MODELING AFFIX GENERATION IN PATH OF EXILE

OWEN VISSER

Abstract. In this paper I construct a statistical model of the Path of Exile's item system to analyze the probability of acquiring desirable items through three major crafting methods; alt-regal, alch-scour, and essence-scour. The full equations that the model puts forth are far to large to be included. However, their foundation and form are explained thoroughly. Time and cost analysis aside, the essence-scour method is the most probable method, following shortly behind, the alch-scour method, and finally the alt-regal method is the least likely to occur.

1. Introduction

Many of our hobbies have deep mathematical roots. Whether it be card games or rolling dice, mathematicians have modeled almost everything you could imagine. However, some online games that have been made in recent years have not been modeled yet. After my four years at the University of North Florida, I've come to understand mathematics far more than I did when I first started playing my favorite game, Path of Exile(PoE). PoE is an Action Role-Playing Game, where a player is pushed to progressively defeat harder content. A character in PoE is not fully defined by the player or whatever abilities the player has chosen, but by the items the character has. A large part of the game is based around acquiring these items. These items are crafted, found, bought, etc. The goal of my project is to model the item system in PoE so that I can optimize my game-play.

After my four year Applied Mathematics degree and personal studies in the field of statistics I've come to see the mathematical fundamentals behind the item system. To begin analyzing items as a whole, we must understand their structure.

An item in PoE consists of modifiers that give bonuses to a characters survivability and power. The amount of modifiers that an item has to choose from varies from the type of item(e.g.Gloves, Boots, Helmet, etc.), and the monster-level of the zone the item was acquired from(level 100 being the highest). These items have modifiers that determine how powerful the items are. A powerful item is what we will consider desirable. The majority of our discussion will be focused on these desirable items and the chance of obtaining them.

2. Items

2.1. Definition of an Item.
An item can be in one of three states; *normal, magic,* or *rare.* Normal items are items without any affixes. A normal item can be upgraded into a magic item, or a rare item. Magic items have up to one prefix and one suffix, then can be upgraded into rare items, or downgraded back to normal items. Rare items have up to three prefixes and three suffixes. Rare items cannot be upgraded further, and can only be downgraded to normal items. [1]

Definition 2.1. Some of the rules associated with items are as follows:

(1) If a suffix exists on an item, it has no effect on a prefix of an item.
(2) If a prefix exists on an item, it has no effect on a suffix of an item.
(3) An item may not have the same two prefixes or suffixes.

To describe an item within the bounds of these rules we will use the notation of rolls (R) and of items that contain rolls, $I\{R_1, R_2, R_3, R_4, R_5, R_6, \}$. The way items are rolled is inherently conditional, because of this we have to enumerate them so that we know which roll comes first. Where R_1 is the first roll on an item, R_2 the second, and so on.

2.2. Affixes: Prefixes and Suffixes.
Let us begin by defining the discrete sets, A and B, as the pool of all possible affixes. We will then call A the set of all possible prefixes of any item, and B the set of all possible suffixes of any item, such that

$$A = \{a_1, a_2, ..., a_n\}$$
$$B = \{b_1, b_2, ..., b_m\}.$$

Due to the nature of the game, not all affixes are desirable. Hence, for some $1 \leqslant k \leqslant n-1$ and some $1 \leqslant l \leqslant m-1$ we define the sets of all desirable prefixes and suffixes by

$$A_G = \{a_{k+1}, a_{k+2}, ..., a_n\}$$
$$B_G = \{b_{l+1}, b_{l+2}, ..., b_m\}.$$

This implies that depending on the item's type and level, an item will have $n-k$ desirable, k non-desirable prefixes, and $m-l$ desirable, l non-desirable suffixes.

2.3. Ranges.
Each prefix and suffix that can be rolled has a *range.* Each range has a starting and ending value. Imagine a prefix that has values from $\{(1 \to 100)\}$, this prefix may have packets of values, $\{(1 \to 11), (12 \to 25), ..., (91 \to 100)\}$. These packets are called *ranges.*

The amount of ranges that a prefix or suffix may have is solely based on the level of the item. A level one item may only have one range within a prefix, where as a level one-hundred item may have α many ranges within a prefix. For example, the prefix a_1 is defined as the set of possible ranges that the prefix a_1 may take,

$$a_1 = \{a_{1,1}, a_{1,2}, .., a_{1,\alpha}\}.$$

Where a_1 signifies the first prefix, and $a_{1,1}$ signifies the first prefix in its first range. Hence, for prefixes, we can write,

$$\begin{aligned} A &= \{(a_{1,1}, ..., a_{1,\alpha}), ..., (a_{n,1}, ..., a_{n,\alpha})\} \\ A_G &= \{(a_{k+1,1}, ..., a_{k+1,\alpha}), ..., (a_{n,1}, ..., a_{n,\alpha})\}, 1 \leqslant k \leqslant n-1. \end{aligned}$$

Similarly for suffixes,

$$\begin{aligned} B &= \{(b_{1,1}, ..., b_{1,\beta}), ..., (b_{m,1}, ..., b_{m,\beta})\} \\ B_G &= \{(b_{l+1,1}, ..., b_{l+1,\beta}), ..., (b_{m,1}, ..., b_{m,\beta})\}, 1 \leqslant l \leqslant m-1. \end{aligned}$$

Like our affixes, these ranges have desirable and non-desirable values. For a max-level item, say a_1, with ranges $a_{1,1}, ..., a_{1,\alpha}$, we would define the desirable ranges of a_1 as $a_{1,\zeta+1}, ..., a_{1,\alpha}$. Where as the non-desirable ranges of a_1 are defined as $a_{1,1}, ..., a_{1,\zeta}$, where $1 \leqslant \zeta \leqslant \alpha - 1$. This process also applies to the suffixes as well, which brings us to our last group of sets:

$$\begin{aligned} Z &= \{(a_{k+1,\zeta+1}, ..., a_{k+1,\alpha}), ..., (a_{n,\zeta+1}, ..., a_{n,\alpha})\}, 1 \leqslant \zeta \leqslant \alpha - 1 \\ Z^c &= \{(a_{k+1,1}, ..., a_{k+1,\zeta}), ..., (a_{n,1}, ..., a_{n,\zeta})\} \\ H &= \{(b_{l+1,\eta+1}, ..., b_{l+1,\beta}), ..., (b_{m,\eta+1}, ..., b_{m,\beta})\}, 1 \leqslant \eta \leqslant \beta - 1 \\ H^c &= \{(b_{l+1,1}, ..., b_{l+1,\eta}), ..., (b_{m,1}, ..., b_{m,\eta})\}, \end{aligned}$$

where Z is the set of all desirable prefixes, $k+1 \to n$, with all desirable ranges, $\zeta + 1 \to \alpha$, and Z^c is all desirable prefixes, $k+1 \to n$, with all non-desirable ranges, $1 \to \zeta$. Similarly, H is the set of all desirable suffixes, $l+1 \to m$, with all desirable ranges, $\eta + 1 \to \beta$. H^c is all desirable suffixes, $l+1 \to m$, with all non-desirable ranges, $1 \to \eta$.

