Supplementary Information

Enhanced Real-life Data Modeling with the Modified Burr III Odds Ratio-G Distribution

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S1 Hazard Rate Function

Remark 1. The hazard rate function of the Burr III Scaled Inverse Odds Ratio-G distribution is given by:

$$h_{\text{B-SIOR-G}}(x) = \frac{kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k}}, \tag{S1}$$

and the reverse hazard rate function is:

$$\tau_{\text{B-SIOR-G}}(x) = kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-1}. \tag{S2}$$

Proof. Let X denote a continuous random variable with pdf $f(x)$, cdf $F(x)$, and survival function $\overline{F}(x) = 1 - F(x)$. The hazard rate function (hrf), mean residual life function (mrlf), and reverse hazard rate function (rhrf) of X are defined as $h_F(x) = \frac{f(x)}{\overline{F}(x)}$, $\sigma_F(x) = \int_x^\infty \frac{\overline{F}(u)}{\overline{F}(x)}du$, and $\tau_F(x) = \frac{f(x)}{F(x)}$, respectively.

In their seminal work, [1] demonstrated the equivalent behavior of $h_F(x)$, $\tau_F(x)$, and $\sigma_F(x)$. In this paper, our focus is on presenting the hazard rate function, while acknowledging that similar derivations can be made for the residual life and reverse hazard functions. The hazard rate function for the

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Burr III Exponentiated Odds Ratio-G distribution is expressed as

$$h_{B-SIOR-G}(x) = \frac{kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k}} \tag{S3}$$

and the reverse hazard rate function is given by

$$\begin{aligned}\tau_{B-SIOR-G}(x) &= \frac{kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k-1}}{\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k}} \\ &= kabg(x,\boldsymbol{\psi})\frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}}\left(1+a\left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-1} \end{aligned} \tag{S4}$$

□

S2 Quantile Function

Remark 2. The quantile function for the Burr III Scaled Inverse Odds Ratio-G distribution is defined as:

$$x_p = G^{-1}(q), \tag{S5}$$

where $0 \leq p \leq 1$ and:

$$q = \frac{1}{1+\left(\frac{p^{-\frac{1}{k}}-1}{a}\right)^{\frac{1}{b}}}. \tag{S6}$$

Proof. Let

$$F_{B-SIOR-G}(x) = \left(1+a\left[\frac{\overline{G}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})}\right]^{b}\right)^{-k} = p$$

for some $0 \leq p \leq 1$. Then we have

$$D(x,\boldsymbol{\psi}) = \frac{1}{1+\left(\frac{p^{-\frac{1}{k}}-1}{a}\right)^{\frac{1}{b}}} := q \tag{S7}$$

As a result, the quantile x_p of the distribution reduces to the quantile x_q of the baseline distribution with cdf $D(x,\boldsymbol{\psi})$, and it is given by the equation

$$x_q = G^{-1}(q) \tag{S8}$$

□

S3 Moments

Remark 3. For a random variable $Y \sim$ *B-SIOR-G*$(x; k, a, b, \boldsymbol{\psi})$, the r^{th} moment of the B-SIOR-G distribution is:

$$E(Y^r) = \sum_{i,j=0}^{\infty} c_{i,j} E(Z_{i,j}^r), \tag{S9}$$

where $Z_{i,j}$ follows the exponentiated generalized distribution with parameter $b^* = i - bj - b$ and $c_{i,j}$ as defined in Theorem 1.

Proof. Based on Theorem 1, we can represent the r^{th} moment of the distribution as

$$\begin{aligned} E(Y^r) &= \int_0^{\infty} y^r f_{B-SIOR-G}(y) dy = \int_0^{\infty} y^r \sum_{i,j=0}^{\infty} c_{i,j} r_{i-bj-b-1}(y, \boldsymbol{\psi}) dy \\ &= \sum_{i,j=0}^{\infty} c_{i,j} \int_0^{\infty} y^r r_{i-bj-b-1}(y, \boldsymbol{\psi}) dy = \sum_{i,j=0}^{\infty} c_{i,j} E(Z_{i,j}^r) \end{aligned} \tag{S10}$$

where $Z_{i,j}$ is the exponentiated generalized distribution with the parameter $b^* = i - bj - b$ and $c_{i,j}$ as being defined in Theorem 1.

□

S4 Incomplete Moments

Remark 4. The incomplete moment for the distribution is formulated as:

$$I_Y(z) = \sum_{i,j=0}^{\infty} c_{i,j} I_{i,j}(y), \tag{S11}$$

where $I_{i,j}(y) = \int_0^z y^s r_{i-bj-b}(x, \boldsymbol{\psi})$.

Proof. The incomplete moment of the distribution is in the form of

$$I_Y(z) = \int_0^z y^s f_{B-SIOR-G}(y)\, dy = \sum_{i,j=0}^{\infty} c_{i,j} I_{i,j}(y) \tag{S12}$$

where $I_{i,j}(y) = \int_0^z y^s r_{i-bj-b}(x, \boldsymbol{\psi})$ and $r_{i-bj-b}(x, \boldsymbol{\psi})$ as being defined in Theorem 1.

□

S5 Moment Generating Functions

Remark 5. The moment generating function (mgf) for the distribution is:

$$M_Y(t) = \sum_{i,j=0}^{\infty} c_{i,j} M_{Z_{i,j}}(t), \tag{S13}$$

where $M_{Z_{i,j}}(t)$ denotes the mgf of the exponentiated generalized distribution with parameter $b^* = i - bj - b$.

Proof. The moment generating function (mgf) of the distribution takes the form

$$M_Y(t) = E(e^{tY}) = \sum_{i,j=0}^{\infty} c_{i,j} E(e^{tZ_{i,j}}) = \sum_{i,j=0}^{\infty} c_{i,j} M_{Z_{i,j}}(t) \tag{S14}$$

□

S6 Stochastic Ordering

Theorem 10. *Given $X_1 \sim$ B-SIOR-G$(x; k, a_1, b, \boldsymbol{\psi})$ and $X_2 \sim$ B-SIOR-G$(x; k, a_2, b, \boldsymbol{\psi})$, the likelihood ratio Λ is:*

$$\Lambda = \frac{a_1}{a_2} \left[\frac{1 + a_2 \left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right]^b}{1 + a_1 \left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right]^b} \right]^{k+1}, \tag{S15}$$

indicating the relative likelihood of outcomes from two distributions based on their parameters.

