Supplementary Information

Enhanced Cancer Data Modeling with the Modified Burr III Odds Ratio-G Distribution

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1 Expansion of pdf

Theorem 1. *The probability density function of the Burr III Scaled Inverse Odds Ratio-G distribution can be expressed as a linear combination of the exponentiated generalized distribution as follows:*

$$f_{\text{B-SIOR-G}}(x) = \sum_{i,j=0}^{\infty} c_{i,j} r_{i-bj-b-1}(x, \boldsymbol{\psi}), \tag{1}$$

where the coefficients $c_{i,j}$ *are given by*

$$c_{i,j} = \frac{kaba^j(-1)^{i+j}\binom{k+1+j-1}{j}\binom{b(j+1)-1}{i}}{i-bj-b}, \tag{2}$$

and the term $r_{i-bj-b-1}(x, \boldsymbol{\psi})$ *is defined as*

$$r_{i-bj-b-1}(x, \boldsymbol{\psi}) = (i-bj-b)g(x, \boldsymbol{\psi})[D(x, \boldsymbol{\psi})]^{i-bj-b-1}, \tag{3}$$

representing the pdf of an exponentiated generalized distribution with parameter $W^* = i - bj - b$.

Proof. Consider the general form of binomial series expansion:

$$\left(1 + a\left[\frac{\overline{D}(x, \boldsymbol{\psi})}{D(x, \boldsymbol{\psi})}\right]^b\right)^{-k-1} = \sum_{j=0}^{\infty} \binom{k+1+j-1}{j} (-1)^j \left(a\left[\frac{\overline{D}(x, \boldsymbol{\psi})}{D(x, \boldsymbol{\psi})}\right]^b\right)^j \tag{4}$$

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Thus, the pdf of the B-EOR-G distribution can be expanded as

$$
\begin{aligned}
f_{B-EOR-G}(x) &= kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}(-1)^j a^j\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{bj} \\
&= kabg(x,\boldsymbol{\psi})\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}(-1)^j a^j\frac{[\overline{D}(x,\boldsymbol{\psi})]^{bj+b-1}}{[D(x,\boldsymbol{\psi})]^{bj+b+1}} \qquad (5)
\end{aligned}
$$

Note that using the definition of odds ratio, the function can be generalized as

$$
\begin{aligned}
[\overline{D}(x,\boldsymbol{\psi})]^{bj+b-1} &= [1-D(x,\boldsymbol{\psi})]^{bj+b-1} \\
&= \sum_{i=0}^{\infty}\binom{b(j+1)-1}{i}(-1)^i[D(x,\boldsymbol{\psi})]^i \qquad (6)
\end{aligned}
$$

Therefore we have the pdf as

$$
\begin{aligned}
f_{B-EOR-G}(x) &= kabg(x,\boldsymbol{\psi})\sum_{j=0}^{\infty}\binom{k+1+j-1}{j}(-1)^j a^j\frac{\sum_{i=0}^{\infty}\binom{b(j+1)-1}{i}(-1)^i[G(x,\boldsymbol{\psi})]^i}{[D(x,\boldsymbol{\psi})]^{bj+b+1}} \\
&= kabg(x,\boldsymbol{\psi})\sum_{i,j=0}^{\infty}\binom{k+1+j-1}{j}(-1)^{i+j}a^j\binom{b(j+1)-1}{i}[D(x,\boldsymbol{\psi})]^{i-bj-b-1} \\
&= \sum_{i,j=0}^{\infty}c_{i,j}r_{i-bj-b-1}(x,\boldsymbol{\psi}) \qquad (7)
\end{aligned}
$$

where

$$
c_{i,j} = \frac{kaba^j(-1)^{i+j}\binom{k+1+j-1}{j}\binom{b(j+1)-1}{i}}{i-bj-b} \qquad (8)
$$

and

$$
r_{i-bj-b-1}(x,\boldsymbol{\psi}) = (i-bj-b)g(x,\boldsymbol{\psi})[D(x,\boldsymbol{\psi})]^{i-bj-b-1} \qquad (9)
$$

which is the pdf of exponentiated generalized distribution with parameter $b^* = i-bj-b$. □

2 Hazard Rate Function

Theorem 2. *The hazard rate function of the Burr III Scaled Inverse Odds Ratio-G distribution is given by:*

$$
h_{\text{B-SIOR-G}}(x) = \frac{kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b\right)^{-k}}, \qquad (10)
$$

and the reverse hazard rate function is:

$$\tau_{B\text{-}SIOR\text{-}G}(x) = kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-1}. \tag{11}$$

Proof. Let X denote a continuous random variable with pdf $f(x)$, cdf $F(x)$, and survival function $\overline{F}(x) = 1 - F(x)$. The hazard rate function (hrf), mean residual life function (mrlf), and reverse hazard rate function (rhrf) of X are defined as $h_F(x) = \frac{f(x)}{\overline{F}(x)}$, $\sigma_F(x) = \int_x^\infty \frac{\overline{F}(u)}{\overline{F}(x)}du$, and $\tau_F(x) = \frac{f(x)}{F(x)}$, respectively.

In their seminal work, [1] demonstrated the equivalent behavior of $h_F(x)$, $\tau_F(x)$, and $\sigma_F(x)$. In this paper, our focus is on presenting the hazard rate function, while acknowledging that similar derivations can be made for the residual life and reverse hazard functions. The hazard rate function for the Burr III Exponentiated Odds Ratio-G distribution is expressed as

$$h_{B-EOR-G}(x) = \frac{kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k}} \tag{12}$$

and the reverse hazard rate function is given by

$$\begin{aligned}\tau_{B-EOR-G}(x) &= \frac{kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k-1}}{\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k}} \\ &= kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-1}\end{aligned} \tag{13}$$

□

3 Quantile Function

Theorem 3. *The quantile function for the Burr III Scaled Inverse Odds Ratio-G distribution is defined as:*

$$x_p = G^{-1}(q), \tag{14}$$

where $0 \leq p \leq 1$ *and:*

$$q = \frac{1}{1+\left(\frac{p^{-\frac{1}{k}}-1}{a}\right)^{\frac{1}{b}}}. \tag{15}$$

Proof. Let

$$F_{B-EOR-G}(x) = \left(1 + a\left[\frac{\overline{G}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b\right)^{-k} = p$$

for some $0 \le p \le 1$. Then we have

$$D(x,\boldsymbol{\psi}) = \frac{1}{1+\left(\frac{p^{-\frac{1}{k}}-1}{a}\right)^{\frac{1}{b}}} := q \tag{16}$$

As a result, the quantile x_p of the distribution reduces to the quantile x_q of the baseline distribution with cdf $D(x,\boldsymbol{\psi})$, and it is given by the equation

$$x_q = G^{-1}(q) \tag{17}$$

□

4 Moments

Theorem 4. *For a random variable $Y \sim$ B-SIOR-G$(x;k,a,b,\boldsymbol{\psi})$, the r^{th} moment of the B-SIOR-G distribution is:*

$$E(Y^r) = \sum_{i,j=0}^{\infty} c_{i,j} E(Z_{i,j}^r), \tag{18}$$

where $Z_{i,j}$ follows the exponentiated generalized distribution with parameter $b^* = i - bj - b$ and $c_{i,j}$ as defined in Theorem 1.

Proof. Based on Theorem 1, we can represent the r^{th} moment of the distribution as

$$\begin{aligned} E(Y^r) &= \int_0^\infty y^r f_{B-EOR-G}(y)dy = \int_0^\infty y^r \sum_{i,j=0}^{\infty} c_{i,j} r_{i-bj-b-1}(y,\boldsymbol{\psi})dy \\ &= \sum_{i,j=0}^{\infty} c_{i,j} \int_0^\infty y^r r_{i-bj-b-1}(y,\boldsymbol{\psi})dy = \sum_{i,j=0}^{\infty} c_{i,j} E(Z_{i,j}^r) \end{aligned} \tag{19}$$

where $Z_{i,j}$ is the exponentiated generalized distribution with the parameter $b^* = i - bj - b$ and $c_{i,j}$ as being defined in Theorem 1.