Because we do not care about the set of non-desirable affixes, with desirable ranges, those sets will not be given any definition.

2.4. **Matrix Notation.** Before we continue, rewriting A and B in the form of matrices will help us simplify the definition of items and form equations. So, we can rewrite A and B as,

$$A = \{(a_{1,1}, ..., a_{1,\alpha}), ..., (a_{n,1}, ..., a_{n,\alpha})\} \equiv \begin{bmatrix} a_{1,1} & a_{2,1} & \dots & a_{n,1} \\ a_{1,2} & a_{2,2} & \dots & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,\alpha} & a_{2,\alpha} & \dots & a_{n,\alpha} \end{bmatrix}$$

and,

$$B = \{(b_{1,1}, ..., b_{1,\beta}), ..., (b_{m,1}, ..., b_{m,\beta})\} \equiv \begin{bmatrix} b_{1,1} & b_{2,1} & \dots & b_{m,1} \\ b_{1,2} & b_{2,2} & \dots & b_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1,\beta} & b_{2,\beta} & \dots & b_{m,\beta} \end{bmatrix}.$$

Taking a closer look at the form of A and B,

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{k+1,1} & \dots & a_{n,1} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{1,\zeta+1} & \dots & a_{k+1,\zeta+1} & \dots & a_{n,\zeta+1} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{1,\alpha} & \dots & a_{k+1,\alpha} & \dots & a_{n,\alpha} \end{bmatrix},$$

$$B = \begin{bmatrix} b_{1,1} & \dots & b_{l+1,1} & \dots & b_{m,1} \\ \vdots & \ddots & \vdots & & \vdots \\ b_{1,\eta+1} & \dots & b_{l+1,\eta+1} & \dots & b_{m,\eta+1} \\ \vdots & & \vdots & \ddots & \vdots \\ b_{1,\beta} & \dots & b_{l+1,\beta} & \dots & b_{m,\beta} \end{bmatrix}, \tag{1}$$

we can see A_G, B_G, Z, and H within the A, B, matrices

$$A_G = \begin{bmatrix} a_{k+1,1} & \cdots & a_{n,1} \\ \vdots & \ddots & \vdots \\ a_{k+1,\zeta+1} & \cdots & a_{n,\zeta+1} \\ \vdots & \ddots & \vdots \\ a_{k+1,\alpha} & \cdots & a_{n,\alpha} \end{bmatrix},$$

$$Z = \begin{bmatrix} a_{k+1,\zeta+1} & \cdots & a_{n,\zeta+1} \\ \vdots & \ddots & \vdots \\ a_{k+1,\alpha} & \cdots & a_{n,\alpha} \end{bmatrix},$$

$$B_G = \begin{bmatrix} b_{l+1,1} & \cdots & b_{m,1} \\ \vdots & \ddots & \vdots \\ b_{l+1,\eta+1} & \cdots & b_{m,\eta+1} \\ \vdots & \ddots & \vdots \\ b_{l+1,\beta} & \cdots & b_{m,\beta} \end{bmatrix},$$

and,

$$H = \begin{bmatrix} b_{l+1,\eta+1} & \cdots & b_{m,\eta+1} \\ \vdots & \ddots & \vdots \\ b_{l+1,\beta} & \cdots & b_{m,\beta} \end{bmatrix}.$$

From this, it is important to conclude that the matrix H is within the matrix B. To create H from B we have to omit $m - \eta$ rows, and $\beta - l$ columns from B. This can also be done with Z and A, and all remaining submatrices

It is important to notice that not all affixes share the same number of ranges. Some item may have a small number of large ranges, and another may have a large number of small ranges. Let us call α_1 and α_2 the number of ranges of a_1 and a_2 respectively. Because of their difference in amount, A will be called an $max\{\alpha_1, \alpha_2, ..., \alpha_n\} \times n$ matrix with the remaining element spots as zeros. Similarly, B is a $max\{\beta_1, \beta_2, .., \beta_m\} \times m$ matrix with the remaining element spots filled with zeros. Furthermore, α_1 may be much smaller than α_2, while all α_1 ranges are in Z, only $(\alpha_2 - \zeta + 1)$ ranges are in Z for a_2. To align the good ranges, we will begin the listing of the columns from the bottom to the top.

Example 2.1. Say there exists an item with four prefixes, a_1, a_2, a_3, and a_4, with their number of ranges being $2, 4, 3$, and 1 respectively. ($\alpha_1 = 2, \alpha_2 = 4, \alpha_3 = 3, \alpha_4 = 1$). The prefix matrix of the item would look like this,

$$\begin{bmatrix} 0 & a_{2,1} & 0 & 0 \\ 0 & a_{2,2} & a_{3,1} & 0 \\ a_{1,1} & a_{2,3} & a_{3,2} & 0 \\ a_{1,2} & a_{2,4} & a_{3,3} & a_{4,1} \end{bmatrix}.$$

3. **Probabilities of Rolls**

3.1. **Probability Matrices and Functions.** We defined the elements of A and B to be the affixes of all possible items. In the same regard, we will define the elements of P_A and P_B to be the probabilities of rolling each affix of all the possible items.

Definition 3.1. Call P_A the *probability matrix of* A, or in this case, the *prefix probability matrix*. Where P_A is a matrix of the same form as A, such that all elements are equal in value to the probabilities of choosing themselves. Similarly, P_B is the *suffix probability matrix*.

Because P_A is the matrix of all the probabilities of each prefixes and their ranges, it makes sense that the sum of it's elements equal one. Furthermore, when we sum across any submatrix of A, we will receive the probability of getting a roll within that submatrix.