Proof. Consider two random variables $X_1 \sim$ *B-SIOR-G*$(x; k, a_1, b, \boldsymbol{\psi})$ and $X_2 \sim$ *B-SIOR-G*$(x; k, a_2, b, \boldsymbol{\psi})$. The likelihood ratio is defined as

$$\begin{aligned} \Lambda = \frac{f_{X_1}(x)}{f_{X_2}(x)} &= \frac{k a_1 b g(x,\boldsymbol{\psi}) \frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}} \left(1 + a_1 \left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right]^b \right)^{-k-1}}{k a_2 b g(x,\boldsymbol{\psi}) \frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}} \left(1 + a_2 \left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right]^b \right)^{-k-1}} \\ &= \frac{a_1}{a_2} \left[\frac{1 + a_2 \left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right]^b}{1 + a_1 \left[\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right]^b} \right]^{k+1} \end{aligned} \tag{S16}$$

Next, we differentiate Eqn. (S16) with respect to x and attain

$$\frac{d\Lambda}{dx} = \frac{(k+1)a_1 b}{a_2} g(x,\boldsymbol{\psi}) \frac{[\overline{D}(x,\boldsymbol{\psi})]^{b-1}}{[D(x,\boldsymbol{\psi})]^{b+1}} \left(a_1 \left[1 + a_1 \left(\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right)^b \right]^{-(k+2)} \right.$$
$$\left. - a_2 \left[1 + a_2 \left(\frac{\overline{D}(x,\boldsymbol{\psi})}{D(x,\boldsymbol{\psi})} \right)^b \right]^k \right). \tag{S17}$$

□

S7 Maximum Likelihood Estimation

The first derivative of the log-likelihood function $\ell(\boldsymbol{\Delta}(\sigma))$ with respect to σ can be calculated as:

$$\frac{\partial \ell}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n} \log \left(1 + a \left[\frac{1 - G(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})} \right]^b \right) \tag{S18}$$

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - (k+1) \sum_{i=1}^{n} \frac{[1 - G(x_i,\boldsymbol{\psi})]^b}{[G(x_i,\boldsymbol{\psi})]^b + a[1 - G(x_i,\boldsymbol{\psi})]^b} \tag{S19}$$

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log[1 - G(x_i,\boldsymbol{\psi})] - \sum_{i=1}^{n} \log[G(x_i,\boldsymbol{\psi})] - a(k+1)$$
$$\times \sum_{i=1}^{n} \frac{[1 - G(x_i,\boldsymbol{\psi})]^b}{[G(x_i,\boldsymbol{\psi})]^b + a[1 - G(x_i,\boldsymbol{\psi})]^b} \log \left[\frac{1 - G(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})} \right] \tag{S20}$$

$$\frac{\partial \ell}{\partial \psi_s} = \sum_{i=1}^{n} \frac{1}{g(x_i,\boldsymbol{\psi})} \frac{\partial g(x_i,\boldsymbol{\psi})}{\partial \psi_s} - (b-1) \sum_{i=1}^{n} \frac{1}{1 - G(x_i,\boldsymbol{\psi})} \frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s}$$
$$- (b+1) \sum_{i=1}^{n} \frac{1}{G(x_i,\boldsymbol{\psi})} \frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} + ab(k+1)$$
$$\times \sum_{i=1}^{n} \frac{[1 - G(x_i,\boldsymbol{\psi})]^{b-1}}{[G(x_i,\boldsymbol{\psi})]^{b+1} + aG(x_i,\boldsymbol{\psi})[1 - G(x_i,\boldsymbol{\psi})]^b} \frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} \tag{S21}$$

where ψ_s is the s^{th} element of the vector parameter $\boldsymbol{\psi}$.

S8 Least Square and Weighted Least Square Estimation

Differentiating the LS equation with respect to σ yields

$$\frac{\partial LS}{\partial k} = -2\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{i}{n+1}\right]\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} \times \log\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right) \tag{S22}$$

$$\frac{\partial LS}{\partial a} = -2k\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{i}{n+1}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b \times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1} \tag{S23}$$

$$\begin{aligned}\frac{\partial LS}{\partial b} &= -2ka\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{i}{n+1}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b \\ &\quad\times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1}\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right] \\ &= a\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]\frac{\partial LS}{\partial a}\end{aligned} \tag{S24}$$

$$\frac{\partial LS}{\partial \psi_s} = 2kab\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{i}{n+1}\right]\frac{[\overline{G}(x_i,\boldsymbol{\psi})]^{b-1}}{[G(x_i,\boldsymbol{\psi})]^{b+1}} \times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} \tag{S25}$$

S9 Maximum Product Spacing Approach of Estimation

Let $\delta_j = 1 + a\left[\frac{\overline{G}(x_j,\boldsymbol{\psi})}{G(x_j,\boldsymbol{\psi})}\right]^b$ for $j = 1, 2, \cdots, n$. The first partial derivatives are provided as follows:

$$\frac{\partial L}{\partial k} = \frac{1}{n+1}\left\{ \frac{(\delta_n)^{-k}\log(\delta_n)}{1-(\delta_n)^{-k}} - \log(\delta_1) \right.$$
$$\left. + \sum_{i=2}^{n} \frac{(\delta_{i-1})^{-k}\log(\delta_{i-1}) - (\delta_i)^{-k}\log(\delta_i)}{(\delta_i)^{-k} - (\delta_{i-1})^{-k}} \right\} \tag{S26}$$

$$\frac{\partial L}{\partial a} = \frac{k}{n+1}\left\{ \frac{(\delta_n)^{-k-1}\left[\frac{\overline{G}(x_n,\boldsymbol{\psi})}{G(x_n,\boldsymbol{\psi})}\right]^b}{1-(\delta_n)^{-k}} - \frac{\left[\frac{\overline{G}(x_1,\boldsymbol{\psi})}{G(x_1,\boldsymbol{\psi})}\right]^b}{\delta_1} \right.$$
$$\left. + \sum_{i=2}^{n} \frac{(\delta_{i-1})^{-k-1}\left[\frac{\overline{G}(x_{i-1},\boldsymbol{\psi})}{G(x_{i-1},\boldsymbol{\psi})}\right]^b - (\delta_i)^{-k-1}\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b}{(\delta_i)^{-k} - (\delta_{i-1})^{-k}} \right\} \tag{S27}$$