□

5 Incomplete Moments

Theorem 5. *The incomplete moment for the distribution is formulated as:*

$$I_Y(z) = \sum_{i,j=0}^{\infty} c_{i,j} I_{i,j}(y), \tag{20}$$

where $I_{i,j}(y) = \int_0^z y^s r_{i-bj-b}(x, \boldsymbol{\psi})$.

Proof. The incomplete moment of the distribution is in the form of

$$I_Y(z) = \int_0^z y^s f_{B-EOR-G}(y)\, dy = \sum_{i,j=0}^{\infty} c_{i,j} I_{i,j}(y) \tag{21}$$

where $I_{i,j}(y) = \int_0^z y^s r_{i-bj-b}(x, \boldsymbol{\psi})$ and $r_{i-bj-b}(x, \boldsymbol{\psi})$ as being defined in Theorem 1.

□

6 Moment Generating Functions

Theorem 6. *The moment generating function (mgf) for the distribution is:*

$$M_Y(t) = \sum_{i,j=0}^{\infty} c_{i,j} M_{Z_{i,j}}(t), \tag{22}$$

where $M_{Z_{i,j}}(t)$ *denotes the mgf of the exponentiated generalized distribution with parameter* $b^* = i - bj - b$.

Proof. The moment generating function (mgf) of the distribution takes the form

$$M_Y(t) = E(e^{tY}) = \sum_{i,j=0}^{\infty} c_{i,j} E(e^{tZ_{i,j}}) = \sum_{i,j=0}^{\infty} c_{i,j} M_{Z_{i,j}}(t) \tag{23}$$

□

7 Rényi Entropy

Theorem 8. *The Rényi Entropy for the B-SIOR-G distribution is calculated as:*

$$I_R(\omega) = (1-\omega)^{-1} \Bigg\{ \omega(\log(k) + \log(a) + \log(b)) + \log \Bigg[\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a^j (-1)^{i+j} \times \binom{\omega(k+1)+j-1}{j} \binom{\omega b - \omega + jb}{i} \left(\frac{\omega}{i - \omega b - jb} \right)^{\omega} \times e^{(i-\omega) I_{REG}} \Bigg] \Bigg\},$$

where $0 < \omega \neq 1$, indicating the diversity of values the distribution can take, with I_{REG} being the Rényi Entropy for the exponentiated generalized distribution parameterized by $b^* = \frac{i-\omega b-jb}{\omega}$.

Proof. The Rényi Entropy of the B-EOR-G distribution is given by

$$\begin{aligned}
I_R(\omega) &= (1-\omega)^{-1}\log\left[\int_{-\infty}^{\infty} f^{\omega}(x)dx\right] \\
&= (1-\omega)^{-1}\Bigg\{\omega(\log(k)+\log(a)+\log(b))+\log\Bigg[\int_{-\infty}^{\infty} g^{\omega}(x,\boldsymbol{\psi}) \\
&\times\frac{[\overline{D}(x,\boldsymbol{\psi})]^{\omega(b-1)}}{[D(x,\boldsymbol{\psi})]^{\omega(b+1)}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-\omega(k+1)}dx\Bigg]\Bigg\}
\end{aligned} \tag{24}$$

where $\omega > 0$ and $\omega \neq 1$. By applying the same expansion technique for the pdf, we obtain

$$\begin{aligned}
I_R(\omega) &= (1-\omega)^{-1}\Bigg\{\omega(\log(k)+\log(a)+\log(b))+\log\Bigg[\int_{-\infty}^{\infty} g^{\omega}(x,\boldsymbol{\psi}) \\
&\times\frac{[\overline{D}(x,\boldsymbol{\psi})]^{\omega(b-1)}}{[D(x,\boldsymbol{\psi})]^{\omega(b+1)}}\sum_{j=0}^{\infty}\binom{\omega(k+1)+j-1}{j}(-1)^j\left(a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{j}dx\Bigg]\Bigg\} \\
&= (1-\omega)^{-1}\Bigg\{\omega(\log(k)+\log(a)+\log(b))+\log\Bigg[\sum_{j=0}^{\infty}\binom{\omega(k+1)+j-1}{j}(-1)^j a^j \\
&\times\int_{-\infty}^{\infty} g^{\omega}(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{\omega(b-1)+jb}}{[D(x,\boldsymbol{\psi})]^{\omega(b+1)+jb}}dx\Bigg]\Bigg\}
\end{aligned} \tag{25}$$

Using the definition of odds ratio mentioned in Eq. 6, we have

$$\begin{aligned}
[\overline{D}(x,\boldsymbol{\psi})]^{\omega(b-1)+jb} &= [1-D(x,\boldsymbol{\psi})]^{\omega(b-1)+jb} \\
&= \sum_{i=0}^{\infty}\binom{\omega b-\omega+jb}{i}(-1)^i[D(x,\boldsymbol{\psi})]^i
\end{aligned} \tag{26}$$

Therefore, the further expansion of the Rényi Entropy can be generalized as

$$
\begin{aligned}
I_R(\omega) &= (1-\omega)^{-1}\Bigg\{\omega(\log(k)+\log(a)+\log(b))+\log\Bigg[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}a^j(-1)^i \\
&\times\binom{\omega(k+1)+j-1}{j}(-1)^j\binom{\omega b-\omega+jb}{i}\int_{-\infty}^{\infty}g^{\omega}(x,\boldsymbol{\psi})[D(x,\boldsymbol{\psi})]^{i-\omega(b+1)-jb}dx\Bigg]\Bigg\} \\
&= (1-\omega)^{-1}\Bigg\{\omega(\log(k)+\log(a)+\log(b))+\log\Bigg[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}a^j(-1)^i \\
&\times\binom{\omega(k+1)+j-1}{j}(-1)^j\binom{\omega b-\omega+jb}{i}\frac{\omega^{\omega}}{[i-\omega(b+1)-jb+\omega]^{\omega}} \\
&\times\int_{-\infty}^{\infty}\left[\frac{i-\omega(b+1)-jb+\omega}{\omega}g(x,\boldsymbol{\psi})[D(x,\boldsymbol{\psi})]^{\frac{i-\omega b-\omega-jb}{\omega}}\right]^{\omega}dx\Bigg]\Bigg\} \\
&= (1-\omega)^{-1}\Bigg\{\omega(\log(k)+\log(a)+\log(b))+\log\Bigg[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}a^j(-1)^{i+j} \\
&\times\binom{\omega(k+1)+j-1}{j}\binom{\omega b-\omega+jb}{i}\left(\frac{\omega}{i-\omega b-jb}\right)^{\omega}\times e^{(i-\omega)I_{REG}}\Bigg]\Bigg\}
\end{aligned}
\tag{27}
$$

where I_{REG} is the Rényi Entropy of the exponentiated generalized distribution with parameter $W^* = \frac{i-\omega b-jb}{\omega}$. □

8 Order Statistics

Theorem 9. *For $X_1, X_2, \ldots, X_n$ as i.i.d. random variables from the B-SIOR-G distribution, the pdf of the j^{th} order statistic $f_{j:n}(x)$ is:*

$$
f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!}\sum_{s=0}^{n-j}\binom{n-j}{s}\frac{(-1)^s}{j+s}f_{B\text{-}SIOR\text{-}G}(x;(j+s)k,a,b), \tag{28}
$$

expressing $f_{j:n}(x)$ as a linear combination of B-SIOR-G with parameters (k^*, a, b), where $k^* = (j+s)k$.

