Supplementary Information

The Lomax Exponentiated Odds Ratio-G Distribution and Its Applications

Sudakshina Singha Roy*† Hannah Knehr*†
Declan McGurk *† Xinyu Chen *† Achraf Cohen *
Shusen Pu*‡

Proofs and Explanations for Theorems and Lemmas

3.1 Expansion of the pdf

Theorem 1. *The probability density function of the L-EOR-G distribution can be expressed in terms of a linear combination of exponentiated generalized distributions as follows:*

$$f_{L-EOR-G}(x) = \sum_{j,m=0}^{\infty} c_{j,m} s_{m+\beta(j+1)}(x, \boldsymbol{\psi}),$$

where the coefficients $c_{j,m}$ are determined by

$$c_{j,m} = \frac{k\alpha\beta(-1)^m \alpha^j}{(m+\beta(j+1))} \binom{-k-1}{j} \binom{\beta(j+1)-1}{m}. \tag{1}$$

The term $s_{m+\beta(j+1)}(x, \boldsymbol{\psi})$ is the exponentiated-G distribution with parameter $m+\beta(j+1)$, defined as

$$s_{m+\beta(j+1)}(x, \boldsymbol{\psi}) = (m+\beta(j+1)) g(x, \boldsymbol{\psi}) G(x, \boldsymbol{\psi})^{m+\beta j+\beta-1}. \tag{2}$$

Proof. Consider the expansion

$$\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k-1} = \sum_{j=0}^{\infty} \binom{-k-1}{j} \alpha^j \left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta j}, \tag{3}$$

*Department of Mathematics and Statistics, University of West Florida, Pensacola, FL, USA.
†These authors contributed equally.
‡Correspondence should be addressed to Shusen Pu, E-mail:spu@uwf.edu

then the pdf of L-EOR-G can be expanded as

$$
\begin{aligned}
f_{L-EOR-G}(x) &= k\alpha\beta g(x,\boldsymbol{\psi})\frac{G(x,\boldsymbol{\psi})^{\beta-1}}{\overline{G}(x,\boldsymbol{\psi})^{\beta+1}}\sum_{j=0}^{\infty}\binom{-k-1}{j}\alpha^{j}\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta j} \\
&= k\alpha\beta g(x,\boldsymbol{\psi})\sum_{j=0}^{\infty}\alpha^{j}\binom{-k-1}{j}\frac{\overline{G}(x,\boldsymbol{\psi})^{-\beta j-\beta-1}}{G(x,\boldsymbol{\psi})^{-\beta j-\beta+1}} \qquad (4)
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\overline{G}(x,\boldsymbol{\psi})^{-\beta j-\beta-1} &= [1-G(x,\boldsymbol{\psi})]^{-\beta j-\beta-1} \\
&= \sum_{m=0}^{\infty}\binom{-\beta(j+1)-1}{m}(-1)^{m}G(x,\boldsymbol{\psi})^{m} \qquad (5)
\end{aligned}
$$

Then,

$$
\begin{aligned}
f_{L-EOR-G}(x) &= k\alpha\beta g(x,\boldsymbol{\psi})\sum_{j=0}^{\infty}\alpha^{j}\binom{-k-1}{j} \\
&\times\sum_{m=0}^{\infty}\binom{-\beta(j+1)-1}{m}(-1)^{m}G(x,\boldsymbol{\psi})^{m+\beta j+\beta-1} \\
&= k\alpha\beta g(x,\boldsymbol{\psi})\sum_{j,m=0}^{\infty}\frac{(-1)^{m}\alpha^{j}}{(m+\beta(j+1))}\binom{-k-1}{j} \\
&\times\binom{-\beta(j+1)-1}{m}(m+\beta(j+1))\,[G(x,\boldsymbol{\psi})]^{m+\beta j+\beta-1} \\
&= \sum_{j,m=0}^{\infty}c_{j,m}s_{m+\beta(j+1)}(x,\boldsymbol{\psi}) \qquad (6)
\end{aligned}
$$

where

$$
c_{j,m} = \frac{k\alpha\beta(-1)^{m}\alpha^{j}}{(m+\beta(j+1))}\binom{-k-1}{j}\binom{-\beta(j+1)-1}{m} \qquad (7)
$$

and

$$
s_{m+\beta(j+1)}(x,\boldsymbol{\psi}) = (m+\beta(j+1))g(x,\boldsymbol{\psi})G(x,\boldsymbol{\psi})^{m+\beta j+\beta-1} \qquad (8)
$$

which is the pdf of exponentiated generalized (EG) distribution with parameter $\beta^{*} = m+\beta(j+1)$. □

3.2 Hazard Rate

Theorem 2. *The hazard rate function (hrf) for the L-EOR-G distribution is given by:*

$$hrf(x) = k\alpha\beta g(x,\boldsymbol{\psi})\frac{G(x,\boldsymbol{\psi})^{\beta-1}}{\overline{G}(x,\boldsymbol{\psi})^{\beta+1}}\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-1},$$

and the reverse hazard rate function (rhrf) is defined as:

$$\tau(x) = \frac{k\alpha\beta g(x,\boldsymbol{\psi})\frac{G(x,\boldsymbol{\psi})^{\beta-1}}{\overline{G}(x,\boldsymbol{\psi})^{\beta+1}}\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k-1}}{1-\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k}}.$$

Proof. For the $hrf(x)$ we can calculate:

$$\begin{aligned} hrf(x) =& \frac{k\alpha\beta g(x,\boldsymbol{\psi})\frac{G(x,\boldsymbol{\psi})^{\beta-1}}{\overline{G}(x,\boldsymbol{\psi})^{\beta+1}}\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k-1}}{\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k}} \\ &= k\alpha\beta g(x,\boldsymbol{\psi})\frac{G(x,\boldsymbol{\psi})^{\beta-1}}{\overline{G}(x,\boldsymbol{\psi})^{\beta+1}}\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-1} \qquad (9) \end{aligned}$$

and for the $\tau(x)$:

$$\tau(x) = \frac{k\alpha\beta g(x,\boldsymbol{\psi})\frac{G(x,\boldsymbol{\psi})^{\beta-1}}{\overline{G}(x,\boldsymbol{\psi})^{\beta+1}}\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k-1}}{1-\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k}} \qquad (10)$$

□

3.3 Quantile function

Theorem 3. *The quantile function for the L-EOR-G distribution is defined as:*

$$Q(u) = G^{-1}\left[\frac{((1-u)^{-\frac{1}{k}}-1)^{\frac{1}{\beta}}}{(\alpha)^{\frac{1}{\beta}}+((1-u)^{-\frac{1}{k}}-1)^{\frac{1}{\beta}}}\right]$$

Proof. We use the cdf of L-EOR-G to derive the quantile function by solving the non-linear Equation:

$$u = 1-\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k} \qquad (11)$$

where $0 < u < 1$. Solving for $G(x, \boldsymbol{\psi})$ we get

$$G(x, \boldsymbol{\psi}) = \frac{((1-u)^{-\frac{1}{k}} - 1)^{\frac{1}{\beta}}}{(\alpha)^{\frac{1}{\beta}} + ((1-u)^{-\frac{1}{k}} - 1)^{\frac{1}{\beta}}} \tag{12}$$

Finally the quantile function is given as

$$Q(u) = G^{-1}\left[\frac{((1-u)^{-\frac{1}{k}} - 1)^{\frac{1}{\beta}}}{(\alpha)^{\frac{1}{\beta}} + ((1-u)^{-\frac{1}{k}} - 1)^{\frac{1}{\beta}}}\right] \tag{13}$$

□

3.4 Moments, Generating functions and Incomplete Moments

3.4.1 Raw Moments

Lemma 4. *The r^{th} order raw moment of the L-EOR-G distribution is expressed as:*

$$\mu'_r = E[Y^r] = \sum_{m,j=0}^{\infty} c_{j,m}(m + \beta(j+1)) \int_0^1 u^{m+\beta(j+1)-1} Q_G(u, \boldsymbol{\psi}) du, \tag{14}$$

where $Y_{m+\beta(j+1)}$ is the Exponentiated-G random variable with parameter $(m + \beta(j+1))$, $Q_G(u, \boldsymbol{\psi})$ is the quantile function of the baseline distribution with the cdf $G(x, \boldsymbol{\psi})$, and $c_{j,m}$ is as defined in Eqn. (7)*.*

Proof. The r^{th} order raw moment of any distribution is given as follows

$$\mu'_r = E(X^r) \tag{15}$$

Let us use the expanded form of the L-EOR-G pdf given by Eqn 4

$$\begin{aligned} E(X^r) &= \sum_{j,m=0}^{\infty} c_{j,m} E(Y_{m+\beta(j+1)}) \\ &= \sum_{m,j=0}^{\infty} c_{j,m}(m + \beta(j+1)) \int_0^1 u^{m+\beta(j+1)-1} Q_G(u, \boldsymbol{\psi}) du \end{aligned} \tag{16}$$

where $Y_{m+\beta(j+1)}$ is the Exponentiated-G random variable with parameter $(m + \beta(j+1))$, $Q_G(u, \boldsymbol{\psi})$ is the quantile function of the baseline distribution with the cdf $G(x, \boldsymbol{\psi})$ and $c_{j,m}$ is as defined in Eqn. (7). □

3.4.2 Central Moments

Lemma 5. *The n^{th} order central moment of the L-EOR-G distribution is articulated as:*

$$\mu_n = \sum_{m,j,r=0}^{\infty} \binom{n}{r} (-\mu_1')^{n-r} c_{j,m} E[Y_{m+\beta(j+1)}],$$

where $E[Y_{m+\beta(j+1)}]$ signifies the expected value of the Exponentiated-G random variable $Y_{m+\beta(j+1)}$, and $c_{j,m}$ is as outlined in Eqn. (7).