Theorem 1. *The probability of receiving a roll within a certain matrix or submatrix is equal to the sum of the elements in the respective matrix or submatrix.*

$$\begin{aligned} P(R \in A) &= \sum_{i=1}^{max\{\alpha_1,\dots,\alpha_n\}} \sum_{j=1}^{n} a_{j,i} = \sum P_A = 1 \\ \Rightarrow P(R \in Z) &= \sum_{i=\zeta+1}^{max\{\alpha_{k+1},\dots,\alpha_n\}} \sum_{j=k+1}^{n} a_{j,i} = \sum P_Z \\ \Rightarrow P(R = a_{j,i}) &= a_{j,i} \end{aligned}$$

Proof. Given some $m \times n$ *probability matrix*, P_w, with elements $w_{i,j}$, where $1 \leq i \leq n, 1 \leq j \leq m$. The probability of rolling any of the elements is,

$$\begin{aligned} &P([R = w_{1,1}] \cup \dots \cup [R = w_{n,m}]) = P(R = w_{1,1}) + \dots + P(R = w_{n,m}) \\ &= w_{1,1} + \dots + w_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} w_{i,j} \end{aligned}$$

□

Much like a deck of cards having a card removed, we cannot remove the same card again, and the total chance of pulling another card out, given we do, is always 100%. Therefore, when a roll is taken by an item it cannot have a duplicate roll. In order to tackle this issue, we must adjust the notation of R,

Definition 3.2. The affix roll of an item is ${}_jR_{\#}$, where j is the column number that the roll was taken from, and # is the order number the roll was taken in.

When we look at $I([{}_jR_1 \in Z], [{}_jR_2 \in B], [{}_jR_3 \in Z])$, it must be take into account that once ${}_jR_1$ is rolled, the probability of getting another roll in the j column is removed. Furthermore, the total probability must be the same if we sum over P_A, because if we roll a prefix again, there has to be a 100% chance we are in P_A. Therefore, after every roll we choose, we must adjust P_A by a factor of the column j.

Theorem 2. *Let, P_A be a $max\{\alpha_1, \alpha_2, ..., \alpha_n\} \times n$ matrix, I_j be an $n \times n$ matrix, and γ_j be a factor based on the j^{th} column of P_A, where I_j and γ_j are defined such that,*

$$I_j = diag(1, 1, ..., 1) - diag(e_j),$$

where e_j is the j-th standard vector in $\mathbb{R}^\alpha$.

$$\gamma_j = \frac{1}{1 - \sum_{i=1}^{\alpha_j} \alpha_{j,i}}, \text{ where } \sum_{i=1}^{\alpha_j} \alpha_{j,i} \neq 1.$$

Given that ${}_jR_{\#}$ was taken from the j^{th} column of P_A. We define the new prefix probability matrix as,

$$\begin{aligned}
\{P_A|{}_jR_{\#} removed\} =& \\
&= \gamma_j \times P_A \times I_j \\
&= \begin{bmatrix} \gamma_j a_{1,1} & \dots & \gamma_j a_{j-1,1} & 0 & \gamma_j a_{j+1,1} & \dots & \gamma_j a_{n,1} \\ \vdots & \ddots & \vdots & 0 & \vdots & \ddots & \vdots \\ \gamma_j a_{1,\alpha} & \dots & \gamma_j a_{j-1,\alpha} & 0 & \gamma_j a_{j+1,\alpha} & \dots & \gamma_j a_{n,\alpha} \end{bmatrix} \\
&= {}_jP_A
\end{aligned}$$

Proof. Let $E = \{e_1, e_2, ..., e_n\}$ be a set of events. Assume the set of events is under the condition that once an event is chosen, it becomes cyclic(i.e that is, if an event is chosen a second time, the choice will be discarded and an event will be chosen again). Therefore, the new chance of getting e_j, after event e_i has occured, is given by the equation below via the geometric series,

$$\begin{aligned} P(e_j)_{new} =& P(e_j) + P(e_j)P(e_i) + P(e_j)P(e_i)^2 + ... \\ =& P(e_j)(1 + P(e_i) + P(e_i)^2 + ...) \\ =& P(e_j)\left(\frac{1}{1 - P(e_i)}\right), \end{aligned}$$

where $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$.
Furthermore, because e_i was chosen, $P(e_i)_{new} = 0$. Therefore to see $P(E)_{new}$, let $\frac{1}{1 - P(e_i)} = \gamma_i$.

$$\begin{aligned} P(E)_{new} =& P(e_1)_{new} + ... + P(e_{i-1})_{new} + 0 + P(e_{i+1})_{new} + ... + P(e_n)_{new} \\ =& P(e_1)\gamma_i + ... + P(e_{i-1})\gamma_i + 0 + P(e_{i+1})\gamma_i + ... + P(e_n)\gamma_i \\ =& [P(e_1) + ... + P(e_{i-1}) + 0 + P(e_{i+1}) + ... + P(e_n)] \times \gamma_i \\ =& [P(e_1) \quad \dots \quad P(e_n)] \times [1_1 \dots 1_{i-1} \quad 0 \quad 1_{i+1} \dots 1_n]^T \times \gamma_i \end{aligned}$$

□

This new ${}_jP_A$ is the *prefix probability matrix* after the prefix j was removed. In addition to the subscript j, we may add more variables to indicate that more than one column was removed. Therefore 2 columns removed for the last roll would look like ${}_{j_1,j_2}P_A$.

Notice that the sum of P_A is the same as the sum of ${}_jP_A$, where they both equal 1. This allows us to still say $a_{j,i} = P(a_{j,i})$, regardless of whether $a_{j,i}$ is from ${}_jP_A$ or P_A This will now allow us to compute multiple rolled items with much easier notation.