Proof. Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variables distributed by B-SIOR-G distribution. The pdf of the j^{th} order statistic

$f_{j:n}(x)$ is given by

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!} f_{B-EOR-G}(x) \left[F_{B-EOR-G}(x)\right]^{j-1} \left[1 - F_{B-EOR-G}(x)\right]^{n-j}$$
$$= \frac{n!}{(j-1)!(n-j)!} f_{B-EOR-G}(x) \sum_{s=0}^{n-j} \binom{n-j}{s} (-1)^s \left[F_{B-EOR-G}(x)\right]^{j+s-1}$$
$$= \frac{n!}{(j-1)!(n-j)!} f_{B-EOR-G}(x) \sum_{s=0}^{n-j} \binom{n-j}{s} (-1)^s \left(1 + a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b\right)^{-k(j+s-1)}$$
$$= \frac{n!}{(j-1)!(n-j)!} \sum_{s=0}^{n-j} \binom{n-j}{s} \frac{(-1)^s}{j+s} f_{B-EOR-G}(x; (j+s)k, a, b) \quad (29)$$

$\square$

9 Stochastic Ordering

Theorem 10. *Given $X_1 \sim$ B-SIOR-G$(x; k, a_1, b, \boldsymbol{\psi})$ and $X_2 \sim$ B-SIOR-G$(x; k, a_2, b, \boldsymbol{\psi})$, the likelihood ratio Λ is:*

$$\Lambda = \frac{a_1}{a_2}\left[\frac{1 + a_2\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b}{1 + a_1\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b}\right]^{k+1}, \quad (30)$$

indicating the relative likelihood of outcomes from two distributions based on their parameters.

Proof. Consider two random variables $X_1 \sim$ *B-EOR-G*$(x; k, a_1, b, \boldsymbol{\psi})$ and $X_2 \sim$ *B-EOR-G*$(x; k, a_2, b, \boldsymbol{\psi})$. The likelihood ratio is defined as

$$\Lambda = \frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{k a_1 b g(x,\boldsymbol{\psi}) \frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}} \left(1 + a_1\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b\right)^{-k-1}}{k a_2 b g(x,\boldsymbol{\psi}) \frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}} \left(1 + a_2\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b\right)^{-k-1}}$$
$$= \frac{a_1}{a_2}\left[\frac{1 + a_2\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b}{1 + a_1\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^b}\right]^{k+1} \quad (31)$$

Next, we differentiate Eqn. (31) with respect to x and attain

$$\frac{d\Lambda}{dx} = \frac{(k+1)a_1 b}{a_2} g(x,\boldsymbol{\psi}) \frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}} \left(a_1 \left[1 + a_1 \left(\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right)^b \right]^{-(k+2)} \right.$$
$$\left. - a_2 \left[1 + a_2 \left(\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right)^b \right]^k \right). \tag{32}$$

$\square$

10 Maximum Likelihood Estimation

The first derivative of the log-likelihood function $\ell(\boldsymbol{\Delta}(\sigma))$ with respect to σ can be calculated as:

$$\frac{\partial \ell}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n} \log \left(1 + a \left[\frac{1 - G(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})} \right]^b \right) \tag{33}$$

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - (k+1) \sum_{i=1}^{n} \frac{[1 - G(x_i,\boldsymbol{\psi})]^b}{[G(x_i,\boldsymbol{\psi})]^b + a[1 - G(x_i,\boldsymbol{\psi})]^b} \tag{34}$$

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log[1 - G(x_i,\boldsymbol{\psi})] - \sum_{i=1}^{n} \log[G(x_i,\boldsymbol{\psi})] - a(k+1)$$
$$\times \sum_{i=1}^{n} \frac{[1 - G(x_i,\boldsymbol{\psi})]^b}{[G(x_i,\boldsymbol{\psi})]^b + a[1 - G(x_i,\boldsymbol{\psi})]^b} \log \left[\frac{1 - G(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})} \right] \tag{35}$$

$$\frac{\partial \ell}{\partial \psi_s} = \sum_{i=1}^{n} \frac{1}{g(x_i,\boldsymbol{\psi})} \frac{\partial g(x_i,\boldsymbol{\psi})}{\partial \psi_s} - (b-1) \sum_{i=1}^{n} \frac{1}{1 - G(x_i,\boldsymbol{\psi})} \frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s}$$
$$- (b+1) \sum_{i=1}^{n} \frac{1}{G(x_i,\boldsymbol{\psi})} \frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} + ab(k+1)$$
$$\times \sum_{i=1}^{n} \frac{[1 - G(x_i,\boldsymbol{\psi})]^{b-1}}{[G(x_i,\boldsymbol{\psi})]^{b+1} + aG(x_i,\boldsymbol{\psi})[1 - G(x_i,\boldsymbol{\psi})]^b} \frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} \tag{36}$$

where ψ_s is the s^{th} element of the vector parameter $\boldsymbol{\psi}$.

11 Least Square and Weighted Least Square Estimation

Differentiating the LS equation with respect to σ yields

$$\frac{\partial LS}{\partial k} = -2\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{i}{n+1}\right]\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} \times \log\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right) \tag{37}$$

$$\frac{\partial LS}{\partial a} = -2k\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{i}{n+1}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b \times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1} \tag{38}$$

$$\begin{aligned}\frac{\partial LS}{\partial b} &= -2ka\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{i}{n+1}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b \\ &\quad\times\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1}\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right] \\ &= a\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]\frac{\partial LS}{\partial a}\end{aligned} \tag{39}$$

$$\frac{\partial LS}{\partial \psi_s} = 2kab\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{i}{n+1}\right]\frac{[\overline{G}(x_i,\boldsymbol{\psi})]^{b-1}}{[G(x_i,\boldsymbol{\psi})]^{b+1}} \times\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} \tag{40}$$

12 Maximum Product Spacing Approach of Estimation

Let $\delta_j = 1 + a\left[\frac{\overline{G}(x_j,\boldsymbol{\psi})}{G(x_j,\boldsymbol{\psi})}\right]^b$ for $j = 1, 2, \cdots, n$. The first partial derivatives are provided as follows:

$$\frac{\partial L}{\partial k} = \frac{1}{n+1}\left\{\frac{(\delta_n)^{-k}\log(\delta_n)}{1-(\delta_n)^{-k}} - \log(\delta_1)\right.$$
$$\left. + \sum_{i=2}^{n}\frac{(\delta_{i-1})^{-k}\log(\delta_{i-1}) - (\delta_i)^{-k}\log(\delta_i)}{(\delta_i)^{-k}-(\delta_{i-1})^{-k}}\right\} \tag{41}$$

$$\frac{\partial L}{\partial a} = \frac{k}{n+1}\left\{\frac{(\delta_n)^{-k-1}\left[\frac{\overline{G}(x_n,\boldsymbol{\psi})}{G(x_n,\boldsymbol{\psi})}\right]^b}{1-(\delta_n)^{-k}} - \frac{\left[\frac{\overline{G}(x_1,\boldsymbol{\psi})}{G(x_1,\boldsymbol{\psi})}\right]^b}{\delta_1}\right.$$
$$\left. + \sum_{i=2}^{n}\frac{(\delta_{i-1})^{-k-1}\left[\frac{\overline{G}(x_{i-1},\boldsymbol{\psi})}{G(x_{i-1},\boldsymbol{\psi})}\right]^b - (\delta_i)^{-k-1}\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b}{(\delta_i)^{-k}-(\delta_{i-1})^{-k}}\right\} \tag{42}$$