$$\frac{\partial L}{\partial b} = \frac{ka}{n+1}\left\{ \frac{(\delta_n)^{-k-1}\left[\frac{\overline{G}(x_n,\boldsymbol{\psi})}{G(x_n,\boldsymbol{\psi})}\right]^b \log\left[\frac{\overline{G}(x_n,\boldsymbol{\psi})}{G(x_n,\boldsymbol{\psi})}\right]}{1-(\delta_n)^{-k}} - \frac{\left[\frac{\overline{G}(x_1,\boldsymbol{\psi})}{G(x_1,\boldsymbol{\psi})}\right]^b \log\left[\frac{\overline{G}(x_1,\boldsymbol{\psi})}{G(x_1,\boldsymbol{\psi})}\right]}{\delta_1} \right.$$
$$\left. + \sum_{i=2}^{n} \frac{(\delta_{i-1})^{-k-1}\left[\frac{\overline{G}(x_{i-1},\boldsymbol{\psi})}{G(x_{i-1},\boldsymbol{\psi})}\right]^b \log\left[\frac{\overline{G}(x_{i-1},\boldsymbol{\psi})}{G(x_{i-1},\boldsymbol{\psi})}\right] - (\delta_i)^{-k-1}\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b \log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]}{(\delta_i)^{-k} - (\delta_{i-1})^{-k}} \right\} \tag{S28}$$

$$\frac{\partial L}{\partial \psi_s} = \frac{kab}{n+1}\left\{ \frac{\frac{\left[\overline{G}(x_1,\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_1,\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_1,\boldsymbol{\psi})}{\partial \psi_s}}{\delta_1} - \frac{(\delta_n)^{-k-1}\frac{\left[\overline{G}(x_n,\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_n,\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_n,\boldsymbol{\psi})}{\partial \psi_s}}{1-(\delta_n)^{-k}} \right.$$
$$\left. + \sum_{i=2}^{n} \frac{(\delta_i)^{-k-1}\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_i,\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} - (\delta_{i-1})^{-k-1}\frac{\left[\overline{G}(x_{i-1},\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_{i-1},\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_{i-1},\boldsymbol{\psi})}{\partial \psi_s}}{(\delta_i)^{-k} - (\delta_{i-1})^{-k}} \right\} \tag{S29}$$

S10 Cramér-von Mises Approach of Estimation

Differentiating $CVM(x, \sigma)$ with respect to σ yields

$$\frac{\partial CVM}{\partial k} = \frac{-2}{n}\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{2i-1}{2n}\right]\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} \times \log\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right) \tag{S30}$$

$$\frac{\partial CVM}{\partial a} = \frac{-2k}{n}\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{2i-1}{2n}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b \times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1} \tag{S31}$$

$$\begin{aligned}\frac{\partial CVM}{\partial b} &= \frac{-2ka}{n}\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{2i-1}{2n}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b \\ &\times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1}\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right] \\ &= a\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]\frac{\partial LS}{\partial a}\end{aligned} \tag{S32}$$

$$\frac{\partial CVM}{\partial \psi_s} = \frac{2kab}{n}\sum_{i=1}^{n}\left[\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k} - \frac{2i-1}{2n}\right]\frac{[\overline{G}(x_i,\boldsymbol{\psi})]^{b-1}}{[G(x_i,\boldsymbol{\psi})]^{b+1}} \times \left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)^{-k-1}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s} \tag{S33}$$

S11 Anderson and Darling Approach of Estimation

We take the first derivatives of $AD(\sigma)$ and acquire

$$\frac{\partial AD}{\partial k} = \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(1+a\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^b\right)\right.$$
$$\left.+\log\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)\frac{\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}{1-\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}\right] \tag{S34}$$

$$\frac{\partial AD}{\partial a} = \frac{k}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b}{\left[G(x_i,\boldsymbol{\psi})\right]^b+a\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b}\right.$$
$$\left.+\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\frac{\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}\right] \tag{S35}$$

$$\frac{\partial AD}{\partial b} = \frac{ka}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b}{\left[G(x_i,\boldsymbol{\psi})\right]^b+a\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b}\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]\right.$$
$$\left.+\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\log\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]\frac{\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}\right] \tag{S36}$$

$$\frac{\partial AD}{\partial \psi_s} = \frac{-kab}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_i,\boldsymbol{\psi})\right]^{b+1}+a\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^b G(x_i,\boldsymbol{\psi})}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s}\right.$$
$$\left.+\frac{\left[\overline{G}(x_{n+1-i},\boldsymbol{\psi})\right]^{b-1}}{\left[G(x_{n+1-i},\boldsymbol{\psi})\right]^{b+1}}\frac{\partial G(x_{n+1-i},\boldsymbol{\psi})}{\partial \psi_s}\frac{\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k-1}}{1-\left(1+a\left[\frac{\overline{G}(x_{n+1-i},\boldsymbol{\psi})}{G(x_{n+1-i},\boldsymbol{\psi})}\right]^b\right)^{-k}}\right] \tag{S37}$$

S12 More Plots for Special Cases

In this section, we offer additional illustrations depicting the skewness and kurtosis for specific instances with set parameters of the B-SIOR-G distribution.

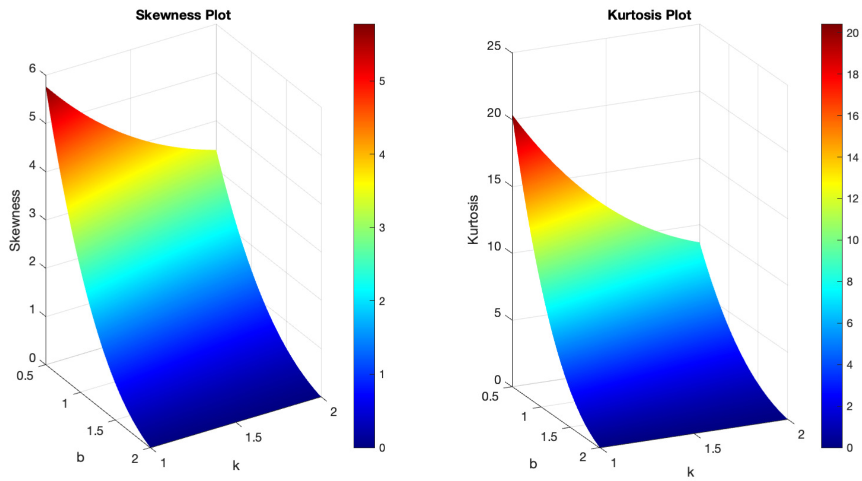

Figure S1: Left: The skewness plot of $B\text{-}SIOR\text{-}E(x; k, a = 1, b, \lambda = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}E(x; k, a = 1, b, \lambda = 1)$.