Example 3.1. Coming back to our four-prefix matrix. Assume we want to find the chance of rolling within a given submatrix two times. Let Q be our main matrix, and q be our submatrix;

$$Q = \begin{bmatrix} 0 & a_{2,1} & 0 & 0 \\ 0 & a_{2,2} & a_{3,1} & 0 \\ a_{1,1} & a_{2,3} & a_{3,2} & 0 \\ a_{1,2} & a_{2,4} & a_{3,3} & a_{4,1} \end{bmatrix}, q = \begin{bmatrix} a_{2,3} & a_{3,2} & 0 \\ a_{2,4} & a_{3,3} & a_{4,1} \end{bmatrix}.$$

Initially we want to find the probability of getting our first roll in q. From Theorem 1 we see that the $P(R_1 \in q) = \sum P_q$. For the second roll, we have to take into account what happens when our first roll is any of the five prefixes in q. Hence,

$$
\begin{aligned}
P(R_1 \in q \cap R_2 \in q) &= P([R_1 = a_{2,3} \cup a_{3,2} \cup a_{2,4} \cup a_{3,3} \cup a_{4,1}] \cap R_2 \in q) \\
&= P([R_1 = a_{2,3} \cap R_2 \in q] \cup [R_1 = a_{3,2} \cap R_2 \in q] \\
&\quad \cup [R_1 = a_{2,4} \cap R_2 \in q] \cup [R_1 = a_{3,3} \cap R_2 \in q] \\
&\quad \cup [R_1 = a_{4,1} \cap R_2 \in q]) \\
&= (a_{2,3} \cdot \sum {}_2P_q) + (a_{3,2} \cdot \sum {}_3P_q) + (a_{2,4} \cdot \sum {}_2P_q) \\
&\quad + (a_{3,3} \cdot \sum {}_3P_q) + (a_{4,1} \cdot \sum {}_4P_q) \\
&= \sum {}_2P_q(a_{2,3} + a_{2,4}) + \sum {}_3P_q(a_{3,2} + a_{3,3}) + \sum {}_4P_q(a_{4,1}) \\
&= \sum_{j=2}^{4} \left[\sum {}_jP_q \cdot \sum_{i \in q} a_{j,i} \right].
\end{aligned}
$$

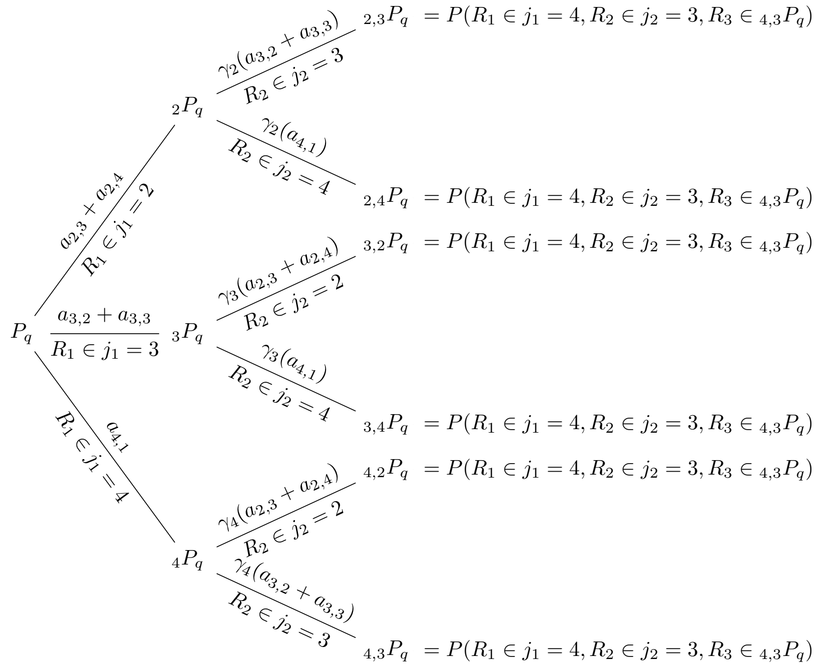
It's important to note, because of the way we have to structure our matrices due to item level constraints, that $i \in q$ is based on each j in particular. Some columns have different row values for their prefixes and suffixes.

This gives us a general formula for the probability of the two sequential rolls in a prefix or suffix submatrix.

$$
P(R_1 \in q_{i \times j} \cap R_2 \in q_{i \times j}) = \sum_{j \in q} \left[\sum {}_jP_q \cdot \sum_{i \in q} a_{j,i} \right] \tag{2}
$$

where, q is the submatrix, i is the row dimension of q, and j is the column dimension of q. In words, the probability of getting two rolls within the same submatrix is based primarily on the column of the roll you chose was from.

Now assume we wanted to know what are the chances of a third roll in q. To visualize the pattern we need to use the following probability tree.



To see the probability of the third roll we must multiply across each branch of the tree from it's starting point. We then get what is on the right of the tree. After summing and algebraic manipulation we recieve the general formula for the probability of the third roll in a prefix or suffix matrix,

$$
\begin{aligned}
&P(R_1 \in q_{i\times j} \cap R_2 \in q_{i\times j} \cap R_3 \in q_{i\times j}) \\
&= \sum_{\substack{j_1 \neq j_2 \\ j_1 \in q \\ j_2 \in q}} \left[\sum {}_{j_1,j_2}P_q \cdot \left(\sum_{i\in q} a_{j_1,i} \cdot \sum_{i \in q} a_{j_2,i} \cdot (\gamma_{j_1}) \right) \right],
\end{aligned}
$$

where, q is the submatrix, i is the row dimension of q, j is the column dimension of q, j_1 is the column of the first roll, and j_2 is the column of the second roll.

3.2. **Prefix and Suffix Discrepancy.** We have to look at each roll on a given item in two parts. The probability that the roll chooses a prefix or suffix, and the probability that the roll is in a submatrix of the respected affix type. Hence,

$$P(R_1 \in Z) = P(R_1 \in A) \cdot \sum P_Z$$

In Definition 2.1, we see that an a magic item can only roll one prefix and one suffix. Therefore, this gives magic item two choices of equal weight. Either the first roll is a prefix, or the first roll is a suffix.

$$P([R_1 \in A], [R_2 \in B]) = P([R_1 \in B], [R_2 \in A]) = 1/2$$

Because the probability of rolling a prefix or a suffix is equal in both ways for a magic item, it's irrelevant which comes first.