Proof. Following Eqn. 16 we derive the n^{th} central moment which is used in obtaining the skewness and kurtosis as

$$\begin{aligned}
\mu_n =& E(X - \mu_1')^n \\
=& E\left[\sum_{r=0}^{\infty} \binom{n}{r} (-\mu_1')^{n-r} X^r\right] \\
=& \sum_{r=0}^{\infty} \binom{n}{r} (-\mu_1')^{n-r} \sum_{m,j,r=0}^{\infty} c_{j,m} E(Y_{m+\beta(j+1)}) \\
=& \sum_{m,j,r=0}^{\infty} \binom{n}{r} (-\mu_1')^{n-r} c_{j,m} E(Y_{m+\beta(j+1)})
\end{aligned} \tag{17}$$

where $E[Y_{m+\beta(j+1)}]$ signifies the expected value of the Exponentiated-G random variable $Y_{m+\beta(j+1)}$, and $c_{j,m}$ is as outlined in Eqn. (7). □

3.4.3 Incomplete Moments

Lemma 6. *The s^{th} incomplete moment of the L-EOR-G distribution is detailed as:*

$$\eta_s(t) = \sum_{j,m=0}^{\infty} c_{j,m} \int_{-\infty}^{t} x^s h_{m+\beta(j+1)}(x, \boldsymbol{\psi}) dx,$$

highlighting the integration of x^s with the function $h_{m+\beta(j+1)}(x, \boldsymbol{\psi})$ over the range to t, and $c_{j,m}$ is as defined in Eqn. (7).

Proof. Incomplete moments are needed for the derivation of Bonferroni, Lorenz and Zenga curves. The s^{th} incomplete moments, denoted as $\eta_s(t)$, is given as

$$\eta_s(t) = \int_{-\infty}^{t} x^s f(x; k, \alpha, \beta, \boldsymbol{\psi}) dx \tag{18}$$

highlighting the integration of x^s with the function $h_{m+\beta(j+1)}(x,\boldsymbol{\psi})$ over the range to t. Using the expanded pdf Eqn. (4).

$$\eta_s(t)=\sum_{j,m=0}^{\infty} c_{j,m}\int_{-\infty}^{t} x^s h_{m+\beta(j+1)}(x,\boldsymbol{\psi})dx \tag{19}$$

where $\int_{-\infty}^{t} x^s h_{m+\beta(j+1)}(x,\boldsymbol{\psi})dx$ is the s^{th} incomplete moment of the Exp-G random variable $Y_{m+\beta(j+1)}$, and $c_{j,m}$ is as outlined in Eqn. (7). □

3.4.4 Moment Generating Functions

Lemma 7. *The moment generating function (MGF) of the L-EOR-G distribution is provided in terms of the MGF of the Exponentiated-G distribution:*

$$M_X(t)=\sum_{m,j,r=0}^{\infty}\binom{n}{r}(-\mu_1')^{n-r}c_{j,m}E[Y_{m+\beta(j+1)}],$$

where $M_{m+\beta(j+1)}(t)$ represents the MGF of the Exponentiated-G random variable $Y_{m+\beta(j+1)}$, and $c_{j,m}$ is as defined in Eqn. (7).

Proof. The moment generating function of the L-EOR-G distribution, say $M_X(t)$, can also be derived using the pdf expansion, Eqn. (4)

$$\begin{aligned} M_X(t) =& E(e^{tX}) \\ =& \sum_{m,j=0}^{\infty} c_{j,m}M_{m+\beta(j+1)}(t) \\ =& \sum_{m,j,r=0}^{\infty}\binom{n}{r}(-\mu_1')^{n-r}c_{j,m}E(Y_{m+\beta(j+1)}) \end{aligned} \tag{20}$$

where $M_{m+\beta(j+1)}(t)$ is the moment generating function of the Exp-G random variable $Y_{m+\beta(j+1)}$ and $c_{j,m}$ is as outlined in Eqn. (7). □

3.5 Renyi Entropy and Order Statistics

Lemma 8. *The Rényi entropy of the L-EOR-G distribution is expressed as follows:*

$$I_R(v)=\frac{1}{1-v}\log\left[\sum_{j,m=0}^{\infty} w_{\frac{m+\beta j}{v}+\beta}\exp\left[(1-v)I_{REG}\right]\right],$$

where I_{REG} is the Rényi entropy of the Exponentiated-G family.

Proof. Let us start with the pdf of L-EOR-G distribution

$$
\begin{aligned}
f^v(x;k,\alpha,\beta,\boldsymbol{\psi}) =&(k\alpha\beta)^v g^v(x,\boldsymbol{\psi})\frac{G(x,\boldsymbol{\psi})^{v(\beta-1)}}{\overline{G}(x,\boldsymbol{\psi})^{v(\beta+1)}} \\
&\times\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{v(-k-1)}
\end{aligned}
\tag{21}
$$

Using generalized binomial series expansion, we get

$$
\begin{aligned}
f^v(x;k,\alpha,\beta,\boldsymbol{\psi}) =&(k\alpha\beta)^v g^v(x,\boldsymbol{\psi})\frac{G(x,\boldsymbol{\psi})^{v(\beta-1)}}{\overline{G}(x,\boldsymbol{\psi})^{v(\beta+1)}} \\
&\times\sum_{j=0}^{\infty}\binom{v(-k-1)}{j}\alpha^j\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta j} \\
=&(k\alpha\beta)^v g^v(x,\boldsymbol{\psi})\sum_{j=0}^{\infty}\binom{v(-k-1)}{j}\alpha^j\frac{\overline{G}(x,\boldsymbol{\psi})^{-\beta j-\beta v-v}}{G(x,\boldsymbol{\psi})^{-\beta j-\beta v+v}} \\
=&(k\alpha\beta)^v g^v(x,\boldsymbol{\psi})\sum_{j,m=0}^{\infty}(-1)^m\alpha^j\binom{v(-k-1)}{j} \\
&\times\binom{-\beta(j+v)-v}{m}G(x,\boldsymbol{\psi})^{m+\beta(j+v)-v}
\end{aligned}
\tag{22}
$$

Thus, the Renyi entropy function is written as

$$
\begin{aligned}
I_R(v) =&\frac{1}{1-v}\log\left[(k\alpha\beta)^v\sum_{j,m=0}^{\infty}\frac{(-1)^m\alpha^j}{\left[\frac{m+\beta j}{v}+\beta\right]^v}\binom{v(-k-1)}{j}\binom{-\beta(j+v)-v}{m}\right. \\
&\times\left.\int_0^{\infty}\left(\left[\frac{m+\beta j}{v}+\beta\right]g(x,\boldsymbol{\psi})G(x,\boldsymbol{\psi})^{\frac{m+\beta j+\beta v}{v}-1}\right)^v dx\right] \\
=&\frac{1}{1-v}log\left[\sum_{j,m=0}^{\infty}w_{\frac{m+\beta j}{v}+\beta}exp\left[(1-v)I_{REG}\right]\right]
\end{aligned}
\tag{23}
$$

where

$$
I_{REG}=\frac{1}{1-v}log\left[\int_0^{\infty}\left(\left[\frac{m+\beta j}{v}+\beta\right]g(x,\boldsymbol{\psi})G(x,\boldsymbol{\psi})^{\frac{m+\beta j+\beta v}{v}-1}\right)^v dx\right] \tag{24}
$$

is the Renyi entropy of Exponentiated-G family with power parameter $\frac{m+\beta j}{v}+\beta$ $\square$

Lemma 9. *Let $X_1, X_2, ..., X_n$ be independent and identically distributed (i.i.d.) random variables from the L-EOR-G distribution. The i^{th} order statistic is articulated as:*

$$
\begin{aligned}
f_{i:n}(x) =& \frac{n!}{(i-1)!(n-i)!} \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m \sum_{j=0}^{i-1+m} \binom{i-1+m}{j} (-1)^j \\
& \times \sum_{z=0}^{-kj} \binom{-kj}{z} \left(\alpha \frac{G(x, \boldsymbol{\psi})}{\overline{G}(x, \boldsymbol{\psi})} \right)^{-\beta kj} f_{L-EOR-G}(x),
\end{aligned}
$$

illustrating the complexity and application potential of the L-EOR-G distribution's order statistics.