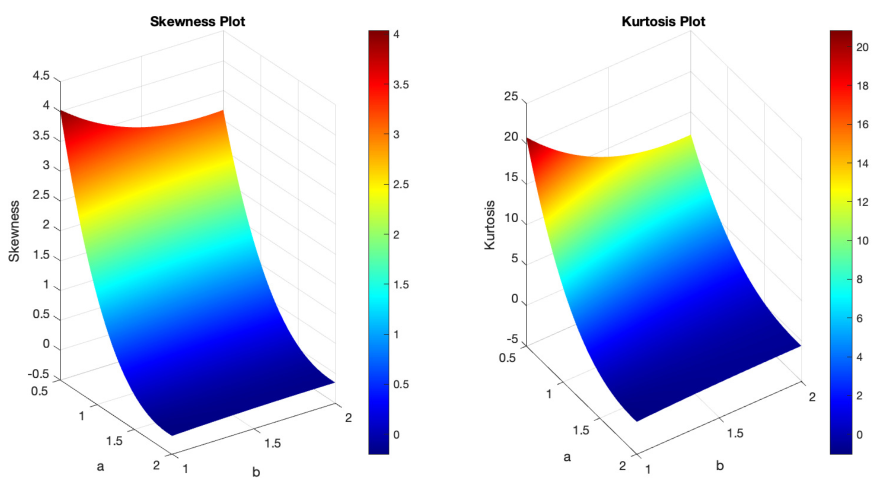

Figure S2: Left: The skewness plot of $B\text{-}SIOR\text{-}E(x; k = 1, a, b, \lambda = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}E(x; k = 1, a, b, \lambda = 1)$.

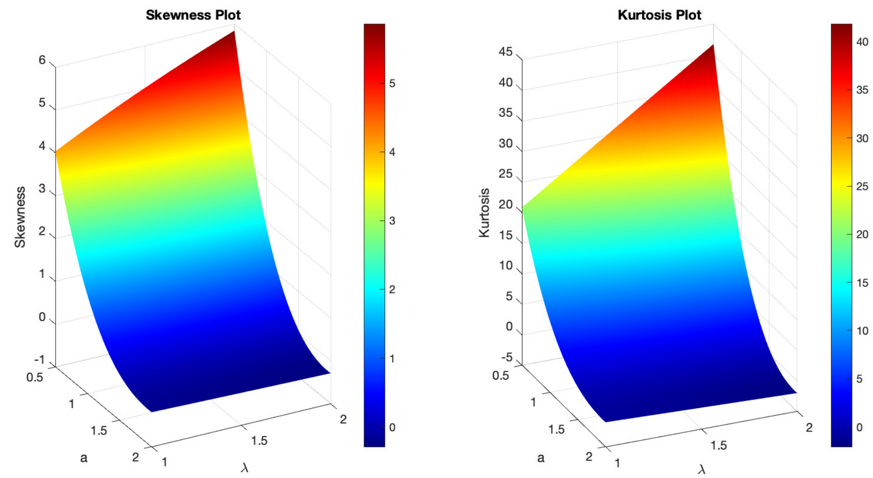

Figure S3: Left: The skewness plot of B-$SIOR$-$E(x; k = 1, a, b = 1, \lambda)$. Right: The kurtosis plot of B-$SIOR$-$E(x; k = 1, a, b = 1, \lambda)$.

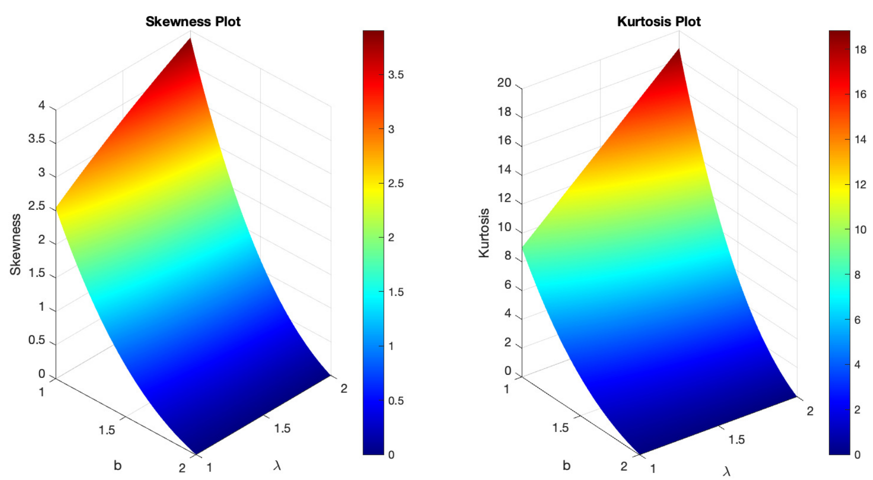

Figure S4: Left: The skewness plot of B-$SIOR$-$E(x; k = 1, a = 1, b, \lambda)$. Right: The kurtosis plot of B-$SIOR$-$E(x; k = 1, a = 1, b, \lambda)$.

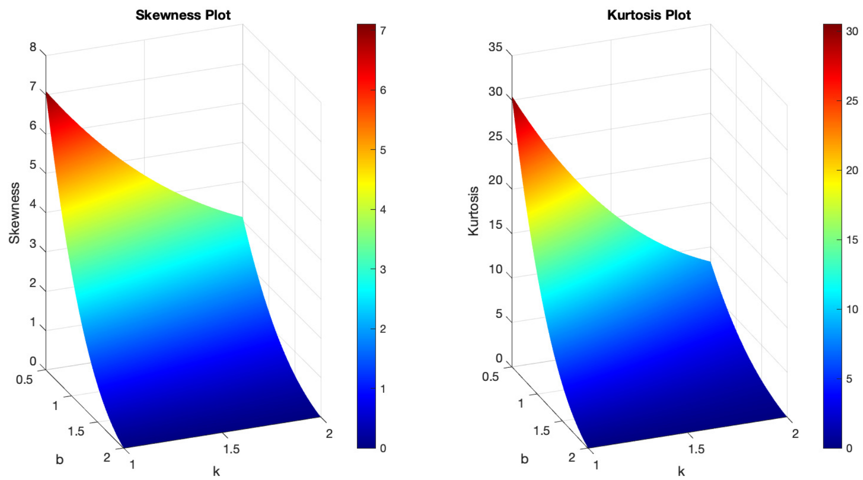

Figure S5: Left: The skewness plot of B-$SIOR$-$U(x; k, a = 0.5, b, \lambda = 1)$. Right: The kurtosis plot of B-$SIOR$-$U(x; k, a = 0.5, b, \lambda = 1)$.