$$\frac{\partial L}{\partial b} = \frac{ka}{n+1}\left\{\frac{(\delta_n)^{-k-1}\left[\frac{\overline{G}(x_n,\boldsymbol{\psi})}{G(x_n,\boldsymbol{\psi})}\right]^b\log\left[\frac{\overline{G}(x_n,\boldsymbol{\psi})}{G(x_n,\boldsymbol{\psi})}\right]}{1-(\delta_n)^{-k}} - \frac{\left[\frac{\overline{G}(x_1,\boldsymbol{\psi})}{G(x_1,\boldsymbol{\psi})}\right]^b\log\left[\frac{\overline{G}(x_1,\boldsymbol{\psi})}{G(x_1,\boldsymbol{\psi})}\right]}{\delta_1}\right.$$
$$\left. + \sum_{i=2}^{n}\frac{(\delta_{i-1})^{-k-1}\left[\frac{\overline{G}(x_{i-1},\boldsymbol{\psi})}{G(x_{i-1},\boldsymbol{\psi})}\right]^b\log\left[\frac{\overline{G}(x_{i-1},\boldsymbol{\psi})}{G(x_{i-1},\boldsymbol{\psi})}\right] - (\delta_i)^{-k-1}\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]}{(\delta_i)^{-k}-(\delta_{i-1})^{-k}}\right\} \tag{43}$$

$$\frac{\partial L}{\partial \psi_s} = \frac{kab}{n+1}\left\{\frac{\frac{\left[\overline{G}(x_1,\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_1,\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_1,\boldsymbol{\psi})}{\partial \psi_s}}{\delta_1} - \frac{(\delta_n)^{-k-1}\frac{\left[\overline{G}(x_n,\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_n,\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_n,\boldsymbol{\psi})}{\partial \psi_s}}{1-(\delta_n)^{-k}}\right.$$
$$\left. + \sum_{i=2}^{n}\frac{(\delta_i)^{-k-1}\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_i,\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} - (\delta_{i-1})^{-k-1}\frac{\left[\overline{G}(x_{i-1},\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_{i-1},\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_{i-1},\boldsymbol{\psi})}{\partial \psi_s}}{(\delta_i)^{-k}-(\delta_{i-1})^{-k}}\right\} \tag{44}$$

13 Cramér-von Mises Approach of Estimation

Differentiating $CVM(x,\sigma)$ with respect to σ yields

$$\frac{\partial CVM}{\partial k} = \frac{-2}{n}\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{2i-1}{2n}\right]\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k}$$
$$\times \log\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right) \qquad (45)$$

$$\frac{\partial CVM}{\partial a} = \frac{-2k}{n}\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{2i-1}{2n}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b$$
$$\times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1} \qquad (46)$$

$$\frac{\partial CVM}{\partial b} = \frac{-2ka}{n}\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{2i-1}{2n}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b$$
$$\times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1}\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]$$
$$= a\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]\frac{\partial LS}{\partial a} \qquad (47)$$

$$\frac{\partial CVM}{\partial \psi_s} = \frac{2kab}{n}\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{2i-1}{2n}\right]\frac{[\overline{G}(x_i,\boldsymbol{\psi})]^{b-1}}{[G(x_i,\boldsymbol{\psi})]^{b+1}}$$
$$\times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} \qquad (48)$$

14 Anderson and Darling Approach of Estimation

We take the first derivatives of $AD(\sigma)$ and acquire

$$\frac{\partial AD}{\partial k}=\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)\right.$$
$$\left.+\log\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)\frac{\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}{1-\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}\right] \tag{49}$$

$$\frac{\partial AD}{\partial a}=\frac{k}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b}{\left[G(x_i,\boldsymbol{\psi})\right]^b+a\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b}\right.$$
$$\left.+\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\frac{\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}\right] \tag{50}$$

$$\frac{\partial AD}{\partial b}=\frac{ka}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b}{\left[G(x_i,\boldsymbol{\psi})\right]^b+a\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b}\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]\right.$$
$$\left.+\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\log\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]\frac{\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}\right] \tag{51}$$

$$\frac{\partial AD}{\partial \psi_s}=\frac{-kab}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_i,\boldsymbol{\psi})\right]^{b+1}+a\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b G(x_i,\boldsymbol{\psi})}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial\psi_s}\right.$$
$$\left.+\frac{\left[\overline{G}(x_{n+1-i},\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_{n+1-i},\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_{n+1-i},\boldsymbol{\psi})}{\partial\psi_s}\frac{\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}\right] \tag{52}$$

15 More Plots for Special Cases

In this section, we offer additional illustrations depicting the skewness and kurtosis for specific instances with set parameters of the B-SIOR-G distribution.

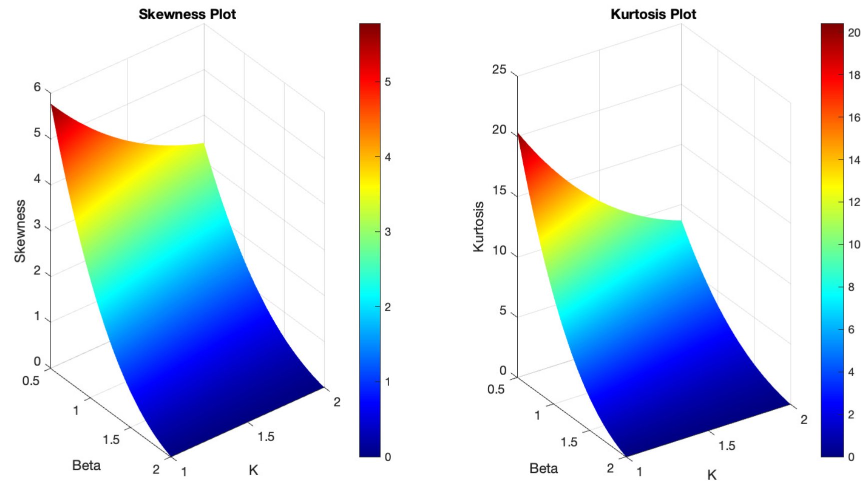

Figure 1: Left: The skewness plot of *B-SIOR-E*$(x; k, a = 1, b, \lambda = 1)$. Right: The kurtosis plot of *B-SIOR-E*$(x; k, a = 1, b, \lambda = 1)$.

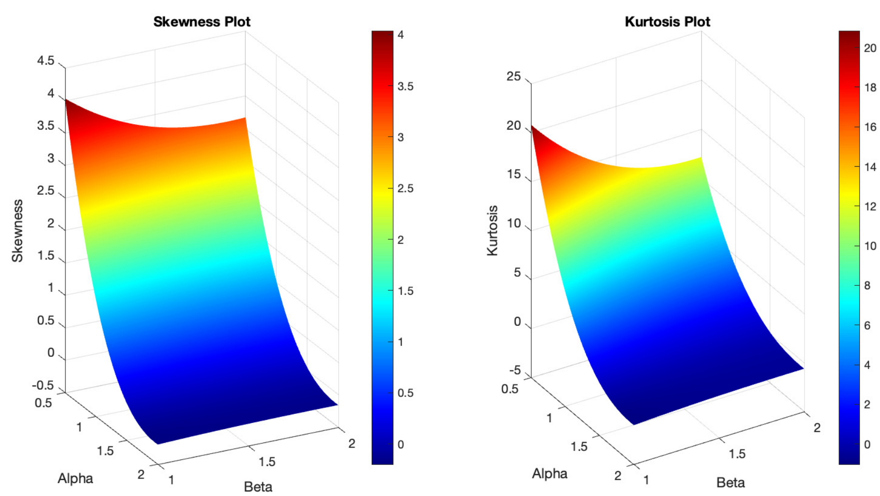

Figure 2: Left: The skewness plot of *B-EOR-E*$(x; k = 1, a, b, \lambda = 1)$. Right: The kurtosis plot of *B-EOR-E*$(x; k = 1, a, b, \lambda = 1)$.