Proof. For a random sample as, the order statistics for that sample have cdf as follows:

$$
F_{X(k)}(x) = \sum_{j=r}^{n} \binom{n}{j} (F_X(x))^j (1 - F_{X(x)})^{n-j} \tag{25}
$$

The corresponding pdf can be derived from the cdf as:

$$
f_{X(k)}(x) = \frac{n!}{((i-1)!(n-i)!} f_X(X)(F_X(x))^{i-1}(1 - F_{X(x)})^{n-i} \tag{26}
$$

Therefore:

$$
\begin{aligned}
f_{i:n}(x) &= \frac{n! f_{L-EOR-G}(x)}{(i-1)!(n-i)!}[F_{L-EOR-G}(x)]^{i-1}[1-F_{L-EOR-G}(x)]^{n-i} \\
&= \frac{n! f_{L-EOR-G}(x)}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m[F_{L-EOR-G}(x)]^{i-1+m} \\
&= \frac{n! f_{L-EOR-G}(x)}{(i-1)!(n-i)!} \\
&\times \sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m\left[1-\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k}\right]^{i-1+m} \\
&= \frac{n! f_{L-EOR-G}(x)}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m \\
&\times \sum_{j=0}^{i-1+m}\binom{i-1+m}{j}(-1)^j\left[\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k}\right]^{j} \\
&= \frac{n! f_{L-EOR-G}(x)}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m \\
&\times \sum_{j=0}^{i-1+m}\binom{i-1+m}{j}(-1)^j\left(1+\alpha\left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right]^{\beta}\right)^{-kj} \\
&= \frac{n! f_{L-EOR-G}(x)}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m\sum_{j=0}^{i-1+m}\binom{i-1+m}{j}(-1)^j \\
&\times \sum_{z=0}^{-kj}\binom{-kj}{z}\left(\alpha\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})}\right)^{-\beta kj} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m \\
&\times \sum_{j=0}^{i-1+m}\binom{i-1+m}{j}(-1)^j f_{L-EOR-G}(x)\left[\overline{F_{L-EOR-G}}(x)\right]^{j} \qquad (27)
\end{aligned}
$$

$\square$

3.6 Probability Weighted Moments

Lemma 10. *The Probability Weighted Moments of the L-EOR-G distribution is given as follows:*

$$PWM(x;p,q) = \int_{-\infty}^{\infty} (F_{B-EOR-G}(x))^p (1 - F_{B-EOR-G}(x))^q f_{B-EOR-G}(x) dx$$

Proof. The r^{th} probability weighted moments of the distribution can be calculated as

$$\begin{aligned} PWM(x;p,q,r) &= E\left[x^r \cdot (F_{B-EOR-G}(x))^p \cdot (1 - F_{B-EOR-G}(x))^q\right] \\ &= \int_{-\infty}^{\infty} x^r (F_{B-EOR-G}(x))^p (1 - F_{B-EOR-G}(x))^q f_{B-EOR-G}(x) dx \end{aligned} \tag{28}$$

for non negative integers p, q, and r.

While introducing multiple parameters to the PWM function can indeed complicate it, there are cases where considering additional parameters can improve its representation of real-world conditions. However, it's crucial to strike a balance between simplicity and representativeness. For the most typical case in hydrological applications, we set $r = 0$ and PWM can be simplified as follows

$$PWM(x;p,q) = \int_{-\infty}^{\infty} (F_{B-EOR-G}(x))^p (1 - F_{B-EOR-G}(x))^q f_{B-EOR-G}(x) dx \tag{29}$$

Here we provide a commonly used instance $PWM_{1,0}$ for clarification

$$\begin{aligned} PWM_{1,0} &= \int_{-\infty}^{\infty} (F_{B-EOR-G}(x)) f_{B-EOR-G}(x) dx \\ &= k\alpha\beta \int_{-\infty}^{\infty} \left[g(x,\boldsymbol{\psi}) \left(1 + \alpha \left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})} \right]^{\beta} \right)^{-k} \frac{[G(x,\boldsymbol{\psi})]^{\beta-1}}{[\overline{G}(x,\boldsymbol{\psi})]^{\beta+1}} \right. \\ &\times \left. \left(1 + \alpha \left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})} \right]^{\beta} \right)^{-k-1} \right] dx \\ &= k\alpha\beta \int_{-\infty}^{\infty} g(x,\boldsymbol{\psi}) \frac{[G(x,\boldsymbol{\psi})]^{\beta-1}}{[\overline{G}(x,\boldsymbol{\psi})]^{\beta+1}} \left(1 + \alpha \left[\frac{G(x,\boldsymbol{\psi})}{\overline{G}(x,\boldsymbol{\psi})} \right]^{\beta} \right)^{-2k-1} dx \end{aligned} \tag{30}$$

for $p = 1$ and $q = 0$. □

4 Additional Plots of Special Cases

Below are the skewness and kurtosis plots for all other possible combinations of parameters for the special cases presented.

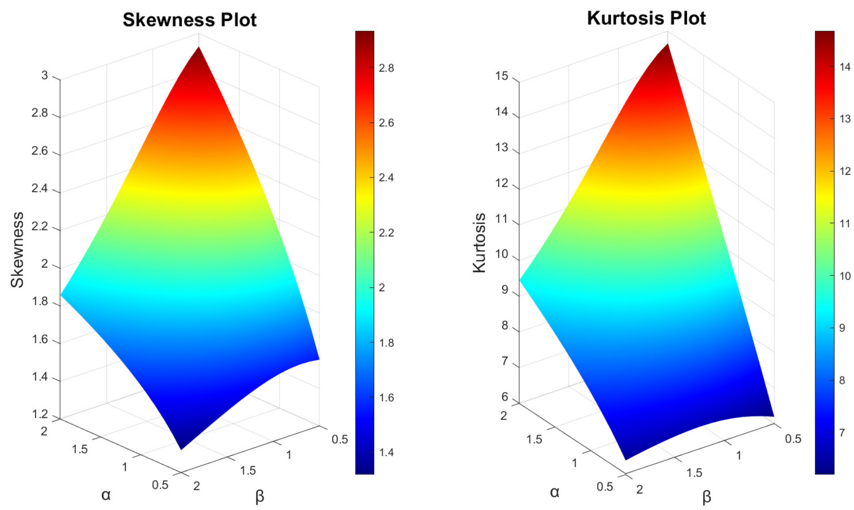

Figure 1: Skewness and Kurtosis plot for the L-EOR-W distribution. The parameters α and β were varied, while $k = 1$, $\gamma = 1$, and $\lambda = 1$.

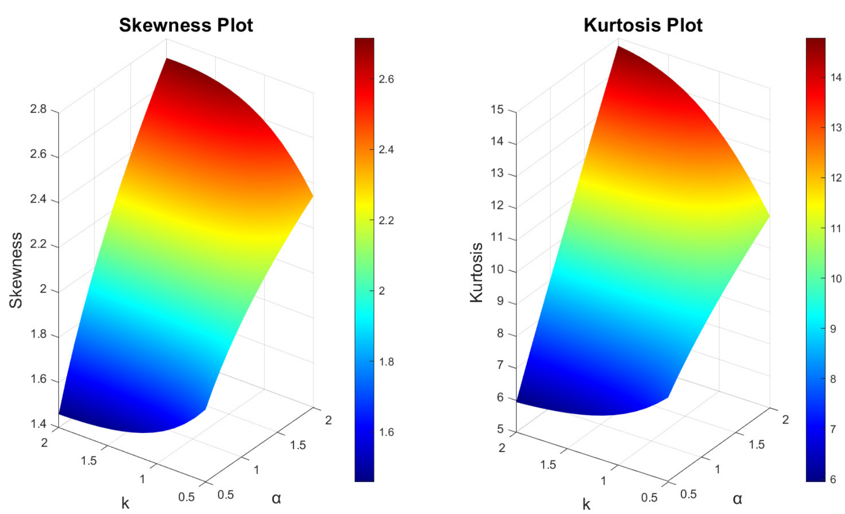

Figure 2: Skewness and Kurtosis plot for the L-EOR-W distribution. The parameters α and k were varied, while $\beta = 1$, $\gamma = 1$, and $\lambda = 1$.

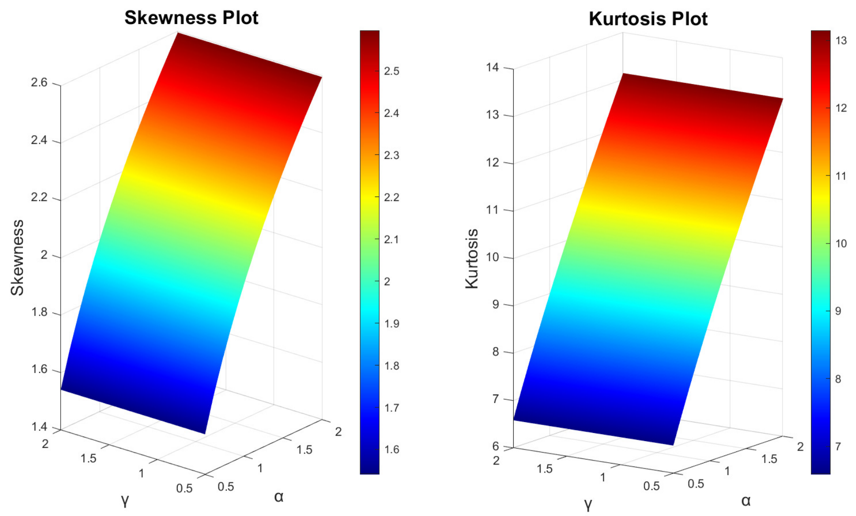

Figure 3: Skewness and Kurtosis plot for the L-EOR-W distribution. The parameters α and γ were varied, while $k = 1$, $\beta = 1$, and $\lambda = 1$.