Example 3.2. Consider two items with three rolls, $I_1 = ([R_1 \in A], [R_2 \in B], [R_3 \in A])$, and $I_2 = ([R_1 \in B], [R_2 \in A], [R_3 \in A])$. Notice,

$$P(I_1) = [P(R_1 \in A) \cdot \sum P_A] \cdot [P(R_2 \in B) \cdot \sum P_B] \cdot [P(R_3 \in A) \cdot \sum{}_{j_1} P_A)]$$
$$P(I_1) = \left[\sum P_B \cdot \sum P_A\right] \cdot [P(R_2 \in B) \cdot P(R_1 \in A)] \cdot [P(R_3 \in A) \cdot \sum{}_{j_1} P_A)]$$
$$P(I_1) = [1] \cdot [\frac{1}{2}] \cdot [P(R_3 \in A) \cdot 1]$$
$$P(I_1) = \frac{1}{2} \cdot [P(R_3 \in A)]$$
$$P(I_2) = [P(R_1 \in B) \cdot \sum P_B] \cdot [P(R_2 \in A) \cdot \sum P_A] \cdot [P(R_3 \in A) \cdot \sum{}_{j_1} P_A)]$$
$$P(I_2) = \left[\sum P_B \cdot \sum P_A\right] \cdot [P(R_2 \in B) \cdot P(R_1 \in A)] \cdot [P(R_3 \in A) \cdot \sum{}_{j_1} P_A)]$$
$$P(I_2) = [1] \cdot [\frac{1}{2}] \cdot [P(R_3 \in A) \cdot 1]$$
$$P(I_2) = \frac{1}{2} \cdot [P(R_3 \in A)]$$
$$P(I_1) = P(I_2)$$

Therefore there is no discrepancy on the order of the first two rolls on a magic item. Whether or not a prefix comes first does not affect the future probabilities of an item. However, the order of rolls of a rare item do alter the probability of the later rolls. Looking at definition 2.1, for a rare item, no rules apply to the choice of affix type. We will assume the choice is random.

Definition 3.3. The probability of rolling a prefix or suffix on a rare item is,

$$P(R \in A) = \frac{\#\text{remaining prefixes}}{\#\text{remaining affixes}}$$

and similarly,

$$P(R \in B) = \frac{\#\text{remaining suffixes}}{\#\text{remaining affixes}}.$$

Furthermore, as we see from the beginning of Section 3, the probability of a roll is based solely on its own matrix and not on the opposing roll types. It would make sense that the order of whether a prefix comes before a suffix or vice-verse is irrelevant to their own probabilities. Suffice to say that the probability of rolling a prefix then a suffix is equal to rolling a suffix then a prefix for the third and fourth roll respectively. However, it is not equally valid to say that the *event* of receiving a prefix for a third roll and a suffix for a fourth roll is the counted as the same event in the opposite direction. They are two separate events in their own right, but they do share a common chance of occurring. Therefore, when calculating the chance of a prefix occurring before a suffix and vice-verse, we cannot ignore the fact that this can happen twice.

4. **Methods of Crafting**

In PoE there are a multitude of methods to craft items. Every method will involve using an in-game *currency item* on a piece of gear. Each of these currencies will augment gear in a different way. A full list of currency items can be found here.

Definition 4.1. [2] The currencies that will be used in the crafting methods are,

- Orb of Alteration: Reforges a magic item with new random affixes.
- Orb of Alchemy: Adds four to six affixes and upgrades a normal item to a rare item.
- Orb of Augmentation: Adds an affix to a magic item, given it has one available affix.
- Orb of Transmutation: Adds one to two affixes and upgrades a normal item to a magic item.
- Regal Orb: Adds an affix to an item and upgrades a magic item to a rare item.
- Essence: An essence adds four to six affixes, with one being fixed, and upgrades a normal item into a rare item.
- Scouring Orb: Downgrades an item, from any rarity, to a normal item and removes all affixes.

Each of these orbs are used in what is commonly referred to as "the three best ways to craft". These three ways are

- "alt-regal": This involves rolling a magic item with alterations and augmentations until it has a desirable prefix and suffix, then using a regal to get a desirable affix of either a prefix or suffix. This results in a 3 affix item where each roll is desirable. In the event that the regal is undesirable, a scouring orb is used and the process repeated.

- "alch-scour": In this method, we take a normal item, and use an orb of alchemy on it. The item will roll anywhere from four to six affixes. The result needed for each of these is different. Given that an item rolls
  - four mods, we need at least three desirable affixes.
  - five mods, we need at least three desirable affixes.
  - six mods, we need at least four desirable affixes.
- "essence-scour": This method uses an essence on a normal item to upgrade it into a rare item already has the desirable roll from the essence. Like our alch-scour, given that an item rolls
  - four mods, we need at least three desirable affixes.
  - five mods, we need at least three desirable affixes.
  - six mods, we need at least four desirable affixes.

4.1. **Method One: "alt-regal".** Much like Example 3.1, we are looking for three desirable affixes on an item in a row. We know that for an item, $I_{\text{alt-regal}}$, crafted in this method to be good, it must take the form of

$$I_{\text{alt-regal}} =([R_1 \in (Z \text{ or } H)], [R_2 \in (Z \text{ or } H)], [R_3 \in ({}_jZ \text{ or } {}_jH)]).$$

Therefore the probability of receiving item $I_{\text{alt-regal}}$ is,

$$\begin{aligned} P(I_{\text{alt-regal}}) =& P([R_1 \in (Z \text{ or } H)], [R_2 \in (Z \text{ or } H)], [R_3 \in ({}_jZ \text{ or } {}_jH)]) \\ =& P(R_1 \in (Z \text{ or } H)) \cdot P(R_2 \in (Z \text{ or } H)) \cdot P(R_3 \in ({}_jZ \text{ or } {}_jH)). \end{aligned}$$

From Example 3.2, the first part of the probability is

$$\begin{aligned} & P(R_1 \in (Z \text{ or } H)) \cdot P(R_2 \in (Z \text{ or } H)) \\ =& [P(R_1 \in Z) + P(R_1 \in H)] \cdot [P(R_2 \in Z) + P(R_2 \in H)] \\ =& 0 + P(R_1 \in Z)P(R_2 \in H) + P(R_2 \in H)P(R_2 \in Z) + 0 \\ =& \frac{1}{2} \sum P_Z \sum P_H + \frac{1}{2} \sum P_H \sum P_Z \\ =& \sum P_Z \sum P_H. \end{aligned}$$

For the remaining part of the $P(I_{\text{alt-regal}})$ we see,

$$\begin{aligned} &P(R_3 \in ({}_jZ \text{ or } {}_jH)) \\ =&P(R_3 \in {}_jZ) + P(R_3 \in {}_jH) \\ =&\sum {}_jP_Z + \sum {}_jP_H. \end{aligned}$$

Therefore the probability of receiving a desirable item with the "alt-regal" method is,

$$P(I_{\text{alt-regal}}) = \left[\sum P_Z \sum P_H\right] \cdot \left[\sum {}_jP_Z + \sum {}_jP_H\right].$$

Although we have the probability of rolling a good item with the alt-regal method, we are not entirely interested in the probability of getting the rolls alone. Rather, we want to find out the average number of attempts, and the associated variance, that it will take to receive an item with only three affixes that are all desirable. To calculate this we must treat the method as a geometric probability distribution.[3]

In geometric probability, the expected value, or expected number of X trials until a success, is

$$E\{X\} = \frac{1}{p}. \tag{3}$$

The variance of a geometric distribution, or σ^2, is defined as

$$Var\{X\} = \frac{1-p}{p^2} \tag{4}$$

where p is the probability of success.