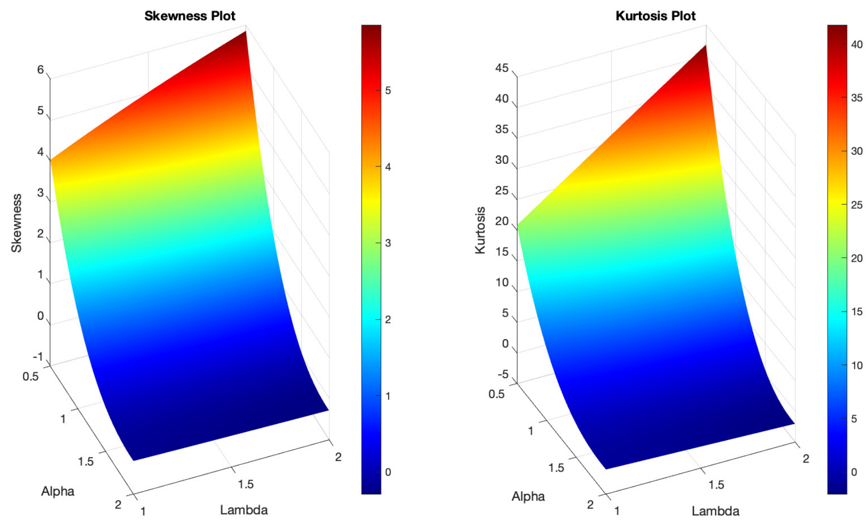

Figure 3: Left: The skewness plot of $B\text{-}EOR\text{-}E(x; k = 1, a, b = 1, \lambda)$. Right: The kurtosis plot of $B\text{-}EOR\text{-}E(x; k = 1, a, b = 1, \lambda)$.

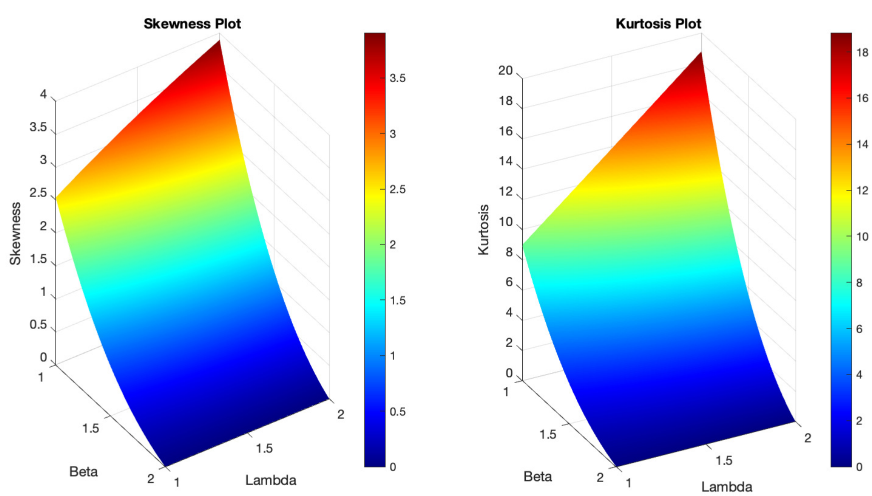

Figure 4: Left: The skewness plot of $B\text{-}EOR\text{-}E(x; k = 1, a = 1, b, \lambda)$. Right: The kurtosis plot of $B\text{-}EOR\text{-}E(x; k = 1, a = 1, b, \lambda)$.

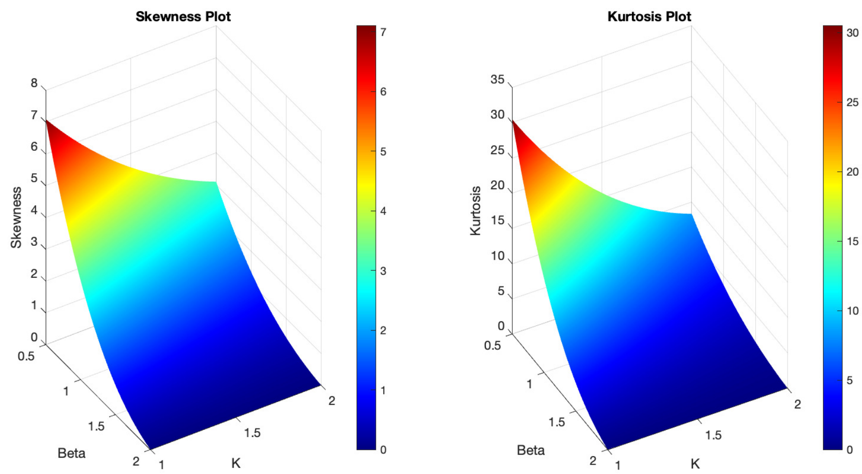

Figure 5: Left: The skewness plot of $B\text{-}SIOR\text{-}U(x; k, a = 0.5, b, \lambda = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}U(x; k, a = 0.5, b, \lambda = 1)$.

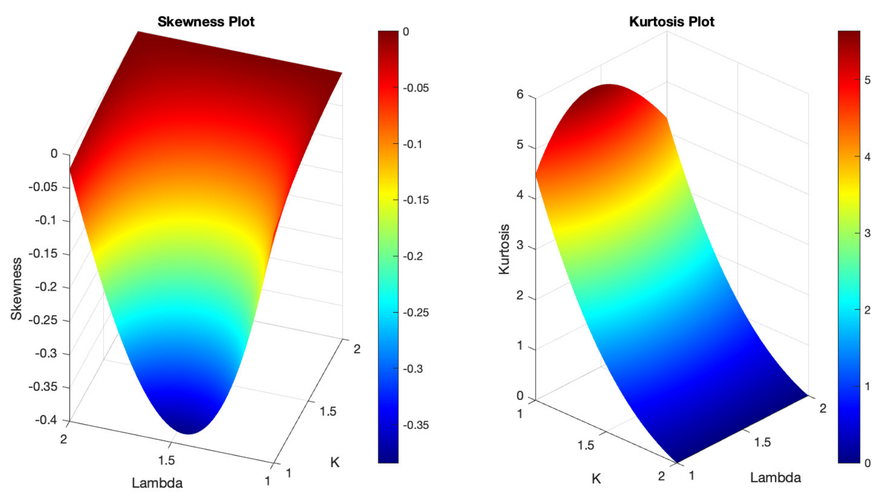

Figure 6: Left: The skewness plot of $B\text{-}SIOR\text{-}U(x; k, a = 1, b = 1, \lambda)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}U(x; k, a = 1, b = 1, \lambda)$.

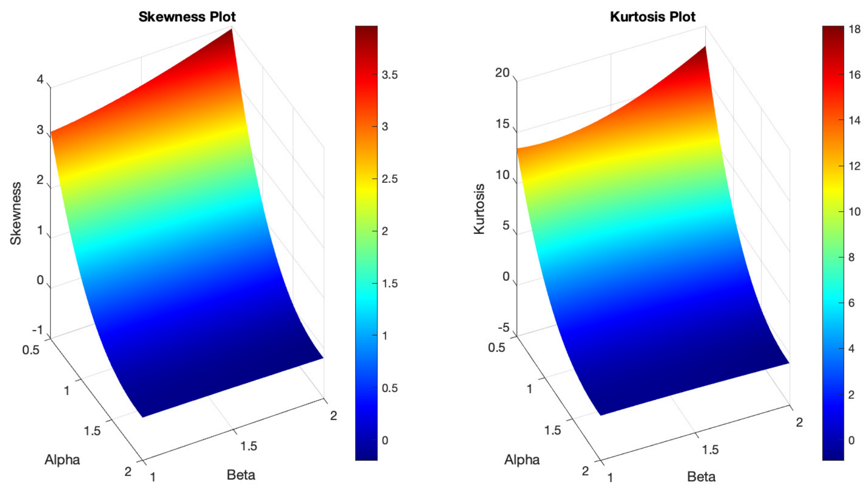

Figure 7: Left: The skewness plot of $B\text{-}SIOR\text{-}U(x; k = 1, a, b, \lambda = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}U(x; k = 1, a, b, \lambda = 1)$.