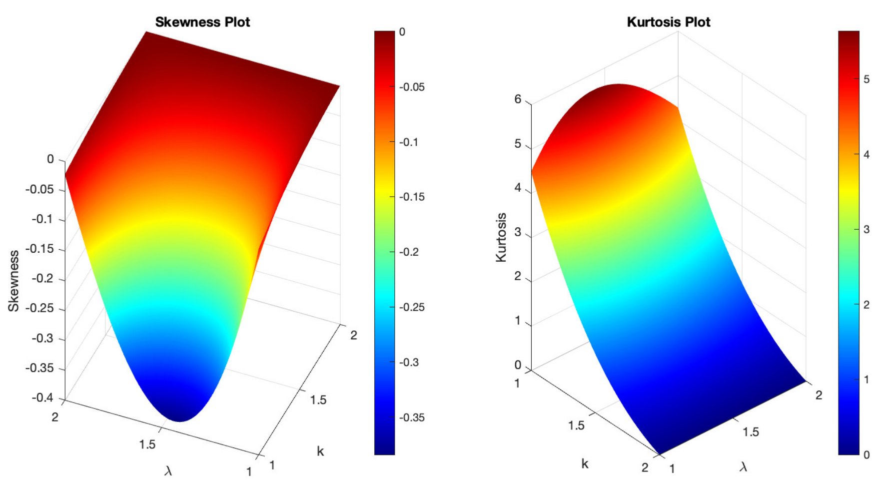

Figure S6: Left: The skewness plot of B-$SIOR$-$U(x; k, a = 1, b = 1, \lambda)$. Right: The kurtosis plot of B-$SIOR$-$U(x; k, a = 1, b = 1, \lambda)$.

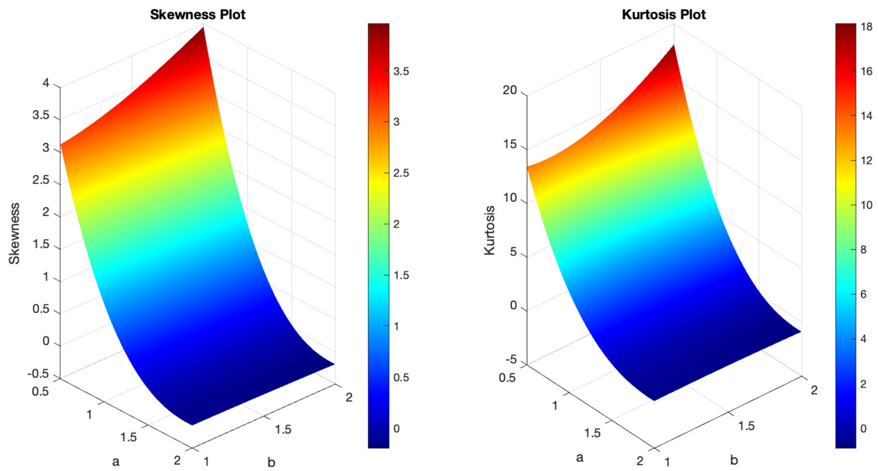

Figure S7: Left: The skewness plot of $B\text{-}SIOR\text{-}U(x; k = 1, a, b, \lambda = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}U(x; k = 1, a, b, \lambda = 1)$.

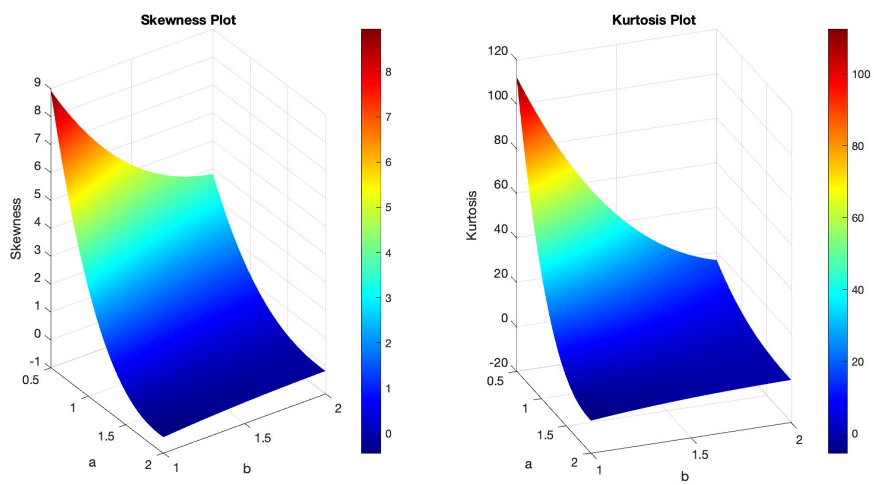

Figure S8: Left: The skewness plot of $B\text{-}SIOR\text{-}U(x; k = 0.5, a, b, \lambda = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}U(x; k = 0.5, a, b, \lambda = 1)$.

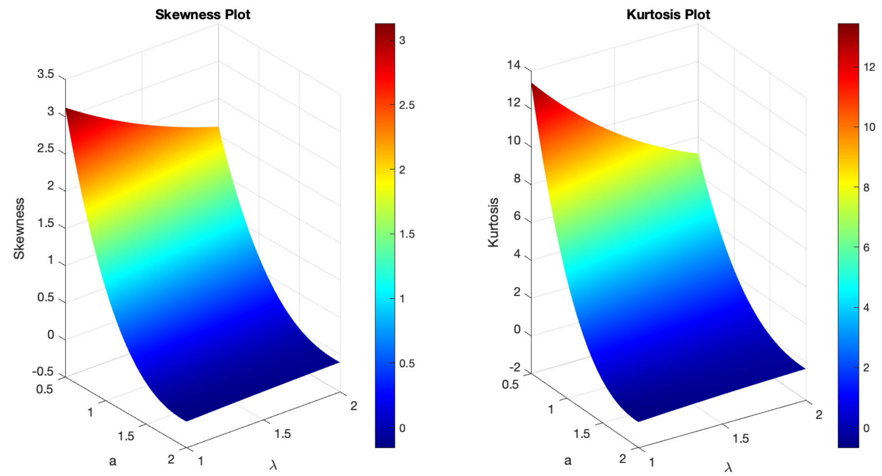

Figure S9: Left: The skewness plot of B-$SIOR$-$U(x; k = 1, a, b = 1, \lambda)$. Right: The kurtosis plot of B-$SIOR$-$U(x; k = 1, a, b = 1, \lambda)$.

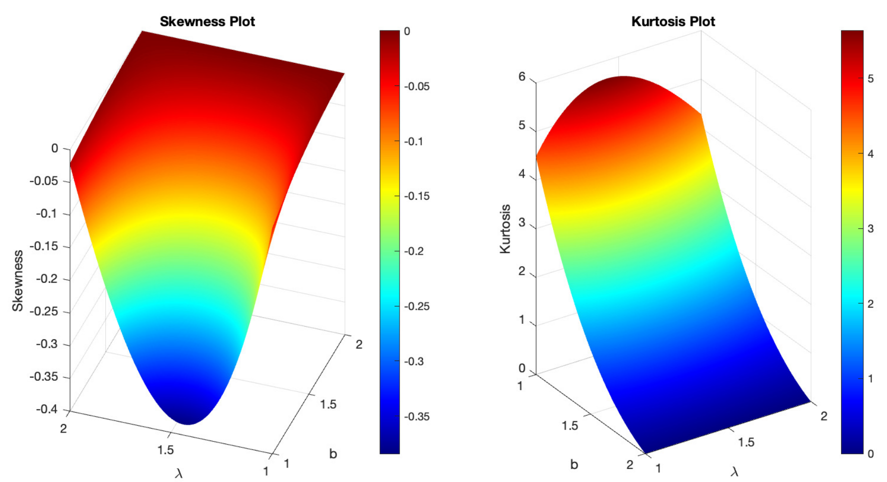

Figure S10: Left: The skewness plot of B-$SIOR$-$U(x; k = 1, a = 1, b, \lambda)$. Right: The kurtosis plot of B-$SIOR$-$U(x; k = 1, a = 1, b, \lambda)$.