Therefore if we consider $P(I_{\text{alt-regal}})$ the probability of a successful craft, then the average number of trials to receive a good item using the alt-regal crafting method is equivalent to

$$E\{X\} = \frac{1}{[\sum P_Z \sum P_H] \cdot [\sum {}_jP_Z + \sum {}_jP_H]},$$

along with the associated variance of

$$Var\{X\} = \frac{1 - [\sum P_Z \sum P_H] \cdot [\sum {}_jP_Z + \sum {}_jP_H]}{([\sum P_Z \sum P_H] \cdot [\sum {}_jP_Z + \sum {}_jP_H])^2}.$$

4.2. **Method Two: "alch-scour".** For not only the alch-scour method, but also the essence-scour method, we must focus on two different aspects of how these rolls can occur. The order of which a good or any roll comes in, and the amount of prefixes and suffixes our item can have.

4.2.1. *Prefix and suffix amount.* Initially we see that our item, I_4 (where the "$_4$" stands for four affixes), can have four scenarios in it's prefix and suffix rolls. Three prefixes and one suffix, two prefixes and two suffixes, and finally one prefix and three suffixes. The associated probabilities with receiving rolls in these orders are,

$$\begin{aligned} P(I_4) =& Pref \cdot Suff \cdot Pref \cdot Pref \equiv 1 \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{6} \\ &+ Pref \cdot Suff \cdot Suff \cdot Pref \equiv 1 \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{3} \\ &+ Pref \cdot Suff \cdot Pref \cdot Suff \equiv 1 \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{3} \\ &+ Pref \cdot Suff \cdot Suff \cdot Suff \equiv 1 \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{6}. \end{aligned}$$

In section 3.2 we saw the discrepancy between the prefixes and suffixes, suffice to say we can combine our middle two terms into one,

$$\rightarrow Pref \cdot Suff \cdot Pref \cdot Suff \equiv 2 \cdot \frac{1}{3} = \frac{2}{3}.$$

This gives us an easy layout for when we calculate the second step,

$$P(I_4) = \frac{1}{6}(P, P, P, S) + \frac{2}{3}(P, P, S, S) + \frac{1}{6}(P, S, S, S),$$

where P is prefix, and S is suffix.

For I_5 on an item this becomes much easier. There exist only two cases, Three prefixes and two suffixes, or two prefixes and three suffixes. We see,

$$\begin{aligned} P(I_5) =& Pref \cdot Suff \cdot Pref \cdot Pref \cdot Suff \\ &+ Pref \cdot Suff \cdot Pref \cdot Suff \cdot Suff \\ &\rightarrow P(I_5) = \frac{1}{2}(P, P, P, S, S) + \frac{1}{2}(P, P, S, S, S). \end{aligned}$$

Finally we have six affixes that can only occur in one way,

$$P(I_6) = Pref \cdot Suff \cdot Pref \cdot Pref \cdot Suff \cdot Suff = (P, P, P, S, S, S).$$

Each of these different affix combinations has a chance to occur when we use an alchemy orb on an item, giving us our total number of possible prefix and affix combinations for an item I.

$$P(I_{\text{alch-scour}}) = \frac{1}{3}P(I_4) + \frac{1}{3}P(I_5) + \frac{1}{3}P(I_6)$$

4.2.2. *Order of the desirable rolls.* Because we have to have at least a certain number of desirable rolls on an item, this leaves room for play. We could end up with an item with four affixes and four desirable rolls, or one bad roll and three desirable rolls. Furthermore, the fourth "good or bad" roll can happen in any order. We will write these rolls as P_y or P_x for prefixes, (S_y and S_x for suffixes) where y is any roll at all (an "anything" roll), and x is a desirable roll. For I_4 we saw that we have three possible combinations of prefixes and suffixes. Each of these combinations has "subpermutations" that are represented as,

$$\begin{aligned}
(P,P,P,S) =&(P_x,P_x,P_x,S_y) + (P_x,P_x,P_y,S_x)\\
&+(P_x,P_y,P_x,S_x) + (P_y,P_x,P_x,S_x)\\
(P,P,S,S) =&(P_x,P_x,S_x,S_y) + (P_x,P_x,S_y,S_x)\\
&+(P_x,P_y,S_x,S_x) + (P_y,P_x,S_x,S_x)\\
(P,S,S,S) =&(P_x,S_x,S_x,S_y) + (P_x,S_x,S_y,S_x)\\
&+(P_x,S_y,S_x,S_x) + (P_y,S_x,S_x,S_x).
\end{aligned}$$

For I_5 we see,

$$\begin{aligned}
(P,P,P,S,S) =&(P_y,P_y,P_x,S_x,S_x) + (P_y,P_x,P_y,S_x,S_x) + (P_y,P_x,P_x,S_y,S_x)\\
&+(P_y,P_x,P_x,S_x,S_y) + (P_x,P_y,P_y,S_x,S_x) + (P_x,P_y,P_x,S_y,S_x)\\
&+(P_x,P_y,P_x,S_x,S_y) + (P_x,P_x,P_y,S_y,S_x) + (P_x,P_x,P_y,S_x,S_y)\\
&+(P_x,P_x,P_x,S_y,S_y)\\
(P,P,S,S,S) =&(P_y,P_y,S_x,S_x,S_x) + (P_y,P_x,S_y,S_x,S_x) + (P_y,P_x,S_x,S_y,S_x)\\
&+(P_y,P_x,S_x,S_x,S_y) + (P_x,P_y,S_y,S_x,S_x) + (P_x,P_y,S_x,S_y,S_x)\\
&+(P_x,P_y,S_x,S_x,S_y) + (P_x,P_x,S_y,S_y,S_x) + (P_x,P_x,S_y,S_x,S_y)\\
&+(P_x,P_x,S_x,S_y,S_y).
\end{aligned}$$