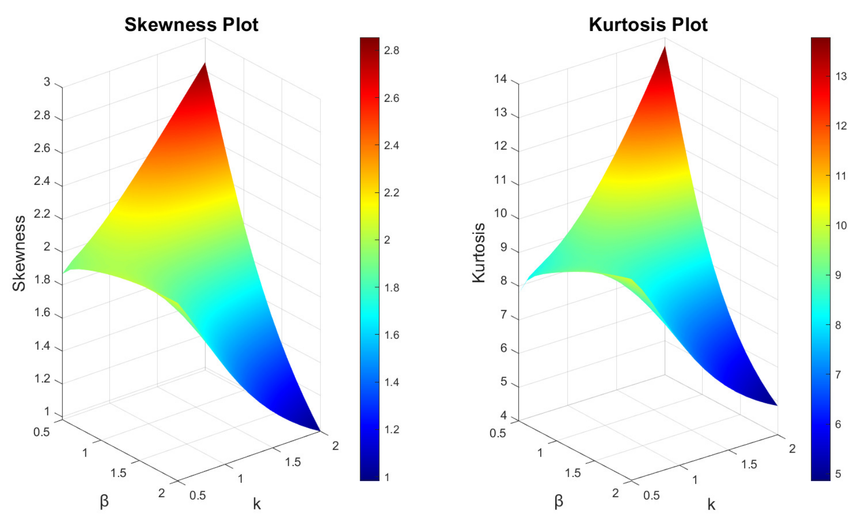

Figure 4: Skewness and Kurtosis plot for the L-EOR-W distribution. The parameters k and β were varied, while $\alpha = 1$, $\gamma = 1$, and $\lambda = 1$.

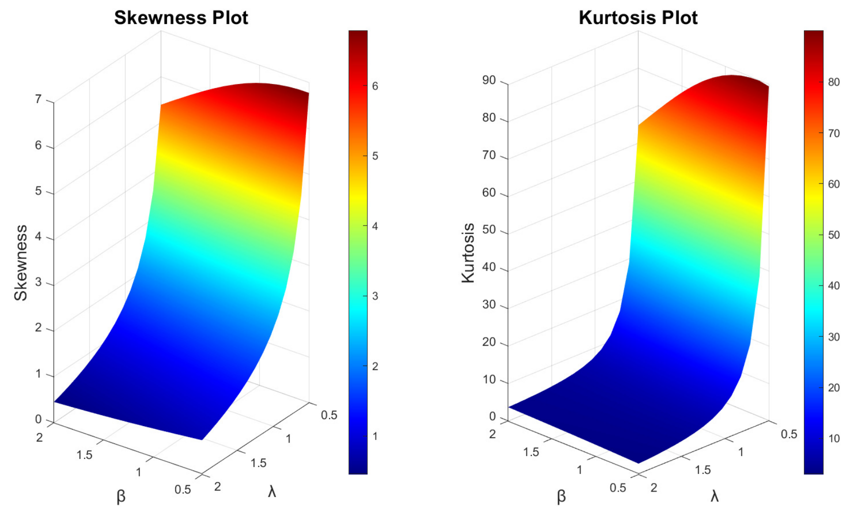

Figure 5: Skewness and Kurtosis plot for the L-EOR-W distribution. The parameters λ and β were varied, while $k = 1$, $\gamma = 1$, and $\alpha = 1$.

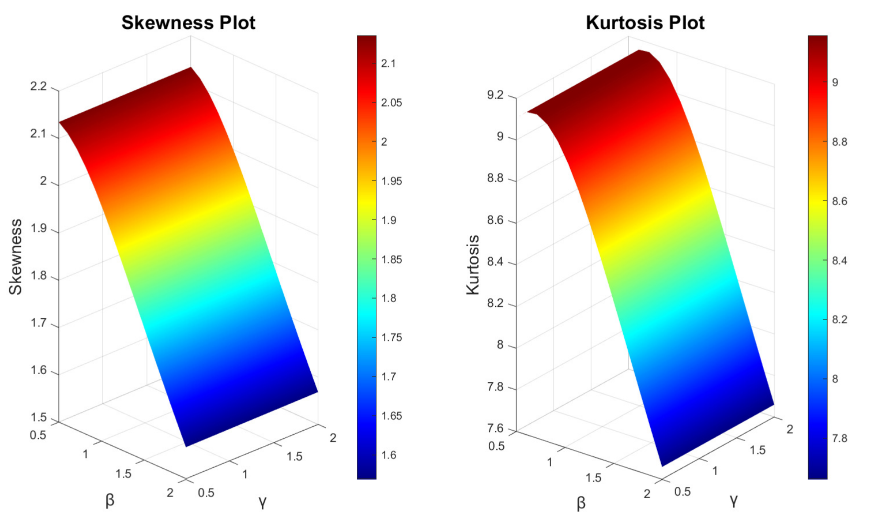

Figure 6: Skewness and Kurtosis plot for the L-EOR-W distribution. The parameters γ and β were varied, while $k = 1$, $\alpha = 1$, and $\lambda = 1$.

5 Derivatives of Estimators

5.1 Maximum Likelihood Estimation

The first derivative of $\ell(f^n_{L-EOR-G}(\boldsymbol{\Delta}))$ with respect to $\boldsymbol{\Delta}$ are shown as following:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - (k+1)\sum_{i=1}^{n} \frac{[G(x_i, \boldsymbol{\psi})]^{\beta}}{[\overline{G}(x_i, \boldsymbol{\psi})]^{\beta} + \alpha[G(x_i, \boldsymbol{\psi})]^{\beta}} \tag{31}$$

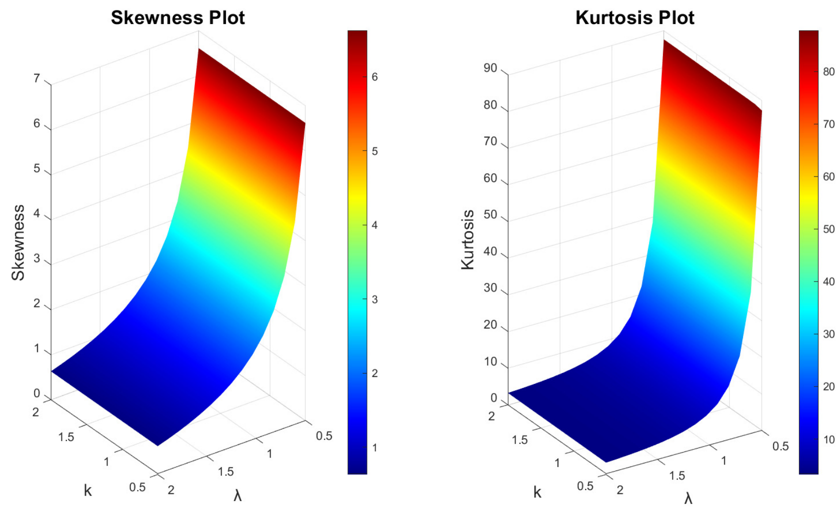

Figure 7: Skewness and Kurtosis plot for the L-EOR-W distribution. The parameters k and λ were varied, while $\alpha = 1$, $\gamma = 1$, and $\beta = 1$.

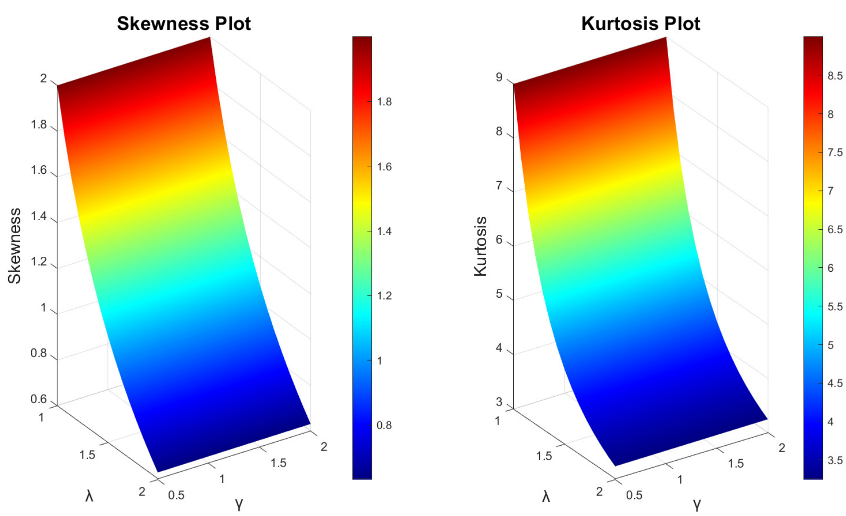

Figure 8: Skewness and Kurtosis plot for the L-EOR-W distribution. The parameters λ and γ were varied, while $k = 1$, $\alpha = 1$, and $\beta = 1$.