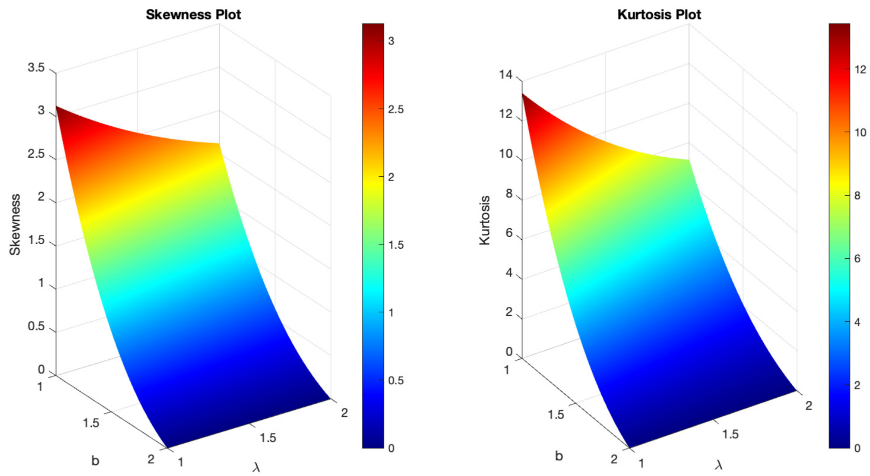

Figure S11: Left: The skewness plot of *B-SIOR-U*$(x; k = 1, a = 0.5, b, \lambda)$. Right: The kurtosis plot of *B-SIOR-U*$(x; k = 1, a = 0.5, b, \lambda)$.

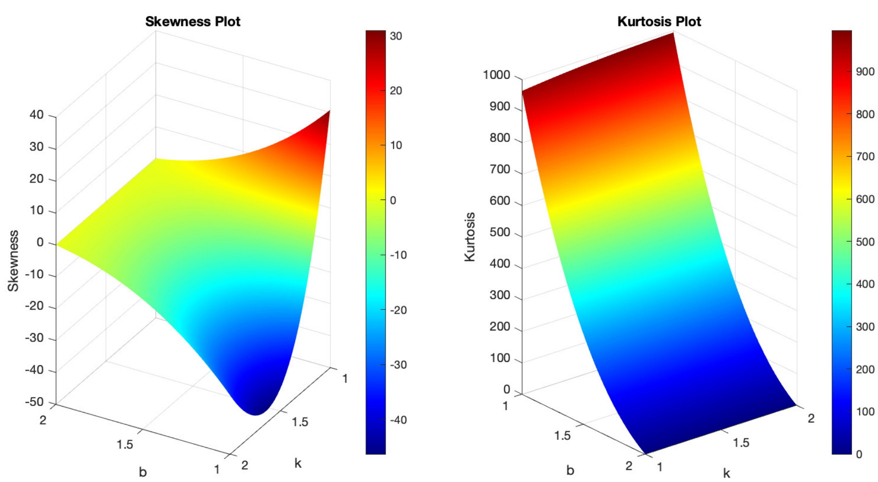

Figure S12: Left: The skewness plot of *B-SIOR-P*$(x; k, a = 1, b, c = 2, \theta = 1)$. Right: The kurtosis plot of *B-SIOR-P*$(x; k, a = 1, b, c = 2, \theta = 1)$.

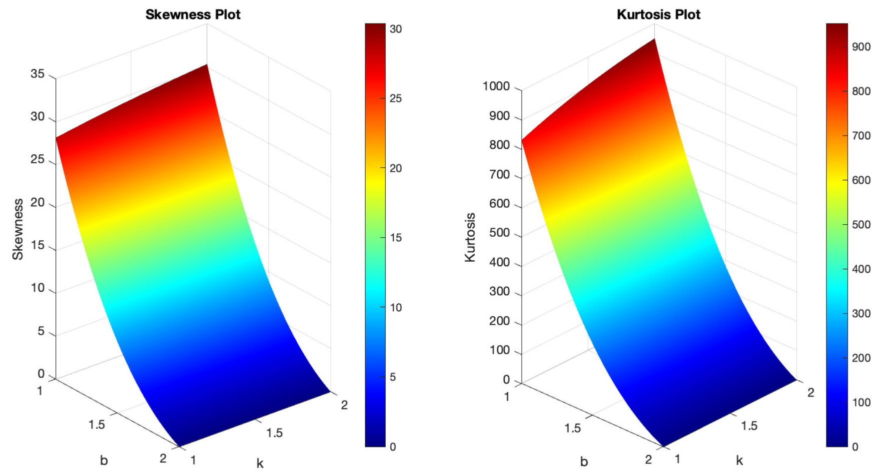

Figure S13: Left: The skewness plot of B-$SIOR$-$P(x;k,a=0.5,b,c=2,\theta=0.5)$. Right: The kurtosis plot of B-$SIOR$-$P(x;k,a=0.5,b,c=2,\theta=0.5)$.

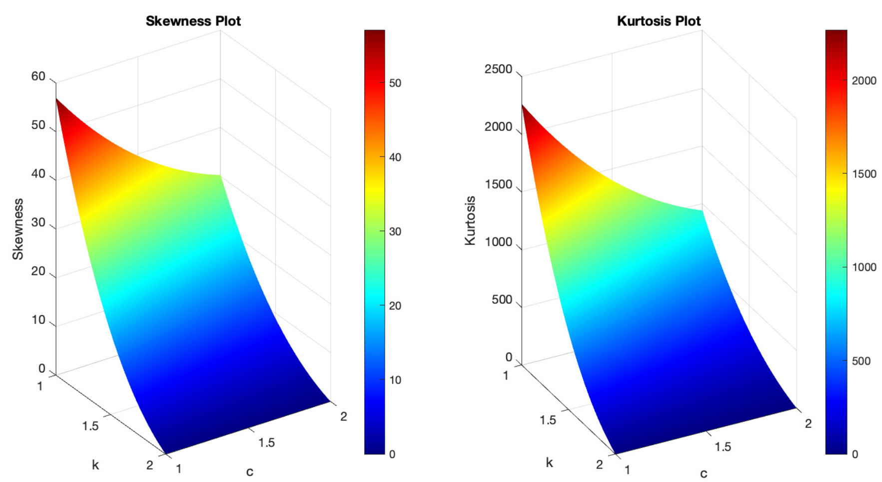

Figure S14: Left: The skewness plot of B-$SIOR$-$P(x;k,a=0.5,b=1,c,\theta=1)$. Right: The kurtosis plot of B-$SIOR$-$P(x;k,a=0.5,b=1,c,\theta=1)$.