Finally for I_6 we have,

$$\begin{aligned}
&(P,P,P,S,S,S) = \\
&\qquad (P_y,P_y,P_x,S_x,S_x,S_x)+(P_y,P_x,P_y,S_x,S_x,S_x)+(P_y,P_x,P_x,S_y,S_x,S_x) \\
&\qquad +(P_y,P_x,P_x,S_x,S_y,S_x)+(P_y,P_x,P_x,S_x,S_x,S_y)+(P_x,P_y,P_y,S_x,S_x,S_x) \\
&\qquad +(P_x,P_y,P_x,S_y,S_x,S_x)+(P_x,P_y,P_x,S_x,S_y,S_x)+(P_x,P_y,P_x,S_x,S_x,S_y) \\
&\qquad +(P_x,P_x,P_y,S_y,S_x,S_x)+(P_x,P_x,P_y,S_x,S_y,S_x)+(P_x,P_x,P_y,S_x,S_x,S_y) \\
&\qquad +(P_x,P_x,P_x,S_y,S_y,S_x)+(P_x,P_x,P_x,S_y,S_x,S_y)+(P_x,P_x,P_x,S_x,S_y,S_y).
\end{aligned}$$

Now that we have a list of each probability that we will need to calculate to sum to a chance of finding a desirable item with the alch-scour method. We need to come up with general formulas for each of these probabilities.

4.2.3. *General equations for the probability of the "alch-scour" method.* Luckily we have already done all of these possible orders of rolls without realizing it. We can convert our Example 3.1 to match each of these possibilities. Notice that our general form for two good rolls in a row,

$$P([R_1 \in q_{i\times j}] \cap [R_2 \in q_{i\times j}]) = \sum_{j\in q}\left[\sum{}_j P_q \;\cdot \sum_{i\in q} a_{j,i}\right],$$

has the sum of the first rolls chosen, times the probability of landing in q after those rolls were chosen. To include an "anything" roll all we must do is adjust this sum.

For an "anything" roll followed by a desirable roll we have,

$$P([R_1 \in M] \cap [R_2 \in q_{i\times j}]) = \sum_{j\in M}\left[\sum{}_j P_q \;\cdot \sum_{i\in M} a_{j,i}\right],$$

for some matrix M with desirable submatrix q. Following this, if we wanted to have a desirable roll followed by an "anything" roll. We simply use the known fact that the sum of an entire probability matrix is equal to one. Hence,

$$\begin{aligned}
P([R_1 \in q_{i\times j}] \cap [R_2 \in M]) &= \sum_{j\in q}\left[\sum{}_j P_M \;\cdot \sum_{i\in q} a_{j,i}\right] \\
&= \sum_{j\in q}\left[1 \cdot \sum_{i\in q} a_{j,i}\right].
\end{aligned}$$

From theorem 1 we see,

$$\sum_{j\in q}\left[1\cdot\sum_{i\in q}a_{j,i}\right]=\sum_{j\in q}\cdot\sum_{i\in q}a_{j,i}=\sum P_q.$$

Furthermore, for an "anything" roll within two other desirable rolls we can see from Example 3.1,

$$\begin{aligned}&P([R_1\in M]\cap[R_2\in q_{i\times j}]\cap[R_3\in q_{i\times j}])\\ &=\sum_{\substack{j_1\neq j_2\\ j_1\in M\\ j_2\in q}}\left[\sum{}_{j_1,j_2}P_q\cdot\left(\sum_{i\in M}a_{j_1,i}\cdot\sum_{i\in q}a_{j_2,i}\cdot(\gamma_{j_1})\right)\right],\end{aligned}$$

and

$$\begin{aligned}&P([R_1\in q_{i\times j}]\cap[R_2\in M]\cap[R_3\in q_{i\times j}])\\ &=\sum_{\substack{j_1\neq j_2\\ j_1\in q\\ j_2\in M}}\left[\sum{}_{j_1,j_2}P_q\cdot\left(\sum_{i\in q}a_{j_1,i}\cdot\sum_{i\in M}a_{j_2,i}\cdot(\gamma_{j_1})\right)\right].\end{aligned}$$

Just as before, the probability of three rolls where the first two are desirable and the last is "anything" is,

$$\begin{aligned}&P([R_1\in q_{i\times j}]\cap[R_2\in q_{i\times j}]\cap[R_3\in M])\\ &=\sum_{\substack{j_1\neq j_2\\ j_1\in q\\ j_2\in q}}\left[\sum{}_{j_1,j_2}P_M\cdot\left(\sum_{i\in q}a_{j_1,i}\cdot\sum_{i\in q}a_{j_2,i}\cdot(\gamma_{j_1})\right)\right]\\ &=\sum_{\substack{j_1\neq j_2\\ j_1\in q\\ j_2\in q}}\left[\sum_{i\in q}a_{j_1,i}\cdot\sum_{i\in q}a_{j_2,i}\cdot(\gamma_{j_1})\right].\end{aligned}$$

Finally we come to having two "anything" rolls before having a single desirable roll.

$$
\begin{aligned}
&P([R_1 \in M] \cap [R_2 \in M] \cap [R_3 \in q_{i\times j}]) \\
&= \sum_{\substack{j_1 \neq j_2 \\ j_1 \in M \\ j_2 \in M}} \left[\sum{}_{j_1,j_2} P_q \cdot \left(\sum_{i\in M} a_{j_1,i} \cdot \sum_{i \in M} a_{j_2,i} \cdot (\gamma_{j_1}) \right) \right].
\end{aligned}
$$

As well, a desirable roll before two "anything" rolls.

$$
\begin{aligned}
&P([R_1 \in q_{i\times j}] \cap [R_2 \in M] \cap [R_3 \in M]) \\
&= \sum_{\substack{j_1 \neq j_2 \\ j_1 \in q \\ j_2 \in M}} \left[\sum{}_{j_1,j_2} P_M \cdot \left(\sum_{i\in q} a_{j_1,i} \cdot \sum_{i \in M} a_{j_2,i} \cdot (\gamma_{j_1}) \right) \right] \\
&= \sum_{\substack{j_1 \neq j_2 \\ j_1 \in q \\ j_2 \in M}} \left[1 \cdot \left(\sum_{i\in q} a_{j_1,i} \cdot \sum_{i \in M} a_{j_2,i} \cdot (\gamma_{j_1}) \right) \right] \\
&= \sum_{\substack{j_1 \neq j_2 \\ j_1 \in q \\ j_2 \in M}} \left[\sum_{i\in q} a_{j_1,i} \left(\sum_{i \in M} a_{j_2,i} \cdot (\gamma_{j_1}) \right) \right] \\
&= \sum_{\substack{j_1 \neq j_2 \\ j_1 \in q \\ j_2 \in M}} \left[\sum_{i\in q} a_{j_1,i}\,(1) \right] = \sum P_q
\end{aligned}
$$