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} =& \frac{n}{\beta} + \sum_{i=1}^{n} \log[G(x_i, \boldsymbol{\psi})] - \sum_{i=1}^{n} \log[\overline{G}(x_i, \boldsymbol{\psi})] \\ & - \alpha(k+1) \times \sum_{i=1}^{n} \frac{[G(x_i, \boldsymbol{\psi})]^{\beta}}{[\overline{G}(x_i, \boldsymbol{\psi})]^{\beta} + \alpha[G(x_i, \boldsymbol{\psi})]^{\beta}} \log\left[\frac{G(x_i, \boldsymbol{\psi})}{\overline{G}(x_i, \boldsymbol{\psi})}\right] \qquad (32)\end{aligned}$$

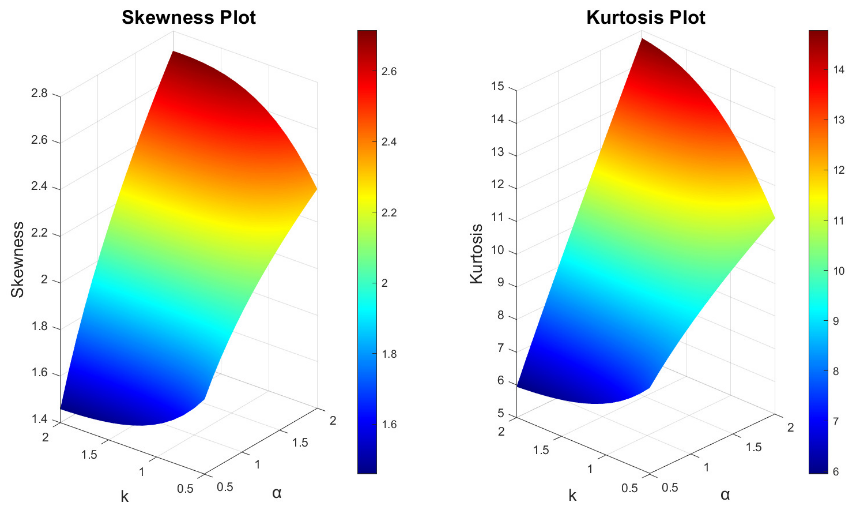

Figure 9: Skewness and Kurtosis plot for the L-EOR-E distribution. The parameters α and k were varied, while $\beta = 1$, and $\gamma = 1$.

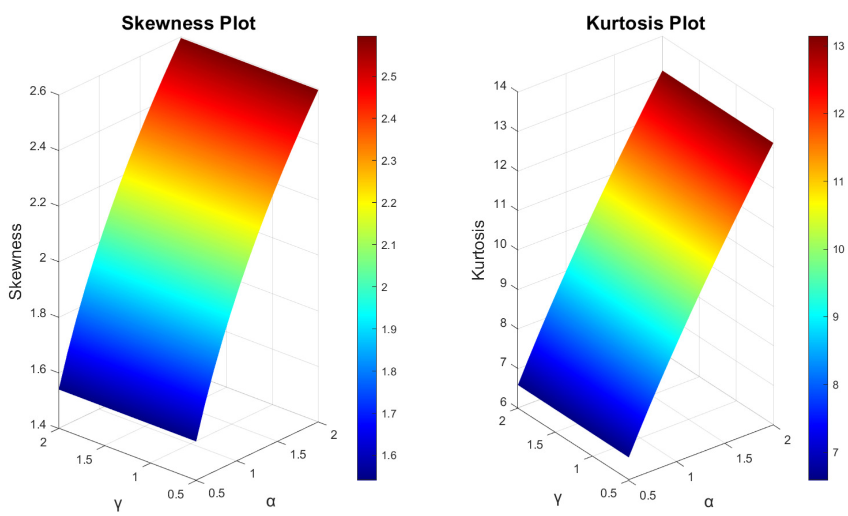

Figure 10: Skewness and Kurtosis plot for the L-EOR-E distribution. The parameters α and γ were varied, while $\beta = 1$, and $k = 1$.

$$\frac{\partial \ell}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n} \log \left(1 + \alpha \left[\frac{G(x_i, \boldsymbol{\psi})}{\overline{G}(x_i, \boldsymbol{\psi})} \right]^{\beta} \right) \tag{33}$$

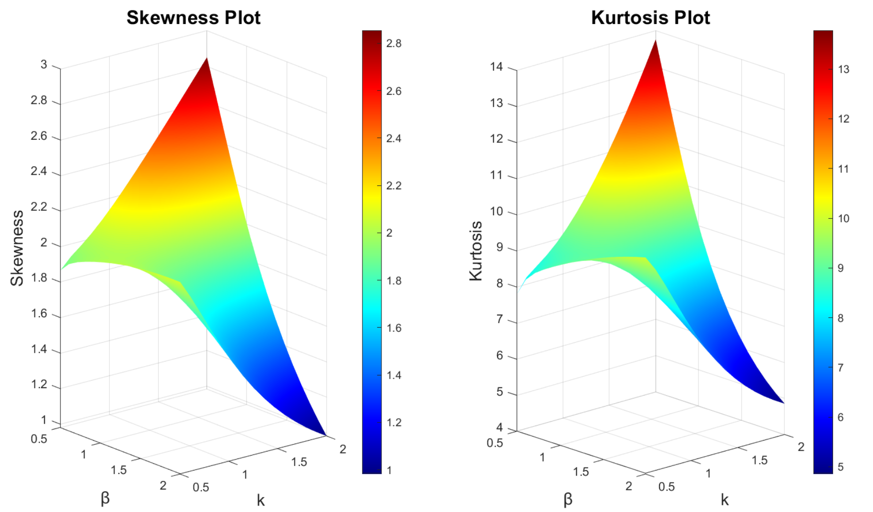

Figure 11: Skewness and Kurtosis plot for the L-EOR-E distribution. The parameters β and k were varied, while $\alpha = 1$, and $\gamma = 1$.

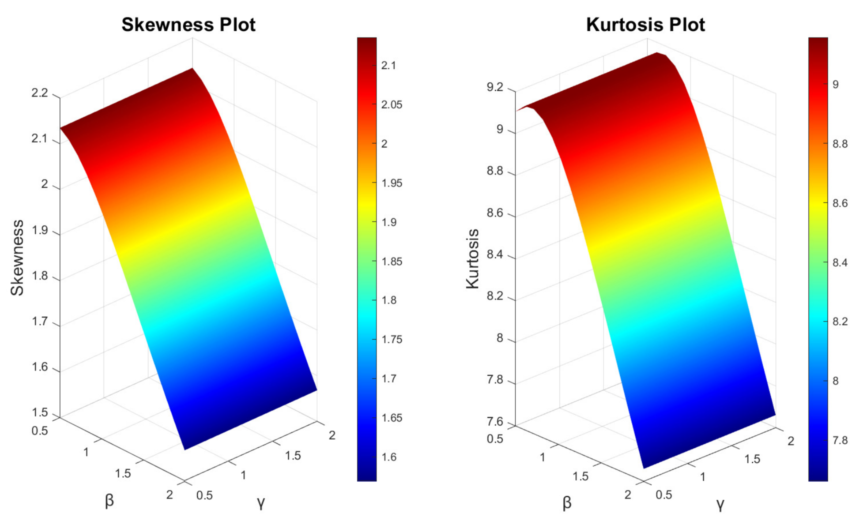

Figure 12: Skewness and Kurtosis plot for the L-EOR-E distribution. The parameters β and γ were varied, while $\alpha = 1$, and $k = 1$.

$$
\begin{aligned}
\frac{\partial \ell}{\partial \psi_s} &= \sum_{i=1}^{n} \frac{1}{g(x_i, \boldsymbol{\psi})} \frac{\partial g(x_i, \boldsymbol{\psi})}{\partial \psi_s} - (\beta - 1) \sum_{i=1}^{n} \frac{1}{G(x_i, \boldsymbol{\psi})} \frac{\partial \overline{G}(x_i, \boldsymbol{\psi})}{\partial \psi_s} \\
&\quad - (\beta + 1) \sum_{i=1}^{n} \frac{1}{\overline{G}(x_i, \boldsymbol{\psi})} \frac{\partial \overline{G}(x_i, \boldsymbol{\psi})}{\partial \psi_s} + \alpha\beta(k+1) \\
&\quad \times \sum_{i=1}^{n} \frac{[G(x_i, \boldsymbol{\psi})]^{\beta-1}}{[\overline{G}(x_i, \boldsymbol{\psi})]^{\beta+1} + \alpha \overline{G}(x_i, \boldsymbol{\psi}) [G(x_i, \boldsymbol{\psi})]^{\beta}} \frac{\partial \overline{G}(x_i, \boldsymbol{\psi})}{\partial \psi_s}
\end{aligned}
\tag{34}
$$

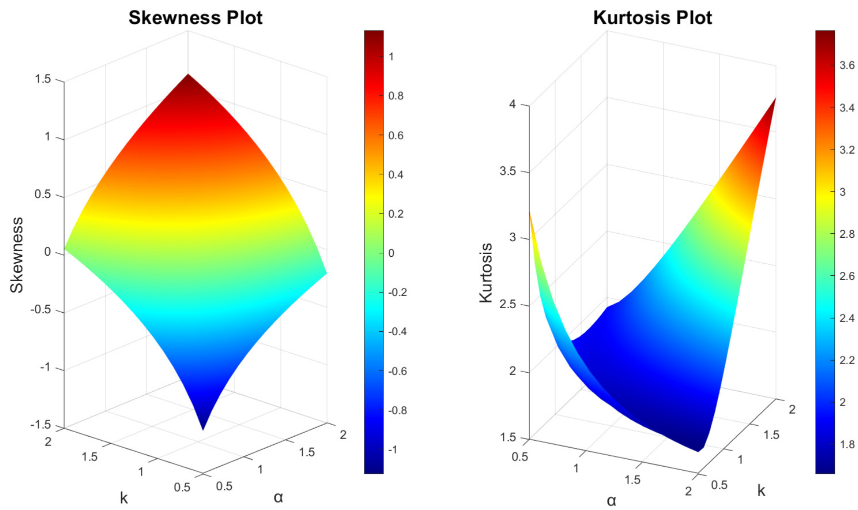

Figure 13: Skewness and Kurtosis plot for the L-EOR-K distribution. The parameters α and k were varied, while $\beta = 1$, $a = 1$, and $b = 1$.