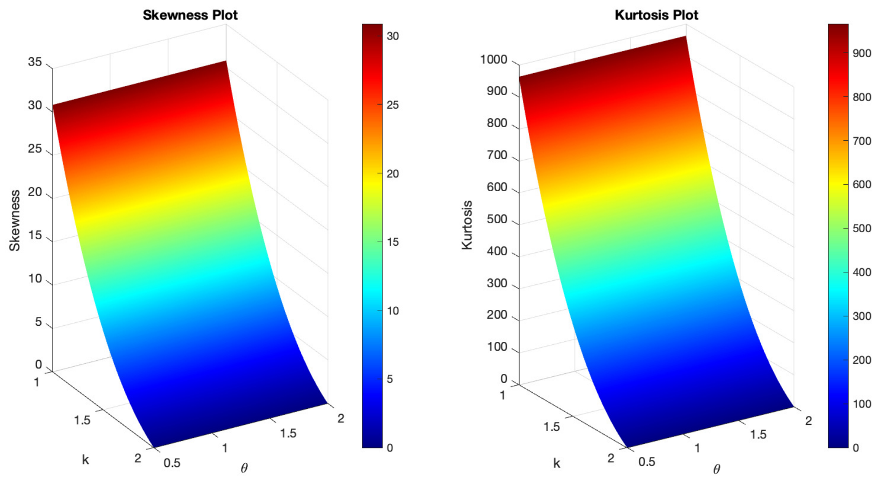

Figure S15: Left: The skewness plot of *B-SIOR-P*$(x; k, a = 1, b = 1, c = 2, \theta)$. Right: The kurtosis plot of *B-SIOR-P*$(x; k, a = 1, b = 1, c = 2, \theta)$.

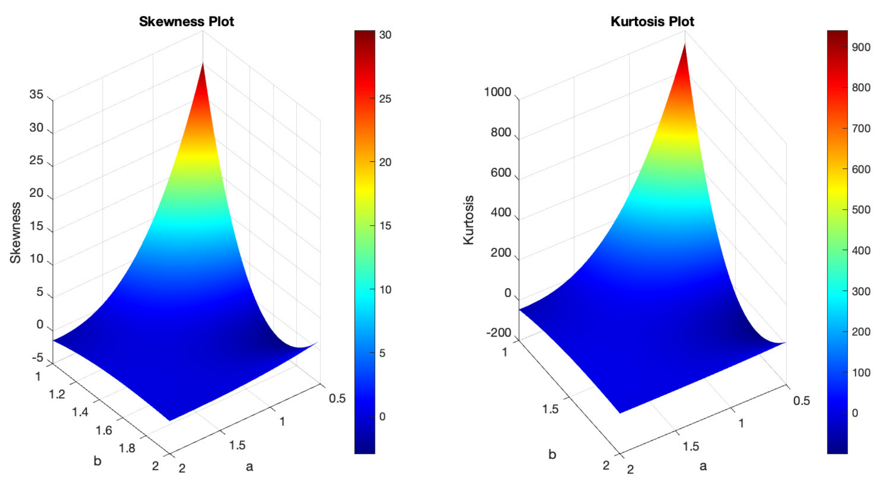

Figure S16: Left: The skewness plot of *B-SIOR-P*$(x; k = 1, a, b, c = 2, \theta = 1)$. Right: The kurtosis plot of *B-SIOR-P*$(x; k = 1, a, b, c = 2, \theta = 1)$.

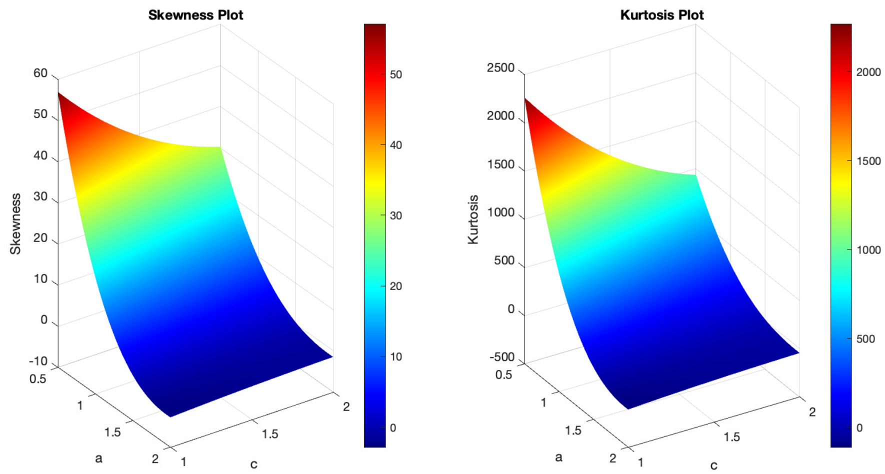

Figure S17: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b = 1, c, \theta = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b = 1, c, \theta = 1)$.

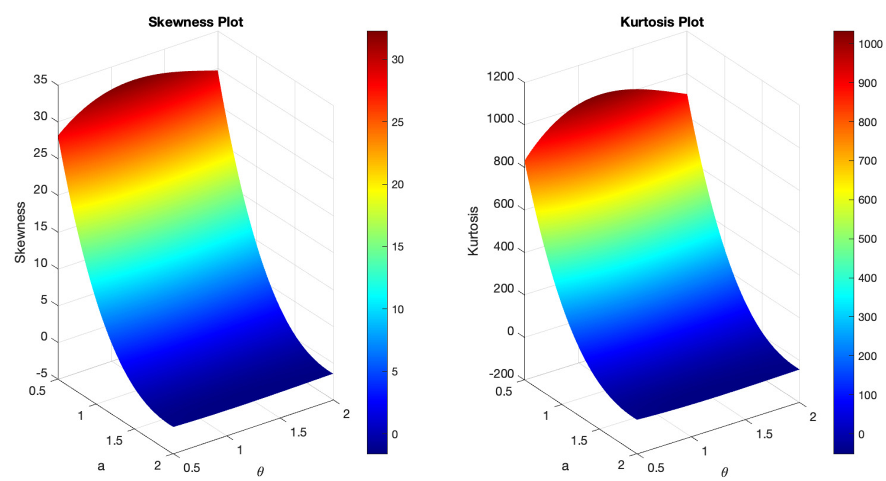

Figure S18: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b = 1, c = 2, \theta)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a, b = 1, c = 2, \theta)$.

