Problems: Probability and Random Processes

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1. *Functions of random variables.* A common model for path loss between a transmitter and receiver is:

$$L = 32.4 + 17.3\log_{10}(d) + 20\log_{10}(f_c) + \xi, \quad \xi \sim \mathcal{N}(0, \sigma^2) \text{ [dB]},$$

   where d is the distance in meters, f_c is the carrier frequency in GHz, and $\sigma = 3.0$ dB. Let $G = 10^{-0.1L}$ be the channel power gain in linear scale.

   (a) What is $\mathbb{E}(L)$, the average path loss in dB?

   (b) What is $-10\log_{10}(\mathbb{E}(G))$, the path loss, averaged in linear scale?

2. *Joint density.* Consider the joint density,

$$p(x, y) = xe^{-x(y+1)}, \quad x, y \geq 0.$$

   (a) Find the marginal density $f(x)$.

   (b) Find the conditional density $f(y|x)$?

   (c) Find the conditional expectation $\mathbb{E}(Y|X = x)$. (Hint: You could use integration by parts, or you may use that $\int_0^\infty x^k e^{-ax}dx = k!/a^{k+1}$ holds for any $a > 0$.)

   (d) Write a few lines of MATLAB code to generate 1000 samples of X and Y from the given PDF.

3. *Conditional densities.* Suppose that a receiver sees a random number K of transmitters. Given K, the received signal is,

$$Y \sim \mathcal{C}N(0, KE_1 + E_0),$$

   where E_0 is the average received energy when there are no transmitters and E_1 is the additional average received energy per transmitter. Suppose that K is Poisson distributed with mean $\bar{K} > 0$.

   (a) What is $P(|Y| \geq t|K = k)$ as a function of t?

   (b) What is $P(|Y| \geq t)$ as a function of t?

   (c) Find $E(|Y|^2)$.

4. *Random processes.* Let $x[n] \in \{0, 1\}$ where $x[n]$ transitions from 0 to 1 with probability p and 1 to 0 with probability q in each time step. Suppose that $x[n]$ is in the stationary distribution.

   (a) Find the $\mu = \mathbb{E}(x[n])$, the stationary mean.

(b) Find the $R[m] = \mathbb{E}(x[n]x[n-m])$, the autocorrelation function.

5. *Phase noise.* Phase noise is a key challenge in receivers, particularly at higher frequencies. One simple model for phase noise is as follows: Phase noise will transform a signal $x(t)$ to a noisy signal $y(t)$ by the relation:

$$y(t) = x(t)e^{i\theta(t)}, \quad \theta(t) = \int_{-\infty}^{t} w(u)du,$$

where $w(u)$ is real-valued WSS Gaussian noise with

$$R_w(\tau) = \sigma^2\delta(\tau), \quad \sigma^2 = 2\pi f_0 10^{-0.1L},$$

where L is the phase noise level in dBc/Hz and f_0 is a reference frequency. Thus, phase noise acts as time-varying phase rotation on the signal.

(a) What is the distribution of $\theta(t) - \theta(s)$ as a function of s, t and σ^2.

(b) What is the autocorrelation function $R_v(\tau)$ of the multiplicative noise factor $v(t) = e^{i\theta(t)}$? You may look and use the *characteristic function* of a Gaussian.

(c) Use MATLAB to plot the auto-correlation function $R_v(\tau)$ for a phase noise of $L = -80$ dBc/Hz at $f_0 = 10$ kHz. At what time τ is $R_v(\tau) = 0.5R_v(0)$?