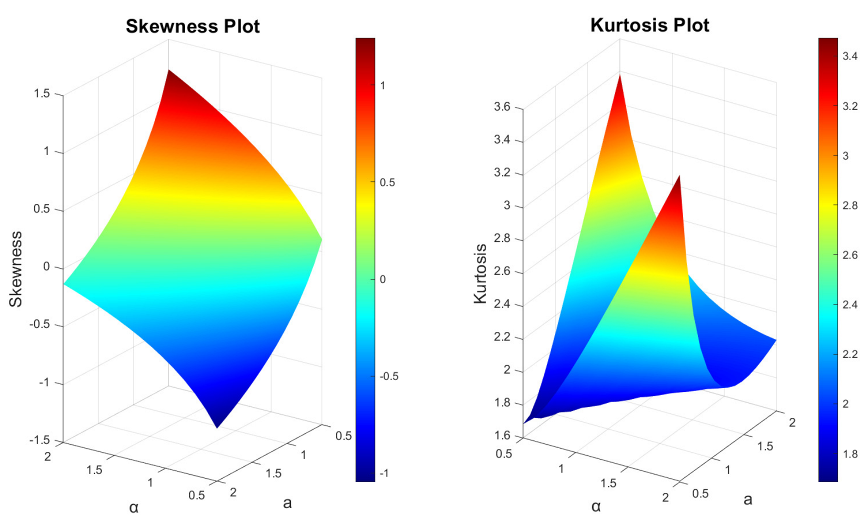

Figure 14: Skewness and Kurtosis plot for the L-EOR-K distribution. The parameters α and a were varied, while $\beta = 1$, $k = 1$, and $b = 1$.

5.2 Least Square and Weighted Least Square Estimation

Differentiating Least Square with respect to σ yields

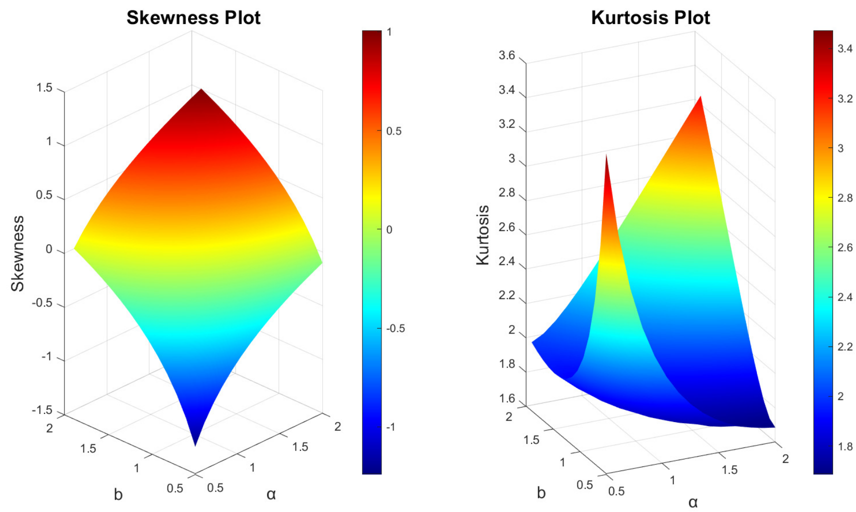

Figure 15: Skewness and Kurtosis plot for the L-EOR-K distribution. The parameters α and b were varied, while $\beta = 1$, $a = 1$, and $k = 1$.

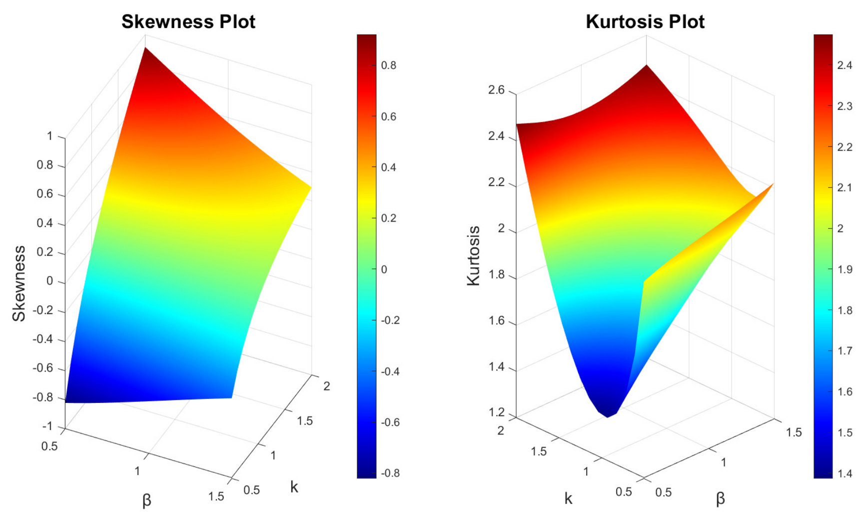

Figure 16: Skewness and Kurtosis plot for the L-EOR-K distribution. The parameters β and k were varied, while $\alpha = 1$, $a = 1$, and $b = 1$.

$$\frac{\partial LS}{\partial k} = 2\sum_{i=1}^{n}\left[1-\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k}-\frac{i}{n+1}\right]\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{\beta}\right)^{-k}$$
$$\times\log\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{\beta}\right) \tag{35}$$

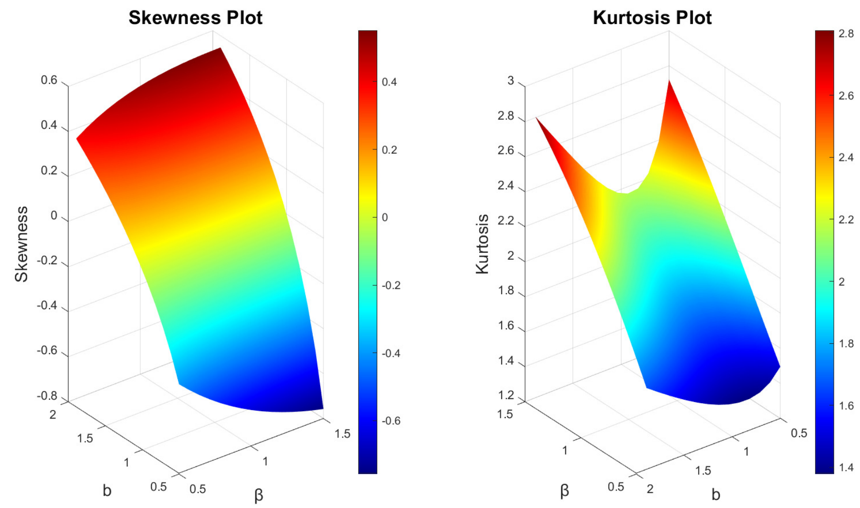

Figure 17: Skewness and Kurtosis plot for the L-EOR-K distribution. The parameters β and b were varied, while $\alpha = 1$, $a = 1$, and $k = 1$.

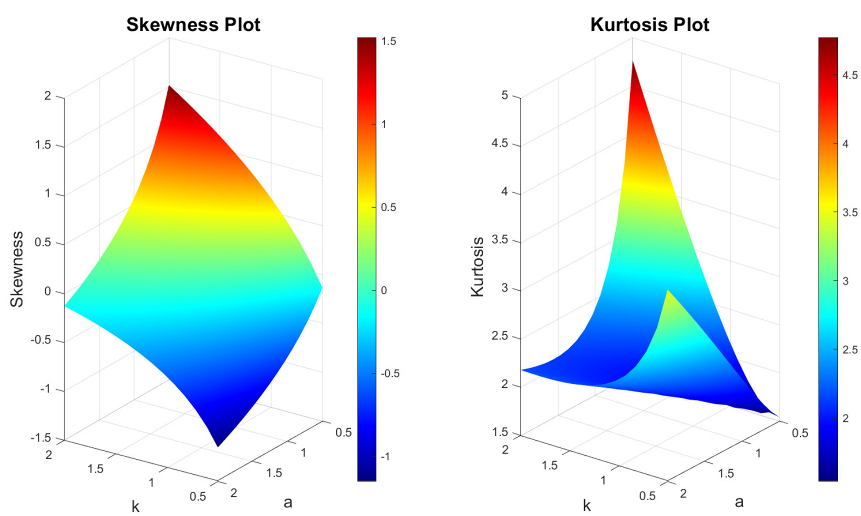

Figure 18: Skewness and Kurtosis plot for the L-EOR-K distribution. The parameters a and k were varied, while $\beta = 1$, $\alpha = 1$, and $b = 1$.

$$\frac{\partial LS}{\partial \alpha} = 2k \sum_{i=1}^{n} \left[1 - \left(1 + \alpha \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right]^{-\beta} \right)^{-k} - \frac{i}{n+1} \right] \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right]^{-\beta} \times \left(1 + \alpha \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right]^{-\beta} \right)^{-k-1} \tag{36}$$