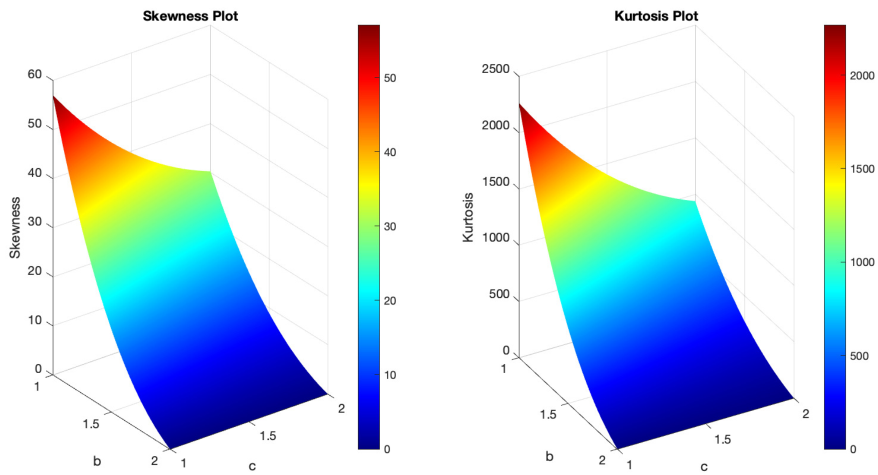

Figure S19: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b, c, \theta = 1)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b, c, \theta = 1)$.

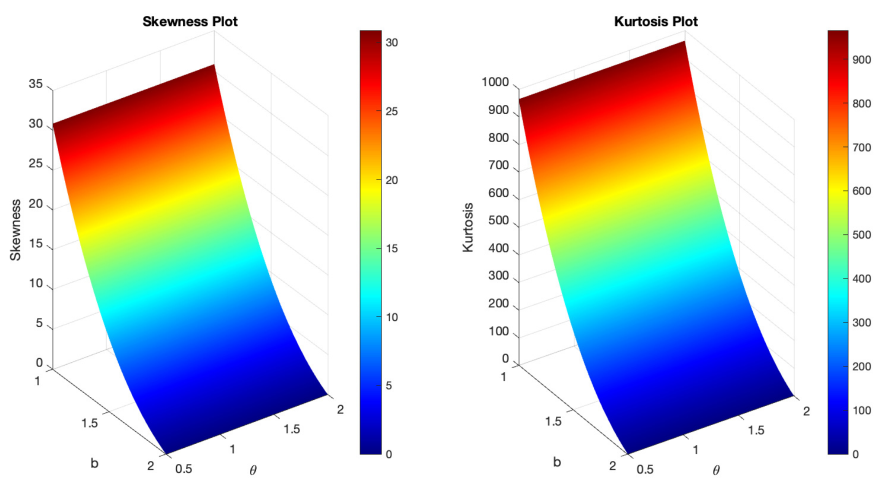

Figure S20: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 1, b, c = 2, \theta)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 1, b, c = 2, \theta)$.

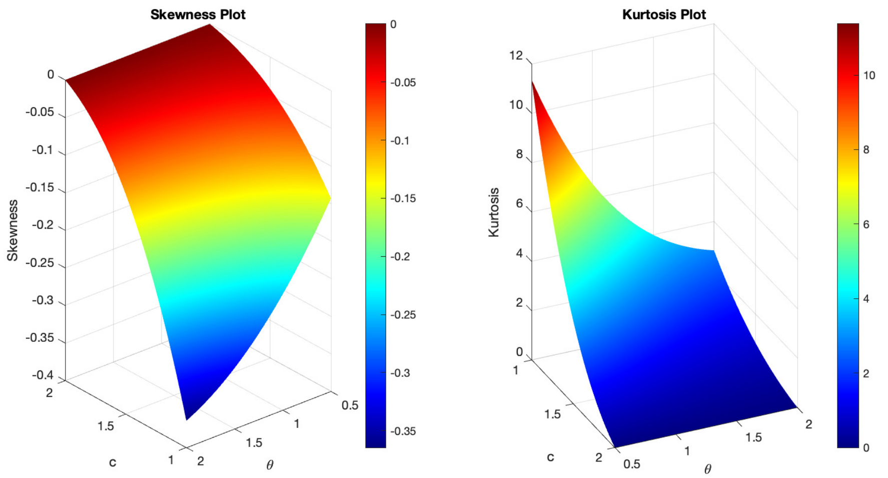

Figure S21: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b = 2, c, \theta)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b = 2, c, \theta)$.

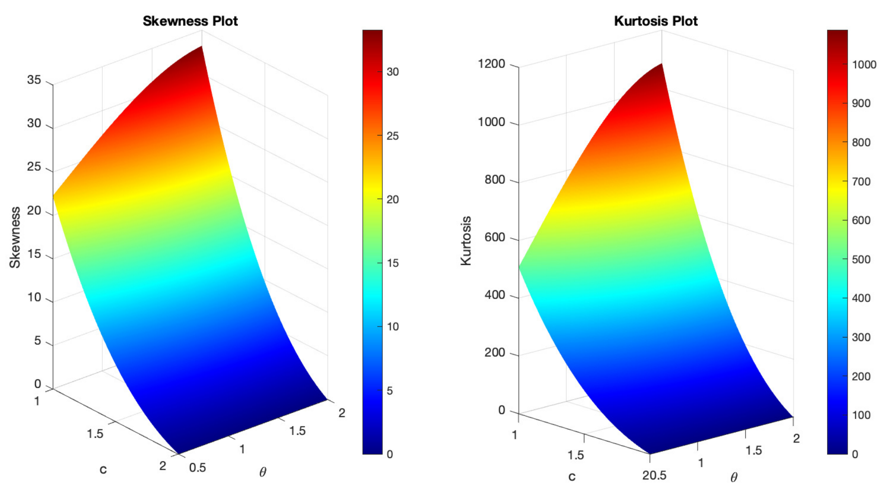

Figure S22: Left: The skewness plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b = 5, c, \theta)$. Right: The kurtosis plot of $B\text{-}SIOR\text{-}P(x; k = 1, a = 0.5, b = 5, c, \theta)$.

References

[1] M. Shaked and J. G. Shanthikumar, *Stochastic orders and their applications.* Academic press, 1994.