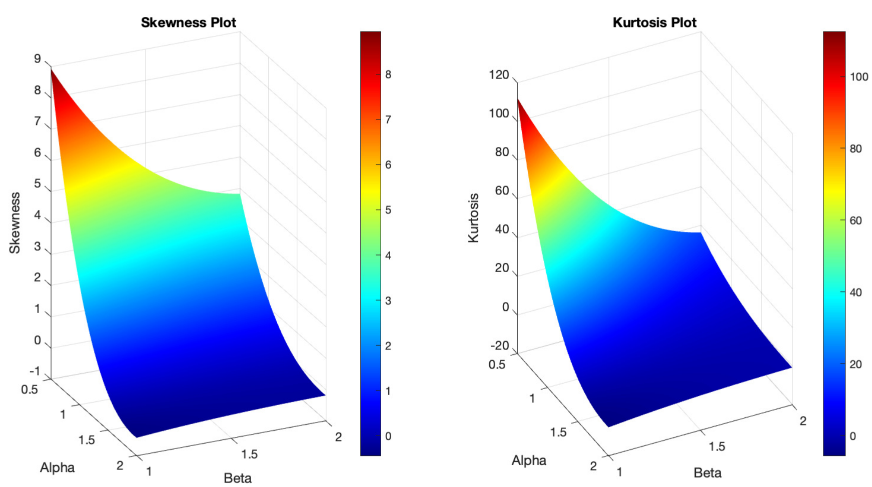

Figure 8: Left: The skewness plot of $B\text{-}SIOR\text{-}U(x; k = 0.5, a, b, \lambda = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}U(x; k = 0.5, a, b, \lambda = 1)$.

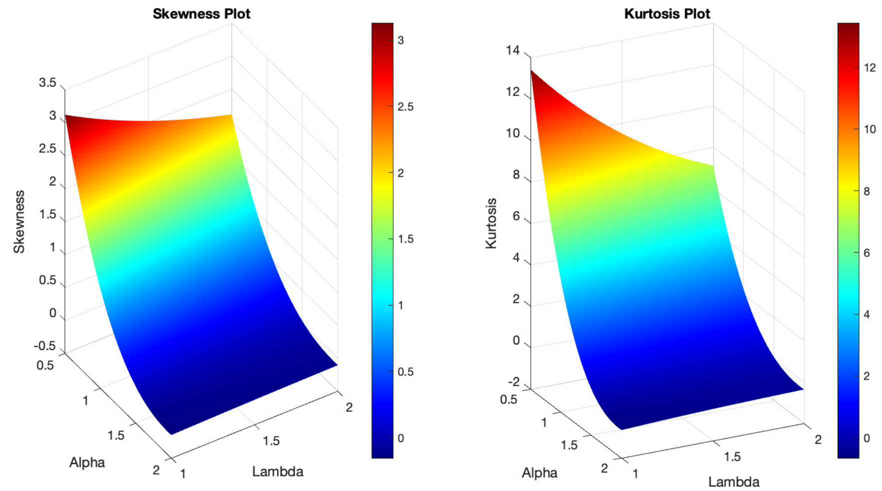

Figure 9: Left: The skewness plot of $B\text{-}SIOR\text{-}U(x; k = 1, a, b = 1, \lambda)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}U(x; k = 1, a, b = 1, \lambda)$.

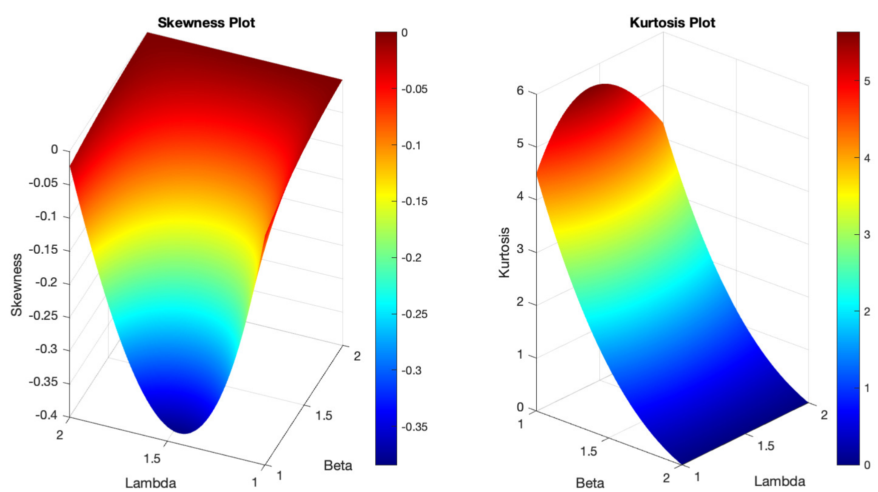

Figure 10: Left: The skewness plot of $B\text{-}SIOR\text{-}U(x; k = 1, a = 1, b, \lambda)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}U(x; k = 1, a = 1, b, \lambda)$.

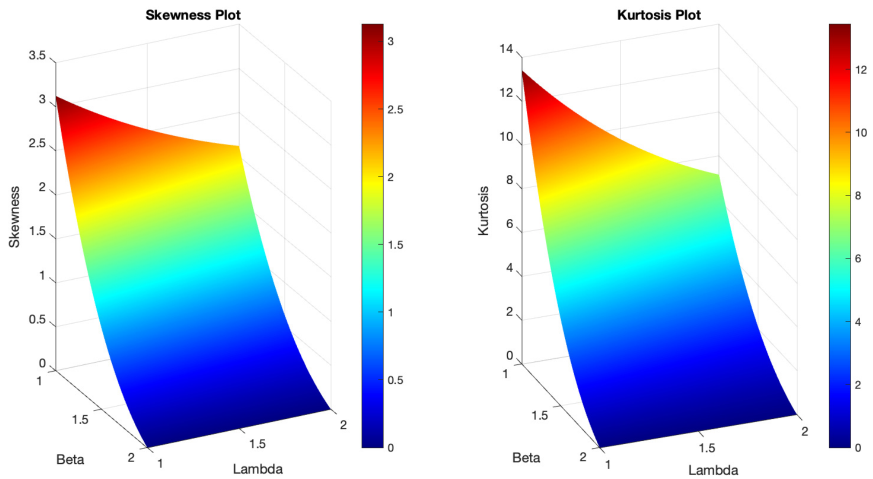

Figure 11: Left: The skewness plot of $B\text{-}SIOR\text{-}U(x; k=1, a=0.5, b, \lambda)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}U(x; k=1, a=0.5, b, \lambda)$.

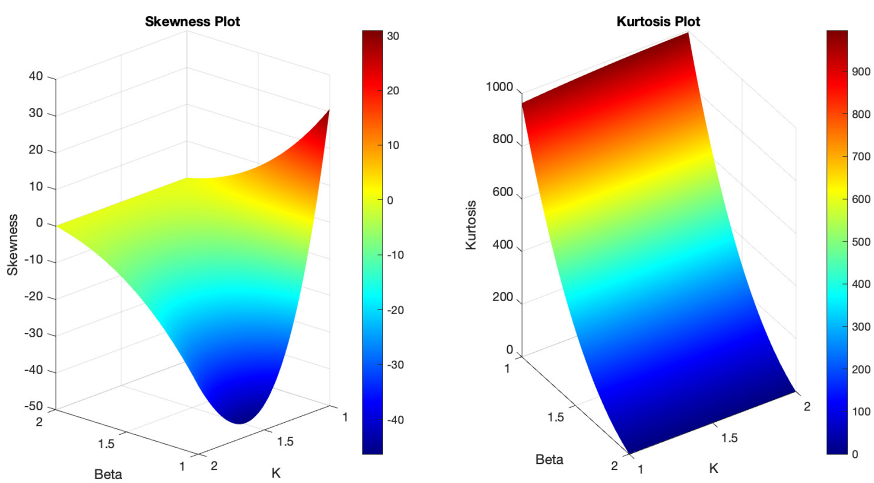

Figure 12: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k, a=1, b, c=2, \theta=1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k, a=1, b, c=2, \theta=1)$.