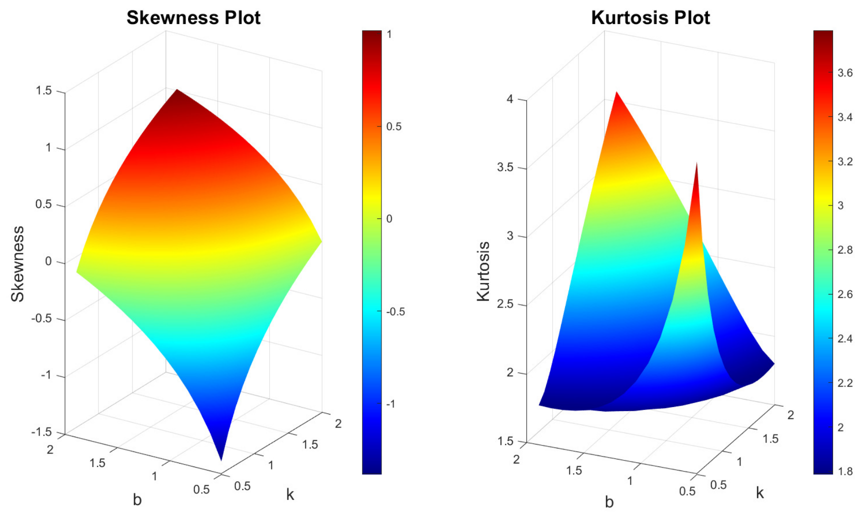

Figure 19: Skewness and Kurtosis plot for the L-EOR-K distribution. The parameters b and k were varied, while $\beta = 1$, $a = 1$, and $\alpha = 1$.

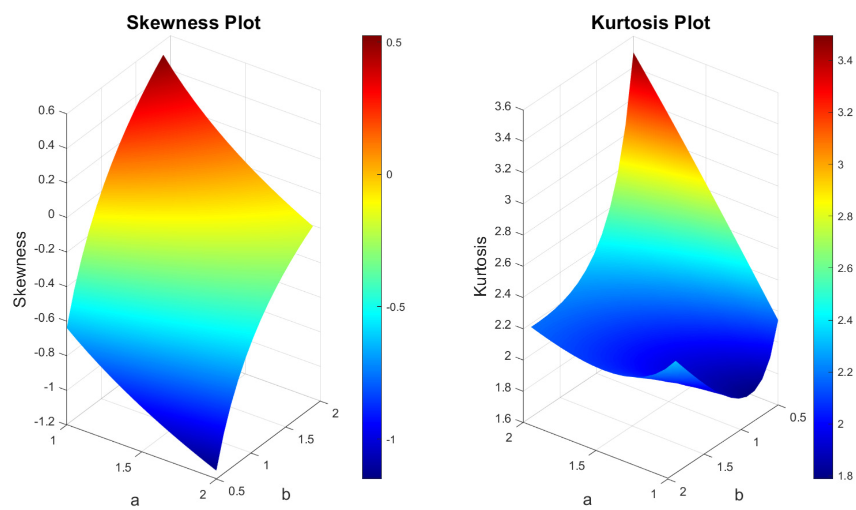

Figure 20: Skewness and Kurtosis plot for the L-EOR-K distribution. The parameters a and b were varied, while $\beta = 1$, $\alpha = 1$, and $k = 1$.

$$
\begin{aligned}
\frac{\partial LS}{\partial \beta} &= 2k\alpha \sum_{i=1}^{n} \left[1 - \left(1 + \alpha \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right]^{-\beta} \right)^{-k} - \frac{i}{n+1} \right] \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right]^{-\beta} \\
&\quad \times \left(1 + \alpha \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right]^{-} -\beta \right)^{-k-1} \log \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right] \\
&= \alpha \log \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right] \frac{\partial LS}{\partial \alpha}
\end{aligned} \tag{37}
$$

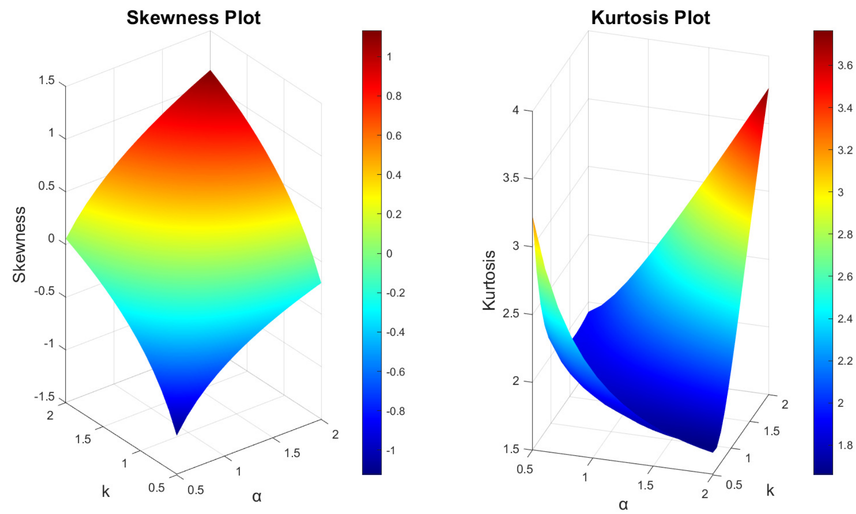

Figure 21: Skewness and Kurtosis plot for the L-EOR-U distribution. The parameters α and k were varied, while $\beta = 1$, and $\theta = 1$.

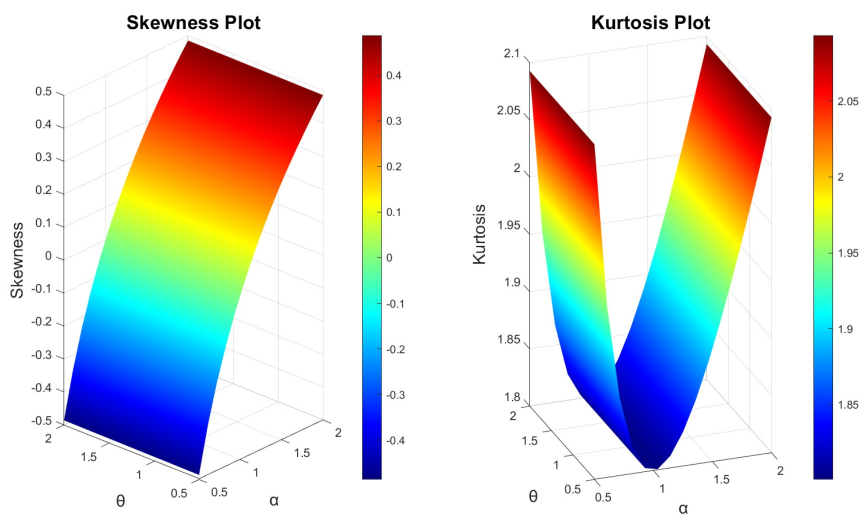

Figure 22: Skewness and Kurtosis plot for the L-EOR-U distribution. The parameters α and θ were varied, while $\beta = 1$, and $t = 1$.

$$\frac{\partial LS}{\partial \psi_s} = -2k\alpha - \beta \sum_{i=1}^{n} \left[1 - \left(1 + \alpha \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right]^{-\beta} \right)^{-k} - \frac{i}{n+1} \right] \frac{[\overline{G}(x_i, \boldsymbol{\psi})]^{-\beta-1}}{[G(x_i, \boldsymbol{\psi})]^{-\beta+1}}$$
$$\times \left(1 + \alpha \left[\frac{\overline{G}(x_i, \boldsymbol{\psi})}{G(x_i, \boldsymbol{\psi})} \right]^{-\beta} \right)^{-k-1} g(x_i, \boldsymbol{\psi}) \tag{38}$$

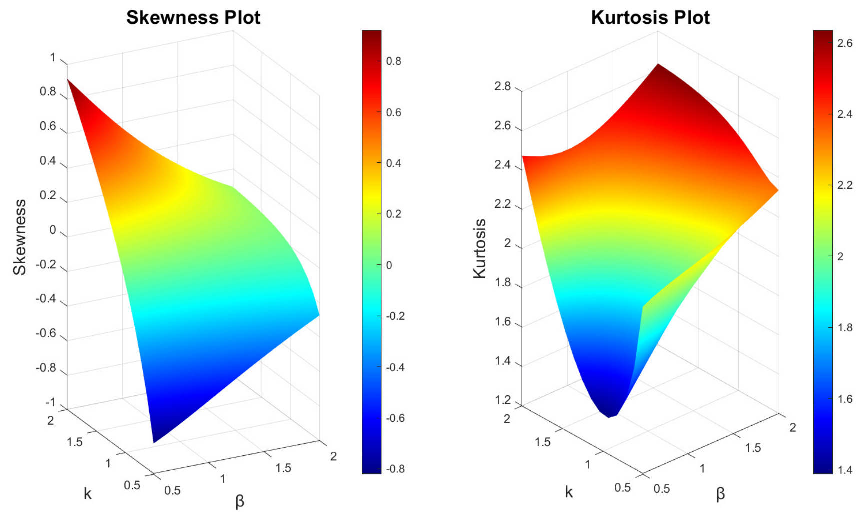

Figure 23: Skewness and Kurtosis plot for the L-EOR-U distribution. The parameters β and k were varied, while $\alpha = 1$, and $\theta = 1$.

