t)} \approx \sum_{j=0}^{k} \alpha_j \phi_j(t)$, $\alpha_j = \langle \phi_j, f \rangle$ is asymptotically exact

Common Orthonormal Sets and Bases

- Set of complex exponentials $\phi_k(t) = \frac{1}{\sqrt{T}} \mathrm{e}^{2\pi jkt/T}$
  - This is a basis of $L_2[0, T]$
- Set of sinc's: $\phi_k(t) = \frac{1}{\sqrt{T}} Sinc\left(\frac{t-nT}{T}\right)$
  - This is a basis of the set of signals band-limited to $|f| < \frac{1}{2T}$
- Set of rectangles: $f_k(t) = \frac{1}{\sqrt{T}} Rect\left(\frac{t-nT}{T}\right)$
  - This is an orthonormal set but not a basis

Energy and an Orthonormal Basis

- Suppose V is a signal space with N DoFs
- We can find an orthonormal basis $\phi_1(t), \ldots, \phi_N(t)$
- Given any $s(t) \in V$, we can write $s(t) = \sum_{j=1}^{N} z_j \phi_j(t)$, $z_j = \langle \phi_j, s \rangle$
- Signal energy is $\|s\|^2 = \int |s(t)|^2 dt$
- Parseval's Theorem: $\|s\|^2 = \|\boldsymbol{z}\|^2 = \sum_{j=1}^{N} \left|z_j\right|^2$
- Proof: On board
- In an orthonormal basis, we can compute energy in the coefficients
  - Call $\left|z_j\right|^2$ the energy in the j th degree of freedom

Example Problem

- Suppose that $s(t)$ has a coefficient representation $s(t) = \sum z_j \phi_j(t)$ in an o.n. basis
- Suppose coefficients are $\boldsymbol{z} = [3 + i, 2 - i, 1]\sqrt{A}$
- What is A if the signal energy is -150 dBmJ?
- Solution:
  - Signal energy is $\|s\|^2 = \int |s(t)|^2 dt = 10^{-15}$ mJ
  - By Parseval's theorem $\|s\|^2 = \|\boldsymbol{z}\|^2 = \sum_{j=1}^{N} \left|z_j\right|^2 = A[(9 + 1) + (4 + 1) + 1] = A(16)$
  - Hence $A = \frac{10^{-15}}{16}$ mJ / DoF