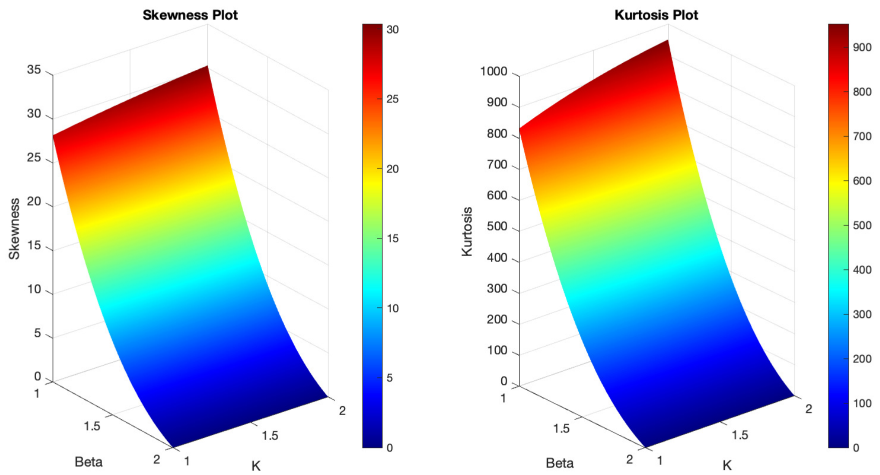

Figure 13: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k, a = 0.5, b, c = 2, \theta = 0.5)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k, a = 0.5, b, c = 2, \theta = 0.5)$.

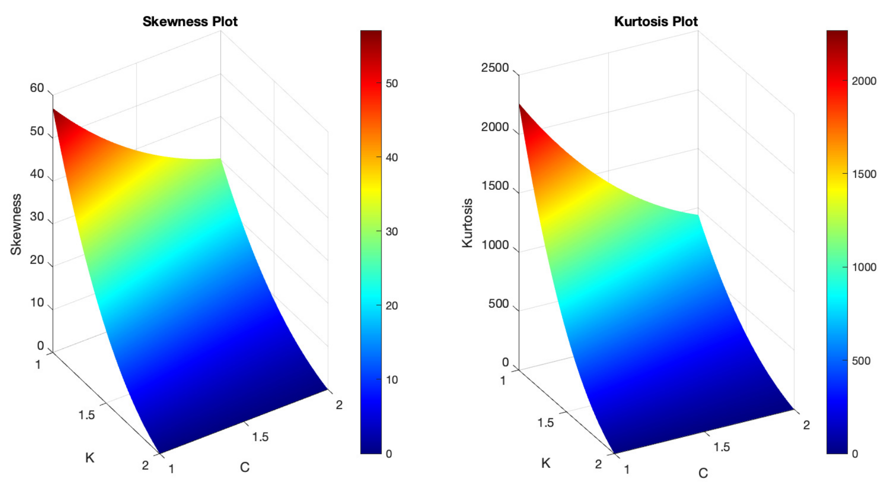

Figure 14: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k, a = 0.5, b = 1, c, \theta = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k, a = 0.5, b = 1, c, \theta = 1)$.

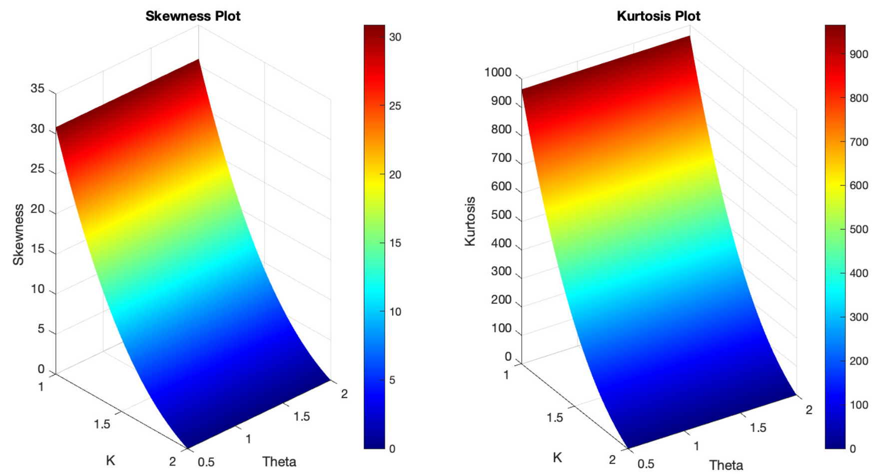

Figure 15: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k, a = 1, b = 1, c = 2, \theta)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k, a = 1, b = 1, c = 2, \theta)$.

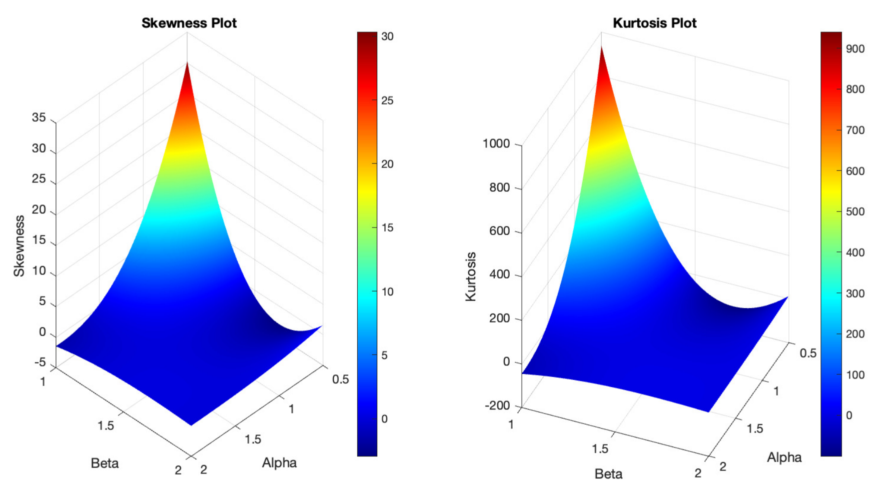

Figure 16: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b, c = 2, \theta = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b, c = 2, \theta = 1)$.

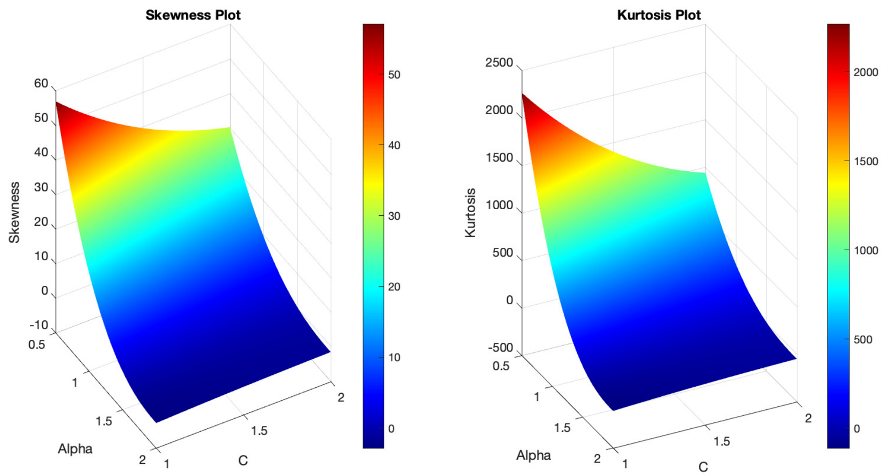

Figure 17: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b = 1, c, \theta = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b = 1, c, \theta = 1)$.