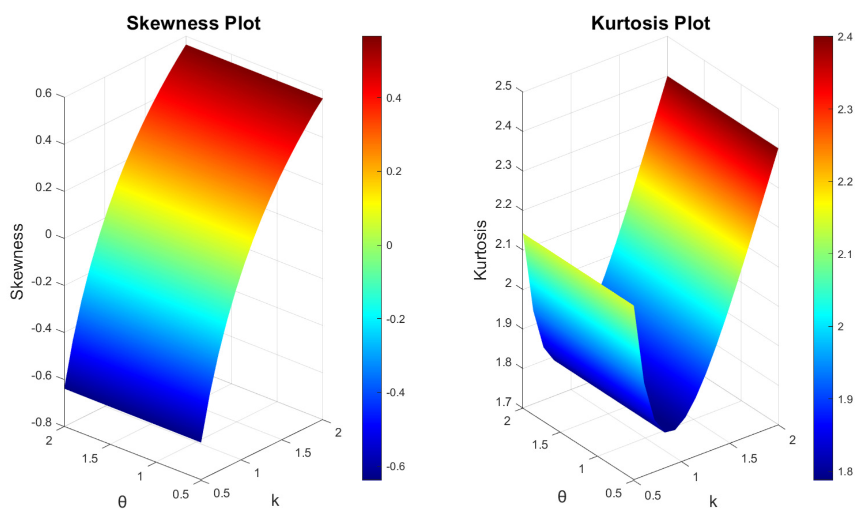

Figure 24: Skewness and Kurtosis plot for the L-EOR-U distribution. The parameters θ and k were varied, while $\beta = 1$, and $\alpha = 1$.

5.3 Maximum Product Spacing Approach of Estimation

Let $\delta_j = 1 + \alpha \left[\frac{\overline{G}(x_j, \boldsymbol{\psi})}{G(x_j, \boldsymbol{\psi})} \right]^{-\beta}$ for $j = 1, 2, \cdots, n$. The first partial derivatives are provided as follows:

$$\frac{\partial L}{\partial k} = \frac{1}{n+1}\left\{\frac{(\delta_1)^{-k}\log(\delta_1)}{1-(\delta_1)^{-k}} - \log(\delta_n) + \sum_{i=2}^{n}\frac{(\delta_i)^{-k}\log(\delta_i)-(\delta_{i-1})^{-k}\log(\delta_{i-1})}{(\delta_{i-1})^{-k}-(\delta_i)^{-k}}\right\} \tag{39}$$

$$\frac{\partial L}{\partial \alpha} = \frac{k}{n+1}\left\{\frac{(\delta_1)^{-k-1}\left[\frac{\overline{G}(x_1,\boldsymbol{\psi})}{G(x_1,\boldsymbol{\psi})}\right]^{-\beta}}{1-(\delta_1)^{-k}} - \frac{\left[\frac{\overline{G}(x_n,\boldsymbol{\psi})}{G(x_n,\boldsymbol{\psi})}\right]^{-\beta}}{\delta_n} + \sum_{i=2}^{n}\frac{(\delta_i)^{-k-1}\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}-(\delta_{i-1})^{-k-1}\left[\frac{\overline{G}(x_{i-1},\boldsymbol{\psi})}{G(x_{i-1},\boldsymbol{\psi})}\right]^{-\beta}}{(\delta_{i-1})^{-k}-(\delta_i)^{-k}}\right\} \tag{40}$$

$$\frac{\partial L}{\partial \beta} = \frac{k\alpha}{n+1}\left\{\frac{(\delta_1)^{-k-1}\left[\frac{\overline{G}(x_1,\boldsymbol{\psi})}{G(x_1,\boldsymbol{\psi})}\right]^{--\beta}\log\left[\frac{\overline{G}(x_1,\boldsymbol{\psi})}{G(x_1,\boldsymbol{\psi})}\right]}{1-(\delta_1)^{-k}} - \frac{\left[\frac{\overline{G}(x_n,\boldsymbol{\psi})}{G(x_n,\boldsymbol{\psi})}\right]^{-\beta}\log\left[\frac{\overline{G}(x_n,\boldsymbol{\psi})}{G(x_n,\boldsymbol{\psi})}\right]}{\delta_n} + \sum_{i=2}^{n}\frac{(\delta_i)^{-k-1}\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]-(\delta_{i-1})^{-k-1}\left[\frac{\overline{G}(x_{i-1},\boldsymbol{\psi})}{G(x_{i-1},\boldsymbol{\psi})}\right]^{-\beta}\log\left[\frac{\overline{G}(x_{i-1},\boldsymbol{\psi})}{G(x_{i-1},\boldsymbol{\psi})}\right]}{(\delta_{i-1})^{-k}-(\delta_i)^{-k}}\right\} \tag{41}$$

$$\frac{\partial L}{\partial \psi_s} = \frac{k\alpha\beta}{n+1}\left\{\frac{\frac{\left[\overline{G}(x_n,\boldsymbol{\psi})\right]^{-\beta-1}}{[G(x_n,\boldsymbol{\psi})]^{-\beta+1}}\frac{\partial G(x_n,\boldsymbol{\psi})}{\partial\psi_s}}{\delta_n} - \frac{(\delta_1)^{-k-1}\frac{\left[\overline{G}(x_1,\boldsymbol{\psi})\right]^{-\beta-1}}{[G(x_1,\boldsymbol{\psi})]^{-\beta+1}}\frac{\partial G(x_1,\boldsymbol{\psi})}{\partial\psi_s}}{1-(\delta_1)^{-k}} + \sum_{i=2}^{n}\frac{(\delta_{i-1})^{-k-1}\frac{\left[\overline{G}(x_{i-1},\boldsymbol{\psi})\right]^{-\beta-1}}{[G(x_{i-1},\boldsymbol{\psi})]^{-\beta+1}}\frac{\partial G(x_{i-1},\boldsymbol{\psi})}{\partial\psi_s}-(\delta_i)^{-k-1}\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^{-\beta-1}}{[G(x_i,\boldsymbol{\psi})]^{-\beta+1}}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial\psi_s}}{(\delta_{i-1})^{-k}-(\delta_i)^{-k}}\right\} \tag{42}$$

5.4 Cramer-von Mises Approach of Estimation

Differentiating Cramer-von Mises with respect to σ yields

$$\frac{\partial CVM}{\partial k} = \frac{2}{n}\sum_{i=1}^{n}\left[1-\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}-\frac{2i-1}{2n}\right]\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}$$
$$\times\log\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right) \tag{43}$$

$$\frac{\partial CVM}{\partial \alpha} = \frac{2k}{n}\sum_{i=1}^{n}\left[1-\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}-\frac{2i-1}{2n}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}$$
$$\times\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k-1} \tag{44}$$

$$\frac{\partial CVM}{\partial \beta} = \frac{2k\alpha}{n}\sum_{i=1}^{n}\left[1-\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}-\frac{2i-1}{2n}\right]\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}$$
$$\times\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k-1}\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]$$
$$=\alpha\log\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]\frac{\partial LS}{\partial \alpha} \tag{45}$$

$$\frac{\partial CVM}{\partial \psi_s} = \frac{-2k\alpha{-}\beta}{n}\sum_{i=1}^{n}\left[1-\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}-\frac{2i-1}{2n}\right]\frac{[\overline{G}(x_i,\boldsymbol{\psi})]^{-\beta-1}}{[G(x_i,\boldsymbol{\psi})]^{-\beta+1}}$$
$$\times\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k-1} g(x_i,\boldsymbol{\psi}) \tag{46}$$

5.5 Anderson and Darling Approach of Estimation

We similarly take the first derivatives of $AD(\sigma)$ and acquire

$$\frac{\partial AD}{\partial k} = \frac{-1}{n}\sum_{i=1}^{n}(2i-1)\log\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)$$
$$\times\left[1+\frac{\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}}{1-\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}}\right] \tag{47}$$

$$\frac{\partial AD}{\partial \alpha} = \frac{-k}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\left[\frac{1}{1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}}\right.$$
$$\left.+\frac{\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k-1}}{1-\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}}\right] \tag{48}$$

$$\frac{\partial AD}{\partial \beta} = \frac{-k\alpha}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\log\left(\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right)\left[\frac{1}{1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}}\right.$$
$$\left.+\frac{\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k-1}}{1-\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}}\right] \tag{49}$$

$$\frac{\partial AD}{\partial \psi_s} = \frac{k\alpha{-}\beta}{n}\sum_{i=1}^{n}(2i-1)\frac{\left[\overline{G}(x_i,\boldsymbol{\psi})\right]^{-\beta-1}}{\left[G(x_i,\boldsymbol{\psi})\right]^{-\beta+1}}\frac{\partial G(x_i,\boldsymbol{\psi})}{\partial \psi_s}\left[\frac{1}{1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}}\right.$$
$$\left.+\frac{\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k-1}}{1-\left(1+\alpha\left[\frac{\overline{G}(x_i,\boldsymbol{\psi})}{G(x_i,\boldsymbol{\psi})}\right]^{-\beta}\right)^{-k}}\right] \tag{50}$$