Now that we have all the general formulas for each style of probability we can finally calculate $P(I_{\text{alch-scour}})$. However, to include the full general form would not be sensical. Therefore we will just keep the probability as the label $P(I_{\text{alch-scour}})$. As for our average number of trials, we come back to geometric distribution to receive,

the expected number of X trials until we roll a desirable item with the alch-scour method is,

$$
E\{X\} = \frac{1}{P(I_{\text{alch-scour}})}
$$

The associated variance is,

$$Var\{X\} = \frac{1 - P(I_{\text{alch-scour}})}{P(I_{\text{alch-scour}})^2}$$

4.3. **Method Three: "essence-scour".** Much like the alch-scour method, we must take into account the number of prefixes and suffixes, as well as the ordering of the desirable vs. "anything" rolls. However, in the case of essences, our initial prefix or suffix is always fixed as a desirable roll. Because the first prefix or suffix can be fixed, we will assume that it is only the suffix that is fixed as that is the most commonly fixed roll for essence crafting. Beginning with I_4, we see that we may remove quite a few possibilities.

$$\begin{aligned}
(P,P,P,S) =& (P_x,P_x,P_y,S_x) + (P_x,P_y,P_x,S_x) \\
&+ (P_y,P_x,P_x,S_x) \\
(P,P,S,S) =& (P_x,P_x,S_x,S_y) + (P_x,P_y,S_x,S_x) \\
&+ (P_y,P_x,S_x,S_x) \\
(P,S,S,S) =& (P_x,S_x,S_x,S_y) + (P_x,S_x,S_y,S_x) \\
&+ (P_y,S_x,S_x,S_x)
\end{aligned}$$

For I_5 we see,

$$\begin{aligned}
(P,P,P,S,S) =& (P_y,P_y,P_x,S_x,S_x) + (P_y,P_x,P_y,S_x,S_x) \\
&+ (P_y,P_x,P_x,S_x,S_y) + (P_x,P_y,P_y,S_x,S_x) \\
&+ (P_x,P_y,P_x,S_x,S_y) + (P_x,P_x,P_y,S_x,S_y) \\
(P,P,S,S,S) =& (P_y,P_y,S_x,S_x,S_x) + (P_y,P_x,S_x,S_y,S_x) \\
&+ (P_y,P_x,S_x,S_x,S_y) + (P_x,P_y,S_x,S_y,S_x) \\
&+ (P_x,P_y,S_x,S_x,S_y) + (P_x,P_x,S_x,S_y,S_y)
\end{aligned}$$

Finally for I_6 we have,

$$\begin{aligned}(P,P,P,S,S,S) = \\ &(P_y,P_y,P_x,S_x,S_x,S_x) + (P_y,P_x,P_y,S_x,S_x,S_x) + (P_y,P_x,P_x,S_x,S_y,S_x) \\ &+(P_y,P_x,P_x,S_x,S_x,S_y) + (P_x,P_y,P_y,S_x,S_x,S_x) + (P_x,P_y,P_x,S_x,S_y,S_x) \\ &+(P_x,P_y,P_x,S_x,S_x,S_y) + (P_x,P_x,P_y,S_x,S_y,S_x) + (P_x,P_x,P_y,S_x,S_x,S_y) \\ &+(P_x,P_x,P_x,S_x,S_y,S_y)\end{aligned}$$

Intuition would say that having one single fixed desirable roll would increase the probability of finding a desirable item, although initially we see fewer probable cases of a desirable item being obtained. In the event that an essence is used and it's fixed roll is gained, we neglected the fact that this fixed roll has a 100% chance of being obtained. For the probability equations for essences, we see that in I_4 we have

$$\begin{aligned}(P_x,P_x,P_y,S_x) = &\qquad \text{Our prefix portion} \quad \cdot \quad \text{Our Suffix portion} \\ &= \sum_{\substack{j_1 \neq j_2 \\ j_1 \in q \\ j_2 \in q}} \left[\sum_{i \in q} a_{j_1,i} \cdot \sum_{i \in q} a_{j_2,i} \cdot (\gamma_{j_1}) \right] \qquad \cdot \qquad 1\end{aligned}$$

We see rather than lowering our probability of obtaining an item of the form (P_x,P_x,P_y,S_x) by multiplying the probability of a desirable suffix, the essence allows us to set the first probability of $S_x = 1$. This method is furthered with each of the other permuations.

Following suit with the alch-scour method, the general formula for $P(I_{\text{essence-scour}})$ is quite large and will be omitted from the paper. As for our average number of trials, we come back to geometric distribution. The expected number of X trials until we roll a desirable item with the essence-scour method is,

$$E\{X\} = \frac{1}{P(I_{\text{essence-scour}})}$$

The associated variance is,

$$Var\{X\} = \frac{1 - P(I_{\text{essence-scour}})}{P(I_{\text{essence-scour}})^2}$$

5. Conclusion

From the equations above we can see generally what method is going to be the most probable. The alt-regal method depends solely on a single "three in a row" case of luck. It seems to be the most unlikely of the three. However, compared to the case of alch-scour and essence-scour in the case of cost, these two methods are far less used. In the event that cost is not an issue, the player would be best off applying these two methods.

However, the cost and time of each method is the least of the concern. The modeling results are more to give the player an idea of what item type would be the best to craft on. Obviously an item with fewer non-desirable mods would be the most optimal. However, from the models we can see that if a substantially larger probability exists within the affix matrix of an item, the data will skew such that if the largest probability is not chosen first the likelihood of falling within a given deviation is swayed toward the large probability. Therefore if the item has a desirable affix that has a large probability compared to that of other desirable affixes, it may be that this item type is more reliable in its crafting than an item with more desirable affixes with consistent smaller probabilities. In digression, we won't know until the equations are calculated in all possible formats(i.e all item types and item levels with each method). Currently I lack the ability to compute this, however in the future I plan to purse this with code.

6. Thoughts and Acknowledgements

Modeling PoE was an educating experience for me personally. As a long time player it felt gratifying to combine my enjoyment for the game and my love of mathematics. I plan on using the same model in future personal projects.

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