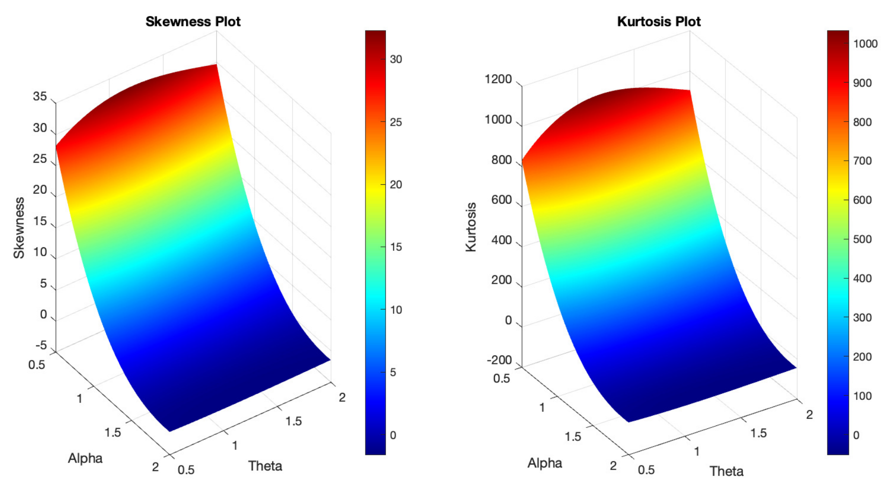

Figure 18: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b = 1, c = 2, \theta)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b = 1, c = 2, \theta)$.

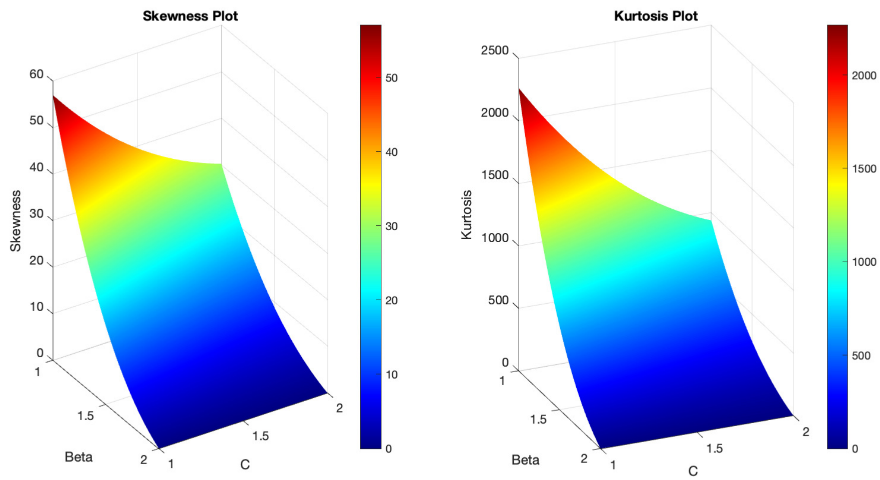

Figure 19: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b, c, \theta = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b, c, \theta = 1)$.

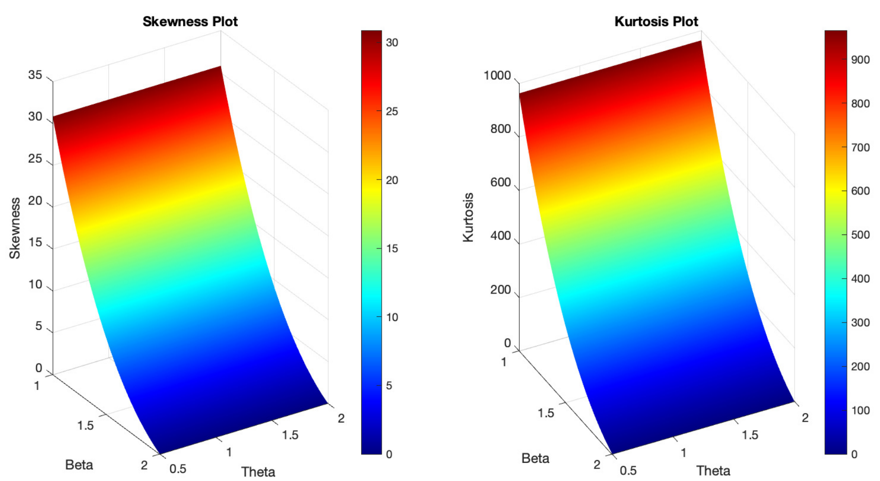

Figure 20: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 1, b, c = 2, \theta)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 1, b, c = 2, \theta)$.

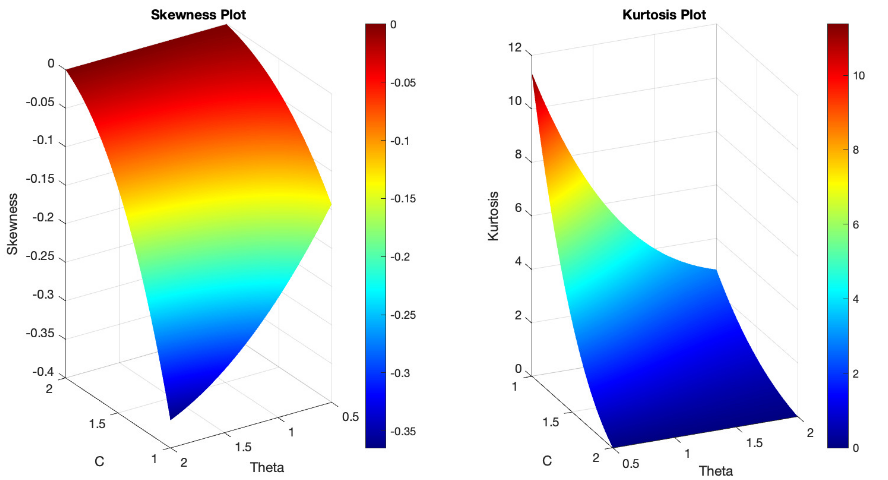

Figure 21: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b = 2, c, \theta)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b = 2, c, \theta)$.

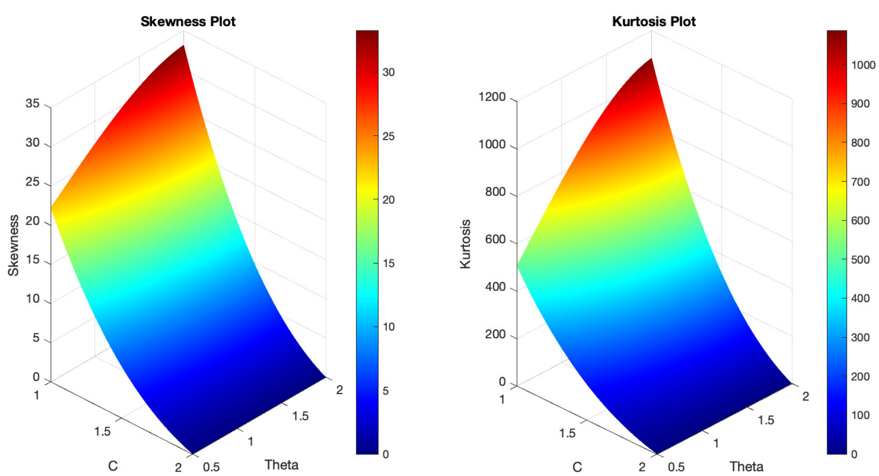

Figure 22: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b = 5, c, \theta)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b = 5, c, \theta)$.

References

[1] M. Shaked and J. G. Shanthikumar, *Stochastic orders and their applications.* Academic press, 1994.